WEIGHTED INEQUALITIES FOR SUPERPOSITION OF OPERATORS

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In this paper we consider the superposition of three operators: Copson, Hardy and Tandori. Denote by $M^+(0,\infty)$ the set of all non-negative measurable functions on $(0,\infty)$.

Let $1 \le p < \infty$, $0 < q < \infty$, u, v and w are weights, (i.e. locally integrable non-negative functions on $(0,\infty)$), φ is strictly increasing function $(0,\infty)$, and $\frac{\varphi}{U}$ is decreasing on $(0,\infty)$, where

$$U(s) = \int_0^s u(t)dt$$

Our goal in this paper is to characterize the following inequality

$$\left(\int_0^\infty \left(\sup_{t<s<\infty} \frac{1}{\varphi(s)} \int_0^s \left(\int\limits_\tau^\infty h(y)dy\right) u(\tau)d\tau\right)^q w(t)dt\right)^{\frac{1}{q}} \le C\Big(\int\limits_0^\infty h^p(s)v(s)ds\Big)^{\frac{1}{p}} \tag{1}$$

for all $h \in M^+(0,\infty)$.

Using the Fubini theorem for non-negative functions, we have

$$\int\limits_0^s u(\tau)\int\limits_\tau^\infty h(t)dtdy \quad = \quad \int\limits_0^s U(\tau)h(\tau) + U(s)\int\limits_s^\infty h(\tau)d\tau.$$

Therefore the inequality (1) is equivalent with following inequality

$$\left(\int\limits_0^\infty \left(\sup_{t<s<\infty} \frac{1}{\varphi(s)} \left(\int\limits_0^s U(\tau)h(\tau)d\tau + U(s)\int\limits_s^\infty h(\tau)d\tau\right)\right)^q w(t)dt\right)^{\frac{1}{q}} \le C\left(\int\limits_0^\infty h^p(\tau)v(\tau)d\tau\right)^{\frac{1}{p}}. \tag{2}$$

Throughout the paper, we always denote by c or C a positive constant which is independent of the main parameters, but it may vary from line to line. However a constant with subscript such as $_1$ does not change in different occurrences.

Theorem 1 *Let $q \in (0,\infty)$, $p \in (1,\infty)$, and let u,v,w be weights on $(0,\infty)$. φ is U -quasiconcave function on $(0,\infty)$, Then there exists a constant $C>0$ such that the inequality (2) holds for all $f \in M^+(0,\infty)$ if and only if one of the following conditions is satisfied:*

(i) $1 < p \le q$,

$$C_1 := \sup_{t\in(0,\infty)} \left(\int_0^\infty \frac{w(s)ds}{(\varphi(s)+\varphi(t))^q}\right)^{\frac{1}{q}} \left(\int_0^\infty \frac{U^{p'}(t)U^{p'}(s)}{U^{p'}(s)+U^{p'}(t)} v^{1-p'}(s)ds\right)^{\frac{1}{p'}} < \infty,$$

(ii) $1 < p, q < p$,

$$C_2 := \left(\int_0^\infty \left(\int_0^\infty \frac{w(s)ds}{(\varphi(s)+\varphi(t))^q}\right)^{\frac{r}{q}} d\nu_p(t)\right)^{\frac{1}{r}} < \infty,$$

where ν_p is the representation measure of

$$\varphi^r(t) \sup_{s\in(t,\infty)} \frac{1}{\varphi^r(s)} \left(\int_0^\infty \frac{U^{p'}(\tau)U^{p'}(s)}{U^{p'}(s)+U^{p'}(\tau)} v^{1-p'}(\tau)d\tau\right)^{\frac{r}{p'}},$$

i.e.

$$\varphi^r(t) \sup_{s\in(t,\infty)} \frac{1}{\varphi^r(s)} \left(\int_0^\infty \frac{U^{p'}(\tau)U^{p'}(s)}{U^{p'}(s)+U^{p'}(\tau)} v^{1-p'}(\tau)d\tau\right)^{\frac{r}{p'}} = \int_0^\infty \frac{\varphi^r(t)}{\varphi^r(s)+\varphi^r(t)} \nu_p(s)ds.$$

Moreover, the best constant in the inequality (2) satisfies

$$C \approx \begin{cases} C_1 & \text{in case (i),} \\ C_2 & \text{in case (ii).} \end{cases}$$

The proof of Theorem (1) is essentially based on the following theorem.

Theorem 2 *Let $q \in (0,\infty)$, $p \in (1,\infty)$, and let u,v,w be weights on $(0,\infty)$. φ is U -quasiconcave function on $(0,\infty)$, Then there exists a constant $C>0$ such that the inequality (2) holds for all $f \in M^+(0,\infty)$ if and only if one of the following conditions is satisfied:*

(i) $1 < p \le q$,

$$A_1 := \sup_{k\in Z} \sup_{x_k\le t\le x_{k+1}} \frac{G(t)}{\varphi(t)} \left(\int_{x_k}^t U^{p'}(s)v^{1-p'}(s)ds + U^{p'}(t)\int_t^{x_{k+1}} v^{1-p'}(s)ds,\right)^{\frac{1}{p'}} < \infty,$$

(ii) $1 < p, q < p$,

$$A_2 := \left(\sum_{k\in Z} \left(\sup_{x_k\le t\le x_{k+1}} \frac{G(t)}{\varphi(t)} \left(\int_{x_k}^t U^{p'}(s)v^{1-p'}(s)ds + U^{p'}(t)\int_t^{x_{k+1}} v^{1-p'}(s)ds\right)^{\frac{1}{p'}}\right)^r\right)^{\frac{1}{r}} < \infty,$$

where $\{x_k\}_{k=N}^{M+1}$ is discretization sequence for G and

$$G(t) = \int_0^t w(s)ds + \varphi(t)\int_t^\infty \varphi^{-1}(s)w(s)ds.$$

Moreover, the best constant in the inequality (2) satisfies

$$C \approx \begin{cases} A_1 & \text{in case (i),} \\ A_2 & \text{in case (ii).} \end{cases}$$

We present the following definitions and auxiliary statements from [1; 2; 3; 4; 5] that are used to prove the indicated theorems.

Definition 1 *[2] Let φ be a continuous strictly increasing function on $[0,\infty)$ such that $\varphi(0)=0$ and $\lim_{t\to\infty}\varphi(t)=\infty$. Then we say that φ is admissible.*

Let φ be an admissible function. We say that a function h is φ-quasiconcave if h is equivalent to an increasing function on $[0,\infty)$ and $\frac{h}{\varphi}$ is equivalent to a decreasing function on $(0,\infty)$. We say that a φ-quasiconcave function h is non-degenerate if

$$\lim_{t\to 0+} h(t) = \lim_{t\to\infty}\frac{1}{h(t)} = \lim_{t\to\infty}\frac{h(t)}{\varphi(t)} = \lim_{t\to 0+}\frac{\varphi(t)}{h(t)} = 0. \tag{3}$$

The family of non-degenerate φ -quasiconcave functions will be denoted by Ω_φ.

Definition 2 *[3] Assume that φ is admissible and $h\in\Omega_\varphi$. We say that $\{\mu_k\}_{k\in Z}$ is a discretizing sequence for h with respect to φ if*

(i) $\mu_0=1$ and $\varphi(\mu_k)$;

(ii) $h(\mu_k)$ and $\frac{h(\mu_k)}{\varphi(\mu_k)}$;

(iii) there is a decomposition $Z=Z_1\cup Z_2$ such that $Z_1\cap Z_2=\emptyset$ an for every $t\in[\mu_k,\mu_{k+1}]$,

$$\begin{aligned} h(\mu_k) &\approx h(t) \quad \textit{if} \quad k\in Z_1, \\ \frac{h(\mu_k)}{\varphi(\mu_k)} &\approx \frac{h(t)}{\varphi(t)} \quad \textit{if} \quad k\in Z_2. \end{aligned}$$

Definition 3 *Let φ be an admissible function and let ν be a non-negative Borel measure on $[0,\infty)$. We say that the function h defined by*

$$h(t)=\varphi(t)\int_{[0,\infty)}\frac{d\nu(s)}{\varphi(s)+\varphi(t)},\;\; t\in(0,\infty),$$

is the fundamental function of the measure ν with respect to φ. We will also say that ν is a representation measure of h with respect to φ.

We say that ν is non-degenerate if the following conditions are satisfied for every $t\in(0,\infty)$:

$$\int_{[0,\infty)}\frac{d\nu(s)}{\varphi(s)+\varphi(t)}<\infty,\; t\in(0,\infty) \quad \textit{and} \quad \int_{[0,1]}\frac{d\nu(s)}{\varphi(s)}=\int_{[1,\infty)}d\nu(s)=\infty.$$

Lemma 1 *Assume that φ is an admissible function, $f\in\Omega_\varphi$, ν is a non-negative non-degenerate Borel measure on $[0,\infty)$ and h is the fundamental function of ν with respect to φ. If $\{x_k\}$ is a discretizing sequence for h with respect to φ, then*

$$\int_{[0,\infty)}\frac{f(t)}{\varphi(t)}d\nu(t)\approx\sum_{k\in Z}\left(\frac{f(x_k)}{\varphi(x_k)}\right)\varphi(x_k).$$

Lemma 2 *Assume that φ is an admissible function, ν is a nondegenerate nonnegative Borel measure on $[0,\infty)$, h is the fundamental function of ν and $f \in M^+(0,\infty)$. If $\{x_k\}$ is a discretizing sequence for h with respect to φ, then*

$$\int\limits_{[0,\infty)} \sup_{y\in(0,\infty)} \frac{|f(y)|}{\varphi(x)+\varphi(y)} d\nu(x) \approx \sum_{k\in Z} \left(\sup_{y\in(0,\infty)} \frac{|f(y)|}{\varphi(x_k)+\varphi(y)} \right) \varphi(x_k)$$

$$\approx \sum_{k\in Z} \left(\varphi^{-1}(x_k) \sup_{x_{k-1}\le y<x_k} |f(y)| + \sup_{x_k\le y<x_{k+1}} |f(y)|\varphi^{-1}(y) \right) h(x_k)$$

$$\approx \sum_{k\in Z} \sup_{x_k\le y<x_{k+1}} |f(y)|\varphi^{-1}(y)h(y).$$

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