Existence of solutions in boundary value problems for nonlinear equations involving q-analogs of fractional derivatives

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This study begins by presenting the key definitions and preliminary concepts, including the fundamental principles of q-calculus, which form the theoretical foundation of the work. Detailed discussions can be found in [1], [2] and [3]. This work operates under the standing assumption that $0 < q < 1$.

Let $\alpha \in R$. Then a q-real number $[\alpha]_q$ is defined by

$$[\alpha]_q = \frac{1-q^\alpha}{1-q},$$

where $\lim\limits_{q \to 1} \frac{1-q^\alpha}{1-q} = \alpha$.

We introduce for $k \in N$:

$$(a;q)_0 = 1,\ (a;q)_n = \prod_{k=0}^{n-1} \left(1 - q^k a\right),\ (a;q)_\infty = \lim_{n\to\infty} (a,q)_n,\ (a;q)_\alpha = \frac{(a;q)_\infty}{(q^\alpha a;q)_\infty}.$$

For any two real numbers α and β, we have

$$(a-b)_q^\alpha \left(a - q^\alpha b\right)_q^\beta = (a-b)_q^{\alpha+\beta}.$$

The gamma function $\Gamma_q(x)$ is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x},$$

for any $x > 0$. Moreover, it yields that

$$\Gamma_q(x)[x]_q = \Gamma_q(x+1).$$

The q-analogue differential operator $D_q f(x)$ is

$$D_q f(x) = \frac{f(x) - f(qx)}{x(1-q)},$$

and the q-derivatives $D_q^n(f(x))$ of higher order are defined inductively as follows:

$$D_q^0(f(x)) = f(x), \quad D_q^n(f(x)) = D_q\left(D_q^{n-1} f(x)\right), (n = 1, 2, 3, \dots).$$

The q-integral of a function f defined in the interval $[0,b]$ is defined by the expression

$$(I_q f)(x) = \int\limits_0^x f(x)\mathrm{d}_q x = x(1-q)\sum_{n=0}^{\infty} f\left(xq^n\right)q^n, \quad x \in [0,b].$$

The q-Beta function is defined for any $\alpha,\beta>0$ as follows:

$$B_q(\alpha,\beta) = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)} = \int\limits_0^1 x^{\alpha-1}(1-qx)_q^{\beta-1}d_q x.$$

Definition 1. [4]. Let $\alpha \geq 0$ and f be a function defined on $[0,1]$. The Riemann-Liouville fractional q-integral is expressed as $\left(I_q^\alpha f\right)(x)$, and are defined by

$$\left(I_{q,0+}^\alpha f\right)(x) = \frac{1}{\Gamma_q(\alpha)}\int\limits_0^x (x-qs)_q^{(\alpha-1)} f(s) d_q s.$$

Definition 2. The Riemann-Liouville fractional q-derivative of order $\alpha \geq 0$ is defined as $\left(D_q^\alpha f\right)(x)$, and

$$\left(D_q^\alpha f\right)(x) = \left(D_q^{[\alpha]} I_q^{[\alpha]-\alpha} f\right)(x), \quad \alpha>0,$$

where $[\alpha]$ is the smallest integer greater than or equal to α.

Lemma 1. [5]. Let $g(x) \in L[0,1]$ and $1<\alpha\leq 2$, then there exists a unique solution for

$$D_{q,0+}^\alpha y(x) + f(x,y(x)) = 0, \quad 0<x<1, \ \ 0<q<1, \tag{1}$$

$$y(0)=0, \quad D_{q,0+}^\beta y(1) = aD_{q,0+}^\beta y(\tau) + \lambda \tag{2}$$

is

$$y(x) = \int\limits_0^1 G_q(x,s) f(x,y(x))\mathrm{d}_q s + \lambda x^{\alpha-1}\frac{\Gamma_q(\alpha+\beta)}{\Gamma_q(\alpha)},$$

where

$$G_q(x,s) = \begin{cases} \frac{dx^{\alpha-1}(1-qs)_q^{(\alpha-\beta-1)} - adx^{\alpha-1}(\tau-qs)_q^{(\alpha-\beta-1)} - (x-qs)_q^{(\alpha-1)}}{\Gamma_q(\alpha)}, & 0\leq s\leq \min\{x,\tau\}<1, \\ \frac{dx^{\alpha-1}(1-qs)_q^{(\alpha-\beta-1)} - (x-qs)_q^{(\alpha-1)}}{\Gamma_q(\alpha)}, & 0\leq\tau\leq s\leq x\leq 1, \\ \frac{dx^{\alpha-1}(1-qs)_q^{(\alpha-\beta-1)} - adx^{\alpha-1}(\tau-qs)_q^{(\alpha-\beta-1)}}{\Gamma_q(\alpha)}, & 0\leq x\leq s\leq\tau\leq 1, \\ \frac{dx^{\alpha-1}(1-qs)_q^{(\alpha-\beta-1)}}{\Gamma_q(\alpha)}, & \max\{x,\tau\}\leq s\leq 1, \end{cases}$$

for which $d = \left(1-a\tau^{\alpha-\beta-1}\right)^{-1}$, $0\leq\beta\leq 1$, $0\leq a\leq 1$, $\lambda\geq\lambda_0>0$, $\tau\in(0,1)$, $a\tau^{\alpha-\beta-2}\leq 1-\beta$, $0\leq\alpha-\beta-1$ and $f:[0,1]\times[0,\infty)\to[0,\infty)$.

In this work, we assume that the function

$$f : [0,1] \times [0,\infty) \to [0,\infty)$$

the following Caratheodory conditions are satisfied

$(H1)$:

(i) For every fixed $t \in [0,\infty)$, the mapping $x \mapsto f(x,t)$ is Lebesgue measurable on $[0,1]$;

(ii) For every fixed $x \in [0,1]$, the mapping $t \mapsto f(x,t)$ is continuous on $[0,\infty)$.

Theorem 1. Assume that (H1) is satisfied and that there exists a real-valued function $h(x) \in L[0,1]$ such that

$$|f(x,y) - f(x,z)| \le h(x)|y - z|,$$

for almost every $x \in [0,1]$ and all $y, z \in [0,\infty)$. If

$$0 < \int_0^1 G_q(s,s)\, h(s)\, d_q s < 1,$$

then, the boundary value problem (BVP) defined by (1) and (2) on $[0,1]$ admits a unique positive solution.

Lemma 2. Assume that condition (H1) holds and that there exist two nonnegative, real-valued functions m and n in $L[0,1]$ such that

$$f(x,t) \le n(x) + m(x)t,$$

for almost all $x \in [0,1]$ and all $t \in [0,\infty)$. Hence, the operator $T : P \to P$, as defined in (Lemma 1), is completely continuous.

Theorem 2. Assume that all the conditions of lemma 2 are satisfied. If

$$\int_0^1 G_q(s,s) m(s)\, \mathrm{d}_q s < 1,$$

then the boundary value problem (1) and (2) admits at least one solution.

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