Schröder T-quasigroups

V.A. Shcherbacov

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Abstract

We prolong research of Schröder quasigroups and Schröder T-quasigroups.

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1 Introduction

This paper is connected in main with the following **Belousov problem 1 [2]** : Find a complete characterization of groups isotopic to quasigroups which satisfy one of the identities: $x(y \cdot yx) = y$, $xy \cdot yx = y$ (Stein 3rd law [15]), $xy \cdot yx = x$ (Schröder 2nd law [15]).

Some of these identities guarantee that a quasigroup is orthogonal to its parastrophe [2, 16].

Here we find conditions imposed not only on a group G over which various Schröder quasigroups are defined, but on the group of automorphisms $Aut\, G$, over which various T-objects (quasigroups and groupoids) are defined.

Necessary definitions can be found in [1, 4, 13, 16].

Definition 1. Binary groupoid $(Q, \circ)$ is called a left quasigroup if for any ordered pair $(a, b) \in Q^2$ there exist the unique solution $x \in Q$ to the equation $a \circ x = b$ [1].

Definition 2. Binary groupoid $(Q, \circ)$ is called a right quasigroup if for any ordered pair $(a, b) \in Q^2$ there exist the unique solution $y \in Q$ to the equation $y \circ a = b$ [1].

Definition 3. A quasigroup $(Q, \cdot)$ with an element $1 \in Q$, such that $1 \cdot x = x \cdot 1 = x$ for all $x \in Q$, is called a *loop*.

Definition 4. Binary groupoid $(Q, \cdot)$ is called medial if this groupoid satisfies the following medial identity:

$$xy \cdot uv = xu \cdot yv \tag{1}$$

for all $x, y, u, v \in Q$ [1].

We recall

Definition 5. Quasigroup $(Q, \cdot)$ is a T-quasigroup if and only if there exists an abelian group $(Q, +)$, its automorphisms φ and ψ and a fixed element $a \in Q$ such that $x \cdot y = \varphi x + \psi y + a$ for all $x, y \in Q$ [9].

A T-quasigroup with the additional condition $\varphi\psi = \psi\varphi$ is medial.

Definition 6. Garrett Birkhoff [3] has defined an equational quasigroup as an algebra with three binary operations $(Q, \cdot, /, \backslash)$ that satisfies the following six identities:

$$x \cdot (x\backslash y) = y, \tag{2}$$

$$(y/x) \cdot x = y, \tag{3}$$

$$x\backslash(x \cdot y) = y, \tag{4}$$

$$(y \cdot x)/x = y, \tag{5}$$

$$x/(y\backslash x) = y, \tag{6}$$

$$(x/y)\backslash x = y. \tag{7}$$

Ernst Schröder (a German mathematician mainly known for his work on algebraic logic) introduced and studied the following identity of generalized associativity [6]:

$$(y \cdot z)\backslash x = z(x \cdot y). \tag{8}$$

See also [7, 8, 10] for detail. Notice, article [10] is well written and very detailed.

In the quasigroup case the identity (8) is equivalent to the following identity [14, 16]:

$$(y \cdot z) \cdot (z \cdot (x \cdot y)) = x \tag{9}$$

If in idempotent quasigroup $(Q; \cdot)$ the identity (9) we put $x = y$, then we obtain the following standard Schröder identity:

$$(x \cdot y) \cdot (y \cdot x) = x. \tag{10}$$

Definition 7. Any quasigroup with the identity (10) is called a Schröder quasigroup.

So we have different objects that have name Schröder. Namely,

(i) the following identity of generalized associativity [6]:

$(y \cdot z)\backslash x = z(x \cdot y)$ (8);

(ii) the Schröder identity of generalized associativity in quasigroups: (9);

(iii) Schröder identity (Schröder 2-nd identity [15]) $(x \cdot y) \cdot (y \cdot x) = x$ (10);

Identity $(x \cdot y) \cdot (y \cdot x) = y$ is named by Albert Sade [15] as Stein 3-rd identity.

Each of these identities deserves a separate study in the class of groupoids, left (right) quasigroups, in the classes of quasigroups and of T-quasigroups.

1.1 Schröder identity of generalized associativity

It is convenient to call this identity the Schröder identity of generalized associativity. Often various variants of associative identity, which are true in a quasigroup, guarantee that this quasigroup is a loop.

It is not so in the case with the identity. We give an example of quasigroup which is not a loop with the identity (9) [14]. See also [16]. A quasigroup from this example does not have left and right identity element.

Quasigroups with Schröder identity of generalized associativity are not necessary idempotent and associative. See the following example.

Table 1

$\cdot$	0	1	2	3	4	5	6	7
0	1	4	7	0	6	5	2	3
1	5	2	3	6	0	1	4	7
2	0	7	4	1	5	6	3	2
3	6	3	2	5	1	0	7	4
4	4	1	0	7	3	2	5	6
5	3	6	5	2	4	7	0	1
6	7	0	1	4	2	3	6	5
7	2	5	6	3	7	4	1	0

The left cancellation (left division) groupoid with the identity (9) and with the identity ($x/x = y/y$) (in a quasigroup this identity guarantees existence of the left identity element) is a commutative group of exponent two [14].

The similar results are true for the right case [14]. In this case we use the identity ($x\backslash x = y\backslash y$).

It is clear that this result is true for any quasigroup with the left or right identity element.

Notice, any 2-group $(G, +)$ (in such group $x + x = 0$ for any $x \in G$) satisfies Schröder identity of generalized associativity.

1.2 Schröder identity $(x \cdot y) \cdot (y \cdot x) = x$

We recall information from [11].

Lemma 1. *In any Schröder quasigroup $(Q, \cdot)$ the equality $x \cdot x = y \cdot y$ implies $x = y$, and the equality $x \cdot y = y \cdot x$ implies $x = y$.*

Proof. Suppose $(x \cdot x) = (y \cdot y)$. Then from the identity (10) we have $x = (x \cdot x) \cdot (x \cdot x) = (y \cdot y) \cdot (y \cdot y) = y$.

Suppose $x \cdot y = y \cdot x$. Then we have $y = (x \cdot y) \cdot (y \cdot x) = (y \cdot x) \cdot (x \cdot y) = x$. □

Theorem 1. *Necessary condition for the existence of an idempotent Schröder quasigroup $(Q, \cdot)$ of order v is that $v = 0$ or $= 1$ pmod 4 [11].*

In the proofs of Theorems 1 are used sufficiently results from [17].

Example 1. Define groupoid $GF(2^r, *)$ over the Galois field $GF(2^r)$ in the following way:

$$x * y = a \cdot x + (a + 1)y, \tag{11}$$

where $x, y \in GF(2^r)$, the element a is a fixed element of the set $(GF(2^r))$, the operations $+$ and $\cdot$ are binary operations of this field.

The groupoid $(GF(2^r), *)$ is an idempotent medial Schröder quasigroup [11].

Example 2. In [11] it is mentioned that do not exist Schröder quasigroups $(Q, \cdot)$ of order 5. Using Mace 5 we construct quasigroups with Stein 3-rd law of order 5 (left loop and idempotent quasigroup).

$\star$	0	1	2	3	4
0	0	1	2	3	4
1	2	3	1	4	0
2	3	0	4	2	1
3	4	2	0	1	3
4	1	4	3	0	2

$\cdot$	0	1	2	3	4
0	0	2	3	4	1
1	3	1	4	2	0
2	4	0	2	1	3
3	1	4	0	3	2
4	2	3	1	0	4

We remark, in quasigroups from Schröder identity does not follow Stein 3-rd (quasigroup $(Q, *)$) and vice.

In quasigroup $(Q, *)$ is true identity (10) and is not true identity (1), in quasigroup $(Q, \circ)$ is true identity (1) and is not true identity (10).

$*$	0	1	2	3
0	0	1	3	2
1	2	3	1	0
2	1	0	2	3
3	3	2	0	1

$\circ$	0	1	2	3
0	0	2	3	1
1	3	1	0	2
2	1	3	2	0
3	2	0	1	3

1.3 Quasigroups with identity $x(y \cdot yx) = y$

Next Cayley tables of quasigroups with identity $x(y \cdot yx) = y$ were obtained using Mace [12].

*:	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

*:	0	1	2	3
0	0	2	3	1
1	1	3	2	0
2	2	0	1	3
3	3	1	0	2

*:	0	1	2	3	4
0	0	2	4	1	3
1	2	1	3	4	0
2	4	3	2	0	1
3	1	4	0	3	2
4	3	0	1	2	4

*:	0	1	2	3	4	5	6
0	0	2	3	1	5	6	4
1	4	1	6	0	3	2	5
2	5	0	2	4	6	1	3
3	6	5	0	3	1	4	2
4	1	6	5	2	4	3	0
5	2	3	4	6	0	5	1
6	3	4	1	5	2	0	6

*:	0	1	2	3	4	5	6	7
0	0	2	4	1	6	3	7	5
1	6	1	5	2	0	7	3	4
2	7	4	2	5	3	6	0	1
3	4	7	0	3	5	1	2	6
4	5	3	6	7	4	2	1	0
5	2	0	7	6	1	5	4	3
6	3	5	1	4	7	0	6	2
7	1	6	3	0	2	4	5	7

*:	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	2	0	1	8	6	4	5	3	7
2	1	2	0	7	5	6	4	8	3
3	4	5	6	0	3	7	8	1	2
4	3	7	8	4	0	1	2	5	6
5	8	3	7	6	2	0	1	4	5
6	7	8	3	5	1	2	0	6	4
7	6	4	5	2	8	3	7	0	1
8	5	6	4	1	7	8	3	2	0

*:	0	1	2	3	4	5	6	7	8	9	10
0	0	2	4	1	5	3	7	9	6	10	8
1	8	1	7	10	0	2	9	6	3	5	4
2	6	8	2	4	9	0	1	10	5	7	3
3	10	6	0	3	1	9	8	4	7	2	5
4	7	5	6	0	4	10	3	2	1	8	9
5	9	0	3	8	7	5	2	1	10	4	6
6	4	10	9	5	8	1	6	3	2	0	7
7	5	3	8	2	10	6	4	7	9	1	0
8	2	7	10	9	3	4	5	0	8	6	1
9	3	4	1	7	6	8	10	5	0	9	2
10	1	9	5	6	2	7	0	8	4	3	10

*:	0	1	2	3	4	5	6	7	8	9	10	11
0	1	0	2	3	4	7	6	5	8	11	10	9
1	0	1	3	2	5	4	7	6	9	8	11	10
2	2	3	0	1	6	5	4	7	10	9	8	11
3	3	2	1	0	7	6	5	4	11	10	9	8
4	4	5	6	7	8	9	10	11	0	1	2	3
5	7	4	5	6	10	8	11	9	2	0	3	1
6	6	7	4	5	9	11	8	10	1	3	0	2
7	5	6	7	4	11	10	9	8	3	2	1	0
8	8	9	10	11	0	2	1	3	4	5	6	7
9	11	8	9	10	1	0	3	2	5	6	7	4
10	10	11	8	9	2	3	0	1	6	7	4	5
11	9	10	11	8	3	1	2	0	7	4	5	6

We recall,

Lemma 2. *If two algebraic systems, say A and B satisfy an identity, then direct product of these systems $A \times B$ satisfy this identity [5, 16].*

Proof. It follows from Definition 6. $\square$

Lemma 3. *Quasigroups with identity $x(y \cdot yx) = y$ there exists for any n, where*

$$n \in \{3^{n_1} \cdot 4^{n_2} \cdot 5^{n_3} \cdot 7^{n_4} \cdot 8^{n_5} \cdot 11^{n_6} \cdot 23^{n_7}\}, \tag{12}$$

$n_i \in \mathbb{N}$, $i \in \overline{\{1, 7\}}$.

Proof. This follows from Lemma 2, given in this section examples and Example 5. $\square$

1.4 Stein 3-rd identity

Next Cayley tables of quasigroups with identity $xy \cdot yx = y$ were obtained using Mace [12].

*:	0	1	2	3
0	0	1	3	2
1	2	3	1	0
2	1	0	2	3
3	3	2	0	1

*:	0	1	2	3	4
0	0	1	2	3	4
1	2	3	1	4	0
2	3	0	4	2	1
3	4	2	0	1	3
4	1	4	3	0	2

*:	0	1	2	3	4	5
0	1	0	2	3	4	5
1	0	1	3	2	5	4
2	2	3	4	5	0	1
3	3	2	5	4	1	0
4	4	5	0	1	2	3
5	5	4	1	0	3	2

*:	0	1	2	3	4	5	6
0	1	0	2	3	4	6	5
1	0	1	3	2	5	4	6
2	2	3	0	6	1	5	4
3	3	2	4	5	6	0	1
4	5	6	1	4	0	2	3
5	4	5	6	1	2	3	0
6	6	4	5	0	3	1	2

*:	0	1	2	3	4	5	6	7
0	1	0	2	3	4	7	6	5
1	0	1	3	2	5	4	7	6
2	2	3	0	1	6	5	4	7
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	7	4	5	6	1	2	3	0
6	6	7	4	5	2	3	0	1
7	5	6	7	4	3	0	1	2

*:	0	1	2	3	4	5	6	7	8
0	1	0	2	3	4	8	7	6	5
1	0	1	3	2	5	4	8	7	6
2	2	3	0	1	6	5	4	8	7
3	3	2	1	0	7	6	5	4	8
4	4	5	6	7	8	1	0	3	2
5	8	4	5	6	0	7	3	2	1
6	7	8	4	5	1	2	6	0	3
7	6	7	8	4	2	3	1	5	0
8	5	6	7	8	3	0	2	1	4

*:	0	1	2	3	4	5	6	7	8	9
0	1	0	2	3	4	7	8	5	6	9
1	0	1	3	2	5	4	9	8	7	6
2	2	3	0	1	6	5	4	9	8	7
3	3	2	1	0	7	6	5	4	9	8
4	4	5	6	7	8	9	0	1	2	3
5	8	7	9	6	0	1	2	3	4	5
6	9	8	7	5	1	2	3	6	0	4
7	5	4	8	9	2	3	6	7	1	0
8	6	9	4	8	3	0	7	2	5	1
9	7	6	5	4	9	8	1	0	3	2

Lemma 4. *Quasigroups with identity $xy \cdot yx = y$ there exists for any n, where*

$$n \in \{4^{n_1} \cdot 5^{n_2} \cdot 6^{n_3} \cdot 7^{n_4} \cdot 8^{n_5} \cdot 9^{n_6} \cdot 10^{n_7} \cdot 13^{n_8} \cdot 17^{n_9} \cdot 29^{n_{10}}\}, \tag{13}$$

$n_i \in \mathbb{N}$, $i \in \overline{\{1, 10\}}$.

Proof. This follows from Lemma 2, given in this paragraph examples and Examples 9, 10, 11. □

2 Schröder identities in T-quasigroups

2.1 Schröder identity of generalized associativity in T-quasigroups

Theorem 2. *In T-quasigroup $(Q, \cdot)$ of the form $x \cdot y = \varphi x + \psi y$ Schröder identity of generalized associativity is true if and only if $\varphi x = \psi^{-2}x$, $\varepsilon = \varphi^7$, $\varepsilon = \psi^{14}$, $\varphi\psi z + \psi\varphi z = 0$.*

Proof. We rewrite identity (9) in the following form:

$$\varphi^2 y + \psi^3 y + \varphi\psi z + \psi\varphi z + \psi^2\varphi x = x. \tag{14}$$

If we substitute in equality (14) $y = z = 0$ then we have

$$\varphi x = \psi^{-2}x. \tag{15}$$

If we substitute in equality (14) $x = z = 0$ then we have

$$\varphi^2 y + \psi^3 y = 0. \tag{16}$$

Taking into consideration equality (15), we can re-write equality (16) in the form

$$\psi^{-4}y + \psi^3 y = 0, \tag{17}$$

or in the form

$$\psi^3 = I\psi^{-4}, \tag{18}$$

where $Ix = -x$ for all $x \in Q$. Notice, the permutation I is an automorphism of the group $(Q, +)$ here. Therefore, we can rewrite previous equalities in the form

$$\varepsilon = I\psi^{-7}, I = \psi^{-7}, \varepsilon = \psi^{-14}, \varepsilon = \psi^{14}, \varepsilon = \varphi^7. \tag{19}$$

If we substitute in equality (14) $x = y = 0$ then we have

$$\varphi\psi z + \psi\varphi z = 0. \tag{20}$$

Converse. If we substitute in identity (9) the expression $x \cdot y = \varphi x + \psi y$, then we obtain equality (14), which is true taking into consideration equalities (15), (16), (20). Then we obtain, that identity (9) is true in this case. □

Corollary 1. *In medial quasigroup $(Q, \cdot)$ of the form $x \cdot y = \varphi x + \psi y$ Schröder identity of generalized associativity is true if and only if the group $(Q, +)$ is an abelian 2-group (i.e. $x + x = 0$ for any $x \in Q$), $\varphi x = \psi^{-2}x$, $\varepsilon = \varphi^7$, $\varepsilon = \psi^{14}$.*

Proof. From the identity of mediality it follows that $\varphi\psi z + \psi\varphi z = 2 \cdot \varphi\psi z = 0$ for all $z \in Q$, i.e., the group $(Q, +)$ is an abelian 2-group. $\square$

Example 3. We start from the group $GL(3, 2) \cong PSL(2, 7)$. This is the group of non-degenerate matrices of size 3×3 over the field of order 2. Or the group of non-degenerate matrices of size 2×2 over the field of order 7. The order of this group is equal to 8, $|Aut(GL(3, 2)| = 168 = 3 \times 7 \times 8$.

We can present elements of the group $(Z_2^3, +)$ in the following form : $1 = (000), 2 = (001), 3 = (010), 4 = (011), 5 = (100), 6 = (101), 7 = (110), 8 = (111)$.

+	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	4	3	6	5	8	7
3	3	4	1	2	7	8	5	6
4	4	3	1	1	8	7	6	5
5	5	6	7	8	1	2	3	4
6	6	5	8	7	2	1	4	3
7	7	8	5	6	3	4	1	2
8	8	7	6	5	4	3	2	1

We have the following automorphisms of the group $(Z_2^3, +)$:

$$\varphi = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \psi = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Notice that $\varphi^7 = \psi^7 = \varepsilon$, $\varphi = \psi^{-2}$, $\varphi\psi = \psi\varphi$.

Therefore Schröder medial quasigroup $(Q, \circ)$ of generalized associativity can have the form $x \circ y = \varphi x + \psi y$:

$\circ$	1	2	3	4	5	6	7	8
1	1	4	8	5	3	2	6	7
2	3	2	6	7	1	4	8	5
3	6	7	3	2	8	5	1	4
4	8	5	1	4	6	7	3	2
5	7	6	2	3	5	8	4	1
6	5	8	4	1	7	6	2	3
7	4	1	5	8	2	3	7	6
8	2	3	7	6	4	1	5	8

2.2 Schröder identity

Theorem 3. *In T-quasigroup $(Q, \cdot)$ of the form $x \cdot y = \varphi x + \psi y$ Schröder identity is true if and only if $\varphi^2 + \psi^2 = \varepsilon$, $\varphi\psi y + \psi\varphi y = 0$.*

Proof. From identity

$$xy \cdot yx = x \tag{21}$$

we have

$$\varphi(\varphi x + \psi y) + \psi(\varphi y + \psi x) = x. \tag{22}$$

If in (22) $y = 0$, then

$$\varphi^2 + \psi^2 = \varepsilon. \tag{23}$$

If in (22) $x = 0$, then

$$\varphi\psi y + \psi\varphi y = 0. \tag{24}$$

Converse. If we substitute in identity (10) the expression $x \cdot y = \varphi x + \psi y$, then we obtain equality (22), which is true taking into consideration equalities (23), (24). Then we obtain, that identity (10) is true in this case. ☐

Corollary 2. *In medial quasigroup $(Q, \cdot)$ of the form $x \cdot y = \varphi x + \psi y$ Schröder identity is true if and only if $\varphi^2 + \psi^2 = \varepsilon$, the group $(Q, +)$ is an abelian 2-group (i.e., $x + x = 0$ for any $x \in Q$).*

Proof. This follows from mediality of quasigroup $(Q, \cdot)$. ☐

2.3 T-quasigroup with identity $x(y \cdot yx) = y$

Theorem 4. *In T-quasigroup $(Q, \cdot)$ of the form $x \cdot y = \varphi x + \psi y$ identity $x(y \cdot yx) = y$ is true if and only if $\varphi = I\psi^3$, $\psi^4 + \psi^5 = I$.*

Proof. From identity

$$x(y \cdot yx) = y \tag{25}$$

we have

$$\varphi x + \psi(\varphi y + \psi(\varphi y + \psi x) = y. \tag{26}$$

$$\varphi x + \psi\varphi y + \psi^2\varphi y + \psi^3 x = y. \tag{27}$$

If in (27) $y = 0$, then

$$\varphi + \psi^3 = 0, \tag{28}$$

$$\varphi = I\psi^3. \tag{29}$$

If in (27) $x = 0$, then

$$\psi\varphi + \psi^2\varphi = \varepsilon. \tag{30}$$

Taking in account (29) we have

$$\psi^4 + \psi^5 = I. \tag{31}$$

Converse. If we substitute in identity (25) the expression $x \cdot y = \varphi x + \psi y$, then we obtain equality (26), which is true taking into consideration equalities (35), (30). Then we obtain, that identity (25) is true in this case. ☐

Corollary 3. *In medial quasigroup $(Q, \cdot)$ of the form $x \cdot y = \varphi x + \psi y$ identity $x(y \cdot yx) = y$ is true if and only if $\varphi = I\psi^3$, $\psi^4 + \psi^5 = I$.*

Proof. This follows from mediality of quasigroup $(Q, \cdot)$. □

Example 4. Suppose we have the group Z_n of residues modulo n. If $\psi = -2$, then $\varphi = I\psi^3 = 8$, $\psi^4 + \psi^5 = 16 - 32 = -16 = -1$. The last is true if $-16 \equiv -1 \pmod{n}$, $15 \equiv 0 \pmod{n}$.

Then quasigroup $(Z_{15}, \circ)$ of the form $x \circ y = 8 \cdot x - 2 \cdot y \pmod{15}$ is medial quasigroup with identity $x(y \cdot yx) = y$.

Check. $8x - 2(8y - 2(8y - 2x)) = y \pmod{15}$, $8x - 16y + 32y - 8x = y \pmod{15}$, $16y = y \pmod{15}$, $y = y \pmod{15}$.

Example 5. Suppose we have the group Z_n of residues modulo n. If $\psi = -3$, then $\varphi = I\psi^3 = 27$, $\psi^4 + \psi^5 = 81 - 243 = -162 = -1$. The last is true if $-162 \equiv -1 \pmod{n}$, $161 \equiv 0 \pmod{n}$.

Quasigroup $(Z_{23}, \circ)$ of the form $x \circ y = 4 \cdot x - 3 \cdot y \pmod{23}$ is medial quasigroup with identity $x(y \cdot yx) = y$.

Check. $4x - 3(4y - 3(4y - 3x)) = y \pmod{23}$, $4x - 12y + 36y - 27x = y \pmod{23}$, $24y = y \pmod{23}$, $y = y \pmod{23}$.

Example 6. Quasigroup $(Z_{161}, \circ)$ of the form $x \circ y = 27 \cdot x - 3 \cdot y \pmod{161}$ is medial quasigroup with identity $x(y \cdot yx) = y$.

Check. $27x - 3(27y - 3(27y - 3x)) = y \pmod{161}$, $27x - 81y + 243y - 27x = y \pmod{161}$, $162y = y \pmod{161}$, $y = y \pmod{161}$.

2.4 T-quasigroups with Stein 3-rd identity

Theorem 5. *In T-quasigroup $(Q, \cdot)$ of the form $x \cdot y = \varphi x + \psi y$ Stein 3-rd identity is true if and only if $\varphi^2 + \psi^2 = 0$, $\varphi\psi y + \psi\varphi y = \varepsilon$.*

Proof. From identity

$$xy \cdot yx = y \tag{32}$$

we have

$$\varphi(\varphi x + \psi y) + \psi(\varphi y + \psi x) = y. \tag{33}$$

If in (33) $y = 0$, then

$$\varphi^2 + \psi^2 = 0. \tag{34}$$

$$\varphi^2 = I\psi^2. \tag{35}$$

If in (33) $x = 0$, then

$$\varphi\psi + \psi\varphi = \varepsilon. \tag{36}$$

Converse. If we substitute in identity (32) the expression $x \cdot y = \varphi x + \psi y$, then we obtain equality (33), which is true taking into consideration equalities (34), (36). Then we obtain, that identity (32) is true in this case. $\square$

Corollary 4. *In medial quasigroup $(Q, \cdot)$ of the form $x \cdot y = \varphi x + \psi y$ Stein 3-rd identity is true if and only if $\varphi^2 + \psi^2 = 0$, $2\varphi\psi = \varepsilon$.*

Proof. This follows from mediality of quasigroup $(Q, \cdot)$. $\square$

Example 7. Suppose we have the group Z_n of residues modulo n. If $\varphi = 1$, $\psi = 3$ then $\varphi^2 + \psi^2 = 1 + 9 = 0 \pmod 5, n = 5$. Further $2\varphi\psi = 2 \cdot 1 \cdot 3 = 6 = \varepsilon = 1 \pmod 5$, $x \cdot y = x + 3y \pmod 5$.

Check. $(x + 3y) + 3(y + 3x) = y \pmod 5$, $x + 3y + 3y + 9x = y \pmod 5$, $y = y \pmod 5$.

Example 8. Suppose we have the group Z_n of residues modulo n. If $\varphi = 4$, $\psi = 2$ then $\varphi^2 + \psi^2 = 16 + 4 = 0 \pmod 5, n = 5$. Further $2\varphi\psi = 2 \cdot 4 \cdot 2 = 16 = \varepsilon = 1 \pmod 5$, $x \cdot y = 4x + 2y \pmod 5$.

Check. $4(4x + 2y) + 2(4y + 2x) = y \pmod 5$, $16x + 8y + 8y + 4x = y \pmod 5$, $y = y \pmod 5$.

Example 9. Suppose we have the group Z_n of residues modulo n. If $\varphi = 2$, $\psi = 10$ then $\varphi^2 + \psi^2 = 4 + 100 = 104 = 0 \pmod{13}, n = 13$. Further $2\varphi\psi = 2 \cdot 2 \cdot 10 = 40 = \varepsilon = 1 \pmod{13}$, $x \cdot y = 2x + 10y \pmod{13}$.

Check. $2(2x + 10y) + 10(2y + 10x) = y \pmod{13}$, $4x + 20y + 20y + 100x = y \pmod{13}$, $y = y \pmod{13}$.

Example 10. Suppose we have the group Z_n of residues modulo n. If $\varphi = 6$, $\psi = 10$ then $\varphi^2 + \psi^2 = 36 + 100 = 136 = 0 \pmod{17}, n = 17$. Further $2\varphi\psi = 2 \cdot 6 \cdot 10 = 120 = \varepsilon = 1 \pmod{17}$, $x \cdot y = 6x + 10y \pmod{17}$.

Check. $6(6x + 10y) + 10(6y + 10x) = y \pmod{17}$, $36x + 60y + 60y + 100x = y \pmod{17}$, $y = y \pmod{17}$.

Example 11. Suppose we have the group Z_n of residues modulo n. If $\varphi = 8$, $\psi = 20$ then $\varphi^2 + \psi^2 = 64 + 400 = 464 = 0 \pmod{29}, n = 29$. Further $2\varphi\psi = 2 \cdot 8 \cdot 20 = 320 = \varepsilon = 1 \pmod{29}$, $x \cdot y = 8x + 20y \pmod{29}$.

Check. $8(8x + 20y) + 20(8y + 20x) = y \pmod{29}$, $64x + 160y + 160y + 400x = y \pmod{29}$, $y = y \pmod{65}$.

Example 12. Suppose we have the group Z_n of residues modulo n. If $\varphi = 3$, $\psi = 11$ then $\varphi^2 + \psi^2 = 9 + 121 = 130 = 0 \pmod{65}, n = 65$. Further $2\varphi\psi = 2 \cdot 3 \cdot 11 = 66 = \varepsilon = 1 \pmod{65}$, $x \cdot y = 3x + 11y \pmod{65}$.

Check. $3(3x + 11y) + 11(3y + 11x) = y \pmod{65}$, $9x + 33y + 33y + 121x = y \pmod{65}$, $y = y \pmod{65}$.

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Victor Shcherbacov
"Vladimir Andrunachievici"
Institute of Mathematics and Computer Science
5 Academiei str., Chişinău MD-2028
Moldova
E-mail: *scerb@math.md*