Homework 4

ISyE 6420
Spring 2021

1. Simple Metropolis. Let the posterior be proportional to a mixture:

$$g(\theta) = 0.6 \times \exp\left\{-\frac{\theta^2}{2}\right\} + 0.4 \times \frac{1}{2}\exp\left\{-\frac{(\theta-5)^2}{2\cdot 2^2}\right\}.$$

(a) Use the Metropolis-Hastings algorithm with normal $\mathcal{N}(0, 4^2)$ independence candidate density to sample from the posterior. Generate a sample of size $N = 10,500$ and plot the histogram of the simulated values minus 500 as the burnin.

(b) Repeat (a) with $\mathcal{N}(0, 0.4^2)$ and $\mathcal{N}(0, 40^2)$ candidate densities. By comparing histograms of simulated values, which one of the three candidate densities would you recommend?

Remark. This example is only to illustrate Metropolis algorithm, in this case "independence" Metropolis. Of course the posterior is simply $\frac{g(\theta)}{\sqrt{2\pi}}$.

2. Beta-Binomial via Gibbs. This is an example from Casella and George (1992). Assume that the posterior for parameter (θ, λ) given the observation X is proportional to

$$\pi(\theta, \lambda) \propto \binom{n}{\theta}\lambda^{\theta+\alpha-1}(1-\lambda)^{n-\theta+\beta-1},$$

where α and β depend on X.

The full conditionals are binomial and beta,

$$[\theta|X,\lambda] \sim \mathcal{B}in(n,\lambda), \quad [\lambda|X,\theta] \sim \mathcal{B}e(\alpha+\theta, \beta+n-\theta),$$

and allow for Gibbs sampling.

(a) Using suggested Gibbs sampler draw 5000 samples from the posterior (θ, λ). Use $n = 54, \alpha = 3.4$, and $\beta = 5.2$.

(b) The theoretical marginal posterior for $[\theta|X]$ is beta-binomial with pdf

$$\pi(\theta|X) = \binom{n}{\theta}\frac{B(\alpha+n, \beta+n-\theta)}{B(\alpha,\beta)},$$

where $B(u, v) = \int_0^1 t^{u-1}(1-t)^{v-1}dt$ is standard beta function. Compare how well the histogram of θ's from (a), fits the theoretical beta-binomial density $\pi(\theta|X)$.