ASSIGNMENT NO. 1

Course : Elasticity (MTH488)

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Question #4: For given matrix / vector pairs, compute a_{ii}, $a_{ij}a_{ij}$, ---- while $a_{ij}a_{jk}$ is product $[a][a]$.

a) $a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}$, $b_i = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$a_{ii} = a_{11} + a_{22} + a_{33} = 1+4+1 = 6$ (scalar)

$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$

$= 1+1+1+0+16+4+0+1+1 = 25 \Rightarrow$ scalar

$a_{ij}a_{jk} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 4 \\ 0 & 18 & 10 \\ 0 & 5 & 3 \end{bmatrix} \Rightarrow$ matrix

$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \Rightarrow$ vector

$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3 + a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3$

$= 1+2+4 = 7 \Rightarrow$ scalar

$b_ib_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix} \Rightarrow$ matrix

$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3 = 1+0+4 = 5 \Rightarrow$ scalar

b) $a_{ij} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}$, $b_i = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

$a_{ii} = a_{11} + a_{22} + a_{33} = 1+2+2 = 5 \Rightarrow$ scalar

$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$

$= 1+4+4+1+16+4 = 30 \Rightarrow$ scalar.

$a_{ij}a_{jk} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 2 \\ 0 & 8 & 4 \\ 0 & 16 & 8 \end{bmatrix} \Rightarrow$ matrix

$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 = \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix} \Rightarrow$ vector

$$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3 + a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3$$

$$= 4+4+2+1+4+2 = 17 \Rightarrow \text{scalar}$$

$$b_ib_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \Rightarrow \text{matrix}$$

$$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3 = 4+1+1 = 6 \rightarrow \text{scalar}$$

c) $$a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}, \quad b_i = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$a_{ii} = a_{11} + a_{22} + a_{33} = 1+0+4 = 5 \Rightarrow \text{scalar}$$

$$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} = 1+1+1+1+4+1+16 = 25 \Rightarrow \text{scalar}$$

$$a_{ij}a_{jk} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 7 \\ 1 & 3 & 9 \\ 1 & 4 & 18 \end{bmatrix} \Rightarrow \text{matrix}$$

$$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \rightarrow \text{vector}$$

$$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_1b_2 + a_{22}b_2b_2 + a_{23}b_2b_3 + a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3$$

$$= 1+1+1 = 3 \Rightarrow \text{scalar}$$

$$b_ib_j = \begin{bmatrix} b_1b_1 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2b_2 & b_2b_3 \\ b_3b_1 & b_3b_2 & b_3b_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{matrix}$$

$$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3 = 1+1+0 = 2 \Rightarrow \text{scalar}$$

Question 1-2: Use decomposition result (1.2.10) to express a_{ij} - - - - - - - - last pagraph of section 1.2.

a) $$a_{ij} = \tfrac{1}{2}(a_{ij} + a_{ji}) + \tfrac{1}{2}(a_{ij} - a_{ji})$$

$$= \tfrac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

clearly $a_{(ij)}$ & $a_{[ij]}$ satisfy appropriate conditions.

b) $a_{ij} = \frac{1}{2}(a_{ij}+a_{ji}) + \frac{1}{2}(a_{ij}-a_{ji})$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}$$

Clearly $a(ij)$ & $a[ij]$ satisfy appropriate conditions.

c) $a_{ij} = \frac{1}{2}(a_{ij}+a_{ji}) + \frac{1}{2}(a_{ij}-a_{ji})$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Clearly $a(ij)$ & $a[ij]$ satisfy appropriate conditions.

Question # 1-3: If a_{ij} is symmetric & b_{ij} is anti-- ------- --- terms from Excercise 1-2.

$a_{ij} b_{ij} = -a_{ji} b_{ji} = -a_{ij} b_{ij} \Rightarrow 2a_{ij}b_{ij} = 0 \Rightarrow a_{ij}b_{ij} = 0$

a) $a(ij)\, a[ij] = \frac{1}{4}\, tr\left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^T\right) = 0$

b) $a(ij)\, a[ij] = \frac{1}{4}\, tr\left(\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}^T\right) = 0$

c) $a(ij)\, a[ij] = \frac{1}{4}\, tr\left(\begin{bmatrix} 2 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 8 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}^T\right) = 0$

Question # 1-4: Verify following properties, explicitly.

a) $\delta_{ij} a_j = a_i$

$$\delta_{ij} a_j = \delta_{i1} a_1 + \delta_{i2} a_2 + \delta_{i3} a_3 = \begin{bmatrix} \delta_{11}a_1 + \delta_{12}a_2 + \delta_{13}a_3 \\ \delta_{21}a_1 + \delta_{22}a_2 + \delta_{23}a_3 \\ \delta_{31}a_1 + \delta_{32}a_2 + \delta_{33}a_3 \end{bmatrix}.$$

$$= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_i$$

b) $\delta_{ij} a_{jk} = a_{ij}$

$$\delta_{ij} a_{jk} = \begin{bmatrix} \delta_{11}a_{11}+\delta_{12}a_{21}+\delta_{13}a_{31} & \delta_{11}a_{12}+\delta_{12}a_{22}+\delta_{13}a_{32} & \delta_{11}a_{13}+\delta_{12}a_{23}+\delta_{13}a_{33} \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{ij} \text{ or } a_{ik}$$

Question # 1-5: Formally expand expression (1.3.4) ---

--------- traditional form for det $[a_{ij}]$.

$$\det(a_{ij}) = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k} = \varepsilon_{123} a_{11}a_{22}a_{33} + \varepsilon_{231} a_{12}a_{23}a_{31}$$
$$+ \varepsilon_{321} a_{13}a_{21}a_{32} + \varepsilon_{321} a_{13}a_{22}a_{31} + \varepsilon_{132} a_{11}a_{23}a_{32} +$$
$$\varepsilon_{213} a_{12}a_{21}a_{33}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} -$$
$$a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31})$$
$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Question # 1-6: Determine components of vector ..

------------- in Example 1-2.

$45°$ rotation about x_1-axis $\Rightarrow Q_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$

from Excercice 1-1 (a): $b_i' = Q_{ij} b_j$

$$b_i' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

$$a_{ij}' = Q_{ip} Q_{jp} Q_{pq}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T = \begin{bmatrix} 1 & \sqrt{2} & 0 \\ 0 & 4 & -1 \\ 0 & -2 & 1 \end{bmatrix}$$

b) $$b_i' = Q_{ij} b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ \sqrt{2} \\ 0 \end{bmatrix}$$

$$a_{ij}' = Q_{ip} Q_{jq} Q_{pq}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T = \begin{bmatrix} 1 & \sqrt{2} & -\sqrt{2} \\ 0 & 4.5 & -1.5 \\ 0 & 1.5 & -0.5 \end{bmatrix}$$

c) $$b_i' = Q_{ij} b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$$

$$a_{ij}' = Q_{ip} Q_{jq} Q_{pq}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2}/2 & 3.5 & 2.5 \\ -\sqrt{2}/2 & 1.5 & 0.5 \end{bmatrix}$$

Question 1-7: Consider two dimensional cordinate ... . Show that transformation matrix is given by.

$$Q_{ij} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad b_i = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad a_{ij} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}.$$

$$Q_{ij} = \begin{bmatrix} \cos(x_1', x_1) & \cos(x_1', x_2) \\ \cos(x_2', x_1) & \cos(x_2', x_2) \end{bmatrix} = \begin{bmatrix} \cos\theta & \cos(90-\theta) \\ \cos(90+\theta) & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$b_i' = Q_{ij} b_j = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1\cos\theta + b_2\sin\theta \\ -b_1\sin\theta + b_2\cos\theta \end{bmatrix}$$

$$a_{ij}' = Q_{ip} Q_{jq} a_{pq} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^T$$

$$= \begin{bmatrix} a_{11}\cos^2\theta + (a_{12}+a_{21})\sin\theta\cos\theta + a_{22}\sin^2\theta & a_{12}\cos^2\theta - (a_{11}-a_{22})\sin\theta\cos\theta - a_{21}\sin^2\theta \\ a_{21}\cos^2\theta - (a_{11}-a_{22})\sin\theta\cos\theta - a_{12}\sin^2\theta & a_{11}\sin^2\theta - (a_{12}+a_{21})\sin\theta\cos\theta + a_{22}\cos^2\theta \end{bmatrix}$$

Question 1-8: Show that second order ------ that an isotropic second order tensor,

$$a'\delta_{ij}' = Q_{ip} Q_{jq}\, a\,\delta_{pq} = a\, Q_{ip} Q_{jp} = a\,\delta_{ij}$$

Question 1-9: Show that second general form of a fourth order ---- general tranformation.

$$\alpha'\delta'_{ij}\delta'_{kl} + \beta'\delta'_{ik}\delta'_{jl} + \gamma'\delta'_{il}\delta'_{jk}$$

$$= Q_{im} Q_{jn} Q_{kp} Q_{lq} (\alpha\,\delta_{mn}\delta_{pq} + \beta\,\delta_{mp}\delta_{nq} + \gamma\,\delta_{mq}\delta_{np})$$

$$= \alpha\, Q_{im} Q_{jm} Q_{kp} Q_{lp} + \beta\, Q_{im} Q_{jn} Q_{km} Q_{ln} + \gamma\, Q_{im} Q_{jn} Q_{kn} Q_{lm} = \alpha\,\delta_{ij}\delta_{kl} + \beta\,\delta_{ik}\delta_{jl} + \gamma\,\delta_{il}\delta_{jk}$$

Question 1-10 For the fourth order tensor given in Ex 1-9 show that if $\beta = \gamma$, then the tensor will have the following symmetry $C_{ijkl} = C_{klij}$.

Proof

from Ex 1-9 we have

$$C_{ijkl} = \alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk}$$

using $\beta = \gamma$

$$C_{ijkl} = \alpha \delta_{ij} \delta_{kl} + \beta (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$
$$= \alpha \delta_{kl} \delta_{ij} + \beta (\delta_{ki} \delta_{lj} + \delta_{kj} \delta_{li})$$
$$= C_{klij}$$

Question 1-11 Show that the fundamental invariants can be expressed in terms of the principal values as given by relations (1.6.5)

sol

If $a = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

then

$$I_a = a_{ii} = \lambda_1 + \lambda_2 + \lambda_3$$

$$II_a = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} + \begin{vmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_3 \end{vmatrix}$$
$$= \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3$$

$$III_a = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix} = \lambda_1 \lambda_2 \lambda_3$$

Question No: 1-12 Determine the invariants & principal values & directions of the following matrices ....

(a) $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Sol Given that $a_{ij} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow \quad I_a = -1$

$II_a = -2 \quad \& \quad III_a = 0$

$\therefore$ characteristic Eq is $-\lambda^3 - \lambda^2 + 2\lambda = 0$

$\Rightarrow \lambda(\lambda^2 + \lambda - 2) = 0 \quad \Rightarrow \lambda(\lambda+2)(\lambda-1) = 0$

$\lambda_1 = -2, \quad \lambda_2 = 0, \quad \lambda_3 = 1$

For case 1 . when $\lambda_1 = -2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{bmatrix} = 0 \quad \Rightarrow \quad \begin{aligned} n_1^{(1)} + n_2^{(1)} &= 0 \\ n_3^{(1)} &= 0 \\ n_1^{(1)^2} + n_2^{(1)^2} + n_3^{(1)^2} &= 1 \end{aligned}$$

$n_1^{(1)} = -n_2^{(1)}$
$= \pm\sqrt{2}/2 \quad , n^{(1)} = \pm(\sqrt{2}/2)(-1, 1, 0)$

For $\lambda_2 = 0$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \quad \Rightarrow \quad \begin{aligned} -n_1^{(2)} + n_2^{(2)} &= 0 \\ n_3^{(2)} &= 0 \end{aligned}$$

$\Rightarrow \quad n_1 = n_2 = \pm\sqrt{2}/2 \quad \Rightarrow n^{(2)} = \pm(\sqrt{2}/2)(1, 1, 0)$

$n_1^{(2)^2} + n_2^{(2)^2} + n_3^{(2)^2} = 1$

For $\lambda_3 = 1$

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0$$

$\Rightarrow \quad -2n_1^{(3)} + n_2^{(3)} = 0 \qquad \Rightarrow \quad n_1 = n_2 = 0 \;,\; n_3^{(3)} = 1$

$n_1^{(3)} - 2n_2^{(3)} = 0 \qquad\qquad n^{(3)} = 1\,(0, 0, 1)$

$n_1^{(3)^2} + n_2^{(3)^2} + n_3^{(3)^2} = 1$

The rotation matrix is given by

$$Q_{ij}^{0} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \quad \&$$

$$a_{ij}' = Q_{ip} Q_{ip}\, a_{pq} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Sol $\quad a_{ij}^{0} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \; I_a = -4 \,,\; II_a = 3$

$III_a = 0$

$\therefore$ Characteristic eq is $-\lambda^3 - 4\lambda^2 - 3\lambda = 0$

$\Rightarrow \lambda(\lambda^2 + 4\lambda + 3) = 0$

$\Rightarrow \lambda(\lambda+3)(\lambda+1) = 0$

$\Rightarrow \lambda_1 = -3 \;,\; \lambda_2 = -1 \;\; \& \;\; \lambda_3 = 0$

For $\lambda_1 = -3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{bmatrix} = 0 \Rightarrow \begin{array}{l} n_1^{(1)} + n_2^{(1)} = 0 \\ n_3^{(1)} = 0 \end{array}$$

$\Rightarrow n_1^{(1)} = -n_2^{(1)} = \pm\sqrt{2}/2$, $n^{(1)} = \pm(\sqrt{2}/2)(-1, 1, 0)$

$n_1^{(1)2} + n_2^{(1)2} + n_3^{(1)2} = 1$

For $\lambda_2 = -1$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} -n_1^{(2)} + n_2^{(2)} = 0 \\ n_3^{(2)} = 0 \end{array}$$

$\Rightarrow n_1 = n_2 = \pm\sqrt{2}/2 \Rightarrow n^{(2)} = \pm\sqrt{2}/2 \; (1, 1, 0)$

$n_1^{(2)2} + n_2^{(2)2} + n_3^{(2)2} = 1$

For $\lambda_3 = 0$ case

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} -2n_1^{(3)} + n_2^{(3)} = 0 \\ n_1^{(3)} - 2n_2^{(3)} = 0 \Rightarrow n_1 = n_2 = 0 \\ n_3^{(3)} = 1 \Rightarrow n^{(3)} = \pm(0, 0, 1) \end{array}$$

$n_1^{(3)2} + n_2^{(3)2} + n_3^{(3)2} = 1$

The rotation matrix will be

$$Q_{ij} = \sqrt{2}/2 \begin{bmatrix} +1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \quad \&$$

$$a'_{ij} = Q_{ip} Q_{jp} a_{pq} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T$$

$$= \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For $\lambda_1 = -3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{bmatrix} = 0 \Rightarrow \begin{array}{l} n_1^{(1)} + n_2^{(1)} = 0 \\ n_3^{(1)} = 0 \end{array}$$

$$\Rightarrow n_1^{(1)} = -n_2^{(1)} = \pm \sqrt{2}/2 \quad , \quad n^{(1)} = \pm(\sqrt{2}/2)(-1, 1, 0)$$

$$n_1^{(1)2} + n_2^{(1)2} + n_3^{(1)2} = 1$$

For $\lambda_2 = -1$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} -n_1^{(2)} + n_2^{(2)} = 0 \\ n_3^{(2)} = 0 \end{array}$$

$$\Rightarrow n_1 = n_2 = \pm\sqrt{2}/2 \Rightarrow n^{(2)} = \pm\sqrt{2}/2 \; (1, 1, 0)$$

$$n_1^{(2)2} + n_2^{(2)2} + n_3^{(2)2} = 1$$

For $\lambda_3 = 0$ case

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow \begin{array}{l} -2n_1^{(3)} + n_2^{(3)} = 0 \\ n_1^{(3)} - 2n_2^{(3)} = 0 \Rightarrow n_1 = n_2 = 0 \\ n_3^{(3)} = 1 \Rightarrow n^{(3)} = \pm(0, 0, 1) \end{array}$$

$$n_1^{(3)2} + n_2^{(3)2} + n_3^{(3)2} = 1$$

The rotation matrix will be:

$$Q_{ij} = \sqrt{2}/2 \begin{bmatrix} +1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \quad \&$$

$$a'_{ij} = Q_{ip} Q_{jp} a_{pq} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T$$

$$= \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) $a_{ij} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow I_a = -2, \; II_a = 0, \; III_a = 0$

$\therefore$ characteristic Eq is $-\lambda^3 - 2\lambda^2 = 0$ or

$\lambda^2(\lambda+2) = 0$

$\Rightarrow \lambda_1 = -2, \; \lambda_2 = \lambda_3 = 0$

For $\lambda_1 = -2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{bmatrix} = 0$$

$\Rightarrow n_1^{(1)} + n_2^{(1)} = 0 \quad \Rightarrow \quad n_1^{(1)} = -n_2^{(1)} = \pm\sqrt{2}/2$

$n_3^{(1)} = 0 \quad\quad n^{(1)} = \pm\sqrt{2}/2\,(-1, 1, 0)$

$n_1^{(1)2} + n_2^{(1)2} + n_3^{(1)2} = 1$

For $\lambda_2 = \lambda_3 = 0$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \Rightarrow -n_1 + n_2 = 0$$

$\Rightarrow n_1 = n_2, \; n_3^2 = 1 - 2n_1^2$

$\Rightarrow n = \pm(k, k, \sqrt{1-2k^2})$

$\Rightarrow n_1^2 + n_2^2 + n_3^2 = 1$

for arbitrary k & thus directions are not uniquely determined. That's why choose $k = \sqrt{2}/2$ & 0 to get $n^{(2)} = \pm\sqrt{2}/2\,(1, 1, 0)$ & $n^{(3)} = \pm(0, 0, 1)$

The rotation matrix is:

$$Q_{ij} = \sqrt{2}/2 \begin{bmatrix} +1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \quad \&$$

$$a'_{ij} = Q_{ip} Q_{jq} a_{pq} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Question No: 1-14 Calculate the quantities $\nabla \cdot u$, $\nabla \times u$, $\nabla^2 u$, ∇u, $tr(\nabla u)$ for the following cartesian vector fields:

(a) $u = x_1 e_1 + x_1 x_2 e_2 + 2x_1 x_2 x_3 e_3$

$\nabla \cdot u = U_{1,1} + U_{2,2} + U_{3,3} = 1 + x_1 + 2x_1 x_2$

$$\nabla \times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1 & x_1 x_2 & 2x_1 x_2 x_3 \end{vmatrix} = 2x_1 x_3 e_1 - 2x_2 x_3 e_2 + x_2 e_3$$

$\nabla^2 u = 0e_1 + 0e_2 + 0e_3 = 0$

$$\nabla u = \begin{bmatrix} 1 & 0 & 0 \\ x_1 & x_2 & 0 \\ 2x_2 x_3 & 2x_1 x_3 & 2x_1 x_2 \end{bmatrix}$$

$tr(\nabla u) = 1 + x_1 + 2x_1 x_2$

(b) $u = x_1^2 e_1 + 2x_1 x_2 e_2 + x_3^3 e_3$

$\nabla \cdot u = U_{1,1} + U_{2,2} + U_{3,3} = 2x_1 + 2x_1 + 3x_3^2$

$$\nabla \times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1^2 & 2x_1 x_2 & x_3^3 \end{vmatrix} = 0e_1 - 0e_2 + 2x_2 e_3$$

$\nabla^2 u = 2e_1 + 0e_2 + 6x_3 e_3 = 0$

$$\nabla u = \begin{bmatrix} 2x_1 & 0 & 0 \\ 2x_2 & 2x_1 & 0 \\ 0 & 0 & 3x_3^2 \end{bmatrix}$$

$tr(\nabla u) = 4x_1 + 3x_3^2$

(c) $u = x_2^2 e_1 + 2x_2 x_3 e_2 + 4x_1^2 e_3$

$\nabla \cdot u = U_{1,1} + U_{2,2} + U_{3,3} = 0 + 2x_3 + 0$

$$\nabla \times u = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_2^2 & 2x_2x_3 & 4x_1^2 \end{vmatrix} = -2x_2 e_1 - 8x_1 e_2 - 2x_2 e_3$$

$\nabla^2 u = 2e_1 + 0e_2 + 8e_3 = 0$

$$\nabla u = \begin{bmatrix} 0 & 2x_2 & 0 \\ 0 & 2x_3 & 2x_2 \\ 8x_1 & 0 & 0 \end{bmatrix}$$

$tr(\nabla u) = 3x_3$

Question No: 1-15 The dual vector a_i of antisymmetric second order tensor a_{ij} is defined by $a_i = -1/2\, \varepsilon_{ijk} a_{jk}$. Show that this expression can be inverted to get $a_{jk} = -\varepsilon_{ijk} a_i$.

Sol

$a_i = -\frac{1}{2} \varepsilon_{ijk} a_{jk}$

$$\varepsilon_{imn} a_i = -\frac{1}{2} \varepsilon_{ijk} \varepsilon_{imn} a_{jk} = -\frac{1}{2} \begin{vmatrix} \delta_{ii} & \delta_{im} & \delta_{in} \\ \delta_{ji} & \delta_{jm} & \delta_{jn} \\ \delta_{ki} & \delta_{km} & \delta_{kn} \end{vmatrix} a_{jk}$$

$= -\frac{1}{2} (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) a_{jk}$

$= -\frac{1}{2} (a_{mn} - a_{nm}) = -\frac{1}{2} (a_{mn} + a_{mn})$

$= -a_{mn}$

$\therefore \quad a_{jk} = -\varepsilon_{ijk} a_i$

Question No: 1-16 Using index notation, explicitly verify the vector identities:

(a) (1.8.5)$_{1,2,3}$

$$\nabla(\phi\psi) = (\phi\psi)_{,k} = \phi\psi_{,k} + \phi_{,k}\psi = \nabla\phi\,\psi + \phi\nabla\psi$$

$$\nabla^2(\phi\psi) = (\phi\psi)_{,kk} = (\phi\psi_{,k} + \phi_{,k}\psi)_{,k}$$
$$= \phi\psi_{,kk} + \phi_{,k}\psi_{,k} + \phi_{,kk}\psi$$
$$= \phi_{,kk}\psi + \phi\psi_{,kk} + 2\phi_{,k}\psi_{,k}$$
$$= (\nabla^2\phi)\psi + \phi(\nabla^2\psi) + 2\nabla\phi\cdot\nabla\psi$$

$$\nabla\cdot(\phi u) = (\phi u_k)_{,k} = \phi u_{k,k} + \phi_{,k}u_k$$
$$= \nabla\phi\cdot u + \phi(\nabla\cdot u)$$

(b) $$\nabla\times(\phi u) = \varepsilon_{ijk}(\phi u_k)_{,j} = \varepsilon_{ijk}(\phi u_{k,j} + \phi_{,j}u_k)$$
$$= \varepsilon_{ijk}\phi_{,j}u_k + \phi\varepsilon_{ijk}u_k = \nabla\phi\times u + \phi(\nabla\times u)$$

$$\nabla\cdot(u\times v) = (\varepsilon_{ijk}u_jv_k)_{,i} = \varepsilon_{ijk}(u_jv_{k,i} + u_{j,i}v_k)$$
$$= v_k\varepsilon_{ijk}u_{j,i} + u_j\varepsilon_{ijk}v_{k,i}$$
$$= v\cdot(\nabla\times u) - u\cdot(\nabla\times v)$$

$$\nabla\times\nabla\phi = \varepsilon_{ijk}(\phi_{,k})_{,j} = \varepsilon_{ijk}\phi_{,kj} = 0$$ cz of symmetry & antisymmetry in jk

$$\nabla\cdot\nabla\phi = (\phi_{,k})_{,k} = \phi_{,kk} = \nabla^2\phi$$

(c) $$\nabla\cdot(\nabla\times u) = (\varepsilon_{ijk}u_{k,j})_{,i} = \varepsilon_{ijk}u_{k,ji} = 0$$ cz of symmetry & antisymmetry in ij

$$\nabla\times(\nabla\times u) = \varepsilon_{mni}(\varepsilon_{ijk}u_{k,j})_{,n} = \varepsilon_{imn}\varepsilon_{ijk}u_{k,jn}$$
$$= (\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj})u_{k,jn} = u_{n,nm} - u_{m,nn}$$
$$= \nabla(\nabla\cdot u) - \nabla^2 u$$

$$u\times(\nabla\times u) = \varepsilon_{ijk}u_j(\varepsilon_{kmn}u_{n,m}) = \varepsilon_{kij}\varepsilon_{kmn}u_ju_{n,m}$$
$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})u_ju_{n,m}$$
$$= u_nu_{n,i} - u_mu_{i,m}$$
$$= \frac{1}{2}\nabla(u\cdot u) - u\cdot\nabla u$$

Question No : 1-17 Extend the results found in example 1-5, & determine the forms of ∇f, $\nabla \cdot u$, $\nabla^2 f$, & $\nabla \times u$ for a 3-dimensional cylindrical coordinate system.

Sol Cylindrical coordinates are : $\xi^1 = r$, $\xi^2 = \theta$, $\xi^3 = z$

$(ds)^2 = (dr)^2 + (r d\theta)^2 + (dz)^2 \Rightarrow h_1 = 1, h_2 = r, h_3 = 1$

$\hat{e}_r = \cos\theta e_1 + \sin\theta e_2$, $\hat{e}_\theta = -\sin\theta e_1 + \cos\theta e_2$, $\hat{e}_z = e_3$

$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta, \quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r, \quad \frac{\partial \hat{e}_r}{\partial r} = \frac{\partial \hat{e}_\theta}{\partial r} = \frac{\partial \hat{e}_z}{\partial r} = \frac{\partial \hat{e}_z}{\partial \theta} = \frac{\partial \hat{e}_z}{\partial z} = 0$$

$$\nabla = \hat{e}_r \partial/\partial r + \hat{e}_\theta \frac{1}{r} \partial/\partial\theta + \hat{e}_z \partial/\partial z$$

$$\nabla f = \hat{e}_r \frac{\partial f}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{e}_z \frac{\partial f}{\partial z}$$

$$\nabla \cdot u = \frac{1}{r} \partial/\partial r\,(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla \times u = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right)\hat{e}_z$$

Question No: 1-18 For the spherical coordinate system (R, ϕ, θ) in figure 1-6, show that ....

Sol Spherical coordinates : $\xi^1 = R$, $\xi^2 = \phi$, $\xi^3 = \theta$

$x^1 = \xi^1 \sin\xi^2 \cos\xi^3$, $x^2 = \xi^1 \sin\xi^2 \sin\xi^3$, $x^3 = \xi^1 \cos\xi^2$

Scale factors :

$$(h_1)^2 = \frac{\partial x^k}{\partial \xi^1}\frac{\partial x^k}{\partial \xi^1} = (\sin\phi\cos\theta)^2 + (\sin\phi\sin\theta)^2 + \cos^2\phi = 1$$

$$(h_2)^2 = \frac{\partial x^k}{\partial \xi^2}\frac{\partial x^k}{\partial \xi^2} = R^2 \Rightarrow h_2 = R$$

$$(h_3)^2 = \frac{\partial x^k}{\partial \xi^3}\frac{\partial x^k}{\partial \xi^3} = R^2\sin^2\phi \Rightarrow h_3 = R\sin\phi$$

Unit vectors:

$$\hat{e}_R = \cos\theta\sin\phi e_1 + \sin\theta\sin\phi e_2 + \cos\phi e_3$$

$$\hat{e}_\phi = \cos\theta\cos\phi e_1 + \sin\theta\cos\phi e_2 - \sin\phi e_3$$

$$\hat{e}_\theta = -\sin\theta e_1 + \cos\theta e_2$$

$$\frac{\partial \hat{e}_R}{\partial R} = 0,\ \frac{\partial \hat{e}_R}{\partial \phi} = \hat{e}_\phi,\ \frac{\partial \hat{e}_R}{\partial \theta} = \sin\phi\, \hat{e}_\theta$$

$$\frac{\partial \hat{e}_\phi}{\partial R} = 0,\ \frac{\partial \hat{e}_\phi}{\partial \phi} = -\hat{e}_r,\ \frac{\partial \hat{e}_\phi}{\partial \theta} = \cos\phi\, \hat{e}_\theta$$

$$\frac{\partial \hat{e}_\theta}{\partial R} = 0,\ \frac{\partial \hat{e}_\theta}{\partial \phi} = 0,\ \frac{\partial \hat{e}_\theta}{\partial \theta} = -\cos\phi\, \hat{e}_\phi$$

Using (1.9.12) − (1.9.16) $\Rightarrow$

$$\nabla = \hat{e}_R \frac{\partial}{\partial R} + \hat{e}_\phi \frac{1}{R}\frac{\partial}{\partial \phi} + \hat{e}_\theta \frac{1}{R\sin\phi}\frac{\partial}{\partial \theta}$$

$$\nabla f = \hat{e}_R \frac{\partial f}{\partial R} + \hat{e}_\phi \frac{1}{R}\frac{\partial f}{\partial \phi} + \hat{e}_z \frac{1}{R\sin\phi}\frac{\partial f}{\partial \theta}$$

$$\nabla\cdot u = \frac{1}{R^2\sin\phi\,\partial R}\frac{\partial}{\partial R}(R^2\sin\phi u_R) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \phi}(R\sin\phi u_\phi) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial}$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 u_R) + \frac{1}{R\sin\phi}\frac{\partial}{\partial \phi}(\sin\phi u_\phi) + \frac{1}{R\sin\phi}\frac{\partial}{\partial \theta}(u_\theta)$$

$$\nabla^2 f = \frac{1}{R^2\sin\phi}\frac{\partial}{\partial R}\left(R^2\sin\phi \frac{\partial f}{\partial R}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial f}{\partial \phi}\right) +$$

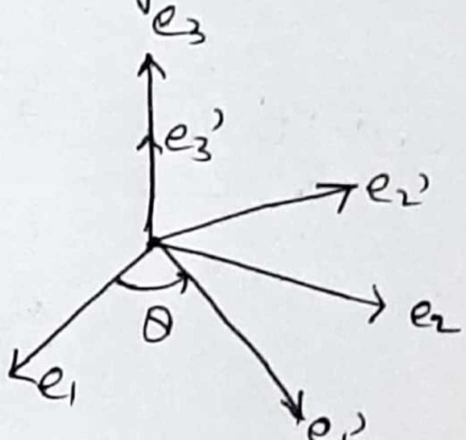
$$\frac{1}{R^2\sin\phi}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\phi}\frac{\partial f}{\partial \theta}\right)$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial f}{\partial R}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial f}{\partial \phi}\right) +$$

$$\frac{1}{R^2\sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}$$

Example Proof: Suppose the basis $\{e_1', e_2', e_3'\}$ is obtained by rotating basis $\{e_1, e_2, e_3\}$ through angle θ about unit vector e_3. Write out rule for 2-tensors explicitly.

$$e_1' = \cos\theta \, e_1 + \sin\theta \, e_2$$
$$e_2' = -\sin\theta \, e_1 + \cos\theta \, e_2$$
$$e_3' = e_3$$



$$[Q] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[A'] = [Q][A][Q^T]$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} A_{11}' & A_{12}' & A_{13}' \\ A_{21}' & A_{22}' & A_{23}' \\ A_{31}' & A_{32}' & A_{33}' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11}\cos\theta + A_{12}\sin\theta & -A_{11}\sin\theta + A_{12}\cos\theta & A_{13} \\ A_{21}\cos\theta + A_{22}\sin\theta & -A_{21}\sin\theta + A_{22}\cos\theta & A_{23} \\ A_{31}\cos\theta + A_{32}\sin\theta & -A_{31}\sin\theta + A_{32}\cos\theta & A_{33} \end{bmatrix}$$

$$R.H.S = \begin{bmatrix} A_{11}\cos^2\theta + A_{12}\sin\theta\cos\theta + A_{21}\cos\theta\sin\theta + A_{22}\sin^2\theta. & -A_{11}\sin\theta\cos\theta + A_{12}\cos^2\theta - A_{21}\sin^2\theta + A_{22}\cos\theta\sin\theta & A_{13}\cos\theta + A_{23}\sin\theta \\ -A_{11}\sin\theta\cos\theta - A_{21}\cos^2\theta + A_{22}\sin\theta\cos\theta. & A_{11}\sin^2\theta - A_{12}\cos\theta\sin\theta - A_{21}\sin\theta\cos\theta + A_{22}\cos^2\theta & \cdot A_{23}\cos\theta - A_{13}\sin\theta \\ A_{31}\cos\theta + A_{32}\sin\theta & + A_{32}\cos\theta - A_{31}\sin\theta & A_{33} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}\cos^2\theta + A_{22}\sin^2\theta + (A_{12}+A_{21})\sin\theta\cos\theta. & A_{12}\cos^2\theta - A_{21}\sin^2\theta + (A_{22}-A_{11})\cos\theta\sin\theta & A_{13}\cos\theta + A_{23}\sin\theta \\ A_{21}\cos^2\theta - A_{12}\sin^2\theta + (A_{22}-A_{11})\sin\theta\cos\theta. & A_{22}\cos^2\theta + A_{11}\sin^2\theta - (A_{12}+A_{21})\cos\theta\sin\theta & A_{23}\cos\theta - A_{13}\sin\theta \\ A_{31}\cos\theta + A_{32}\sin\theta. & A_{32}\cos\theta - A_{31}\sin\theta & A_{33} \end{bmatrix}$$

Using half-angle identities : $\sin^2\theta = \frac{1-\cos 2\theta}{2}$, $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, $\sin\theta\cos\theta = \frac{\sin 2\theta}{2}$

$$= \begin{bmatrix} \left(\frac{A_{11}+A_{22}}{2}\right) + \left(\frac{A_{11}-A_{22}}{2}\right)\cos 2\theta + \left(\frac{A_{12}+A_{21}}{2}\right)\sin 2\theta & \left(\frac{A_{12}-A_{21}}{2}\right) + \left(\frac{A_{12}+A_{21}}{2}\right)\cos 2\theta + \left(\frac{A_{22}-A_{11}}{2}\right)\sin 2\theta. & A_{13}\cos\theta + A_{23}\sin\theta \\ \left(\frac{A_{21}-A_{12}}{2}\right) + \left(\frac{A_{21}+A_{12}}{2}\right)\cos 2\theta + \left(\frac{A_{22}-A_{11}}{2}\right)\sin 2\theta. & \left(\frac{A_{22}+A_{11}}{2}\right) + \left(\frac{A_{22}-A_{11}}{2}\right)\cos 2\theta - \left(\frac{A_{12}+A_{21}}{2}\right)\sin 2\theta. & A_{23}\cos\theta - A_{13}\sin\theta \\ A_{31}\cos\theta + A_{32}\sin\theta. & A_{32}\cos\theta - A_{31}\sin\theta & A_{33} \end{bmatrix}$$

comparing both sides , we get

$$A_{11}' = \frac{A_{11}+A_{22}}{2} + \frac{A_{11}-A_{22}}{2}\cos 2\theta + \frac{A_{12}+A_{21}}{2}\sin 2\theta$$

$$A_{12}' = \frac{A_{12}-A_{21}}{2} + \frac{A_{12}+A_{21}}{2}\cos 2\theta + \frac{A_{22}-A_{11}}{2}\sin 2\theta$$

$$A_{13}' = A_{13}\cos\theta + A_{23}\sin\theta$$

$$A_{21}' = \frac{A_{21}-A_{12}}{2} + \frac{A_{21}+A_{12}}{2}\cos 2\theta + \frac{A_{22}-A_{11}}{2}\sin 2\theta$$

$$A_{22}' = \frac{A_{22}+A_{11}}{2} + \frac{A_{22}-A_{11}}{2}\cos 2\theta - \frac{A_{12}+A_{21}}{2}\sin 2\theta$$

$$A_{23}' = A_{23}\cos\theta - A_{13}\sin\theta$$

$$A_{31}' = A_{31}\cos\theta + A_{32}\sin\theta$$

$$A_{32}' = A_{32}\cos\theta - A_{31}\sin\theta.$$

$$A_{33}' = A_{33}$$

In the special case when $[A]$ is symmetric in addition $A_{13} = A_{23} = 0$, so nine equations simplify to

$$A_{11}' = \frac{A_{11}+A_{22}}{2} + \frac{A_{11}-A_{22}}{2}\cos 2\theta + A_{12}\sin 2\theta.$$

$$A_{22}' = \frac{A_{11}+A_{22}}{2} - \frac{A_{11}-A_{22}}{2}\cos 2\theta - A_{12}\sin 2\theta$$

$$A_{12}' = -\frac{A_{11}-A_{22}}{2}\sin 2\theta$$

together with $A_{13}' = A_{23}' = 0$ and $A_{33}' = A_{33}$. They are well known equations underlying the Mohr's circle for transforming 2- tensors in 2D.

Strain Displacment Relation :

$u = r\cos\theta$.

$\Rightarrow$ Cartesian Coordinates $\quad r = \sqrt{x^2+y^2}$, $v = r\sin\theta$.

$u = \sqrt{x^2+y^2}\cos\theta$, $v = \sqrt{x^2+y^2}\sin\theta$, $w = z$

$$e_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2}\cos\theta\right) = \frac{2x}{\sqrt{x^2+y^2}}\cos\theta.$$

$$e_y = \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(\sqrt{x^2+y^2}\sin\theta\right) = \frac{2y}{\sqrt{x^2+y^2}}\sin\theta$$

$$e_z = \frac{\partial w}{\partial z} = \frac{\partial}{\partial z}(z) = 1.$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left(\frac{2y}{\sqrt{x^2+y^2}}\cos\theta + \frac{2x}{\sqrt{x^2+y^2}}\sin\theta.\right)$$

$$= \frac{x\sin\theta + y\cos\theta}{\sqrt{x^2+y^2}}$$

$$e_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}(0+) = 0$$

$$e_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \frac{1}{2}(0+0) = 0.$$

$\Rightarrow$ Cylindrical Coordinates

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

$u_r = r\cos\theta$, $u_\theta = r\sin\theta$, $u_z = z$

$$e_r = \frac{\partial u_r}{\partial r} = \frac{\partial}{\partial r}(r\cos\theta) = \cos\theta.$$

$$e_\theta = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right) = \frac{1}{r}(r\cos\theta + r\cos\theta.) = 2\cos\theta.$$

$$e_z = \frac{\partial u_z}{\partial z} = \frac{\partial}{\partial z}(z) = 1$$

$$e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right).$$

$$= \frac{1}{2}\left[\frac{1}{\cancel{r}}(-\cancel{r}\sin\theta.) + \sin\theta. - \frac{\cancel{r}\sin\theta}{\cancel{r}}\right]$$

$$= \frac{1}{2}(-\sin\theta + \sin\theta - \sin\theta.) = -\frac{1}{2}\sin\theta.$$

$$e_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)$$

$$= \frac{1}{2}\left(0 + \frac{1}{r}0\right) = 0$$

$$e_{zr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) = 0$$

$\Rightarrow$ Spherical Coordinates:

$x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\phi$

suppose $R = \rho$

$u_R = R\cos\theta\sin\phi$, $u_\theta = R\sin\theta\sin\phi$, $u_\phi = R\cos\phi$.

$$e_R = \frac{\partial u_R}{\partial R} = \cos\theta\sin\phi$$

$$e_\phi = \frac{1}{R}\left(u_R + \frac{\partial u_\phi}{\partial \phi}\right) = \frac{1}{R}\left[R\cos\theta\sin\phi + (-R\sin\phi)\right]$$

$$= \sin\phi(\cos\theta - 1).$$

$$e_\theta = \frac{1}{R\sin\phi}\left(\frac{\partial u_\theta}{\partial \theta} + \sin\phi\, u_R + \cos\phi\, u_\phi\right)$$

$$= \frac{1}{R\sin\phi}\left[R\cos\theta\sin\phi + \sin\phi\, R\cos\theta\sin\phi + \cos\phi\, R\cos\phi\right]$$

$$= \frac{1}{\sin\phi}\left(\cos\theta\sin\phi + \sin^2\phi\cos\theta + \cos^2\phi\right).$$

$$= \cos\theta + \cos\theta\sin\phi + \frac{\cos^2\phi}{\sin\phi}.$$

$$= \cos\theta + \cos\theta\sin\phi + \cot\phi\cos\phi.$$