INTRODUCTION TO THE OPTIMAL CONTROL THEORY AND SOME APPLICATIONS

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Abstract. This paper aims to give a brief introduction to the optimal control theory and attempts to derive some of the central results of the subject, including the Hamilton-Jacobi-Bellman PDE and the Pontryagin Maximal Principle. Along the way, some of the more rigorous mathematical tools, such as Hamilton-Jacobi equations, viscosity solutions for PDEs, and the method of characteristics, will be introduced. Finally, some particular examples will be studied at the end of the paper using the developed theorems.

Contents

1. Introduction

We begin by defining control function, controlled dynamical systems, and the payoff functional of a controlled dynamical system.

Definition 1.1. A control function is a function $\alpha(t)$ of time that maps time $t \in [0, +\infty)$ to a set $\mathbf{A}$, where $\mathbf{A}$ is called the set of admissible controls.

Definition 1.2. A controlled dynamical system can be characterized by its dynamics

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \alpha(t), t)$$

and the initial state of the system

$$\mathbf{x}(0) = x_0$$

where $\mathbf{x} \in \mathbf{R}^n$, and the function $\mathbf{f}$ maps $\mathbf{R}^n \times \mathbf{A} \times [0, +\infty)$ to $\mathbf{R}^n$.

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A dynamical system is called autonomous if it doesn't explicitly depend on time, i.e. its dynamics can be written as

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \alpha(t)).$$

In this case, the function $\mathbf{f}$ maps $\mathbf{R}^n \times \mathbf{A}$ to $\mathbf{R}^n$.

Definition 1.3. The payoff functional of a dynamical system is of the form

$$\mathbf{P}[\alpha] = \int_0^T r(\mathbf{x}(t), \alpha(t))\,\mathrm{d}t + g(\mathbf{x}(T))$$

where the function $r(\mathbf{x}, \alpha)$ is associated with the "running payoff" of the system and $g(\mathbf{x})$ is related to the "terminal payoff" of the system.

The optimal control theory aims to solve the problem of finding a control for a certain autonomous dynamical system that will make the payoff functional of the system $\mathbf{P}[\alpha]$ attains its maximum value. Written out in mathematical notation, given a controlled dynamical system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \alpha(t)) \\ \mathbf{x}(0) = x_0 \end{cases}$$

we want to find such an $\alpha^*(t)$ so that for all possible control functions $\alpha(t)$, we have

$$\mathbf{P}[\alpha^*(t)] \geq \mathbf{P}[\alpha(t)].$$

It is worth noting that in practice, usually we will have more requirements on the regularity for $\mathbf{f}(x, a)$, $r(x, a)$, and $g(x)$, like being Lipschitz continuous.

Readers may better understand the concepts by viewing a toy example. Suppose there is a person originally at the origin of the number line, and this person will be awarded if he stays close to the origin by the end of 1 minute of time. If this person has maximal speed 1 meter per minute, then what is his best strategy to get awarded the most?

In mathematical language, the dynamical system will be

$$\begin{cases} \dot{\mathbf{x}}(t) = \alpha(t) \\ \mathbf{x}(0) = 0. \end{cases}$$

The admissible set of control is $\mathbf{A} = B_1(0)$; the running payoff function $r(x, a) = 0$; the terminal payoff function is $g(x) = -|x|$. Thus, the payoff functional is

$$\mathbf{P}[\alpha] = -|\mathbf{x}(1)|,$$

and we want to maximize this functional.

Therefore, in general terms, the key idea of optimal control is to find a function that maximizes a certain functional, i.e. an infinite dimensional optimization problem. Usually this type of problem can be resolved using a tool called the calculus of variations, so in Section 2 to Section 5, we will introduce the basic ideas in the calculus of variations, namely the Euler-Lagrange equations, the Hamilton equations, and the Hamilton-Jacobi equations. In Section 5 and 6, we will introduce a type of weak solutions for the Hamilton-Jacobi PDEs called "viscosity solutions" and then build the connection between viscosity solutions and optimal control theory and derive a sufficient condition for a system to have an optimal control. In Section 7, we will use a method called the method of characteristics to obtain necessary conditions for a control system to have optimal control, namely the Pontryagin

Maximum Principle. Finally, we will apply these results to solve a toy example of an optimal control problem.

2. Calculus of Variations and the Euler-Lagrange Equations

Suppose we are given a functional

$$\mathbf{I}[x(t)] = \int_0^T L(\dot{\mathbf{x}}(t), \mathbf{x}(t), t)\,\mathrm{d}t \tag{2.1}$$

where we assume $\mathbf{x}$ is a C^1 function that maps time $t \in [0, T]$ to $\mathbf{R}^n$, and $\mathbf{L}$ is called the *Lagrangian* which is a real-valued function of variables $\dot{\mathbf{x}}$, $\mathbf{x}$, and t, with $\dot{\mathbf{x}}(t)$ being the time derivative of $\mathbf{x}(t)$, i.e. $L(\dot{\mathbf{x}}, \mathbf{x}, t) := \mathbf{R}^n \times \mathbf{R}^n \times [0, T] \mapsto \mathbf{R}$. Note that $\mathbf{x}$ is the second variable that gets plugged into L and also a path.

If we want to find the extremal of a classical real value function with finite dimensional domain, one thing that we will usually do is to calculate the directional derivatives of that function and then find the point that makes all the directional derivatives be 0. In the infinite dimensional case, the procedure will be fairly similar, and that leads to the following theorem.

Theorem 2.2 (Euler-Lagrange Equations)**.** *Suppose L is C^2, and $\mathbf{x} : [0, T] \to \mathbf{R}^n$. Then if $\mathbf{x}(t)$ is an extremal of $\mathbf{I}(\cdot)$, we have the following equation*

$$-\frac{d}{dt}[\nabla_{\dot{x}} L(\dot{\mathbf{x}}(t), \mathbf{x}(t), t)] + \nabla_x L(\dot{\mathbf{x}}(t), \mathbf{x}(t), t) = 0.$$

Proof. Suppose we have a smooth test function $\phi(t) \in C^\infty$, where $\phi(0) = \phi(T) = 0$. Because $\mathbf{x}$ is a local extremal of $\mathbf{I}(\cdot)$, then if we perturb the original function $\mathbf{x}$ by a small function $k\phi$ with k small, the resulting functional $\mathbf{I}[\mathbf{x} + k\phi]$ should be roughly unchanged. In other words, we should have

$$\frac{d}{dk}[\mathbf{I}(\mathbf{x} + k\phi)]\bigg|_{k=0} = 0.$$

Now, we just compute the derivative, and by Leibniz's rule we may exchange the integration sign with the derivative sign. Thus, we have for each coordinate of $\mathbf{x}$

$$\int_0^T \frac{dL(\dot{\mathbf{x}}_i + k\dot{\phi}_i, \mathbf{x}_i + k\phi_i, t)}{dk}\,\mathrm{d}t = 0 \text{ for each i.}$$

Hence, by the fact that we evaluate at $k = 0$, we get

$$\int_0^T \frac{\partial L}{\partial \dot{\mathbf{x}}_i}\dot{\phi}_i + \frac{\partial L}{\partial \mathbf{x}_i}\phi_i\,\mathrm{d}t = 0.$$

Then, we can use the integration by parts formula and the fact that ϕ is smooth and vanishes at the boundary to get

$$0 = \frac{\partial L}{\partial \dot{\mathbf{x}}_i}\phi_i\bigg|_0^T - \int_0^T \frac{d}{dt}(\frac{\partial L}{\partial \dot{\mathbf{x}}_i})\phi_i\,\mathrm{d}t + \int_0^T \frac{\partial L}{\partial \mathbf{x}_i}\phi_i\,\mathrm{d}t.$$

Consequently,

$$\int_0^T \phi_i(-\frac{d}{dt}(\frac{\partial L}{\partial \dot{\mathbf{x}}_i}) + \frac{\partial L}{\partial \mathbf{x}_i})\,\mathrm{d}t = 0.$$

Also, by the fact that the choice of the test function is arbitrary, we have that for each coordinate

$$-\frac{d}{dt}(\frac{\partial L}{\partial \dot{\mathbf{x}}_i}) + \frac{\partial L}{\partial \mathbf{x}_i} = 0.$$

Finally, we have the Euler-Lagrange equation

$$-\frac{d}{dt}[\nabla_{\dot{x}} L(\dot{\mathbf{x}}(t), \mathbf{x}(t), t)] + \nabla_x L(\dot{\mathbf{x}}(t), \mathbf{x}(t), t) = 0.$$

□

We notice that the Euler-Lagrange equation is a second-order partial differential equation, which is somehow more challenging to handle compared with a system of ordinary differential equations. Therefore, it will be of our benefit to transform the Euler-Lagrange equation to a system of ordinary differential equations. In the next section we shall show how to obtain the desired system of first order ordinary differential equations.

3. Hamilton Equations

We first begin this section by defining the conjugate variable $\mathbf{p}(t)$ and the *Hamiltonian* $\mathbf{H}$.

Definition 3.1. The conjugate variable $\mathbf{p}(t)$ is defined to be

$$\mathbf{p}(t) = \nabla_{\dot{x}} L(\dot{\mathbf{x}}(t), \mathbf{x}(t), t).$$

From now on, we assume that the function $\dot{\mathbf{x}}(t)$ can be express as a function of $\mathbf{p}$, $\mathbf{x}$ and t. The rigorous proof of this fact can be found in Evans' book "Partial Differential Equations", and the intuition here is that $\dot{\mathbf{x}}$ is defined implicitly by $\mathbf{p}$, $\mathbf{x}$ and t.

Definition 3.2. The Hamiltonian $\mathbf{H}$ is defined to be

$$H(\mathbf{p}, \mathbf{x}, t) = \mathbf{p} \cdot \dot{\mathbf{x}} - L(\dot{\mathbf{x}}, \mathbf{x}, t)$$

with $\dot{\mathbf{x}}$ being a function of $\mathbf{p}$, $\mathbf{x}$ and t.

We have the following theorem for the following system of ordinary differential equations.

Theorem 3.3 (Hamilton Equations)**.** *Define the conjugate variable* $\mathbf{p}$ *and the Hamiltonian* $\mathbf{H}$ *as above. Then these two variables satisfy the following equations.*

$$\begin{cases} \dot{\mathbf{x}}(t) = \nabla_p H(\mathbf{p}(t), \mathbf{x}(t), t) \\ \dot{\mathbf{p}}(t) = -\nabla_x H(\mathbf{p}(t), \mathbf{x}(t), t). \end{cases}$$

Also, we have

$$\frac{d}{dt} H(\mathbf{p}(t), \mathbf{x}(t)) \equiv 0$$

if H doesn't explicitly depend on t.

Proof. We first compute $\nabla_p H$, and along a specific trajectory we get

$$\begin{aligned} \nabla_p H(\mathbf{p}(t), \mathbf{x}(t), t) &= \nabla_p [\mathbf{p}(t) \cdot \dot{\mathbf{x}}(p, x, t) - L(\dot{\mathbf{x}}(p, x, t), \mathbf{x}(t), t)] \\ &= \dot{\mathbf{x}}(p, x, t) + \mathbf{p}(t) \cdot \nabla_p \dot{\mathbf{x}}(p, x, t) - \nabla_{\dot{x}} L \cdot \nabla_p \dot{\mathbf{x}}(p, x, t). \end{aligned}$$

However, recall that $\mathbf{p}(t)$ is defined to be $\nabla_{\dot{x}} L$, so we have

$$\nabla_p H(\mathbf{p}(t), \mathbf{x}(t), t) = \dot{\mathbf{x}}(p, x, t) = \dot{\mathbf{x}}(t).$$

Then we compute $\nabla_x H$, which yields

$$\begin{aligned}\nabla_x H(\mathbf{p}(t), \mathbf{x}(t), t) &= \nabla_x[\mathbf{p}(t) \cdot \dot{\mathbf{x}}(p, x, t) - L(\dot{\mathbf{x}}(p, x, t), \mathbf{x}(t), t)] \\ &= \mathbf{p}(t) \cdot \nabla_x \dot{\mathbf{x}}(p, x, t) - \nabla_{\dot{x}} L \cdot \nabla_x \dot{\mathbf{x}}(p, x, t) - \nabla_x L \\ &= -\nabla_x L.\end{aligned}$$

To complete the proof, it suffices to show that $\dot{\mathbf{p}}(t) = \nabla_x L$, and this fact follows immediately from the Euler-Lagrange Equations

$$-\frac{d}{dt}\nabla_{\dot{x}} L + \nabla_x L = 0$$

because $\dot{\mathbf{p}}(t) = \frac{d}{dt}\nabla_{\dot{x}} L$.

Finally, if $\mathbf{H}$ doesn't explicitly depend on t, we have

$$\begin{aligned}\frac{d}{dt} H(\mathbf{p}(t), \mathbf{x}(t)) &= \nabla_p H \cdot \dot{\mathbf{p}}(t) + \nabla_x H \cdot \dot{\mathbf{x}}(t) \\ &= -\nabla_p H \cdot \nabla_x H + \nabla_x H \cdot \nabla_p H \\ &= 0\end{aligned}$$

as desired. □

4. Hamilton-Jacobi Equations

Now after we have obtained the Hamilton Equations, we still want to know if there is an even simpler treatment for these types of problems. For example, if we can somehow get a system to have time invariant $\mathbf{x}$ and $\mathbf{p}$, then the problem will become far simpler. In this section we will introduce the notions of canonical transformations and Hamilton-Jacobi Equations. The key idea is to find a canonical transformation that makes the new corresponding Hamilton system particularly simple. Then after we solve the Hamilton equations in the new system, we can find the solutions for the Hamilton equations in the original system by doing the inverse canonical transformation.

Definition 4.1. A canonical transformation is a transformation of the form

$$\mathbf{X} = \mathbf{X}(\mathbf{p}, \mathbf{x}, t)$$

and

$$\mathbf{P} = \mathbf{P}(\mathbf{p}, \mathbf{x}, t)$$

such that these two new variables ($\mathbf{X}$ and $\mathbf{K}$) correspond to a new Hamiltonian $\mathbf{K} = K(\mathbf{P}(t), \mathbf{X}(t), t)$, and still satisfy the relationship

$$\begin{cases}\dot{\mathbf{X}}(t) = \nabla_P K(\mathbf{P}(t), \mathbf{X}(t), t) \\ \dot{\mathbf{P}}(t) = -\nabla_X K(\mathbf{P}(t), \mathbf{X}(t), t).\end{cases}$$

In other words, we did a coordinate transformation in the phase space that will preserve the Hamilton structure in the original system, i.e. the new trajectories $\mathbf{X}(t)$ and $\mathbf{P}(t)$ solve the Hamilton equations in the new coordinates if and only if the original trajectories $\mathbf{x}(t)$ and $\mathbf{p}(t)$ solve the Hamilton equations in the original coordinates.

Now we prove a lemma concerning about the relationship between the *Hamiltonian* and the *Lagrangian*.

Theorem 4.2 (Legendre transform)**.** *We can obtain the Lagrangian* $\mathbf{L}$ *from Hamiltonian* $\mathbf{H}$ *by doing the Legendre transform, i.e.*

$$L(\dot{\mathbf{x}}, \mathbf{x}, t) = \mathbf{p} \cdot \dot{\mathbf{x}} - H(\mathbf{p}, \mathbf{x}, t)$$

Proof. By Theorem 3.3, we have

$$\dot{\mathbf{x}}(t) = \nabla_p H(\mathbf{p}(t), \mathbf{x}(t), t)$$

so $\dot{\mathbf{x}}(t)$ can be obtained if H and $\mathbf{p}$ are given; then by Definition 3.2, we have the result above. Thus, we say H and L are dual functions. □

If we do a canonical transformation to the original system, the corresponding *Lagrangian* will have the same extremals. The following lemma shows a situation where this condition can be satisfied.

Theorem 4.3. *If two Lagrangians are different from each other by some function's total derivative with respect to time* $\frac{dU}{dt}$*, then these two two Lagrangians have the same extremals, i.e.*

$$\mathbf{I}_1[x(t)] = \int_0^T L(\dot{\mathbf{x}}(t), \mathbf{x}(t), t)\mathrm{d}t$$

and

$$\mathbf{I}_2[x(t)] = \int_0^T L(\dot{\mathbf{x}}(t), \mathbf{x}(t), t) + \frac{dU}{dt}\mathrm{d}t$$

have same $x(t)$ *that let them attain their extremum values.*

Proof. To prove the theorem, it suffices to show that the Euler-Lagrange equation are satisfied for every smooth function U's total derivative with respect to time $\frac{dU}{dt}$.

Assuming U is a function of t and $\mathbf{x}$, then we have

$$\frac{d}{dt}U(\mathbf{x}, t) = \frac{\partial U}{\partial t} + \sum_{i=1}^{n} \frac{\partial U}{\partial \mathbf{x}_i}\dot{\mathbf{x}}_i.$$

Thus, we have

$$-\frac{d}{dt}\left(\frac{\partial(\frac{d}{dt}U(\mathbf{x}, t))}{\partial \dot{\mathbf{x}}_i}\right) = -\frac{d}{dt}\left(\frac{\partial U}{\partial \mathbf{x}_i}\right)$$

and

$$\frac{\partial(\frac{d}{dt}U(\mathbf{x}, t))}{\partial \mathbf{x}_i} = \frac{\partial U}{\partial \mathbf{x}_i}\frac{d}{dt} = \frac{d}{dt}\left(\frac{\partial U}{\partial \mathbf{x}_i}\right)$$

Hence, two equations above will sum up to 0, and the Euler-Lagrange equation is satisfied. Therefore, having $\frac{dU}{dt}$ in the intergrand won't change the variational result. $\mathbf{I}_1$ and $\mathbf{I}_2$ will still have the same Euler-Lagrange equations. □

Therefore, we now have two equivalent systems of Hamilton equations, and to simplify the expression we want to make the new Hamiltonian to be constant 0 with respect to the new conjugate variables. The following theorem gives us a mathematical description of the transformation.

Theorem 4.4 (Hamilton-Jacobi Equations)**.** *Assume that transformed Hamiltonian* $\mathbf{K}$ *is constant 0 with respect to the new conjugate variables* $\mathbf{X}$ *and* $\mathbf{P}$*. Then we have a generating function* $U(\mathbf{x}, \mathbf{X}, t)$*, such that*

$$\frac{\partial U}{\partial t} - H(\nabla_{\mathbf{x}} U, \mathbf{x}, t) = 0.$$

Proof. By assumption, the system after the canonical transformation should have the same extremals for its Lagrangian. Thus by the lemmas above, we have

$$\mathbf{P}\cdot\dot{\mathbf{X}} - K(\mathbf{P},\mathbf{X},t) = \mathbf{p}\cdot\dot{\mathbf{x}} - H(\mathbf{p},\mathbf{x},t) + \frac{dU}{dt}.$$

Moving terms around and multiplying both sides by dt, we have

$$\mathbf{P}\cdot d\mathbf{X} + \mathbf{p}\cdot d\mathbf{x} + (H(\mathbf{p},\mathbf{x},t) - K(\mathbf{P},\mathbf{X},t))dt = dU$$

Thus, conclude that

$$\begin{cases} \mathbf{P} = \nabla_X U(\mathbf{x},\mathbf{X},t) \\ \mathbf{p} = \nabla_x U(\mathbf{x},\mathbf{X},t) \\ H(\mathbf{p},\mathbf{x},t) - K(\mathbf{P},\mathbf{X},t) = \frac{\partial U}{\partial t}. \end{cases}$$

Now with $\mathbf{K}\equiv 0$, the third equation becomes

$$H(\mathbf{p},\mathbf{x},t) = \frac{\partial U}{\partial t}.$$

Then by the second equation, we get

$$\frac{\partial U}{\partial t} - H(\nabla_{\mathbf{x}}U,\mathbf{x},t) = 0$$

which is the Hamilton-Jacobi equation. □

5. Viscosity Solution for the Hamilton-Jacobi Equation

At this point we have found the mathematical expression for the Hamilton-Jacobi PDE, and the next move shall be solving this equation. However, unfortunately, it is usually impossible to find a classical solution for the Hamilton-Jacobi equation, and typical requirements for weak solutions, like Lipschitz continuity, fail to provide a unique solution. Thus, a new notion of weak solution needs to be produced. Most of the derivations in this section follows Evans' book "Partial Differential Equations" pretty closely, and readers can check Chapter 10 of his book for direct reference.

One of the usual motivations for the viscosity solution is that, provided a Hamilton-Jacobi PDE with initial condition

$$\begin{cases} \frac{\partial U}{\partial t} + H(\nabla_{\mathbf{x}}U,\mathbf{x},t) = 0 & \text{in } \mathbf{R}^n\times(0,\infty) \\ U = g & \text{on } \mathbf{R}^n\times\{t=0\} \end{cases} \tag{5.1}$$

we want to add a viscosity term $\epsilon\Delta U$ to the equation, so that it becomes

$$\begin{cases} \frac{\partial U^\epsilon}{\partial t} + H(\nabla_{\mathbf{x}}U^\epsilon,\mathbf{x},t) - \epsilon\Delta U = 0 & \text{in } \mathbf{R}^n\times(0,\infty) \\ U^\epsilon = g & \text{on } \mathbf{R}^n\times\{t=0\} \end{cases} \tag{5.2}$$

and when we have $\epsilon\to 0$, we hope that U^ϵ will converge to some weak solution U. This U will then be called a viscosity solution for the Hamilton-Jacobi equation. The following definition will be the actual definition of the viscosity solutions, and it can be shown that these two ways of defining viscosity solutions are consistent with each other.

Definition 5.3. Suppose U is a bounded and uniformly continuous function on $\mathbf{R}^n\times(0,T)$, for every $T>0$. We say U is a viscosity solution of the Hamilton-Jacobi equation (5.1) if

(i) $U = g$ on $\mathbf{R}^n\times\{t=0\}$

(ii)for every smooth test function $v\in C^\infty(\mathbf{R}^n\times(0,\infty))$ we have

① if $U - v$ has a local maximum at $(x_0, t_0) \in \mathbf{R}^n \times (0, \infty)$, then

$$\frac{\partial v}{\partial t}(x_0, t_0) + H(\nabla_x v(x_0, t_0), x_0, t_0) \leq 0$$

and

② if $U - v$ has a local minimum at $(x_0, t_0) \in \mathbf{R}^n \times (0, \infty)$, then

$$\frac{\partial v}{\partial t}(x_0, t_0) + H(\nabla_x v(x_0, t_0), x_0, t_0) \geq 0.$$

Theorem 5.4. *The viscosity solution defined in Definition 5.2 can be obtained by the vanishing viscosity method as in equation (5.2), given that $\{U^{\epsilon_i}\}$ converges uniformly to some function U as $\epsilon \to 0$.*

Proof. First, we need to establish some convergence behavior of U^ϵ as $\epsilon \to 0$, and we can do that by taking advantage of the Arzela-Ascoli compactness criterion, because for a compact subset of our domain, in practice it can be often shown that the family of functions $\{U^\epsilon\}$ will be bounded and equicontinuous. Therefore, by the criterion we will at least have some local uniform convergence for a subsequence $\{U^{\epsilon_i}\}$ of $\{U^\epsilon\}$. Hence, the limit U shall exist, and it is easy to see that if it is a solution it has to satisfy (i).

Thus, we now consider condition (ii). For ①, first suppose $U - v$ has a strict local maximum at $(x_0, t_0) \in \mathbf{R}^n \times (0, \infty)$. Then, we have for all other pairs (x, t) in the local neighborhood of (x_0, t_0)

$$(U - v)(x_0, t_0) > (U - v)(x, t).$$

By the Arzela-Ascoli argument above, we know that for $U^{\epsilon_i} \in \{U^{\epsilon_i}\}$, there exists a point $(x_{\epsilon_i}, t_{\epsilon_i})$ such that $U^{\epsilon_i} - v$ attains maximum at that point, with $(x_{\epsilon_i}, t_{\epsilon_i})$ converging to (x_0, t_0).

Moreover, we have

$$\nabla_x U^{\epsilon_i}(x_{\epsilon_i}, t_{\epsilon_i}) = \nabla_x v(x_{\epsilon_i}, t_{\epsilon_i})$$

and

$$\frac{\partial U^{\epsilon_i}}{\partial t}(x_{\epsilon_i}, t_{\epsilon_i}) = \frac{\partial v}{\partial t}(x_{\epsilon_i}, t_{\epsilon_i})$$

and

$$-\Delta U^{\epsilon_i}(x_{\epsilon_i}, t_{\epsilon_i}) \geq -\Delta v(x_{\epsilon_i}, t_{\epsilon_i})$$

since U^{ϵ_i} is smooth.

We then calculate

$$\begin{aligned}\frac{\partial v}{\partial t}(x_{\epsilon_i}, t_{\epsilon_i}) + H(\nabla_x v(x_{\epsilon_i}, t_{\epsilon_i}), x_{\epsilon_i}, t_{\epsilon_i}) &= \frac{\partial U^{\epsilon_i}}{\partial t}(x_{\epsilon_i}, t_{\epsilon_i}) + H(\nabla_x U^{\epsilon_i}(x_{\epsilon_i}, t_{\epsilon_i}), x_{\epsilon_i}, t_{\epsilon_i}) \\ &= \epsilon_i \Delta U^{\epsilon_i}(x_{\epsilon_i}, t_{\epsilon_i}) \\ &\leq \epsilon_i \Delta v(x_{\epsilon_i}, t_{\epsilon_i})\end{aligned}$$

Thus, as $\epsilon_i \to 0$, we conclude

$$\frac{\partial v}{\partial t}(x_0, t_0) + H(\nabla_x v(x_0, t_0), x_0, t_0) \leq 0.$$

Now suppose $U - v$ has a local maximum at (x_0, t_0), not necessarily strict. Then we can create a new smooth function $\tilde{v}(x, t)$, where

$$\tilde{v}(x, t) = v(x, t) + \delta(|x - x_0|^2 + (t - t_0)^2)$$

for some $\delta > 0$. Then it is obvious that $U - \tilde{v}$ has a strict local maximum at (x_0, t_0). We then repeat the argument above with $\tilde{v}$ replacing v and obtain the same result.

Similarly, for ② we can apply the same argument only with the inequality sign reversed. □

Therefore, we have two consistent notions of the viscosity solution. We now show that viscosity solutions solve that Hamilton-Jacobi PDE in the classical sense at points where they are differentiable.

Theorem 5.5. *Suppose U is a viscosity solution for equation (5.1). Then if U is differentiable at $(x_0, t_0) \in \mathbf{R}^n \times (0, \infty)$, we have*

$$\frac{\partial U}{\partial t}(x_0, t_0) + H(\nabla_x U(x_0, t_0), x_0, t_0) = 0.$$

Proof. In the view of Theorem 5.4, we know that if we have a smooth C^∞ function v and $U - v$ attains strict local maximum(minimum) at (x_0, t_0), then we can deduce the fact that

$$\frac{\partial v}{\partial t}(x_0, t_0) + H(\nabla_x v(x_0, t_0), x_0, t_0) \leq 0$$

and

$$\frac{\partial v}{\partial t}(x_0, t_0) + H(\nabla_x v(x_0, t_0), x_0, t_0) \geq 0.$$

Now because U is differentiable at (x_0, t_0), where $U - v$ attains local maximum or minimum, the first order derivatives for U and v should coincide, then we can replace v with U and obtain the equality.

Thus, what's left to do is to construct a smooth v for which $U - v$ attains strict local maximum(minimum) at (x_0, t_0).

We know that if we can find a C^1 function v that makes $U - v$ attain a strict local maximum(minimum) at (x_0, t_0), then we shall be able to find a C^∞ function v^ϵ for which $U - v^\epsilon$ attains a strict local maximum(minimum) at (x_ϵ, t_ϵ), with (x_ϵ, t_ϵ) converging to (x_0, t_0). We shall get this v^ϵ by taking the convolution between v and the usual mollifier φ_ϵ.

$$v^\epsilon = \varphi_\epsilon * v.$$

Thus, we have

$$\begin{cases} v^\epsilon \to v \\ \nabla_x v^\epsilon \to \nabla_x v \\ \frac{\partial v^\epsilon}{\partial t} \to \frac{\partial v}{\partial t} \end{cases}$$

near (x_0, t_0). Then from Theorem 5.4 we have

$$\frac{\partial v^\epsilon}{\partial t}(x_\epsilon, t_\epsilon) + H(\nabla_x v^\epsilon(x_\epsilon, t_\epsilon), x_\epsilon, t_\epsilon) \leq 0$$

and consequently,

$$\frac{\partial v}{\partial t}(x_0, t_0) + H(\nabla_x v(x_0, t_0), x_0, t_0) \leq 0$$

where v needs only to be a C^1 function.

Hence, we now only need to construct a C^1 function v that lets $U - v$ attains strict local maximum(minimum) at (x_0, t_0).

For the sake of simplicity use x_0 to denote (x_0, t_0). Without loss of generality, we can assume that

$$x_0 = 0,\ U(0) = \nabla_x U(0) = 0$$

because otherwise we need only consider

$$\tilde{U}(x) = U(x + x_0) - U(x_0) - \nabla_x U(x_0) \cdot x.$$

Let

$$U(x) = |x|\rho_1(x)$$

where $\rho_1(x)$ is continuous and $\rho_1(0) = 0$. Also, let

$$\rho_2(r) = \max_{x \in B_r(0)} |\rho_1(x)|.$$

Then $\rho_2(r)$ is a continuous increasing function with $\rho_2(0) = 0$.

Now we can construct v by

$$v(x) = \int_{|x|}^{2|x|} \rho_2(r)\mathrm{d}t + |x|^2.$$

Then we have

$$v(0) = \nabla_x v(0) = 0$$

and for $x \neq 0$

$$\nabla_x v(x) = \frac{2x}{|x|}\rho_2(2|x|) - \frac{x}{|x|}\rho_2(|x|) + 2x.$$

Therefore, conclude v is C^1. Finally, for $x \neq 0$

$$\begin{aligned}
U(x) - v(x) &= |x|\rho_1(x) - (\int_{|x|}^{2|x|} \rho_2(r)\mathrm{d}t + |x|^2) \\
&\leq |x|\rho_2(x) - (\int_{|x|}^{2|x|} \rho_2(r)\mathrm{d}t + |x|^2) \\
&\leq -|x|^2 \\
&< 0 = U(0) - v(0)
\end{aligned}$$

so $U - v$ has strict minimum at 0. Therefore, we can indeed find a C^1 function v such that $U - v$ attains strict local maximum(minimum) at (x_0, t_0), and thus

$$\begin{cases} \nabla_x U(x_0, t_0) = \nabla_x v(x_0, t_0) \\ \frac{\partial U}{\partial t}(x_0, t_0) = \frac{\partial v}{\partial t}(x_0, t_0). \end{cases}$$

Then, replacing v with U, we have

$$\frac{\partial U}{\partial t}(x_0, t_0) + H(\nabla_x U(x_0, t_0), x_0, t_0) = 0.$$

□

The following theorem shows that under certain conditions, there will be a unique viscosity solution for the Hamilton-Jacobi equations.

Theorem 5.6. *If the Hamiltonian* **H** *is Lipschitz continuous, i.e.*

$$\begin{cases} |H(p, x, t) - H(q, x, t)| \leq C|p - q| \\ |H(p, x, t) - H(p, y, s)| \leq C(|x - y| + |t - s|)(1 + |p|), \end{cases} \tag{5.7}$$

then there exists at most one viscosity solution for the Hamilton-Jacobi equation (5.1).

Proof. Argue by contradiction. Suppose we have two different viscosity solutions u and v, and

$$\sup(u-v)=\sigma>0.$$

Then we choose $0<\varepsilon,\lambda<1$, and introduce the function $\Phi(x,y,t,s)$, such that

$$\Phi(x,y,t,s)=u(x,t)-v(y,s)-\lambda(t+s)-\varepsilon(|x|^2+|y|^2)-\frac{1}{\varepsilon^2}(|t-s|^2+|x-y|^2).$$

Because of the last few terms of the function, this function should admit a global maximum at some point (x_0,y_0,t_0,s_0), i.e.

$$\Phi(x_0,y_0,t_0,s_0)=\max_{\mathbf{R}^{2n}\times[0,T]^2}\Phi(x,y,t,s).$$

Also, we can let ε,λ to be small, so that we have

$$\Phi(x_0,y_0,t_0,s_0)\geq\sup_{\mathbf{R}^n\times[0,T]}\Phi(x,x,t,t)\geq\frac{\sigma}{2}.$$

Now, observe that the map $(x,t)\mapsto\Phi(x,y_0,t,s_0)$ has maximum at (x_0,t_0). Thus, let $\varphi(x,t)$ be

$$\varphi(x,t)=v(y_0,s_0)+\lambda(t+s_0)+\varepsilon(|x|^2+|y_0|^2)+\frac{1}{\varepsilon^2}(|t-s_0|^2+|x-y_0|^2)$$

and notice that $u(x,t)-\varphi(x,t)$ attains its maximal value at (x_0,t_0). Then by the definition of viscosity solutions, we have

$$\frac{\partial\varphi}{\partial t}(x_0,t_0)+H(\nabla_x\varphi(x_0,t_0),x_0,t_0)\leq 0.$$

In other words

$$\lambda+\frac{2(t_0-s_0)}{\varepsilon^2}+H(2\varepsilon x_0+\frac{2(x_0-y_0)}{\varepsilon^2},x_0,t_0)\leq 0.$$

Similarly, the map $(y,s)\mapsto-\Phi(x_0,y,t_0,s)$ has a minimum at (y_0,s_0). Thus, let $\psi(y,s)$ be

$$\psi(y,s)=u(x_0,t_0)-\lambda(t_0+s)-\varepsilon(|x_0|^2+|y|^2)-\frac{1}{\varepsilon^2}(|t_0-s|^2+|x_0-y|^2).$$

Then $v(y,s)-\psi(y,s)$ attains its minimal value at (y_0,s_0). Then by the definition of viscosity solutions, we have

$$\frac{\partial\psi}{\partial t}(y_0,s_0)+H(\nabla_x\psi(y_0,s_0),y_0,s_0)\geq 0,$$

i.e.

$$-\lambda+\frac{2(t_0-s_0)}{\varepsilon^2}+H(-2\varepsilon y_0+\frac{2(x_0-y_0)}{\varepsilon^2},y_0,s_0)\geq 0.$$

Combining the two equations we have

$$2\lambda\leq H(-2\varepsilon y_0+\frac{2(x_0-y_0)}{\varepsilon^2},y_0,s_0)-H(2\varepsilon x_0+\frac{2(x_0-y_0)}{\varepsilon^2},x_0,t_0).$$

By Lipschitz condition (5.7), we have

$$2\lambda\leq C\varepsilon(|x_0+y_0|)+C(|x_0-y_0|+|t_0-s_0|)\left(1+\frac{|x_0-y_0|}{\varepsilon^2}+\varepsilon(|x_0|+|y_0|)\right).\tag{5.8}$$

Finally, to prove contradiction we just show that the right hand side approaches 0 as $\varepsilon\to 0$. Because u,v are bounded, using the fact that $\Phi(x_0,y_0,t_0,s_0)\geq\Phi(0,0,0,0)$, we deduce that

$$|t_0-s_0|,|x_0-y_0|=O(\varepsilon)$$

and

$$|x_0| + |y_0| = O(\sqrt{\varepsilon}).$$

Then, using the fact that $\Phi(x_0, y_0, t_0, s_0) \geq \Phi(x_0, x_0, t_0, t_0)$, we have

$$u(x_0, t_0) - v(y_0, s_0) - \lambda(t_0 + s_0) - \varepsilon(|x_0|^2 + |y_0|^2) - \frac{1}{\varepsilon^2}(|t_0 - s_0|^2 + |x_0 - y_0|^2) \\ \geq u(x_0, t_0) - v(x_0, t_0) - 2\lambda t_0 - 2\varepsilon|x_0|^2.$$

Thus

$$\frac{1}{\varepsilon^2}(|t_0 - s_0|^2 + |x_0 - y_0|^2) \leq v(x_0, t_0) - v(y_0, s_0) + \lambda(t_0 - s_0) + \varepsilon(|x_0|^2 - |y_0|^2).$$

Because v as a viscosity solution is uniformly continuous, the left hand side goes to 0 as $\varepsilon \to 0$, so

$$|t_0 - s_0|, |x_0 - y_0| = o(\varepsilon).$$

Finally, we apply these estimations to equation (5.8), and conclude that the right hand side will go to 0 as $\varepsilon \to 0$. This yields the contradiction as we desired. □

6. The Optimal Control Theory and the Hamilton-Jacobi-Bellman Equation

Having developed the theory of Hamilton-Jacobi PDE and viscosity solutions, we are now going to see how it can be applied to the optimal control theory. In the introduction, we stated the central question of the optimal control theory, i.e. given a controlled dynamical system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \alpha(t)) \\ \mathbf{x}(0) = x_0 \end{cases} \tag{6.1}$$

and a payoff functional

$$\mathbf{P}[\alpha] = \int_0^T r(\mathbf{x}(t), \alpha(t))\mathrm{d}t + g(\mathbf{x}(T)) \tag{6.2}$$

find an $\alpha^*(t)$ so that for all possible control functions $\alpha(t)$, we have

$$\mathbf{P}[\alpha^*(t)] \geq \mathbf{P}[\alpha(t)].$$

We start now by defining the value function, which is the best possible payoff starting from a fixed position and time.

Definition 6.3. For $x \in \mathbf{R}^n$, $t \in [0, T]$, the *value function* is defined to be the greatest possible payoff starting at $x \in \mathbf{R}^n$ at time t, i.e.

$$v(x, t) = \sup_{\alpha(t)} \mathbf{P}_{x,t}[\alpha(t)].$$

Observe that by Definition 6.3, we have

$$v(x, T) = g(x). \tag{6.4}$$

This will serve as our boundary condition.

It can be shown that the value function will be bounded and lipschitz continuous given that the function $\mathbf{f}(x, a)$ is lipschitz continuous. Due to the length limit of this paper, we are not going to show the proof of that theorem in this paper, and readers can refer to Section 10.3.3 of Evans' book "Partial Differential Equations" for the complete proof.

To establish the Hamilton-Jacobi equation for the optimal control theory, we also need a definition for the control theory Hamiltonian.

Definition 6.5. The control theory Hamiltonian $H(\nabla_x v, \mathbf{x}, t)$ is defined by

$$H(\nabla_x v, \mathbf{x}, t) = \max_{a \in A} H(\nabla_x v, \mathbf{x}, t, a) = \max_{a \in A} \big\{ \mathbf{f}(x, a) \cdot \nabla_x v + r(x, a) \big\},$$

because the dynamical system is autonomous, we can also write the Hamiltonian as $H(\nabla_x v, \mathbf{x})$.

We have the following theorem regarding the control theory Hamilton-Jacobi PDE, namely the Hamilton-Jacobi-Bellman(HJB) equation.

Theorem 6.6. *If the Hamiltonian $H(\nabla_x v, \mathbf{x})$ satisfies the Lipschitz continuity conditions as described in Theorem 5.6, then the value function $v(x, t)$ is the unique viscosity solution for the following Hamilton-Jacobi-Bellman PDE.*

$$\begin{cases} \frac{\partial v}{\partial t} + H(\nabla_x v, \mathbf{x}) = 0 & \text{in } \mathbf{R}^n \times [0, T] \\ v = g & \text{on } \mathbf{R}^n \times \{t = T\} \end{cases} \tag{6.7}$$

Proof. First, the boundary condition of the Hamilton-Jacobi-Bellman PDE is clearly satisfied by the definition of the value function.

Now to show v is a viscosity solution we need to check condition (ii) in Definition 5.3. However, we need to be careful here, because equation (6.7) has terminal boundary condition instead of initial boundary condition. As a consequence, the inequality signs in condition (ii) need to be reversed, i.e.

for φ smooth,

① if $v - \varphi$ has a local maximum at $(x_0, t_0) \in \mathbf{R}^n \times (0, T)$, then

$$\frac{\partial \varphi}{\partial t}(x_0, t_0) + H(\nabla_x \varphi(x_0, t_0), x_0, t_0) \geq 0,$$

② if $v - \varphi$ has a local minimum at $(x_0, t_0) \in \mathbf{R}^n \times (0, T)$, then

$$\frac{\partial \varphi}{\partial t}(x_0, t_0) + H(\nabla_x \varphi(x_0, t_0), x_0, t_0) \leq 0.$$

This reversal of inequality signs can be justified because we can just construct another function $w(x, t)$, such that

$$w(x, t) = v(x, T - t).$$

In that case, we have

$$\nabla_x w(x, t) = \nabla_x v(x, T - t)$$

and

$$\frac{\partial w}{\partial t}(x, t) = -\frac{\partial v}{\partial t}(x, T - t).$$

Therefore, the Hamilton-Jacobi-Bellman equation with initial condition becomes

$$\frac{\partial w}{\partial t} - H(\nabla_x w, \mathbf{x}) = 0$$

and by Definition 5.3 and the vanishing viscosity construction, if there is a smooth function $\psi(x, t)$ such that $w - \psi$ attains its maximal value at (x_0, t_0), we have

$$\begin{aligned} \frac{\partial \psi}{\partial t}(x_{\epsilon_i}, t_{\epsilon_i}) - H(\nabla_x \psi(x_{\epsilon_i}, t_{\epsilon_i}), x_{\epsilon_i}, t_{\epsilon_i}) &= \frac{\partial w^{\epsilon_i}}{\partial t}(x_{\epsilon_i}, t_{\epsilon_i}) - H(\nabla_x w^{\epsilon_i}(x_{\epsilon_i}, t_{\epsilon_i}), x_{\epsilon_i}, t_{\epsilon_i}) \\ &= \epsilon_i \Delta w^{\epsilon_i}(x_{\epsilon_i}, t_{\epsilon_i}) \\ &\leq \epsilon_i \Delta \psi(x_{\epsilon_i}, t_{\epsilon_i}). \end{aligned}$$

Thus, we get

$$\frac{\partial \psi}{\partial t}(x_0, t_0) - H(\nabla_x \psi(x_0, t_0), x_0, t_0) \leq 0.$$

If we then construct another smooth function $\varphi(x, t)$, such that

$$\varphi(x, t) = \psi(x, T - t),$$

then the function $v - \varphi$ attains its maximal value at $(x_0, T - t_0)$, and we have

$$-\frac{\partial \varphi}{\partial t}(x_0, T - t_0) - H(\nabla_x \psi(x_0, T - t_0), x_0, T - t_0) \leq 0.$$

After rearranging the terms, we shall find that we have the inequality sign reversed for ①, and the justification for the inequality sign reversal for ② will be exactly the same.

Now we prove ①. Argue by contradiction. Supposing this is false, then there exist some $\delta > 0$, and a function φ, such that at a point (x_0, t_0)

$$\frac{\partial \varphi}{\partial t}(x_0, t_0) + H(\nabla_x \varphi(x_0, t_0), x_0, t_0) \leq -\delta < 0.$$

For a point (x, t) sufficiently close to (x_0, t_0), we have

$$(v - \varphi)(x, t) \leq (v - \varphi)(x_0, t_0).$$

Suppose ε is sufficiently small, so that for all possible controls $\alpha(t)$, we will have $(\mathbf{x}(t_0 + \varepsilon), t_0 + \varepsilon)$ sufficiently close to (x_0, t_0).

Thus, we get

$$\begin{aligned} v(\mathbf{x}(t_0 + \varepsilon), t_0 + \varepsilon) - v(x_0, t_0) &\leq \varphi(\mathbf{x}(t_0 + \varepsilon), t_0 + \varepsilon) - \varphi(x_0, t_0) \\ &= \int_{t_0}^{t_0 + \varepsilon} \frac{\partial \varphi}{\partial t}(\mathbf{x}(s), s) + \nabla_x v(\mathbf{x}(s), s) \cdot \mathbf{f}(\mathbf{x}(s), \alpha(s)) \mathrm{d}s. \end{aligned}$$

Also, by the definition of the value function, we can choose some specific control $\alpha^*(t) \in A$, and have

$$v(x_0, t_0) \leq \int_{t_0}^{t_0 + \varepsilon} r(\mathbf{x}(s), \alpha^*(s)) \mathrm{d}s + v(\mathbf{x}(t_0 + \varepsilon), t_0 + \varepsilon) + \frac{\delta \varepsilon}{2}.$$

Combining the two equations we get

$$-\frac{\delta \varepsilon}{2} \leq \int_{t_0}^{t_0 + \varepsilon} \frac{\partial \varphi}{\partial t}(\mathbf{x}(s), s) + \nabla_x v(\mathbf{x}(s), s) \cdot \mathbf{f}(\mathbf{x}(s), \alpha^*) + r(\mathbf{x}(s), \alpha^*) \, \mathrm{d}s \leq -\delta \varepsilon.$$

This yields a contradiction, so we need to have

$$\frac{\partial \varphi}{\partial t}(x_0, t_0) + H(\nabla_x \varphi(x_0, t_0), x_0, t_0) \geq 0$$

if $v - \varphi$ attains a local maximum at (x_0, t_0).

For ② the proof is similar. Argue be contradiction. Supposing this is false, there exists some $\delta > 0$, such that

$$\frac{\partial \varphi}{\partial t}(x_0, t_0) + H(\nabla_x \varphi(x_0, t_0), x_0, t_0) \geq \delta > 0.$$

For all point (x, t) sufficiently close to (x_0, t_0), we have

$$(v - \varphi)(x, t) \geq (v - \varphi)(x_0, t_0)$$

and suppose we have time difference ε sufficiently small, so that for all constant control $a \in A$, we will have $(\mathbf{x}(t_0 + \varepsilon), t_0 + \varepsilon)$ sufficiently close to (x_0, t_0).

Thus, we get

$$\begin{aligned} v(\mathbf{x}(t_0+\varepsilon), t_0+\varepsilon) - v(x_0, t_0) &\geq \varphi(\mathbf{x}(t_0+\varepsilon), t_0+\varepsilon) - \varphi(x_0, t_0) \\ &= \int_{t_0}^{t_0+\varepsilon} \frac{\partial \varphi}{\partial t}(\mathbf{x}(s), s) + \nabla_x v(\mathbf{x}(s), s) \cdot \mathbf{f}(\mathbf{x}(s), a) \mathrm{d}s. \end{aligned}$$

Also, by the definition of the value function, for all control $a \in A$, we have

$$v(x_0, t_0) \geq \int_{t_0}^{t_0+\varepsilon} r(\mathbf{x}(s), a)\mathrm{d}s + v(\mathbf{x}(t_0+\varepsilon), t_0+\varepsilon).$$

Combining the two equations we get

$$0 \geq \int_{t_0}^{t_0+\varepsilon} \frac{\partial \varphi}{\partial t}(\mathbf{x}(s), s) + \nabla_x v(\mathbf{x}(s), s) \cdot \mathbf{f}(\mathbf{x}(s), a) + r(\mathbf{x}(s), a)\, \mathrm{d}s \geq \delta\varepsilon.$$

This is a contradiction, so we need to have

$$\frac{\partial \varphi}{\partial t}(x_0, t_0) + H(\nabla_x \varphi(x_0, t_0), x_0, t_0) \leq 0$$

if $v - \varphi$ attains a local minimum at (x_0, t_0).

Therefore, we have checked the conditions for v to be a viscosity solution of equation (6.7), and because the Hamiltonian is Lipschitz continuous, by Theorem 5.6 we have the value function v to be the unique viscosity solution for the Hamilton-Jacobi-Bellman equation. □

7. The Pontryagin Maximum Principle

In addition to the HJB equations for the optimal control theory, there is actually another way of finding the optimal control of a control system, which is called the Pontryagin Maximum Principle. In this section, we are going to see that this principle is really a special case of the Hamilton-Jacobi-Bellman equation, and build the equivalence between two criteria using what's called the method of characteristics.

Theorem 7.1 (Pontryagin Maximum Principle). *Suppose the control $\alpha^*(t)$ is optimal for the control system (6.1), and $\mathbf{x}^*(t)$ is the trajectory corresponding to this optimal control. Furthermore, assume the value function of this control system $v(x,t)$ is C^2.*

Then there exists a function $\mathbf{p}^*(t)$, such that

$$\begin{cases} \dot{\mathbf{x}}^*(t) = \nabla_p H(\mathbf{p}^*(t), \mathbf{x}^*(t), \alpha^*(t)) \\ \dot{\mathbf{p}}^*(t) = -\nabla_x H(\mathbf{p}^*(t), \mathbf{x}^*(t), \alpha^*(t)) \end{cases}$$

and

$$H(\mathbf{p}^*(t), \mathbf{x}^*(t), \alpha^*(t)) = \max_{a \in A} H(\mathbf{p}^*(t), \mathbf{x}^*(t), a)$$

and the map

$$t \mapsto H(\mathbf{p}^*(t), \mathbf{x}^*(t), \alpha^*(t)) \text{ is constant.}$$

Finally,

$$\mathbf{p}^*(T) = \nabla g(\mathbf{x}^*(T)).$$

Actually, we have

$$\mathbf{p}^*(t) = \nabla_x v(\mathbf{x}^*(t), t).$$

Proof. First, define
$$\mathbf{p}^*(t) = \nabla_x v(\mathbf{x}^*(t), t).$$
Then it is clear that the Hamiltonian in the Pontryagin Maximal Principle is consistent with the Hamiltonian in the Hamilton-Jacobi-Bellman equation, since both Hamiltonians attain maximal value when $\alpha^*(t)$ is optimal.

Then after the definition of the Hamiltonian is justified, we can directly obtain the equation
$$\dot{\mathbf{x}}^*(t) = \nabla_p H(\mathbf{p}^*(t), \mathbf{x}^*(t), \alpha^*(t))$$
by taking the derivative with respect to $\nabla_x v(\mathbf{x}^*(t), t)$ along the trajectory.

Thus, the important step now is to show that
$$\dot{\mathbf{p}}^*(t) = -\nabla_x H(\mathbf{p}^*(t), \mathbf{x}^*(t), \alpha^*(t)).$$
Because $v(x,t)$ is C^2, we take its partial derivative with respect to the components of $\mathbf{x}$, and it yields
$$\frac{\partial^2 v}{\partial x_i \partial x_j} = \frac{\partial p_i}{\partial x_j} = \frac{\partial p_j}{\partial x_i}.$$
Also, we have
$$\frac{\partial p_i}{\partial t} = \frac{\partial^2 v}{\partial x_i \partial t} = -\frac{\partial H}{\partial x_i}(p, x) = -\sum_{j=1}^{n} \frac{\partial H}{\partial p_j} \cdot \frac{\partial p_i}{\partial x_j} - \frac{\partial H}{\partial x_i}.$$
Therefore, along the curve $t \mapsto \mathbf{x}^*(T)$
$$\begin{aligned} \frac{d}{dt} p_i(x(t), t) &= \sum_{j=1}^{n} \frac{\partial p_i}{\partial x_j} \cdot \dot{x_j} + \frac{\partial p_i}{\partial t} \\ &= -\frac{\partial H}{\partial x_i} + \sum_{j=1}^{n} \frac{\partial p_i}{\partial x_j} \cdot (\dot{x_j} - \frac{\partial H}{\partial p_j}) \\ &= -\frac{\partial H}{\partial x_i} \end{aligned}$$
because the last term sums up to 0 along the curve. Thus, conclude that
$$\dot{\mathbf{p}}^*(t) = -\nabla_x H(\mathbf{p}^*(t), \mathbf{x}^*(t), \alpha^*(t)).$$

Now, we want to show that the Hamiltonian is unchanging with respect to time, and we can do this by using the Legendre transformation and the Euler-Lagrange equations. After doing the Legendre transformation to the Hamiltonian, we get
$$L(\dot{\mathbf{x}}^*(t), \mathbf{x}^*(t), \alpha^*(t)) = \dot{\mathbf{x}}^*(t) \cdot \mathbf{p}^*(\dot{x}, x, t) - H(\mathbf{p}^*(\dot{x}, x, t), \mathbf{x}^*(t), \alpha^*(t)).$$
It is then clear that
$$\frac{\partial L}{\partial \alpha^*} = -\frac{\partial H}{\partial \alpha^*}.$$
Recall that $\alpha^*(t)$ makes $L(\dot{\mathbf{x}}^*(t), \mathbf{x}^*(t), \alpha^*(t))$ attains its stationary point, so we have
$$-\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}^*} + \frac{\partial L}{\partial \alpha^*} = 0$$
based on the Euler-Lagrange equations. However, because $L(\dot{\mathbf{x}}^*(t), \mathbf{x}^*(t), \alpha^*(t))$ doesn't depend on $\dot{\alpha}^*$ by definition, we get
$$\frac{\partial L}{\partial \dot{\alpha}^*} = 0,$$

and as a result, we have

$$\frac{\partial H}{\partial \alpha^*} = -\frac{\partial L}{\partial \alpha^*} = -\frac{d}{dt}(0) = 0.$$

Therefore, when we calculate $\frac{d}{dt}H(\mathbf{p}^*(\dot{x}, x, t), \mathbf{x}^*(t), \alpha^*(t))$ along the trajectory, we get

$$\begin{aligned}\frac{d}{dt}H(\mathbf{p}^*(t), \mathbf{x}^*(t), \alpha^*(t)) &= \nabla_p H \cdot \dot{\mathbf{p}^*} + \nabla_x H \cdot \dot{\mathbf{x}^*} + \frac{\partial H}{\partial \alpha^*} \cdot \dot{\alpha^*} \\ &= \dot{\mathbf{x}^*} \cdot \dot{\mathbf{p}^*} - \dot{\mathbf{p}^*} \cdot \dot{\mathbf{x}^*} + 0 \\ &= 0\end{aligned}$$

and we have the Hamiltonian to be a constant along the trajectory as desired.

Finally, to check

$$\mathbf{p}^*(T) = \nabla g(\mathbf{x}^*(T))$$

we just recall the fact that

$$v(x, T) = g(x)$$

and then take the derivative with respect to x. □

8. Applications

8.1. Example: The Game "Tag". The playground game"tag" is a game that often involves two or more players, with one player selected to be"it". The "it" player will chase the other players, attempting to get close enough to "tag" one of them (touching them with a hand) while the other players try to escape. In the case of a two player game, it is interesting to consider the optimal strategy for the chaser. Having studied the central results of the optimal control theory, it is interesting to consider this problem under this framework.

Suppose we have Player A, also called the "runner", keeps running in one direction with the constant velocity. Then what is going to be the optimal strategy of Player B, i.e. the "chaser"? We also assume that the maximal velocity for the chaser is greater than the velocity of the runner.

Thus, the mathematical model comes down to that for all t, Player A will have velocity $\overrightarrow{a} = (a_1, a_2)$, with $|\overrightarrow{a}| < r$, and the admissible control for the system (the possible velocity of Player B) is $B_r(0)$. Also, we have the initial position difference between two players $\overrightarrow{BA} = x_0$.

Therefore, the dynamical system can be written as

$$\begin{cases}\dot{\mathbf{x}}(t) = a - \beta \\ \mathbf{x}(0) = x_0\end{cases}$$

with $\beta(t) \in B_r(0)$. The pay-off functional will be

$$\mathbf{P}[\beta] = -|\mathbf{x}(T)|.$$

Now we suppose T_1 is the shortest time such that Player A and B will meet each other under all circumstances. We let $0 < \varepsilon \leq T_1$ be a very short time, and then we apply the Pontryagin Maximal Principle. Hence, we have

$$\begin{aligned}\mathbf{p}^*(T_1-\varepsilon) &= \nabla g(\mathbf{x}^*(T_1-\varepsilon)) \\ &= -\nabla|\mathbf{x}^*(T_1-\varepsilon)| \\ &= -\frac{\mathbf{x}^*(T_1-\varepsilon)}{|\mathbf{x}^*(T_1-\varepsilon)|}.\end{aligned}$$

By the definition of the Hamiltonian, we have

$$\begin{aligned}H(\mathbf{p}^*(t),\mathbf{x}^*(t),\beta^*(t)) &= \max_{b\in B_r(0)} H(\mathbf{p}^*(t),\mathbf{x}^*(t),b) \\ &= \max_{b\in B_r(0)} (\mathbf{p}^*(t)\cdot(a-b)).\end{aligned}$$

Thus, at time $t=T_1-\varepsilon$, we have

$$\begin{aligned}H(\mathbf{p}^*(t),\mathbf{x}^*(t),\beta^*(t)) &= \max_{b\in B_r(0)} (\mathbf{p}^*(t)\cdot(a-b)) \\ &= \max_{b\in B_r(0)} \left(-\frac{\mathbf{x}^*(T_1-\varepsilon)}{|\mathbf{x}^*(T_1-\varepsilon)|}\cdot(a-b)\right).\end{aligned}$$

Then, we can deduce that at this time $(b-a)$ should be parallel to $\mathbf{x}^*(T_1-\varepsilon)$, and their product should be maximal. This condition is possible because we have $r>|a|$, and thus we need to have $|b|=r$ due to requirement for a maximal.

Furthermore, for all t, we have

$$\begin{aligned}\dot{\mathbf{p}^*}(t) &= -\nabla_x H(\mathbf{p}^*(t),\mathbf{x}^*(t),\beta^*(t)) \\ &= 0.\end{aligned}$$

Thus, we have $\dot{\mathbf{p}^*}(t)$ to be constant, which implies that $(b-a)$ is parallel to $\mathbf{x}^*(T_1-\varepsilon)$ for all t, and from this result we conclude that $(b-a)$ must be parallel to x_0.

Therefore, in case the Player A is running in a straight line in the game "tag", the optimal strategy for Player B is to run with his maximal speed in a direction such that we have $a-b\|x_0$, i.e. the velocity vector of Player A minus the velocity vector of Player B should be parallel to the initial displacement vector x_0. In other words, Player B should run in a straight line as well in order to intercept Player A in Player A's future path, and both players should arrive at the interception point at the same time.

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