Applied Mathematic

IV semester

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IOE All Subject Notes
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Formula (1)

Some Useful formula required here:

1. $e^{i\theta} = \cos\theta + isin\theta$
2. $e^{-i\theta} = \cos\theta - isin\theta$
3. $cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$
4. $sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$
5. $\sin(i\theta) = isinh\theta$
6. $\sinh(i\theta) = isin\theta$
7. $\cos(i\theta) = cosh\theta$
8. $\cosh(i\theta) = cos\theta$
9. $\sin(A+B).\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$
10. $\cos(A+B).\cos(A-B) = \cos^2 A - \sin^2 B = \sin^2 B - \sin^2 A$
11. $2\sin A.\sin B = \cos(A-B) - \cos(A+B)$
12. $2\cos A.\cos B = \cos(A-B) + \cos(A+B)$
13. $2\sin A.\cos B = \sin(A+B) + \sin(A-B)$
14. $2\cos A.\sin B = \sin(A+B) - \sin(A-B)$

दिएको f(x) लाई solve गरेर यी formula को help ले $x + iy$ को form मा separate गरेर $u + iv$ संग compare गर्ने । ***You will get u and v***

Complex Analysis:

Function of complex variable: Let $z = x + iy$ and $w = u + iv$ be two complex number. w is said to be function of z i.e. $w = f(z)$. If each value of z there is corresponding value of w.

Analytical function: a complex function which is differential at a point and in its neighborhood (n b d) is said to be an analytic at the point.

Singular point: a point Zo in the complex plane is said to be a singular point or a singularity of a function $f(z)$ if $f(z)$ is not analytic at $Z = Zo$ but it is analytic at every point in its (nbd).

Cauchy Riemann Equations ☹[CR –equation]:-

(Necessary condition of function for analytic)

Theorem 1:- If a function $\boldsymbol{w = f(z) = u + iv}$ is analytical in a region D of the complex plane then $\boldsymbol{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$; $\boldsymbol{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$

Proof:

Let δx and δy be incensement in x and y respectively. Corresponding incensement in u ,v and z be $\delta u, \delta v, \delta z$ respectively. so $z = x + iy \; and \; w = f(z) = u + iv$

$f(z + \delta z) = (u + \delta u) + i(v + \delta v)$

$z + \delta z = x + \delta x + i(y + \delta y)$

$\delta z = \delta x + i\delta y$

since $f(z)$ is analytical in the region D so $f'(z)$ exist at every point of region D

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

$$f'(z) = \lim_{\delta z \to 0} \frac{(u + \delta u) + i(v + \delta v) - u - iv}{\delta z}$$

$$f'(z) = lim_{\delta z \to 0} \frac{\delta u + i\delta v}{\delta z}$$

$$f'(z) = lim_{\delta z \to 0} \left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}\right)$$

The above limit must exist and unique for any path through which $\delta z \to 0$.

let $\delta z \to 0$ alone the line parallel to x-axis so that $\delta y = 0$ and $\delta z = \delta x$

$$f'(z) = lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right)$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \text{-------------} (1)$$

let $\delta z \to 0$ alone the line parallel to y-axis so that $\delta x = 0$ and $\delta z = i\delta y$

$$f'(z) = \ lim_{i\delta y \to 0}\left(\frac{\delta u}{i\delta y} + i\frac{\delta v}{i\delta y}\right)$$
$$= lim_{\delta y \to 0}\left(-\frac{i^2\delta u}{i\delta y} + \frac{\delta v}{\delta y}\right)$$
$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} - - - - - - - - - - - - - - - - - - (2)$$

Here eq(1) and eq(2) are same Thus,

$\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

on comparing we get, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Note: - If $f(z) = u + iv$ possess continuous first order partial derivative $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ satisfying C-R equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ **and** $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at every point of D then $f(z)$ is analytical in the region. (Sufficient condition)

C-R equation in polar form:-

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}; \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

To show a function is analytical:

- satisfy C-R equation in Cartesian $\left[\text{i.e.} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right]$

Also,

To show continuous $f(z)$ function:

- $\lim_{x,y \to 0} u = \lim_{x,y \to 0} v$ सबै zero आउछ।
- $\lim_{z \to 0} f(z)$

- satisfy C-R equation in polar form $\left[\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}; \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}\right]$

$f(z) = (rcos\theta + irsin\theta)^{-n}$ यहां imaginary र real part छुटयाउन गाह्रो छ ।

तर !

By using Demovre's Theorem

$f(z) = r^{-n}(\cos n\theta - i\sin n\theta)$ अब सजिलो भो ।☺

Laplace equation मा

- ⇨ Polar form :- $r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$
- ⇨ Cartesian Co-ordinate :- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Harmonic function

- ⇨ A function ϕ=ϕ(x,y) is said to be harmonic function if it satisfy Laplace equation. (i.e. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$)

Theorem 1:- If $\boldsymbol{f(z) = u + iv}$ is analytical then its components $\boldsymbol{u}$ and $\boldsymbol{v}$ are harmonics.

Note (1):- If u is harmonic of analytic function $f(z) = u + iv$ then v is conjugate harmonic of u.

Note (2):- $f(z) = u + iv$ is analytic iff v is conjugate harmonic of u.

Method to construct analytical function (Type-1)

To construct analytical function when $\boldsymbol{u}$ is given

- ⇨ find $\frac{\partial u}{\partial x}$ or $\frac{\partial u}{\partial r}$ $= value$ ---------- (1) & $\frac{\partial u}{\partial y}$ or $\frac{\partial u}{\partial \theta}$ ← this value will be used later
- ⇨ assign $\frac{\partial v}{\partial y}$ or $\frac{\partial v}{\partial \theta}$ $= value$ of $\frac{\partial u}{\partial x}$ obtained from $eq(1)$ ∴ from C-R equation
- ⇨ integrating both side $w.r.to.\ y$ $or\ \theta$ [hence you will get $v = \cdots \ldots \ldots \ldots \ldots + \phi(x) or f(r)$ ------- (3)($\phi(x) or f(r)$ *is integration constant*)

- ⇨ find $\frac{\partial v}{\partial x}$ or $\frac{\partial v}{\partial r}$ & use C-R equation i.e. $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ or. $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

After this step you will have some value of $\phi(x)$ or f(r); mainly $\phi(x)$ or f(r) = c

- ⇨ Now we put $\phi(x)$ or $f(r)$ value in equation (3) and get $\boldsymbol{v} = -----------$ ($proper\ value$)
- ⇨ Now put all value of $\boldsymbol{u}$ and $\boldsymbol{v}$ in $u + iv$ and separate real & imaginary part that will be required analytical function.

Method to construct analytical function (Type-2 Mini Thomson method)

To construct analytical function when $\boldsymbol{u} - \boldsymbol{v}$ is given

step1. Differentiate w.r.to $\boldsymbol{x}$ and eliminate the term containing v ($i.e. \frac{\partial v}{\partial x}$) with the help of **C-R equation** $---(1)$

step2. Differentiate w.r.to $\boldsymbol{y}$ and eliminate the term containing v ($i.e. \frac{\partial v}{\partial y}$) with the help of **C-R equation** $---(2)$

step3. Now perform following two task

Task1: **adding** $eq(1)$ and $eq(2)$ Hence you will get $\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{y}}$

Task2: **Subtracting** $eq(1)$ from $eq(2)$ Hence you will get $\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}$

You can also verify harmonic function as you have $\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{y}}$ & $\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}$ **i.e.** $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

step4. *Here we use mini Thomson method to construct analytical function*

$$f(z) = u + iv$$

Differentiate w.r.to. x

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \phi_1(x,y) + i\phi_2(x,y)$$

By using mini Thomson method $put\ x = z\ ; y = 0\ so\ dx = dz\ \&\ dy = 0$

$$f'(z) = \phi_1(z,0) + i\phi_2(z,0)$$

$$f(z) = \int \phi_1(z,0) + i\int \phi_2(z,0)$$

$\phi_1(z,0)$ र $\phi_2(z,0)$ $function$ भनेको $\frac{\partial u}{\partial x}$ र $\frac{\partial v}{\partial x}$ हो ।
अनि यहां functional variable replace गर्नें
$x = z\ ; y = 0$ $to\ get\ value\ of$ $\frac{\partial v}{\partial x}$ use $\boldsymbol{C-R}$ $equation.$

Do not forget to

Keep integration constant(c) in answer.

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