Vector Calculus (SKP)

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
import sympy as smp
from sympy import *
from sympy.vector import *
from sys import displayhook
from tkinter.tix import DisplayStyle
```

Vector derivatives

In [2]:
```
x, y, z, t = smp.symbols('x y z t')
```

In [3]:
```
# input the vector
r = smp.Matrix([4*t,6*smp.cos(5*t),t**3])
display('r', r)
diffr = smp.diff(r,t)
display('dr/dt', diffr)
```

'r'

$$\begin{bmatrix} 4t \\ 6\cos(5t) \\ t^3 \end{bmatrix}$$

'dr/dt'

$$\begin{bmatrix} 4 \\ -30\sin(5t) \\ 3t^2 \end{bmatrix}$$

Example: Find the angle between the velocity and acceleration as a function of time $\theta(t)$ and also find the angle at $t = 4s$. Plot t vs $\theta(t)$ graph.

In [4]:
```
v = smp.diff(r,t)
a = smp.diff(v,t)
theta = smp.acos(v.dot(a)/(v.norm()*a.norm()))
display('theta',theta.simplify())
display('theta at t=4', theta.subs(t,4).evalf())
```

'theta'

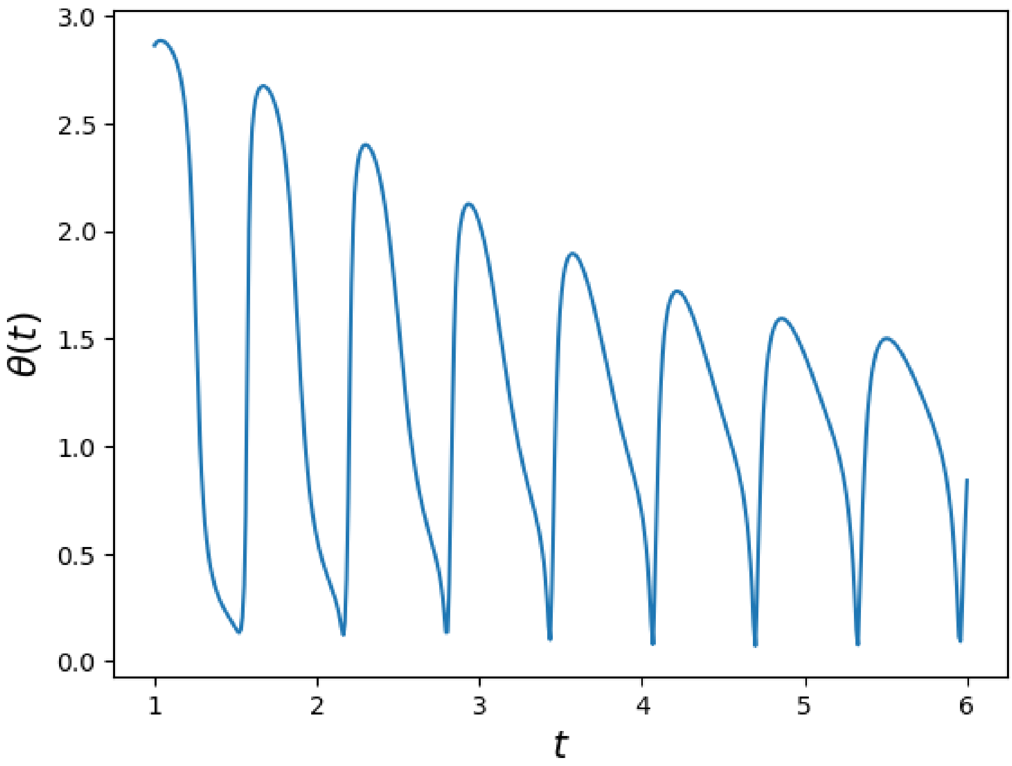
$$\operatorname{acos}\left(\frac{3\left(t^3 + 125\sin(10t)\right)}{\sqrt{|t|^2 + 625|\cos(5t)|^2}\sqrt{9|t^2|^2 + 900|\sin(5t)|^2 + 16}}\right)$$

'theta at t=4'

0.681852695830224

In [5]:
```
thetaf = smp.lambdify([t], theta)  # function

tt = np.linspace(1,6,500)
tht = thetaf(tt)
plt.plot(tt,tht)
plt.xlabel('$t$', fontsize=15)
plt.ylabel(r'$\theta(t)$', fontsize=15)
plt.show()
```



Vector Integrals

In [6]:
```
# input the matrix
r = smp.Matrix([smp.exp(-t**3), smp.sin(t), 5*t**3 + 4*t])
I = smp.Integral(r,t)
display(I)
display(I.doit())
```

$$\int \begin{bmatrix} e^{-t^3} \\ \sin(t) \\ 5t^3 + 4t \end{bmatrix} dt$$

$$\begin{bmatrix} \frac{\Gamma\left(\frac{1}{3}\right)\gamma\left(\frac{1}{3}, t^3\right)}{9\Gamma\left(\frac{4}{3}\right)} \\ -\cos(t) \\ \frac{5t^4}{4} + 2t^2 \end{bmatrix}$$

Some cases integrals can't be solved analytically. We need to solve them *numerically*.

In [7]:
```
# input the matrix
r1 = smp.Matrix([smp.exp(-t**2)*smp.cos(t)**3, smp.exp(-t**4), 1/(3+t**2)])
I1 = smp.Integral(r1, (t,0,1))
display(I1)
```

$$\int_0^1 \begin{bmatrix} e^{-t^2}\cos^3(t) \\ e^{-t^4} \\ \frac{1}{t^2+3} \end{bmatrix} dt$$

In [8]:
```
from scipy.integrate import quad_vec
r1f = smp.lambdify([t],r1)
quad_vec(r1f,0,1)
```

Out[8]:
```
(array([[0.53525785],
        [0.84483859],
        [0.30229989]]),
 3.5151979041265046e-14)
```

Gradients (∇f)

Here we need to work with some particular co-ordinate system.

In [9]:
```
C = CoordSys3D('')

# write the function
f1 = C.x*smp.cos(C.y)
display('function',f1)
gradf1 = gradient(f1)
gradf1m = gradf1.to_matrix(C)
display('gradient', gradf1, gradf1m)
```

```
'function'
```

$\mathbf{x}\cos(\mathbf{y})$

```
'gradient'
```

$(\cos(\mathbf{y}))\hat{\mathbf{i}} + (-\mathbf{x}\sin(\mathbf{y}))\hat{\mathbf{j}}$

$$\begin{bmatrix} \cos(\mathbf{y}) \\ -\mathbf{x}\sin(\mathbf{y}) \\ 0 \end{bmatrix}$$

Directional Derivatives

$$D_u f = \nabla f \cdot u$$

In [10]:
```
# write the function
f1 = C.x*smp.cos(C.y)
display('function',f1)

# write the vector
uvec = 6*C.i +3*C.j -5*C.k
u1 = uvec.normalize()  # making unit vector
display('unit vector', u1)

Du1f1 = gradient(f1).dot(u1)
display('directional derivative', Du1f1)
```

'function'

$$\mathbf{x}\cos(\mathbf{y})$$

'unit vector'

$$\left(\frac{3\sqrt{70}}{35}\right)\hat{\mathbf{i}} + \left(\frac{3\sqrt{70}}{70}\right)\hat{\mathbf{j}} + \left(-\frac{\sqrt{70}}{14}\right)\hat{\mathbf{k}}$$

'directional derivative'

$$-\frac{3\sqrt{70}\mathbf{x}\sin(\mathbf{y})}{70} + \frac{3\sqrt{70}\cos(\mathbf{y})}{35}$$

Line Integrals (Scalar)

Given curve, $\vec{r}(t) = \langle g(t), h(t), k(t)\rangle$. The line integral of $f(x, y, z)$ along the curve is,

$$\int_C f(x, y, z)ds = \int_a^b f(g(t), h(t), k(t))|d\vec{r}/dt|dt$$

In [11]:
```
t = smp.symbols('t', real=True)
x,y,z,f,a,b = smp.symbols('x y z f a b', cls=smp.Function, real=True)
x = x(t)
y = y(t)
z = z(t)
f = f(x,y,z)
r = smp.Matrix([x,y,z])

linints = f*r.diff(t).norm()
smp.Integral(linints, (t, a, b))
```

Out[11]:

$$\int\limits_a^b \sqrt{\left|\frac{d}{dt}x(t)\right|^2 + \left|\frac{d}{dt}y(t)\right|^2 + \left|\frac{d}{dt}z(t)\right|^2} f(x(t), y(t), z(t))\, dt$$

Write the curve $\vec{r}$ and the function $f(x, y, z)$,

1. $\vec{r}(t) = \langle\cos(t), \sin(t), t\rangle$;   (Helix)
2. $f(x, y, z) = 2xy + \sqrt{z}$

We are going from $t = 0$ to $t = 2\pi$.

In [12]:
```
f1s = 2*x*y + smp.sqrt(z)  # input f(x,y,z)
x1s = smp.cos(t)  # input x(t)
y1s = smp.sin(t)  # input y(t)
z1s = t   # input z(t)
a1s, b1s = 0, 2*smp.pi   # input limits of t
linints1 = linints.subs([(f,f1s),(x,x1s),(y,y1s),(z,z1s)]).doit().simplify()
display(smp.Integral(linints1, (t,a1s,b1s)))
smp.integrate(linints1, (t,a1s, b1s)).simplify()
```

$$\int_{0}^{2\pi} \sqrt{2}\left(\sqrt{t} + \sin(2t)\right)\, dt$$

Out[12]:

$$\frac{8\pi^{\frac{3}{2}}}{3}$$

We also need the numerical solution as most of the cases can't be solved symbolically. For integration we can use `quad` function of `scipy.integrate`.

Write the curve $\vec{r}$ and the function $f(x, y, z)$,

1. $\vec{r}(t) = \left\langle 3\cos(t), 2\sin(t), e^{t/4} \right\rangle$
2. $f(x, y, z) = 2xy + \sqrt{z}$

We are going from $t = 0$ to $t = 2\pi$.

In [13]:
```
f2s = 2*x*y + smp.sqrt(z)  # input f(x,y,z)
x2s = 3*smp.cos(t)    # input x(t)
y2s = 2*smp.sin(t)    # input y(t)
z2s = smp.exp(t/4)    # input z(t)
a2s, b2s = 0, 2*smp.pi   # input limits of t
linints2 = linints.subs([(f,f2s),(x,x2s),(y,y2s),(z,z2s)]).doit().simplify()
display(smp.Integral(linints2, (t,a2s,b2s)))
#smp.integrate(linints2, (t,a2s, b2s)).simplify()
```

$$\int_{0}^{2\pi} \frac{\left(e^{\frac{t}{8}} + 6\sin(2t)\right)\sqrt{e^{\frac{t}{2}} + 80\sin^{2}(t) + 64}}{4}\, dt$$

In [14]:
```
from scipy.integrate import quad
linints2f = smp.lambdify([t], linints2)
quad(linints2f, a2s, b2s)[0]
```

Out[14]: 24.294733741870633

Line Integrals (Vector)

Given, $\vec{r}(t) = \langle g(t), h(t), k(t)\rangle$. The line integral of $\vec{F}(x, y, z)$ along the curve is;

$$\int_C \vec{F}(x, y, z) \cdot d\vec{r} = \int_a^b \vec{F}(g(t), h(t), k(t)) \cdot \frac{d\vec{r}}{dt} dt$$

In [15]:
```python
t, a, b = smp.symbols('t a b', real=True)
x,y,z,F1,F2,F3 = smp.symbols('x y z F_1 F_2 F_3',cls=smp.Function,real=True)
x, y, z = x(t), y(t), z(t)
F1, F2, F3 = F1(x,y,z), F2(x,y,z), F3(x,y,z)
r = smp.Matrix([x, y, z])
F = smp.Matrix([F1, F2, F3])

linintv = F.dot(r.diff(t))
display(smp.Integral(linintv, (t,a,b)).simplify())
```

$$\int_a^b \left(\mathrm{F}_1\left(x(t), y(t), z(t)\right)\frac{d}{dt}x(t) + \mathrm{F}_2\left(x(t), y(t), z(t)\right)\frac{d}{dt}y(t) + \mathrm{F}_3\left(x(t), y(t), z(t)\right)\frac{d}{dt}z(t)\right)\, dt$$

Write the curve $\vec{r}(t)$ and the Function $\vec{F}(x, y, z)$;

1. Curve: $\vec{r}(t) = \langle t, t^2, t^4 \rangle$
2. Function: $\vec{F} = \langle \sqrt{z}, -2x, \sqrt{y} \rangle$

Limit: from $t = 0$ to $t = 1$.

In [16]:
```python
F11s = smp.sqrt(z)
F21s = -2*x
F31s = smp.sqrt(y)
x1s = t
y1s = t**2
z1s = t**4
a1s, b1s = 0, 1
linintv1 = linintv.subs([(F1,F11s),(F2,F21s),(F3,F31s),
            (x,x1s),(y,y1s),(z,z1s)]).doit().simplify()
display(smp.Integral(linintv1, (t,a1s,b1s)))
smp.integrate(linintv1, (t,a1s,b1s))
```

$$\int_0^1 t^2 \cdot (4t\,|t| - 3)\, dt$$

Out[16]: $-\frac{1}{5}$

Many of the integrals can't be solved symbolically and we must do that numerically.

Write the curve $\vec{r}(t)$ and the Function $\vec{F}(x, y, z)$;

1. Curve: $\vec{r}(t) = \langle 3\cos^2(t), t^2, 2\sin(t) \rangle$
2. Function: $\vec{F} = \langle \sqrt{|z|}, -2x, \sqrt{|y|} \rangle$

Limit: from $t = 0$ to $t = 2\pi$.

In [17]:
```
F12s = smp.sqrt(smp.Abs(z))
F22s = -2*x
F32s = smp.sqrt(smp.Abs(y))
x2s = 3*smp.cos(t)**2
y2s = t**2
z2s = 2*smp.sin(t)
a2s, b2s = 0, 2*smp.pi
linintv2 = linintv.subs([(F1,F12s),(F2,F22s),(F3,F32s),
            (x,x2s),(y,y2s),(z,z2s)]).doit().simplify()
display(smp.Integral(linintv2, (t,a2s,b2s)))
#smp.integrate(linintv2, (t,a2s,b2s)) # takes much time
```

$$\int_0^{2\pi} 2\left(-6t\cos(t) - 3\sqrt{2}\sin(t)\sqrt{|\sin(t)|} + |t|\right)\cos(t)\,dt$$

In [18]:
```
from scipy.integrate import quad
linintv2f = smp.lambdify([t], linintv2)
quad(linintv2f, a2s, b2s)[0]
```

Out[18]: `-118.4352528130723`

Surface Integrals (Scalar)

Area of a surface is given by;

$$A = \iint_S \left|\frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv}\right| dudv; \quad where\ \vec{r} = \vec{r}(u, v)$$

$\vec{r}$ denotes the surface and it's a function of 2 variables.

The surface integral of a scalar function $G(\vec{r})$ is given by;

$$\iint_S G(\vec{r}(u, v)) \left|\frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv}\right| dudv$$

In [19]:
```
#  r = r(u=rho, v=theta)
rho, th = smp.symbols('\\rho \\theta', pos=True, real=True)
x, y, z, G = smp.symbols('x y z G', cls=smp.Function, real=True)
x, y, z = x(rho,th), y(rho,th), z(rho,th)
G = G(x,y,z)
r = smp.Matrix([x,y,z])

surints = G* r.diff(rho).cross(r.diff(th)).norm()
surints
```

Out[19]:

$$\sqrt{\left|\frac{\partial}{\partial\rho}x(\rho,\theta)\frac{\partial}{\partial\theta}y(\rho,\theta) - \frac{\partial}{\partial\theta}x(\rho,\theta)\frac{\partial}{\partial\rho}y(\rho,\theta)\right|^2 + \left|\frac{\partial}{\partial\rho}x(\rho,\theta)\frac{\partial}{\partial\theta}z(\rho,\theta) - \frac{\partial}{\partial\theta}x(\rho,\theta)\frac{\partial}{\partial\rho}z(\rho,\theta)\right|^2 +}$$

Write the curve $\vec{r}(\rho, \theta)$ and the Function $G(x, y, z)$;

1. Surface: $\vec{r}(x, y) = \left\langle x, y, x^2 + y^2 \right\rangle$ and thus $\vec{r}(\rho, \theta) = \left\langle \rho\cos\theta, \rho\sin\theta, \rho^2 \right\rangle$
2. Function: $G(x, y, z) = x^2 + y^2$

Limits: ($\rho = 0$ to $\rho = 1$) and ($\theta = 0$ to $\theta = 2\pi$).

In [20]:
```
G1s = x**2 +y**2
x1s = rho*smp.cos(th)
y1s = rho*smp.sin(th)
z1s = rho**2
rhomin1, rhomax1 = 0, 1
thmin1, thmax1 = 0, 2*smp.pi

surints1 = surints.subs([(G,G1s),(x,x1s),(y,y1s),(z,z1s)]).doit().simplify()
display(smp.Integral(surints1,(rho,rhomin1,rhomax1),(th,thmin1,thmax1)))
smp.integrate(surints1,(rho,rhomin1,rhomax1),(th,thmin1,thmax1))
```

$$\int_{0}^{2\pi}\int_{0}^{1} \rho^2 \sqrt{4\rho^2 + 1}\,|\rho|\; d\rho\, d\theta$$

Out[20]:

$$2\pi\left(\frac{1}{120} + \frac{5\sqrt{5}}{24}\right)$$

Complicated integrals can be solved numerically.

Write the curve $\vec{r}(\rho, \theta)$ and the Function $G(x, y, z)$;

1. Surface: $\vec{r}(x, y) = \left\langle x, y, x^2 + y^2 \right\rangle$ and thus $\vec{r}(\rho, \theta) = \left\langle \rho\cos\theta, \rho\sin\theta, \rho^2 \right\rangle$
2. Function: $G(x, y, z) = (x^2 + y^2)e^z$

Limits: ($\rho = 0$ to $\rho = 1$) and ($\theta = 0$ to $\theta = 2\pi$).

In [21]:
```
G2s = (x**2+y**2)*smp.exp(z)
x2s = rho*smp.cos(th)
y2s = rho*smp.sin(th)
z2s = rho**2
rhomin2, rhomax2 = 0, 1
thmin2, thmax2 = 0, 2*smp.pi

surints2 = surints.subs([(G,G2s),(x,x2s),(y,y2s),(z,z2s)]).doit().simplify()
display(smp.Integral(surints2,(rho,rhomin2,rhomax2),(th,thmin2,thmax2)))
#smp.integrate(surints2,(rho,rhomin2,rhomax2),(th,thmin2,thmax2)).evalf()
```

$$\int_{0}^{2\pi}\int_{0}^{1} \rho^2 \sqrt{4\rho^2 + 1}e^{\rho^2}\,|\rho|\; d\rho\, d\theta$$

In [22]:
```
import numpy as np
from scipy.integrate import dblquad
surints2f = smp.lambdify([rho,th],surints2)
dblquad(surints2f,thmin2,thmax2,rhomin2,rhomax2)[0]
```

Out[22]:
```
6.13723773063216
```

Surface Integtals(Vectors)

The surface integral of a vector function $\vec{G}(\vec{r})$ is given by;

$$\iint_S \vec{G}(\vec{r}(u,v)) \cdot \left(\frac{d\vec{r}}{du} \times \frac{d\vec{r}}{dv}\right) dudv$$

This is also known as flux of the vector field $\vec{G}$ through the surface $\vec{r}$.

In [23]:

```
# u = rho and v = theta
rho, th = smp.symbols('\\rho \\theta', pos=True, real=True)
x,y,z,G1,G2,G3 = smp.symbols('x y z G_1 G_2 G_3',cls=smp.Function,real=True)
x, y, z = x(rho,th), y(rho,th), z(rho,th)
G1, G2, G3 = G1(x,y,z), G2(x,y,z), G3(x,y,z)
r = smp.Matrix([x,y,z])
G = smp.Matrix([G1,G2,G3])

surintv = G.dot(r.diff(rho).cross(r.diff(th)))
surintv
```

Out[23]:

$$\left(\frac{\partial}{\partial\rho}x(\rho,\theta)\frac{\partial}{\partial\theta}y(\rho,\theta)-\frac{\partial}{\partial\theta}x(\rho,\theta)\frac{\partial}{\partial\rho}y(\rho,\theta)\right)\mathrm{G}_3\left(x(\rho,\theta),y(\rho,\theta),z(\rho,\theta)\right)+\left(-\frac{\partial}{\partial\rho}x(\rho,\theta)\frac{\partial}{\partial\theta}z(\rho,\right.$$

$$\left(x(\rho,\theta),y(\rho,\theta),z(\rho,\theta)\right)+\left(\frac{\partial}{\partial\rho}y(\rho,\theta)\frac{\partial}{\partial\theta}z(\rho,\theta)-\frac{\partial}{\partial\theta}y(\rho,\theta)\frac{\partial}{\partial\rho}z(\rho,\theta)\right)\mathrm{G}_1\left(x(\rho,\theta),y(\rho,\theta),z(\rho,\theta\right.$$

Write the curve $\vec{r}(\rho,\theta)$ and the Function $\vec{G}(x,y,z)$;

1. Surface: $\vec{r}(x,y)=\left\langle x,y,x^2+y^2\right\rangle$ and thus $\vec{r}(\rho,\theta)=\left\langle\rho\cos\theta,\rho\sin\theta,\rho^2\right\rangle$
2. Function: $\vec{G}(x,y,z)=\left\langle y^2,z,0\right\rangle$

Limits: ($\rho=0$ to $\rho=1$) and ($\theta=0$ to $\theta=\pi$).

In [24]:

```
G11s = y**2
G21s = z
G31s = 0
x1s = rho*smp.cos(th)
y1s = rho*smp.sin(th)
z1s = rho**2
rhomin1, rhomax1 = 0, 1
thmin1, thmax1 = 0, smp.pi

surintv1 = surintv.subs([(G1,G11s),(G2,G21s),(G3,G31s),
            (x,x1s),(y,y1s),(z,z1s)]).doit().simplify()
display(smp.Integral(surintv1,(rho,rhomin1,rhomax1),(th,thmin1,thmax1)))
smp.integrate(surintv1,(rho,rhomin1,rhomax1),(th,thmin1,thmax1))
```

$$\int_0^{\pi}\int_0^{1}\left(-2\rho^4\left(\frac{\sin(2\theta)}{2}+1\right)\sin(\theta)\right)d\rho\,d\theta$$

Out[24]:

$$-\frac{4}{5}$$

Complicated integrals can be solved numerically.

Write the curve $\vec{r}(\rho,\theta)$ and the Function $\vec{G}(x,y,z)$;

1. Surface: $\vec{r}(x,y)=\left\langle x,y,x^2+y^2\right\rangle$ and thus $\vec{r}(\rho,\theta)=\left\langle\rho\cos\theta,\rho\sin\theta,\rho^2\right\rangle$
2. Function: $\vec{G}(x,y,z)=\left\langle y^2e^z,x\sin(z),x^4\right\rangle$

Limits: ($\rho=0$ to $\rho=1$) and ($\theta=0$ to $\theta=\pi$).

In [25]:
```
G12s = y**2*smp.exp(z)
G22s = x*smp.sin(z)
G32s = x**4
x2s = rho*smp.cos(th)
y2s = rho*smp.sin(th)
z2s = rho**2
rhomin2, rhomax2 = 0, 1
thmin2, thmax2 = 0, smp.pi

surintv2 = surintv.subs([(G1,G12s),(G2,G22s),(G3,G32s),
            (x,x2s),(y,y2s),(z,z2s)]).doit().simplify()
display(smp.Integral(surintv2,(rho,rhomin2,rhomax2),(th,thmin2,thmax2)))
#smp.integrate(surintv2,(rho,rhomin2,rhomax2),(th,thmin2,thmax2)).evalf()
```

$$\int_0^{\pi}\int_0^{1} \rho^3 \left(\rho^2 \cos^3(\theta) - 2\rho e^{\rho^2} \sin^2(\theta) - 2\sin\left(\rho^2\right)\sin(\theta)\right)\cos(\theta)\,d\rho\,d\theta$$

In [26]:
```
import numpy as np
from scipy.integrate import dblquad
surintv2f = smp.lambdify([rho,th],surintv2)
dblquad(surintv2f,thmin2,thmax2,rhomin2,rhomax2)[0]
```

Out[26]:
```
0.19634954084936201
```

In [ ]: