Mathematical Physics - III Practical

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Contents

1 Differential Equations

1.1 Euler's Method

$$\frac{dy}{dx} = f(x,y)$$

For n intervals between the integration limits (x_0, x),

$$x_n = x_0 + nh \quad ;(n = 1, 2, 3, ...)$$

By Euler's Formula,

$$y_{n+1} = y_n + hf(x_n, y_n)$$

Algorithm:

1. Define the function $f(x, y)$.
2. Set interval and initial values of x and y.
3. Update,

$$y = y + hf(x,y)$$
$$x = x + h$$

   Iterate this in a loop.
4. Collect the (x,y) data.
5. Plot the graph.

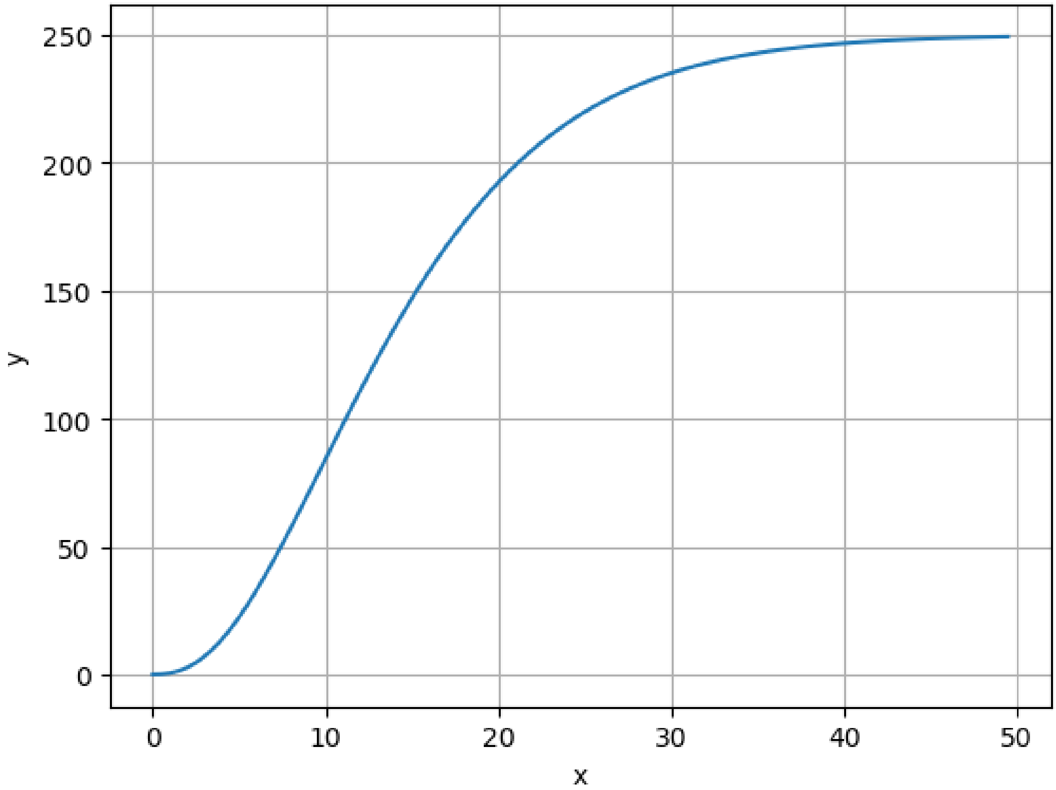
Question: Plot $x - y$ graph for the differential equation,

$$\frac{dy}{dx} = x^2 e^{-x/5}$$

Python Program:

[1]:
```
import numpy as np
import matplotlib.pyplot as plt

def dydx(x,y):
    return x**2 * np.exp(-x/5)

x, y, h = 0, 0, 0.5
xx, yy = [], []

for i in range (100):
```

```
    xx.append(x)
    yy.append(y)
    x += h
    y += h*dydx(x,y)

plt.plot(xx,yy)
plt.xlabel('x')
plt.ylabel('y')
plt.grid()
plt.show()
```



1.2 Modified Euler's Method

To get a better approximation by *trapezoidal rule*,

$$y_{n+1} = y_0 + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$$

By applying iteration on this, we can get a better solution.

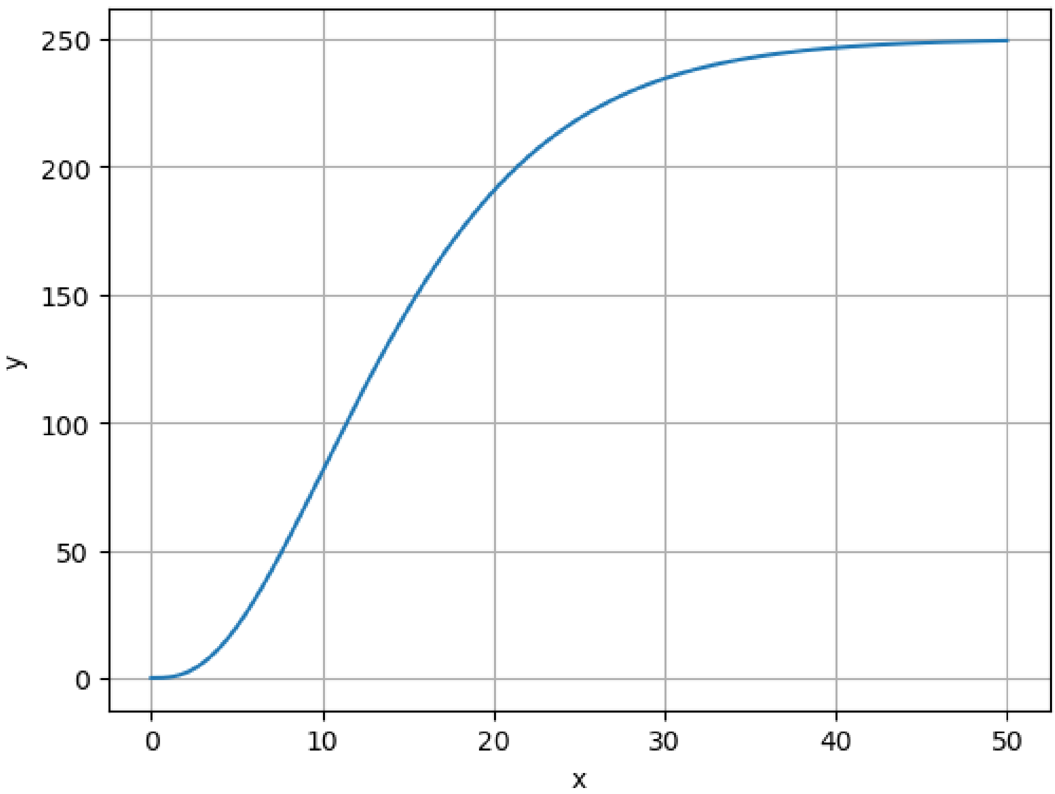
Question: Plot $x - y$ graph for the differential equation,

$$\frac{dy}{dx} = x^2 e^{-x/5}$$

Python Program:

[2]:
```
import numpy as np
import matplotlib.pyplot as plt
```

```
def dydx(x,y):
    return x**2 * np.exp(-x/5)

xm = 50       # upper limit of x
x, y = 0, 0
h = 0.005
xx, yy = [], []

while abs(x) < abs(xm):
    x += h
    dy = (h/2)*(dydx(x,y) + dydx(x + h, y + h*dydx(x,y)))
    y += dy
    xx.append(x), yy.append(y)

plt.plot(xx,yy)
plt.xlabel('x')
plt.ylabel('y')
plt.grid()
plt.show()
```



1.3 Runge - Kutta Method

Here, the change of y is further modified. Let h and k be the changes in x and y.

$$k_1 = hf(x, y)$$
$$k_2 = hf(x + \frac{h}{2}, y + \frac{k_1}{2})$$
$$k_3 = hf(x + \frac{h}{2}, y + \frac{k_2}{2})$$
$$k_4 = hf(x + h, y + k_3)$$

At last, y should be,

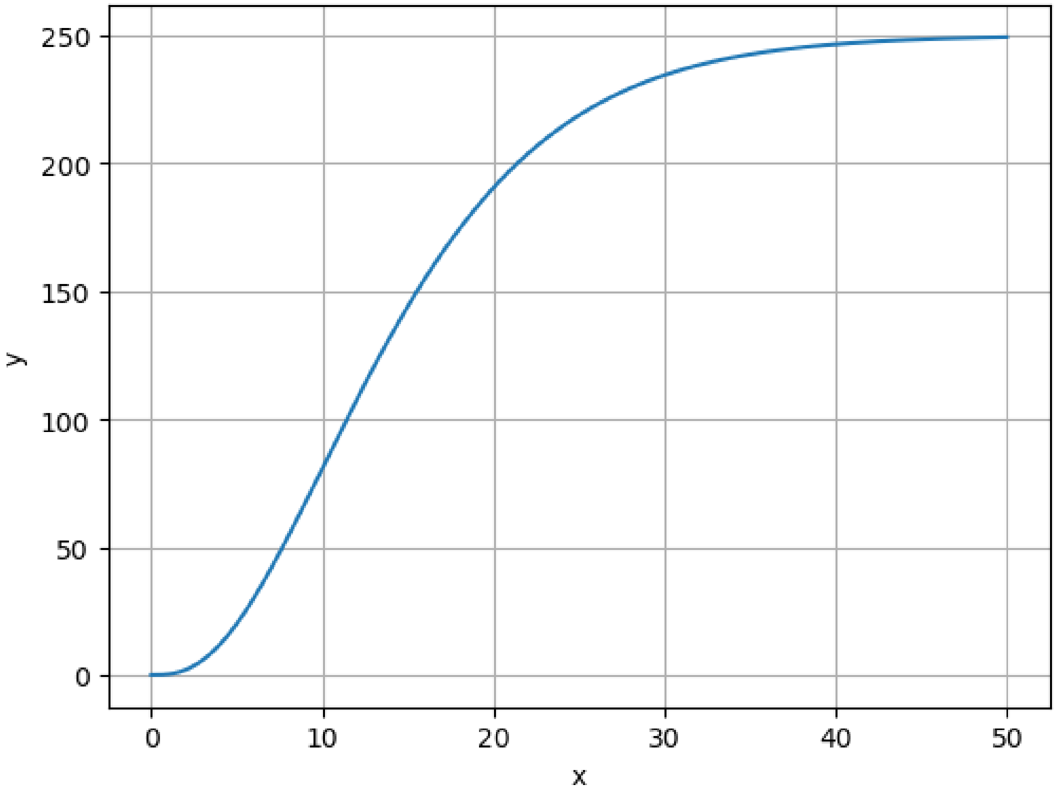
$$y = y + \frac{1}{6}[k_1 + 2(k_2 + k_3) + k_4]$$

Question: Plot $x - y$ graph for the differential equation,

$$\frac{dy}{dx} = x^2 e^{-x/5}$$

Python Program:

[3]:
```
import numpy as np
import matplotlib.pyplot as plt

def dydx(x,y):
    return x**2 * np.exp(-x/5)

xm = 50
x, y = 0, 0
h = 0.005
xx, yy = [], []

while abs(x) < abs(xm):
    xx.append(x), yy.append(y)
    x += h
    k1 = h * dydx(x,y)
    k2 = h * dydx(x + (h/2), y + (k1/2))
    k3 = h * dydx(x + (h/2), y + (k2/2))
    k4 = h * dydx(x + h, y + k3)
    y += (1/6)*(k1 + 2*(k2 + k3) + k4)

plt.plot(xx,yy)
plt.xlabel('x')
plt.ylabel('y')
plt.grid()
plt.show()
```



1.4 2nd Order Differential Equations

$$ay'' + by' + cy = 0 \quad ; y' = \frac{dy}{dx}$$

Let,

$$y' = z \quad ....(1)$$

So,

$$az' + bz + cy = 0 \quad ....(2)$$

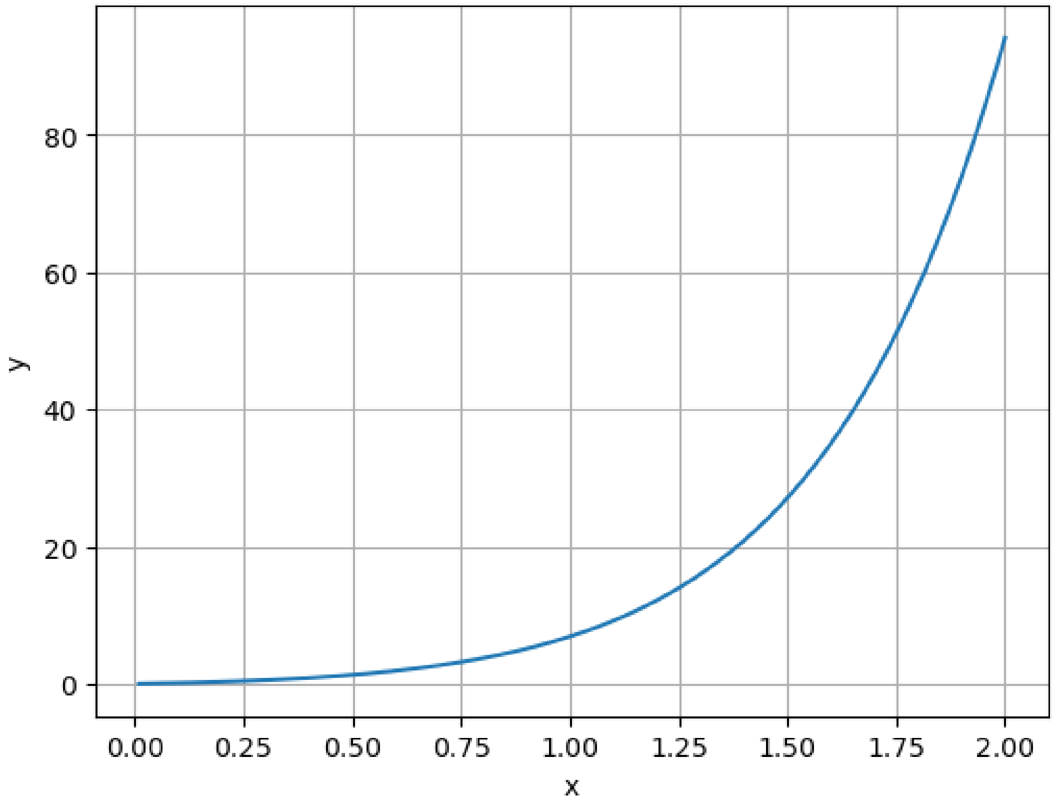
Question: Plot $x - y$ graph for the differential equation,

$$y'' - 4y' + 4y = 0$$

Python Program:

[4]:
```
import matplotlib.pyplot as plt

pr = [1,-4,4]    # parameters = [a,b,c]
x,y,z = 0,0,1
xm = 2
dx = 0.01
xx, yy, zz = [], [], []

def dydx(x,y,z):
    return z
def dzdx(x,y,z):
    return (-1/pr[0]) * (pr[1]*z + pr[2]*y)
```

```
while abs(x) < abs(xm):
    x = x + dx
    y = y + dx * dydx(x,y,z)
    z = z + dx * dzdx(x,y,z)

    xx.append(x)
    yy.append(y)
    zz.append(z)

plt.plot(xx,yy)
plt.xlabel('x')
plt.ylabel('y')
plt.grid()
plt.show()
```



1.5 3rd Order ODE

Question:

$$y''' - 2y'' - y' + 2y = x^2$$

1.5.1 Numerical Solution by different methods

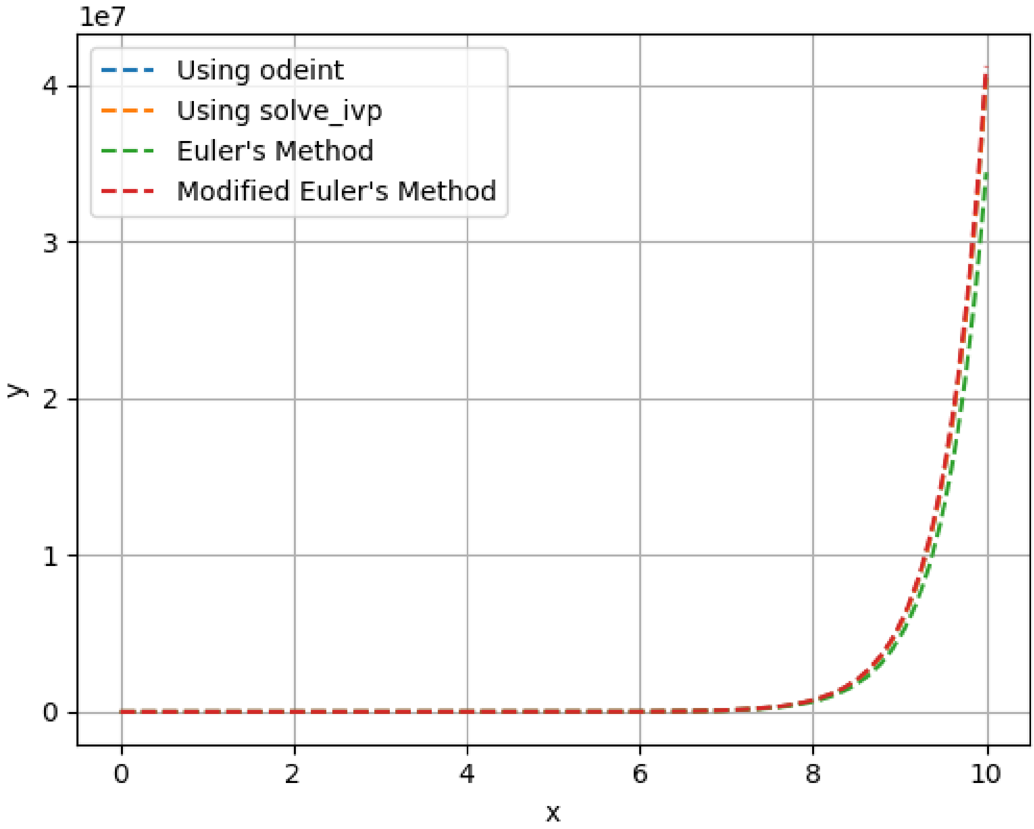
[5]:
```
from scipy.integrate import odeint, solve_ivp
```

[6]:
```
# Write the differential equation. (dy/dx = yp, d2y/dx2 = ypp)

def dSdx(x,S):
    y, yp, ypp = S
```

```
    return [yp, ypp, 2*ypp + yp - 2*y + x**2]

def dydx(x,y,yp,ypp):
    return yp
def dypdx(x,y,yp,ypp):
    return ypp
def dyppdx(x,y,yp,ypp):
    return 2*ypp + yp - 2*y + x**2

x_0, y_0, yp_0, ypp_0 = 0, 0, 0, 0  # initial conditions
x_min, x_max = x_0, 10     # lower and upper limit of x
dx = (x_max-x_0)/1000     # infinitesimal length
```

[7]:
```
# ALL IN ONE
# Using odeint
y0, yp0, ypp0 = y_0, yp_0, ypp_0
S0 = (y0,yp0,ypp0)
x = np.linspace(x_min, x_max,200)
sol = odeint(dSdx, y0=S0, t=x, tfirst=True)
y1 = sol.T[0]
plt.plot(x,y1, '--', label='Using odeint')

# Using solve_ivp
y0, yp0, ypp0 = y_0, yp_0, ypp_0
S0 = (y0,yp0,ypp0)
x = np.linspace(x_min, x_max,200)
sol = solve_ivp(dSdx, t_span=(min(x), max(x)), y0=S0, t_eval=x)
y1 = sol.y[0]
plt.plot(sol.t,y1, '--', label='Using solve_ivp')

# Euler's Method
x, y, yp, ypp = x_0, y_0, yp_0, ypp_0
xmax = x_max
h = dx
xx, yy, yyp, yypp = [], [], [], []
while abs(x) < abs(xmax):
    xx.append(x)
    yy.append(y)
    yyp.append(yp)
    yypp.append(ypp)
    x += h
    y += h*dydx(x,y,yp,ypp)
    yp += h*dypdx(x,y,yp,ypp)
    ypp += h*dyppdx(x,y,yp,ypp)
plt.plot(xx,yy, '--', label='Euler\'s Method')

# Modified Euler's Method
x, y, yp, ypp = x_0, y_0, yp_0, ypp_0
xmax = x_max
h = dx
xx, yy, yyp, yypp = [], [], [], []
while abs(x) < abs(xmax):
    xx.append(x)
    yy.append(y)
    yyp.append(yp)
    yypp.append(ypp)
    x += h
    dy = (h/2) * (dydx(x,y,yp,ypp) +
dydx(x+h, y+h*dydx(x,y,yp,ypp), yp+h*dypdx(x,y,yp,ypp), ypp+h*dyppdx(x,y,yp,ypp)))
    dyp = (h/2) * (dypdx(x,y,yp,ypp) +
```

```
dypdx(x+h, y+h*dydx(x,y,yp,ypp), yp+h*dypdx(x,y,yp,ypp), ypp+h*dyppdx(x,y,yp,ypp)))
    dypp = (h/2) * (dyppdx(x,y,yp,ypp) +
dyppdx(x+h, y+h*dydx(x,y,yp,ypp), yp+h*dypdx(x,y,yp,ypp), ypp+h*dyppdx(x,y,yp,ypp)))
    y += dy
    yp += dyp
    ypp += dypp
plt.plot(xx,yy, '--', label='Modified Euler\'s Method')

plt.xlabel('x')
plt.ylabel('y')
plt.legend()
plt.grid()
plt.show()
```



1.6 Problems

1.6.1 Question-1.1

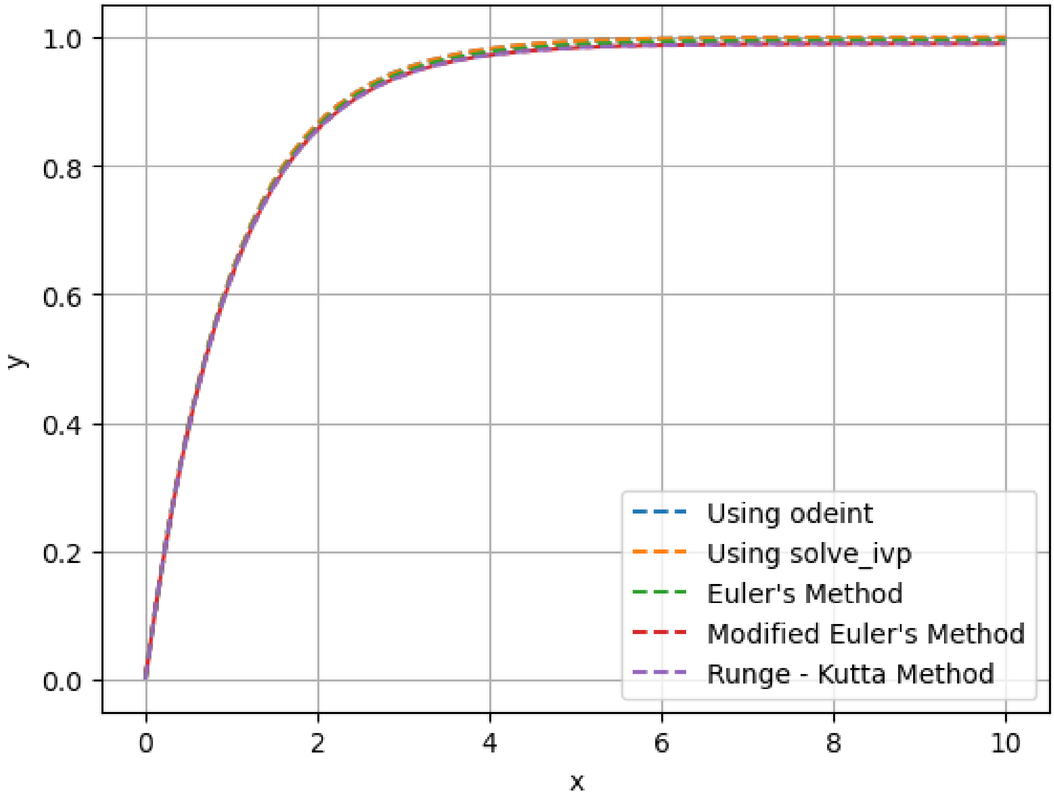
$$\frac{dy}{dx} = e^{-x}$$

Initial condition: $y = 0$ for $x = 0$.

[8]:
```
import numpy as np
import matplotlib.pyplot as plt
import scipy as sp
from scipy.integrate import odeint
from scipy.integrate import solve_ivp
```

```
def dydx(x,y):                    # Write the differential equation.
    return np.exp(-x)

y_0 = 0                 # initial condition
x_min, x_max = 0, 10    # lower and upper limit of x

# Using odeint
y0 = y_0
x = np.linspace(x_min, x_max,500)
sol = odeint(dydx, y0=y0, t=x, tfirst=True)
y1 = sol.T[0]
plt.plot(x,y1, '--', label='Using odeint')

# Using solve_ivp
y0 = y_0
x = np.linspace(x_min, x_max,500)
sol = solve_ivp(dydx, t_span=(min(x), max(x)), y0=[y0], t_eval=x)
y1 = sol.y[0]
plt.plot(x,y1, '--', label='Using solve_ivp')

x_0, y_0 = 0, 0           # initial condition
x_max = 10              # upper limit of x

# Euler's Method
x, y = x_0, y_0
xmax = x_max
h = 0.01
xx, yy = [], []
while abs(x) < abs(xmax):
    xx.append(x)
    yy.append(y)
    x += h
    y += h*dydx(x,y)
plt.plot(xx,yy, '--', label='Euler\'s Method')

# Modified Euler's Method
x, y = x_0, y_0
xmax = x_max
h = 0.01
xx, yy = [], []
while abs(x) < abs(xmax):
    x += h
    dy = (h/2)*(dydx(x,y) + dydx(x + h, y + h*dydx(x,y)))
    y += dy
    xx.append(x), yy.append(y)
plt.plot(xx,yy, '--', label='Modified Euler\'s Method')

# Runge - Kutta Method
x, y = x_0, y_0
xmax = x_max
h = 0.01
xx, yy = [], []
while abs(x) < abs(xmax):
    xx.append(x), yy.append(y)
    x += h
    k1 = h * dydx(x,y)
    k2 = h * dydx(x + (h/2), y + (k1/2))
    k3 = h * dydx(x + (h/2), y + (k2/2))
    k4 = h * dydx(x + h, y + k3)
    y += (1/6)*(k1 + 2*(k2 + k3) + k4)
```

```
plt.plot(xx,yy, '--', label='Runge - Kutta Method')

plt.xlabel('x')
plt.ylabel('y')
plt.legend()
plt.grid()
plt.show()
```



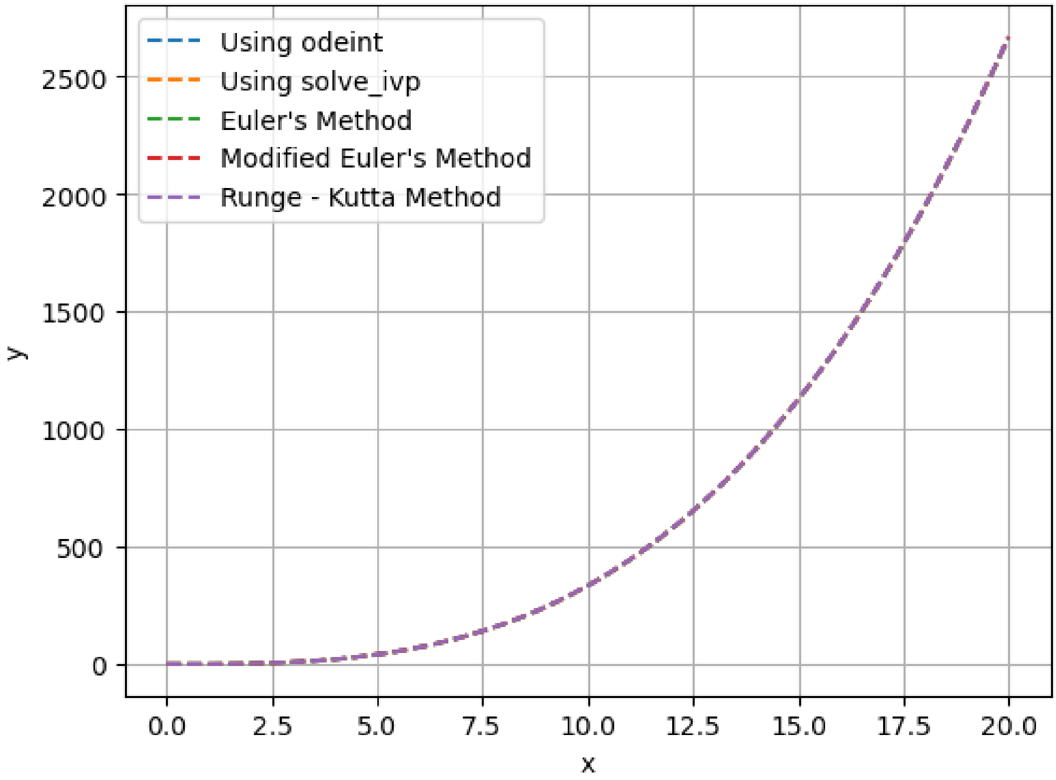
1.6.2 Question-1.2

$$\frac{dy}{dx} + e^{-x} = x^2$$

Initial condition: $y = 0$ for $x = 0$.

[9]:
```
import numpy as np
import matplotlib.pyplot as plt
import scipy as sp
from scipy.integrate import odeint
from scipy.integrate import solve_ivp

def dydx(x,y):                    # Write the differential equation.
    return -np.exp(-x) + x**2

y_0 = 0                 # initial condition
x_min, x_max = 0, 20     # lower and upper limit of x

# Using odeint
y0 = y_0
x = np.linspace(x_min, x_max,500)
sol = odeint(dydx, y0=y0, t=x, tfirst=True)
```

```
y1 = sol.T[0]
plt.plot(x,y1, '--', label='Using odeint')

# Using solve_ivp
y0 = y_0
x = np.linspace(x_min, x_max,500)
sol = solve_ivp(dydx, t_span=(min(x), max(x)), y0=[y0], t_eval=x)
y1 = sol.y[0]
plt.plot(x,y1, '--', label='Using solve_ivp')

x_0, y_0 = 0, 0          # initial condition
x_max = 20              # upper limit of x

# Euler's Method
x, y = x_0, y_0
xmax = x_max
h = 0.01
xx, yy = [], []
while abs(x) < abs(xmax):
    xx.append(x)
    yy.append(y)
    x += h
    y += h*dydx(x,y)
plt.plot(xx,yy, '--', label='Euler\'s Method')

# Modified Euler's Method
x, y = x_0, y_0
xmax = x_max
h = 0.01
xx, yy = [], []
while abs(x) < abs(xmax):
    x += h
    dy = (h/2)*(dydx(x,y) + dydx(x + h, y + h*dydx(x,y)))
    y += dy
    xx.append(x), yy.append(y)
plt.plot(xx,yy, '--', label='Modified Euler\'s Method')

# Runge - Kutta Method
x, y = x_0, y_0
xmax = x_max
h = 0.01
xx, yy = [], []
while abs(x) < abs(xmax):
    xx.append(x), yy.append(y)
    x += h
    k1 = h * dydx(x,y)
    k2 = h * dydx(x + (h/2), y + (k1/2))
    k3 = h * dydx(x + (h/2), y + (k2/2))
    k4 = h * dydx(x + h, y + k3)
    y += (1/6)*(k1 + 2*(k2 + k3) + k4)
plt.plot(xx,yy, '--', label='Runge - Kutta Method')

plt.xlabel('x')
plt.ylabel('y')
plt.legend()
plt.grid()
plt.show()
```



1.6.3 Question-1.3

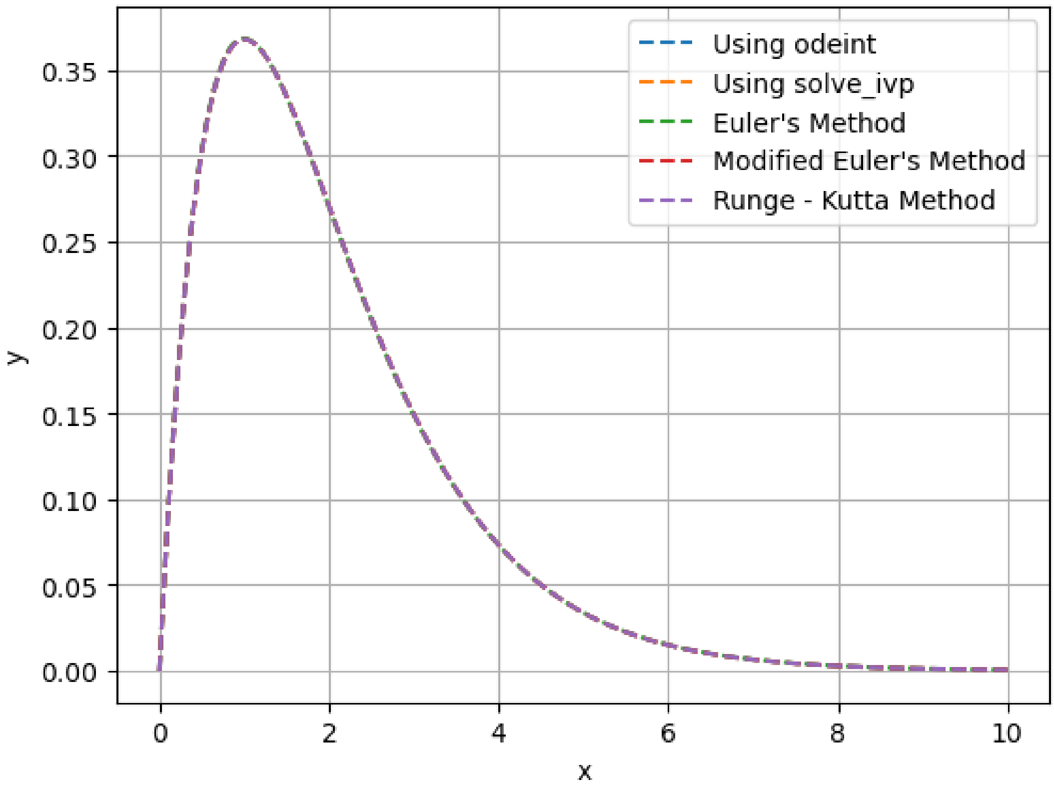
$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

Initial condition: $y = 1$ for $x = 0$.

[10]:
```
import numpy as np
import matplotlib.pyplot as plt
import scipy as sp
from scipy.integrate import odeint
from scipy.integrate import solve_ivp

def dSdx(x,S):         # Write the differential equation. (dy/dx = yp)
    y, yp = S
    return [yp, -2*yp - y]

def dydx(x,y,yp):    # Write the differential equation. (dy/dx = yp)
    return yp
def dypdx(x,y,yp):
    return -2*yp - y

y_0, yp_0 = 0, 1       # initial condition for y and dy/dx
x_min, x_max = 0, 10    # lower and upper limit of x

# Using odeint
y0, yp0 = y_0, yp_0
S0 = (y0,yp0)
x = np.linspace(x_min, x_max,200)
sol = odeint(dSdx, y0=S0, t=x, tfirst=True)
y1 = sol.T[0]
plt.plot(x,y1, '--', label='Using odeint')
```

```
# Using solve_ivp
y0, yp0 = y_0, yp_0
S0 = (y0,yp0)
x = np.linspace(x_min, x_max,200)
sol = solve_ivp(dSdx, t_span=(min(x), max(x)), y0=S0, t_eval=x)
y1 = sol.y[0]
plt.plot(x,y1, '--', label='Using solve_ivp')

x_0, y_0, yp_0 = 0, 0, 1  # initial condition for y and dy/dx
x_max = 10               # upper limit of x

# Euler's Method
x, y, yp = x_0, y_0, yp_0
xmax = x_max
h = 0.01
xx, yy, yyp = [], [], []
while abs(x) < abs(xmax):
    xx.append(x)
    yy.append(y)
    yyp.append(yp)
    x += h
    y += h*dydx(x,y,yp)
    yp += h*dypdx(x,y,yp)
plt.plot(xx,yy, '--', label='Euler\'s Method')

# Modified Euler's Method
x, y, yp = x_0, y_0, yp_0
xmax = x_max
h = 0.01
xx, yy, yyp = [], [], []
while abs(x) < abs(xmax):
    xx.append(x)
    yy.append(y)
    yyp.append(yp)
    x += h
    dy = (h/2)*(dydx(x,y,yp) +
dydx(x + h, y + h*dydx(x,y,yp), yp + h*dypdx(x,y,yp)))
    dyp = (h/2)*(dypdx(x,y,yp) +
dypdx(x + h, y + h*dydx(x,y,yp), yp + h*dypdx(x,y,yp)))
    y += dy
    yp += dyp
plt.plot(xx,yy, '--', label='Modified Euler\'s Method')

# Runge - Kutta Method
x, y, yp = x_0, y_0, yp_0
xmax = x_max
h = 0.01
xx, yy, yyp = [], [], []
while abs(x) < abs(xmax):
    xx.append(x), yy.append(y), yyp.append(yp)
    x += h
    k1 = h * dydx(x,y,yp)
    l1 = h * dypdx(x,y, yp)
    k2 = h * dydx(x + (h/2), y + (k1/2), yp + (l1/2))
    l2 = h * dypdx(x + (h/2), y + (k1/2), yp + (l1/2))
    k3 = h * dydx(x * (h/2), y + (k2/2), yp + (l2/2))
    l3 = h * dypdx(x + (h/2), y + (k2/2), yp + (l2/2))
    k4 = h * dydx(x + h, y + k3, yp + l3)
    l4 = h * dypdx(x + h, y + k3, yp + l3)
```

```
    y += (1/6)*(k1 + 2*(k2 + k3) + k4)
    yp += (1/6)*(l1 + 2*(l2 + l3) + l4)
plt.plot(xx,yy, '--', label='Runge - Kutta Method')

plt.xlabel('x')
plt.ylabel('y')
plt.legend()
plt.grid()
plt.show()
```



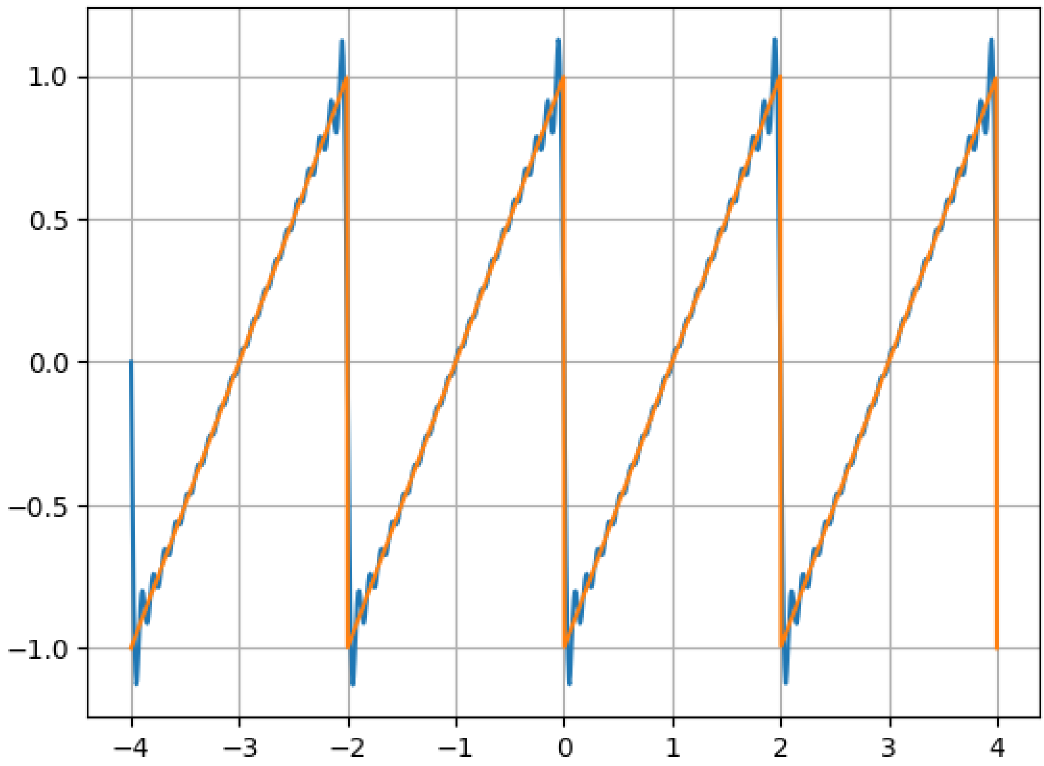
[ ]:

2 Fourier Series

2.1 Sawtooth signal

We can call the function from **scipy.signal**. Here an alternative way to create the function is shown.

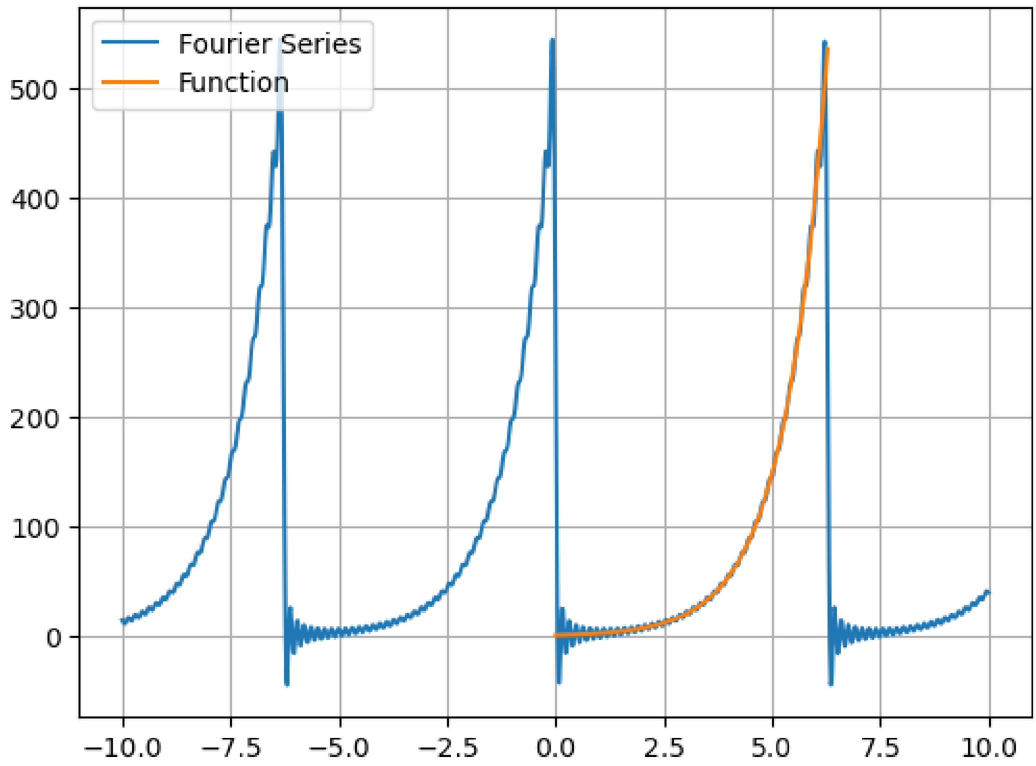
[1]:
```
# alternative method for sawtooth signal
import numpy as np
import matplotlib.pyplot as plt
import scipy as sp
from scipy.integrate import simps

L = 1
x = np.linspace(-L,L,1000)  # Full period
xp = 4*x
f = lambda xp: xp%(2*L) - L

a0 = 1.0/L*simps(f(x),x)
an = lambda n: 1.0/L * simps(f(x)*np.cos(n*np.pi*x/L), x)
bn = lambda n: 1.0/L * simps(f(x)*np.sin(n*np.pi*x/L), x)

S = a0/2 + sum([an(n)*np.cos(n*np.pi*xp/L) +
                bn(n)*np.sin(n*np.pi*xp/L) for  n in range(1,20)])

plt.plot(xp,S,xp,f(xp))
plt.grid()
plt.show()
```



Example:

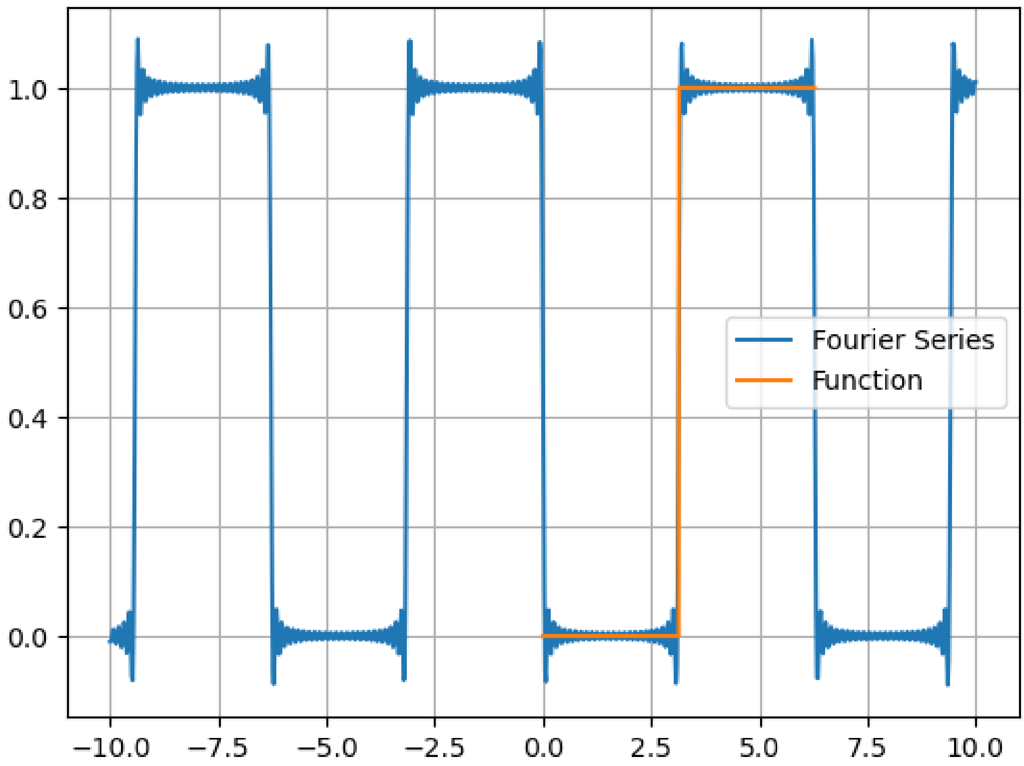
$$f(x) = e^x \quad ; (0 < x < 2\pi)$$

[2]:
```
import numpy as np
import matplotlib.pyplot as plt
import scipy as sp
from scipy.integrate import simps

x = np.linspace(0, 2*np.pi, 1000)    # period of x
f = lambda x: np.exp(x)    # function

a0 = 1/np.pi * simps(f(x),x)
an = lambda n: (1/np.pi) * simps(f(x)*np.cos(n*x), x)
bn = lambda n: (1/np.pi) * simps(f(x)*np.sin(n*x), x)

L1, L2 = -10, 10      # length of the signal
N = 40            # no. of terms in Fourier Series
xp = np.linspace(L1,L2,1000)
S = a0 * 0.5 + sum([an(n)* np.cos(n*xp) +
                    bn(n)*np.sin(n*xp) for n in range (1,N)])

import matplotlib.pyplot as plt
plt.plot(xp, S, label='Fourier Series')
plt.plot(x, f(x), label='Function')
plt.legend()
plt.grid()
plt.show()
```



2.2 Step Function

[3]:
```
import numpy as np
import matplotlib.pyplot as plt
import scipy as sp
from scipy.integrate import simps

# step function
f = lambda x: np.array([0 if 0<=i<np.pi else 1 for i in x])
x = np.linspace(0, 2*np.pi, 1000)        # period of x

a0 = 1/np.pi * simps(f(x),x)
an = lambda n: (1/np.pi) * simps(f(x)*np.cos(n*x), x)
bn = lambda n: (1/np.pi) * simps(f(x)*np.sin(n*x), x)

L = 10      # length of the signal
N = 50      # no. of terms in Fourier Series
xp = np.linspace(-L,L,1000)
S = a0 * 0.5 + sum([an(n)* np.cos(n*xp) +
                    bn(n)*np.sin(n*xp) for n in range (1,N)])

import matplotlib.pyplot as plt
plt.plot(xp, S, label='Fourier Series')
plt.plot(x, f(x), label='Function')
plt.legend()
plt.grid()
plt.show()
```

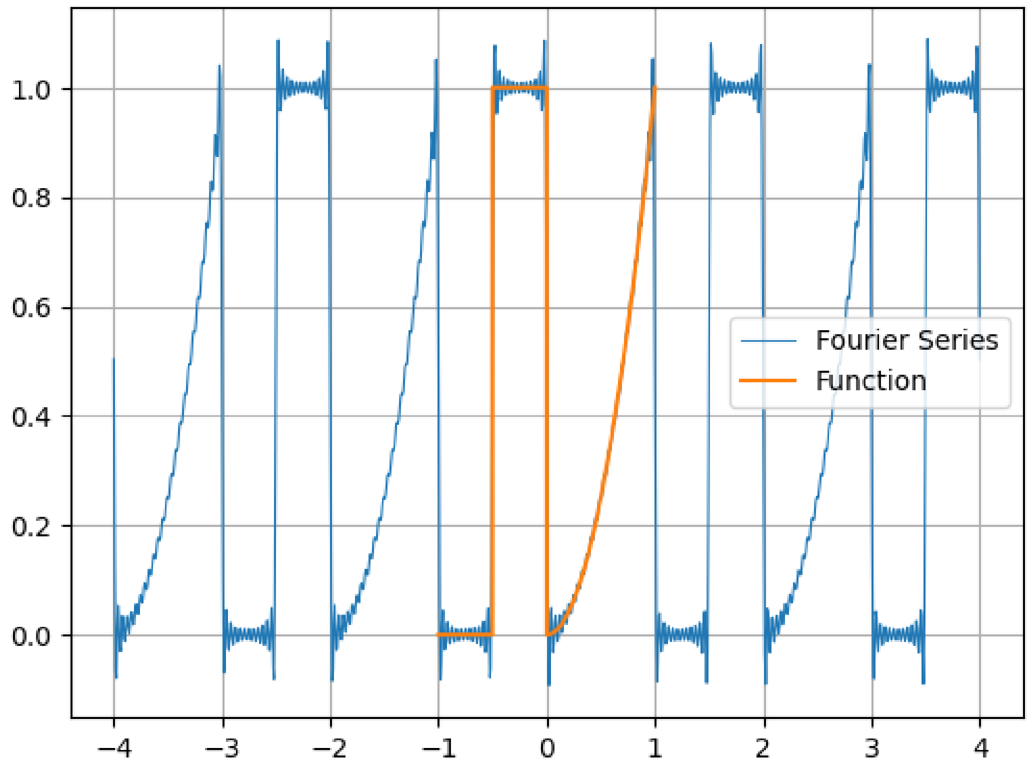


Example

$$\begin{aligned} f(x) &= 0 \quad ;(-1 \leq x < -0.5) \\ &= 1 \quad ;(-0.5 \leq x < 0) \\ &= x^2 \quad ;(0 \leq x < 1) \end{aligned}$$

[4]:
```
import numpy as np
import matplotlib.pyplot as plt
import scipy as sp
from scipy.integrate import simps

M1, M2 = -1, 1          # period of x
# function
x = np.linspace(M1, M2, 1000)
f = lambda x: np.array([0 if M1<=i<-0.5
                        else 1 if -0.5<=i<0 else i**2 for i in x])

a0 = 2/(M2-M1) * simps(f(x), x)
an = lambda n: (2/(M2-M1)) * simps(f(x)*np.cos(n*np.pi*x*2/(M2-M1)), x)
bn = lambda n: (2/(M2-M1)) * simps(f(x)*np.sin(n*np.pi*x*2/(M2-M1)), x)

L1, L2 = -4, 4      # length of the signal
N = 50      # no. of terms in Fourier Series
xp = np.linspace(L1,L2,1000)
S = a0 * 0.5 + sum([an(n)* np.cos(n*np.pi*xp*2/(M2-M1)) +
                    bn(n)*np.sin(n*np.pi*xp*2/(M2-M1)) for n in range (1,N)])

import matplotlib.pyplot as plt
plt.plot(xp, S, lw=0.8, label='Fourier Series')
plt.plot(x, f(x), label='Function')
```

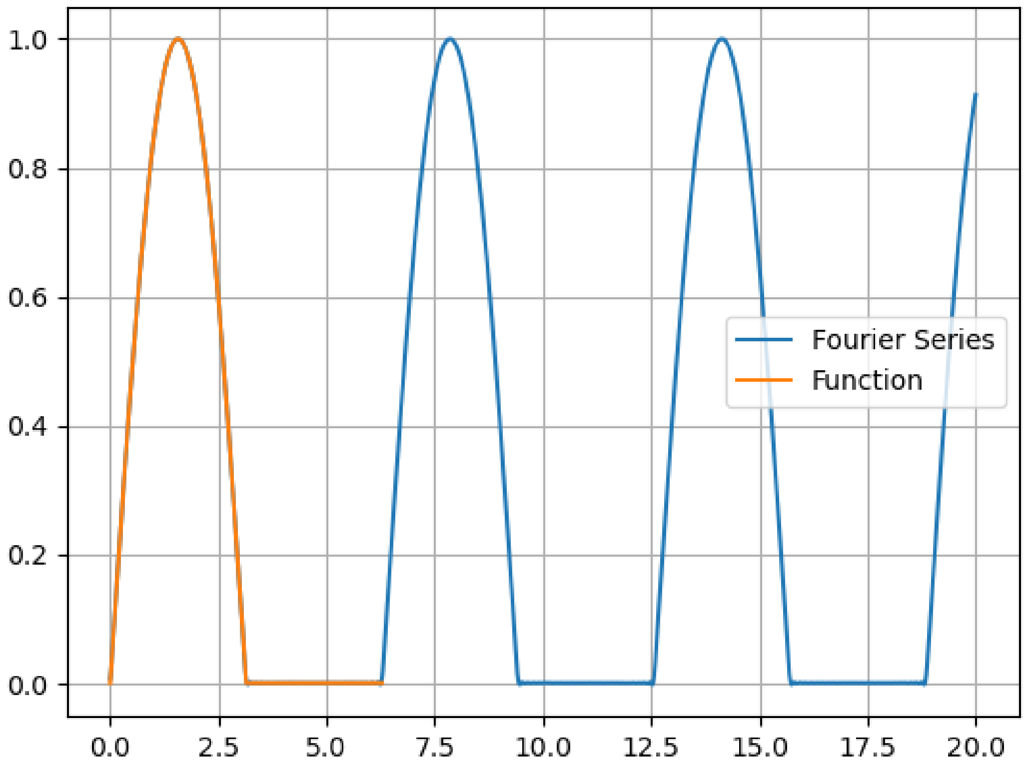
```
plt.legend()
plt.grid()
plt.show()
```



2.3 Half Wave Rectifier

[5]:
```
import numpy as np
import matplotlib.pyplot as plt
import scipy as sp
from scipy.integrate import simps

M1, M2 = 0, 2*np.pi          # period of x
# function
x = np.linspace(M1, M2, 1000)
f = lambda x: np.array([np.sin(i) if M1<=i<(M2-M1)/2
                        else 0 for i in x])

a0 = 2/(M2-M1) * simps(f(x), x)
an = lambda n: (2/(M2-M1)) * simps(f(x)*np.cos(n*np.pi*x*2/(M2-M1)), x)
bn = lambda n: (2/(M2-M1)) * simps(f(x)*np.sin(n*np.pi*x*2/(M2-M1)), x)

L1, L2 = 0, 20       # length of the signal
N = 50       # no. of terms in Fourier Series
xp = np.linspace(L1,L2,1000)
S = a0 * 0.5 + sum([an(n)* np.cos(n*np.pi*xp*2/(M2-M1)) +
                    bn(n)*np.sin(n*np.pi*xp*2/(M2-M1)) for n in range (1,N)])

import matplotlib.pyplot as plt
plt.plot(xp, S, label='Fourier Series')
plt.plot(x, f(x), label='Function')
```

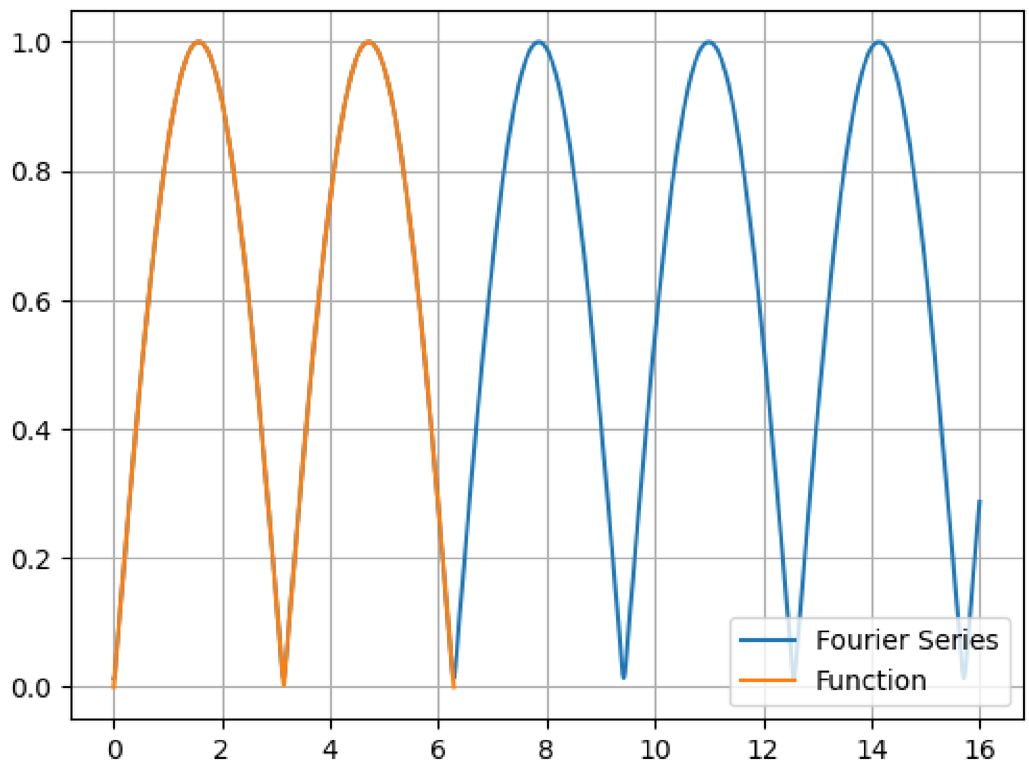
```
plt.legend()
plt.grid()
plt.show()
```



2.4 Full Wave Rectifier

[6]:
```
import numpy as np
import matplotlib.pyplot as plt
import scipy as sp
from scipy.integrate import simps

M1, M2 = 0, 2*np.pi          # period of x
# function
x = np.linspace(M1, M2, 1000)
f = lambda x: abs(np.sin(x))

a0 = 2/(M2-M1) * simps(f(x), x)
an = lambda n: (2/(M2-M1)) * simps(f(x)*np.cos(n*np.pi*x*2/(M2-M1)), x)
bn = lambda n: (2/(M2-M1)) * simps(f(x)*np.sin(n*np.pi*x*2/(M2-M1)), x)

L1, L2 = 0, 16      # length of the signal
N = 50      # no. of terms in Fourier Series
xp = np.linspace(L1,L2,1000)
S = a0 * 0.5 + sum([an(n)* np.cos(n*np.pi*xp*2/(M2-M1)) +
                    bn(n)*np.sin(n*np.pi*xp*2/(M2-M1)) for n in range (1,N)])

import matplotlib.pyplot as plt
plt.plot(xp, S, label='Fourier Series')
plt.plot(x, f(x), label='Function')
plt.legend()
```

```
plt.grid()
plt.show()
```



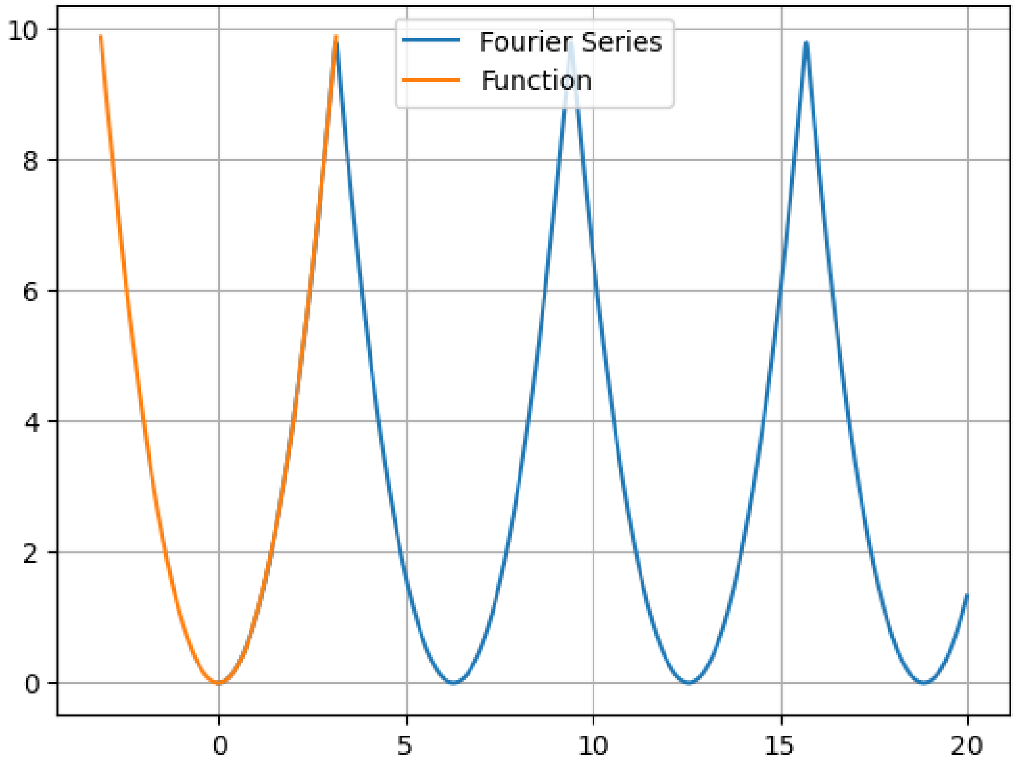
Example

$$f(x) = x^2 \quad ; (-\pi < x < \pi)$$

By using this series, Riemann Zeta 2 function can be obtained.

[7]:
```
import numpy as np
import matplotlib.pyplot as plt
import scipy as sp
from scipy.integrate import simps

M1, M2 = -np.pi, np.pi          # period of x
# function
x = np.linspace(M1, M2, 1000)
f = lambda x: x**2

a0 = 2/(M2-M1) * simps(f(x), x)
an = lambda n: (2/(M2-M1)) * simps(f(x)*np.cos(n*np.pi*x*2/(M2-M1)), x)
bn = lambda n: (2/(M2-M1)) * simps(f(x)*np.sin(n*np.pi*x*2/(M2-M1)), x)

L1, L2 = 0, 20      # length of the signal
N = 50      # no. of terms in Fourier Series
xp = np.linspace(L1,L2,1000)
S = a0 * 0.5 + sum([an(n)* np.cos(n*np.pi*xp*2/(M2-M1)) +
                    bn(n)*np.sin(n*np.pi*xp*2/(M2-M1)) for n in range (1,N)])

import matplotlib.pyplot as plt
plt.plot(xp, S, label='Fourier Series')
```

```
plt.plot(x, f(x), label='Function')
plt.legend()
plt.grid()
plt.show()
```

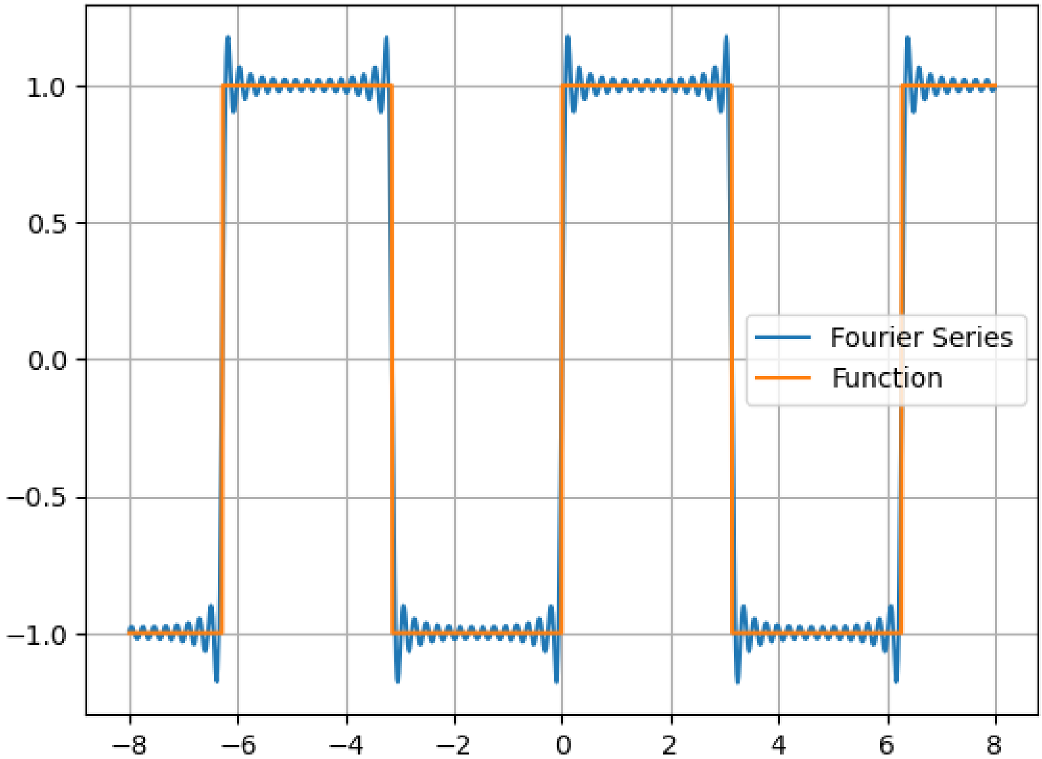


Similarly, Riemann Zeta 4 function can be obtained by using $f(x) = x^4\,; (-\pi < x < \pi)$.

2.5 Question-3: Fourier Series of Square Wave

[8]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import simps
from scipy.signal import square
import scipy as sp

x = np.linspace(-np.pi, np.pi, 1000)
f = square (x)
a0 = 1/np.pi * simps(f,x)
an = lambda n: 1/np.pi * simps(f*np.cos(n*x), x)
bn = lambda n: 1/np.pi * simps(f*np.sin(n*x), x)

L = 8       # length of the signal
N = 30      # number of sine and cosine terms
xp = np.linspace(-L,L,1000)
S = a0 * 0.5 + sum([an(n)* np.cos(n*xp)+
                    bn(n)*np.sin(n*xp) for n in range (1,N)])

plt.plot(xp, S, label='Fourier Series')
plt.plot(xp, square(xp), label='Function')
```

```
plt.legend()
plt.grid()
plt.show()
```



[ ]:

3 Fourier Transforms

3.1 Discrete Fourier Transform (DFT)

Discrete Fourier Transform (DFT):

$$F_m = \sum_{n=0}^{N-1} f_n e^{-2\pi jmn/N} \quad ;(m = 0, 1, 2, 3, ..., N-1)$$

Discrete Inverse Fourier Transform (DIFT):

$$f_n = \sum_{m=0}^{N-1} F_m e^{2\pi jmn/N} \quad ;(n = 0, 1, 2, 3, ..., N-1)$$

[1]:
```
import numpy as np
```

Defining `dft` function for both DFT and DIFT:

[2]:
```
def dft(ft, isg):
    N = len(ft)
    Fs = []
    for m in range(N):
        Fk = 0
        for n in range(N):
            Fk += ft[n] * np.exp(-isg*2*np.pi*1j*m*n/N)
        if isg == 1:
            Fs.append(Fk)
        elif isg == -1:
            Fs.append(Fk/N)
    return Fs
```

Defining `cntdft` function for getting centered or two-sided transform instead of one-sided transform:

[3]:
```
def cntdft(ft, isg):
    N = len(ft)
    a = (N-1)/2
    exft = [ft[i]*np.exp(2*np.pi*1j*a*i/N) for i in range(N)] # pre-transform
    Fs = dft(exft, isg)
    Fs = [Fs[i]*np.exp(2*np.pi*1j*a*(i-a)/N) for i in range(N)] # post-transform
    return Fs
```

Defining `fourspc` function for Fourier space co-ordinates:

$$\delta x \delta k = \frac{2\pi}{N}\,; \quad k_{mx} = \left(1 - \frac{1}{N}\right)\frac{\pi}{\delta x}$$

[4]:
```
def fourspc(x):
    N = len(x)
    dx = x[1]-x[0]
    dk = 2*np.pi/(N*dx)
    kmx = (1 -1/N)*np.pi/dx
    k = [-kmx + i*dk for i in range(N)]
    return k
```

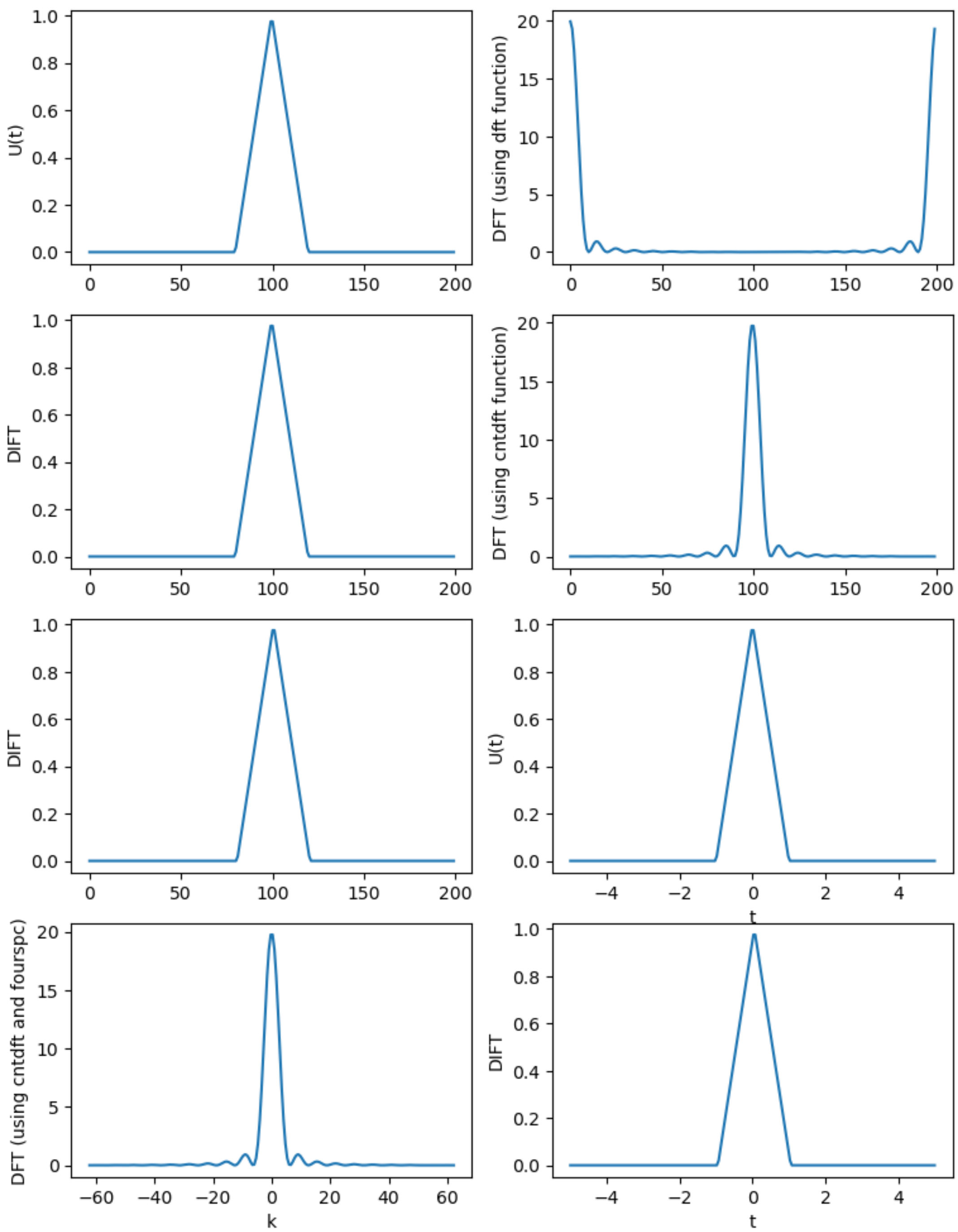
3.1.1 Examples

Example 1

$$\begin{aligned} U(t) &= 1+t \quad ;(-1 \le t < 0) \\ &= 1-t \quad ;(0 \le t < 1) \\ &= 0 \quad otherwise \end{aligned}$$

[5]:
```
import matplotlib.pyplot as plt

def u(t):
    if -1<=t<0:
        ut = 1+t
    elif 0<=t<1:
        ut = 1-t
    else:
        ut = 0
    return ut

tmn, tmx = -5, 5  # time bounds
N = 200     # no. of samples
dt = (tmx-tmn)/(N-1)

t = [tmn + i*dt for i in range(N)] # time samples
k = fourspc(t)
ut = [u(tt) for tt in t] # discrete signal

Fs1 = dft(ut, 1)  # DFT
IFs1 = dft(Fs1, -1)  # DIFT
Fs2 = cntdft(ut, 1)
IFs2 = cntdft(Fs2, -1)

plt.figure(figsize=(9,12))
plt.subplot(4,2,1)
plt.plot(ut)
plt.ylabel('U(t)')

plt.subplot(4,2,2)
plt.plot(np.abs(Fs1))
plt.ylabel('DFT (using dft function)')
plt.subplot(4,2,3)
plt.plot(np.abs(IFs1))
plt.ylabel('DIFT')

plt.subplot(4,2,4)
plt.plot(np.abs(Fs2))
plt.ylabel('DFT (using cntdft function)')
plt.subplot(4,2,5)
plt.plot(np.abs(IFs2))
```

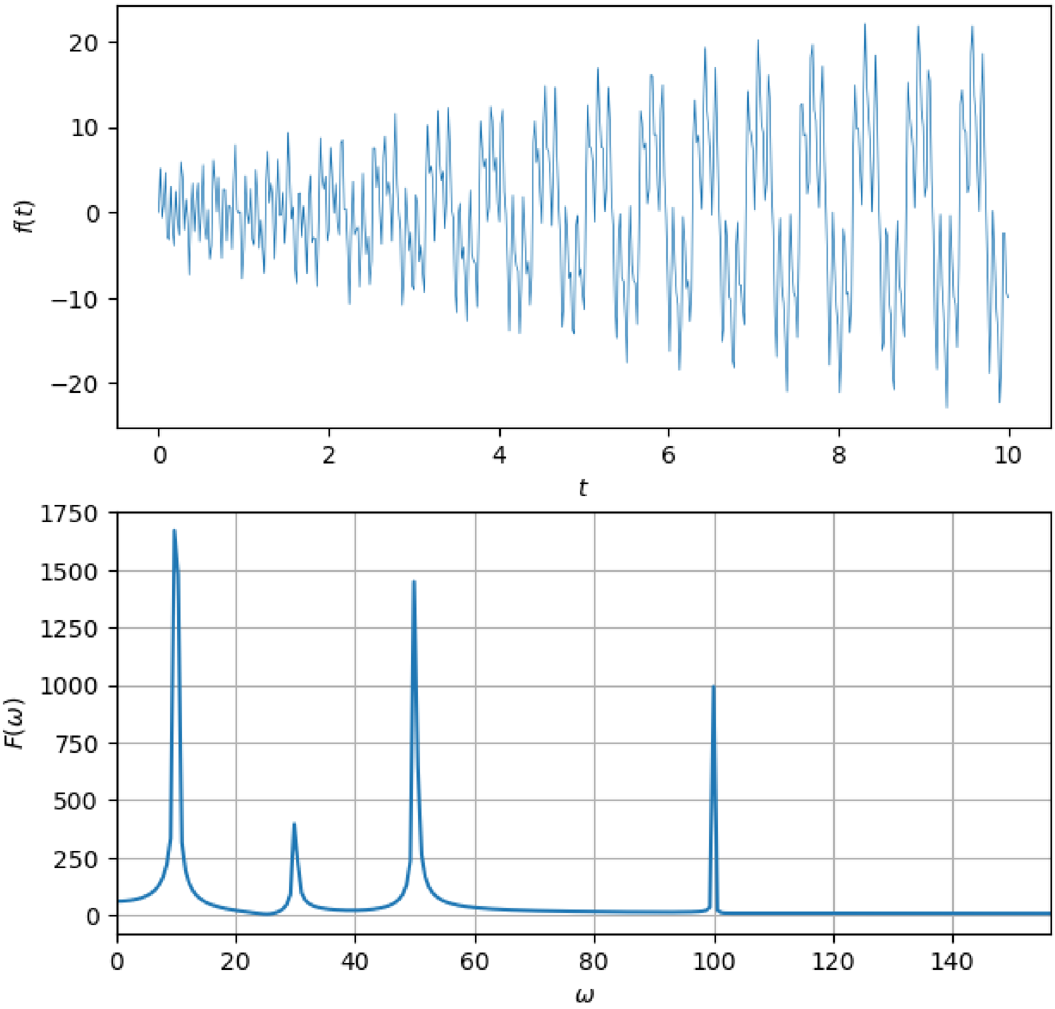
```
plt.ylabel('DIFT')

plt.subplot(4,2,6)
plt.plot(t, ut)
plt.xlabel('t')
plt.ylabel('U(t)')
plt.subplot(4,2,7)
plt.plot(k, np.abs(Fs2))
plt.xlabel('k')
plt.ylabel('DFT (using cntdft and fourspc)')
plt.subplot(4,2,8)
plt.plot(t, np.abs(IFs2))
plt.xlabel('t')
plt.ylabel('DIFT')
plt.show()
```



Example 2

Fourier transform of a signal which is superpositions of a number of signals with different frequencies and amplitudes.

[6]:
```
def f(t):
    freqs = [10, 30, 50, 100] # frequencies
    amps = [t**2*np.exp(-t/5), 2, 3*t**0.5, 4]  # amplitudes
    phis = [0, 0, 0, 0]  # initial phases
    ft = 0
    for i in range(len(freqs)):
        ft += amps[i]*np.sin(freqs[i]*t +phis[i])
    return ft

N = 500
tmn, tmx = 0, 10
dt = (tmx-tmn)/(N-1)
t = [tmn+i*dt for i in range(N)]
w = fourspc(t)
ft = [f(tt) for tt in t]
Fw = cntdft(ft, 1)

plt.figure(figsize=(7, 7))
plt.subplot(211)
plt.plot(t, ft, lw=0.5)
plt.xlabel('$t$')
plt.ylabel('$f(t)$')
plt.subplot(212)
plt.plot(w, np.abs(Fw))
plt.xlim(0, np.max(w))
plt.xlabel('$\omega$')
plt.ylabel('$F(\omega)$')
plt.grid()
plt.show()
```



3.2 Fast Fourier Transform (FFT)

[7]:
```
import numpy as np
```

[8]:
```
def fft2(ft, isg):
    N = len(ft)
    if N==1:
        F = ft # dft is length 1
    else:
        # divide the dft into 2 using radix-2 Cooley-Tukey
        Am = fft2(ft[::2], isg)
        Bm = fft2(ft[1::2], isg)
        # combine with appropriate weights
        m = np.arange(N/2)
        W = np.exp(-isg*2*np.pi*1j*m/N)
        F = np.concatenate([Am + W*Bm, Am - W*Bm])
    return F
```

[9]:
```
def fft(ft, isg):
    N = len(ft)
    if isg==1:
        return fft2(ft, isg)
    elif isg==-1:
```

```
        return fft2(ft, isg)/N
```

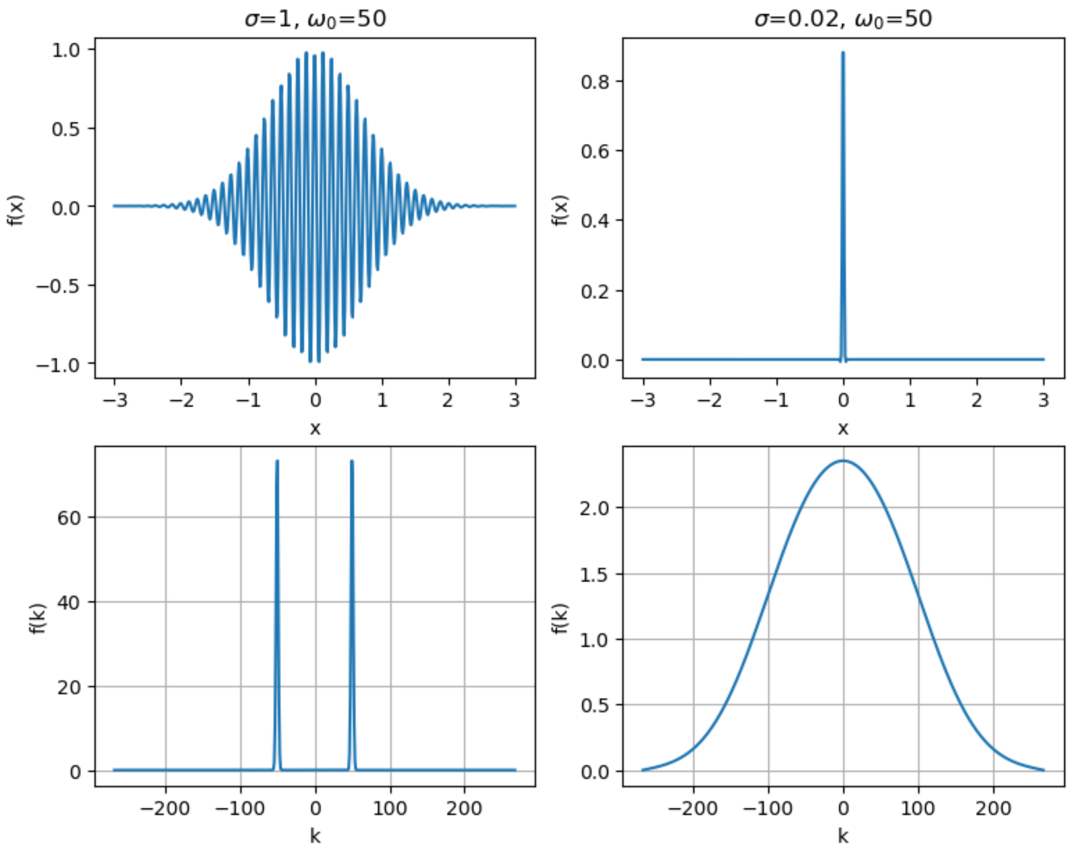
[10]:
```
def cntfft(ft, isg):
    N = len(ft)
    a = (N-1)/2
    exft = [ft[i]*np.exp(2*np.pi*1j*a*i/N) for i in range(N)] # pre-transform
    Fs = fft(exft, isg)
    Fs = [Fs[i]*np.exp(2*np.pi*1j*a*(i-a)/N) for i in range(N)] # post-transform
    return Fs
```

3.2.1 Example

Real part of Gaussian function:

$$f(x) = e^{-\dfrac{x^2}{\sigma^2}} \cos \omega_0 x$$

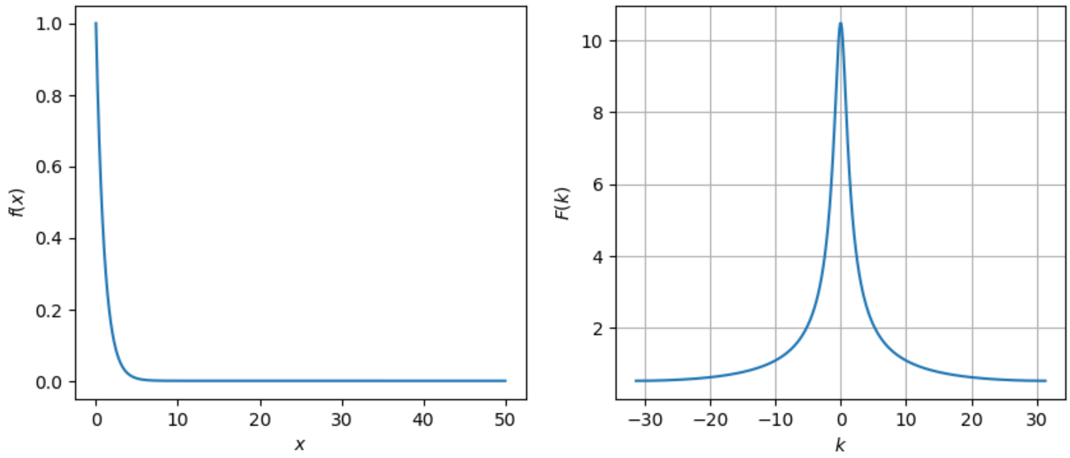
[11]:
```
def f(pr, x):
    sig, w0 = pr
    return np.exp(-x**2/sig**2)*np.cos(w0*x)

N = 512  # N should be in the form 2**n
xmn, xmx = -3, 3
dt = (xmx-xmn)/(N-1)
x = [xmn+i*dt for i in range(N)]
k = fourspc(x)

sig1, w01 = 1, 50  # parameters
pr1 = [sig1, w01]
fx1 = [f(pr1, xx) for xx in x]
fk1 = cntfft(fx1, 1)
sig2, w02 = 0.02, 50  # parameters
pr2 = [sig2, w02]
fx2 = [f(pr2, xx) for xx in x]
fk2 = cntfft(fx2, 1)

plt.figure(figsize=(9, 7))
plt.subplot(221)
plt.title(f'$\sigma$={sig1}, $\omega_0$={w01}')
plt.plot(x, fx1)
plt.xlabel('x')
plt.ylabel('f(x)')
plt.subplot(223)
plt.plot(k, np.abs(fk1))
plt.xlabel('k')
plt.ylabel('f(k)')
plt.grid()

plt.subplot(222)
plt.title(f'$\sigma$={sig2}, $\omega_0$={w02}')
plt.plot(x, fx2)
plt.xlabel('x')
plt.ylabel('f(x)')
plt.subplot(224)
plt.plot(k, np.abs(fk2))
plt.xlabel('k')
plt.ylabel('f(k)')
plt.grid()
plt.show()
```



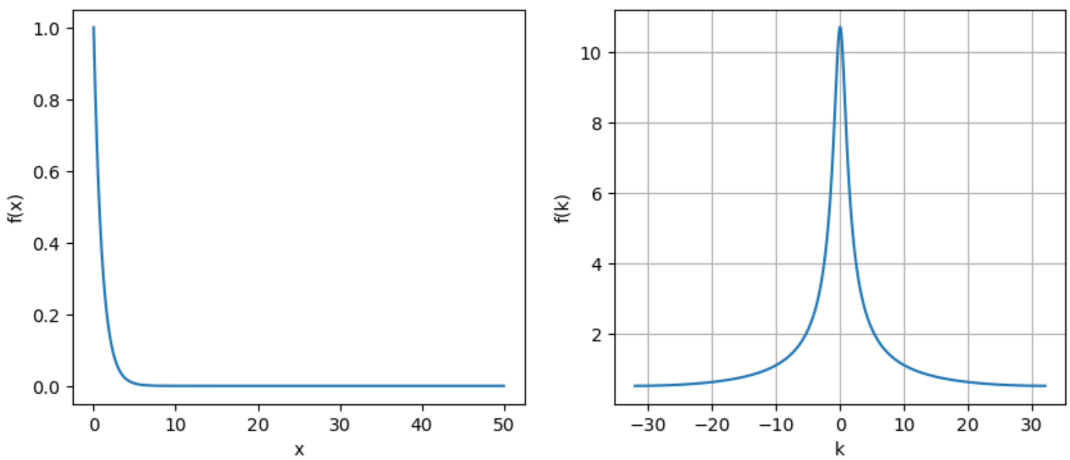
3.2.2 Question-10

Find FFT of e^{-x}.

[12]:
```
def f(x):
    return np.exp(-x)

N = 500
xmn, xmx = 0, 50
dx = (xmx-xmn)/(N-1)
x = [xmn+i*dx for i in range(N)]
k = fourspc(x)
fx = [f(xx) for xx in x]
Fk = cntdft(fx, 1)

plt.figure(figsize=(10, 4))
plt.subplot(121)
plt.plot(x, fx)
plt.xlabel('$x$')
plt.ylabel('$f(x)$')
plt.subplot(122)
plt.plot(k, np.abs(Fk))
# plt.xlim(0, np.max(k))
plt.xlabel('$k$')
plt.ylabel('$F(k)$')
plt.grid()
plt.show()
```



[13]:
```
def f(x):
    return np.exp(-x)
N = 512  # N should be in the form 2**n
xmn, xmx = 0, 50
dx = (xmx-xmn)/(N-1)
x = [xmn+i*dx for i in range(N)]
k = fourspc(x)

fx = [f(xx) for xx in x]
fk = cntfft(fx, 1)

plt.figure(figsize=(10, 4))
plt.subplot(121)
plt.plot(x, fx)
plt.xlabel('x')
plt.ylabel('f(x)')
plt.subplot(122)
plt.plot(k, np.abs(fk))
plt.xlabel('k')
plt.ylabel('f(k)')
plt.grid()
plt.show()
```



[ ]:
```

```

4 Bessel Functions

Bessel's Differential Equation:

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

Solution (Bessel Functions):

$$j_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{(2m+n)}$$

$$j_{-n}(x) = (-1)^n j_n(x)$$

from `scipy.special`

[1]:
```
from scipy.special import *
jn(3,2)
```

[1]: 0.12894324947440208

By calculations

[2]:
```
import numpy as np
from scipy.special import factorial

def besselfn(n,x):
    return sum([(-1)**m*(x/2)**(2*m+n)/(factorial(m)*factorial(m+n))
                for m in range(95)])
besselfn(3,2)
```

[2]: 0.1289432494744021

We can get,

$$j_n(x) = \sum_{m=0}^{\infty} t_m$$

Where, $t_m = -\frac{1}{m(m+n)}(\frac{x}{2})^2 t_{m-1}$ and $t_0 = \frac{(x/2)^n}{n!}$.

[3]:
```
def fact(n):
    fct = 1
```

```
    for i in range(2,n+1):
        fct *= i
    return fct
```

[4]:
```
def besfn(nn,x):
    n = abs(nn)
    tol = 1e-5
    t = (x/2)**n/fact(n)   # t_0
    sm = t
    m = 1
    while True:
        sm1 = sm
        t = -(x/2)**2*t/(m*(m+n))
        sm += t
        m += 1
        if abs(sm - sm1) < tol:
            break
    if nn < 0:     # j_-n(x)
        sm = (-1)**n*sm
    return sm
```

[5]:
```
besfn(3,2)
```

[5]: 0.1289434523809524

By recurrence formula

$$j_{n+1}(x) = \frac{2x}{n} j_n(x) - j_{n-1}(x)$$

For given $j_0(x)$ and $j_1(x)$.

[6]:
```
from scipy.special import *

def recjn(n,x):
    if n == 0:
        return jn(0,x)
    elif n == 1:
        return jn(1,x)
    elif n >= 2:
        jn0, jnmin1 = jn(1,x), jn(0,x)
        for i in range(2,n+1):
            jn1 = 2*(i-1)/x*jn0 - jnmin1
            jnmin1 = jn0
            jn0 = jn1
        return jn1
```

[7]:
```
recjn(3,2)
```

[7]: 0.1289432494744024

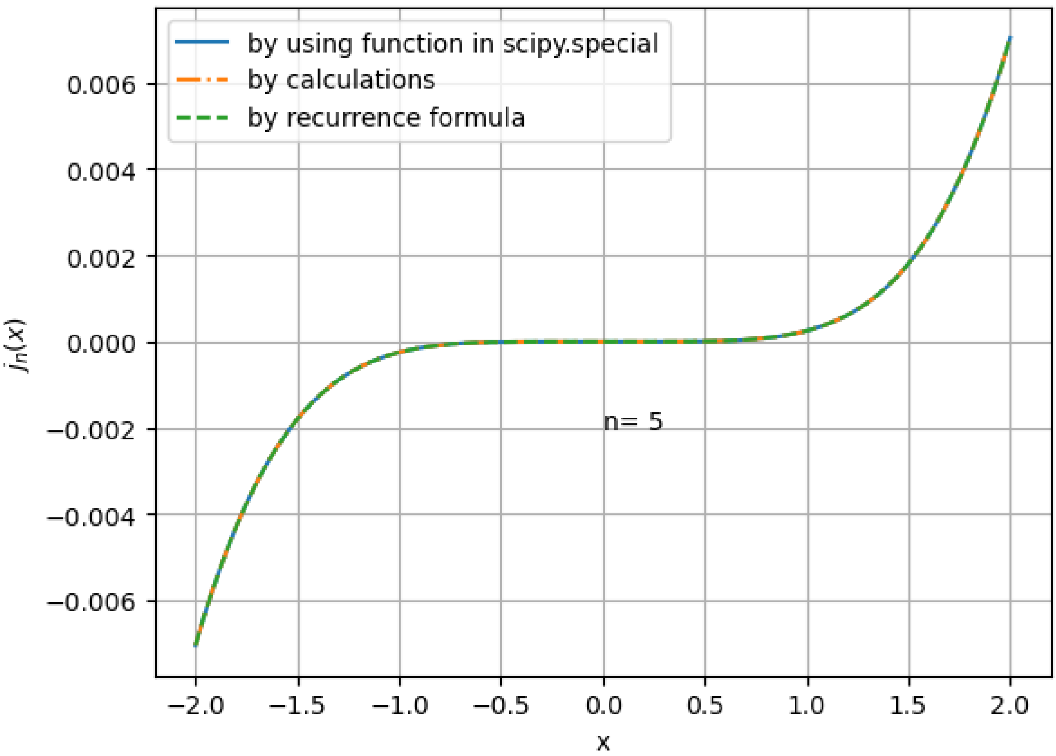
Verification

[8]:
```
n1, x1 = 5,-2  # input values
print('by using function in scipy.special\n\t',jn(n1,x1))
print('by calculations\n\t',besfn(n1,x1))
print('by recurrence formula\n\t',recjn(n1,x1))
```

```
by using function in scipy.special
         -0.007039629755871686
by calculations
         -0.007039517195767195
```

```
by recurrence formula
         -0.007039629755874244
```

[9]:
```
import numpy as np
import matplotlib.pyplot as plt
x = np.linspace(-2,2,100)
n = 5  # input the value
jn1 = jn(n,x)
besfn1 = [besfn(n,x[i]) for i in range(len(x))]
recjn1 = [recjn(n,x[i]) for i in range(len(x))]

plt.plot(x,jn1,'-',label='by using function in scipy.special')
plt.plot(x,besfn1,'-.',label='by calculations')
plt.plot(x,recjn1,'--',label='by recurrence formula')
plt.text((max(x)-abs(min(x)))/2,(max(jn1)-abs(min(jn1)))/3-2e-3,f'n= {n}')
plt.legend()
plt.xlabel('x')
plt.ylabel('$j_n(x)$')
plt.grid()
plt.show()
```



[ ]:

5

Legendre Polynomials

Legendre's Differential Equation:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

Solution (Legendre Polynomials):

$$P_n(x) = \sum_{k=0}^{m} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}$$

For $n = even$, $m = \frac{n}{2}$ and for $n = odd$, $m = \frac{n-1}{2}$.

from scipy.special

[1]:
```
from scipy.special import legendre
legendre(3)(5)
```

[1]: 305.0

By calculations

[2]:
```
def fact(n):
    fct = 1
    for i in range(2,n+1):
        fct *= i
    return fct
```

[3]:
```
def Pn(n,x):
    trm = 0
    m = n//2   # for both even and odd n
    for k in range(0,m+1):
        trm += (-1)**k*fact(2*n-2*k)*x**(n-2*k)/(2**n*fact(k)*fact(n-k)*fact(n-2*k))
    return trm
```

[4]:
```
Pn(3,5)
```

[4]: 305.0

By recurrence formula

$$P_n(x) = \frac{2n-1}{n} x P_{n-1}(x) - \frac{n-1}{n} P_{n-2}(x)$$

Given, $P_0(x) = 1$ and $P_1(x) = x$.

[5]:
```
def recPn(n,x):
    P0 = 1
    P1 = x
    if n==0:
        Pn = P0
    elif n==1:
        Pn = P1
    else:
        Pn_1, Pn_2 = P1, P0
    for i in range(2,n+1):
        Pn = (2*i-1)/i*x*Pn_1 - (i-1)/i*Pn_2
        Pn_1, Pn_2 = Pn, Pn_1
    return Pn
```

[6]:
```
recPn(3,5)
```

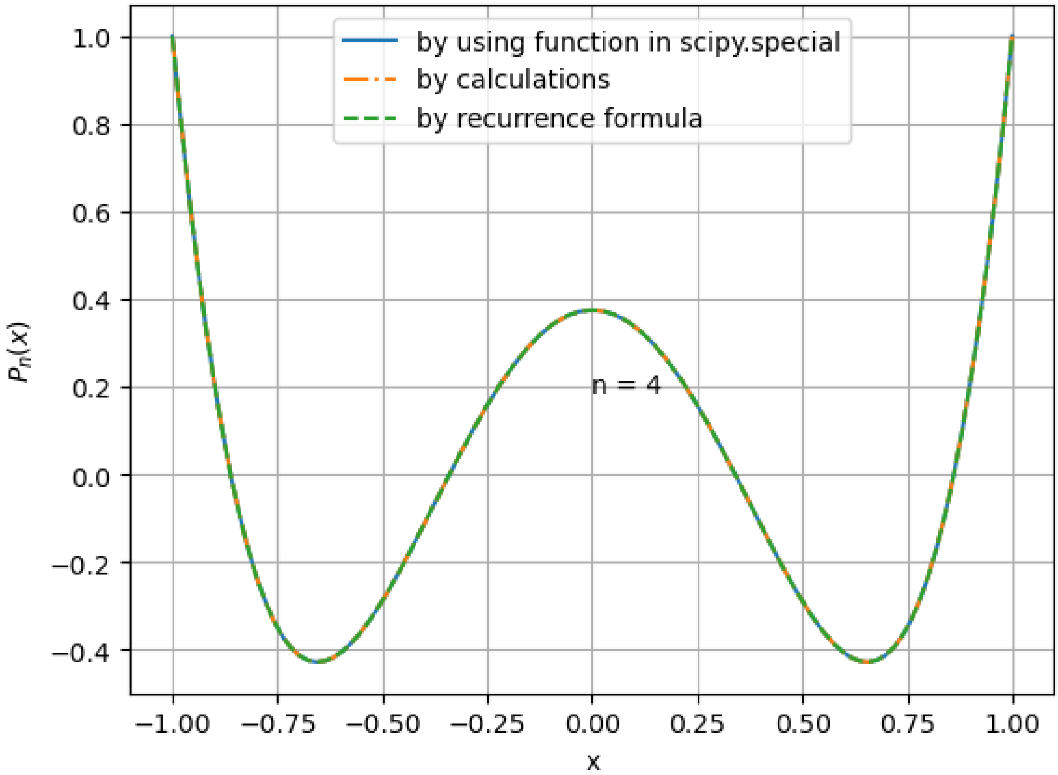
[6]: 305.00000000000006

Verification

[7]:
```
n1, x1 = 5,-1.5  # input values
print('by using function in scipy.special\n\t',legendre(n1)(x1))
print('by calculations\n\t',Pn(n1,x1))
print('by recurrence formula\n\t',recPn(n1,x1))
```

```
by using function in scipy.special
         -33.08203125000001
by calculations
         -33.08203125
by recurrence formula
         -33.08203125
```

[8]:
```
import numpy as np
import matplotlib.pyplot as plt

n = 4   # input degree of the polynomial
x = np.linspace(-1,1,100)
legplot = legendre(n)(x)
Pnplot = [Pn(n,x[i]) for i in range(len(x))]
recPnplot = [recPn(n,x[i]) for i in range(len(x))]

plt.plot(x,legplot,label='by using function in scipy.special')
plt.plot(x,Pnplot,'-.',label='by calculations')
plt.plot(x,recPnplot,'--',label='by recurrence formula')
plt.text((max(x)-abs(min(x)))/2,(max(legplot)-abs(min(legplot)))/3-2e-3,f'n = {n}')
plt.legend()
plt.xlabel('x')
plt.ylabel('$P_n(x)$')
plt.grid()
plt.show()
```


by using function in scipy.special
by calculations
by recurrence formula
n = 4
$P_n(x)$
x
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00

6 Orthogonality of Legendre Polynomials

$$\int_{-1}^{+1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{mn}$$

Short method

[9]:
```
import numpy as np
from scipy.special import legendre

def pmpn(m,n):
    pm = legendre(m)
    pn = legendre(n)
    return pm*pn

# Integration by Simpson's 1/3rd Rule
def simp13x(f,a,b,n):
    h = float(b-a)/n
    x0 = np.arange(a+h,b,2*h)
    xe = np.arange(a+2*h,b,2*h)
    val = h/3*(f2(a) + 4*sum(f2(x0)) + 2*sum(f2(xe)) + f2(b))
    return val

# Check Orthogonality for different values of m and n.
m = 3
n = 3

f2 = pmpn(m,n)
intg = simp13x(f2,-1,1,1000)  # integration

if m==n:
    dmn = 1
else:
    dmn = 0
res = (2/(2*n + 1))* dmn  # result
print('m =',m,'n =',n,'\n','RHS =',intg,'\t','LHS =',res)
# compare the values of intg and res
```

```
m = 3 n = 3
 RHS = 0.28571428581562036        LHS = 0.2857142857142857

C:\ProgramData\Anaconda3\lib\site-packages\numpy\lib\polynomial.py:1329:
FutureWarning: In the future extra properties will not be copied across when
```

```
constructing one poly1d from another
  other = poly1d(other)
```

Detailed method

[10]:
```
# Simpson's 1/3 rule for integration ( with parameters)
def simp13pr(f, pr, a, b, tol):
    n = 10
    I1 = 0
    while True:
        h = (b-a)/n
        I2 = 0
        for i in range(n+1):
            if i==0 or i==n:
                I2 += f(pr, a+i*h)
            elif i%2==0:
                I2 += 2*f(pr, a+i*h)
            else:
                I2 += 4*f(pr, a+i*h)
        I2 = h*I2/3
        if abs(I2-I1) <= tol:
            break
        else:
            I1 = I2
            n += 10
    return I2
```

[11]:
```
# integrand
def PmPn(pr, x):   # pr[0]=m and pr[1]=n
    return recPn(pr[0], x)*recPn(pr[1], x)

tol = 1e-6
n = 6
m = 4
for i in range(3, n+1):
    I = simp13pr(PmPn, [m,i], -1,1, tol)
    print('(2 X %d +1)/2 P%d P%d = %f' %(i,m,i,I*(2*i+1)/2))
```

```
(2 X 3 +1)/2 P4 P3 = 0.000000
(2 X 4 +1)/2 P4 P4 = 1.000008
(2 X 5 +1)/2 P4 P5 = -0.000000
(2 X 6 +1)/2 P4 P6 = 0.000018
```

[ ]:
```
```

7 Complex Integration

19th May, 2023

The integration is done by Simpson's 1/3 rule.

$$\int_a^b f(x)dx = \frac{h}{3}[f(a) + 4(f(a+h) + f(a+3h) + ...) + 2(f(a+2h) + f(a+4h) + ...) + f(b)]$$

[1]:
```
def simp13z(f, pr, a, b, tol):
    n = 10
    I1 = 0
    while True:
        h = (b-a)/n
        I2 = 0
        for i in range(n+1):
            if i==0 or i==n:
                I2 += f(pr, a+i*h)
            elif (i%2)==0:
                I2 += 2*f(pr, a+i*h)
            else:
                I2 += 4*f(pr, a+i*h)
        I2 = (h/3)*I2
        if abs(I2-I1) <= tol:
            break
        else:
            I1 = I2
            n += 10
    return I2
```

Example:

$$\int_0^{\pi+2j} \cos(\frac{z}{2})dz$$

[2]:
```
from cmath import *
def f1(pr, z):
    return cos(z/2)   # function
tol = 1e-6
intgsp1 = simp13z(f1, None, 0, pi +2j, tol)
print(intgsp1)

import numpy as np
def f1(pr, z):
```

```
    return np.cos(z/2)  # function
tol = 1e-6
intgsp2 = simp13z(f1, None, 0, np.pi +2j, tol)
print(intgsp2)
```

```
(3.086161217931016+6.174435784878085e-08j)
(3.086161217931016+6.174435762673625e-08j)
```

Example:

$$\int_0^j \frac{z^2+1}{z+1} dz$$

[3]:
```
import numpy as np
def f2(pr, z):
    return (z**2 +1)/(z +1) # function
tol = 1e-6
intgsp2 = simp13z(f2, None, 0, 1j, tol)
print(intgsp2)
```

```
(0.1931472836593496+0.5707963267676798j)
```

7.1 Contour Integration

We need to evaluate We need to evaluate $\oint_c f(z)dz$ from $z = z_0$ to $z = z_1$ along the curve c and $z = c(t)$.

We can get the integration as, $(z = c(t) = x(t) + jy(t))$

$$\int_{t_0}^{t_1} f(x(t) + jy(t))(x'(t) + jy'(t))dt = \int_{t_0}^{t_1} F(t)dt$$

Differentiation (3 points):

$$\frac{df}{dx} = \frac{f(x+h) - f(x-h)}{2h}$$

[4]:
```
def dfdz3(f, pr, z, tol):
    h = 0.1
    ch = complex(h,h)
    dfdz1 = (f(pr, z+ch) - f(pr, z-ch))/(2*ch)
    while True:
        h = h/2
        ch = complex(h,h)
        dfdz2 = (f(pr, z+ch) - f(pr, z-ch))/(2*ch)
        if abs(dfdz2 -dfdz1) <= tol:
            break
        else:
            dfdz1 = dfdz2
    return dfdz2
```

Formation of integrand and integration:

[5]:
```
def fzdz(fnpr, t):
    f, prf, c, prc, tol = fnpr
    z = c(prc, t)
    Ft = f(prf, z)* dfdz3(c, prc, t, tol)
    return Ft

def simp13cont(f, prf, c, prc, t0, t1, tol):
```

```
    fnpr = [f, prf, c, prc, tol]
    contintg = simp13z(fzdz, fnpr, t0, t1, tol)
    return contintg
```

7.2 Examples

Example 1: $f(z) = \pi exp(\pi \bar{z})$ and c is the boundary of square with vertices $0, 1, 1+j, j$ in anticlockwise direction.

Solution: We have the paths, 1. $c_1 : (z_0 = 0, z_1 = 1)$. 2. $c_2 : (z_0 = 1, z_1 = 1+j)$. 3. $c_3 : (z_0 = 1+j, z_1 = j)$. 4. $c_4 : (z_0 = j, z_1 = 0)$.

[6]:
```
import numpy as np
def f(prf, z):
    return np.pi*np.exp(np.pi*z.conjugate()) # input the function

def c1(prc, z):  # curve (path) 1
    return z
def c2(prc, z):  # curve (path) 2
    return z
def c3(prc, z):  # curve (path) 3
    return z
def c4(prc, z):  # curve (path) 4
    return z
```

[7]:
```
tol = 1e-6
prf, prc = None, None

intg1 = simp13cont(f, prf, c1, prc, 0, 1, tol)
print('I_c1 =', intg1)
intg2 = simp13cont(f, prf, c2, prc, 1, 1+1j, tol)
print('I_c2 =', intg2)
intg3 = simp13cont(f, prf, c3, prc, 1+1j, 1j, tol)
print('I_c3 =', intg3)
intg4 = simp13cont(f, prf, c4, prc, 1j, 0, tol)
print('I_c4 =', intg4)

intg = intg1 + intg2 + intg3 + intg4
print('result I_c =', intg)
```

```
I_c1 = (22.14069355699097-5.899692905713894e-15j)
I_c2 = (46.281386308945336-1.827109891954543e-14j)
I_c3 = (22.14069355699098+6.033653069004802e-15j)
I_c4 = (-2.000000423093183+1.4802973661668754e-17j)
result I_c = (88.56277299983411-1.8122335782592855e-14j)
```

Example 2: $f(z) = \frac{1}{(z-z_0)^n}$, $(n = 2, 3, 4, ...)$; $c(\theta) = Re^{j\theta}$ and $z_0 = \frac{R}{2}exp(\frac{j\pi}{4}), (R = 1)$.

Solution:

[8]:
```
import numpy as np
def f(prf, z):
    z0, n = prf
    return 1/(z-z0)**n # input the function

def c(prc, th):
    R = prc
    return R*np.exp(th*1j)   # input the curve
```

[9]:
```
tol = 1e-6
R = 1
z0 = (R/2)* np.exp(1j*np.pi/4)
for n in range(2,5):
    intg = simp13cont(f, [z0,n], c, R, 0, 2*np.pi, tol)
    print('n = %d, I =' %(n), intg)
```

```
n = 2, I = (4.266343353926582e-09+1.391659530705444e-13j)
n = 3, I = (-5.943228934898735e-09+1.241528591044285e-13j)
n = 4, I = (6.757301247986022e-09-4.962590971092578e-14j)
```

Example 3: $f(z) = \sqrt{z}$ and c is the boundary broken into 3 parts c_1, c_2, c_3 in anticlockwise direction. 1. $c_1 : z = re^0; (0 \leq r \leq 1)$. 2. $c_2 : z = 1e^{j\theta}; (0 \leq \theta \leq \pi)$. 3. $-c_3 : z = re^0; (0 \leq r \leq 1)$.

Solution:

[10]:
```
import numpy as np
def f(prf, z):
    return z**0.5 # input the function

def c1(prc, r):  # curve (path) 1
    return r
def c2(prc, th):  # curve (path) 2
    return np.exp(1j*th)
def c3(prc, r):  # curve (path) 3
    return -r
```

[11]:
```
tol = 1e-6
intg1 = simp13cont(f, None, c1, None, 0, 1, tol)
intg2 = simp13cont(f, None, c2, None, 0, np.pi, tol)
intg3 = simp13cont(f, None, c3, None, 1, 0, tol)
intg = intg1 +intg2 +intg3
print(intg)
```

```
(-1.7539275942214797e-05-1.7268008757120867e-05j)
```

8 Other Topics

8.1 Dirac Delta Function

Gaussian Function,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

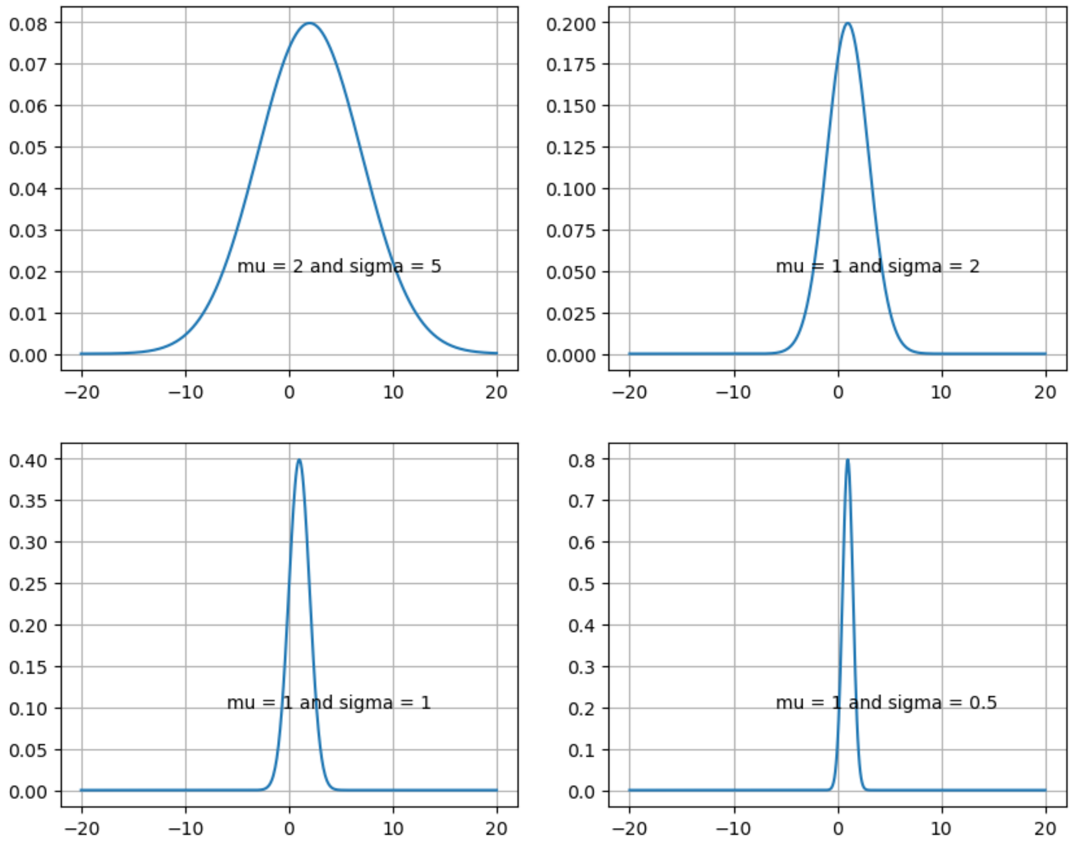
Dirac Delta Function,

$$\begin{aligned}\delta(x - x_0) &= \infty; \quad x = x_0 \\ &= 0; \quad x \neq x_0\end{aligned}$$

We can obtain Dirac Delta function from Gaussian function for some certain range of values of μ and σ.

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import quad

f = lambda x: (1/(sigma*(2*np.pi)**0.5))*np.exp(-(x-mu)**2/(2*sigma**2))
x1 = np.linspace(-20,20,500)

fig, ax = plt.subplots(2,2, figsize=(10,8))
ax1, ax2, ax3, ax4 = ax[0,0], ax[0,1], ax[1,0], ax[1,1]

mu, sigma = 2, 5             # input values
ax1.plot(x1,f(x1))
ax1.text(mu-7, max(f(x1))/4, f'mu = {mu} and sigma = {sigma}')
ar1 = quad(f,-np.inf,np.inf)[0]
# ax1.text(mu-15, max(f(x1))/2, f'area under curve = {ar1}')
ax1.grid()

mu, sigma = 1, 2             # input values
ax2.plot(x1,f(x1))
ax2.text(mu-7, max(f(x1))/4, f'mu = {mu} and sigma = {sigma}')
ar2 = quad(f,-np.inf,np.inf)[0]
# ax2.text(mu-15, max(f(x1))/2, f'area under curve = {ar2}')
ax2.grid()

mu, sigma = 1, 1             # input values
ax3.plot(x1,f(x1))
ax3.text(mu-7, max(f(x1))/4, f'mu = {mu} and sigma = {sigma}')
ar3 = quad(f,-np.inf,np.inf)[0]
```

```
# ax3.text(mu-15, max(f(x1))/2, f'area under curve = {ar3}')
ax3.grid()

mu, sigma = 1, 0.5              # input values
ax4.plot(x1,f(x1))
ax4.text(mu-7, max(f(x1))/4, f'mu = {mu} and sigma = {sigma}')
ar4 = quad(f,-np.inf,np.inf)[0]
# ax4.text(mu-15, max(f(x1))/2, f'area under curve = {ar4}')
ax4.grid()
plt.show()
# change values of mu and sigma to adjust the graphs
```



8.2 Question-2

$\frac{1}{\sqrt{2\pi\sigma^2}}\int_0^\infty e^{-\frac{(2-x)^2}{2\sigma^2}}(x+3)dx$; for $\sigma = 1, 0.1, 0.01$ and show that the value tends to 5.

[2]:
```
import numpy as np
import scipy as sp
from scipy.integrate import simps

sigmas = [1, 0.1, 0.01]  # change the value
for sigma in sigmas:
    xmax = 100  # set the upper limit
    x = np.linspace(0, xmax, int(xmax/sigma))  # set no. of points
    f1 = (1/np.sqrt(2*np.pi*sigma**2))*np.exp(-(2-x)**2/(2*sigma**2))*(x+3)
    f2 = simps(f1,x)
    print(f'sigma = {sigma} \t value = {f2} \t error = {5-f2}')
```

```
sigma = 1        value = 4.978300045718836        error = 0.021699954281164224
sigma = 0.1      value = 5.000000028048047        error = -2.8048047440165647e-08
sigma = 0.01     value = 5.000000026689211        error = -2.668921084136855e-08
```

8.3 Question-3

Program to sum:

$$\sum_{n=1}^{\infty}(0.2)^n$$

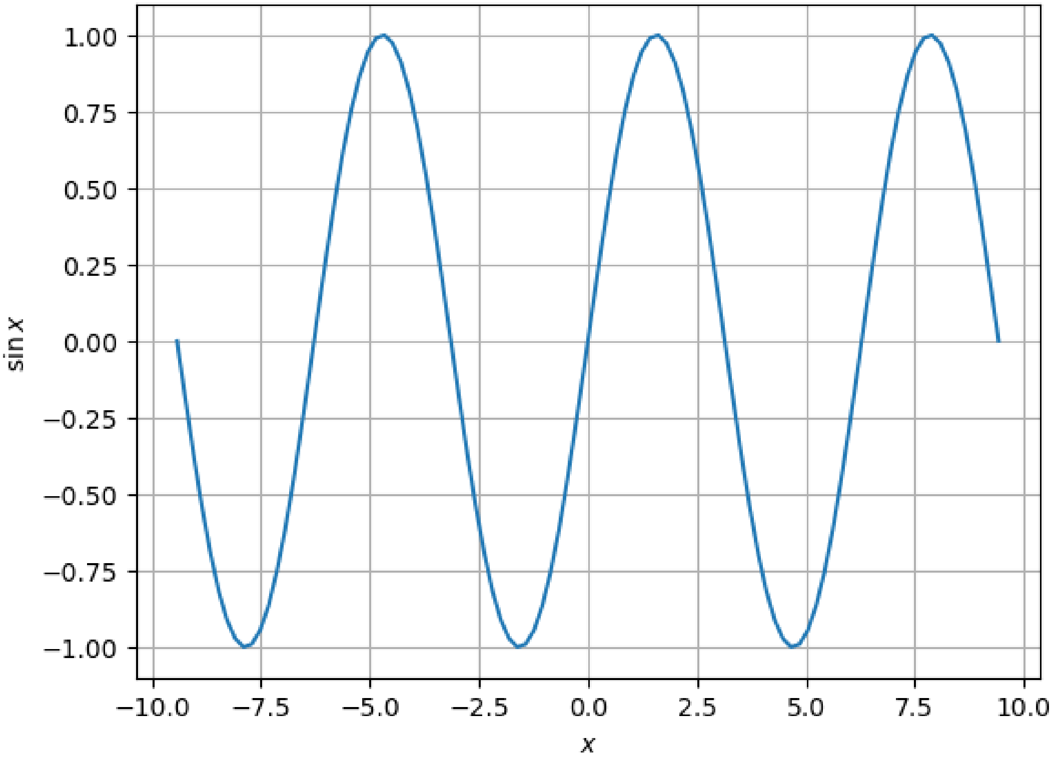
[3]:
```
nmax = 1000    # the upper limit of sum
a = 0
for i in range(1,nmax+1):
    a = a + 0.2**i

print(f'Sum of the series = {a}. (Using {nmax} no. of terms)')
```

```
Sum of the series = 0.2500000000000001. (Using 1000 no. of terms)
```

8.4 Question-7

Sine function from Bessel's function at N points:

[4]:
```
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace(-3*np.pi, 3*np.pi, 100)
a = 1.0
n = 100
sum = 1.0
for i in range(1,n+1):
    a = (-1)*(x**2)*a/(2*i*(2*i+1))
    sum += a
sum = sum*x

plt.plot(x,sum)
plt.xlabel('$x$')
plt.ylabel('$\sin x$')
plt.grid()
plt.show()
```



Computation of sin(6):

[5]:
```
x = 6
a = 1.0
nmax = 100
sum = 1.0
for n in range(1,nmax+1):
    a = (-1)*(x**2)*a/(2*n*(2*n+1))
    # a is the (n+1)th term in lhs nth term in rhs
    sum += a
sum = sum*x

print(f'Value of sin6 by using recurrence formula: {sum}. ({nmax} no. of terms␣
 ↪used)')
print(f'Value of sin6 by using numpy function: {np.sin(6)}')
```

```
Value of sin6 by using recurrence formula: -0.27941549819892436. (100 no. of
terms used)
Value of sin6 by using numpy function: -0.27941549819892586
```

8.5 Question-8: n-th Root of Unity

$$x^n = 1 = e^{i(2\pi k)}$$

$$x = e^{i\left(\frac{2\pi k}{n}\right)} = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$$

[6]:
```
import numpy as np
```

```
def croot(k,n):
    if n<=0:
        return None
    return np.exp(1j*(2*np.pi*k)/n)
n = 4   # put an integer
for k in range(n):
    print(croot(k,n))
```

```
(1+0j)
(6.123233995736766e-17+1j)
(-1+1.2246467991473532e-16j)
(-1.8369701987210297e-16-1j)
```

8.6 Question-9: Square Root of Complex Numbers

$$\sqrt{x+iy} = \sqrt{r}\cos(\theta/2) + i\sqrt{r}\sin(\theta/2) = \sqrt{r}e^{i\theta/2}$$

where $r = \sqrt{x^2+y^2}$ and $\tan(\theta) = \frac{y}{x}$

[7]:
```
import numpy as np

# Values of x and y
x, y = -5, 12

r = (x**2 + y**2)**0.5
th = np.arctan(y/x)
root = r**0.5 * np.exp(1j*th/2)
root
```

[7]: (2.9999999999999996-2j)