The Goldman bracket characterizes homeomorphisms between non-compact surfaces

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Topological Rigidity

$\rightsquigarrow$ Is every homotopy equivalence between n-manifolds homotopic to a homeomorphism?

- True for closed surfaces.
- False for compact surfaces with boundary or non-compact surfaces.
- False for dimensions three and above.

Characterization of homeomorphisms between non-compact surfaces

Theorem [Das-Gadgil-Nair] Let $f : \Sigma' \to \Sigma$ be a homotopy equivalence between two non-compact oriented surfaces without boundary, where $\Sigma \neq \mathbb{R}^2, \mathbb{S}^1 \times \mathbb{R}$. Then,

- f is homotopic to a homeomorphism if and only if f preserves the geometric intersection number.

- f is homotopic to an *orientation-preserving* homeomorphism if and only if f preserves the Goldman bracket.

Sumanta Das, Siddhartha Gadgil, and Ajay Kumar Nair. *The Goldman bracket characterizes homeomorphisms between non-compact surfaces.* Algebraic & Geometric Topology (to appear).

The Goldman Lie algebra

Let S be an oriented surface. The Goldman bracket $[\cdot,\cdot]\colon \mathbb{Z}\left[\mathbb{S}^1,\mathrm{S}\right] \times \mathbb{Z}\left[\mathbb{S}^1,\mathrm{S}\right] \to \mathbb{Z}\left[\mathbb{S}^1,\mathrm{S}\right]$ of $x, y \in \left[\mathbb{S}^1,\mathrm{S}\right]$ is defined as follows:

Pick $\varphi \in x$ and $\psi \in y$ such that φ and ψ intersect tranversally at double points. Then

$$[x,y] \coloneqq \sum_{p\in\varphi\cap\psi} \varepsilon_p \cdot [\varphi *_p \psi]\,,$$

where $[\varphi *_p \psi]$ is the free homotopy class of the based loop product, and $\varepsilon_p \in \{\pm 1\}$.

- If $[x,y]=0$ and x has a *simple representative*, then x and y have disjoint representatives.

W. M. Goldman, *Invariant functions on Lie groups and Hamiltonian flows of surface group representations*, Inventiones Mathematicae (1986).

Strong topological rigidity and an idea of the proof

Proof. To prove the "If" directions, homotope f to a proper map and then apply the following theorem. $\square$

Theorem [Das] Let $f\colon \Sigma' \to \Sigma$ be a homotopy equivalence between two orientable non-compact surfaces without boundary, where $\Sigma \neq \mathbb{R}^2, \mathbb{S}^1 \times \mathbb{R}$. If f is a *proper map*, then f is properly homotopic to a homeomorphism.

Sumanta Das. *Strong Topological Rigidity of Non-Compact Orientable Surfaces.* Algebraic & Geometric Topology (to appear).

Characterization of the image of the Dehn-Nielsen-Baer map

Corollary If $\Sigma \neq \mathbb{R}^2, \mathbb{S}^1 \times \mathbb{R}$ is an oriented surface without boundary, *possibly of infinite type*, then each element of the image of

- $\mathrm{MCG}^{\pm}(\Sigma) \to \mathrm{Out}\big(\pi_1(\Sigma)\big)$ is induced by a self-homotopy equivalence of Σ that preserves the geometric intersection number.
- $\mathrm{MCG}^{+}(\Sigma) \to \mathrm{Out}\big(\pi_1(\Sigma)\big)$ is induced by a self-homotopy equivalence of Σ that preserves the Goldman bracket.

Thank You