

MA 403-2025-1 | Real Analysis

Complete Homework Solutions

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Homework 1

Theorem 1. *If $n \in \mathbb{N}$ is not a perfect square, then \sqrt{n} is irrational.*

Proof. Suppose, for contradiction, that \sqrt{n} is rational. Then we can write

$$\sqrt{n} = \frac{m}{d},$$

where $m, d \in \mathbb{Z}$, $d \neq 0$, and $\gcd(m, d) = 1$.

Squaring both sides gives

$$m^2 = nd^2.$$

Let

$$n = \prod_{i=1}^k p_i^{a_i}, \quad m^2 = \prod_{i=1}^k p_i^{2b_i}, \quad d^2 = \prod_{i=1}^k p_i^{2c_i}$$

be the prime factorizations of n , m^2 , and d^2 .

From $m^2 = nd^2$, we get

$$\prod_{i=1}^k p_i^{2b_i} = \left(\prod_{i=1}^k p_i^{a_i} \right) \left(\prod_{i=1}^k p_i^{2c_i} \right) = \prod_{i=1}^k p_i^{a_i + 2c_i}.$$

Comparing exponents gives

$$2b_i = a_i + 2c_i \implies a_i = 2(b_i - c_i)$$

for each i .

Hence each a_i is even, which means $n = \prod_{i=1}^k p_i^{a_i}$ is a perfect square.

But this contradicts the assumption that n is not a perfect square.

Therefore, our assumption that \sqrt{n} is rational is false, and \sqrt{n} is irrational. \square

Theorem 2. *The number $\sqrt{2} + \sqrt{3}$ is irrational.*

Problem 1

Proof. Suppose, for the sake of contradiction, that $\sqrt{2} + \sqrt{3}$ is rational. Then there exists $r \in \mathbb{Q}$ such that

$$\sqrt{2} + \sqrt{3} = r.$$

Rewriting, we have

$$\sqrt{3} = r - \sqrt{2}.$$

Squaring both sides gives

$$3 = (r - \sqrt{2})^2 = r^2 - 2r\sqrt{2} + 2.$$

Simplifying, we get

$$1 - r^2 = -2r\sqrt{2} \implies \sqrt{2} = \frac{r^2 - 1}{2r}.$$

But the right-hand side is rational, which contradicts the fact that $\sqrt{2}$ is irrational; see [Theorem 1](#). Hence, our assumption is false. \square

Theorem 3. Let $r \in \mathbb{Q}$, $r \neq 0$, and $x \notin \mathbb{Q}$. Then $r + x$ and rx are irrational.

Problem 2

Proof. (i) Suppose $r + x$ is rational, say $r + x = s$ with $s \in \mathbb{Q}$. Then

$$x = s - r \in \mathbb{Q},$$

contradicting x being irrational. Hence $r + x$ is irrational.

(ii) Suppose rx is rational, say $rx = t$ with $t \in \mathbb{Q}$. Then

$$x = \frac{t}{r} \in \mathbb{Q},$$

contradicting x being irrational. Hence rx is irrational. \square

Theorem 4. Given any real number $x > 0$, there exists an irrational number in $(0, x)$.

Problem 3

Proof. We consider two cases depending on whether x is rational or irrational.

Case 1: x is rational. Let $x = r \in \mathbb{Q}$. Consider

$$z = \frac{r}{\sqrt{2}}.$$

Since $r \neq 0$ and $\sqrt{2}$ is irrational, z is irrational. Moreover,

$$0 < z = \frac{r}{\sqrt{2}} < r = x.$$

Hence z is an irrational number in $(0, x)$.

Case 2: x is irrational. Then $x/2$ is positive and irrational. Clearly,

$$0 < \frac{x}{2} < x,$$

so $x/2$ is an irrational number in $(0, x)$.

In either case, there exists an irrational number in $(0, x)$. □

Theorem 5. Suppose $x, y \in \mathbb{R}$ and for each $\varepsilon > 0$, $|x - y| \leq \varepsilon$. Then $x = y$.

Problem 4

Proof. Assume $x \neq y$. Take $\varepsilon = \frac{|x-y|}{2} > 0$. Then

$$|x - y| \leq \varepsilon = \frac{|x - y|}{2},$$

which is impossible. Hence $x = y$. □

Example 6. Consider the set

$$S = (0, 1] = \{x \in \mathbb{R} : 0 < x \leq 1\}.$$

Notice that S is bounded above and below. We have

$$\sup S = 1 \in S, \quad \text{however,} \quad \inf S = 0 \notin S.$$

Problem 5

Theorem 7. Suppose $A, B \subset \mathbb{R}$ such that A is bounded above and B is bounded below. Then the intersection $A \cap B$ is bounded both above and below.

Problem 6

Proof. Since A is bounded above, there exists $M \in \mathbb{R}$ such that

$$a \leq M \quad \forall a \in A.$$

For any $x \in A \cap B$, we have $x \in A$, hence

$$x \leq M.$$

Thus M is an upper bound for $A \cap B$.

Since B is bounded below, there exists $m \in \mathbb{R}$ such that

$$b \geq m \quad \forall b \in B.$$

For any $x \in A \cap B$, we have $x \in B$, hence

$$x \geq m.$$

Thus m is a lower bound for $A \cap B$.

Therefore, $A \cap B$ is bounded both above and below. □

Theorem 8. Let $S \subset \mathbb{R}$ be a nonempty set such that $\sup S$ and $\inf S$ exist. Then $\sup S$ and $\inf S$ are uniquely determined.

Problem 7

Proof. Supremum uniqueness: Suppose u_1 and u_2 are both suprema of S . We want to show $u_1 = u_2$.

By definition of supremum, for any $\varepsilon > 0$, there exist $s_1, s_2 \in S$ such that

$$u_1 - \varepsilon < s_1 \leq u_1 \quad \text{and} \quad u_2 - \varepsilon < s_2 \leq u_2.$$

Take $\varepsilon = |u_1 - u_2|/2$. Without loss of generality, assume $u_1 < u_2$. Then

$$u_2 - \varepsilon = u_2 - \frac{u_2 - u_1}{2} = \frac{u_1 + u_2}{2} > u_1.$$

But there exists $s_2 \in S$ such that $s_2 > u_2 - \varepsilon > u_1$, contradicting that u_1 is an upper bound of S . Hence $u_1 = u_2$.

Infimum uniqueness: Suppose l_1 and l_2 are both infima of S . For any $\varepsilon > 0$, there exist $s_1, s_2 \in S$ such that

$$l_1 \leq s_1 < l_1 + \varepsilon \quad \text{and} \quad l_2 \leq s_2 < l_2 + \varepsilon.$$

Take $\varepsilon = |l_1 - l_2|/2$. Without loss of generality, assume $l_1 < l_2$. Then

$$l_1 + \varepsilon = l_1 + \frac{l_2 - l_1}{2} = \frac{l_1 + l_2}{2} < l_2.$$

But there exists $s_1 \in S$ such that $s_1 < l_1 + \varepsilon < l_2$, contradicting that l_2 is a lower bound of S . Hence $l_1 = l_2$. □

Theorem 9. Let A and B be sets of positive numbers which are bounded above. Let

$$a = \sup A, \quad b = \sup B,$$

and define

$$C = \{xy : x \in A, y \in B\}.$$

Then

$$\sup C = ab.$$

Problem 8

Proof. Let $c \in C$. Then $c = xy$ for some $x \in A$ and $y \in B$. Since $x \leq a$ and $y \leq b$, we have

$$c = xy \leq ab.$$

Hence ab is an upper bound for C .

Let $\varepsilon > 0$ be arbitrary. Since $a = \sup A$, there exists $x_\varepsilon \in A$ such that

$$a - \frac{\varepsilon}{2b} < x_\varepsilon \leq a.$$

Similarly, since $b = \sup B$, there exists $y_\varepsilon \in B$ such that

$$b - \frac{\varepsilon}{2a} < y_\varepsilon \leq b.$$

Consider $c_\varepsilon = x_\varepsilon y_\varepsilon \in C$. Then

$$\begin{aligned} ab - c_\varepsilon &= ab - x_\varepsilon y_\varepsilon \\ &= ab - ay_\varepsilon + ay_\varepsilon - x_\varepsilon y_\varepsilon \\ &= a(b - y_\varepsilon) + y_\varepsilon(a - x_\varepsilon) \\ &< a \cdot \frac{\varepsilon}{2a} + b \cdot \frac{\varepsilon}{2b} = \varepsilon. \end{aligned}$$

Hence, for any $\varepsilon > 0$, there exists $c_\varepsilon \in C$ such that

$$ab - \varepsilon < c_\varepsilon \leq ab.$$

Since ab is an upper bound of C and for every $\varepsilon > 0$ there exists $c_\varepsilon \in C$ with $ab - \varepsilon < c_\varepsilon$, it follows that

$$\sup C = ab.$$

□

Homework 2

Theorem 10. Let $S = \{x \in \mathbb{R} : 3x^2 - 10x + 3 < 0\}$. Then $\inf S = \frac{1}{3}$ and $\sup S = 3$.

Problem 1

Proof. We first consider the general case.

Let

$$q(x) = ax^2 + bx + c, \quad a \neq 0, \quad \Delta = b^2 - 4ac.$$

Then

$$q(x) = a \left(x + \frac{b}{2a} \right)^2 + \left(c - \frac{b^2}{4a} \right) = a \left(x + \frac{b}{2a} \right)^2 - \frac{\Delta}{4a}.$$

Define $S := \{x \in \mathbb{R} : q(x) < 0\}$. We consider three cases.

Case A: $\Delta < 0$

- If $a > 0$: $-\frac{\Delta}{4a} > 0$, and $a(x + b/2a)^2 \geq 0$, so $q(x) > 0$ for all x . Hence $S = \emptyset$.
- If $a < 0$: $-\frac{\Delta}{4a} < 0$, and $a(x + b/2a)^2 \leq 0$, so $q(x) < 0$ for all x . Hence $S = \mathbb{R}$.

Case B: $\Delta = 0$, root $r = -b/(2a)$

- If $a > 0$: $q(x) = a(x - r)^2 \geq 0$, equality at $x = r$. So $S = \emptyset$.
- If $a < 0$: $q(x) = a(x - r)^2 \leq 0$, equality at $x = r$. So $S = \mathbb{R} \setminus \{r\}$.

Case C: $\Delta > 0$, distinct roots $r_1 = \frac{-b-\sqrt{\Delta}}{2a}$, $r_2 = \frac{-b+\sqrt{\Delta}}{2a}$, with $\alpha = \min(r_1, r_2)$, $\beta = \max(r_1, r_2)$

$$q(x) = a(x - r_1)(x - r_2) = a(x - \alpha)(x - \beta).$$

- If $a > 0$: $(x - \alpha)(x - \beta) < 0$ for $\alpha < x < \beta$, so $S = (\alpha, \beta)$.
- If $a < 0$: $(x - \alpha)(x - \beta) < 0$ for $x < \alpha$ or $x > \beta$, so $S = (-\infty, \alpha) \cup (\beta, \infty)$.

$\inf S$ and $\sup S$:

- $\Delta < 0$:
 - $a > 0$: $S = \emptyset$, $\inf S = +\infty$, $\sup S = -\infty$.
 - $a < 0$: $S = \mathbb{R}$, $\inf S = -\infty$, $\sup S = +\infty$.
- $\Delta = 0$:
 - $a > 0$: $S = \emptyset$, $\inf S = +\infty$, $\sup S = -\infty$.
 - $a < 0$: $S = \mathbb{R} \setminus \{r\}$, $\inf S = -\infty$, $\sup S = +\infty$.
- $\Delta > 0$, roots $\alpha < \beta$:
 - $a > 0$: $S = (\alpha, \beta)$, $\inf S = \alpha$, $\sup S = \beta$.
 - $a < 0$: $S = (-\infty, \alpha) \cup (\beta, \infty)$, $\inf S = -\infty$, $\sup S = +\infty$.

If $q(x) = 3x^2 - 10x + 3$, then $S = (\frac{1}{3}, 3)$. Hence, $\inf S = \frac{1}{3}$ and $\sup S = 3$. \square

Theorem 11 (Lagrange's Identity). *For all real numbers a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n , we have*

$$\left(\sum_{k=1}^n a_k b_k \right)^2 = \left(\sum_{k=1}^n a_k^2 \right) \left(\sum_{k=1}^n b_k^2 \right) - \sum_{1 \leq k < j \leq n} (a_k b_j - a_j b_k)^2.$$

Problem 2

Proof. Notice that

$$\left(\sum_{i=1}^n x_i \right)^2 = \sum_{i=1}^n x_i^2 + 2 \sum_{1 \leq i < j \leq n} x_i x_j.$$

Now let

$$A := \sum_{i=1}^n a_i^2, \quad B := \sum_{i=1}^n b_i^2, \quad C := \sum_{i=1}^n a_i b_i.$$

Then

$$AB = \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{j=1}^n b_j^2 \right) = \sum_{i=1}^n \sum_{j=1}^n a_i^2 b_j^2 = \sum_{i=1}^n a_i^2 b_i^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^n a_i^2 b_j^2.$$

Using the expansion with $x_i = a_i b_i$, we obtain

$$C^2 = \left(\sum_{i=1}^n a_i b_i \right)^2 = \sum_{i=1}^n a_i^2 b_i^2 + 2 \sum_{1 \leq i < j \leq n} a_i a_j b_i b_j.$$

Subtracting,

$$\begin{aligned} AB - C^2 &= \left[\sum_{i=1}^n a_i^2 b_i^2 + \sum_{i \neq j} a_i^2 b_j^2 \right] - \left[\sum_{i=1}^n a_i^2 b_i^2 + 2 \sum_{i < j} a_i a_j b_i b_j \right], \\ &= \sum_{i \neq j} a_i^2 b_j^2 - 2 \sum_{i < j} a_i a_j b_i b_j. \end{aligned}$$

Grouping the $i \neq j$ terms:

$$\sum_{i \neq j} a_i^2 b_j^2 = \sum_{i < j} a_i^2 b_j^2 + \sum_{j < i} a_i^2 b_j^2 = \sum_{i < j} (a_i^2 b_j^2 + a_j^2 b_i^2).$$

Hence,

$$AB - C^2 = \sum_{i < j} (a_i^2 b_j^2 + a_j^2 b_i^2 - 2a_i a_j b_i b_j) = \sum_{i < j} (a_i b_j - a_j b_i)^2. \quad \square$$

Corollary 12 (Cauchy–Schwarz Inequality). For all real numbers a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n , we have

$$\left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{i=1}^n b_i^2 \right) \geq \left(\sum_{i=1}^n a_i b_i \right)^2.$$

Problem 2

Theorem 13. Let $f: S \rightarrow T$ be a function. The following statements are equivalent:

Problem 3

(a) f is one-to-one on S .

(b) $f^{-1}(f(A)) = A$ for every subset A of S .

(c) For all subsets $A, B \subseteq S$ with $B \subseteq A$, we have

$$f(A \setminus B) = f(A) \setminus f(B).$$

Problem 3

Proof. (a) \Rightarrow (b): Assume f is one-to-one on S . Let $A \subseteq S$. If $a \in A$, then $f(a) \in f(A)$, so $a \in f^{-1}(f(A))$. Hence $A \subseteq f^{-1}(f(A))$.

Conversely, let $x \in f^{-1}(f(A))$. Then $f(x) \in f(A)$, so there exists $a \in A$ such that $f(x) = f(a)$. Since f is injective, $x = a \in A$. Thus $f^{-1}(f(A)) \subseteq A$, and we conclude $f^{-1}(f(A)) = A$.

(b) \Rightarrow (c): Assume $f^{-1}(f(X)) = X$ for every $X \subseteq S$. Let $A, B \subseteq S$ with $B \subseteq A$.

First, if $y \in f(A \setminus B)$, then $y = f(x)$ for some $x \in A \setminus B$. Clearly $y \in f(A)$. If $y \in f(B)$, then $f(x) = f(b)$ for some $b \in B$, implying $x \in f^{-1}(f(B)) = B$, a contradiction. Hence $y \notin f(B)$, and $y \in f(A) \setminus f(B)$. Thus $f(A \setminus B) \subseteq f(A) \setminus f(B)$.

Conversely, if $y \in f(A) \setminus f(B)$, then $y = f(a)$ for some $a \in A$ but $y \notin f(B)$. If $a \in B$, then $f(a) \in f(B)$, contradiction. Thus $a \in A \setminus B$, and $y \in f(A \setminus B)$. Hence $f(A) \setminus f(B) \subseteq f(A \setminus B)$, giving equality.

(c) \Rightarrow (a): Assume (c) holds. Suppose f is not one-to-one. Then there exist distinct $x, y \in S$ with $f(x) = f(y)$. Let $A = \{x, y\}$ and $B = \{y\}$, so $B \subseteq A$. Then (c) gives

$$f(A \setminus B) = f(A) \setminus f(B).$$

Now $A \setminus B = \{x\}$, so $f(A \setminus B) = \{f(x)\}$. Also $f(A) = \{f(x)\}$ and $f(B) = \{f(y)\} = \{f(x)\}$, hence $f(A) \setminus f(B) = \emptyset$. Thus $\{f(x)\} = \emptyset$, impossible. Therefore, f must be one-to-one.

Since (a) \Rightarrow (b), (b) \Rightarrow (c), and (c) \Rightarrow (a), the three statements are equivalent. \square

Problem 14. Let $S \subseteq \mathbb{R} \times \mathbb{R}$ be the relation defined in each case below.

(a) $S = \{(x, y) \in \mathbb{R}^2 : x \leq y\}.$

(b) $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$

For each case determine whether S is reflexive, symmetric, and/or transitive.

Problem 4

Solution. (a) $S = \{(x, y) : x \leq y\}.$

Reflexive. For every $x \in \mathbb{R}$ we have $x \leq x$, so $(x, x) \in S$. Thus S is reflexive.

Symmetric. If $(x, y) \in S$ then $x \leq y$. This does not imply $y \leq x$ in general (take $x = 0, y = 1$), so S is not symmetric.

Transitive. If $(x, y) \in S$ and $(y, z) \in S$ then $x \leq y$ and $y \leq z$, hence $x \leq z$, so $(x, z) \in S$. Thus S is transitive.

(b) $S = \{(x, y) : x^2 + y^2 = 1\}.$

Reflexive. Reflexivity would require $(x, x) \in S$ for every x , i.e. $2x^2 = 1$ for all x , which is false (for example $(0, 0) \notin S$). Hence S is not reflexive.

Symmetric. The defining equation is symmetric in x and y : if $x^2 + y^2 = 1$ then $y^2 + x^2 = 1$, so $(y, x) \in S$. Thus S is symmetric.

Transitive. Transitivity fails. For example $(1, 0) \in S$ and $(0, 1) \in S$, but $(1, 1) \notin S$ since $1^2 + 1^2 = 2 \neq 1$. Therefore S is not transitive. \square

Theorem 15. The set of all circles in \mathbb{R}^2 whose centers have rational coordinates and whose radii are rational (positive) numbers is countable.

Problem 5 (a)

Proof. A circle in the plane is determined uniquely by its center and its radius. Let

$$\mathcal{C} = \{C((p, q), r) : (p, q) \in \mathbb{Q}^2, r \in \mathbb{Q}_{>0}\},$$

where $C((p, q), r)$ denotes the circle with center (p, q) and radius r . Consider the map

$$\varphi: \mathbb{Q}^2 \times \mathbb{Q}_{>0} \longrightarrow \mathcal{C}, \quad \varphi((p, q), r) = C((p, q), r).$$

This map is surjective by definition and injective because distinct triples $((p, q), r)$ determine distinct circles. Hence \mathcal{C} is in bijection with the set $\mathbb{Q}^2 \times \mathbb{Q}_{>0}$.

Since \mathbb{Q} is countable and any finite Cartesian product of countable sets is countable, the set $\mathbb{Q}^2 \times \mathbb{Q}_{>0}$ is countable. Therefore, \mathcal{C} is countable. This completes the proof. \square

Theorem 16. *Any collection \mathcal{I} of pairwise disjoint intervals in \mathbb{R} , each of positive length, is at most countable (i.e., finite or countably infinite).*

Problem 5 (b)

Proof. Let \mathcal{I} be such a collection. For each interval $I \in \mathcal{I}$ its length $\ell(I) > 0$, so I contains more than one point. Since the rationals \mathbb{Q} are dense in \mathbb{R} , every nondegenerate interval I contains at least one rational number. Choose and fix, for each $I \in \mathcal{I}$, a rational number $q_I \in I$.

We claim the map $I \mapsto q_I$ is injective. Indeed, if $I \neq J$ are two distinct intervals in \mathcal{I} then, because the intervals are pairwise disjoint, $I \cap J = \emptyset$. Hence $q_I \in I$ and $q_J \in J$ cannot be equal. Thus distinct intervals are assigned distinct rationals.

Therefore the set $\{q_I : I \in \mathcal{I}\}$ is an injective image of \mathcal{I} and is a subset of \mathbb{Q} . Since \mathbb{Q} is countable, every subset of \mathbb{Q} is at most countable. It follows that \mathcal{I} is at most countable. \square

Theorem 17. *The set of real numbers \mathbb{R} is uncountable.*

Problem 6

Proof. We show that the set of real numbers \mathbb{R} is uncountable using the Cantor's diagonal argument.

Recall that a *decimal expansion* of a real number x is a representation of the form

$$x = d_0.d_1d_2d_3\ldots := d_0 + \sum_{i=1}^{\infty} d_i 10^{-i},$$

where d_0 is the integer part of x , and each $d_i \in \{0, 1, 2, \dots, 9\}$ is a decimal digit. For numbers in $[0, 1)$, the expansion is of the form $x = 0.d_1d_2d_3\ldots$. Some numbers have two decimal expansions (e.g., $0.5 = 0.5000\ldots = 0.4999\ldots$).

It suffices to prove that the interval $[0, 1) \subset \mathbb{R}$ is uncountable. Assume, for contradiction, that $[0, 1)$ is countable. Suppose that all numbers in $[0, 1)$ can be listed in a sequence:

$$x_1, x_2, x_3, \dots$$

To avoid ambiguity from numbers with two expansions, we adopt the convention: choose the decimal expansion *not ending with infinitely many 9's*. Under this rule, every number in $[0, 1)$ has a unique decimal expansion. Using this convention, write the sequence as:

$$\begin{array}{rcll} x_1 & = & 0. & d_{11} & d_{12} & d_{13} & \dots, \\ x_2 & = & 0. & d_{21} & d_{22} & d_{23} & \dots, \\ & & & \vdots & & & \end{array}$$

Define a number

$$y = 0. \quad a_1 \quad a_2 \quad a_3 \quad \dots$$

by choosing each digit a_i as

$$a_i \neq d_{ii}, \quad a_i \in \{1, 2, \dots, 8\}.$$

This ensures that y differs from x_i in the i -th decimal place. Since we avoided 0 and 9, y does not create ambiguity with decimal expansions.

By construction, $y \in [0, 1)$. However, $y \neq x_i$ for all $i \in \mathbb{N}$, so y is *not* in the list. This contradicts the assumption that all numbers in $[0, 1)$ were listed. Therefore, $[0, 1)$ is uncountable. Consequently, \mathbb{R} is uncountable. \square

Homework 3

Theorem 18. Let $S \subset \mathbb{R}^n$. Then $\text{int } S$ (the interior of S) is an open set.

Problem 1

Proof. Recall that $x \in \text{int } S$ iff there exists $\varepsilon > 0$ such that the open ball $B_\varepsilon(x) = \{y \in \mathbb{R}^n : \|y - x\| < \varepsilon\}$ is contained in S .

Let $x \in \text{int } S$. By definition, choose $\varepsilon > 0$ with $B_\varepsilon(x) \subset S$. We claim $B_\varepsilon(x) \subset \text{int } S$, which will show that $\text{int } S$ is a neighborhood of each of its points and hence open.

Indeed, let $y \in B_\varepsilon(x)$. Then $\|y - x\| < \varepsilon$. Put $\delta = \varepsilon - \|y - x\| > 0$. For any $z \in \mathbb{R}^n$ with $\|z - y\| < \delta$ we have

$$\|z - x\| \leq \|z - y\| + \|y - x\| < \delta + \|y - x\| = \varepsilon,$$

so $z \in B_\varepsilon(x) \subset S$. Thus $B_\delta(y) \subset S$, hence $y \in \text{int } S$. This proves $B_\varepsilon(x) \subset \text{int } S$.

Since every $x \in \text{int } S$ has an open ball around it contained in $\text{int } S$, the set $\text{int } S$ is open. \square

Problem 19. Do S and \overline{S} always have the same interiors? Do S and $\text{int}(S)$ always have the same closures?

Problem 2

Solution. (1) Since $S \subseteq \overline{S}$, the monotonicity of the interior operator gives

$$\text{int}(S) \subseteq \text{int}(\overline{S}).$$

However, equality need not hold.

Counterexample: Let $S = (0, 1) \cup (1, 2) \subset \mathbb{R}$. Then

$$\text{int}(S) = (0, 1) \cup (1, 2), \quad \overline{S} = [0, 2], \quad \text{int}(\overline{S}) = (0, 2),$$

so $\text{int}(S) \neq \text{int}(\overline{S})$.

(2) Since $\text{int}(S) \subseteq S$, taking closures gives

$$\overline{\text{int}(S)} \subseteq \overline{S}.$$

Again, equality need not hold.

Counterexample: Let $S = [0, 1] \cup \{2\} \subset \mathbb{R}$. Then

$$\text{int}(S) = (0, 1), \quad \overline{\text{int}(S)} = [0, 1], \quad \overline{S} = [0, 1] \cup \{2\},$$

so $\overline{\text{int}(S)} \neq \overline{S}$. □

Theorem 20. *The set \mathbb{Z} has no accumulation points. Thus, \mathbb{Z} is closed. However, \mathbb{Z} is not open.*

Problem 3 (a)

Proof. Let $x \in \mathbb{R}$.

Case 1. If $x = k \in \mathbb{Z}$, choose $\varepsilon = \frac{1}{2}$. Then

$$(k - \frac{1}{2}, k + \frac{1}{2}) \cap \mathbb{Z} = \{k\}.$$

Hence the punctured neighborhood $(k - \varepsilon, k + \varepsilon) \setminus \{k\}$ contains no point of \mathbb{Z} ; thus k is not an accumulation point.

Case 2. If $x \notin \mathbb{Z}$, let $d = \inf\{|x - n| : n \in \mathbb{Z}\} > 0$ be the distance from x to the nearest integer. Take $\varepsilon = \frac{d}{2}$. Then $(x - \varepsilon, x + \varepsilon)$ contains no integer, so it contains no point of \mathbb{Z} . Hence x is not an accumulation point.

Therefore \mathbb{Z} has no accumulation points.

Now, \mathbb{Z} is not open (no nonempty interval lies entirely inside \mathbb{Z}) and closed, since it contains all of its accumulation points (vacuously, because there are none). □

Theorem 21. *Every real number is an accumulation point of \mathbb{Q} . Moreover, \mathbb{Q} is neither open nor closed.*

Problem 3 (b)

Proof. Let $x \in \mathbb{R}$ and $\varepsilon > 0$ be arbitrary. Choose an integer N such that $\frac{1}{N} < \varepsilon$. There exists an integer k with

$$\frac{k}{N} \leq x < \frac{k+1}{N}.$$

Then $\frac{k}{N}$ is rational and lies in $[x - \frac{1}{N}, x] \subset (x - \varepsilon, x + \varepsilon)$. If $\frac{k}{N} \neq x$, we are done. If $\frac{k}{N} = x$, then

$$0 < \frac{k+1}{N} - x < \frac{1}{N} < \varepsilon,$$

so $\frac{k+1}{N} \in (x - \varepsilon, x + \varepsilon)$ and $\frac{k+1}{N} \neq x$. Thus every punctured neighborhood of x contains a rational distinct from x , and hence x is an accumulation point of \mathbb{Q} .

Now, \mathbb{Q} is not open (every interval contains irrationals) and not closed (irrationals are accumulation points not in \mathbb{Q}). \square

Theorem 22. *Let*

$$S = \left\{ \frac{1}{n} + \frac{1}{m} : m, n \in \mathbb{Z}_+ \right\}.$$

Then the accumulation points of S are precisely

$$\{0\} \cup \left\{ \frac{1}{k} : k \in \mathbb{Z}_+ \right\}.$$

Moreover, S is neither open nor closed.

Problem 3 (c)

Proof. For $m, n \in \mathbb{Z}_+$, define $s_{n,m} := \frac{1}{n} + \frac{1}{m}$.

(1) *0 is an accumulation point:* Let $\varepsilon > 0$. Choose N such that $\frac{2}{N} < \varepsilon$. Then for all $m, n \geq N$,

$$0 < s_{n,m} \leq \frac{1}{N} + \frac{1}{N} = \frac{2}{N} < \varepsilon.$$

Hence $s_{n,m} \in (0 - \varepsilon, 0 + \varepsilon)$ and $s_{n,m} \neq 0$. Thus every punctured neighborhood of 0 contains a point of S , so 0 is an accumulation point.

(2) *Each $\frac{1}{k}$ is an accumulation point:* Fix $k \in \mathbb{Z}_+$ and let $\varepsilon > 0$. Choose M such that $\frac{1}{M} < \varepsilon$. Then

$$s_{k,M} = \frac{1}{k} + \frac{1}{M} \in \left(\frac{1}{k} - \varepsilon, \frac{1}{k} + \varepsilon \right),$$

and $s_{k,M} \neq \frac{1}{k}$. Hence each punctured neighborhood of $\frac{1}{k}$ contains a point of S , so $\frac{1}{k}$ is an accumulation point.

(3) *No other accumulation points exist:* Let $y \in \mathbb{R}$ and suppose y is an accumulation point of S . We will show that $y = 0$ or $y = \frac{1}{k}$ for some $k \in \mathbb{Z}_+$.

First observe that $S \subset (0, 2]$, so any accumulation point y must satisfy $0 \leq y \leq 2$. If $y = 0$, we are done. Assume $y > 0$.

Because y is an accumulation point, for every $\varepsilon > 0$, the punctured neighborhood $(y - \varepsilon, y + \varepsilon) \setminus \{y\}$ contains some $s_{n,m} \neq y$. Consider the set of index pairs

$$P(\varepsilon) = \{(n, m) \in \mathbb{Z}_+^2 : s_{n,m} \in (y - \varepsilon, y + \varepsilon)\}.$$

Suppose, for contradiction, that both coordinates n and m are bounded on $P(\varepsilon_0)$ for some sufficiently small $\varepsilon_0 > 0$. That is, there exist integers N_0, M_0 such that whenever $(n, m) \in P(\varepsilon_0)$, we have $n \leq N_0$ and $m \leq M_0$. Then the set of possible values

$$F = \{s_{n,m} : 1 \leq n \leq N_0, 1 \leq m \leq M_0\}$$

is finite.

If $y \notin F$, let

$$\delta = \min\{|y - f| : f \in F\} > 0,$$

and choose $\varepsilon < \frac{\delta}{2}$. Then $(y - \varepsilon, y + \varepsilon) \cap F = \emptyset$, contradicting $P(\varepsilon_0) \neq \emptyset$.

If $y \in F$, let

$$\delta = \min\{|y - f| : f \in F, f \neq y\} > 0,$$

and take $\varepsilon < \frac{\delta}{2}$. Then the punctured neighborhood $(y - \varepsilon, y + \varepsilon) \setminus \{y\}$ contains no element of F , again contradicting $P(\varepsilon) \neq \emptyset$.

Therefore, it is impossible that both coordinates are bounded for arbitrarily small ε . Therefore, at least one coordinate is unbounded among pairs (n, m) whose sums $s_{n,m}$ lie arbitrarily close to y .

Case A: both coordinates can be made arbitrarily large.

Then for any $\varepsilon > 0$ we can find n, m so large that

$$s_{n,m} = \frac{1}{n} + \frac{1}{m} < \varepsilon.$$

(Choose N with $\frac{2}{N} < \varepsilon$ and take $n, m \geq N$.) Hence, we must have $y = 0$. But we assumed $y > 0$, so this case cannot occur for $y > 0$.

Case B: exactly one coordinate is unbounded while the other takes only finitely many values.

Then there exists some fixed $k \in \mathbb{Z}_+$ and arbitrarily large integers m (or vice versa) such that $s_{k,m}$ lies within any given ε -neighborhood of y . But for every $\varepsilon > 0$ there exists M with $|s_{k,m} - \frac{1}{k}| < \varepsilon$ for all $m \geq M$. By the punctured-neighborhood definition, this forces $y = \frac{1}{k}$.

Combining the impossibility of Case A for $y > 0$ and the conclusion of Case B, we find that any positive accumulation point y must be equal to some $\frac{1}{k}$.

Thus the only accumulation points are 0 and the numbers $\frac{1}{k}$ for $k \in \mathbb{Z}_+$.

Now, S is not open (its points are isolated in the sense that for a fixed (n, m) we can choose ε small enough to exclude all other $s_{n',m'}$), and not closed because 0 (and the points $\frac{1}{k}$) are accumulation points not in S . \square

Theorem 23. *The set of accumulation points of $S = \{(x, y) \in \mathbb{R}^2 : x \geq 0\} \subset \mathbb{R}^2$ is $\{(x, y) \in \mathbb{R}^2 : x \geq 0\}$. Thus, S is closed. However, S is not open.*

Proof. Let $p = (x, y) \in \mathbb{R}^2$.

(i) If $x > 0$. Fix $\varepsilon > 0$. Take $q = (x', y')$ with $x' = x + \min\{\varepsilon/2, x/2\} > 0$ and $y' = y$. Then $\|q - p\| = |x' - x| < \varepsilon$ and $q \in S$, $q \neq p$. Thus every punctured neighborhood of p meets S ; so p is an accumulation point.

(ii) If $x = 0$. Fix $\varepsilon > 0$. Let $q = (\varepsilon/2, y)$. Then $\|q - p\| = \varepsilon/2 < \varepsilon$ and $q \in S$. Hence $(0, y)$ is an accumulation point.

(iii) If $x < 0$. Put $\varepsilon = -x/2 > 0$. If $\|q - p\| < \varepsilon$ then the first coordinate x' of q satisfies $|x' - x| < \varepsilon$, so $x' \leq x + \varepsilon = x/2 < 0$. Thus no point of S lies in $B_\varepsilon(p)$. Hence p is not an accumulation point.

Combining (i)–(iii) gives that the accumulation points are exactly those with $x \geq 0$. Therefore, S is closed.

Now, we show S is not open. Let $p = (0, 1) \in S$. Take $\varepsilon > 0$. If $q = (-\varepsilon/2, 1)$, then $\|q - p\| < \varepsilon$, i.e., $q \in B_\varepsilon(p)$. However, $q \notin S$.

Therefore every neighborhood of p contains a point of $\mathbb{R}^2 \setminus S$, so S is not open. \square

Theorem 24. *The set of accumulation points of $S = \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 < 1\}$ is $\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 \leq 1\}$. Moreover, S is open but not closed.*

Problem 3 (e)

Proof. Let $p = (x, y) \in \mathbb{R}^2$. Define $g(x, y) = x^2 - y^2$.

(i) If $g(x, y) < 1$. Set $\Delta := 1 - g(x, y) > 0$. Choose

$$\delta = \min \left\{ 1, \frac{\Delta}{4(|x| + |y| + 1)} \right\} > 0.$$

If $\|(x', y') - (x, y)\| < \delta$ then in particular $|x' - x| < \delta$ and $|y' - y| < \delta$. Now

$$|x'^2 - x^2| = |x' - x| |x' + x| \leq \delta(2|x| + \delta) \leq \delta(2|x| + 1),$$

and similarly

$$|y'^2 - y^2| \leq \delta(2|y| + 1).$$

Hence

$$\begin{aligned} |g(x', y') - g(x, y)| &\leq |x'^2 - x^2| + |y'^2 - y^2| \\ &\leq \delta(2(|x| + |y|) + 2) \\ &\leq 2\delta(|x| + |y| + 1). \end{aligned}$$

By the choice of δ we have $2\delta(|x| + |y| + 1) \leq \Delta/2$, so $|g(x', y') - g(x, y)| < \Delta/2$. Therefore

$$g(x', y') < g(x, y) + \Delta/2 = 1 - \Delta/2 < 1.$$

Thus every punctured neighborhood of p contains points of S ; so p is an accumulation point (and an interior point).

(ii) If $g(x, y) = 1$. Note $x \neq 0$ (otherwise $-y^2 = 1$ impossible). Fix $\varepsilon > 0$. Choose $\delta > 0$ with $\delta|x| < \varepsilon$, for example $\delta = \min\{\varepsilon/(2|x|), 1/2\}$. Let $x' = (1 - \delta)x$, $y' = y$. Then

$$\|(x', y') - (x, y)\| = |x - x'| = \delta|x| < \varepsilon,$$

and

$$g(x', y') = (1 - \delta)^2 x^2 - y^2 = x^2 - y^2 - 2\delta x^2 + \delta^2 x^2 = 1 - 2\delta x^2 + \delta^2 x^2 < 1.$$

Thus every punctured neighborhood of a boundary point (x, y) meets S , so every boundary point is an accumulation point (but not in S).

(iii) If $g(x, y) > 1$. Put $\Gamma := g(x, y) - 1 > 0$. Choose

$$\delta = \min \left\{ 1, \frac{\Gamma}{4(|x| + |y| + 1)} \right\} > 0.$$

Arguing as in (i) we obtain

$$|g(x', y') - g(x, y)| \leq 2\delta(|x| + |y| + 1) \leq \Gamma/2,$$

whenever $\|(x', y') - (x, y)\| < \delta$. Hence for such (x', y') ,

$$g(x', y') > g(x, y) - \Gamma/2 = 1 + \Gamma/2 > 1,$$

so no point of S lies in $B_\delta(p)$. Thus p is not an accumulation point.

Combining (i)–(iii) shows the accumulation points are exactly those with $x^2 - y^2 \leq 1$.

Now, we show S is open. Let $p = (x, y) \in S$ and define $\Delta = 1 - (x^2 - y^2)$. Then $\Delta > 0$. Choose

$$\delta = \min \left\{ 1, \frac{\Delta}{4(|x| + |y| + 1)} \right\} > 0.$$

If $\|(x', y') - (x, y)\| < \delta$, then $|x' - x| < \delta$, $|y' - y| < \delta$, so

$$|x'^2 - x^2| \leq \delta(2|x| + 1), \quad |y'^2 - y^2| \leq \delta(2|y| + 1).$$

Hence

$$|(x'^2 - y'^2) - (x^2 - y^2)| \leq 2\delta(|x| + |y| + 1) \leq \Delta/2,$$

so $x'^2 - y'^2 < 1$. Thus $B_\delta(p) \subset S$, and S is open.

Boundary points (where $x^2 - y^2 = 1$) are accumulation points not in S , so S is not closed. \square

Theorem 25. Every point of \mathbb{R}^n is an accumulation point of \mathbb{Q}^n . Moreover, \mathbb{Q}^n is neither open nor closed.

Problem 3 (f)

Proof. Fix $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\varepsilon > 0$. Choose a positive integer N with

$$\frac{1}{N} < \frac{\varepsilon}{\sqrt{n}}.$$

For each coordinate x_i choose an integer k_i with

$$\frac{k_i}{N} \leq x_i < \frac{k_i + 1}{N}.$$

Set $q_i = \frac{k_i}{N}$ for $i = 1, \dots, n$ and $q = (q_1, \dots, q_n)$. Then each $q_i \in \mathbb{Q}$ and

$$|x_i - q_i| < \frac{1}{N} < \frac{\varepsilon}{\sqrt{n}}.$$

Therefore

$$\|x - q\| = \sqrt{\sum_{i=1}^n (x_i - q_i)^2} < \sqrt{n \cdot \frac{\varepsilon^2}{n}} = \varepsilon.$$

If $q \neq x$ we are done. If $q = x$ (this can only occur when $x \in \mathbb{Q}^n$), then modify one coordinate slightly: replace q_1 by $q_1 + \frac{1}{N}$ (which is rational and still satisfies $|x_1 - (q_1 + 1/N)| \leq 1/N < \varepsilon/\sqrt{n}$), so the modified rational vector $q' \in \mathbb{Q}^n$ satisfies $\|x - q'\| < \varepsilon$ and $q' \neq x$. Hence every punctured ball around x contains a rational point distinct from x , proving the claim.

But \mathbb{Q}^n has no interior points, since every ball contains irrationals. Therefore \mathbb{Q}^n is neither open nor closed. \square

Theorem 26. Let

$$S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \left\{ 1 + \frac{1}{n} : n \in \mathbb{N} \right\} \cup \left\{ 2 + \frac{1}{n} : n \in \mathbb{N} \right\}.$$

Then the set of accumulation points of S is exactly $\{0, 1, 2\}$.

Problem 4

Proof. Let $a \in \{0, 1, 2\}$ and let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Then the point $a + \frac{1}{N} \in S$ (for $a = 0$ we interpret this as $\frac{1}{N} \in S$) satisfies

$$0 < |a + \frac{1}{N} - a| = \frac{1}{N} < \varepsilon.$$

Hence every punctured neighborhood $(a - \varepsilon, a + \varepsilon) \setminus \{a\}$ contains points of S . Thus a is an accumulation point of S .

Let $y \in \mathbb{R} \setminus \{0, 1, 2\}$. Define

$$d = \min\{|y - 0|, |y - 1|, |y - 2|\} > 0, \quad r = \frac{d}{2}.$$

Let $F = \{s \in S : |s - y| < r\}$. Suppose for contradiction that F is infinite. Then there exists $i \in \{0, 1, 2\}$ and an infinite subset $\mathcal{A} \subseteq \mathbb{N}$ such that

$$\left| i + \frac{1}{n} - y \right| < r \quad \text{for all } n \in \mathcal{A}.$$

Fix $n \in \mathcal{A}$ with $n > \frac{2}{d}$ (such an n exists because \mathcal{A} is infinite). Then $\frac{1}{n} < \frac{d}{2}$ and

$$|y - i| \leq \left| y - \left(i + \frac{1}{n} \right) \right| + \frac{1}{n} < r + \frac{1}{n} = \frac{d}{2} + \frac{1}{n} < \frac{d}{2} + \frac{d}{2} = d,$$

which contradicts the definition of d (since $|y - i| \geq d$). Hence F must be finite.

If $F = \emptyset$, then $B_r(y)$ contains no point of S and we are done. Otherwise, set

$$\varepsilon = \min \left\{ r, \frac{1}{2} \min_{s \in F} |s - y| \right\} > 0.$$

Then no point of S (other than possibly y itself, but $y \notin S$) lies in $(y - \varepsilon, y + \varepsilon)$. Hence the punctured neighborhood $(y - \varepsilon, y + \varepsilon) \setminus \{y\}$ contains no point of S , so y is not an accumulation point.

Therefore, the set of accumulation points of S is exactly $\{0, 1, 2\}$. \square

Theorem 27. Let $S \subset \mathbb{R}^n$. The closure \bar{S} is the intersection of all closed

subsets of \mathbb{R}^n that contain S , i.e.,

$$\overline{S} = \bigcap \{ F \subset \mathbb{R}^n : F \text{ is closed and } S \subset F \}.$$

Problem 5

Proof. Let $\mathcal{F} = \{ F \subset \mathbb{R}^n : F \text{ is closed and } S \subset F \}$ and set

$$K := \bigcap_{F \in \mathcal{F}} F.$$

We will show $\overline{S} = K$.

(1) $\overline{S} \subset K$. By definition \overline{S} is a closed set containing S . Since K is the intersection of *all* closed sets that contain S , every such closed set in particular contains \overline{S} . Hence $\overline{S} \subset F$ for every $F \in \mathcal{F}$, and therefore $\overline{S} \subset K$.

(2) $K \subset \overline{S}$. Suppose $x \notin \overline{S}$. By the definition of closure there exists $\varepsilon > 0$ such that the open ball $B_\varepsilon(x)$ satisfies

$$B_\varepsilon(x) \cap S = \emptyset.$$

Equivalently, $S \subset \mathbb{R}^n \setminus B_\varepsilon(x)$. The complement $\mathbb{R}^n \setminus B_\varepsilon(x)$ is closed and contains S , but it does not contain x . Thus $\mathbb{R}^n \setminus B_\varepsilon(x) \in \mathcal{F}$ and $x \notin \bigcap_{F \in \mathcal{F}} F = K$. Hence every $x \notin \overline{S}$ is also not in K , so $K \subset \overline{S}$.

Combining (1) and (2) yields $\overline{S} = K$, which proves the claim. \square

Theorem 28. *Let*

$$\mathcal{F} = \left\{ \left(\frac{1}{n}, \frac{2}{n} \right) : n \in \mathbb{Z}_+ \right\}.$$

Then \mathcal{F} is an open cover of $(0, 1)$, but no finite subcollection of \mathcal{F} covers $(0, 1)$.

Problem 6

Proof. Each set $(1/n, 2/n)$ is open, so \mathcal{F} is a collection of open sets. Let $x \in (0, 1)$ be arbitrary. Then $1/x > 1$, hence

$$\frac{2}{x} - \frac{1}{x} = \frac{1}{x} > 1,$$

so the open interval $(1/x, 2/x)$ has length $1/x > 1$ and therefore contains at least one integer. Thus there exists $n \in \mathbb{Z}_+$ with

$$\frac{1}{x} < n < \frac{2}{x}.$$

Rewriting the inequalities gives

$$\frac{1}{n} < x < \frac{2}{n},$$

so $x \in (1/n, 2/n) \in \mathcal{F}$. Since x was arbitrary, $\bigcup \mathcal{F} = (0, 1)$, i.e., \mathcal{F} is an open cover of $(0, 1)$.

We show that no finite subcollection of \mathcal{F} covers $(0, 1)$. Suppose, for contradiction, that a finite subcollection $\{(1/n_i, 2/n_i) : i = 1, \dots, k\} \subset \mathcal{F}$ covers $(0, 1)$. Let $N = \max\{n_1, \dots, n_k\}$. Consider the point

$$x = \frac{1}{N+1} \in (0, 1).$$

For any chosen index i we have $n_i \leq N$, hence

$$\frac{1}{n_i} \geq \frac{1}{N} > \frac{1}{N+1} = x,$$

so $x \notin (1/n_i, 2/n_i)$. Thus x is not contained in any of the finitely many chosen intervals, contradicting the assumption that the finite subcollection covers $(0, 1)$. Therefore no finite subcollection of \mathcal{F} can cover $(0, 1)$.

This completes the proof. □

Theorem 29. *Let*

$$\mathcal{B} = \{ B((q, q), q) : q \in \mathbb{Q}_{>0} \},$$

where $B((q, q), q) = \{(u, v) \in \mathbb{R}^2 : \sqrt{(u - q)^2 + (v - q)^2} < q\}$. Then \mathcal{B} is a countable collection and

$$\bigcup_{q \in \mathbb{Q}_{>0}} B((q, q), q) = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}.$$

In particular \mathcal{B} is a countable cover of the open first quadrant.

Problem 7

Proof. The set $\mathbb{Q}_{>0}$ of positive rationals is countable, hence the indexed family \mathcal{B} is countable.

Let (a, b) be an arbitrary point with $a > 0$ and $b > 0$. Define the function

$$F(r) = (a - r)^2 + (b - r)^2 - r^2.$$

A point (a, b) lies in $B((r, r), r)$ precisely when $F(r) < 0$. Expand and simplify:

$$F(r) = (a^2 + b^2) - 2(a + b)r + r^2.$$

Thus $F(r) < 0$ is equivalent to

$$r^2 - 2(a + b)r + (a^2 + b^2) < 0.$$

The quadratic on the left has discriminant

$$\Delta = 4(a + b)^2 - 4(a^2 + b^2) = 8ab > 0,$$

so the inequality holds exactly for r lying between the two real roots

$$r_{\pm} = (a + b) \pm \sqrt{2ab}.$$

Hence

$$F(r) < 0 \iff r \in (r_-, r_+).$$

Note that $r_- > 0$ because $(a + b)^2 - 2ab = a^2 + b^2 > 0$, so the open interval (r_-, r_+) is a nonempty interval contained in $(0, \infty)$.

By density of the rationals there exists some $q \in \mathbb{Q}_{>0} \cap (r_-, r_+)$. For such a rational q we have $F(q) < 0$, i.e.

$$\sqrt{(a - q)^2 + (b - q)^2} < q,$$

so $(a, b) \in B((q, q), q)$. Since (a, b) was an arbitrary point of the first quadrant, every such point is contained in some ball from \mathcal{B} .

Combining the two parts, \mathcal{B} is a countable cover of $\{(x, y) : x > 0, y > 0\}$. □

Theorem 30. Let \mathcal{U} be a collection of pairwise disjoint *nonempty* open subsets of \mathbb{R}^n . Then \mathcal{U} is at most countable.

Proof. The set \mathbb{Q}^n of points with rational coordinates is countable. Enumerate $\mathbb{Q}^n = \{q_1, q_2, q_3, \dots\}$.

By [Theorem 25](#), for each $U \in \mathcal{U}$ the intersection $U \cap \mathbb{Q}^n$ is nonempty. Define an assignment $f: \mathcal{U} \rightarrow \mathbb{Q}^n$ by letting $f(U)$ be the first rational q_i (with smallest index i) that lies in U . This is well defined because each U contains at least one rational and our enumeration gives a least index.

We claim f is injective. Indeed, if $U \neq V$ are two distinct sets in \mathcal{U} then $U \cap V = \emptyset$ by hypothesis; hence no rational point can lie in both U and V . Therefore the first rational in U cannot equal the first rational in V , so $f(U) \neq f(V)$.

Since f injects \mathcal{U} into the countable set \mathbb{Q}^n , the collection \mathcal{U} must itself be at most countable. \square

Remark 31. The hypothesis “nonempty” is essential: the empty set is open and many copies of it would be pairwise disjoint but not interesting.

Example 32. The family of singletons

$$\mathcal{C} = \{\{x\} : x \in [0, 1]\}$$

is an uncountable collection of pairwise disjoint closed subsets of \mathbb{R} . Each $\{x\}$ is closed in \mathbb{R} , distinct singletons are disjoint, and the indexing set $[0, 1]$ is uncountable, so \mathcal{C} is uncountable.

Problem 8

Homework 4

Problem 33. Which the following subsets of \mathbb{R}^2 are compact?

- (a) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$
- (b) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1\}$
- (c) $\{(x, y) \in \mathbb{R}^2 : x, y \in \mathbb{Q}, x^2 + y^2 \leq 1\}$

Problem 1

Solution. By the Heine–Borel theorem, a subset of \mathbb{R}^2 is compact if and only if it is *closed* and *bounded*.

(a) The set

$$A = \{(x, y) : x^2 + y^2 = 1\}$$

is the unit circle. It is bounded, since $x^2 + y^2 = 1$ implies $\sqrt{x^2 + y^2} = 1$. Now we show A is closed.

Let $p = (x, y) \in \mathbb{R}^2$ be any point not on the circle A , so $r := x^2 + y^2 \neq 1$. Consider two cases.

Case 1: $r > 1$. Put $\delta = \sqrt{r} - 1 > 0$. If $q = (u, v)$ satisfies $\|(u, v) - (x, y)\| < \delta$, then by the triangle inequality,

$$\|(u, v)\| \geq \|(x, y)\| - \|(u, v) - (x, y)\| > \sqrt{r} - \delta = 1,$$

hence $u^2 + v^2 > 1$. So every point q in the open ball $B_\delta(p)$ satisfies $u^2 + v^2 > 1$, i.e. $B_\delta(p)$ is contained in $\mathbb{R}^2 \setminus A$.

Case 2: $r < 1$. Put $\delta = 1 - \sqrt{r} > 0$. If $\|(u, v) - (x, y)\| < \delta$, then

$$\|(u, v)\| \leq \|(x, y)\| + \|(u, v) - (x, y)\| < \sqrt{r} + \delta = 1,$$

so $u^2 + v^2 < 1$. Thus again $B_\delta(p)$ is contained in $\mathbb{R}^2 \setminus A$.

In either case $\mathbb{R}^2 \setminus A$ is a neighborhood of p . Since p was an arbitrary point outside A , the complement $\mathbb{R}^2 \setminus A$ is open, so A is closed.

(b) The set

$$B = \{(x, y) : x^2 + y^2 \geq 1\}$$

is closed (as $\mathbb{R}^2 \setminus B = B_1(0)$), but not bounded. Therefore B is not compact.

(c) The set

$$C = \{(x, y) : x, y \in \mathbb{Q}, x^2 + y^2 \leq 1\}$$

is bounded but not closed in \mathbb{R}^2 , because there exist sequences of rational points inside C that converge to irrational points still satisfying $x^2 + y^2 \leq 1$. For example, $(r_n, 0) \in C$ with $r_n \rightarrow \frac{\sqrt{2}}{2}$ converge to $(\frac{\sqrt{2}}{2}, 0) \notin C$. Hence C is not compact. \square

Theorem 34. *There exists a countable open cover \mathcal{F} of $\mathbb{Z} \subset \mathbb{R}$ which has no finite subcover.*

Problem 2

Proof. For each integer $k \in \mathbb{Z}$ define the open interval

$$U_k = (k - \frac{1}{2}, k + \frac{1}{2}).$$

Let $\mathcal{F} = \{U_k : k \in \mathbb{Z}\}$. Clearly \mathcal{F} is countable and each U_k is open in \mathbb{R} . Since $k \in U_k$ for every k , the family \mathcal{F} covers \mathbb{Z} .

Suppose, for contradiction, that a finite subcollection $\{U_{k_1}, \dots, U_{k_m}\} \subset \mathcal{F}$ already covers \mathbb{Z} . Let

$$K = \max\{|k_1|, \dots, |k_m|\}.$$

Then every interval U_{k_j} contains only integers in the range $[k_j - 1/2, k_j + 1/2]$, hence the finite union $\bigcup_{j=1}^m U_{k_j}$ contains only integers n with $|n| \leq K$. But the integer $K + 1$ is not contained in this union, contradicting the assumption that the finite subcollection covers \mathbb{Z} . Therefore no finite subcollection of \mathcal{F} covers \mathbb{Z} . \square

Lemma 35. Let $f: [0, \infty) \rightarrow [0, 1)$ be defined by

$$f(t) = \frac{t}{1+t}.$$

Then f is increasing on $[0, \infty)$ and is sub-additive:

$$f(a+b) \leq f(a) + f(b) \quad \text{for all } a, b \geq 0.$$

Proof. Fix $0 \leq s < t$. Compute the difference

$$f(t) - f(s) = \frac{t}{1+t} - \frac{s}{1+s} = \frac{t(1+s) - s(1+t)}{(1+t)(1+s)} = \frac{t-s}{(1+t)(1+s)}.$$

Since $t-s > 0$ and $(1+t)(1+s) > 0$, we get $f(t) - f(s) > 0$. Thus $f(t) > f(s)$ whenever $t > s$, so f is strictly increasing on $[0, \infty)$.

For $a, b \geq 0$ consider

$$\Delta := f(a) + f(b) - f(a+b) = \frac{a}{1+a} + \frac{b}{1+b} - \frac{a+b}{1+a+b}.$$

Bring the terms to a common denominator and simplify (algebraic manipulation):

$$\Delta = \frac{a(1+b)(1+a+b) + b(1+a)(1+a+b) - (a+b)(1+a)(1+b)}{(1+a)(1+b)(1+a+b)}.$$

Expanding and cancelling terms in the numerator (a straightforward calculation) yields

$$\Delta = \frac{ab(a+b+2)}{(1+a)(1+b)(1+a+b)}.$$

Every factor in the denominator is positive and the numerator $ab(a+b+2)$ is nonnegative for $a, b \geq 0$. Hence $\Delta \geq 0$, i.e.

$$f(a+b) \leq f(a) + f(b).$$

Equality holds precisely when $ab = 0$ (so one of a, b is zero). This completes the proof. \square

Problem 36. Which of the following functions define a metric on \mathbb{R} ?

(a) $d(x, y) = (x - y)^2$

(b) $\tilde{d}(x, y) = |x - 2y|$

(c) $d^*(x, y) = \frac{|x - y|}{1 + |x - y|}$

Problem 3

Solution. Recall that a function $d: \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$ is a metric if it satisfies:

1. $d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$,
2. $d(x, y) = d(y, x)$ (symmetry),
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

(a) For $d(x, y) = (x - y)^2$: Non-negativity and symmetry hold, and $d(x, y) = 0$ iff $x = y$. However, the triangle inequality fails. For example, take $x = 0, y = 1, z = 2$:

$$d(0, 2) = 4, \quad d(0, 1) + d(1, 2) = 1 + 1 = 2.$$

Since $4 \not\leq 2$, the triangle inequality fails. Hence, d is not a metric.

(b) For $\tilde{d}(x, y) = |x - 2y|$: We have $\tilde{d}(x, y) \geq 0$ and symmetry fails because

$$\tilde{d}(x, y) = |x - 2y| \neq |y - 2x| = \tilde{d}(y, x)$$

in general. For example, $\tilde{d}(0, 1) = 2$ while $\tilde{d}(1, 0) = 1$. Hence, \tilde{d} is not symmetric, so it is not a metric.

(c) For $d^*(x, y) = \frac{|x - y|}{1 + |x - y|}$: Clearly, $d^*(x, y) \geq 0$ and $d^*(x, y) = 0$ iff $x = y$. Symmetry holds since $|x - y| = |y - x|$. Now, we show d^* satisfies the triangle inequality. Let $f(t) = \frac{t}{1+t}$ for $t \in [0, \infty)$. Then by [Lemma 35](#), we have

$$\begin{aligned} d^*(x, z) &= f(|x - z|) \\ &\leq f(|x - y| + |y - z|) \\ &\leq f(|x - y|) + f(|y - z|) \\ &= d^*(x, y) + d^*(y, z). \end{aligned}$$

Therefore, d^* satisfies all metric properties. □

Theorem 37. Let $d_1: \mathbb{R}^n \times \mathbb{R}^n \rightarrow [0, \infty)$ be defined by

$$d_1(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|,$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. Then d_1 is a metric on \mathbb{R}^n .

Problem 4

Proof. We check the three metric axioms. Let $x, y, z \in \mathbb{R}^n$.

For each i , $|x_i - y_i| \geq 0$, so the maximum of these finitely many nonnegative numbers is nonnegative: $d_1(x, y) \geq 0$. If $d_1(x, y) = 0$, then $\max_{1 \leq i \leq n} |x_i - y_i| = 0$, so for every i we have $|x_i - y_i| = 0$, hence $x_i = y_i$ for all i , i.e. $x = y$. Conversely, if $x = y$ then each $|x_i - y_i| = 0$ and thus $d_1(x, y) = 0$. Therefore $d_1(x, y) = 0$ iff $x = y$.

For each i , $|x_i - y_i| = |y_i - x_i|$. Taking maxima on both sides gives

$$d_1(x, y) = \max_{1 \leq i \leq n} |x_i - y_i| = \max_{1 \leq i \leq n} |y_i - x_i| = d_1(y, x).$$

For each coordinate i the usual triangle inequality in \mathbb{R} gives

$$|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|.$$

Therefore for every i ,

$$|x_i - z_i| \leq \max_{1 \leq j \leq n} |x_j - y_j| + \max_{1 \leq j \leq n} |y_j - z_j| = d_1(x, y) + d_1(y, z).$$

Taking the maximum over $i = 1, \dots, n$ on the left-hand side yields

$$d_1(x, z) = \max_{1 \leq i \leq n} |x_i - z_i| \leq d_1(x, y) + d_1(y, z).$$

Since all three metric properties hold, d_1 is a metric on \mathbb{R}^n . □

Theorem 38. Let $d_2: \mathbb{R}^n \times \mathbb{R}^n \rightarrow [0, \infty)$ be defined by

$$d_2(x, y) = \sum_{i=1}^n |x_i - y_i|,$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. Then d_2 is a metric on \mathbb{R}^n .

Problem 4

Proof. We verify the metric axioms. Let $x, y, z \in \mathbb{R}^n$.

For each i we have $|x_i - y_i| \geq 0$, so the finite sum $d_2(x, y) = \sum_{i=1}^n |x_i - y_i| \geq 0$. If $d_2(x, y) = 0$, then every term $|x_i - y_i| = 0$, hence $x_i = y_i$ for all i , so $x = y$. Conversely, if $x = y$ then every term is 0 and $d_2(x, y) = 0$. Thus $d_2(x, y) = 0$ iff $x = y$.

For each i , $|x_i - y_i| = |y_i - x_i|$. Summing gives

$$d_2(x, y) = \sum_{i=1}^n |x_i - y_i| = \sum_{i=1}^n |y_i - x_i| = d_2(y, x).$$

For each coordinate i the real-line triangle inequality yields

$$|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|.$$

Summing these inequalities over $i = 1, \dots, n$ gives

$$\begin{aligned} d_2(x, z) &= \sum_{i=1}^n |x_i - z_i| \\ &\leq \sum_{i=1}^n (|x_i - y_i| + |y_i - z_i|) \\ &= \sum_{i=1}^n |x_i - y_i| + \sum_{i=1}^n |y_i - z_i| \\ &= d_2(x, y) + d_2(y, z). \end{aligned}$$

Since all three metric properties hold, d_2 is a metric on \mathbb{R}^n . □

Theorem 39. Let $a = (a_1, a_2) \in \mathbb{R}^2$ and $r > 0$. Then the open ball

$$B_{d_1}(a; r) = \{x \in \mathbb{R}^2 : d_1(x, a) < r\},$$

where $d_1(x, a) = \max\{|x_1 - a_1|, |x_2 - a_2|\}$, is an open square with sides parallel to the coordinate axes, and

$$B_{d_2}(a; r) = \{x \in \mathbb{R}^2 : d_2(x, a) < r\},$$

where $d_2(x, a) = |x_1 - a_1| + |x_2 - a_2|$, is a square whose diagonals are parallel to the coordinate axes (a diamond).

Proof. **(a)** By definition

$$B_{d_1}(a; r) = \{(x_1, x_2) \in \mathbb{R}^2 : \max\{|x_1 - a_1|, |x_2 - a_2|\} < r\}.$$

The inequality $\max\{|x_1 - a_1|, |x_2 - a_2|\} < r$ is equivalent to the pair of inequalities

$$|x_1 - a_1| < r \quad \text{and} \quad |x_2 - a_2| < r,$$

which in turn are equivalent to

$$a_1 - r < x_1 < a_1 + r \quad \text{and} \quad a_2 - r < x_2 < a_2 + r.$$

Thus $B_{d_1}(a; r) = (a_1 - r, a_1 + r) \times (a_2 - r, a_2 + r)$, an open square (a Cartesian product of two open intervals) whose sides are parallel to the coordinate axes and of side length $2r$. The boundary (where $\max\{|x_1 - a_1|, |x_2 - a_2|\} = r$) is the usual axis-parallel square of side $2r$ centered at a .

(b) By definition

$$B_{d_2}(a; r) = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - a_1| + |x_2 - a_2| < r\}.$$

Consider the boundary curve given by the equality $|x_1 - a_1| + |x_2 - a_2| = r$. Break into the four sign-regions for the two absolute values. For instance, in the region $x_1 \geq a_1, x_2 \geq a_2$ the equality becomes

$$(x_1 - a_1) + (x_2 - a_2) = r \quad \Longleftrightarrow \quad x_2 = -x_1 + (a_1 + a_2 + r),$$

a straight line of slope -1 . Doing the same in the other three regions yields four line segments:

$$\begin{aligned} x_2 &= -x_1 + (a_1 + a_2 + r) && (\text{for } x_1 \geq a_1, x_2 \geq a_2), \\ x_2 &= x_1 + (a_2 - a_1 - r) && (\text{for } x_1 \leq a_1, x_2 \geq a_2), \\ x_2 &= -x_1 + (a_1 + a_2 - r) && (\text{for } x_1 \leq a_1, x_2 \leq a_2), \\ x_2 &= x_1 + (a_2 - a_1 + r) && (\text{for } x_1 \geq a_1, x_2 \leq a_2). \end{aligned}$$

These four line segments join at the four points

$$(a_1 \pm r, a_2) \quad \text{and} \quad (a_1, a_2 \pm r),$$

forming a convex quadrilateral whose sides have slopes ± 1 . That quadrilateral is a square (all four sides have equal length $r\sqrt{2}$) with its diagonals lying on the coordinate directions (the diagonals connect $(a_1 - r, a_2)$ to $(a_1 + r, a_2)$ and $(a_1, a_2 - r)$ to $(a_1, a_2 + r)$). The interior of this quadrilateral is exactly $\{(x_1, x_2) : |x_1 - a_1| + |x_2 - a_2| < r\}$, so $B_{d_2}(a; r)$ is that diamond-shaped square (i.e. a square rotated 45° relative to the axes) centered at a . \square

Homework 5

Theorem 40. Let $x, y \in \mathbb{R}^n$ and write $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$. Define

$$\begin{aligned} d_1(x, y) &:= \max_{1 \leq i \leq n} |x_i - y_i|, \\ \|x - y\| &:= \left(\sum_{i=1}^n |x_i - y_i|^2 \right)^{1/2}, \\ d_2(x, y) &:= \sum_{i=1}^n |x_i - y_i|. \end{aligned}$$

Then for every $x, y \in \mathbb{R}^n$ the following inequalities hold:

$$d_1(x, y) \leq \|x - y\| \leq d_2(x, y) \text{ and } d_2(x, y) \leq \sqrt{n} \|x - y\| \leq n d_1(x, y).$$

Problem 1

Proof. Put $a_i := |x_i - y_i| \geq 0$ for $i = 1, \dots, n$. Then

$$d_1(x, y) = \max_{1 \leq i \leq n} a_i, \quad \|x - y\| = \left(\sum_{i=1}^n a_i^2 \right)^{1/2}, \quad d_2(x, y) = \sum_{i=1}^n a_i.$$

(1) $d_1(x, y) \leq \|x - y\|$:

Let k be an index with $a_k = \max_i a_i = d_1(x, y)$. Then

$$\|x - y\| = \left(\sum_{i=1}^n a_i^2 \right)^{1/2} \geq (a_k^2)^{1/2} = a_k = d_1(x, y),$$

so $d_1(x, y) \leq \|x - y\|$.

(2) $\|x - y\| \leq d_2(x, y)$:

Each $a_i^2 \leq a_i \cdot d_2(x, y)$ (since $a_i \leq d_2(x, y)$), hence

$$\sum_{i=1}^n a_i^2 \leq \sum_{i=1}^n a_i d_2(x, y) = d_2(x, y)^2,$$

and taking square roots gives $\|x - y\| \leq d_2(x, y)$.

$$(3) \quad d_2(x, y) \leq \sqrt{n} \|x - y\|:$$

By Theorem 12,

$$d_2(x, y) = \sum_{i=1}^n a_i \cdot 1 \leq \left(\sum_{i=1}^n a_i^2 \right)^{1/2} \left(\sum_{i=1}^n 1^2 \right)^{1/2} = \|x - y\| \sqrt{n}.$$

$$(4) \quad \sqrt{n} \|x - y\| \leq n d_1(x, y):$$

Since $\sum_{i=1}^n a_i^2 \leq \sum_{i=1}^n d_1(x, y)^2 = n d_1(x, y)^2$, taking square roots,

$$\|x - y\| \leq \sqrt{n} d_1(x, y).$$

Multiplying both sides by \sqrt{n} gives

$$\sqrt{n} \|x - y\| \leq n d_1(x, y).$$

Combining (1) and (2), we get $d_1(x, y) \leq \|x - y\| \leq d_2(x, y)$, and combining (3) and (4), we get $d_2(x, y) \leq \sqrt{n} \|x - y\| \leq n d_1(x, y)$. \square

Theorem 41. Let (M, d) be a metric space and let $S, T \subseteq M$ with $S \subseteq T$. Then

$$(a) \quad \overline{S} \subseteq \overline{T}.$$

$$(b) \quad \text{int}(S) \subseteq \text{int}(T).$$

Problem 2

Proof. **(a)** Let $x \in \overline{S}$. By definition of closure, every open ball $B(x, r)$ (with $r > 0$) meets S . Since $S \subseteq T$, the same ball meets T . Hence every open ball about x meets T , so $x \in \overline{T}$. Thus $\overline{S} \subseteq \overline{T}$.

(b) Let $x \in \text{int}(S)$. Then by definition there exists $r > 0$ such that the open ball $B(x, r) \subseteq S$. Using $S \subseteq T$ we get

$$B(x, r) \subseteq S \subseteq T,$$

so $B(x, r) \subseteq T$. Therefore $x \in \text{int}(T)$. As x was arbitrary, $\text{int}(S) \subseteq \text{int}(T)$. \square

Theorem 42. Let (S, d) be a metric space, and let $A, B, C \subseteq S$ be such that A is dense in B and B is dense in C . Then A is dense in C .

Problem 3

Proof. Recall that for any subset $X \subseteq S$, its closure \overline{X} is the set of all points $p \in S$ such that every open ball around p intersects X . A subset X is dense in $Y \subseteq S$ if $\overline{X} \supseteq Y$.

We are given:

$$\overline{A} \supseteq B \quad \text{and} \quad \overline{B} \supseteq C.$$

Since \overline{A} is closed and $B \subseteq \overline{A}$, it follows that

$$\overline{B} \subseteq \overline{\overline{A}} = \overline{A}.$$

Combining with $\overline{B} \supseteq C$, we obtain

$$\overline{A} \supseteq \overline{B} \supseteq C.$$

Thus $\overline{A} \supseteq C$, which means that A is dense in C . □

Theorem 43. There exists a metric space (M, d) and subsets $A, B \subseteq M$ such that:

- (a) $\text{int}(\partial A) = M$;
- (b) $\text{int}(A) = \text{int}(B) = \emptyset$ but $\text{int}(A \cup B) = M$.

Problem 4

Proof. Let $M = \mathbb{R}$ with the usual Euclidean metric $d(x, y) = |x - y|$.

(a) Let $A = \mathbb{Q}$, the set of all rational numbers.

Every nonempty open interval in \mathbb{R} contains both rational and irrational numbers, hence

$$\overline{\mathbb{Q}} = \mathbb{R} \quad \text{and} \quad \overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}.$$

Therefore the boundary of A is

$$\partial A = \overline{A} \cap \overline{M \setminus A} = \mathbb{R} \cap \mathbb{R} = \mathbb{R}.$$

Thus,

$$\text{int}(\partial A) = \text{int}(\mathbb{R}) = \mathbb{R} = M.$$

(b) Again let $M = \mathbb{R}$ and define

$$A = \mathbb{Q}, \quad B = \mathbb{R} \setminus \mathbb{Q}.$$

Each of A and B has empty interior, since any open interval in \mathbb{R} contains both rationals and irrationals:

$$\text{int}(A) = \text{int}(B) = \emptyset.$$

However,

$$A \cup B = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R} = M,$$

so

$$\text{int}(A \cup B) = \text{int}(\mathbb{R}) = \mathbb{R} = M. \quad \square$$

Theorem 44. *If $0 \leq r < 1$, then $\lim_{n \rightarrow \infty} r^n = 0$.*

Proof. Write $r = \frac{1}{1 + \delta}$ with $\delta > 0$. By the binomial theorem,

$$(1 + \delta)^n = \sum_{k=0}^n \binom{n}{k} \delta^k = 1 + n\delta + \sum_{k=2}^n \binom{n}{k} \delta^k.$$

All terms in the last sum are nonnegative, so

$$(1 + \delta)^n \geq 1 + n\delta.$$

Taking reciprocals (all quantities positive) yields

$$r^n = \frac{1}{(1 + \delta)^n} \leq \frac{1}{1 + n\delta}.$$

Now let $\varepsilon > 0$ be given. Choose an integer

$$N > \frac{1}{\delta} \left(\frac{1}{\varepsilon} - 1 \right) \quad (\text{for instance } N = \left\lceil \frac{1/\varepsilon - 1}{\delta} \right\rceil).$$

Then for every $n \geq N$,

$$r^n \leq \frac{1}{1 + n\delta} \leq \frac{1}{1 + N\delta} < \varepsilon.$$

Thus by the ε -definition, $r^n \rightarrow 0$ as $n \rightarrow \infty$. \square

Theorem 45. For any fixed real number x , we have

$$\frac{x^n}{n!} \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Problem 5 (a)

Proof. We shall use the ε -definition of limit.

Let $\varepsilon > 0$. Set $M = \lceil |x| \rceil$ (an integer satisfying $M \geq |x|$). Then $M + 1 > |x|$, so the number

$$r := \frac{|x|}{M + 1}$$

satisfies $0 \leq r < 1$.

For every $n \geq M$, we can write

$$n! = M!(M + 1)(M + 2) \cdots n.$$

Hence

$$\frac{|x|^n}{n!} = \frac{|x|^M}{M!} \cdot \frac{|x|^{n-M}}{(M + 1)(M + 2) \cdots n} \leq \frac{|x|^M}{M!} \cdot \frac{|x|^{n-M}}{(M + 1)^{n-M}} = C r^{n-M},$$

where $C := \frac{|x|^M}{M!}$ is a fixed positive constant.

Since $0 \leq r < 1$, the geometric sequence $C r^{n-M}$ tends to 0 as $n \rightarrow \infty$; see [Theorem 44](#). Choose $N \geq M$ such that

$$C r^{N-M} < \varepsilon.$$

Then for every $n \geq N$,

$$\left| \frac{x^n}{n!} \right| \leq C r^{n-M} \leq C r^{N-M} < \varepsilon.$$

Thus, for every $\varepsilon > 0$, there exists N such that for all $n \geq N$, $\left| \frac{x^n}{n!} \right| < \varepsilon$.

Therefore,

$$\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0.$$

□

Theorem 46. If (x_n) is a sequence of nonnegative real numbers such that $x_n \rightarrow a$, then $\sqrt{x_n} \rightarrow \sqrt{a}$.

Problem 5 (b)

Proof. First, assume $a > 0$. Let $\varepsilon > 0$. Since $x_n \rightarrow a$, there exists N such that $|x_n - a| < \varepsilon\sqrt{a}$ for all $n \geq N$. Thus, for $n \geq N$,

$$|\sqrt{x_n} - \sqrt{a}| = \frac{|x_n - a|}{\sqrt{x_n} + \sqrt{a}} \leq \frac{|x_n - a|}{\sqrt{a}} < \varepsilon.$$

Therefore, $\sqrt{x_n} \rightarrow \sqrt{a}$ if $a > 0$.

Now, assume $a = 0$. Let $\varepsilon > 0$. Since $x_n \rightarrow 0$, there exists N such that $|x_n - 0| < \varepsilon^2$ for all $n \geq N$. Since $x_n \geq 0$, we get $|\sqrt{x_n} - \sqrt{0}| < \varepsilon$ for $n \geq N$. Therefore, $\sqrt{x_n} \rightarrow \sqrt{a}$ if $a = 0$. \square

Theorem 47. Let (S, d) be a metric space. If $x_n \rightarrow x$ and $y_n \rightarrow y$ in S , then

$$d(x_n, y_n) \rightarrow d(x, y).$$

Problem 6

Proof. Fix $\varepsilon > 0$. Since $x_n \rightarrow x$ and $y_n \rightarrow y$, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$d(x_n, x) < \frac{\varepsilon}{2} \quad \text{for all } n \geq N_1, \quad d(y_n, y) < \frac{\varepsilon}{2} \quad \text{for all } n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$.

Using the triangle inequality twice, we estimate:

$$|d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y).$$

Indeed,

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x) + d(x, y_n) \\ &\leq d(x_n, x) + d(x, y) + d(y, y_n), \end{aligned}$$

and also

$$\begin{aligned} d(x, y) &\leq d(x, y_n) + d(y_n, y) \\ &\leq d(x_n, y_n) + d(x_n, x) + d(y_n, y), \end{aligned}$$

which together yield the desired inequality.

Therefore, for all $n \geq N$,

$$|d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$d(x_n, y_n) \longrightarrow d(x, y).$$

□

Theorem 48. *Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function such that $f(x) = 0$ whenever x is rational. Then $f(x) = 0$ for every $x \in [a, b]$.*

Problem 7

Proof. Since the set of rational numbers \mathbb{Q} is dense in \mathbb{R} , for any $x \in [a, b]$ there exists a sequence (r_n) of rational numbers such that $r_n \rightarrow x$ as $n \rightarrow \infty$.

By continuity of f at x , we have

$$\lim_{n \rightarrow \infty} f(r_n) = f(x).$$

But each r_n is rational, and hence $f(r_n) = 0$ for all n . Therefore,

$$f(x) = \lim_{n \rightarrow \infty} f(r_n) = \lim_{n \rightarrow \infty} 0 = 0.$$

Since $x \in [a, b]$ was arbitrary, it follows that $f(x) = 0$ for all $x \in [a, b]$. □

Theorem 49. *Define functions $f, g: [0, 1] \rightarrow \mathbb{R}$ by*

$$f(x) = \begin{cases} 0 & \text{if } x \text{ irrational,} \\ 1 & \text{if } x \text{ rational,} \end{cases} \quad g(x) = \begin{cases} 0 & \text{if } x \text{ irrational,} \\ x & \text{if } x \text{ rational.} \end{cases}$$

Then f is discontinuous at every point of $[0, 1]$, and g is continuous exactly at $x = 0$ and discontinuous at every $x \in (0, 1]$.

Problem 8

Proof. We use the sequential characterization: a function h is continuous at x_0 iff for every sequence (x_n) with $x_n \rightarrow x_0$ we have $h(x_n) \rightarrow h(x_0)$.

We first show that f is not continuous at any $x_0 \in [0, 1]$. Fix $x_0 \in [0, 1]$. Because rationals and irrationals are both dense in \mathbb{R} , there exist sequences (q_n) of rationals and (r_n) of irrationals with $q_n \rightarrow x_0$ and $r_n \rightarrow x_0$. Then

$$f(q_n) = 1 \text{ for all } n \implies f(q_n) \rightarrow 1,$$

while

$$f(r_n) = 0 \text{ for all } n \implies f(r_n) \rightarrow 0.$$

Since these two possible sequence-limits differ, there is no single value $L = f(x_0)$ to which $f(x_n)$ must converge for every sequence $x_n \rightarrow x_0$. Thus, by the sequential criterion, f is not continuous at x_0 . As x_0 was arbitrary, f is discontinuous everywhere.

Now, we show that g is continuous at 0. Let $\varepsilon > 0$. Define $\delta := \varepsilon$. If $x \in (0 - \delta, 0 + \delta)$, then either $g(x) = x$ or $g(x) = 0$. Thus,

$$|g(x) - g(0)| = |g(x)| < \delta = \varepsilon \quad \text{if } x \in (0 - \delta, 0 + \delta).$$

Therefore, g is continuous at 0.

Finally, we show that g is discontinuous at every $x_0 \in (0, 1]$. Fix $x_0 \in (0, 1]$. Again use density to choose a rational sequence (q_n) and an irrational sequence (r_n) with $q_n \rightarrow x_0$ and $r_n \rightarrow x_0$. Then

$$g(q_n) = q_n \rightarrow x_0, \quad g(r_n) = 0 \text{ for all } n \implies g(r_n) \rightarrow 0.$$

Because $x_0 > 0$ the two limits x_0 and 0 are different, so there exists sequences approaching x_0 whose g -images have different limits. By the sequential criterion g is not continuous at x_0 .

Therefore, g is continuous only at 0. □

Midterm

Theorem 50. *There is no continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $f(\mathbb{R}) = \mathbb{Q}$.*

Problem 1 (a)

Proof. Suppose $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous with $f(\mathbb{R}) = \mathbb{Q}$. Then f is not constant, so pick x_1, x_2 with $a := f(x_1) < b := f(x_2)$ (both rational). Choose any irrational $s \in (a, b)$ (every nonempty open interval contains irrationals), for example, we may take $s := a + \frac{b-a}{\sqrt{2}}$. By the Intermediate Value Theorem there exists $c \in (x_1, x_2)$ with $f(c) = s$, contradicting $f(\mathbb{R}) = \mathbb{Q}$. Thus no such continuous f exists. \square

Theorem 51. *There is a continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $f((0, 1)) = (0, 1]$.*

Problem 1 (b)

Proof. Consider $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) := \begin{cases} 0, & x \leq 0, \\ 2x, & 0 < x \leq \frac{1}{2}, \\ 2(1-x), & \frac{1}{2} < x < 1, \\ 0, & x \geq 1. \end{cases}$$

Then f is continuous and $f((0, 1)) = (0, 1]$. \square

Theorem 52. *Consider the function $d: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ defined by*

$$d(x, y) = |2x - y|.$$

Then d is not a metric on \mathbb{R} .

Problem 1 (c)

Proof. To be a metric, d must satisfy the following for all $x, y, z \in \mathbb{R}$:

(a) $d(x, y) \geq 0$

(b) $d(x, y) = 0$ if and only if $x = y$.

(c) $d(x, y) = d(y, x)$.

(d) $d(x, y) \leq d(x, z) + d(z, y)$

Notice that (a) holds. However, (b) does not hold in general; for instance, $d(1, 2) = |2 - 2| = 0$, but $1 \neq 2$. Similarly, (c) also does not hold: $d(1, 2) = |2 - 2| = 0$, but $d(2, 1) = |4 - 1| = 3 \neq 0$. Hence, d is not a metric on \mathbb{R} . \square

Theorem 53. *The set $\mathbb{Z} \subset \mathbb{R}$ has no accumulation point.*

Problem 1 (d)

Proof. Let $x \in \mathbb{R}$. We consider two cases:

Case 1: $x \in \mathbb{Z}$.

Take $\varepsilon = \frac{1}{4}$. Then the interval $(x - \varepsilon, x + \varepsilon)$ contains no integer other than x itself. By the definition of an accumulation point, x would need to have an integer in every interval around it different from x . Since $(x - \varepsilon, x + \varepsilon)$ contains no such point, x is not an accumulation point.

Case 2: $x \notin \mathbb{Z}$.

Let $n = [x]$ be the greatest integer less than x , and let $d := \min\{x - n, (n + 1) - x\} > 0$ be the distance from x to the nearest integer. Take $\varepsilon = \frac{d}{2}$. Then the interval $(x - \varepsilon, x + \varepsilon)$ contains no integers at all. Hence, by the definition, x is not an accumulation point.

Since $x \in \mathbb{R}$ was arbitrary, no point of \mathbb{R} is an accumulation point of \mathbb{Z} . Therefore, \mathbb{Z} has no accumulation points. \square

Theorem 54. *Let $S \subset \mathbb{R}$ be nonempty with $b = \sup S$. Then for every $\varepsilon > 0$ there exists $x \in S$ satisfying $x \leq b < x + \varepsilon$.*

Problem 1 (e)

Proof. Let $\varepsilon > 0$ be given. By the definition of supremum, b is the least upper bound of S , so $b - \varepsilon < b$ is not an upper bound of S . Hence, there exists $x \in S$ such that $b - \varepsilon < x \leq b$. Adding ε to the left inequality, we get $x \leq b < x + \varepsilon$.

This proves that for every $\varepsilon > 0$, there exists $x \in S$ satisfying $x \leq b < x + \varepsilon$. \square

Theorem 55. Let $\mathcal{F} := \{I_\alpha : \alpha \in A\}$ be a family of *non-empty* open intervals in \mathbb{R} which are pairwise disjoint, i.e., $I_\alpha \cap I_\beta = \emptyset$ whenever $\alpha \neq \beta$. Then A is a countable set.

Problem 1 (f)

Proof. Since each $I_\alpha = (a_\alpha, b_\alpha)$ is nonempty and open, by the density of rationals in \mathbb{R} , there exists a rational number $q_\alpha \in I_\alpha$.

Because the intervals are pairwise disjoint, $q_\alpha \neq q_\beta$ whenever $\alpha \neq \beta$. Thus the map

$$\alpha \mapsto q_\alpha$$

is injective from A into \mathbb{Q} .

Since \mathbb{Q} is countable, it follows that A is at most countable. \square

Theorem 56. Let $S \subset \mathbb{R}^n$ be open and $x_0 \in \mathbb{R}^n$ be fixed. Define

$$T := \{x_0 + y : y \in S\}.$$

Then T is an open set.

Problem 2 (a)

Proof. Take any $t \in T$. Then there exists $y \in S$ such that $t = x_0 + y$. Since S is open, there exists $\varepsilon > 0$ such that

$$B(y, \varepsilon) := \{w \in \mathbb{R}^n : \|w - y\| < \varepsilon\} \subset S.$$

Now consider

$$B(t, \varepsilon) := \{z \in \mathbb{R}^n : \|z - t\| < \varepsilon\}.$$

For any $z \in B(t, \varepsilon)$, let $w := z - x_0$. Then

$$\|w - y\| = \|(z - x_0) - y\| = \|z - t\| < \varepsilon,$$

so $w \in B(y, \varepsilon) \subset S$. Hence $z = x_0 + w \in T$.

This shows $B(t, \varepsilon) \subset T$. Since $t \in T$ was arbitrary, T is open. \square

Theorem 57. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x^2$. Then f is not uniformly continuous on \mathbb{R} .

Problem 2 (b)

Proof. Suppose, for contradiction, that f is uniformly continuous on \mathbb{R} . Consider any $\varepsilon > 0$. Then, there exists $\delta > 0$ such that for all $x, y \in \mathbb{R}$,

$$|x - y| < \delta \implies |x^2 - y^2| < \varepsilon.$$

Let N be a positive integer. Now take $x = \delta N$ and $y = x + \frac{\delta}{2}$. Then $|x - y| = \frac{\delta}{2} < \delta$ and

$$|x^2 - y^2| = |x - y| |x + y| = \frac{\delta}{2} \left(2\delta N + \frac{\delta}{2} \right) = \delta^2 N + \frac{\delta^2}{4}.$$

By taking N large enough, for instance,

$$N := \left\lceil \frac{|\varepsilon - \frac{\delta^2}{4}|}{\delta^2} \right\rceil + 1,$$

we can make $|x^2 - y^2| > \varepsilon$, contradicting the uniform continuity condition.

Hence, $f(x) = x^2$ is not uniformly continuous on \mathbb{R} . □

Theorem 58 (Heine–Borel). A subset $K \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded. Equivalently, every open cover of K has a finite sub-cover.

Problem 3 (a)

Remark 59. Let $X = \mathbb{R}$ with the discrete metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

Consider $S = [-1, 1] \subset X$.

Then S is bounded, since $d(x, y) \leq 1$ for all $x, y \in S$, and S is closed (all subsets of a discrete metric space are closed).

However, S is not compact. Consider the open cover

$$\{\{x\} : x \in [-1, 1]\}.$$

No finite subcollection covers S , so S is not compact.

Hence, in this metric space, a set can be closed and bounded but not compact. Therefore, the Heine–Borel theorem does not hold in general metric spaces.

Theorem 60. *Let $a \in \mathbb{R}^n$ and $r > 0$. Then $\overline{B}(a; r) := \{x \in \mathbb{R}^n : \|x - a\| \leq r\}$ is a closed set.*

Problem 3 (b)

Proof. Consider the complement

$$\mathbb{R}^n \setminus \overline{B}(a; r) = \{x \in \mathbb{R}^n : \|x - a\| > r\}.$$

Take any $x \in \mathbb{R}^n \setminus \overline{B}(a; r)$. Then $\|x - a\| > r$, and let

$$\varepsilon := \|x - a\| - r > 0.$$

For any $y \in \mathbb{R}^n$ with $\|y - x\| < \varepsilon$, the triangle inequality gives

$$\|y - a\| \geq \|x - a\| - \|y - x\| > \|x - a\| - \varepsilon = r.$$

Hence $y \in \mathbb{R}^n \setminus \overline{B}(a; r)$, showing that the complement is open.

Since the complement of $\overline{B}(a; r)$ is open, $\overline{B}(a; r)$ is closed. □

Theorem 61. *Let S be a bounded subset of \mathbb{R}^n . Let $\varepsilon > 0$. Then S can be covered by a finite number of balls of radius ε .*

Problem 3 (c)

Proof. Let $S \subset \mathbb{R}^n$ be bounded. Then there exists $a \in \mathbb{R}^n$ and $r > 0$ such that

$$S \subset \overline{B}(a, r) := \{x \in \mathbb{R}^n : \|x - a\| \leq r\}.$$

The closure \overline{S} of S satisfies

$$\overline{S} \subseteq \overline{B}(a, r),$$

so \overline{S} is bounded. By definition, \overline{S} is also closed.

By the Heine–Borel theorem, a set in \mathbb{R}^n is compact if and only if it is closed and bounded. Hence \overline{S} is compact.

Let $\varepsilon > 0$. Consider the open cover

$$\{B(x, \varepsilon) : x \in \overline{S}\}.$$

By compactness, there exists a finite subcollection of balls that covers \overline{S} . These balls also cover $S \subset \overline{S}$.

Therefore, S can be covered by finitely many balls of radius ε . \square

Theorem 62. *Let $S \subset \mathbb{R}^n$ be bounded. Then for every $\varepsilon > 0$ there exist finitely many points $x_1, x_2, \dots, x_m \in S$ such that*

$$S \subset \bigcup_{i=1}^m B(x_i, \varepsilon).$$

In other words, every bounded subset of \mathbb{R}^n is totally bounded, and the covering balls of fixed radius ε may be chosen with centers in S .

Proof. Suppose, for contradiction, that $S \subset \mathbb{R}^n$ is bounded but not totally bounded. Then there exists some $\varepsilon > 0$ such that no finite collection of ε -balls centered at points of S covers S .

Pick any $x_1 \in S$. Since $\{B(x_1, \varepsilon)\}$ does not cover S , we may choose $x_2 \in S \setminus B(x_1, \varepsilon)$. Inductively, having chosen $x_1, \dots, x_k \in S$, the finite union $\bigcup_{i=1}^k B(x_i, \varepsilon)$ does not cover S , so we may pick

$$x_{k+1} \in S \setminus \bigcup_{i=1}^k B(x_i, \varepsilon).$$

This produces an infinite sequence $(x_m)_{m \geq 1} \subset S$ with the property that

$$\|x_i - x_j\| > \varepsilon \quad \text{for all } i \neq j.$$

Since S is bounded, the sequence (x_m) is bounded. By the Bolzano–Weierstrass theorem, there exists a subsequence (x_{m_k}) converging to some limit $x \in \mathbb{R}^n$. Choose K such that for all $k \geq K$,

$$\|x_{m_k} - x\| < \frac{\varepsilon}{2}.$$

Then for $k, \ell \geq K$ we have

$$\|x_{m_k} - x_{m_\ell}\| \leq \|x_{m_k} - x\| + \|x_{m_\ell} - x\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

contradicting the fact that all pairwise distances exceed ε . Therefore, our assumption was false, and S must be totally bounded. \square

Definition 63. Let (M, d) be a metric space. A sequence $\{x_n\}_{n \geq 1}$ in M is said to *converge* to a point $p \in M$ if for every $\varepsilon > 0$, there exists an integer $N \geq 1$ such that

$$d(x_n, p) < \varepsilon \quad \text{for all } n \geq N.$$

In symbols, we write

$$x_n \rightarrow p \quad \text{as } n \rightarrow \infty.$$

Problem 4 (a)

Theorem 64. Let $x \in \mathbb{R}$. Let $\{x_n\}_{n \geq 1}$ be a sequence of real numbers such that $x_n \rightarrow x$. Consider the sequence of arithmetic means $\{s_n\}_{n \geq 1}$, defined by

$$s_n := \frac{1}{n} \sum_{k=1}^n x_k.$$

Then $\{s_n\}_{n \geq 1}$ also converges to x .

Problem 4 (b)

Proof. Fix $\varepsilon > 0$. Since $x_n \rightarrow x$, there exists a positive integer n_0 such that

$$|x_n - x| < \frac{\varepsilon}{2} \quad \text{for all } n \geq n_0.$$

Then for $n > n_0$, we can write

$$s_n - x = \frac{1}{n} \sum_{k=1}^n (x_k - x) = \frac{1}{n} \sum_{k=1}^{n_0-1} (x_k - x) + \frac{1}{n} \sum_{k=n_0}^n (x_k - x).$$

For the first sum,

$$\left| \frac{1}{n} \sum_{k=1}^{n_0-1} (x_k - x) \right| \leq \frac{1}{n} \sum_{k=1}^{n_0-1} |x_k - x| = \frac{C}{n},$$

where $C := \sum_{k=1}^{n_0-1} |x_k - x|$. Note that C does not depend on n .

For the second sum,

$$\left| \frac{1}{n} \sum_{k=n_0}^n (x_k - x) \right| \leq \frac{1}{n} \sum_{k=n_0}^n |x_k - x| \leq \frac{n - n_0 + 1}{n} \cdot \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2}.$$

Hence,

$$|s_n - x| \leq \frac{C}{n} + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{2} < \varepsilon \quad \text{as } n \geq \max \left\{ n_0, \frac{4C}{\varepsilon} \right\}.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $s_n \rightarrow x$. □

Consider the metric on \mathbb{R}^n given by

$$d(x, y) := \max_{1 \leq i \leq n} |x_i - y_i|,$$

and let

$$\|x\| := \sqrt{x_1^2 + \cdots + x_n^2}$$

denote the Euclidean norm on \mathbb{R}^n , where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are two points in \mathbb{R}^n .

Write $B_d(a; r)$ for an open ball in the metric space (\mathbb{R}^n, d) , i.e.,

$$B_d(a; r) := \{x \in \mathbb{R}^n : d(a, x) < r\},$$

and write $B(a; r)$ for an open ball in \mathbb{R}^n with the Euclidean norm, i.e.,

$$B(a; r) := \{x \in \mathbb{R}^n : \|x - a\| < r\}.$$

Theorem 65. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be two points in \mathbb{R}^n . Then

$$d(x, y) \leq \|x - y\| \leq \sqrt{n} d(x, y).$$

Problem 5 (a)

Proof. There exists $k \in \{1, \dots, n\}$ such that $d(x, y) = |x_k - y_k|$. Then

$$\|x - y\|^2 = \sum_{i=1}^n |x_i - y_i|^2 \geq |x_k - y_k|^2 = d(x, y)^2.$$

Furthermore,

$$\|x - y\|^2 = \sum_{i=1}^n |x_i - y_i|^2 \leq \sum_{i=1}^n d(x, y)^2 = n d(x, y)^2.$$

Taking square roots gives the desired inequalities. □

Theorem 66. *Let $a \in \mathbb{R}^n$ and $r > 0$. Then*

$$B_d(a; r) \subset B(a; \sqrt{n} r) \quad \text{and} \quad B(a; r) \subset B_d(a; r).$$

Problem 5 (b)

Proof. If $x \in B_d(a; r)$, then $d(a, x) < r$. By [Theorem 65](#),

$$\|x - a\| \leq \sqrt{n} d(a, x) < \sqrt{n} r,$$

so $x \in B(a; \sqrt{n} r)$.

If $x \in B(a; r)$, then $\|x - a\| < r$. By [Theorem 65](#),

$$d(a, x) \leq \|x - a\| < r,$$

so $x \in B_d(a; r)$. □

Theorem 67. *Let $S \subset \mathbb{R}^n$. Then S is open in \mathbb{R}^n with respect to the Euclidean norm if and only if S is open in the metric space (\mathbb{R}^n, d) .*

Problem 5 (c)

Proof. Suppose S is open in the Euclidean norm. For any $x \in S$, there exists $r > 0$ such that $B(x; r) \subset S$. By [Theorem 66](#), $B_d(x; r) \subset B(x; r) \subset S$. Hence S is open in d .

Conversely, suppose S is open in d . For $x \in S$, there exists $r > 0$ such that $B_d(x; r) \subset S$. By [Theorem 66](#), $B(x; r) \subset B_d(x; r) \subset S$. Hence S is open in the Euclidean norm. □

Homework 6

Theorem 68. Let S be a non-empty closed subset of \mathbb{R} , and let $f: S \rightarrow \mathbb{R}$ be continuous. Define

$$A := \{x \in S : f(x) = 0\}.$$

Then A is a closed subset of \mathbb{R} .

Problem 1

Proof. Consider the complement

$$\mathbb{R} \setminus A = (\mathbb{R} \setminus S) \cup \{x \in S : f(x) \neq 0\}.$$

Since S is closed, $\mathbb{R} \setminus S$ is open. Let

$$B := \{x \in S : f(x) \neq 0\}.$$

Take any $x \in B$. Since f is continuous at x and $f(x) \neq 0$, there exists $\varepsilon > 0$ such that

$$|f(y) - f(x)| < |f(x)| \quad \text{for all } y \in S \text{ with } |y - x| < \varepsilon.$$

Then

$$|f(y)| \geq |f(x)| - |f(y) - f(x)| > 0,$$

so $y \in B$. Therefore, B is open in \mathbb{R} .

Hence,

$$\mathbb{R} \setminus A = (\mathbb{R} \setminus S) \cup B$$

is a union of open sets, and thus open. Therefore, A is closed in \mathbb{R} . \square

Theorem 69. Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous and suppose $x_1, x_2 \in [a, b]$ with $x_1 < x_2$ are local maxima of f . Then there exists $c \in (x_1, x_2)$ such that $f(c)$ is a local minimum.

Problem 2

Proof. Consider the interval $[x_1, x_2]$. By the Extreme Value Theorem, f attains a minimum on $[x_1, x_2]$, say

$$f(c) = \inf_{x \in [x_1, x_2]} f(x)$$

for some $c \in [x_1, x_2]$.

Since x_1 and x_2 are local maxima, this minimum cannot occur at the endpoints x_1 or x_2 . Hence $c \in (x_1, x_2)$.

By definition of the minimum on $[x_1, x_2]$, there exists $\delta > 0$ such that

$$f(c) \leq f(x) \quad \text{for all } x \in (c - \delta, c + \delta) \subset (x_1, x_2),$$

so f has a local minimum at c . □

Theorem 70. *There is a continuous function from $(0, 1)$ onto $(0, 1]$.*

Problem 3 (a)

Proof. Consider $f: (0, 1) \rightarrow (0, 1]$ defined by

$$f(x) := \begin{cases} 2x, & 0 < x \leq \frac{1}{2}, \\ 2(1 - x), & \frac{1}{2} < x < 1. \end{cases}$$

Then f is continuous and $f((0, 1)) = (0, 1]$. □

Theorem 71. *There is no continuous function from $(0, 1)$ onto $(0, 1) \cup (1, 2)$.*

Problem 3 (b)

Proof. The domain $(0, 1)$ is connected, but the range is disconnected. The continuous image of a connected set must be connected. □

Theorem 72. *There is no continuous function from \mathbb{R} onto \mathbb{Q} .*

Problem 3 (c)

Proof. See [Theorem 50](#). □

Theorem 73. *There is no continuous function from $[0, 1] \times [0, 1]$ onto \mathbb{R}^2 .*

Problem 3 (d)

Proof. The domain $[0, 1]^2$ is compact, and the continuous image of a compact set is compact, but \mathbb{R}^2 is not compact. \square

Theorem 74. *There is a continuous function from $(0, 1) \times (0, 1)$ onto \mathbb{R}^2 .*

Problem 3 (e)

Proof. Define

$$f: (0, 1) \rightarrow \mathbb{R}, \quad f(x) := \tan(\pi(x - 1/2)).$$

- f is continuous on $(0, 1)$ because \tan is continuous on $(-\pi/2, \pi/2)$.
- $\lim_{x \rightarrow 0^+} f(x) = -\infty$, $\lim_{x \rightarrow 1^-} f(x) = +\infty$.
- Therefore, $f((0, 1)) = \mathbb{R}$, i.e., f is surjective.

Similarly, for a continuous surjection $g: (0, 1)^2 \rightarrow \mathbb{R}^2$, define

$$g(x, y) := (\tan(\pi(x - 1/2)), \tan(\pi(y - 1/2))).$$

Then g is continuous and $g((0, 1)^2) = \mathbb{R}^2$. \square

Theorem 75. *Let $f: (S, d_S) \rightarrow (T, d_T)$ be a function between metric spaces. Then*

$$f \text{ is continuous on } S \iff f(\overline{A}) \subseteq \overline{f(A)} \text{ for all } A \subseteq S.$$

Problem 4

Proof. (\Rightarrow) Suppose f is continuous and let $x \in \overline{A}$. Then there exists a sequence $(x_n) \subset A$ with $x_n \rightarrow x$. By continuity, $f(x_n) \rightarrow f(x)$. Since each $f(x_n) \in f(A)$ and $\overline{f(A)}$ is closed, it follows that $f(x) \in \overline{f(A)}$. Hence $f(\overline{A}) \subseteq \overline{f(A)}$.

(\Leftarrow) Suppose $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subseteq S$. Assume, for contradiction, that f is not continuous at some $x_0 \in S$. Then there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$ there exists $x \in S$ with $d_S(x, x_0) < \delta$ but $d_T(f(x), f(x_0)) \geq \varepsilon_0$.

Construct a sequence $(x_n) \subset S$ such that $d_S(x_n, x_0) < 1/n$ and $d_T(f(x_n), f(x_0)) \geq \varepsilon_0$. Let $A = \{x_n : n \geq 1\}$. Then $x_0 \in \overline{A}$, so $f(x_0) \in \overline{f(A)} \subset f(\overline{A}) \subset \overline{f(A)}$.

By definition of closure, there exists a subsequence $(f(x_{n_k})) \subset \overline{f(A)}$ such that $f(x_{n_k}) \rightarrow f(x_0)$. This is impossible, because by construction $d_T(f(x_n), f(x_0)) \geq \varepsilon_0$ for all n , so no subsequence can converge to $f(x_0)$.

This contradiction shows that f must be continuous at x_0 . Since x_0 was arbitrary, f is continuous on S . \square

Alternative Proof. (\Rightarrow) Suppose f is continuous. Let $y \in f(\overline{A})$, so $y = f(x)$ with $x \in \overline{A}$. For any open neighborhood V of y in T , $f^{-1}(V)$ is open in S and contains x . Since $x \in \overline{A}$, we have $f^{-1}(V) \cap A \neq \emptyset$, i.e., $V \cap f(A) \neq \emptyset$. Hence $y \in \overline{f(A)}$. Therefore $f(\overline{A}) \subseteq \overline{f(A)}$.

(\Leftarrow) Suppose $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subset S$. Let $U \subset T$ be open. Set $A = S \setminus f^{-1}(U)$. Then $f(A) \subset T \setminus U$, which is closed, so $\overline{f(A)} \subset T \setminus U$. By assumption, $f(\overline{A}) \subseteq \overline{f(A)} \subset T \setminus U$, hence $\overline{A} \subset S \setminus f^{-1}(U)$, so $S \setminus f^{-1}(U)$ is closed. Thus $f^{-1}(U)$ is open. Since U was arbitrary, f is continuous. \square

Theorem 76. Let (S, d) be a metric space. Then S is connected if and only if the only subsets of S which are both open and closed (clopen) are \emptyset and S .

Problem 5

Proof. (\Rightarrow) Suppose S is connected. Assume for contradiction that there exists $A \subset S$ with $A \neq \emptyset$, $A \neq S$, and A both open and closed. Then $S \setminus A$ is also nonempty and open. Thus $S = A \cup (S \setminus A)$ is a union of two nonempty disjoint open sets, which is a separation of S . This contradicts the connectedness of S . Hence, the only clopen sets are \emptyset and S .

(\Leftarrow) Suppose the only clopen subsets of S are \emptyset and S . Assume for contradiction that S is not connected. Then there exists a separation $S = U \cup V$ with U, V nonempty, disjoint, and open. Then U is open and $S \setminus U = V$ is also open, so U is clopen. This is a nonempty proper clopen subset, contradicting the assumption. Hence S must be connected. \square

Theorem 77. *Let S be a connected subset of a metric space (X, d) , and let T satisfy*

$$S \subset T \subset \overline{S}.$$

Then T is connected. In particular, the closure \overline{S} of a connected set is connected.

Problem 6

Proof. Suppose, for contradiction, that T is not connected. Then there exists a separation $T = U \cup V$ where U and V are nonempty, disjoint, and open in the subspace topology of T . Define

$$U_S := U \cap S, \quad V_S := V \cap S.$$

Then U_S and V_S are open in the subspace topology of S , disjoint, and

$$U_S \cup V_S = (U \cup V) \cap S = T \cap S = S.$$

We need to show that U_S and V_S are nonempty. Suppose, for contradiction, that $U_S = \emptyset$. Then $U \subset T \setminus S \subset \overline{S} \setminus S$. But U is open in T , so there exists $u \in U$ and $\varepsilon > 0$ such that $B_\varepsilon(u) \cap T \subset U$. Since $u \in T \subset \overline{S}$, any neighborhood of u intersects S , so $B_\varepsilon(u) \cap T \cap S \neq \emptyset$. This contradicts $U_S = \emptyset$. Similarly, $V_S \neq \emptyset$.

Thus U_S and V_S are nonempty, disjoint, open in S , and cover S . This is a separation of S , contradicting its connectedness. Therefore, T must be connected.

In particular, taking $T = \overline{S}$, we conclude that the closure of a connected set is connected. \square

Theorem 78. *Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x^2$. Then f is not uniformly continuous on \mathbb{R} .*

Problem 7

Proof. See [Theorem 57](#). □

Theorem 79. *Let $f: (S, d_S) \rightarrow (T, d_T)$ be uniformly continuous on S . If $\{x_n\} \subset S$ is a Cauchy sequence, then $\{f(x_n)\} \subset T$ is also a Cauchy sequence.*

Problem 8

Proof. Let $\{x_n\}$ be a Cauchy sequence in S . We need to show that $\{f(x_n)\}$ is a Cauchy sequence in T . Let $\varepsilon > 0$ be given. By uniform continuity of f , there exists $\delta > 0$ such that

$$d_S(x, y) < \delta \implies d_T(f(x), f(y)) < \varepsilon \quad \text{for all } x, y \in S.$$

Since $\{x_n\}$ be a Cauchy sequence in S , there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$d_S(x_m, x_n) < \delta.$$

Then, for all $m, n \geq N$,

$$d_T(f(x_m), f(x_n)) < \varepsilon.$$

Hence $\{f(x_n)\}$ is a Cauchy sequence in T . □

Theorem 80. *The connected subsets of \mathbb{R} are exactly the empty set, singletons, and intervals (open, closed, half-open, or infinite).*

Problem 9

Proof. The empty set \emptyset and singletons $\{x_0\}$ are trivially connected.

Let $I \subset \mathbb{R}$ be an interval. Suppose, for contradiction, that I is not connected. Then there exists a separation $I = U \cup V$, where U and V are nonempty, disjoint, and open in the subspace topology of I . Pick $u \in U$ and $v \in V$ with $u < v$, and define

$$S := \{x \in [u, v] \cap I : [u, x] \subset U\}.$$

Then S is nonempty since $u \in S$. Let $s = \sup S$. If $s \in U$, then by openness of U in I , there exists $\varepsilon > 0$ such that $[s, s + \varepsilon) \cap I \subset U$, contradicting the definition of s as a supremum. If $s \in V$, then $s \in$

$[u, v] \cap I$ but $s \notin U$, also contradicting the definition of s . In both cases we get a contradiction. Therefore, I cannot be separated, and hence I is connected.

Finally, let $S \subset \mathbb{R}$ be any connected subset. If $|S| \leq 1$, then S is either empty or a singleton. Suppose $|S| \geq 2$ and pick $x, y \in S$ with $x < y$. If there exists $z \in (x, y)$ with $z \notin S$, then

$$U := S \cap (-\infty, z), \quad V := S \cap (z, \infty)$$

are nonempty, disjoint, open subsets of S , and $S = U \cup V$, which is a separation of S . This contradicts the connectedness of S . Therefore, S contains all points between any two of its points, and hence S is an interval.

Combining all cases, the connected subsets of \mathbb{R} are exactly the empty set, singletons, and intervals. \square

Homework 7

Theorem 81. Let $f: \mathbb{R} \rightarrow \mathbb{R}$, and suppose that

$$|f(x) - f(y)| \leq (x - y)^2 \quad \text{for all } x, y \in \mathbb{R}.$$

Then f is constant.

Problem 1

Proof. Fix $a, b \in \mathbb{R}$, and for an integer $n \geq 1$ partition the interval from a to b into n equal sub-intervals:

$$x_k = a + k \frac{b - a}{n}, \quad k = 0, 1, \dots, n.$$

By the triangle inequality and the given hypothesis, we have

$$\begin{aligned} |f(b) - f(a)| &= \left| \sum_{k=0}^{n-1} (f(x_{k+1}) - f(x_k)) \right| \\ &\leq \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \\ &\leq \sum_{k=0}^{n-1} (x_{k+1} - x_k)^2 \\ &= n \left(\frac{b - a}{n} \right)^2 \\ &= \frac{(b - a)^2}{n}. \end{aligned}$$

Since this holds for every n , letting $n \rightarrow \infty$ gives

$$|f(b) - f(a)| \leq 0 \quad \implies \quad f(b) = f(a).$$

Thus f is constant on \mathbb{R} . □

Lemma 82. Let $m \in \mathbb{N} \cup \{0\}$. Then $\lim_{x \rightarrow 0} |x|^{-m} e^{-1/x^2} = 0$.

Proof. For $t \geq 0$ the exponential series gives

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \geq \frac{t^{k+1}}{(k+1)!} \quad (k \in \mathbb{N} \cup \{0\}).$$

Hence for $t > 0$

$$\frac{t^k}{e^t} \leq \frac{(k+1)!}{t} \xrightarrow{t \rightarrow \infty} 0,$$

so $\lim_{t \rightarrow \infty} \frac{t^k}{e^t} = 0$.

Now let $m \geq 0$ be an integer and put $t = 1/x^2$ for $x \neq 0$. Then for $t \geq 1$, we have

$$\frac{e^{-1/x^2}}{|x|^m} = t^{m/2} e^{-t} \leq t^{[m/2]} e^{-t} \xrightarrow{t \rightarrow \infty} 0,$$

which shows e^{-1/x^2} tends to 0 faster than any power of $|x|$ as $x \rightarrow 0$. \square

Theorem 83. Define $f: \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then

- (a) f is continuous for all $x \in \mathbb{R}$.
- (b) For every $n \geq 1$, the derivative $f^{(n)}$ exists and is continuous on \mathbb{R} , and $f^{(n)}(0) = 0$.

Problem 2

Proof of (a). If $x \neq 0$, then f is the composition of the continuous functions $\mathbb{R} \setminus \{0\} \ni x \mapsto -1/x^2 \in \mathbb{R} \setminus \{0\}$ and $\mathbb{R} \setminus \{0\} \ni t \mapsto e^t \in \mathbb{R} \setminus \{0\}$, so f is continuous at every nonzero point. It remains to check continuity at 0. By [Lemma 82](#), $\lim_{x \rightarrow 0} e^{-1/x^2} = 0$. Hence f is continuous at 0. Combining this with continuity away from 0 gives continuity on \mathbb{R} . \square

Lemma 84. Let $f(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$ be a polynomial of degree m . Then

$$|f(x)| \leq |x|^m (|a_m| + |a_{m-1}| + \cdots + |a_0|)$$

for $|x| \geq 1$.

Proof. Let x be a real number such that $|x| \geq 1$. Then

$$\begin{aligned} |f(x)| &= |a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0| \\ &= |x|^m \left| a_m + a_{m-1} \frac{1}{x} + \cdots + a_1 \frac{1}{x^{m-1}} + a_0 \frac{1}{x^m} \right| \\ &\leq |x|^m \left(|a_m| + |a_{m-1}| \frac{1}{|x|} + \cdots + |a_1| \frac{1}{|x|^{m-1}} + |a_0| \frac{1}{|x|^m} \right) \\ &\leq |x|^m (|a_m| + |a_{m-1}| + \cdots + |a_0|). \quad \square \end{aligned}$$

Proof of (b). We first prove by induction that for each $n \geq 1$ there exists a polynomial P_n (with real coefficients) such that for every $x \neq 0$

$$f^{(n)}(x) = P_n(1/x) e^{-1/x^2}. \quad (1)$$

For $n = 0$ take $P_0 \equiv 1$. Suppose (1) holds for some n . Differentiate (for $x \neq 0$):

$$f^{(n+1)}(x) = (P_n(1/x))' e^{-1/x^2} + P_n(1/x) (e^{-1/x^2})'.$$

Since $(e^{-1/x^2})' = \frac{2}{x^3} e^{-1/x^2}$ and $(P_n(1/x))'$ is again a rational function which can be written as a polynomial in $1/x$ (times a power of x^{-1}), we see that $f^{(n+1)}(x)$ can be written in the form

$$f^{(n+1)}(x) = P_{n+1}(1/x) e^{-1/x^2}$$

for some polynomial P_{n+1} . This completes the induction.

Now fix $n \geq 0$. From (1) we have for $x \neq 0$

$$|f^{(n)}(x)| = |P_n(1/x)| e^{-1/x^2}.$$

The polynomial $|P_n(1/x)|$ grows at most like a fixed power of $|x|^{-1}$; hence, by Lemma 84, there exist constants $C > 0$ and $m \geq 0$ such that

$$|f^{(n)}(x)| \leq C |x|^{-m} e^{-1/x^2} \quad \text{for } |x| \leq 1.$$

As in part (a), with $t = 1/x^2$ we get

$$|x|^{-m} e^{-1/x^2} = t^{m/2} e^{-t} \rightarrow 0 \quad \text{as } x \rightarrow 0.$$

Thus $\lim_{x \rightarrow 0} f^{(n)}(x) = 0$. Define $f^{(n)}(0) := 0$. The preceding limit shows that this value agrees with the limit of $f^{(n)}(x)$ as $x \rightarrow 0$, so $f^{(n)}$ is continuous at 0. Together with smoothness on $\mathbb{R} \setminus \{0\}$, this proves $f^{(n)}$ exists and is continuous on all of \mathbb{R} , and $f^{(n)}(0) = 0$.

Finally, to see explicitly that the derivatives at 0 computed via the difference quotient agree with 0, one can check by induction that

$$\frac{d^n f}{dx^n}(0) = \lim_{x \rightarrow 0} \frac{f^{(n-1)}(x) - f^{(n-1)}(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{f^{(n-1)}(x)}{x} = 0,$$

using the fact already established that $f^{(n-1)}(x)$ tends to 0 faster than any power of x . This gives another direct verification that all derivatives at 0 are 0. \square

Theorem 85. *Let*

$$f_n(x) = \begin{cases} x^n \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then f_1 is continuous but not differentiable at 0. Also, f_2 is differentiable but not of class C^1 . In general, $f_n \in C^k$ at 0 if and only if $n \geq k + 1$.

Problem 3

Proof. For $n = 1$, we have

$$f_1(x) = \begin{cases} x \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then

$$\lim_{x \rightarrow 0} f_1(x) = \lim_{x \rightarrow 0} x \sin(1/x).$$

Since $|\sin(1/x)| \leq 1$, we have $|x \sin(1/x)| \leq |x| \rightarrow 0$ as $x \rightarrow 0$. Hence f_1 is continuous at 0. Now,

$$\lim_{x \rightarrow 0} \frac{f_1(x) - f_1(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x \sin(1/x)}{x} = \lim_{x \rightarrow 0} \sin(1/x),$$

which does not exist due to oscillation. Therefore f_1 is not differentiable at 0.

Next, for $n = 2$, we have

$$f_2(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then

$$f_2'(0) = \lim_{x \rightarrow 0} \frac{x^2 \sin(1/x) - 0}{x} = \lim_{x \rightarrow 0} x \sin(1/x) = 0.$$

Hence f_2 is differentiable at 0. Moreover, for $x \neq 0$, we have

$$f_2'(x) = \frac{d}{dx} (x^2 \sin(1/x)) = 2x \sin(1/x) - \cos(1/x).$$

Now,

$$\lim_{x \rightarrow 0} f_2'(x) = \lim_{x \rightarrow 0} (2x \sin(1/x) - \cos(1/x))$$

does not exist because $\cos(1/x)$ oscillates. Hence, f_2' is not continuous at 0, so $f_2 \notin C^1$.

Finally, we assume $n \geq 3$. For $x \neq 0$,

$$f_n'(x) = nx^{n-1} \sin(1/x) - x^{n-2} \cos(1/x).$$

To have $f_n'(0)$ exist, the term $x^{n-2} \cos(1/x)$ must vanish as $x \rightarrow 0$. This requires $n - 2 > 0 \implies n \geq 3$.

Hence the general pattern:

- f_n is continuous at 0 for all $n \geq 1$.
- f_n is differentiable at 0 if $n \geq 2$.
- $f_n \in C^1$ (i.e., derivative continuous at 0) if $n \geq 3$.

Now, we show that $f_n \in C^k$ at 0 if and only if $n \geq k + 1$. For $x \neq 0$,

$$f_n'(x) = nx^{n-1} \sin(1/x) - x^{n-2} \cos(1/x).$$

The first term $nx^{n-1} \sin(1/x)$ vanishes as $x \rightarrow 0$ if $n - 1 > 0$. The second term $-x^{n-2} \cos(1/x)$ vanishes as $x \rightarrow 0$ if $n - 2 > 0$. Hence the term with the smallest power of x dominates the behavior near 0.

After taking k derivatives, the most singular term behaves like

$$x^{n-k} \cdot (\sin(1/x) \text{ or } \cos(1/x)).$$

This term determines whether $f_n^{(k)}(x)$ can extend continuously to 0.

For $f_n^{(k)}$ to be continuous at 0, we require

$$\lim_{x \rightarrow 0} x^{n-k} (\sin(1/x) \text{ or } \cos(1/x)) = 0,$$

which holds if and only if

$$n - k > 0 \quad \implies \quad n \geq k + 1.$$

Then we define $f_n^{(k)}(0) = 0$ to make it continuous. □

Theorem 86. Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) . If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.

Problem 4

Proof. Take any $x, y \in [a, b]$ with $x < y$. By the Mean Value Theorem there exists $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}.$$

Since $f'(c) = 0$ by hypothesis, it follows that $f(y) - f(x) = 0$, so $f(y) = f(x)$. Because x, y were arbitrary points of $[a, b]$, the function f is constant on $[a, b]$. □

Theorem 87. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be continuous, $f'(x)$ exists for all $x \neq 0$, and

$$\lim_{x \rightarrow 0} f'(x) = 3.$$

Then $f'(0)$ exists and $f'(0) = 3$.

Problem 5

Proof. For $x \neq 0$, apply the Mean Value Theorem on $[0, x]$ (if $x > 0$) or $[x, 0]$ (if $x < 0$). There exists c_x between 0 and x such that

$$f(x) - f(0) = f'(c_x) x.$$

Dividing by x gives

$$\frac{f(x) - f(0)}{x} = f'(c_x).$$

As $x \rightarrow 0$, the point c_x lies between 0 and x , so $c_x \rightarrow 0$. By hypothesis,

$$\lim_{x \rightarrow 0} f'(c_x) = 3.$$

Hence

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = 3,$$

so $f'(0)$ exists and $f'(0) = 3$. □

Theorem 88 (Banach Fixed-Point). *Let (X, d) be a complete metric space, and let $T : X \rightarrow X$ satisfy*

$$d(T(x), T(y)) \leq \alpha d(x, y) \quad \text{for all } x, y \in X,$$

for some $0 \leq \alpha < 1$. Then T has a unique fixed point $x^ \in X$. Moreover, for any $x_0 \in X$, the sequence defined by $x_{n+1} = T(x_n)$ converges to x^* .*

Proof. Let $x_0 \in X$ and define $x_{n+1} = T(x_n)$ for $n \geq 0$. For $n \geq 1$,

$$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) \leq \alpha d(x_n, x_{n-1}).$$

By induction,

$$d(x_{n+1}, x_n) \leq \alpha^n d(x_1, x_0).$$

For $m > n$, by the triangle inequality,

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{k=n}^{m-1} d(x_{k+1}, x_k) \\ &\leq d(x_1, x_0) \sum_{k=n}^{m-1} \alpha^k \\ &\leq \frac{\alpha^n}{1 - \alpha} d(x_1, x_0) \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

Hence (x_n) is Cauchy.

Since X is complete, there exists $x^* \in X$ with $x_n \rightarrow x^*$. By continuity of T ,

$$T(x^*) = T\left(\lim_{n \rightarrow \infty} x_n\right) = \lim_{n \rightarrow \infty} T(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = x^*.$$

If $y^* \in X$ is another fixed point, then

$$d(x^*, y^*) = d(T(x^*), T(y^*)) \leq \alpha d(x^*, y^*) \implies d(x^*, y^*) = 0.$$

Thus $x^* = y^*$. □

Theorem 89. Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) , with

$$a \leq f(x) \leq b \quad \text{for all } x \in [a, b],$$

and

$$|f'(x)| \leq \alpha < 1 \quad \text{for all } x \in (a, b).$$

Then f has a unique fixed point in $[a, b]$.

Problem 6

Proof. We first show that f is a contraction. For any $x, y \in [a, b]$, $x \neq y$, by the Mean Value Theorem there exists c between x and y such that

$$f(x) - f(y) = f'(c)(x - y),$$

so

$$|f(x) - f(y)| = |f'(c)||x - y| \leq \alpha|x - y|.$$

Hence f is a contraction with constant $\alpha < 1$.

Since $[a, b]$ is a closed interval in \mathbb{R} (a complete metric space), the Banach fixed-point theorem guarantees that f has a unique fixed point $x^* \in [a, b]$. □

Homework 8

Theorem 90. Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} x + 2x^2 \sin \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

- (a) Then $f'(0) = 1$ and $f'(x) = 1 - 2 \cos(1/x) + 4x \sin(1/x)$ for $x \neq 0$.
(b) There exists a sequence of points $\{x_n\}$ with $x_n \neq 0$, $x_n \rightarrow 0$, and $f'(x_n) < 0$.

Problem 1

Proof. (a) For $x \neq 0$, we have

$$f(x) = x + 2x^2 \sin \frac{1}{x}, \quad f(0) = 0.$$

Then

$$\begin{aligned} f'(0) &= \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h + 2h^2 \sin(1/h)}{h} \\ &= \lim_{h \rightarrow 0} (1 + 2h \sin(1/h)) \\ &= 1. \end{aligned}$$

Hence $f'(0) = 1 > 0$.

For $x \neq 0$, differentiating directly gives

$$\begin{aligned} \frac{d}{dx} (2x^2 \sin(1/x)) &= 4x \sin(1/x) + 2x^2 \cos(1/x) \left(-\frac{1}{x^2} \right) \\ &= 4x \sin(1/x) - 2 \cos(1/x). \end{aligned}$$

Therefore,

$$f'(x) = 1 - 2 \cos(1/x) + 4x \sin(1/x), \quad x \neq 0.$$

(b) We seek a sequence $\{x_n\}$ with $x_n \rightarrow 0$ and $f'(x_n) < 0$. Choose x_n such that $\cos(1/x_n) = 1$ and $\sin(1/x_n) = 0$, for example,

$$x_n = \frac{1}{2\pi n}, \quad n = 1, 2, 3, \dots$$

Then $1/x_n = 2\pi n$, so $\cos(1/x_n) = 1$ and $\sin(1/x_n) = 0$. Substituting into the formula for $f'(x)$,

$$f'(x_n) = 1 - 2 \cdot 1 + 4x_n \cdot 0 = -1 < 0.$$

Hence $x_n \neq 0$, $x_n \rightarrow 0$, and $f'(x_n) = -1 < 0$.

Although $f'(0) = 1 > 0$, there are points arbitrarily close to 0 where $f'(x) < 0$. Thus, there is no open interval around 0 on which f is increasing. \square

Theorem 91. Suppose $f: (a, b) \rightarrow \mathbb{R}$ is r -th order differentiable at x . If $P(h)$ and $Q(h)$ are two polynomials of degree $\leq r$ such that

$$\lim_{h \rightarrow 0} \frac{f(x+h) - P(h)}{h^r} = 0 = \lim_{h \rightarrow 0} \frac{f(x+h) - Q(h)}{h^r},$$

then $Q = P$.

Problem 2

Proof. Set $S(h) := P(h) - Q(h)$. Then

$$\lim_{h \rightarrow 0} \frac{S(h)}{h^r} = 0.$$

Suppose S is not the zero polynomial. Then we can write

$$\frac{S(h)}{h^r} = h^{m-r} (d_m + d_{m+1}h + \dots + d_r h^{r-m})$$

for some $m \leq r$ and some $d_m \neq 0$. Let $\varphi(h) := d_m + d_{m+1}h + \dots + d_r h^{r-m}$. Then $\lim_{h \rightarrow 0} \varphi(h) = d_m$. Therefore, if $m < r$, then $|h^{m-r}| \rightarrow \infty$ as $h \rightarrow 0$, contradicting that the limit above equals 0. On the other hand, if $m = r$, then $\frac{S(h)}{h^r} \rightarrow d_m$ as $h \rightarrow 0$, so the limit is $d_m \neq 0$, again a contradiction. Hence no such m exists and all $d_k = 0$; therefore $S \equiv 0$ and $P(h) = Q(h)$. \square

Theorem 92 (Peano form of the Taylor approximation). Let $f: (a, b) \rightarrow \mathbb{R}$ be r -times differentiable at x . Define the r -th order Taylor polynomial of f at x by

$$P_r(h) := f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \cdots + \frac{f^{(r)}(x)}{r!}h^r.$$

Then the remainder

$$R(h) := f(x+h) - P_r(h)$$

satisfies

$$\frac{R(h)}{h^r} \longrightarrow 0 \quad \text{as } h \rightarrow 0,$$

i.e., $R(h)$ is r -th order flat at 0.

Proof. By the definition of the Taylor polynomial, $P_r(h)$ matches the first r derivatives of f at x . Therefore, for the remainder $R(h) = f(x+h) - P_r(h)$,

$$R(0) = R'(0) = \cdots = R^{(r)}(0) = 0.$$

By the Mean Value Theorem, there exists $\theta_1 \in (0, h)$ such that

$$R(h) - R(0) = R'(\theta_1)h \implies R(h) = R'(\theta_1)h.$$

Apply the Mean Value Theorem to $R'(\theta_1) - R'(0)$: there exists $\theta_2 \in (0, \theta_1)$ such that

$$R'(\theta_1) - R'(0) = R''(\theta_2)\theta_1 \implies R'(\theta_1) = R''(\theta_2)\theta_1.$$

Substituting back gives

$$R(h) = R''(\theta_2)\theta_1 h.$$

Repeating this process $(r-1)$ times, we obtain

$$R(h) = R^{(r-1)}(\theta_{r-1})\theta_{r-2} \cdots \theta_1 h,$$

where

$$0 < \theta_{r-1} < \cdots < \theta_1 < h.$$

Thus, when $0 < h < 1$,

$$\left| \frac{R(h)}{h^r} \right| = \left| \frac{R^{(r-1)}(\theta_{r-1})\theta_{r-2} \cdots \theta_1 h}{h^r} \right| \leq \left| \frac{R^{(r-1)}(\theta_{r-1}) - 0}{\theta_{r-1}} \right| \rightarrow 0.$$

as $h \rightarrow 0+$. Hence,

$$\frac{R(h)}{h^r} \rightarrow 0 \quad \text{as } h \rightarrow 0+.$$

If $-1 < h < 0$, the same is true with

$$h < \theta_1 < \theta_2 < \cdots < \theta_{r-1} < 0.$$

Therefore, $R(h)$ is r -th order flat at 0. □

Theorem 93. Suppose f is defined in an open interval containing a , and suppose $f''(a)$ exists. Then

$$f''(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}.$$

Problem 3 (a)

Proof. Since $f''(a)$ exists, by [Theorem 92](#), we can write the Taylor expansions for small h :

$$f(a+h) = f(a) + f'(a)h + \frac{1}{2}f''(a)h^2 + o(h^2),$$

$$f(a-h) = f(a) - f'(a)h + \frac{1}{2}f''(a)h^2 + o(h^2),$$

where $o(h^2)$ denotes a term such that $\frac{o(h^2)}{h^2} \rightarrow 0$ as $h \rightarrow 0$.

Form the symmetric difference quotient:

$$f(a+h) - 2f(a) + f(a-h) = f''(a)h^2 + o(h^2).$$

Divide both sides by h^2 :

$$\frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = f''(a) + \frac{o(h^2)}{h^2}.$$

Taking the limit as $h \rightarrow 0$, we get

$$\lim_{h \rightarrow 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = f''(a). \quad \square$$

Remark 94. Here is an example where the limit exists but $f''(a)$ does not. Consider

$$f(x) = x|x|, \quad a = 0.$$

The symmetric difference quotient is

$$\frac{f(h) - 2f(0) + f(-h)}{h^2} = \frac{h|h| + (-h)|-h| - 0}{h^2} = \frac{h^2 - h^2}{h^2} = 0.$$

Therefore, the limit exists and equals 0:

$$\lim_{h \rightarrow 0} \frac{f(h) - 2f(0) + f(-h)}{h^2} = 0.$$

However, the second derivative $f''(0)$ does not exist, because

$$f''(x) = \begin{cases} 2 & x > 0, \\ -2 & x < 0, \end{cases}$$

so the left and right second derivatives at 0 are different.

Hence this function satisfies the required conditions.

Problem 3 (b)

Theorem 95 (Taylor's theorem (degree n with Lagrange remainder)).
If g is C^{n+1} on an interval containing 0 and t , then there exists ξ between 0 and t such that

$$g(t) = g(0) + g'(0)t + \frac{g''(0)}{2!}t^2 + \cdots + \frac{g^{(n)}(0)}{n!}t^n + \frac{g^{(n+1)}(\xi)}{(n+1)!}t^{n+1}.$$

Theorem 96. *Let*

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1,$$

and the point $x = 0$ is a removable discontinuity of f (since $f(0) = 0 \neq 1$). Re-defining $f(0) := 1$ makes f continuous at 0.

Problem 4 (a)

Proof. We use Taylor's theorem with the Lagrange form of the remainder for the function $g(t) = \sin t$ about $t = 0$.

Since $g(0) = 0$, $g'(0) = 1$, and $g''(u) = -\sin u$, for each x there exists ξ between 0 and x with

$$\sin x = 0 + 1 \cdot x + \frac{-\sin \xi}{2} x^2 = x - \frac{\sin \xi}{2} x^2.$$

For $x \neq 0$ divide both sides by x to obtain

$$\frac{\sin x}{x} = 1 - \frac{\sin \xi}{2} x,$$

where ξ lies between 0 and x .

Since $|\sin \xi| \leq 1$ for all real ξ , we have the estimate

$$\left| \frac{\sin x}{x} - 1 \right| = \left| \frac{\sin \xi}{2} x \right| \leq \frac{|x|}{2}.$$

As $x \rightarrow 0$ the right-hand side $\frac{|x|}{2} \rightarrow 0$, therefore

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

The two-sided limit $\lim_{x \rightarrow 0} \frac{\sin x}{x}$ exists and equals 1, while the function value given is $f(0) = 0$. Hence the limit and the value differ:

the discontinuity at 0 is *removable*. If we redefine

$$\tilde{f}(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0, \\ 1, & x = 0, \end{cases}$$

then \tilde{f} is continuous at 0. □

Theorem 97. *Let*

$$f(x) = \begin{cases} e^{1/x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then

$$\lim_{x \rightarrow 0^+} f(x) = +\infty, \quad \lim_{x \rightarrow 0^-} f(x) = 0,$$

and the discontinuity of f at $x = 0$ is essential (equivalently: an infinite/non-removable discontinuity).

Problem 4 (b)

Proof. We shall use Taylor's theorem (Lagrange form of the remainder) for the function $g(t) = e^t$ about $t = 0$, for which $g^{(k)}(u) = e^u$ for all k and u .

(i) *We claim that* $\lim_{x \rightarrow 0^+} e^{1/x} = +\infty$.

Put $t = \frac{1}{x}$. When $x \rightarrow 0^+$ we have $t \rightarrow +\infty$. Apply Taylor's theorem with $n = 1$ to $g(t) = e^t$ at 0: for each $t > 0$ there exists $\xi \in (0, t)$ such that

$$e^t = g(0) + g'(0)t + \frac{g''(\xi)}{2}t^2 = 1 + t + \frac{e^\xi}{2}t^2.$$

Since $e^\xi > 0$, the remainder term $\frac{e^\xi}{2}t^2$ is positive, so for every $t > 0$

$$e^t = 1 + t + \frac{e^\xi}{2}t^2 > 1 + t > t.$$

Now let $M > 0$ be arbitrary. Choose $T > M$. For $t > T$ we have $e^t > t > T > M$. Translating back to x : choose $\delta = \frac{1}{T}$. Then if

$0 < x < \delta$ we get $t = \frac{1}{x} > T$ and hence $e^{1/x} > M$. Since M was arbitrary this proves $\lim_{x \rightarrow 0^+} e^{1/x} = +\infty$.

(ii) We claim that $\lim_{x \rightarrow 0^-} e^{1/x} = 0$.

For $x \rightarrow 0^-$ set $t = \frac{1}{x}$; then $t \rightarrow -\infty$. Write $t = -s$ with $s \rightarrow +\infty$. Then

$$e^{1/x} = e^t = e^{-s} = \frac{1}{e^s}.$$

It suffices to show $e^s \rightarrow +\infty$ as $s \rightarrow +\infty$. Apply Taylor's theorem with $n = 2$ to $g(s) = e^s$ at 0: for each $s > 0$ there exists $\eta \in (0, s)$ such that

$$e^s = 1 + s + \frac{s^2}{2} e^\eta.$$

Since $e^\eta \geq 1$ for $\eta \geq 0$, we have

$$e^s \geq 1 + s + \frac{s^2}{2}.$$

The right-hand side tends to $+\infty$ as $s \rightarrow +\infty$, hence $e^s \rightarrow +\infty$. Therefore

$$e^{1/x} = e^{-s} = \frac{1}{e^s} \longrightarrow 0 \quad \text{as } s \rightarrow +\infty,$$

i.e. $\lim_{x \rightarrow 0^-} e^{1/x} = 0$.

We have $\lim_{x \rightarrow 0^-} f(x) = 0 = f(0)$, while $\lim_{x \rightarrow 0^+} f(x) = +\infty$. Thus the two one-sided limits are not both finite and equal (indeed the right-hand limit diverges to $+\infty$). Consequently the two-sided limit $\lim_{x \rightarrow 0} f(x)$ does not exist as a finite real number, and the point $x = 0$ is not removable. Because one one-sided limit is infinite, the usual real-analysis terminology classifies this as an *essential* (or *infinite* / *non-removable*) discontinuity at $x = 0$. \square

Theorem 98. Let f be an increasing function on $[a, b]$, and let $x_1, \dots, x_n \in (a, b)$ with

$$a < x_1 < x_2 < \dots < x_n < b.$$

1. Then

$$\sum_{k=1}^n [f(x_k^+) - f(x_k^-)] \leq f(b) - f(a).$$

2. For each $m \in \mathbb{Z}^+$, let

$$S_m = \{x \in [a, b] : f(x^+) - f(x^-) > 1/m\}.$$

Then S_m is finite.

3. Thus, the set of discontinuities of f is countable.

Problem 5

Proof. Since f is increasing, the total change from a to b can be written as the sum of the continuous increases between the points and the jumps at the points:

$$\begin{aligned} f(b) - f(a) &= [f(x_1^-) - f(a)] + [f(x_1^+) - f(x_1^-)] \\ &\quad + [f(x_2^-) - f(x_1^+)] + [f(x_2^+) - f(x_2^-)] \\ &\quad + \cdots \\ &\quad + [f(x_n^-) - f(x_{n-1}^+)] + [f(x_n^+) - f(x_n^-)] \\ &\quad + [f(b) - f(x_n^+)]. \end{aligned}$$

By considering jumps at x_k , we immediately get:

$$\sum_{k=1}^n [f(x_k^+) - f(x_k^-)] \leq f(b) - f(a),$$

as required. This completes the proof of 1.

Suppose, for some $m \in \mathbb{Z}^+$, that S_m has infinitely many points. Let $l \in \mathbb{N}$ be such that $\frac{l}{m} > f(b) - f(a)$, and choose x_1, \dots, x_l distinct points from S . Then

$$\sum_{k=1}^l [f(x_k^+) - f(x_k^-)] > \#S_m \cdot \frac{l}{m} > f(b) - f(a),$$

which contradicts part 1. Therefore, S_m must be finite. This completes the proof of 2.

Let D be the set of discontinuities of f in $[a, b]$. Each discontinuity corresponds to a jump, so for each $x \in D$, there exists some $m \in \mathbb{Z}^+$ such that the jump at x is greater than $1/m$. Therefore, we can write

$$D = \bigcup_{m=1}^{\infty} S_m,$$

where each S_m is finite by part 2. A countable union of finite sets is countable. Hence, the set of discontinuities D is countable. \square

Homework 9

Definition 99. A function $f: [a, b] \rightarrow \mathbb{R}$ is said to satisfy a *uniform Lipschitz condition of order $\alpha > 0$* on $[a, b]$ if there exists a constant $M > 0$ such that

$$|f(x) - f(y)| \leq M|x - y|^\alpha, \quad \forall x, y \in [a, b].$$

Theorem 100. Let $f: [a, b] \rightarrow \mathbb{R}$ be a function that satisfy a uniform Lipschitz condition of order $\alpha > 0$ on $[a, b]$.

1. If $\alpha > 1$, then f is constant on $[a, b]$.
2. If $\alpha = 1$, then f is of bounded variation on $[a, b]$.

Problem 2

Proof of 1. For $x \neq y$,

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M|x - y|^{\alpha-1}.$$

Since $\alpha - 1 > 0$,

$$\lim_{y \rightarrow x} \left| \frac{f(x) - f(y)}{x - y} \right| \leq \lim_{y \rightarrow x} M|x - y|^{\alpha-1} = 0.$$

Therefore,

$$f'(x) = \lim_{y \rightarrow x} \frac{f(y) - f(x)}{y - x} = 0 \quad \forall x \in [a, b].$$

Since $f'(x) = 0$ for all $x \in [a, b]$, the Mean Value Theorem implies that f is constant on $[a, b]$. \square

Proof of 2. For any partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$,

$$\sum_{i=1}^n |f(x_i) - f(x_{i-1})| \leq \sum_{i=1}^n M|x_i - x_{i-1}| = M \sum_{i=1}^n (x_i - x_{i-1}) = M(b - a).$$

Since this bound holds for any partition P , we have

$$V_a^b(f) \leq M(b - a) < \infty,$$

so f is of bounded variation on $[a, b]$. \square

Theorem 101. *Let*

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then f is Lipschitz continuous on $[0, 1]$ with Lipschitz constant $L = 3$.

Problem 1 (a)

Proof. We need to show that there exists a constant $L > 0$ such that for all $x, y \in [0, 1]$,

$$|f(x) - f(y)| \leq L|x - y|.$$

First suppose $x, y \neq 0$. By the Mean Value Theorem, there exists c between x and y such that

$$f(x) - f(y) = f'(c)(x - y).$$

Hence,

$$\begin{aligned} |f(x) - f(y)| &= |f'(c)| |x - y| \\ &= |2c \sin(1/c) - \cos(1/c)| |x - y| \\ &\leq (2|c| + |\cos(1/c)|) |x - y| \\ &\leq 3 |x - y|. \end{aligned}$$

Now, suppose one of the points is 0. Without loss of generality, let $x = 0$ and $y \neq 0$. Then

$$|f(y) - f(0)| = |y^2 \sin(1/y) - 0| \leq y^2 \leq |y - 0|.$$

The same estimate holds if $y = 0$ and $x \neq 0$.

Combining both cases, we obtain for all $x, y \in [0, 1]$:

$$|f(x) - f(y)| \leq 3|x - y|.$$

Therefore, f is Lipschitz continuous on $[0, 1]$ with Lipschitz constant $L = 3$. □

Theorem 102. *Let*

$$f(x) = \begin{cases} \sqrt{x} \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then f is not of bounded variation on $[0, 1]$.

Problem 1 (b)

Proof. Consider the sequence

$$x_n = \frac{1}{n\pi + \pi/2}, \quad n = 0, 1, 2, \dots$$

Then $f(x_n) = \sqrt{x_n} \sin(1/x_n) = (-1)^n \sqrt{x_n}$.

Let $P_N = \{0, x_N, x_{N-1}, \dots, x_1, x_0, 1\}$. This is an increasing sequence from left to right (toward 0). The total variation along P_N is

$$\begin{aligned} V(f, P_N) &= |f(0) - f(x_N)| + |f(x_0) - f(1)| + \sum_{n=1}^N |f(x_n) - f(x_{n-1})| \\ &\geq \sum_{n=1}^N |f(x_n) - f(x_{n-1})| \\ &= \sum_{n=1}^N |(-1)^n \sqrt{x_n} - (-1)^{n-1} \sqrt{x_{n-1}}| \\ &= \sum_{n=1}^N (\sqrt{x_n} + \sqrt{x_{n-1}}) \\ &\geq \sum_{n=1}^N \sqrt{x_n}, \end{aligned}$$

which goes to ∞ as $N \rightarrow \infty$

Since there exists a sequence of partitions $\{P_N\}$ with total variation tending to ∞ , the function f is not of bounded variation on $[0, 1]$. \square

Definition 103. A function $f: [a, b] \rightarrow \mathbb{R}$ is said to be *absolutely continuous* if: For every $\epsilon > 0$, there exists $\delta > 0$ such that for any finite collection

of pairwise disjoint open sub-intervals $(a_k, b_k) \subset [a, b]$, $k = 1, 2, \dots, n$, with

$$\sum_{k=1}^n (b_k - a_k) < \delta,$$

we have

$$\sum_{k=1}^n |f(b_k) - f(a_k)| < \epsilon.$$

Theorem 104. *Let $f: [a, b] \rightarrow \mathbb{R}$ is an absolutely continuous function. Then f is continuous on $[a, b]$.*

Problem 3

Proof. Fix $\epsilon > 0$. By absolute continuity, there exists $\delta > 0$ such that for any finite collection of disjoint intervals with total length less than δ , the sum of the function differences is less than ϵ . In particular, consider a single interval (x, y) with $|y - x| < \delta$. Then,

$$|f(y) - f(x)| < \epsilon.$$

This is exactly the definition of continuity at every point $x \in [a, b]$. \square

Proposition 105. The function

$$f(x) = \begin{cases} x \sin(1/x), & x \neq 0, \\ 0, & x = 0 \end{cases}$$

is continuous on $[0, 1]$ but not absolutely continuous.

Proof. Clearly, f is continuous on $[0, 1]$.

Define

$$x_n := \frac{1}{n\pi + \pi}, \quad y_n := \frac{1}{n\pi + \pi/2}, \quad n = 1, 2, 3, \dots$$

The intervals $[x_n, y_n]$ are disjoint because

$$y_n = \frac{1}{n\pi + \pi/2} < \frac{1}{(n-1)\pi + \pi} = x_{n-1}$$

for $n \geq 2$. Moreover, we have

$$y_n - x_n = \frac{1}{n\pi + \pi/2} - \frac{1}{n\pi + \pi} = \frac{\pi/2}{(n\pi + \pi/2)(n\pi + \pi)} < \frac{1}{2n^2}.$$

Hence, for large enough N , the total length

$$\sum_{n=N}^{\infty} (y_n - x_n) < \delta$$

for any given $\delta > 0$.

On each interval $[x_n, y_n]$,

$$|f(y_n) - f(x_n)| = |y_n \cdot 1 - 0| = \frac{1}{n\pi + \pi/2}.$$

Thus, for $n \geq N$,

$$\sum_{n=N}^{\infty} |f(y_n) - f(x_n)| \geq \sum_{n=N}^{\infty} \frac{1}{2n\pi} = \infty.$$

Let $\varepsilon = 1$ and choose any $\delta > 0$. Then, as above, we can select large N such that the sum of interval lengths $\sum_{n=N}^{\infty} (y_n - x_n) < \delta$. However, the total change in f over these intervals is infinite, which exceeds ε . This contradicts the definition of absolute continuity.

Therefore, f is continuous but not absolutely continuous. \square

Theorem 106. Let $f: [a, b] \rightarrow \mathbb{R}$ is an absolutely continuous function. Then f is a bounded variation on $[a, b]$.

Problem 3

Proof. Fix $\epsilon = 1$. Since f is absolutely continuous, there exists $\delta > 0$ such that for any finite collection of pairwise disjoint sub-intervals $(x_1, y_1), \dots, (x_m, y_m)$ of $[a, b]$ with $\sum_{k=1}^m (y_k - x_k) < \delta$, we have

$$\sum_{k=1}^m |f(y_k) - f(x_k)| < \epsilon = 1.$$

Next, divide $[a, b]$ into sub-intervals of length at most $\delta/2$ by defining the partition

$$P^* = \{a_0 = a < a_1 < \cdots < a_N = b\}, \quad a_i - a_{i-1} \leq \frac{\delta}{2}.$$

Then the number of sub-intervals satisfies

$$N \leq \frac{2(b-a)}{\delta} + 1.$$

Now, take any partition $P = \{a = x_0 < x_1 < \cdots < x_s = b\}$ of $[a, b]$ and consider its refinement

$$P' = P \cup P^* = \{a = z_0 < z_1 < \cdots < z_m = b\}.$$

For each $i = 1, \dots, N$, let $a_{i-1} = y_{i,1} < y_{i,2} < \cdots < y_{i,k_i} = a_i$ denote all the points of $P' \cap [a_{i-1}, a_i]$.

By construction, each sub-interval $[a_{i-1}, a_i]$ has length $\leq \delta/2 < \delta$. Therefore, applying absolute continuity to the points in $P' \cap [a_{i-1}, a_i]$ gives

$$\sum_{l=1}^{k_i-1} |f(y_{i,l}) - f(y_{i,l+1})| < 1.$$

Summing over all $i = 1, \dots, n$, we obtain

$$\begin{aligned} V(P, f) &= \sum_{j=1}^s |f(c_j) - f(c_{j-1})| \\ &\leq \sum_{i=1}^m |f(z_i) - f(z_{i-1})| && \text{as } P \subseteq P' \\ &= \sum_{i=1}^N \sum_{l=1}^{k_i-1} |f(y_{i,l}) - f(y_{i,l+1})| \\ &\leq N. \end{aligned}$$

Since n is finite and depends only on $b - a$ and δ , we conclude that

$$V_a^b(f) := \sup_P \sum_{j=1}^{|P|} |f(c_j) - f(c_{j-1})| \leq N < \infty.$$

Thus, f is of bounded variation on $[a, b]$ □

Remark 107. The Cantor function $c: [0, 1] \rightarrow [0, 1]$ is a continuous, non-decreasing function which is not absolutely continuous. In particular, the Cantor function is of bounded variation on $[0, 1]$.

Theorem 108. Let $f: [a, b] \rightarrow \mathbb{R}$ be integrable and let $c \in \mathbb{R}$. Then cf is integrable and

$$\int_a^b cf = c \int_a^b f.$$

Problem 4

Proof. Let $\epsilon > 0$. Since f is integrable, there exists a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \begin{cases} \epsilon/|c|, & \text{if } c \neq 0, \\ \epsilon, & \text{if } c = 0. \end{cases}$$

If $c = 0$, then $cf = 0$ is constant and hence integrable, with $\int_a^b 0 = 0$. So suppose $c \neq 0$.

Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$. For each i define

$$M_i = \sup_{[x_{i-1}, x_i]} f, \quad m_i = \inf_{[x_{i-1}, x_i]} f.$$

Then for cf ,

$$\sup_{[x_{i-1}, x_i]} cf = \begin{cases} cM_i, & \text{if } c > 0, \\ cm_i, & \text{if } c < 0, \end{cases} \quad \inf_{[x_{i-1}, x_i]} cf = \begin{cases} cm_i, & \text{if } c > 0, \\ cM_i, & \text{if } c < 0. \end{cases}$$

Hence,

$$U(P, cf) - L(P, cf) = |c| (U(P, f) - L(P, f)) < |c| \cdot \frac{\epsilon}{|c|} = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, cf is integrable.

Finally, for $c > 0$, $L(P, cf) = cL(P, f)$ and $U(P, cf) = cU(P, f)$, while for $c < 0$, $L(P, cf) = cU(P, f)$ and $U(P, cf) = cL(P, f)$. Using $I_f = \int_a^b f = \sup_P L(P, f) = \inf_P U(P, f)$, we obtain

$$\int_a^b cf = \sup_P L(P, cf) = \begin{cases} c \sup_P U(P, f) = c \int_a^b f & \text{if } c > 0 \\ c \inf_P L(P, f) = c \int_a^b f & \text{if } c < 0. \end{cases} \quad \square$$

Theorem 109. Let $f, g: [a, b] \rightarrow \mathbb{R}$ be integrable functions. Then $f + g$ is integrable and

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

Problem 5

Proof. Let $\epsilon > 0$. Since f and g are integrable, there exist partitions P_f and P_g of $[a, b]$ such that

$$U(P_f, f) - L(P_f, f) < \frac{\epsilon}{2}, \quad U(P_g, g) - L(P_g, g) < \frac{\epsilon}{2}.$$

Let $P_0 = P_f \cup P_g$ be the common refinement. By the refinement property,

$$U(P_0, f) - L(P_0, f) < \frac{\epsilon}{2}, \quad U(P_0, g) - L(P_0, g) < \frac{\epsilon}{2}.$$

Write P_0 as $\{a = x_0 < \cdots < x_n = b\}$ and let

$$M_i^f = \sup_{[x_{i-1}, x_i]} f, \quad m_i^f = \inf_{[x_{i-1}, x_i]} f,$$

$$M_i^g = \sup_{[x_{i-1}, x_i]} g, \quad m_i^g = \inf_{[x_{i-1}, x_i]} g.$$

Then for each i ,

$$\sup_{[x_{i-1}, x_i]} (f + g) \leq M_i^f + M_i^g, \quad \inf_{[x_{i-1}, x_i]} (f + g) \geq m_i^f + m_i^g.$$

Hence the upper and lower sums satisfy

$$\begin{aligned} L(P_0, f) + L(P_0, g) &\leq L(P_0, f + g) \\ &\leq U(P_0, f + g) \\ &\leq U(P_0, f) + U(P_0, g), \end{aligned}$$

which implies

$$\begin{aligned} U(P_0, f + g) - L(P_0, f + g) &\leq (U(P_0, f) - L(P_0, f)) \\ &\quad + (U(P_0, g) - L(P_0, g)) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $f + g$ is integrable.

Let $I_f = \int_a^b f$ and $I_g = \int_a^b g$. Then,

$$I_f = \sup_P L(P, f) = \inf_P U(P, f)$$

and

$$I_g = \sup_P L(P, g) = \inf_P U(P, g).$$

Therefore,

$$\begin{aligned} I_f - \frac{\epsilon}{2} + I_g - \frac{\epsilon}{2} &\leq U(P_0, f) - \frac{\epsilon}{2} + U(P_0, g) - \frac{\epsilon}{2} \\ &< L(P_0, f) + L(P_0, g) \\ &\leq L(P_0, f + g) \\ &\leq U(P_0, f + g) \\ &\leq U(P_0, f) + U(P_0, g) \\ &< L(P_0, f) + \frac{\epsilon}{2} + L(P_0, g) + \frac{\epsilon}{2} \\ &\leq I_f + \frac{\epsilon}{2} + I_g + \frac{\epsilon}{2}. \end{aligned}$$

Thus,

$$\int_a^b (f + g) = \inf_P U(P, f + g) \leq U(P_0, f + g) \leq I_f + I_g + \epsilon$$

and

$$\int_a^b (f + g) = \sup_P L(P, f + g) \geq L(P_0, f + g) \geq I_f + I_g - \epsilon.$$

Since ϵ is arbitrary,

$$\int_a^b (f + g) = I_f + I_g = \int_a^b f + \int_a^b g. \quad \square$$

Theorem 110. Let $f, g: [a, b] \rightarrow \mathbb{R}$ be integrable functions such that

Problem 6

$f(x) \geq g(x)$ for all $x \in [a, b]$. Then

$$\int_a^b f \geq \int_a^b g.$$

Problem 6

Proof. Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$, with $\Delta x_i = x_i - x_{i-1}$. Define the upper and lower sums for f :

$$U(f, P) = \sum_{i=1}^n M_i^f \Delta x_i, \quad M_i^f = \sup_{x \in [x_{i-1}, x_i]} f(x),$$

$$L(f, P) = \sum_{i=1}^n m_i^f \Delta x_i, \quad m_i^f = \inf_{x \in [x_{i-1}, x_i]} f(x),$$

and similarly for g :

$$U(g, P) = \sum_{i=1}^n M_i^g \Delta x_i, \quad L(g, P) = \sum_{i=1}^n m_i^g \Delta x_i.$$

Since $f(x) \geq g(x)$ for all x , we have for each interval $[x_{i-1}, x_i]$:

$$m_i^f \geq m_i^g \quad \text{and} \quad M_i^f \geq M_i^g.$$

Hence, for any partition P ,

$$L(f, P) \geq L(g, P) \quad \text{and} \quad U(f, P) \geq U(g, P).$$

Taking the supremum of lower sums (or infimum of upper sums) over all partitions, and using Riemann integrability of f and g , we get

$$\int_a^b f = \sup_P L(f, P) \geq \sup_P L(g, P) = \int_a^b g. \quad \square$$

Theorem 111. Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous and non-negative

Problem 7

$(f(x) \geq 0 \text{ for all } x \in [a, b])$. If

$$\int_a^b f = 0,$$

then $f(x) = 0$ for all $x \in [a, b]$.

Problem 7

Proof. Suppose, for contradiction, that f is not identically zero. Then there exists $x_0 \in [a, b]$ such that

$$f(x_0) > 0.$$

Since f is continuous at x_0 , for $\varepsilon = \frac{f(x_0)}{2}$, there exists $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon \quad \text{for all } x \in I := [x_0 - \delta, x_0 + \delta] \cap [a, b].$$

That is,

$$f(x_0) - f(x) = |f(x_0)| - |f(x)| \leq |f(x_0) - f(x)| < \frac{f(x_0)}{2} \quad \text{for all } x \in I.$$

Equivalently,

$$\frac{f(x_0)}{2} < f(x) \quad \text{for all } x \in I.$$

By the properties of the integral over sub-intervals:

$$\int_a^b f \geq \int_I f \geq \int_I \frac{f(x_0)}{2} = \frac{f(x_0)}{2} \cdot \text{length}(I) > 0.$$

This contradicts the assumption that $\int_a^b f = 0$. Hence no such x_0 exists, and we must have

$$f(x) = 0 \quad \text{for all } x \in [a, b].$$

□

Homework 10

Theorem 112. Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous and suppose that

$$\int_a^b f = 0.$$

Then there exists a point $c \in [a, b]$ such that $f(c) = 0$.

Problem 1

Proof. Since f is continuous on the compact interval $[a, b]$, it attains both a minimum and a maximum on $[a, b]$.

Suppose, for contradiction, that $f(x) \neq 0$ for every $x \in [a, b]$. By continuity, f cannot change sign without vanishing, so it must have a constant sign on $[a, b]$. Hence either

1. $f(x) > 0$ for all $x \in [a, b]$, or
2. $f(x) < 0$ for all $x \in [a, b]$.

In the first case, let $m = \min_{[a, b]} f > 0$. Then

$$\int_a^b f \geq \int_a^b m = m(b - a) > 0,$$

contradicting the hypothesis $\int_a^b f = 0$. In the second case, let $M = \max_{[a, b]} f < 0$. Then

$$\int_a^b f \leq \int_a^b M = M(b - a) < 0,$$

again a contradiction.

Therefore our assumption was false, and there exists $c \in [a, b]$ such that $f(c) = 0$. \square

Theorem 113 (Mean Value Theorem for Integrals). Let $f: [a, b] \rightarrow \mathbb{R}$

Problem 2

be continuous. Then there exists $c \in [a, b]$ such that

$$\int_a^b f = (b - a)f(c).$$

Problem 2

Proof. If $a = b$ the identity is trivial (take $c = a$). Assume $a < b$. By continuity on the compact interval $[a, b]$, f attains a minimum m and a maximum M on $[a, b]$, so

$$m \leq f(x) \leq M \quad \text{for all } x \in [a, b].$$

By [Theorem 110](#),

$$m(b - a) \leq \int_a^b f \leq M(b - a).$$

Dividing by $b - a > 0$ yields

$$m \leq \frac{1}{b - a} \int_a^b f \leq M.$$

Since f attains every value between m and M (Intermediate Value Theorem), there exists $c \in [a, b]$ with

$$f(c) = \frac{1}{b - a} \int_a^b f,$$

and multiplying by $b - a$ gives the result. \square

Definition 114. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of the interval $[a, b]$. A *sub-interval* of P is a closed interval $[x_{i-1}, x_i]$ for some $i = 1, \dots, n$.

Theorem 115. Let $f, g: [a, b] \rightarrow \mathbb{R}$ be bounded functions that are equal except at finitely many points. Then f is Riemann integrable if and only

Problem 3

if g is Riemann integrable, and in that case

$$\int_a^b f = \int_a^b g.$$

Problem 3

Proof. Set $h := f - g$. By hypothesis, there exists a finite subset $\mathcal{F} \subset [a, b]$ such that $h(x) = 0$ for all $x \in [a, b] \setminus \mathcal{F}$. Define

$$M := \max_{x \in \mathcal{F}} |h(x)|,$$

which is finite.

For any integer $n \geq 1$, let P_n be the partition of $[a, b]$ into n equal sub-intervals, each of length $(b - a)/n$. Denote by \mathcal{I}_n the set of all sub-intervals of P_n , and let

$$\mathcal{A} := \{I \in \mathcal{I}_n : I \cap \mathcal{F} \neq \emptyset\}.$$

Then we have the following:

- $|\mathcal{A}| \leq 2|\mathcal{F}|$.
- If $I \in \mathcal{A}$, then $-M \leq \inf_I h \leq \sup_I h \leq M$.
- If $I \in \mathcal{I}_n \setminus \mathcal{A}$, then $\inf_I h = 0 = \sup_I h$.

Hence,

$$U(h, P_n) = \sum_{I \in \mathcal{I}_n} \frac{b-a}{n} \sup_I h = \sum_{I \in \mathcal{A}} \frac{b-a}{n} \sup_I h \leq 2|\mathcal{F}| \cdot \frac{b-a}{n} \cdot M,$$

and

$$L(h, P_n) = \sum_{I \in \mathcal{I}_n} \frac{b-a}{n} \inf_I h = \sum_{I \in \mathcal{A}} \frac{b-a}{n} \inf_I h \geq 2|\mathcal{F}| \cdot \frac{b-a}{n} \cdot -M.$$

Therefore, for every n ,

$$-2|\mathcal{F}| \frac{b-a}{n} M \leq L(h, P_n) \leq \int_a^b h \leq \int_a^b h \leq U(h, P_n) \leq 2|\mathcal{F}| \frac{b-a}{n} M.$$

Letting $n \rightarrow \infty$ gives

$$0 \leq \int_a^b h \leq \overline{\int_a^b h} \leq 0,$$

so the upper and lower integrals coincide and equal 0. Thus h is Riemann integrable and

$$\int_a^b h = 0.$$

The final statements follow immediately: if one of f, g is integrable then the other is (since they differ by the integrable function h ; see [Theorem 109](#)), and

$$\int_a^b f = \int_a^b (g + h) = \int_a^b g + \int_a^b h = \int_a^b g.$$

This completes the proof. □

Theorem 116. Let $f: [0, 1] \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0, & x = 0 \text{ or } x \text{ irrational,} \\ \frac{1}{q}, & x = \frac{p}{q} \in \mathbb{Q} \setminus \{0\} \text{ written in lowest terms, } q > 0. \end{cases}$$

Then f is Riemann integrable on $[0, 1]$ and

$$\int_0^1 f = 0.$$

Problem 4

Proof. First note that every subinterval of $[0, 1]$ contains irrational points; hence on any subinterval the infimum of f is 0. Therefore every lower sum is 0, so the lower integral satisfies

$$\underline{\int_0^1 f} = 0.$$

It remains to show that the upper integral is also 0.

Let $\varepsilon > 0$. Choose an positive integer N with

$$\frac{1}{N} < \frac{\varepsilon}{2}.$$

If x is a element of $\mathbb{Q} \cap (0, 1]$ such that $x = \frac{p}{q}$ for some positive integers p and q with $\gcd(p, q) = 1$, then the following are equivalent:

- $q \geq N + 1$.
- $f(x) < \varepsilon/2$

Let \mathcal{F} denote the following set

$$\left\{ x \in (0, 1] : x = \frac{p}{q} \text{ for some } p, q \in \mathbb{N} \text{ with } \gcd(p, q) = 1 \text{ and } q \leq N \right\}.$$

Then \mathcal{F} is a finite set.

Choose a partition $P = \{0 = x_0 < x_1 < \cdots < x_k = 1\}$ such that

$$\max_i (x_i - x_{i-1}) < \frac{\varepsilon}{4|\mathcal{F}|}.$$

Denote by \mathcal{I} the set of all sub-intervals of P , and let

$$\mathcal{A} := \{I \in \mathcal{I} : I \cap \mathcal{F} \neq \emptyset\}.$$

Then we have the following:

- $|\mathcal{A}| \leq 2|\mathcal{F}|$.
- If $I \in \mathcal{A}$, then $\sup_I f \leq 1$.
- If $I \in \mathcal{I} \setminus \mathcal{A}$, then $\sup_I f < \frac{\varepsilon}{2}$.
- $\sum_{I \in \mathcal{I} \setminus \mathcal{A}} |I| \leq |[0, 1]| = 1$, since the elements of $\mathcal{I} \setminus \mathcal{A}$ are sub-intervals of $[0, 1]$ with pairwise disjoint interiors.

Therefore,

$$\begin{aligned}
U(P, f) &= \sum_{I \in \mathcal{J}}^k |I| \sup_I f \\
&= \sum_{I \in \mathcal{A}} |I| \sup_I f + \sum_{I \in \mathcal{F} \setminus \mathcal{A}} |I| \sup_I f \\
&< \sum_{I \in \mathcal{A}} \frac{\varepsilon}{4|\mathcal{F}|} \cdot 1 + \sum_{I \in \mathcal{F} \setminus \mathcal{A}} |I| \cdot \frac{\varepsilon}{2} \\
&= |\mathcal{A}| \frac{\varepsilon}{4|\mathcal{F}|} + \frac{\varepsilon}{2} \sum_{I \in \mathcal{F} \setminus \mathcal{A}} |I| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the infimum of the upper sums is 0:

$$\int_0^1 f = 0.$$

Combining the lower and upper integrals gives

$$\underline{\int_0^1} f = \overline{\int_0^1} f = 0,$$

so f is Riemann integrable on $[0, 1]$ and $\int_0^1 f = 0$. □

Theorem 117. *Every monotone function $f: [a, b] \rightarrow \mathbb{R}$ is integrable.*

Proof. Suppose f is monotone increasing on $[a, b]$. (The argument for decreasing f is similar.)

Let $\epsilon > 0$. Choose $P = \{x_0 = a < x_1 < \cdots < x_n = b\}$ a partition of

$[a, b]$ such that $\|P\| := \max_{i=1}^n (x_i - x_{i-1}) < \frac{\epsilon}{f(b) - f(a)}$. Then

$$\begin{aligned}
 U(P, f) - L(P, f) &= \sum_{i=1}^n \left(\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f \right) \cdot (x_i - x_{i-1}) \\
 &= \sum_{i=1}^n (f(x_i) - f(x_{i-1})) \cdot (x_i - x_{i-1}) \\
 &\leq \|P\| \cdot \sum_{i=1}^n (f(x_i) - f(x_{i-1})) \\
 &= \|P\| \cdot (f(b) - f(a)) \\
 &< \epsilon.
 \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, f is integrable.

For a monotone decreasing function, the same argument applies with $f(x_{i-1})$ and $f(x_i)$ interchanged. Hence, every monotone function on $[a, b]$ is integrable. \square

Theorem 118. *Every piecewise-monotone function $f: [a, b] \rightarrow \mathbb{R}$ is integrable.*

Problem 5

Proof. By definition, f is piecewise-monotone if there exists a partition

$$P = \{a = x_0 < x_1 < \cdots < x_N = b\}$$

such that on each sub-interval of P , f is either increasing or decreasing.

On each sub-interval $[x_{i-1}, x_i]$, f is monotone. Every monotone function on a closed interval is integrable; see [Theorem 117](#). That is, for any $\epsilon > 0$, there exists a partition Q_i of $[x_{i-1}, x_i]$ such that

$$U(Q_i, f) - L(Q_i, f) < \frac{\epsilon}{N}.$$

Let $Q = \bigcup_{i=1}^N Q_i$ be the union of all refinements. Then Q is a partition of $[a, b]$, and

$$U(Q, f) - L(Q, f) = \sum_{i=1}^N (U(Q_i, f) - L(Q_i, f)) < \sum_{i=1}^N \frac{\epsilon}{N} = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, f is integrable on $[a, b]$. □

Problem 119. Give an example of a bounded function $f: [a, b] \rightarrow \mathbb{R}$ such that $|f|$ is Riemann-integrable but for which $\int_a^b f$ does not exist.

Problem 6

Solution. Define

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [a, b], \\ -1 & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [a, b]. \end{cases}$$

Every sub-interval of $[a, b]$ contains both rational and irrational numbers, so f is well defined and bounded with $|f(x)| \leq 1$ for all $x \in [a, b]$.

For every $x \in [a, b]$, $|f(x)| = 1$. Thus $|f|$ is the constant function 1, which is integrable, and

$$\int_a^b |f| = \int_a^b 1 = b - a.$$

Now, we show that f is not integrable.

Let $P = \{x_0, x_1, \dots, x_n\}$ be any partition of $[a, b]$. On each sub-interval $[x_{i-1}, x_i]$, since the rationals and irrationals are both dense in \mathbb{R} , we have

$$\sup_{[x_{i-1}, x_i]} f = 1, \quad \inf_{[x_{i-1}, x_i]} f = -1.$$

Hence,

$$U(P, f) = \sum_{i=1}^n (x_i - x_{i-1}) \cdot 1 = b - a$$

and

$$L(P, f) = \sum_{i=1}^n (x_i - x_{i-1}) \cdot (-1) = -(b - a).$$

Therefore,

$$U(P, f) - L(P, f) = 2(b - a) > 0$$

for every partition P . Consequently,

$$\sup_P L(P, f) \neq \inf_P U(P, f),$$

and f is not integrable.

Thus, $|f|$ is integrable but $\int_a^b f$ does not exist.

□

Homework 11

Theorem 120. Let $f: (0, 1] \rightarrow \mathbb{R}$ be a function such that f is integrable on $[c, 1]$ for each $c > 0$, and define the improper integral

$$\int_0^1 f := \lim_{c \rightarrow 0^+} \int_c^1 f,$$

if the limit exists and is finite. Then:

- (a) If f is integrable on $[0, 1]$, then this definition agrees with the usual Riemann integral.
- (b) There exists a function f for which the above improper integral exists, but the integral of $|f|$ does not exist.

Problem 1

Proof of (a). Suppose f is integrable on $[0, 1]$. For any $c > 0$, by additivity of the integral we have

$$\int_0^1 f = \int_0^c f + \int_c^1 f.$$

Rewriting, we get

$$\int_c^1 f = \int_0^1 f - \int_0^c f.$$

Now, since f is integrable on $[0, 1]$, it is bounded, say $|f(x)| \leq M$ for all $x \in [0, 1]$. Hence, for any $c > 0$,

$$\left| \int_0^c f \right| \leq \int_0^c |f| \leq \int_0^c M = M \cdot c.$$

As $c \rightarrow 0^+$, we have $M \cdot c \rightarrow 0$. Therefore,

$$\lim_{c \rightarrow 0^+} \int_c^1 f = 0.$$

Substituting this into the previous equality gives

$$\lim_{c \rightarrow 0^+} \int_c^1 f = \int_0^1 f,$$

which shows that the improper integral definition agrees with the usual integral. □

Theorem 121 (Alternating Series Test (Leibniz)). *Let (a_n) be a sequence of positive real numbers such that*

1. $a_{n+1} \leq a_n$ for all sufficiently large n , and
2. $\lim_{n \rightarrow \infty} a_n = 0$.

Then the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges.

Proof. Let $S_n = a_1 - a_2 + a_3 - \cdots + (-1)^{n+1} a_n$. Then

$$S_{2k+1} - S_{2k-1} = a_{2k} - a_{2k+1} \geq 0,$$

$$S_{2k+2} - S_{2k} = -a_{2k+1} + a_{2k+2} \leq 0.$$

Hence the sequence (S_{2k}) is increasing, and (S_{2k+1}) is decreasing. Since $S_{2k} \leq S_{2k+1}$ for all k , both are bounded and monotone, so each converges. Moreover,

$$S_{2k+1} - S_{2k} = a_{2k+1} \rightarrow 0,$$

so both converge to the same limit. Thus S_n converges, and the alternating series converges. □

Theorem 122. *The geometric series*

$$\sum_{n=0}^{\infty} r^n$$

converges if and only if $|r| < 1$. In that case,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Proof. Let $S_N = 1 + r + r^2 + \cdots + r^N$. Multiplying both sides by r gives

$$rS_N = r + r^2 + \cdots + r^{N+1}.$$

Subtracting, we obtain

$$S_N - rS_N = 1 - r^{N+1},$$

so

$$S_N = \frac{1 - r^{N+1}}{1 - r}.$$

If $|r| < 1$, then $r^{N+1} \rightarrow 0$ as $N \rightarrow \infty$, giving

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}.$$

Now, if $|r| \geq 1$, then r^n does not tend to zero as $n \rightarrow \infty$, so the series diverges. \square

Theorem 123 (p-series test). *For $p > 0$ the series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$.

Proof. Split the positive integers into dyadic blocks

$$B_k = \{2^k + 1, 2^k + 2, \dots, 2^{k+1}\} \quad (k = 0, 1, 2, \dots).$$

Each block B_k contains exactly 2^k integers.

(1) *If $p > 1$ then the series converges.*

For $n \in B_k$ we have $n > 2^k$, hence

$$\frac{1}{n^p} \leq \frac{1}{(2^k)^p}.$$

Summing over the 2^k members of B_k ,

$$\sum_{n \in B_k} \frac{1}{n^p} \leq 2^k \cdot \frac{1}{(2^k)^p} = 2^{k(1-p)}.$$

Thus

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{k=0}^{\infty} \sum_{n \in B_k} \frac{1}{n^p} \leq \sum_{k=0}^{\infty} 2^{k(1-p)}.$$

The right-hand side is a geometric series with ratio $2^{1-p} < 1$ (since $p > 1$), so it converges; see [Theorem 122](#). Hence the p-series converges.

(2) If $0 < p \leq 1$ then the series diverges.

For $n \in B_k$ we have $n \leq 2^{k+1}$, hence

$$\frac{1}{n^p} \geq \frac{1}{(2^{k+1})^p}.$$

Summing over the 2^k members of B_k ,

$$\sum_{n \in B_k} \frac{1}{n^p} \geq 2^k \cdot \frac{1}{(2^{k+1})^p} = 2^{-p} 2^{k(1-p)}.$$

If $0 < p < 1$ then $1 - p > 0$, so $2^{k(1-p)} \rightarrow \infty$ and the lower bounds on block-sums form a divergent geometric-type sequence; summing over blocks shows the whole series diverges.

If $p = 1$ we get the constant lower bound

$$\sum_{n \in B_k} \frac{1}{n} \geq 2^k \cdot \frac{1}{2^{k+1}} = \frac{1}{2}$$

for every k , so the series certainly diverges (its partial sums grow by at least $1/2$ in each block).

Combining (1) and (2) completes the proof. □

Proof of (b). Define $f: (0, 1] \rightarrow \mathbb{R}$ by

$$f(x) = (-1)^n n \quad \text{if } \frac{1}{n+1} < x \leq \frac{1}{n}, \quad n \in \mathbb{N}.$$

For $c \in (1/(N+1), 1/N]$, we have

$$\begin{aligned}
 \int_c^1 f &= \int_{1/2}^1 f + \int_{1/3}^{1/2} f + \cdots + \int_{1/(N-1)}^{1/N} f + \int_{1/N}^c f \\
 &= \int_{1/2}^1 (-1)^1 \cdot 1 + \int_{1/3}^{1/2} (-1)^2 \cdot 2 + \cdots \\
 &\quad + \int_{1/N}^{1/(N-1)} (-1)^{N-1} \cdot (N-1) + \int_c^{1/N} (-1)^N \cdot N \\
 &= \sum_{n=1}^{N-1} (-1)^n n \left(\frac{1}{n} - \frac{1}{n+1} \right) + (-1)^N N \left(\frac{1}{N} - c \right) \\
 &= \sum_{n=1}^{N-1} \frac{(-1)^n}{n+1} + (-1)^N N \left(\frac{1}{N} - c \right).
 \end{aligned}$$

Now, $1 - \frac{1}{N+1} = \frac{N}{N+1} \leq Nc \leq 1$. Hence, taking the limit as $c \rightarrow 0^+$ (equivalently, $N \rightarrow \infty$) gives

$$\int_0^1 f = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1},$$

which converges by [Theorem 121](#).

However,

$$\int_0^1 |f| = \sum_{n=1}^{\infty} n \left(\frac{1}{n} - \frac{1}{n+1} \right) = \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty.$$

Thus, the improper integral of f exists, but the integral of $|f|$ diverges. □

Theorem 124. Let $\gamma_1: [a, b] \rightarrow \mathbb{R}^k$ be a path, and let $\phi: [c, d] \rightarrow [a, b]$ be a continuous, 1-1, onto map such that $\phi(c) = a$. Define the reparametrized curve

$$\gamma_2(s) := \gamma_1(\phi(s)), \quad s \in [c, d].$$

Then:

(a) γ_2 is rectifiable if and only if γ_1 is rectifiable.

(b) If the curves are rectifiable, they have the same length, i.e.,

$$L(\gamma_2) = L(\gamma_1).$$

Problem 2

Proof (a). Let $P = \{c = s_0 < s_1 < \cdots < s_n = d\}$ be a partition of $[c, d]$. Consider the corresponding points in $[a, b]$:

$$t_i := \phi(s_i), \quad i = 0, \dots, n.$$

Since ϕ is 1-1 and onto, $Q = \{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition of $[a, b]$.

The polygonal sum for γ_2 is

$$\sum_{i=1}^n \|\gamma_2(s_i) - \gamma_2(s_{i-1})\| = \sum_{i=1}^n \|\gamma_1(t_i) - \gamma_1(t_{i-1})\|.$$

Every partition of $[c, d]$ corresponds to a partition of $[a, b]$, and conversely, since ϕ is onto. Taking the supremum over all partitions gives

$$\sup_{P \subset [c, d]} \sum_{i=1}^n \|\gamma_2(s_i) - \gamma_2(s_{i-1})\| = \sup_{Q \subset [a, b]} \sum_{i=1}^n \|\gamma_1(t_i) - \gamma_1(t_{i-1})\|.$$

Hence, γ_2 is rectifiable if and only if γ_1 is rectifiable. \square

Proof of (b). By the calculation above, the polygonal sums of γ_2 and γ_1 are identical for corresponding partitions. Therefore, taking the supremum over all partitions,

$$\begin{aligned} L(\gamma_2) &= \sup_{P \subset [c, d]} \sum_{i=1}^n \|\gamma_2(s_i) - \gamma_2(s_{i-1})\| \\ &= \sup_{Q \subset [a, b]} \sum_{i=1}^n \|\gamma_1(t_i) - \gamma_1(t_{i-1})\| \\ &= L(\gamma_1). \end{aligned}$$

This shows that reparametrization via a continuous, 1-1, onto map preserves rectifiability and length. \square

Theorem 125. Let $\{a_n\}$ and $\{b_n\}$ be two real sequences which are bounded below. Then

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n.$$

Problem 3

Proof. Recall that for a real sequence $\{x_n\}$, the *lim sup* is defined as

$$\limsup_{n \rightarrow \infty} x_n := \lim_{n \rightarrow \infty} \sup_{k \geq n} x_k.$$

Since $\{a_n\}$ and $\{b_n\}$ are bounded below, their suprema over tails are finite.

Define for each $n \in \mathbb{N}$:

$$A_n := \sup_{k \geq n} a_k, \quad B_n := \sup_{k \geq n} b_k, \quad S_n := \sup_{k \geq n} (a_k + b_k).$$

For each fixed n , and for all $k \geq n$,

$$a_k + b_k \leq \sup_{j \geq n} a_j + \sup_{j \geq n} b_j = A_n + B_n.$$

Taking the supremum over $k \geq n$ on the left-hand side gives

$$S_n = \sup_{k \geq n} (a_k + b_k) \leq A_n + B_n.$$

The sequences $\{A_n\}$ and $\{B_n\}$ are non-increasing and bounded below, so the limits exist:

$$\lim_{n \rightarrow \infty} A_n = \limsup_{n \rightarrow \infty} a_n, \quad \lim_{n \rightarrow \infty} B_n = \limsup_{n \rightarrow \infty} b_n.$$

From the inequality $S_n \leq A_n + B_n$, we get

$$\lim_{n \rightarrow \infty} S_n \leq \lim_{n \rightarrow \infty} (A_n + B_n) = \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n.$$

By the definition of *lim sup*,

$$\limsup_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} S_n \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n.$$

This completes the proof. □

Theorem 126. Let $\{a_n\}$ be a sequence of real numbers. Then:

(a) $\liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n$.

(b) The sequence $\{a_n\}$ converges if and only if $\limsup_{n \rightarrow \infty} a_n$ and $\liminf_{n \rightarrow \infty} a_n$ are both finite and equal. In this case,

$$\lim_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n.$$

Problem 4

Proof of (a). For each $n \in \mathbb{N}$, define

$$A_n := \sup_{k \geq n} a_k, \quad B_n := \inf_{k \geq n} a_k.$$

Then $B_n \leq A_n$ for all n , and the sequences $\{A_n\}$ and $\{B_n\}$ are non-increasing and non-decreasing respectively. Taking limits gives

$$\liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} B_n \leq \lim_{n \rightarrow \infty} A_n = \limsup_{n \rightarrow \infty} a_n. \quad \square$$

Proof of (b). Suppose $\{a_n\}$ converges to $L \in \mathbb{R}$. Then for any $\varepsilon > 0$, there exists N such that for all $n \geq N$, $L - \varepsilon < a_n < L + \varepsilon$. This implies

$$\inf_{k \geq n} a_k \geq L - \varepsilon, \quad \sup_{k \geq n} a_k \leq L + \varepsilon \quad \forall n \geq N.$$

Taking limits as $n \rightarrow \infty$, we obtain

$$\liminf_{n \rightarrow \infty} a_n \geq L - \varepsilon \quad \text{and} \quad \limsup_{n \rightarrow \infty} a_n \leq L + \varepsilon.$$

By (a),

$$L - \varepsilon \leq \liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n \leq L + \varepsilon.$$

Since ε is arbitrary positive,

$$\liminf_{n \rightarrow \infty} a_n = L = \limsup_{n \rightarrow \infty} a_n$$

Next, suppose $\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = L$ (finite). Let $\varepsilon > 0$. There exists N_1 such that for all $n \geq N_1$, $\inf_{k \geq n} a_k > L - \varepsilon$, and

N_2 such that for all $n \geq N_2$, $\sup_{k \geq n} a_k < L + \varepsilon$. Let $N = \max(N_1, N_2)$. Then for all $n \geq N$,

$$L - \varepsilon < a_n < L + \varepsilon \implies |a_n - L| < \varepsilon.$$

Hence $\{a_n\}$ converges to L , which equals both the \limsup and \liminf . □

Theorem 127. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers such that

$$a_n \leq b_n \quad \text{for all } n \in \mathbb{N}.$$

Then

$$\limsup_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} b_n \quad \text{and} \quad \liminf_{n \rightarrow \infty} a_n \leq \liminf_{n \rightarrow \infty} b_n.$$

Problem 5

Proof. Define for each $n \in \mathbb{N}$:

$$A_n := \sup_{k \geq n} a_k, \quad B_n := \sup_{k \geq n} b_k.$$

Since $a_k \leq b_k$ for all $k \geq n$, we have $A_n \leq B_n$ for all n .

Taking the limit as $n \rightarrow \infty$, we obtain

$$\limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} A_n \leq \lim_{n \rightarrow \infty} B_n = \limsup_{n \rightarrow \infty} b_n.$$

Similarly, define

$$C_n := \inf_{k \geq n} a_k, \quad D_n := \inf_{k \geq n} b_k.$$

Then $C_n \leq D_n$ for all n , and taking limits gives

$$\liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} C_n \leq \lim_{n \rightarrow \infty} D_n = \liminf_{n \rightarrow \infty} b_n.$$

This proves the theorem. □

Theorem 128 (Comparison Test). Let $\sum a_n$ and $\sum b_n$ be series with $a_n, b_n \geq 0$ for all n . If $a_n \leq b_n$ for all sufficiently large n , and $\sum b_n$ converges, then $\sum a_n$ also converges.

Proof. Assume $a_n \leq b_n$ and $\sum b_n$ converges. Let $A_N = \sum_{n=1}^N a_n$ and $B_N = \sum_{n=1}^N b_n$. Then $A_N \leq B_N$ for every N . Since (B_N) converges, it is bounded above. Hence (A_N) is increasing and bounded above, so it also converges. Thus $\sum a_n$ converges. \square

Remark 129. The converse also holds: if $a_n \leq b_n$ for all sufficiently large n , and $\sum a_n$ diverges, then $\sum b_n$ also diverges.

Theorem 130. The series $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$ diverges.

Problem 6 (a)

Proof. Using the identity

$$\sqrt{n+1} - \sqrt{n} = \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}},$$

we have, for all $n \geq 1$,

$$\frac{1}{2\sqrt{n+1}} \leq \frac{1}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{2\sqrt{n}}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (see Theorem 123), by Theorem 128,

$$\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n}) \text{ diverges.}$$

\square

Theorem 131. The series $\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n}$ converges.

Problem 6 (b)

Proof. Let $a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n}$. Then

$$a_n = \frac{1}{n(\sqrt{n+1} + \sqrt{n})}.$$

Then for all $n \geq 1$,

$$\frac{1}{2n\sqrt{n+1}} \leq a_n \leq \frac{1}{2n\sqrt{n}}.$$

Compare with the p -series $\sum \frac{1}{n^{3/2}}$, which converges since $p = 3/2 > 1$; see [Theorem 123](#). Hence, by [Theorem 128](#),

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n} \text{ converges.}$$

□

Theorem 132. *The series*

$$\sum_{n=1}^{\infty} (\sqrt[n]{n} - 1)^n$$

converges.

Problem 6 (c)

Proof. Let $a_n = \sqrt[n]{n} - 1$. Then $(1 + a_n)^n = n$, and since $a_n > 0$,

$$n = (1 + a_n)^n = 1 + na_n + \frac{n(n-1)}{2}a_n^2 + \cdots \geq 1 + na_n + \frac{n(n-1)}{2}a_n^2.$$

Hence

$$1 + na_n + \frac{n(n-1)}{2}a_n^2 \leq n,$$

so

$$na_n + \frac{n(n-1)}{2}a_n^2 \leq n - 1.$$

Dropping the nonnegative term na_n gives

$$\frac{n(n-1)}{2}a_n^2 \leq n - 1,$$

and thus, for all $n \geq 2$,

$$a_n \leq \sqrt{\frac{2}{n}}.$$

Therefore

$$a_n^n \leq \left(\frac{2}{n}\right)^{n/2}.$$

For $n \geq 8$, we have $\frac{2}{n} \leq \frac{1}{2}$, so

$$a_n^n \leq \left(\frac{1}{2}\right)^{n/2}.$$

Hence

$$\sum_{n=8}^{\infty} a_n^n \leq \sum_{n=8}^{\infty} \left(\frac{1}{2}\right)^{n/2},$$

and the right-hand side is a convergent series; see [Theorem 122](#). By [Theorem 128](#),

$$\sum_{n=1}^{\infty} (\sqrt[n]{n} - 1)^n$$

converges. □

Homework 12

Problem 133. Find the radius of convergence of each of the following power series using the root test only:

$$(a) \sum_{n=0}^{\infty} 3^n x^n, \quad (b) \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n.$$

Problem 1

Solution. We use the formula for the radius of convergence of a power series

$$R = \frac{1}{\limsup_{n \rightarrow \infty} |a_n|^{1/n}}.$$

(a) Here $a_n = 3^n$. Then

$$|a_n|^{1/n} = (3^n)^{1/n} = 3.$$

Thus

$$\limsup_{n \rightarrow \infty} |a_n|^{1/n} = 3,$$

and hence

$$R = \frac{1}{3}.$$

(b) Here $a_n = \frac{2^n}{n!}$. Then

$$|a_n|^{1/n} = \left(\frac{2^n}{n!} \right)^{1/n} = \frac{2}{(n!)^{1/n}}.$$

We first show that $(n!)^{1/n} \rightarrow \infty$.

If $n = 2k$ for some $k \in \mathbb{N}$, then

$$\begin{aligned}
(n!)^{1/n} &= (1 \cdot 2 \cdots (k-1) \cdot k \cdot (k+1) \cdots (2k))^{1/n} \\
&\geq (k \cdot (k+1) \cdots (2k))^{1/n} \\
&\geq (k \cdot k \cdots k)^{1/n} \\
&= k^{\frac{k+1}{2k}} \\
&\geq k^{1/2} \\
&= \sqrt{\frac{n}{2}},
\end{aligned}$$

which diverges to ∞ as $n \rightarrow \infty$.

On the other hand, if $n = 2k + 1$ for some $k \in \mathbb{N}$, then

$$\begin{aligned}
(n!)^{1/n} &= (1 \cdot 2 \cdots (k-1) \cdot k \cdot (k+1) \cdots (2k+1))^{1/n} \\
&\geq ((k+1) \cdot (k+2) \cdots (2k+1))^{1/n} \\
&\geq ((k+1) \cdot (k+1) \cdots (k+1))^{1/n} \\
&= (k+1)^{\frac{k+1}{2k+1}} \\
&\geq (k+1)^{1/2} \\
&\geq \sqrt{\frac{n}{2}},
\end{aligned}$$

which diverges to ∞ as $n \rightarrow \infty$.

Thus, in either case, $(n!)^{1/n} \rightarrow \infty$. Hence,

$$\lim_{n \rightarrow \infty} \left(\frac{2^n}{n!} \right)^{1/n} = 0.$$

Therefore,

$$\limsup_{n \rightarrow \infty} |a_n|^{1/n} = 0, \quad \text{and} \quad R = \infty. \quad \square$$

Lemma 134. Let $(x_n)_{n \geq 1}$ be a sequence of non-negative reals with finite $L := \limsup_{n \rightarrow \infty} x_n$. Let $f: [0, \infty) \rightarrow \mathbb{R}$ be a continuous non-decreasing function. Then

$$\limsup_{n \rightarrow \infty} f(x_n) = f\left(\limsup_{n \rightarrow \infty} x_n\right) = f(L).$$

Proof. For $N \geq 1$ set $S_N := \sup_{n \geq N} x_n$. The sequence $(S_N)_{N \geq 1}$ is non-increasing and $\lim_{N \rightarrow \infty} S_N = \inf_{N \geq 1} S_N = L$. Fix N . Since f is non-decreasing, for every $n \geq N$ we have $f(x_n) \leq f(S_N)$, hence

$$\sup_{n \geq N} f(x_n) \leq f(S_N).$$

Conversely, by definition of supremum for every $\varepsilon > 0$ there exists some $n \geq N$ with $x_n > S_N - \varepsilon$. By monotonicity, $f(x_n) \geq f(S_N - \varepsilon)$, so

$$\sup_{n \geq N} f(x_n) \geq f(S_N - \varepsilon).$$

Letting $\varepsilon \downarrow 0$ and using continuity of f at S_N gives $\sup_{n \geq N} f(x_n) \geq f(S_N)$. Thus

$$\sup_{n \geq N} f(x_n) = f(S_N) \quad \text{for every } N \geq 1.$$

Taking infimum over N on both sides yields

$$\inf_{N \geq 1} \sup_{n \geq N} f(x_n) = \inf_{N \geq 1} f(S_N).$$

The left-hand side is $\limsup_{n \rightarrow \infty} f(x_n)$. Since $S_{N+1} \leq S_N$ for all N and f is non-decreasing, we have

$$f(S_{N+1}) \leq f(S_N),$$

so the sequence $(f(S_N))$ is non-increasing. Therefore,

$$\inf_N f(S_N) = \lim_{N \rightarrow \infty} f(S_N).$$

By continuity of f at L , we have $\lim_{N \rightarrow \infty} f(S_N) = f(L)$. Combining these equalities gives the desired identity

$$\limsup_{n \rightarrow \infty} f(x_n) = \inf_{N \geq 1} \sup_{n \geq N} f(x_n) = \inf_{N \geq 1} f(S_N) = f(L).$$

This completes the proof. □

Theorem 135. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R = 2$. Fix an integer $k \geq 1$ and consider the power series

$$\sum_{n=0}^{\infty} a_n^k x^n.$$

Then the radius of convergence of this new series is $R' = 2^k$.

Problem 2 (a)

Proof. Put

$$L := \limsup_{n \rightarrow \infty} |a_n|^{1/n}.$$

By the root-test formula for radii of convergence we have $R = 1/L$. Since $R = 2$ we get $L = \frac{1}{2}$.

Define $b_n := a_n^k$. To find the radius R' of $\sum b_n x^n$ apply the root test:

$$\begin{aligned} R' &= \frac{1}{\limsup_{n \rightarrow \infty} |b_n|^{1/n}} \\ &= \frac{1}{\limsup_{n \rightarrow \infty} |a_n|^{k/n}} \\ &= \frac{1}{\limsup_{n \rightarrow \infty} f(|a_n|^{1/n})} && \text{(where } f(t) = t^k \text{)} \\ &= \frac{1}{f\left(\limsup_{n \rightarrow \infty} |a_n|^{1/n}\right)} && \text{(by Lemma 134)} \\ &= \frac{1}{f\left(\frac{1}{2}\right)} \\ &= 2^k. \end{aligned}$$

□

Lemma 136. Let (a_n) be a real sequence such that

$$\rho = \limsup_{n \rightarrow \infty} |a_n|^{1/n}$$

is a positive real number. Define a new sequence (c_m) by

$$c_m = \begin{cases} a_n & \text{if } m = n^2, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\limsup_{m \rightarrow \infty} |c_m|^{1/m} = 1.$$

Proof. For $m = n^2$ we have

$$|c_{n^2}|^{1/n^2} = |a_n|^{1/n^2} = \left(|a_n|^{1/n}\right)^{1/n}.$$

For m not a perfect square, $c_m = 0$, so $|c_m|^{1/m} = 0$.

Hence the values of $|c_m|^{1/m}$ are either 0 or $(|a_n|^{1/n})^{1/n}$. Since

$$\inf_{N \in \mathbb{N}} \sup_{n \geq N} |a_n|^{1/n} = \limsup_{n \rightarrow \infty} |a_n|^{1/n}$$

is a positive real, the sequence $(|a_n|^{1/n})$ is bounded. Thus, there exists $M > 0$ such that

$$|a_n|^{1/n} \leq M \quad \text{for all } n.$$

Let $\varepsilon > 0$ be arbitrary. Since $M^{1/n} \rightarrow 1$ as $n \rightarrow \infty$, there exists n_0 such that

$$M^{1/n} < 1 + \varepsilon \quad \text{for all } n \geq n_0.$$

Now for any $m \geq n_0^2$, either

$$|c_m|^{1/m} = 0 < 1 + \varepsilon,$$

or $m = n^2$ with $n \geq n_0$, in which case

$$|c_m|^{1/m} = (|a_n|^{1/n})^{1/n} \leq M^{1/n} < 1 + \varepsilon.$$

Therefore, for all $m \geq n_0^2$, we have $|c_m|^{1/m} < 1 + \varepsilon$. Hence

$$\limsup_{m \rightarrow \infty} |c_m|^{1/m} = \inf_{M \in \mathbb{N}} \sup_{m \geq M} |c_m|^{1/m} \leq \sup_{m \geq n_0^2} |c_m|^{1/m} \leq 1 + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$\limsup_{m \rightarrow \infty} |c_m|^{1/m} \leq 1.$$

Now, we show that $\limsup_{m \rightarrow \infty} |c_m|^{1/m} \geq 1$. By assumption,

$$\limsup_{n \rightarrow \infty} |a_n|^{1/n} = \inf_{N \in \mathbb{N}} \sup_{n \geq N} |a_n|^{1/n} = \rho$$

is a positive real. Then by the definition of \limsup , for every $\varepsilon > 0$ there exist infinitely many n such that

$$|a_n|^{1/n} > \rho - \varepsilon.$$

Fix an $\varepsilon \in (0, \rho)$ and pick a corresponding strictly increasing subsequence (n_k) with this property. Then for each k ,

$$|c_{n_k^2}|^{1/n_k^2} = (|a_{n_k}|^{1/n_k})^{1/n_k} > (\rho - \varepsilon)^{1/n_k}.$$

Since $(\rho - \varepsilon)^{1/n_k} \rightarrow 1$ as $k \rightarrow \infty$, there exists K such that

$$(\rho - \varepsilon)^{1/n_k} > 1 - \varepsilon \quad \text{for all } k \geq K.$$

Hence for all large k ,

$$|c_{n_k^2}|^{1/n_k^2} > 1 - \varepsilon.$$

Therefore, for every $\varepsilon > 0$ and every large enough index, there exist infinitely many $m = n_k^2$ satisfying $|c_m|^{1/m} > 1 - \varepsilon$. This means

$$\limsup_{m \rightarrow \infty} |c_m|^{1/m} = \inf_{M \in \mathbb{N}} \sup_{m \geq M} |c_m|^{1/m} \geq 1 - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude

$$\limsup_{m \rightarrow \infty} |c_m|^{1/m} \geq 1.$$

Combining the two inequalities gives

$$\limsup_{m \rightarrow \infty} |c_m|^{1/m} = 1.$$

□

Theorem 137. Let the power series $\sum_{n=0}^{\infty} a_n x^n$ have radius of convergence $R = 2$. Then the series

$$\sum_{n=0}^{\infty} a_n x^{n^2}$$

has radius of convergence $R' = 1$.

Problem 2 (b)

Proof. Define a new sequence (c_m) by

$$c_m = \begin{cases} a_n & \text{if } m = n^2, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\limsup_{n \rightarrow \infty} |a_n|^{1/n} = 1/2$, by [Lemma 136](#), the radius of convergence R' of $\sum_{m=0}^{\infty} c_m x^m$ is given by

$$R' = \frac{1}{\limsup_{m \rightarrow \infty} |c_m|^{1/m}} = \frac{1}{1} = 1. \quad \square$$

Theorem 138. Every uniformly convergent sequence of bounded functions $\{f_n\}$ on a common domain S is uniformly bounded. That is, there exists $M > 0$ such that

$$|f_n(x)| \leq M \quad \text{for all } x \in S \text{ and all } n \in \mathbb{Z}^+.$$

Problem 3

Proof. Let (f_n) be a sequence of bounded functions on S , and assume $f_n \rightarrow f$ uniformly on S .

Since the convergence is uniform, for $\varepsilon = 1$ there exists $N \in \mathbb{Z}^+$ such that

$$|f_n(x) - f(x)| < 1 \quad \text{for all } n \geq N, x \in S. \quad (1)$$

For each $k = 1, 2, \dots, N$, the function f_k is bounded, so define

$$M_k = \sup_{x \in S} |f_k(x)| < \infty.$$

From (1) with $n = N$, we obtain for every $x \in S$,

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| < 1 + M_N.$$

Hence f is bounded with $\sup_{x \in S} |f(x)| \leq 1 + M_N$.

For $n \geq N$ and all $x \in S$, we have

$$|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)| < 1 + (1 + M_N) = 2 + M_N.$$

Finally, let

$$M = \max\{M_1, M_2, \dots, M_{N-1}, 2 + M_N\}.$$

Then $|f_n(x)| \leq M$ for all $x \in S$ and all $n \in \mathbb{Z}^+$, so the sequence $\{f_n\}$ is uniformly bounded. \square

Theorem 139. *There exists a sequence of functions $\{f_n\}$ on \mathbb{R} such that $f_n(x) \rightarrow 0$ for every $x \in \mathbb{R}$, but each f_n is unbounded on \mathbb{R} .*

Problem 4

Proof. Define $f_n: \mathbb{R} \rightarrow \mathbb{R}$ by

$$f_n(x) = \frac{x}{n} \quad (n \in \mathbb{Z}^+, x \in \mathbb{R}).$$

Fix $x \in \mathbb{R}$. As $n \rightarrow \infty$, $\frac{x}{n} \rightarrow 0$. Hence $f_n(x) \rightarrow 0$ for every fixed x , so the convergence is pointwise.

Now fix n . For any $M > 0$ choose $x = n(M + 1)$. Then

$$|f_n(x)| = \left| \frac{n(M + 1)}{n} \right| = M + 1 > M.$$

Since this holds for every $M > 0$, $\sup_{x \in \mathbb{R}} |f_n(x)| = +\infty$. Thus each f_n is unbounded on \mathbb{R} . \square

Theorem 140. *Let $\{f_n\}$ and $\{g_n\}$ be sequences of functions on a set E . If $f_n \rightarrow f$ uniformly on E and $g_n \rightarrow g$ uniformly on E , then $f_n + g_n \rightarrow f + g$ uniformly on E .*

Problem 5

Proof. Fix $\varepsilon > 0$. Since $f_n \rightarrow f$ uniformly, there exists N_1 such that for all $n \geq N_1$ and all $x \in E$,

$$|f_n(x) - f(x)| < \varepsilon/2.$$

Since $g_n \rightarrow g$ uniformly, there exists N_2 such that for all $n \geq N_2$ and all $x \in E$,

$$|g_n(x) - g(x)| < \varepsilon/2.$$

Let $N = \max\{N_1, N_2\}$. Then for $n \geq N$ and all $x \in E$,

$$|(f_n + g_n)(x) - (f + g)(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus $f_n + g_n \rightarrow f + g$ uniformly on E . □

Theorem 141. Let $\{f_n\}$ and $\{g_n\}$ be sequences of bounded functions on a set E . If $f_n \rightarrow f$ uniformly on E and $g_n \rightarrow g$ uniformly on E . Then $f_n g_n \rightarrow f g$ uniformly on E .

Problem 5

Proof. By [Theorem 138](#), there exist constants

$$M_f = \sup_{n \in \mathbb{Z}^+} \sup_{x \in E} |f_n(x)| < \infty, \quad M_g = \sup_{n \in \mathbb{Z}^+} \sup_{x \in E} |g_n(x)| < \infty.$$

In particular, for each $n \in \mathbb{N}$, we have $g_n(x) \in [-M_g, M_g]$ whenever $x \in E$. Thus, $|g(x)| \leq M_g$ for all $x \in E$ since $g_n(x) \rightarrow g(x)$ for all $x \in E$.

Let A be a positive number such that

$$A > \max\{M_f, M_g\}.$$

Let $\varepsilon > 0$. Since $f_n \rightarrow f$ uniformly and $g_n \rightarrow g$ uniformly, there exists N such that for all $n \geq N$ and all $x \in E$,

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2A} \quad \text{and} \quad |g_n(x) - g(x)| < \frac{\varepsilon}{2A}.$$

Therefore, for any $n \geq N$ and any $x \in E$, we have

$$\begin{aligned}
 |f_n(x)g_n(x) - f(x)g(x)| &= |f_n(x)(g_n(x) - g(x)) + (f_n(x) - f(x))g(x)| \\
 &\leq |f_n(x)| |g_n(x) - g(x)| + |g(x)| |f_n(x) - f(x)| \\
 &\leq M_f \cdot \frac{\varepsilon}{2A} + M_g \cdot \frac{\varepsilon}{2A} \\
 &< A \cdot \frac{\varepsilon}{2A} + A \cdot \frac{\varepsilon}{2A} \\
 &= \varepsilon.
 \end{aligned}$$

Hence $f_n g_n \rightarrow fg$ uniformly on E . □

Theorem 142. *There exist sequences $\{f_n\}$ and $\{g_n\}$ on $E = \mathbb{R}$ such that $f_n \rightarrow f$ and $g_n \rightarrow g$ uniformly on \mathbb{R} , the product sequence $f_n g_n$ converges pointwise on \mathbb{R} , but $f_n g_n$ does not converge uniformly on \mathbb{R} .*

Problem 6

Proof. Define

$$f_n(x) = \frac{1}{n}, \quad g_n(x) = x \quad (n \in \mathbb{Z}^+, x \in \mathbb{R}).$$

The sequence f_n converges to the zero function $f \equiv 0$. For every n and every $x \in \mathbb{R}$,

$$|f_n(x) - 0| = \frac{1}{n},$$

so $\sup_{x \in \mathbb{R}} |f_n(x) - 0| = 1/n \rightarrow 0$. Thus $f_n \rightarrow 0$ uniformly.

The sequence g_n is constant in n (each $g_n(x) = x$), so it converges to $g(x) = x$ and for all n ,

$$\sup_{x \in \mathbb{R}} |g_n(x) - g(x)| = 0.$$

Hence $g_n \rightarrow g$ uniformly.

For each fixed $x \in \mathbb{R}$,

$$(f_n g_n)(x) = \frac{x}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

so $f_n g_n$ converges pointwise to the zero function.

Suppose, for contradiction, that $f_n g_n$ converges uniformly to 0. Then given $\varepsilon = 1$ there would exist N such that for all $n \geq N$,

$$\sup_{x \in \mathbb{R}} \left| \frac{x}{n} \right| < 1.$$

But for any fixed n and any $M > 0$ choose $x = nM$. Then

$$\left| \frac{x}{n} \right| = \left| \frac{nM}{n} \right| = M,$$

and since M is arbitrary the supremum $\sup_{x \in \mathbb{R}} |x/n|$ is $+\infty$, never < 1 . This contradiction shows $f_n g_n$ does not converge uniformly on \mathbb{R} . \square

Theorem 143. Let $f_n(x) = \frac{1}{nx+1}$ for $x \in (0, 1)$ and $n = 1, 2, \dots$. Then $\{f_n\}$ converges pointwise but not uniformly on $(0, 1)$.

Problem 7 (a)

Proof. Fix $x \in (0, 1)$. For $n \rightarrow \infty$ we have $nx + 1 \rightarrow \infty$, hence

$$f_n(x) = \frac{1}{nx+1} \rightarrow 0,$$

so $f_n \rightarrow 0$ pointwise on $(0, 1)$.

To show the convergence is not uniform, take $\varepsilon = \frac{1}{2}$. If $f_n \rightarrow 0$ uniformly there would exist N such that for all $n \geq N$ and all $x \in (0, 1)$,

$$|f_n(x) - 0| = \frac{1}{nx+1} < \frac{1}{2}.$$

But for any n choose $x = \frac{1}{2n} \in (0, 1)$. Then

$$f_n\left(\frac{1}{2n}\right) = \frac{1}{n \cdot \frac{1}{2n} + 1} = \frac{1}{\frac{1}{2} + 1} = \frac{2}{3} > \frac{1}{2},$$

contradicting the above. Thus convergence is not uniform on $(0, 1)$. \square

Theorem 144. Let $g_n(x) = \frac{x}{nx+1}$ for $x \in (0, 1)$ and $n = 1, 2, \dots$. Then $\{g_n\}$ converges to 0 uniformly on $(0, 1)$.

Problem 7 (b)

Proof. Fix $x \in (0, 1)$. For each fixed x ,

$$g_n(x) = \frac{x}{nx+1} \longrightarrow 0 \quad (n \rightarrow \infty),$$

so pointwise limit is 0. For $x \in (0, 1)$ and $n \geq 1$, we have

$$\frac{x}{nx+1} \leq \frac{1}{n+1},$$

because cross-multiplying (all denominators are positive)

$$\frac{x}{nx+1} \leq \frac{1}{n+1} \iff (n+1)x \leq nx+1 \iff x \leq 1,$$

which is true. Equality occurs at $x = 1$. Hence

$$\sup_{x \in (0,1)} \left| \frac{x}{nx+1} - 0 \right| = \frac{1}{n+1} \longrightarrow 0 \quad (n \rightarrow \infty).$$

Therefore $g_n \rightarrow 0$ uniformly on $(0, 1)$. □

Homework 13

Theorem 145. *The set*

$$\mathcal{C}([0, 2\pi]) = \{f: [0, 2\pi] \rightarrow \mathbb{R} \mid f \text{ is continuous on } [0, 2\pi]\}$$

is a vector space over the field \mathbb{R} when equipped with the operations

$$(f + g)(x) := f(x) + g(x), \quad (\alpha f)(x) := \alpha f(x),$$

for all $f, g \in \mathcal{C}([0, 2\pi])$, $\alpha \in \mathbb{R}$, and $x \in [0, 2\pi]$.

Problem 1

Proof. We must check two things: (A) the operations are well-defined (closure), and (B) the eight vector-space axioms hold.

(A) *Closure (well-definedness):* We show that if $f, g \in \mathcal{C}([0, 2\pi])$ and $\alpha \in \mathbb{R}$, then $f + g$ and αf are in $\mathcal{C}([0, 2\pi])$.

- *Continuity of $f + g$:* Fix $x_0 \in [0, 2\pi]$. Since f and g are continuous at x_0 , for every $\varepsilon > 0$ there exist $\delta_f > 0$ and $\delta_g > 0$ such that

$$|x - x_0| < \delta_f \implies |f(x) - f(x_0)| < \varepsilon/2,$$

$$|x - x_0| < \delta_g \implies |g(x) - g(x_0)| < \varepsilon/2.$$

Let $\delta := \min\{\delta_f, \delta_g\}$. Then for $|x - x_0| < \delta$,

$$\begin{aligned} |(f + g)(x) - (f + g)(x_0)| &= |f(x) - f(x_0) + g(x) - g(x_0)| \\ &\leq |f(x) - f(x_0)| + |g(x) - g(x_0)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus $f + g$ is continuous at x_0 . Because x_0 was arbitrary, $f + g$ is continuous on $[0, 2\pi]$, hence $f + g \in \mathcal{C}([0, 2\pi])$.

- *Continuity of αf* : Fix $x_0 \in [0, 2\pi]$ and let $\varepsilon > 0$. Since f is continuous at x_0 , there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon/|\alpha|$ if $\alpha \neq 0$. Then for $|x - x_0| < \delta$,

$$|(\alpha f)(x) - (\alpha f)(x_0)| = |\alpha| |f(x) - f(x_0)| < |\alpha| \cdot \frac{\varepsilon}{|\alpha|} = \varepsilon.$$

If $\alpha = 0$, then αf is the zero function, which is continuous. Hence $\alpha f \in \mathcal{C}([0, 2\pi])$.

Therefore the set is closed under the given addition and scalar multiplication.

(B) *Verification of vector-space axioms*: We now verify the axioms. All equalities are checked pointwise: an equality of functions $h_1 = h_2$ means $h_1(x) = h_2(x)$ for every $x \in [0, 2\pi]$.

- *Associativity of addition*: For all $f, g, h \in \mathcal{C}([0, 2\pi])$ and for every x ,

$$\begin{aligned} ((f + g) + h)(x) &= (f + g)(x) + h(x) \\ &= (f(x) + g(x)) + h(x) \\ &= f(x) + (g(x) + h(x)) \\ &= (f + (g + h))(x). \end{aligned}$$

Thus $(f + g) + h = f + (g + h)$.

- *Commutativity of addition*: For all f, g and every x ,

$$(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x),$$

so $f + g = g + f$.

- *Additive identity*: Define the zero function $0: [0, 2\pi] \rightarrow \mathbb{R}$ by $0(x) = 0$ for all x . It is continuous and for every f and x ,

$$(f + 0)(x) = f(x) + 0 = f(x),$$

hence $f + 0 = f$.

- *Additive inverse:* For each f define $(-f)(x) := -f(x)$. The function $-f$ is continuous and

$$(f + (-f))(x) = f(x) - f(x) = 0 = (0)(x),$$

so $f + (-f) = 0$.

- *Compatibility of scalar multiplication with field multiplication:* For $\alpha, \beta \in \mathbb{R}$ and $f \in \mathcal{C}([0, 2\pi])$, for every x ,

$$((\alpha\beta)f)(x) = (\alpha\beta)f(x) = \alpha(\beta f(x)) = (\alpha(\beta f))(x).$$

Thus $(\alpha\beta)f = \alpha(\beta f)$.

- *Identity scalar:* For $1 \in \mathbb{R}$ and any f ,

$$(1 \cdot f)(x) = 1 \cdot f(x) = f(x),$$

so $1 \cdot f = f$.

- *Distributivity over vector addition:* For $f, g \in \mathcal{C}([0, 2\pi])$ and $\alpha \in \mathbb{R}$, for each x ,

$$\begin{aligned} (\alpha(f + g))(x) &= \alpha(f + g)(x) \\ &= \alpha(f(x) + g(x)) \\ &= \alpha f(x) + \alpha g(x) \\ &= (\alpha f + \alpha g)(x). \end{aligned}$$

Hence $\alpha(f + g) = \alpha f + \alpha g$.

- *Distributivity over field addition:* For $\alpha, \beta \in \mathbb{R}$ and $f \in \mathcal{C}([0, 2\pi])$, for every x ,

$$((\alpha + \beta)f)(x) = (\alpha + \beta)f(x) = \alpha f(x) + \beta f(x) = (\alpha f + \beta f)(x),$$

so $(\alpha + \beta)f = \alpha f + \beta f$.

All vector-space axioms are satisfied and closure was shown above, therefore $\mathcal{C}([0, 2\pi])$ is a vector space over \mathbb{R} . \square

Theorem 146. *Let*

$$\mathcal{C}([0, 2\pi]) = \{f: [0, 2\pi] \rightarrow \mathbb{R} \mid f \text{ continuous}\}.$$

Define for $f \in \mathcal{C}([0, 2\pi])$

$$\|f\|_{\infty} := \sup\{|f(x)| : x \in [0, 2\pi]\}.$$

Then $\|\cdot\|_{\infty}$ *is a norm on the vector space* $\mathcal{C}([0, 2\pi])$.

Problem 2

Proof. We must check the three norm axioms: (i) positive definiteness, (ii) absolute homogeneity, and (iii) the triangle inequality.

First note that each $f \in \mathcal{C}([0, 2\pi])$ is continuous on the compact interval $[0, 2\pi]$, so $|f|$ attains a maximum on $[0, 2\pi]$. Therefore the supremum defining $\|f\|_{\infty}$ is finite and in fact is a maximum:

$$\|f\|_{\infty} = \max_{x \in [0, 2\pi]} |f(x)| < \infty.$$

(i) *Positive definiteness:* For any f and any x , $|f(x)| \geq 0$, hence $\|f\|_{\infty} \geq 0$. If $\|f\|_{\infty} = 0$ then $\max_x |f(x)| = 0$, so $|f(x)| = 0$ for all x , which implies $f(x) = 0$ for all x , i.e., $f \equiv 0$. Conversely if $f \equiv 0$ then $\|f\|_{\infty} = 0$. Thus $\|f\|_{\infty} = 0 \iff f \equiv 0$.

(ii) *Absolute homogeneity (positive homogeneity):* Let $\alpha \in \mathbb{R}$ and $f \in \mathcal{C}([0, 2\pi])$. For every x ,

$$|(\alpha f)(x)| = |\alpha| |f(x)|.$$

Taking suprema over $x \in [0, 2\pi]$ yields

$$\|\alpha f\|_{\infty} = \sup_x |(\alpha f)(x)| = |\alpha| \sup_x |f(x)| = |\alpha| \|f\|_{\infty}.$$

(iii) *Triangle inequality:* For all x ,

$$|(f + g)(x)| = |f(x) + g(x)| \leq |f(x)| + |g(x)|.$$

Taking suprema over x gives

$$\begin{aligned}
 \|f + g\|_\infty &= \sup_x |f(x) + g(x)| \\
 &\leq \sup_x (|f(x)| + |g(x)|) \\
 &\leq \sup_x |f(x)| + \sup_x |g(x)| \\
 &= \|f\|_\infty + \|g\|_\infty,
 \end{aligned}$$

where we used that $\sup(A + B) \leq \sup A + \sup B$ for bounded real-valued functions.

Having verified positive definiteness, absolute homogeneity and the triangle inequality, $\|\cdot\|_\infty$ is a norm on $\mathcal{C}([0, 2\pi])$. \square

Theorem 147. *Let $\mathcal{C}([0, 2\pi])$ be the vector space of all continuous functions $f: [0, 2\pi] \rightarrow \mathbb{R}$. Consider the metric*

$$d_{\text{sup}}(f, g) := \|f - g\|_{\text{sup}}, \quad \text{where } \|h\|_{\text{sup}} = \sup_{x \in [0, 2\pi]} |h(x)|.$$

If $f(x) = \sin x$ and $g(x) = \cos x$, then

$$(a) \quad \|f\|_{\text{sup}} = 1,$$

$$(b) \quad \|g\|_{\text{sup}} = 1,$$

$$(c) \quad d_{\text{sup}}(f, g) = \sqrt{2}.$$

Problem 3

Proof. **(a)** By definition,

$$\|f\|_{\text{sup}} = \sup_{x \in [0, 2\pi]} |\sin x|.$$

Since $|\sin x| \leq 1$ for all x and equals 1 at $x = \pi/2$, we obtain $\|f\|_{\text{sup}} = 1$.

(b) Similarly,

$$\|g\|_{\text{sup}} = \sup_{x \in [0, 2\pi]} |\cos x| = 1,$$

because $|\cos x|$ attains its maximum value 1 at $x = 0$ and $x = 2\pi$.

(c) The distance is

$$d_{\sup}(f, g) = \|f - g\|_{\sup} = \sup_{x \in [0, 2\pi]} |\sin x - \cos x|.$$

Let $x \in [0, 2\pi]$. Using the trigonometric identity,

$$\sin x - \cos x = \sqrt{2} \sin\left(x - \frac{\pi}{4}\right),$$

we have

$$|\sin x - \cos x| = \sqrt{2} \left| \sin\left(x - \frac{\pi}{4}\right) \right| \leq \sqrt{2},$$

and equality occurs when $\sin\left(x - \frac{\pi}{4}\right) = \pm 1$, that is, for $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$. Hence

$$d_{\sup}(f, g) = \sqrt{2}.$$

Therefore, $\|f\|_{\sup} = 1$, $\|g\|_{\sup} = 1$, and $d_{\sup}(f, g) = \sqrt{2}$. □

Theorem 148. Let (f_n) be a sequence of functions on an interval I . Fix $x \in I$ and assume each f_n is continuous at x . Suppose $f_n \rightarrow f$ uniformly on I . If $(x_n) \subset I$ satisfies $x_n \rightarrow x$, then $f_n(x_n) \rightarrow f(x)$.

Proof. The function f is continuous at x : Let $\varepsilon > 0$. Pick N with $\sup_{t \in I} |f_N(t) - f(t)| < \varepsilon/3$. The continuity of f_N at x gives $\delta > 0$ such that whenever $t \in (x - \delta, x + \delta) \cap I$, we have $|f_N(t) - f_N(x)| < \varepsilon/3$. Therefore, for every $t \in (x - \delta, x + \delta) \cap I$, we have

$$\begin{aligned} |f(x) - f(t)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(t)| + |f_N(t) - f(t)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &= \varepsilon. \end{aligned}$$

Since f is continuous at x and $x_n \rightarrow x$, we have $f(x_n) \rightarrow f(x)$. Next, uniform convergence gives

$$\sup_{t \in I} |f_n(t) - f(t)| \xrightarrow{n \rightarrow \infty} 0,$$

so in particular $|f_n(x_n) - f(x_n)| \leq \sup_{t \in I} |f_n(t) - f(t)| \rightarrow 0$. Therefore

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \rightarrow 0 + 0 = 0,$$

and hence $f_n(x_n) \rightarrow f(x)$. □

Example 149. Let $I = [0, 1]$, $f_n(t) = t^n$, $x = 1$, and $x_n = 1 - \frac{1}{n}$. Each f_n is continuous at 1 and $x_n \rightarrow 1$, but $f_n(x_n) = \left(1 - \frac{1}{n}\right)^n \rightarrow e^{-1} \neq 1 = f(1)$, because f_n does not converge uniformly to f on $[0, 1]$.

Problem 4

Theorem 150. *There exists a sequence (f_n) of continuous functions on $[0, \infty)$ such that each f_n vanishes outside a bounded interval, $\int_0^\infty f_n(t) dt = 1$ for every n , and $f_n \rightarrow 0$ uniformly on $[0, \infty)$. Thus the theorem that uniform limits of Riemann-integrable functions can be integrated term-wise on a compact interval does not extend to improper integrals on $[0, \infty)$.*

Problem 5

Proof. Define for $n \in \mathbb{N}$

$$f_n(x) = \frac{1}{n} \max\left(0, 1 - \frac{|x - n|}{n}\right), \quad x \in [0, \infty).$$

Each f_n is continuous, supported in $[0, 2n]$, and attains its maximum value $1/n$ at $x = n$.

Since f_n is a triangle with base $2n$ and height $1/n$, its integral is

$$\int_0^\infty f_n(t) dt = \frac{1}{2} \cdot (2n) \cdot \frac{1}{n} = 1,$$

for all n .

The uniform norm of f_n is $\|f_n\|_\infty = 1/n \rightarrow 0$, hence $f_n \rightarrow 0$ uniformly on $[0, \infty)$.

Therefore the sequence has the required properties: each f_n vanishes outside a bounded interval, $\int_0^\infty f_n = 1$, and $f_n \rightarrow 0$ uniformly, so the limit of the integrals (which equals 1) does not equal the integral of the uniform limit (which equals 0). \square

Theorem 151. *Let*

$$f_n(x) = \sqrt{x^2 + \frac{1}{n}}, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}.$$

Then:

(a) f_n converges pointwise to $f(x) = |x|$ on \mathbb{R} .

(b) $f_n \rightarrow f$ uniformly on \mathbb{R} ; in fact

$$\sup_{x \in \mathbb{R}} |f_n(x) - f(x)| = \frac{1}{\sqrt{n}} \xrightarrow{n \rightarrow \infty} 0.$$

(c) The limit $f(x) = |x|$ is not differentiable at $x = 0$ (though it is differentiable for $x \neq 0$). Each f_n is C^∞ and

$$f'_n(x) = \frac{x}{\sqrt{x^2 + \frac{1}{n}}}.$$

The derivatives f'_n converge pointwise on \mathbb{R} to the function

$$g(x) = \begin{cases} \operatorname{sgn}(x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

but the convergence $f'_n \rightarrow g$ is not uniform (and g is discontinuous at 0).

Problem 6

Proof. We prove the three parts in order.

(a) Fix $x \in \mathbb{R}$. Since $\frac{1}{n} \rightarrow 0$, we have

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \sqrt{x^2 + \frac{1}{n}} = \sqrt{x^2} = |x|.$$

Thus the pointwise limit is $f(x) = |x|$.

(b) We show $\sup_{x \in \mathbb{R}} |f_n(x) - f(x)| \rightarrow 0$.

For any $x \in \mathbb{R}$,

$$f_n(x) - f(x) = \sqrt{x^2 + \frac{1}{n}} - |x| \geq 0,$$

so $|f_n(x) - f(x)| = f_n(x) - f(x)$. Rationalize the difference:

$$f_n(x) - f(x) = \frac{\left(\sqrt{x^2 + \frac{1}{n}} - |x|\right) \left(\sqrt{x^2 + \frac{1}{n}} + |x|\right)}{\sqrt{x^2 + \frac{1}{n}} + |x|} = \frac{1/n}{\sqrt{x^2 + \frac{1}{n}} + |x|}.$$

Hence

$$0 \leq f_n(x) - f(x) \leq \frac{1/n}{\sqrt{x^2 + \frac{1}{n}}} = \frac{1}{\sqrt{n} \sqrt{1 + nx^2}} \leq \frac{1}{\sqrt{n}}.$$

The upper bound $1/\sqrt{n}$ is attained at $x = 0$, since

$$f_n(0) - f(0) = \sqrt{\frac{1}{n}} - 0 = \frac{1}{\sqrt{n}}.$$

Therefore

$$\sup_{x \in \mathbb{R}} |f_n(x) - f(x)| = \frac{1}{\sqrt{n}} \xrightarrow{n \rightarrow \infty} 0,$$

which proves uniform convergence $f_n \rightarrow f$.

(c) Each f_n is C^∞ on \mathbb{R} (composition of smooth functions). Differentiate:

$$f'_n(x) = \frac{1}{2} \left(x^2 + \frac{1}{n}\right)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2 + \frac{1}{n}}}.$$

For $x \neq 0$,

$$\lim_{n \rightarrow \infty} f'_n(x) = \lim_{n \rightarrow \infty} \frac{x}{\sqrt{x^2 + \frac{1}{n}}} = \frac{x}{|x|} = \text{sgn}(x).$$

At $x = 0$ we have $f'_n(0) = 0$ for every n , so the pointwise limit of f'_n is the function

$$g(x) = \begin{cases} \text{sgn}(x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

The limit function $f(x) = |x|$ is differentiable for $x \neq 0$ with derivative $f'(x) = \text{sgn}(x)$, but it is not differentiable at $x = 0$ because the left and right derivatives at 0 are -1 and $+1$, respectively:

$$\lim_{h \rightarrow 0^-} \frac{|h| - |0|}{h} = \lim_{h \rightarrow 0^-} \frac{-h}{h} = -1, \quad \lim_{h \rightarrow 0^+} \frac{|h| - |0|}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = 1.$$

Since these one-sided limits differ, f is not differentiable at 0.

Finally, the pointwise limit g of the derivatives is discontinuous at 0 (left limit -1 , right limit $+1$, but $g(0) = 0$), so convergence of f'_n to g cannot be uniform: a uniform limit of continuous functions must be continuous. We give a concrete lower bound showing non-uniformity. Set

$$x_n = \frac{1}{\sqrt{n}} > 0.$$

Then

$$f'_n(x_n) = \frac{1/\sqrt{n}}{\sqrt{1/n + 1/n}} = \frac{1/\sqrt{n}}{\sqrt{2/n}} = \frac{1}{\sqrt{2}},$$

while $g(x_n) = \text{sgn}(x_n) = 1$. Hence for every n ,

$$\sup_{x \in \mathbb{R}} |f'_n(x) - g(x)| \geq |f'_n(x_n) - g(x_n)| = 1 - \frac{1}{\sqrt{2}} \approx 0.2929 \dots,$$

so the supremum does not tend to 0. Therefore f'_n does not converge to g uniformly.

This completes the proof of all claims. □