Maps Between Non-compact Surfaces

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by

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Declaration

I hereby declare that the work reported in this thesis is entirely original and has been carried out by me under the supervision of Prof. Siddhartha Gadgil at the Department of Mathematics, Indian Institute of Science, Bangalore. I further declare that this work has not been the basis for the award of any degree, diploma, fellowship, associateship, or similar title of any University or Institution.

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Prof. Siddhartha Gadgil
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To my parents,

thank you for your endless support ...

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Abstract

This thesis focuses on studying proper maps between two non-compact surfaces, with a particular emphasis on the topological rigidity and the Hopfian property.

Topological rigidity is the property that every homotopy equivalence between two closed n-manifolds is homotopic to a homeomorphism. This property refines the notion of homotopy equivalence, implying homeomorphism for a particular class of spaces. According to Nielsen's results from the 1920s, compact surfaces exhibit topological rigidity. However, topological rigidity fails in dimensions three and above, as well as for compact bordered surfaces.

We prove that all non-compact surfaces are properly rigid. In fact, we prove a stronger result: if a homotopy equivalence between any two non-compact surfaces is a proper map, then it is properly homotopic to a homeomorphism, provided that the surfaces are neither the plane nor the punctured plane. As an application, we also prove that any π_1-injective proper map between two non-compact surfaces is properly homotopic to a finite-sheeted covering map, given that the surfaces are neither the plane nor the punctured plane.

An oriented manifold M is said to be Hopfian if every self-map $f\colon M \to M$ of degree one is a homotopy equivalence. This is the natural topological analogue of Hopfian groups. H. Hopf questioned whether every closed, oriented manifold is Hopfian. We prove that every oriented infinite-type surface is non-Hopfian. Consequently, an oriented surface S is of finite type if and only if every proper self-map of S of degree one is homotopic to a homeomorphism.

Table of contents

List of figures

Chapter 1

Introduction

This thesis focuses on the study of proper maps between two non-compact surfaces, with a particular emphasis on facts stemming from two fundamental questions in topology: whether every homotopy equivalence between two n-manifolds is homotopic to a homeomorphism, and whether every degree-one self-map of an oriented manifold is a homotopy equivalence.

The mapping class group of a surface is defined as the group of homeomorphisms of the surface up to isotopy. It was pioneered by Dehn and Nielsen in the 20th century and extensively developed by Thurston in the 1970s for surfaces of finite type. Conversely, the mapping class group of an infinite-type surface is known as the big mapping class group, and the study of big mapping class groups is a rapidly expanding field. Most of the results to date have focused on exploring the algebraic, topological, and geometric properties of big mapping class groups, alongside comparing them to mapping class groups of finite-type surfaces in terms of their similarities and differences. However, the proper homotopy classification of proper maps of non-zero degree between oriented infinite-type surfaces remains relatively underexplored despite the thorough homotopy classification of non-zero degree maps between closed, oriented surfaces by Dehn-Nielsen [25, 82], Edmonds-Skora [26, 99], and Gabai-Kazez [40]. This thesis aims to contribute a small step toward comprehensively classifying all proper maps between two infinite-type surfaces upto proper homotopy.

The first part of this thesis aims to identify proper homotopy-theoretical conditions that lead to conclusions of a homeomorphism-type. In this direction, there exists a rich collection of works contributed by Siebenmann [98, 97], Edwards [27], and Brown-Tucker [12]. The main theorem of this part states that if a homotopy equivalence between any two infinite-type surfaces is a proper map, then it is properly homotopic to a homeomorphism, i.e., all infinite-type surfaces are topologically rigid in a strong sense; for details, refer to Theorem I. The significant implication of the strong topological rigidity is that any π_1-injective proper map between two infinite-type surfaces is properly homotopic to a finite-sheeted covering map; see Theorem 2.8.1.1. Analog theorems for closed surfaces, closed Haken 3-manifolds, and end-irreducible 3-manifolds are well-established results credited to Nielsen [82], Waldhausen [107], and Brown-Tucker [12], respectively. Consequently, every infinite-type surface exhibits proper rigidity, meaning any proper homotopy equivalence between two infinite-type surfaces is properly homotopic to a homeomorphism; see Theorem 2.8.1.15. It's important to note that all these (proper) rigidity phenomena fall under the (proper) Borel Conjecture, a highly celebrated conjecture in high-dimensional topology. See Section 2.1 for further details.

An additional application of strong topological rigidity is worth mentioning, although its proof is not included in this thesis. In the field of mapping class groups for finite-type surfaces, several pivotal theorems—such as the Dehn-Nielsen-Baer theorem [25, 82, 6, 73], Kerckhoff's Nielsen Realization Theorem [65], the Nielsen-Thurston classification [83, 102], Ivanov's theorem on automorphisms of the curve graph [63], and Masur-Minsky's hyperbolicity of the curve graph [75]—have significantly advanced our understanding of small mapping class groups and their properties. Many of these theorems have also been extended to infinite-type surfaces; for instance, refer to [1], [10], and [56, 9] for analogues of the second, third, and fourth theorems, respectively. However, the analogue of the Dehn-Nielsen-Baer theorem for infinite-type surfaces remained undiscovered until recently. In a recent paper, [22], we showed that the Goldman bracket characterizes the image of the Dehn-Nielsen-Baer map for an infinite-type surface as an application of strong topological rigidity. For further discussion, refer to Section 2.8.2.

The concluding part of this thesis delves into a characterization of oriented surfaces of finite type. Around 1951, H. Hopf posed a long-standing question and gave rise to the concept of Hopfian and non-Hopfian groups [80]: Given a closed, oriented manifold M, is every self-map $f\colon M \to M$ of degree ± 1 a homotopy equivalence? Affirmative answers have been established in specific cases, detailed in Section 3.1. Notably, as of now, no counterexample to this problem has been identified. However, upon extending this question to non-compact manifolds, counterexamples arise. The main theorem of this part asserts that every infinite-type oriented surface admits a non π_1-injective proper map of degree one; see Theorem II. This counterexample primarily arises due to the additional feature of non-compact manifolds in contrast to compact ones, specifically the space of ends. Consequently, an oriented surface S is of finite type if and only if every proper self-map of S of degree one is homotopic to a homeomorphism.

Publications

The content of Chapter 2, except Section 2.8.1, has been accepted for publication in Algebraic & Geometric Topology [20], and the content of Chapter 3 has been published in Comptes Rendus Mathématique [21].

The remainder of this chapter is dedicated to gathering various results, providing readers with enough information to access the content of subsequent chapters.

1.1 Conventions

All manifolds will be assumed to be second countable and Hausdorff. A *bordered surface* (resp. *surface*) is a connected, orientable two-dimensional manifold with a non-empty (resp. an empty) boundary. For integers $g \geq 0,\ b \geq 0, p \geq 0$, denote the connected, orientable 2-manifold of genus g with b boundary components by $S_{g,b}$; and let $S_{g,b,p}$ be the 2-manifold after removing p points from $\mathrm{int}(S_{g,b})$. Note that for a manifold M, we use $\mathrm{int}(M)$ to denote the interior of M. Sometimes, $S_{0,1}$, $S_{0,2}$, $S_{0,3}$, $S_{1,2}$, and $S_{0,1,1}$ will be called a disk, an annulus, a pair of pants, a torus with two disks removed, and a punctured disk, respectively.

We say a connected 2-manifold with or without boundary is of *infinite-type* if its fundamental group is not finitely generated; otherwise, we say it is of *finite-type*.

1.2 Simple closed curves on 2-manifolds

Definition 1.2.1 Let $\mathbf{S}$ be a connected, orientable two-dimensional manifold with or without boundary. A *circle* (resp. *smoothly embedded circle*) on $\mathbf{S}$ is the image of an embedding (resp. a smooth embedding) of $\mathbb{S}^1$ into $\mathbf{S}$. We say a circle $\mathcal{C}$ on $\mathbf{S}$ is a *trivial circle* if there is an embedded disk $\mathcal{D}$ in $\mathbf{S}$ such that $\partial\mathcal{D} = \mathcal{C}$; and, we say a circle $\mathcal{C}$ on $\mathbf{S}$ is a *primitive circle* if it is not a trivial circle.

The following theorem justifies naming a non-disk bounding circle as a primitive circle: a primitive circle represents a primitive element of the fundamental group. Recall that an element g of a group G is *primitive* if there does not exist any $h \in G$ so that $g = h^k$, where $|k| > 1$.

Theorem 1.2.2 [28, Theorems 1.7. and 4.2.] Let $\mathbf{S}$ be a connected, orientable two-dimensional manifold with or without boundary. Let $\mathcal{C}$ be a primitive circle on $\mathbf{S}$, and let $f\colon \mathbb{S}^1 \hookrightarrow \mathbf{S}$ be an embedding with $f(\mathbb{S}^1) = \mathcal{C}$. Then $[f] \in \pi_1(\mathbf{S})$ is a primitive element. In particular, $[f]$ is a non-trivial element of $\pi_1(\mathbf{S})$.

Recall that for a path-connected space X, the natural map from the set of all conjugacy classes of $\pi_1(X, *)$ to the set of all free homotopy classes of maps $\mathbb{S}^1 \to X$ is a bijection [51, Exercise 6 of Section 1.1]. The next theorem says that two pairwise disjoint freely homotopic primitive circles on a two-manifold co-bound an annulus.

Theorem 1.2.3 [28, Lemma 2.4.] Let $\mathbf{S}$ be a connected, orientable two-dimensional manifold with or without boundary, and let $\ell_0, \ell_1\colon \mathbb{S}^1 \hookrightarrow \mathbf{S}$ be two embeddings such that $\ell_0(\mathbb{S}^1)$ is a smoothly embedded submanifold of $\mathbf{S}$ and $\ell_0(\mathbb{S}^1) \cap \ell_1(\mathbb{S}^1) = \varnothing$. If ℓ_0 and ℓ_1 represent the same non-trivial conjugacy class in $\pi_1(\mathbf{S}, *)$, then there is a embedding $\mathcal{L}\colon \mathbb{S}^1 \times [0,1] \hookrightarrow \mathbf{S}$ so that $\mathcal{L}(-,0) = \ell_0$ and $\mathcal{L}(-,1) = \ell_1$.

1.3 Goldman's inductive procedure of constructing all non-compact surfaces

A non-compact surface Σ_{std} is said to be in *standard form* if it is built up from four building blocks, $S_{0,1}$, $S_{0,2}$, $S_{0,3}$, and $S_{1,2}$, in the following inductive manner: Start with $S_{0,1}$. Suppose the i-th step of the induction has already been done. Let K_i be the compact bordered subsurface of Σ_{std} after the i-th step of induction. In particular, $K_1 \cong S_{0,1}$. Now, to obtain K_{i+1} from K_i, consider one of the last three building blocks, say $\mathcal{S}$ (i.e., $\mathcal{S}$ is homeomorphic to either $S_{0,2}$, $S_{0,3}$, or $S_{1,2}$); finally, suitably identify one boundary circle of $\mathcal{S}$ with a boundary circle of K_i (see Figure 1.3.1).

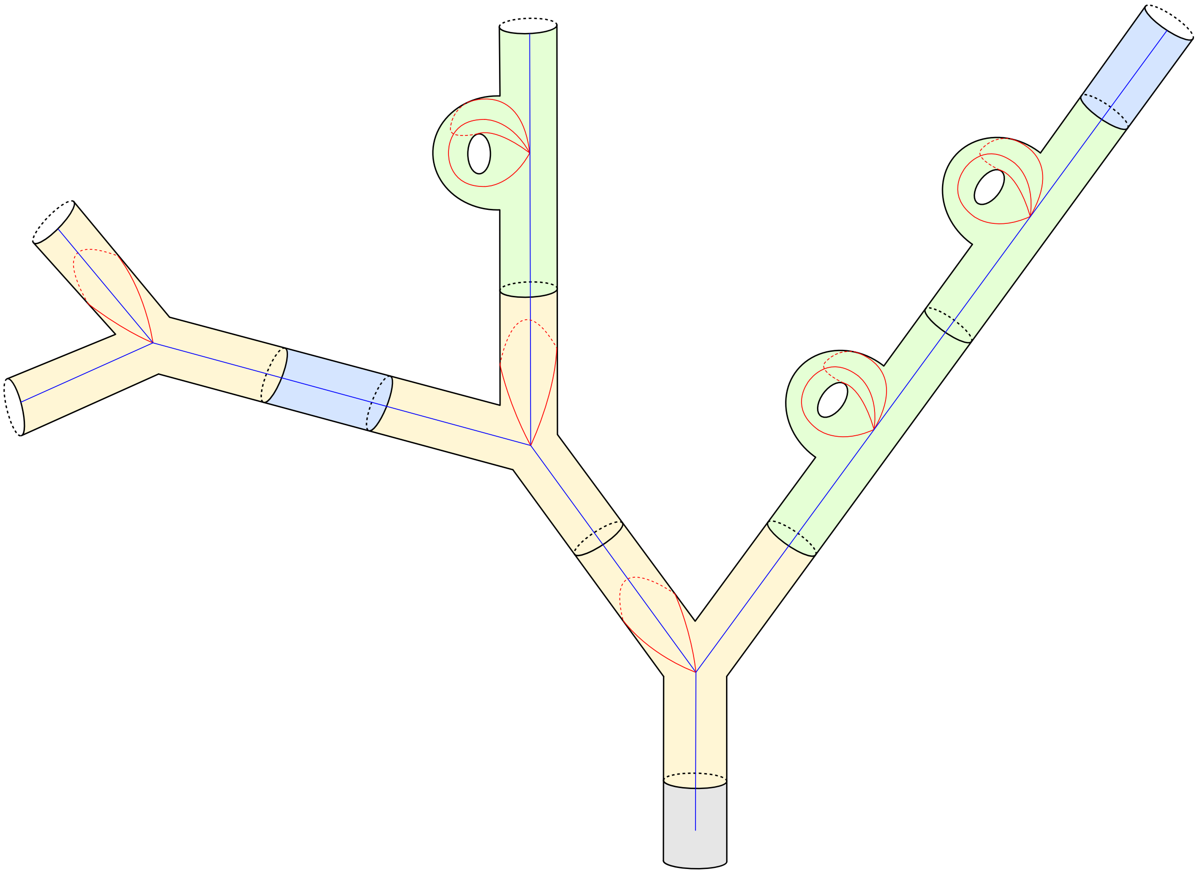

Fig. 1.3.1 The inductive construction of any non-compact surface Σ and its spine uses copies of four compact bordered surfaces: disk, annulus, pair of pants, and torus with two disks removed.

Theorem 1.3.1 [45, Section 2.6.] and [105, page 173] Let Σ be a non-compact surface. Then Σ is homeomorphic to a non-compact surface Σ_{std} in standard form. Thus, every non-compact surface is homeomorphic to a non-compact surface constructed using an inductive procedure, though two non-compact surfaces obtained from two different inductive procedures may be homeomorphic.

Theorem 1.3.2 [45, Section 2.6. and Section 7.3.] [95, Lemma 3.2.2] The graph in Figure 1.3.1 consisting of blue straight line segments and red circles is a deformation retract of the non-compact surface Σ. Thus, Σ is homotopy equivalent to the wedge of at most countably many circles. In particular, $\pi_1(\Sigma)$ is free.

1.4 Ends of spaces

Let X be a connected, separable, locally compact, locally connected Hausdorff ANR (absolute neighbourhood retract) space. For example, X can be any connected topological manifold. We say X admits an *efficient exhaustion by compacta* if there is a nested sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact, connected subsets of X such that $\cup_i K_i = X$, $K_i \subseteq \mathrm{int}(K_{i+1})$ for each i, $\cap_i (X \setminus K_i) = \varnothing$, and the closure of each component of any $X \setminus K_i$ is non-compact. For the existence of efficient exhaustion of X by compacta, see [49, Exercise 3.3.4].

Let $\mathrm{Ends}(X)$ be the set of all sequences $(V_1, V_2, ...)$, where V_i is a component of $X \setminus K_i$ and $V_1 \supseteq V_2 \supseteq \cdots$. Give $X^\dagger := X \cup \mathrm{Ends}(X)$ with the topology generated by the basis consisting of all open subsets of X, and all sets $V_i^\dagger$, where

$$V_i^\dagger := V_i \cup \{(V_1', V_2', ...) \in \mathrm{Ends}(X) | V_i' = V_i\}.$$

Then $X^\dagger$ is separable, compact, and metrizable such that X is an open dense subset of $X^\dagger$; it is known as the *Freudenthal compactification* of X (recall that we say a space $X_{\mathbf{c}}$ is a *compactification* of X if $X_{\mathbf{c}}$ is compact Hausdorff space, and X is a dense subset of $X_{\mathbf{c}}$). The subspace $\mathrm{Ends}(X)$ of $X^\dagger$ is a totally-disconnected space; hence $\mathrm{Ends}(X)$ is a closed subset of the Cantor set.

The Freudenthal compactification *dominates* any other compactification in the following sense: If $\widetilde{X}$ is a compactification of X such that $\widetilde{X} \setminus X$ is totally-disconnected, then there exists a map $f \colon X^\dagger \to \widetilde{X}$ extending Id_X. Moreover, the Freudenthal compactification is *unique* in the following sense: If $X^{\dagger\dagger}$ is a compactification of X such that $X^{\dagger\dagger} \setminus X$ is totally-disconnected and $X^{\dagger\dagger}$ dominates any other compactification, then there exists a homeomorphism $X^{\dagger\dagger} \to X^\dagger$ extending Id_X; see [46, Theorem 3.1]. Thus, the definition of $\mathrm{Ends}(X)$ is independent of the choice of efficient exhaustion of X by compacta.

1.5 Kerékjártó's classification theorem and Ian Richard's representation theorem

Let Σ be a non-compact surface with an efficient exhaustion $\{K_i\}_1^\infty$. Let $e := (V_1, V_2, ...) \in \mathrm{Ends}(\Sigma)$ be an end, where V_i is a component of $X \setminus K_i$. We say e is a *planar end* if V_i is embeddable in $\mathbb{R}^2$ for some positive integer i. An end is said to be *non-planar* if it is not planar. Denote the subspace of $\mathrm{Ends}(\Sigma)$ consisting of all planar (resp. non-planar) ends by $\mathrm{Ends}_{\mathrm{p}}(\Sigma)$ (resp. $\mathrm{Ends}_{\mathrm{np}}(\Sigma)$). Note that $\mathrm{Ends}_{\mathrm{p}}(\Sigma)$ is an open subset of $\mathrm{Ends}(\Sigma)$. Define the *genus* of Σ as $g(\Sigma) := \sup g(\mathcal{S})$, where $\mathcal{S}$ is a compact bordered subsurface of Σ. Therefore, the genus counts the number of handles of a surface, i.e., the number of embedded copies of $S_{1,1}$ in a surface, which may be any non-negative integer or countably infinite.

Theorem 1.5.1 (Kerékjártó's classification of non-compact surfaces [93, Theorem 1]) Let Σ and Σ' be non-compact surfaces of genus g, g', respectively. Then Σ is homeomorphic to Σ' if and only if $g = g'$ and there is homeomorphism $\varphi \colon \mathrm{Ends}(\Sigma) \to \mathrm{Ends}(\Sigma')$ with $\varphi(\mathrm{Ends}_{\mathrm{np}}(\Sigma)) = \mathrm{Ends}_{\mathrm{np}}(\Sigma')$.

Theorem 1.5.2 (Realization of ends and representation of a non-compact surface [93, Theorems 2, 3]) Let $\mathscr{E}_{\mathrm{np}} \subseteq \mathscr{E}$ be two closed totally-disconnected subsets of $\mathbb{S}^1$, and let $\mathscr{G}$ be an at most countable set with the following properties: $\mathscr{E} \neq \varnothing$, and $\mathscr{E}_{\mathrm{np}} \neq \varnothing$ if and only if $\mathscr{G}$ is infinite. Define $\mathbb{D} := \{z \in \mathbb{C} : 0 \leq |z| \leq 1\}$. Then there exists a pairwise disjoint collection $\{\mathcal{D}_i : i \in \mathscr{G}\}$ of disks in $\mathrm{int}(\mathbb{D})$ such that a point $p \in \mathbb{D}$ is an element of $\mathscr{E}_{\mathrm{np}}$ if and only if every neighbourhood of p in $\mathbb{D}$ contains infinitely many elements of $\{\mathcal{D}_i : i \in \mathscr{G}\}$. Moreover, $\mathbf{S} := (\mathbb{D} \setminus \mathscr{E}) \setminus \cup_{i \in \mathscr{G}} \mathrm{int}(\mathcal{D}_i)$ is a non-compact bordered surface, and

$$D\mathbf{S} := \frac{(\mathbf{S} \times 0) \sqcup (\mathbf{S} \times 1)}{(p,0) \sim (p,1),\ p \in \partial \mathbf{S}}$$

is a $|\mathscr{G}|$-genus non-compact surface with $\mathrm{Ends}(D\mathbf{S}) \cong \mathscr{E}$ and $\mathrm{Ends}_{\mathrm{np}}(D\mathbf{S}) \cong \mathscr{E}_{\mathrm{np}}$. Thus, given any non-compact surface Σ, in this procedure, if we assume $\mathscr{E}_{\mathrm{np}} \subseteq \mathscr{E}$ is homeomorphic to the pair $\mathrm{Ends}_{\mathrm{np}}(\Sigma) \subseteq \mathrm{Ends}(\Sigma)$, and $|\mathscr{G}|$ is equal to $g(\Sigma)$; then $D\mathbf{S} \cong \Sigma$, by Theorem 1.5.1.

Pick an orientation for Σ and write Σ as the double of a bordered surface $\mathbf{S}$ using Theorem 1.5.2. Then the homeomorphism $\varphi \colon \Sigma \to \Sigma$ defined by sending $[p,t] \in \Sigma$ to $[p, 1-t] \in \Sigma$ for all $(p,t) \in \mathbf{S} \times \{0,1\}$ is orientation-reversing. Thus, we have the following.

Corollary 1.5.3 (Non-compact oriented surfaces are amphichiral) Every non-compact oriented surface admits an orientation-reversing homeomorphism.

Remark 1.5.4 The classification of non-compact bordered surfaces is also possible: When the boundary is compact, it follows from Theorem 1.5.1 together with [103, Proposition A.3.]. When each boundary component is compact, this follows from [7] (based on the classification of their interiors) or [103, Theorem A.7] (based on the classification of non-compact surfaces obtained from gluing a disk along each boundary component). For arbitrary boundary, see [14, Theorem 2.2.].

1.6 Proper maps

Recall that a map from a space X to a space Y is called a *proper map* if the inverse image of each compact subset of Y is a compact subset of X. The following theorem is crucial for assessing the properness of a map. For that, define the notion of divergent sequence in a space as follows: if X is a topological space, a sequence $\{x_n\}$ in X is said to *diverge to infinity* if for every compact set $K \subseteq X$ there are at most finitely many values of n for which $x_n \in K$.

Theorem 1.6.1 [69, Propositions 4.92. and 4.93.(b)] Suppose $f \colon X \to Y$ is map from a second countable Hausdorff space X to a space Y. Then f is a proper map if and only if f takes sequences diverging to infinity in X to sequences diverging to infinity in Y.

The following theorem tells that a proper map is a closed map when the co-domain is a topological manifold or a CW-complex or, more generally, a compactly generated Hausdorff space. Recall that a topological space Y is said to be *compactly generated* if it has the following property: if B is any subset of Y whose intersection with each compact subset $K \subseteq Y$ is closed in K, then B is closed in Y.

Theorem 1.6.2 [69, Theorem 4.95] [87] A proper map from a space to a compactly generated Hausdorff space is a closed map.

The *proper category* is the category whose objects are topological spaces and whose morphisms are proper maps. The analogues of homotopy, homotopy equivalence, etc., in the proper category, can be defined in the following way. If a homotopy $\mathcal{H}\colon X \times [0,1] \to Y$ is a proper map, then we call $\mathcal{H}$ a *proper homotopy*. Two proper maps from X to Y are said to be *properly homotopic* if there is a proper homotopy between them. It is worth noting that homotopy through proper maps is a weaker notion than proper homotopy. For example, consider $H\colon \mathbb{C} \times [0,1] \to \mathbb{C}$ given by $H(z,t) := tz^2 - z$. Being a polynomial, each $H(-,t)$ is proper. But, H itself is not proper as $H\left(n, \frac{1}{n}\right) = 0$ for all integers $n \geq 1$. However, a homotopy through homeomorphisms of a manifold is proper homotopy. More generally,

Theorem 1.6.3 [18, Theorem 1.2] Let X be a locally compact Hausdorff space. Suppose $\mathcal{H}\colon X \times [0,1] \to X$ is homotopy through homeomorphisms, i.e., $\mathcal{H}(-,t)\colon X \to X$ is a homeomorphism for each $t \in [0,1]$. Then $X \times [0,1] \ni (x,t) \longmapsto \big(\mathcal{H}(x,t), t\big) \in X \times [0,1]$ is a homeomorphism, and thus $\mathcal{H}$ is a proper map.

We say that a proper map $f\colon X \to Y$ is a *proper homotopy equivalence* if there exists a proper map $g\colon Y \to X$ such that both $g \circ f$ and $f \circ g$ are properly homotopic to the identity maps (when such a g exists, we say that g is a *proper homotopy inverse* of f.). Two spaces X and Y are said to have the same *proper homotopy type* if there is a proper homotopy equivalence between them.

We conclude this section by considering the following relationship between Ends and proper maps. For further information about the ends of spaces and proper homotopy, interested readers can refer to [88] and [62].

Proposition 1.6.4 [49, Proposition 3.3.12] Let X and Y be two connected, separable, locally compact, locally connected Hausdorff ANRs. Then we have the following:

(1) Every proper map $f\colon X \to Y$ induces a map $\mathrm{Ends}(f)\colon \mathrm{Ends}(X) \to \mathrm{Ends}(Y)$ that can be used to extend $f\colon X \to Y$ to a map $f^\dagger\colon X^\dagger \to Y^\dagger$ between the Freudenthal compactifications.

(2) If two proper maps $f_0, f_1\colon X \to Y$ are properly homotopic, then $\mathrm{Ends}(f_0) = \mathrm{Ends}(f_1)$.

(3) Ends is a functor in the following sense: the induced map of the identity is the identity, and the induced map of a composition of two proper maps is the composition of their induced maps.

(4) If $f\colon X \to Y$ is a proper homotopy equivalence, then $\mathrm{Ends}(f)\colon \mathrm{Ends}(X) \to \mathrm{Ends}(Y)$ is a homeomorphism.

1.7 The degree of a proper map

We use singular cohomology with compact support to define the notion of the degree of a proper map. Recall that for a topological manifold X, the r-th singular cohomology with compact support $H^r_{\mathbf{c}}(X, \partial X; \mathbb{Z})$ is equal to the direct limit $\varinjlim H^r\big(X, \partial X \cup (X \setminus K); \mathbb{Z}\big)$, where K is a compact subset of X and the various maps defining this direct system are induced by inclusions. Hence, for a compact subset K of X, the definition of direct limit yields an *obvious map* $H^r\big(X, \partial X \cup (X \setminus K); \mathbb{Z}\big) \to H^r_{\mathbf{c}}(X, \partial X; \mathbb{Z})$. It is worth noting that when X is compact topological manifold, $H^r_{\mathbf{c}}(X, \partial X; \mathbb{Z}) = H^r(X, \partial X; \mathbb{Z})$ for all r.

Let X and Y be two topological manifolds. If $f\colon X \to Y$ is a proper map with $f(\partial X) \subseteq \partial Y$, then for each r, f induces a map $H^r_{\mathbf{c}}(f)\colon H^r_{\mathbf{c}}(Y, \partial Y; \mathbb{Z}) \to H^r_{\mathbf{c}}(X, \partial X; \mathbb{Z})$ so that $H^r_{\mathbf{c}}$ becomes a functor in the following sense: the induced map of the identity is the identity, and the induced map of a (well-defined) composition of two proper maps (each of which sends boundary into boundary) is the composition of their induced maps. Moreover, if $\mathcal{H}\colon X \times [0,1] \to Y$ is a proper homotopy such that $\mathcal{H}(\partial X, t) \subseteq \partial Y$ for each $t \in [0,1]$, then $H^r_{\mathbf{c}}\big(\mathcal{H}(-,0)\big) = H^r_{\mathbf{c}}\big(\mathcal{H}(-,1)\big)$ for all r. For more details, see [100, pages 320, 322, 323, 339, 341].

Let M be a connected, orientable, topological n-manifold. Then $H^n_{\mathbf{c}}(M, \partial M; \mathbb{Z})$ is an infinite cyclic group; see [100, page 342]. If we choose an orientation of M (i.e., M is oriented), then there exists a unique element $[M] \in H^n_{\mathbf{c}}(M, \partial M; \mathbb{Z})$ such that the following hold:

- $[M]$ generates $H^n_{\mathbf{c}}(M, \partial M; \mathbb{Z})$, and
- for each $x \in M \setminus \partial M$, the unique generator of $H^n(M, M \setminus x; \mathbb{Z})$, which comes from the chosen orientation of M, is sent to $[M]$ by the isomorphism $H^n(M, M \setminus x; \mathbb{Z}) \to H^n_{\mathbf{c}}(M, \partial M; \mathbb{Z})$; see [29, Proof of Lemma 2.1].

Thus, if $f\colon M \to N$ is a proper map between two connected, oriented, topological n-manifolds with $f(\partial M) \subseteq \partial N$, then the *(compactly supported cohomological) degree* of f is the unique integer $\deg(f)$ defined as follows: $H^n_{\mathbf{c}}(f)\big([N]\big) = \deg(f) \cdot [M]$. Therefore, we have the following:

1. If manifolds are compact, then the notion of compactly supported cohomological degree agrees with the notion of the usual degree defined by singular cohomology.
2. The degree is proper homotopy invariant: If $f, g\colon M \to N$ are proper maps between two connected, oriented, topological n-manifolds with $f(\partial M) \cup g(\partial M) \subseteq \partial N$ such that there is a proper homotopy $\mathcal{H}\colon M \times [0,1] \to N$ with $\mathcal{H}\big(\partial M \times [0,1]\big) \subseteq \partial N$ from f to g, then $\deg(f) = \deg(g)$.
3. The degree is multiplicative: The degree of the (well-defined) composition of two proper maps (each of which sends boundary into boundary) is the product of their degrees.

Therefore, the degree of a proper homotopy equivalence between two oriented, connected, boundaryless n-manifolds is ± 1 due to 2. and 3. above.

We use the following well-known characterizations of the degree of a map. In the below two theorems, D is a disk in a smooth n-manifold X means D is the image of $\{z \in \mathbb{R}^n : |z| \leq 1\}$ under a smooth embedding $\{z \in \mathbb{R}^n : |z| \leq 2\} \hookrightarrow X$.

Theorem 1.7.1 [29, Lemma 2.1b.] Let $f\colon M \to N$ be a proper map between two connected, oriented, smooth manifolds of the same dimension such that $f^{-1}(\partial N) = \partial M$. Suppose there exists a disk D in $\mathrm{int}(N)$ with the property that $f^{-1}(D)$ is the pairwise disjoint union of disks $D_1, \ldots, D_k$ in $\mathrm{int}(M)$ (k must be a non-negative integer as f is proper) such that f maps each $f^{-1}(D_i)$ homeomorphically onto D. For each i, let ε_i be $+1$ or -1 according as the homeomorphism $f|f^{-1}(D_i) \to D$ is orientation-preserving or orientation-reversing. Then $\deg(f) = \sum_{i=1}^{k} \varepsilon_i$. In particular, if $f^{-1}(D) = \varnothing$, then $\deg(f) = 0$.

The proof of Theorem 1.7.1 is almost the same as that of proof of its analogue statement in the compact setting, and it can be used to calculate the degree of a proper map. For example, Theorem 1.7.1 can be used to show that if p is a d-fold covering map between two connected, oriented, topological manifolds of the same dimension, where d is a positive integer, then $\deg(p) = \pm d$. Notice also that Theorem 1.7.1 specifically tells that if the degree of a proper map f is ± 1, there exists a nice disk in the codomain, which is homeomorphic to its own inverse image via f. The following theorem, due to Hopf [58, 59] and Epstein [29], asserts that for a map of degree ± 1, it is possible to attain such a nice disk after a proper homotopy.

Theorem 1.7.2 [29, Theorems 3.1 and 4.1] Let $f\colon M \to N$ be a proper map between two connected, oriented, smooth manifolds of the same dimension such that $f^{-1}(\partial N) \subseteq \partial M$. Let $\ell := |\deg(f)|$. Then there is a proper map $g\colon M \to N$ with $g(\partial M) \subseteq \partial N$ and a homotopy $\mathcal{H}\colon M \times [0,1] \to N$ from f to g with the following properties:

- There exists a compact subset $K \subseteq \mathrm{int}(M)$ such that $\mathcal{H}(x,t) = f(x)$ for all $(x,t) \in (M \setminus K) \times [0,1]$. In particular, $\mathcal{H}$ is a proper homotopy relative to ∂M.
- There exists a disk $D \subseteq \mathrm{int}(N)$ such that $g^{-1}(D)$ is the pairwise disjoint union of disks $D_1, \ldots, D_\ell$ in $\mathrm{int}(M)$ and g maps each $g^{-1}(D_i)$ homeomorphically onto D.

Therefore, if the degree of g, which is the same as the degree of f, is positive (resp. negative), then $g|g^{-1}(D_i) \to D$ is an orientation-preserving (resp. orientation-reserving) homeomorphism for each $i = 1, \ldots, \ell$. In particular, if $\ell = 0$, then f can be properly homotoped to a non-surjective map, relative to the complement of a compact subset of $\mathrm{int}(M)$.

The theorem below, due to Olum [84] and Epstein [29, Corollary 3.4.], roughly states that when there is a degree one map, the domain is more massive than the co-domain.

Theorem 1.7.3 Let $f\colon M \to N$ be a proper map between two connected, oriented, topological manifolds of the same dimension such that $f(\partial M) \subseteq \partial N$. If $\deg(f) \neq 0$, then the index $[\pi_1(N) : \mathrm{im}\ \pi_1(f)]$ divides $\deg(f)$. In particular, if $\deg(f) = \pm 1$, then $\pi_1(f)$ is surjective.

Note that Theorem 1.7.3 appears as a corollary [29, Corollary 3.4.] of a theorem [29, Theorem 3.1.] of Epstein that states the absolute degree is the same as the degree when manifolds are oriented. Epstein credits Olum [84] for the proof of [29, Corollary 3.4.]. Since [29, Corollary 3.4.] comes without proof and follows from general theory, we will supply a proof of Theorem 1.7.3 using Lemma 2.8.1 in Section 2.8.1.

Chapter 2

Strong topological rigidity

2.1 Background and motivation

A fundamental question in topology is whether two closed n-manifolds that are homotopy-equivalent are homeomorphic. This question has a positive answer in dimension two, as two closed surfaces having isomorphic fundamental groups are homeomorphic. However, the same doesn't hold true in other dimensions: the classification of lens spaces up to homotopy [110, Theorem 10] and homeomorphism [11, page 181] show that there are many non-homeomorphic lens spaces of the same homotopy type, for example, $L(7,1)$ and $L(7,2)$. Another interesting class of examples of such pairs arises from the classification theorems of simply connected closed topological 4-manifolds with odd intersection forms up to homotopy [77, Theorem 3] and homeomorphism [35, Theorem 1.5.].

A closed topological n-manifold M is said to be *topologically rigid* if any homotopy equivalence $N \to M$, with a closed topological n-manifold N as the source, is homotopic to a homeomorphism. In dimension two, every closed surface is topologically rigid; this result is known as the Dehn-Nielsen-Baer theorem [25, Appendix]. Waldhausen proved that every closed, orientable, irreducible Haken 3-manifold ($\neq \mathbb{S}^3$) is topologically rigid [107, Theorem 6.1.]. Moreover, any closed, orientable, hyperbolic 3-manifold is known to be topologically rigid [41, Theorem 0.1. i)]. For dimension at least 5, any closed, complete Riemannian manifold whose all sectional curvatures are non-positive is topologically rigid [32, Theorem 3.2]. In this direction, a famous conjecture called the *Borel Conjecture* [94, Conjecture (A. Borel)] asserts that every closed aspherical (i.e., $\pi_k = 0$ if $k \neq 1$) manifold is topologically rigid. The Borel Conjecture is true in dimension two by the Dehn-Nielsen-Baer theorem. In dimension three, the Borel Conjecture is true by Thurston's Geometrization Conjecture (nowadays, a theorem due to the epochal work of Perelman) along with the aforementioned result of Waldhausen and [104]. Several other cases where the Borel Conjecture is known to be true can be found in the survey [72].

However, non-compact manifolds are not rigid in the above sense. For instance, Whitehead's contractible open 3-manifold [109] is homotopy equivalent to $\mathbb{R}^3$ but not homeomorphic to $\mathbb{R}^3$. In fact, for each $n \geq 3$, there exist uncountably many contractible open n-manifolds up to homeomorphism; see [76, Theorem 2], [44, Theorem 3], and [19, Theorem 5.1] for $n = 3$, $n = 4$, and $n \geq 5$, respectively. Similarly, when considering non-compact surfaces, we encounter several examples. In the case of finite-type surfaces, for example, we may consider the once-punctured torus and thrice-punctured sphere, which are homotopy equivalent but non-homeomorphic, as any homomorphism preserves the cardinality of the puncture set as well as the genus. On the other hand, up to homotopy equivalence, there is precisely one infinite-type surface, but up to homeomorphism, there are $2^{\aleph_0}$ many infinite-type surfaces (see Proposition 2.3.11).

It is natural to ask what happens if we move into the proper category. First, let's recall the analogue of topological rigidity in the proper category: A non-compact topological manifold M without boundary is said to be *properly rigid* if, whenever N is another boundaryless topological manifold of the same dimension and $h\colon N \to M$ is a proper homotopy equivalence, then h is properly homotopic to a homeomorphism. The analogue of the Borel Conjecture in the proper category, often called *proper Borel Conjecture* [16, Conjecture 3.1.], asserts that every non-compact aspherical topological manifold without boundary is properly rigid.

It is known that non-compact finite-type surfaces are properly rigid. Further, using the algebraic tools of classification of non-compact surfaces [45, Theorem 4.1.], Goldman showed that two non-compact surfaces of the same proper homotopy type are homeomorphic; see [46, Corollary 11.1]. We show that infinite-type surfaces are also properly rigid. In fact, we show the rigidity of all non-compact surfaces, except for the plane and the punctured plane, under a weaker assumption, namely only assuming the existence of homotopy inverse, which a priori may or may not be proper. For brevity, define a weaker version of proper homotopy equivalence:

Definition A homotopy equivalence is said to be *pseudo proper homotopy equivalence* if it is proper.

Indeed, a proper homotopy equivalence is a pseudo proper homotopy equivalence, though not conversely: a pseudo proper homotopy equivalence has an "ordinary" homotopy inverse but may not have a proper homotopy inverse, for example, consider below given φ and ψ. Our main theorem is the following:

Theorem I (Strong topological rigidity) Let $f\colon \Sigma' \to \Sigma$ be a pseudo proper homotopy equivalence between two non-compact surfaces. Then Σ' is homeomorphic to Σ. If we further assume that Σ is homeomorphic to neither the plane nor the punctured plane, then f is a proper homotopy equivalence, and there exists a homeomorphism $f_{\text{homeo}}\colon \Sigma \to \Sigma'$ as a proper homotopy inverse of f.

The reason for the exclusion of the plane and the punctured plane from the hypothesis is almost immediate; for example, consider $\varphi\colon \mathbb{C} \ni z \longmapsto z^2 \in \mathbb{C}$ and $\psi\colon \mathbb{S}^1 \times \mathbb{R} \ni (z, x) \longmapsto \big(z, |x|\big) \in \mathbb{S}^1 \times \mathbb{R}$; each of these proper maps is a homotopy equivalence, but none of them is a proper homotopy equivalence as the degree of a proper homotopy equivalence is ± 1 (see Section 1.7), though $\deg(\varphi) = \pm 2$ (as φ is a two-fold branched covering; see Theorem 1.7.1) and $\deg(\psi) = 0$ (as ψ is not surjective; see Theorem 2.6.3.1).

In general, additional assumptions must be imposed on a pseudo proper homotopy equivalence to become a proper homotopy equivalence. For example, using the binary symmetry of the Cantor tree $\mathcal{T}_{\text{Cantor}}$, we have a two-fold branched covering $f_{\text{Cantor}}\colon \mathcal{T}_{\text{Cantor}} \to \mathcal{T}_{\text{Cantor}}$, which is undoubtedly a pseudo proper homotopy equivalence (trees are contractible) but not a proper homotopy equivalence (the induced map on $\text{Ends}(\mathcal{T}_{\text{Cantor}})$ by f_{Cantor} is non-injective; see part (1) and (3) of Proposition 1.6.4). Here is another example. Let M be a connected, non-compact, contractible, boundaryless manifold of dimension $n \geq 2$; and let $f\colon M \to M$ be the composition of the following two proper maps: a proper map $M \to [0, \infty)$ (using partition of unity) and a non-surjective proper map $[0, \infty) \to M$ corresponding to an end of M (using compact exhaustion by connected codimension 0-submanifolds; see [49, Exercise 3.3.18]). Then f is a pseudo proper homotopy equivalence (M is contractible) but not a proper homotopy equivalence (a proper homotopy equivalence is a surjective map as its degree is ± 1; see Theorem 2.6.3.1).

Brown showed that a pseudo proper homotopy equivalence between two connected, finite-dimensional, locally finite simplicial complexes is a proper homotopy equivalence if and only if it induces a homeomorphism on the spaces of ends and isomorphisms on all proper homotopy groups; see [13, Whitehead theorem]. In [33, Corollary 4.10.], the authors have shown that if $f\colon M \to N$ is a pseudo proper homotopy equivalence between two simply-connected, non-compact, boundaryless n-dimensional smooth manifolds, where both M and N both are simply-connected at infinity, then f is a proper homotopy equivalence if and only if $\deg(f) = \pm 1$. Brown and Tucker [12, Theorem 4.2] showed that if $\mathfrak{f}\colon \mathfrak{N} \to \mathfrak{M}$ is a pseudo proper homotopy equivalence between two connected, non-compact, orientable, irreducible, end-irreducible, boundaryless 3-manifolds such that $\pi_1(\mathfrak{N})$ is not isomorphic to the fundamental group of any compact surface, then $\mathfrak{f}$ is properly homotopic to a homeomorphism, and thus $\mathfrak{f}$ is a proper homotopy equivalence. Another interesting statement in this context is that a proper map $f\colon X \to Y$ between two locally finite, infinite, connected, 1-dimensional CW-complexes is a proper homotopy equivalence if $\text{Ends}(f)$ is a homeomorphism and f is an extension of a proper homotopy equivalence $X_g \to Y_g$ (where X_g (resp. Y_g) denotes the smallest connected sub-complex of X (resp. Y) that contains all immersed loops of X (resp. Y)); see [2, Corollary 3.7.].

We conclude this section by citing a few more related results of two different flavors: when does a proper homotopy equivalence exist, and if it does exist, whether it determines the space up to homeomorphism. Similar to Kerékjártó's classification theorem (see Theorem 1.5.1), there exists a classification of graphs up to proper homotopy type: Two locally finite, infinite, connected, 1-dimensional CW-complexes X and Y have the same proper homotopy type if and only if $\text{rank}\big(\pi_1(X)\big) = \text{rank}\big(\pi_1(Y)\big)$ and there exists a homeomorphism $\varphi\colon \text{Ends}(X) \to \text{Ends}(Y)$ with $\varphi\big(\text{Ends}(X_g)\big) = \text{Ends}(Y_g)$; see [5, Theorem 2.7.].

As stated earlier, any two non-compact surfaces of the same proper homotopy type are homeomorphic. Sometimes this happens also in other dimensions; for instance, a boundaryless topological manifold of dimension $n \geq 3$ with the same proper homotopy type of $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$; see [27, Theorem 1] for $n = 3$, [35, Corollary 1.2.] for $n = 4$, and [96, Corollary 1.4.] for $n \geq 5$. Thus, an application of Theorem 1.7.2 together with Alexander trick (see Theorem 2.6.2.4) tells us that $\mathbb{R}^n$ is properly rigid for every n. In contrast, there are exotic pairs, for example, two non-compact, connected, boundaryless manifolds N and

M of the dimension $n \geq 5$ exist, where N is smoothable, and M is a nonuniform arithmetic manifold, such that M and N has the same proper homotopy type, but M is not homeomorphic to N; see [15, Theorem 2.6.] with [16, pages 137 and 138]. On the other hand, a non-compact, boundaryless, connected, orientable complete hyperbolic manifold with finite volume whose dimension is different from 3, 4, and 5 is properly rigid [31, Corollary 10.5.].

2.2 Outline of the proof of Theorem I

The next couple of sections will be devoted to proving our main theorem, namely Theorem I, but here, we want to provide a quick sketch of its proof for better understanding. So, let $f\colon \Sigma' \to \Sigma$ be a pseudo proper homotopy equivalence between two non-compact oriented surfaces. Suppose Σ is homeomorphic to neither $\mathbb{R}^2$ nor $\mathbb{S}^1 \times \mathbb{R}$.

2.2.1 Decomposition and transversality

Let $\mathscr{C}$ be a locally finite pairwise disjoint collection of smoothly embedded circles on Σ such that $\mathscr{C}$ decomposes Σ into bordered sub-surfaces, and a complementary component of this decomposition is homeomorphic to either the torus with a disk removed or the pair of pants or the punctured disk (see Theorem 2.3.5).

Properly homotope f to make it smooth as well as transverse to $\mathscr{C}$. Thus, $f^{-1}(\mathcal{C})$ is either *empty* or a pairwise disjoint finite collection of smoothly embedded circles on Σ' for each component $\mathcal{C}$ of $\mathscr{C}$ (see Theorem 2.4.3).

2.2.2 Removing unnecessary circles

Now, following the three steps given below, we properly homotope f further so that for each component of $\mathcal{C}$ of $\mathscr{C}$, either $f^{-1}(\mathcal{C})$ is empty or $f|f^{-1}(\mathcal{C}) \to \mathcal{C}$ is a homeomorphism.

(1) Notice that $f^{-1}(\mathscr{C})$ may have infinitely many disk-bounding components. But, in such a case, an arbitrarily large disk in Σ' bounded by a component of the locally finite collection $f^{-1}(\mathscr{C})$ is not possible as $\Sigma' \not\approx \mathbb{R}^2$ (see Lemma 2.5.1.1), i.e., there always exists an "outermost disk" bounded by some component of $f^{-1}(\mathscr{C})$. Now, properly homotope f to remove all disk bounding components of $f^{-1}(\mathscr{C})$ upon considering all these outermost disks simultaneously (see Theorem 2.5.1.5).

(2) Thereafter, using π_1-bijectivity of f, properly homotope f to map each (primitive) component of $f^{-1}(\mathscr{C})$ onto a component of $\mathscr{C}$ homeomorphically (see Theorem 2.5.2.3).

(3) Since f has homotopy left inverse, any two components of $f^{-1}(\mathscr{C})$ co-bound an annulus in Σ' if and only if their f-images are the same, i.e., arbitrarily large annulus in Σ' co-bounded by two components of $f^{-1}(\mathscr{C})$ is impossible. So, considering all these "outermost annuli" simultaneously, we complete the goal, as stated in the beginning (see Theorem 2.5.3.5).

2.2.3 Showing f is a degree ± 1 map (see Theorem 2.6.3.4)

To rule out the possibility that $f^{-1}(\mathcal{C})$ is empty, where $\mathcal{C}$ is a component of $\mathscr{C}$, we prove deg(f) $= \pm 1$; this is because deg(f) remains the same after any proper homotopy of f, and a map of non-zero degree is surjective; see Theorem 2.6.3.1 and Corollary 2.6.3.2. Our aim is to properly homotope f to obtain a closed disk $\mathcal{D} \subseteq \Sigma$ so that $f|f^{-1}(\mathcal{D}) \to \mathcal{D}$ becomes a homeomorphism, and thus we show deg(f) $= \pm 1$; see Theorem 1.7.1. The argument is based on finding a smoothly embedded finite-type bordered surface $\mathbf{S}$ in Σ such that for each component c of $\partial\mathbf{S}$; we have $f^{-1}(c) \neq \varnothing$, even after any proper homotopy of f. Depending on the nature of $\mathbf{S}$, we consider two cases.

(1) If Σ is either an infinite-type surface or any $S_{g,0,p}$ with high complexity (i.e., $g + p \geq 4$ or $p \geq 6$), then using π_1-surjectivity of f, we can choose $\mathbf{S}$ as a smoothly embedded pair of pants in Σ such that $\Sigma \setminus \mathbf{S}$ has at least two components and every component of $\Sigma \setminus \mathbf{S}$ has a non-abelian fundamental group; see Lemma 2.6.1.3 and Lemma 2.6.1.5. Properly homotope f so that it becomes transverse to $\partial\mathbf{S}$. Then remove unnecessary components from the transversal pre-image $f^{-1}(\partial\mathbf{S})$, i.e., after a proper homotopy, we may assume $f|f^{-1}(c) \to c$ is a homeomorphism for each component c of $\partial\mathbf{S}$. Now, since f is π_1-injective, by the rigidity of pair of pants (see Theorem 2.6.1.10), after a proper homotopy, one can show that $f|f^{-1}(\mathbf{S}) \to \mathbf{S}$ is a homeomorphism; see Lemma 2.6.1.11. Therefore, the required $\mathcal{D}$ can be any disk in int($\mathbf{S}$).

(2) If Σ is a finite-type surface, then we choose a smoothly embedded punctured disk in Σ as $\mathbf{S}$ so that the puncture of $\mathbf{S}$ is an end $e \in \operatorname{im}(\operatorname{Ends}(f)) \subseteq \operatorname{Ends}(\Sigma)$. By Theorem 2.6.2.1, it means every deleted neighbourhood of e in Σ intersects im(f), even after any proper homotopy of f. Now, properly homotope f so that it becomes transverse to $\partial\mathbf{S}$. Then remove unnecessary components from the transversal pre-image $f^{-1}(\partial\mathbf{S})$, i.e., after a proper homotopy, we may assume $f|f^{-1}(\partial\mathbf{S}) \to \partial\mathbf{S}$ is a homeomorphism (as $\Sigma \not\cong \mathbb{S}^1 \times \mathbb{R}$, the fundamental group of $\Sigma \setminus \mathbf{S}$ is non-abelian; and so π_1-surjectivity of f says $f^{-1}(\partial\mathbf{S}) \neq \varnothing$, even after any proper homotopy of f). Now, since f is π_1-injective, by the proper rigidity of the punctured disk (see Theorem 2.6.2.4), after a proper homotopy, one can show that $f|f^{-1}(\mathbf{S}) \to \mathbf{S}$ is a homeomorphism; see Lemma 2.6.2.3. Therefore, the required $\mathcal{D}$ can be any disk in int($\mathbf{S}$).

2.2.4 Inverse decomposition

By the last three parts, after a proper homotopy, removing unnecessary components from the transversal pre-image $f^{-1}(\mathscr{C})$, we may assume that $f|f^{-1}(\mathcal{C}) \to \mathcal{C}$ is a homeomorphism for each component $\mathcal{C}$ of $\mathscr{C}$. Thus, $\mathscr{C}$ and $f^{-1}(\mathscr{C})$ decompose Σ, Σ', respectively; and there is a shape-preserving bijective-correspondence between these two collections of complementary components (see Lemma 2.6.1.11 and Lemma 2.6.2.3). On each complementary component, apply either the rigidity of compact bordered surfaces (see Theorem 2.6.1.10) or the proper rigidity of the punctured disk (see Theorem 2.6.2.4). Thus, we have a collection of boundary-relative proper homotopies so that by pasting them, a proper homotopy from f to a homeomorphism $\Sigma' \to \Sigma$ can be constructed; see the proof of Theorem I in Section 2.7.

2.3 Decomposition of a non-compact surface into pair of pants and punctured disks

Every compact surface of genus $g \geq 2$ is the union (with pairwise disjoint interiors) of $2g-2$ many copies of the pair of pants. But the same thing doesn't happen for non-compact surfaces; for example, the thrice punctured sphere is not a union (with pairwise disjoint interiors) of copies of the pair of pants; we need copies of the punctured disk. The main aim of this section is to prove that every non-compact surface, except the plane and the once punctured torus, decomposes into copies of the pair of pants and copies of the punctured disk when we cut it along a collection of circles, where each circle of this collection has an open neighbourhood that does not intersect with any other circles of this collection.

First, we define a few terminologies.

Definition 2.3.1 Let X be a space, and let $\{X_\alpha : \alpha \in \mathscr{I}\}$ be a collection of subsets of X. We say $\{X_\alpha : \alpha \in \mathscr{I}\}$ is a *locally finite collection* and write $X_\alpha \to \infty$ if, for each compact subset K of X, $X_\alpha \cap K = \varnothing$ for all but finitely many $\alpha \in \mathscr{I}$.

Definition 2.3.2 Let $\mathscr{A}$ be a pairwise disjoint collection of smoothly embedded circles on a surface Σ. We say $\mathscr{A}$ is a *locally finite curve system* (in short, LFCS) on Σ if $\mathscr{A}$ is a locally finite collection.

Remark 2.3.3 Let $\mathscr{A}$ be an LFCS on a surface Σ. Notice that $\cup\mathscr{A}$ (i.e., the union of all elements of $\mathscr{A}$) is a closed subset of Σ as well as a smoothly embedded submanifold of Σ so that the set of all components of $\cup\mathscr{A}$ is $\mathscr{A}$. But to avoid too many notations, whenever needed, we will think of $\mathscr{A}$ and $\cup\mathscr{A}$ as the same without any harm.

Definition 2.3.4 Let $\mathscr{A}$ be an LFCS on a surface Σ. Suppose there exists an at most countable collection $\{\Sigma_n\}$ of bordered sub-surfaces of Σ such that the following hold: (1) each Σ_n is a closed subset of Σ; (2) $\text{int}(\Sigma_n) \cap \text{int}(\Sigma_m) = \varnothing$ if $n \neq m$; (3) $\cup_n \Sigma_n = \Sigma$; and (4) $\cup_n \partial\Sigma_n = \cup\mathscr{A}$. In this case, we say $\mathscr{A}$ *decomposes* Σ *into bordered sub-surfaces*, where *complementary components* are $\{\Sigma_n\}$. Also, we call each component of $\mathscr{A}$ a *decomposition circle*.

The following theorem asserts that any non-compact surface other than the plane has a decomposition, where each complementary part is either a pair of pants, a torus with a disk removed, or a punctured disk. This way of decomposition of the co-domain of a pseudo proper homotopy equivalence will be used in all cases.

Theorem 2.3.5 Let Σ be a non-compact surface not homeomorphic to $\mathbb{R}^2$. Then there is an LFCS $\mathscr{C}$ on Σ such that $\mathscr{C}$ decomposes Σ into bordered sub-surfaces, and a complementary component of this decomposition is homeomorphic to either $S_{1,1}$ (used at most once), $S_{0,3}$, or $S_{0,1,1}$.

Proof. It is enough to find a collection $\{\Sigma_n\}$ of bordered sub-surfaces of Σ with four properties, as mentioned in Definition 2.3.4, so that each Σ_n is homeomorphic to either $S_{0,3}$, $S_{1,1}$, or $S_{0,1,1}$.

For that, consider an inductive construction of Σ; see Theorem 1.3.1. Now, a finite sequence of annuli, when added to the compact bordered surface used just before it, can be ignored. Thus, we may assume $S_{0,3}$ or $S_{1,2}$ is used after $S_{0,1}$ without loss of generality because of $\Sigma \not\cong \mathbb{R}^2$, and hence pushing $S_{0,1}$ into $\operatorname{int}(S_{0,3})$ or $\operatorname{int}(S_{1,2})$, we end up with $S_{0,2}$ (which can be ignored) or $S_{1,1}$. Now, the proof will be completed by observing the following: $S_{1,2}$ can be decomposed into two copies of $S_{0,3}$, and $S_{0,1,1}$ is the union (with pairwise disjoint interiors) of countably many copies of $S_{0,2}$. □

Remark 2.3.6 A statement closely related to Theorem 2.3.5 is in [3, Theorem 1.1.], which says that *"every surface except for the sphere, the plane, and the torus is the union (with pairwise disjoint interiors) of copies of the pair of pants and copies of the punctured disk"*. But due to the part (4) of Definition 2.3.4, if we want that any complementary component is homeomorphic to only either $S_{0,3}$ or $S_{0,1,1}$, then $\Sigma \not\cong S_{1,0,1}$ also needs to consider; see Figure 2.3.1 and Theorem 2.3.7 below.

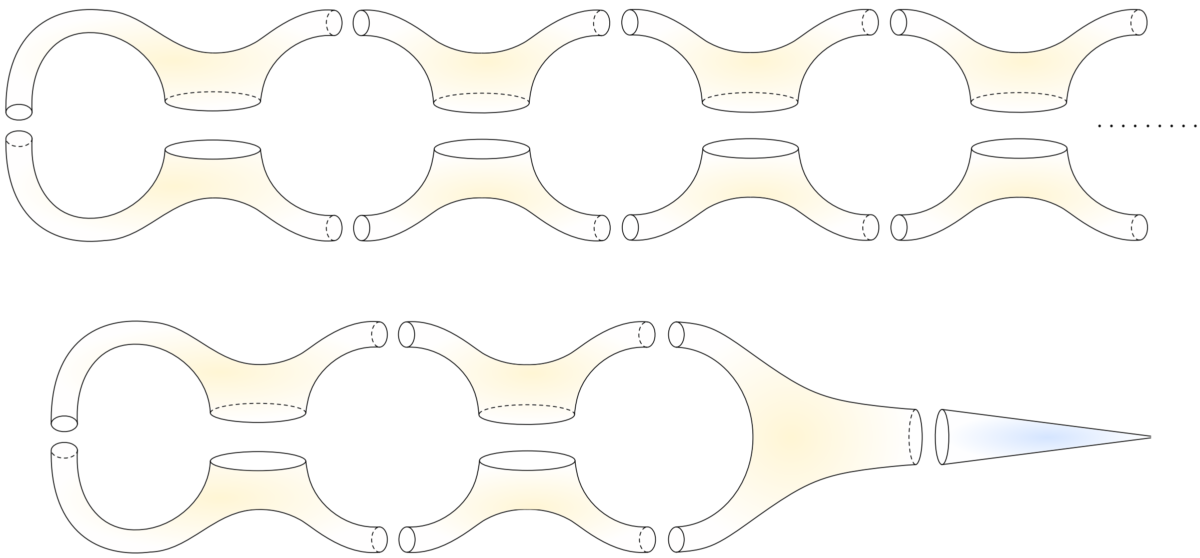

Fig. 2.3.1 On the top: Decomposition of Loch Ness Monster into countably infinitely many copies of the pair of pants. At the bottom: Decomposition of $S_{3,0,1}$ into five copies of the pair of pants and a copy of the punctured disk.

Theorem 2.3.7 Let Σ be a non-compact surface that is not homeomorphic to either $\mathbb{R}^2$ or $S_{1,0,1}$. Then there is an LFCS $\mathscr{C}'$ on Σ such that $\mathscr{C}'$ decomposes Σ into bordered sub-surfaces, and a complementary component of this decomposition is homeomorphic to either $S_{0,3}$ or $S_{0,1,1}$.

To prove Theorem 2.3.7, we need Lemma 2.3.8 below, which says that in an inductive construction of a non-compact surface, interchanging the positions of the compact bordered surfaces used in the first few inductive steps doesn't change the homeomorphism type, and its proof is based on the observation that the portions outside compact subsets determine the space of ends.

Lemma 2.3.8 Let Σ be a non-compact surface with some inductive construction $\mathscr{I}$. Denote the compact bordered subsurface of Σ after the i-th step of $\mathscr{I}$ by K_i. Suppose $\{\mathcal{B}_1, ..., \mathcal{B}_n :$ each $\mathcal{B}_\ell$ is homeomorphic to either $S_{0,2}$, $S_{0,3}$, or $S_{1,2}\}$ is a finite collection of compact bordered surfaces such that $\mathcal{B}_\ell$ is used to construct $K_{i_\ell+1}$ from K_{i_ℓ} for each $\ell = 1, ..., n$. Then there exists a non-compact surface Σ' with an inductive construction $\mathscr{I}'$ such that $\Sigma' \cong \Sigma$ and $\mathcal{B}_\ell$ is used to construct $K'_{\ell+1}$ from K'_ℓ for each $\ell = 1, ..., n$; where K'_i denotes the compact bordered subsurface of Σ after the i-th step of $\mathscr{I}'$.

Proof. Let n_0 be a positive integer such that K_{n_0} contains each $\mathcal{B}_\ell$. Define $\mathbf{S} := \Sigma \setminus \operatorname{int}(K_{n_0})$. Thus $\mathbf{S}$ is a bordered subsurface of Σ with $\partial\mathbf{S} = \partial K_{n_0}$. Now, consider all copies of different building blocks used up to the n_0-th step of $\mathscr{I}$, and inside K_{n_0} interchange them so that $\mathcal{B}_1, ..., \mathcal{B}_n$ comes just after the initial disk K_1 one by one following the increasing order of their indices. Denote the resultant of this interchange process by K'_{n_0}. So $K_{n_0} \cong K'_{n_0}$ as $g(K_{n_0}) = g(K'_{n_0})$ and $\partial K_{n_0} \cong \partial K'_{n_0}$. Define a non-compact surface Σ' as $\Sigma' := K'_{n_0} \cup_{\partial K'_{n_0} \equiv \partial \mathbf{S}} \mathbf{S}$. Therefore, $\Sigma \setminus K_{n_0} = \operatorname{int}(\mathbf{S}) = \Sigma' \setminus K'_{n_0}$ (notice that we are thinking $\mathbf{S}$ as a subset of Σ' using the obvious embedding $\mathbf{S} \hookrightarrow \Sigma'$).

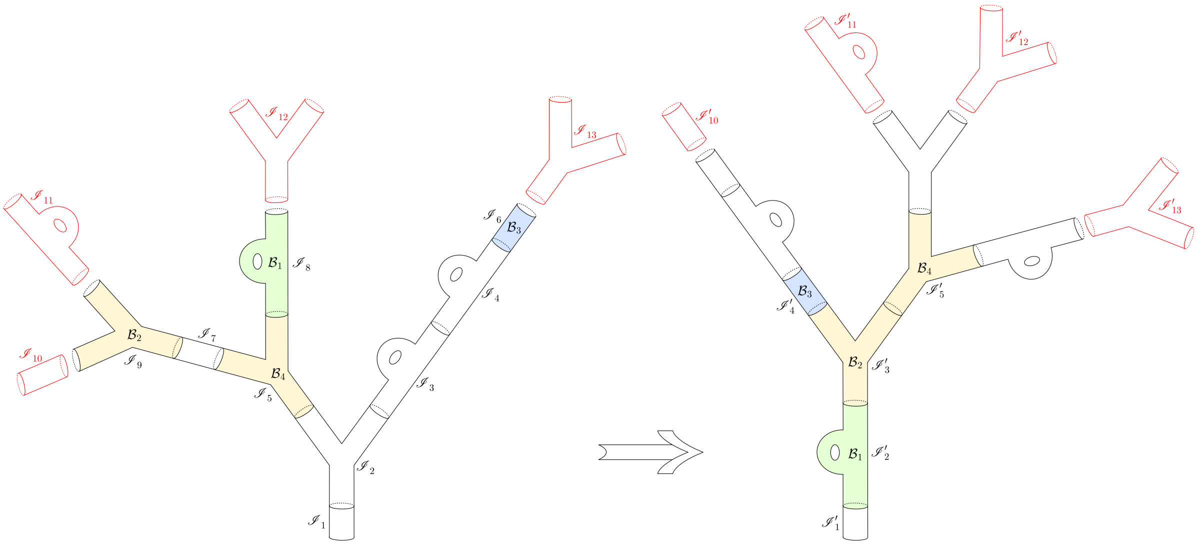

Fig. 2.3.2 $\mathscr{I}_r$ (resp. $\mathscr{I}'_r$) denotes the r-th step of $\mathscr{I}$ (resp. $\mathscr{I}'$). Notice that here, $n_0 = 9$ and $n = 4$. Also, the red-coloured compact bordered surfaces are the portions of $\mathbf{S}$, and the inductive construction of $\mathbf{S}$ given by $\mathscr{I}'$ is inherited from the inductive construction of $\mathbf{S}$ given by $\mathscr{I}$.

Choose an inductive construction $\mathscr{I}'_{\leq n_0}$ of K'_{n_0} such that i-th element of the ordered sequence $K_1, \mathcal{B}_1, ..., \mathcal{B}_\ell$ is used in the i-th step of $\mathscr{I}'_{\leq n_0}$. Also, $\mathscr{I}$ gives a truncated inductive construction $\mathscr{I}|\mathbf{S}$ on $\mathbf{S}$ starting from the (n_0+1)-th step. Now, $\mathscr{I}'_{\leq n_0}$ followed by $\mathscr{I}|\mathbf{S}$ together gives an inductive construction $\mathscr{I}'$ of Σ'. Roughly, it means $\mathscr{I}'$ is the same as the inductive construction of Σ, except for the first few steps. Denote the compact bordered subsurface of Σ' after the i-th step of $\mathscr{I}'$ by K'_i. To complete the proof, we show $\Sigma' \cong \Sigma$ using Theorem 1.5.1.

Consider the efficient exhaustion $\{K_i\}$ (resp. $\{K'_i\}$) of Σ (resp. Σ') by compacta to define $\operatorname{Ends}(\Sigma)$ (resp. $\operatorname{Ends}(\Sigma')$). Recall that the space of ends remains the same up to homeomorphism even if we choose a different efficient exhaustion by compacta; see Section 1.4. By

$\Sigma \setminus K_{n_0} = \text{int}(\mathbf{S}) = \Sigma' \setminus K'_{n_0}$, for every sequence $(V_1, V_2, ...) \in \text{Ends}(\Sigma)$, there exists a unique sequence $(V'_1, V'_2, ...) \in \text{Ends}(\Sigma')$ such that $V_i = V'_i$ for all integer $i \geq n_0$, and conversely. Thus, there exists a homeomorphism $\varphi\colon \text{Ends}(\Sigma) \to \text{Ends}(\Sigma')$ with $\varphi(\text{Ends}_{\text{np}}(\Sigma)) = \text{Ends}_{\text{np}}(\Sigma')$. Also, $\Sigma \setminus K_{n_0} = \text{int}(\mathbf{S}) = \Sigma' \setminus K'_{n_0}$ and $K_{n_0} \cong K'_{n_0}$ together imply $g(\Sigma) = g(\Sigma')$. Therefore, $\Sigma' \cong \Sigma$ by Theorem 1.5.1. □

Proof of Theorem 2.3.7. It is enough to find a collection $\{\Sigma_n\}$ of bordered sub-surfaces of Σ with four properties, as mentioned in Definition 2.3.4, so that each Σ_n is homeomorphic to either $S_{0,3}$ or $S_{0,1,1}$. For that, consider an inductive construction of Σ; see Theorem 1.3.1. We will divide the whole proof into two cases, depending on whether Σ has at least two ends.

At first, suppose the number of ends of Σ is at least two. Now, the definition of the space of ends tells us that we need to use at least one pair of pants in the inductive construction of Σ. By Lemma 2.3.8, we may assume that in this inductive construction, a pair of pants is used just after the disk. Now, an argument similar to before (see the proof of Theorem 2.3.5) concludes this case.

Next, consider the case when the number of ends of Σ is precisely one. That is, Σ can be either Loch Ness Monster (the infinite genus surface with one end) or $S_{g,0,1}$ with $g \geq 2$. Loch Ness Monster decomposes into countably infinitely many copies of the pair of pants, and $S_{g,0,1}$ with $g \geq 2$ decomposes into $2g-1$ many copies of the pair of pants and a copy of the punctured disk (see Figure 2.3.1). □

The spine construction of Goldman's inductive procedure shows that every non-compact surface Σ (Σ may be of infinite-type) is the interior of a bordered surface: consider the graph $\mathcal{G}$ consisting of blue straight line segments and red circles, as given in Figure 1.3.1. Any thickening [45, Definition 7.2.] of $\mathcal{G}$ in Σ is the interior of a bordered subsurface $\mathbf{S}$ of Σ. Now, [45, Corollary 7.2. and section 7.3.] says that $\text{int}(\mathbf{S}) \cong \Sigma$. When Σ is of finite-type, we prove the same thing differently in the following theorem.

Theorem 2.3.9 A non-compact finite-type surface is the interior of a compact bordered surface. In particular, if a non-compact surface has infinite cyclic (resp. trivial) fundamental group, then it is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$ (resp. $\mathbb{R}^2$).

Proof. Let Σ be a finite-type non-compact surface. Consider an inductive construction of Σ; see Theorem 1.3.1. Since $\pi_1(\Sigma)$ is finitely generated, Theorem 1.3.2 says that Σ is homotopy equivalent to $\bigvee^{2r+s} \mathbb{S}^1$, where in this inductive construction, $r \in \mathbb{N}$ is the total number of copies of $S_{1,2}$, and $s \in \mathbb{N}$ is the total number of copies of $S_{0,3}$; see Figure 1.3.1. Thus there is an integer n such that $\overline{\Sigma \setminus K_n}$ (where K_n is the compact bordered subsurface of Σ after n-th inductive step) is a finite collection of punctured disks. Now, $g(\Sigma) = g\big(\text{int}(K_n)\big)$. Also, each end of Σ (resp. $\text{int}(K_n)$) is planar, and the total number of ends of Σ (resp. $\text{int}(K_n)$) is the same as the number of components of ∂K_n. By Theorem 1.5.1, $\Sigma \cong \text{int}(K_n)$.

If Σ is a non-compact surface with the infinite-cyclic fundamental group, then any inductive construction of Σ contains no copy of $S_{1,2}$ but precisely one copy of $S_{0,3}$, i.e., $\Sigma \cong \mathbb{S}^1 \times \mathbb{R}$. Similarly, if Σ is a non-compact surface with the trivial fundamental group, then any inductive construction of Σ has no copy of $S_{0,3}$ as well as no copy of $S_{1,2}$, i.e., $\Sigma \cong \mathbb{R}^2$. □

The proposition below follows directly from Goldman's inductive construction, so we quote it without proof. It says that an infinite-type surface has a finite genus only if it has infinitely many ends. On the other hand, Theorem 1.5.2 guarantees the existence of an infinite-type surface of the infinite genus with infinitely many ends.

Proposition 2.3.10 A non-compact surface is of a finite genus if and only if the total number of copies of $S_{1,2}$ used in any inductive construction of Σ is finite. Thus, if an infinite-type surface has a finite genus, then it must have infinitely many ends.

This section's final fact (as promised on Page 12) says that the fundamental group alone can't detect the homeomorphism type of an infinite-type surface.

Proposition 2.3.11 Up to homotopy equivalence, there is exactly one infinite-type surface, but up to homeomorphism, there are $2^{\aleph_0}$ many infinite-type surfaces.

Proof. Any infinite-type surface is homotopy equivalent to the wedge of countably infinitely many circles; see Theorem 1.3.2. Therefore, any two infinite-type surfaces are homotopy equivalent.

Now, we prove that up to homeomorphism, there are $2^{\aleph_0}$ many infinite-type surfaces. Notice that except for the first step, in each step of Goldman's inductive procedure, we use either $S_{0,3}$, or $S_{0,2}$, or $S_{1,2}$. Thus, we have at most $3^{\aleph_0} = 2^{\aleph_0}$ many non-compact surfaces, up to homeomorphism. Therefore, it is enough to show that this upper bound is reachable. Let τ be a non-empty closed subset of the Cantor set. By Theorem 1.5.2, there exists an infinite genus surface Σ_τ such that $\text{Ends}(\Sigma_\tau) = \text{Ends}_{\text{np}}(\Sigma_\tau) \cong \tau$. Therefore, if τ_1 and τ_2 are two non-homeomorphic non-empty closed subsets of the Cantor set, then Σ_{τ_1} is not homeomorphic to Σ_{τ_2} by Theorem 1.5.1. Now, [92, Theorem 2] says that up to homeomorphism, there are $2^{\aleph_0}$ many closed subsets of the Cantor set. So, we are done. □

2.4 Transversality of a proper map with respect to all decomposition circles

In the previous section, the co-domain of a pseudo proper homotopy equivalence has been decomposed into finite-type bordered surfaces by a locally finite pairwise disjoint collection of circles. This section aims to properly homotope the pseudo proper homotopy equivalence to make it transverse to each decomposition circle.

The theorem below follows from the theory developed in the Appendix A. We aim to use it to impose a one-dimensional submanifold structure on the inverse image of each decomposition circle.

Theorem 2.4.1 Let $f\colon \Sigma' \to \Sigma$ be a proper map between non-compact surfaces, and let $\mathscr{A}$ be an LFCS on Σ. Then f can be properly homotoped to make it smooth as well as transverse to the manifold $\mathscr{A}$.

Proof. Using Theorem A.1, after a proper homotopy, we may assume that f is a smooth proper map. After that, using Theorem A.2, properly homotope f so that it becomes transverse to $\mathscr{A}$. □

Remark 2.4.2 Note that in Theorem 2.4.1, we have no control over those proper homotopies, which make the proper map f as smooth as well as transversal to $\mathscr{A}$, i.e., after these proper homotopies, $f^{-1}(\mathscr{A})$ can be empty, even if these proper homotopies start with a surjective proper map. A remedy for this is: Assume $\deg(f) \neq 0$; this is because the degree is invariant under proper homotopy, and a map of non-zero degree is surjective; see Theorem 2.6.3.1 and Corollary 2.6.3.2. If f is a proper homotopy equivalence, then f has a proper homotopy inverse; hence, $\deg(f) \neq 0$ (see Section 1.7). But if f is a pseudo proper homotopy equivalence, then we don't know (at least till this stage) whether f has a proper homotopy inverse or not (though it has a homotopy inverse). Later in Section 2.6, using π_1-bijectivity, we will show that most pseudo proper homotopy equivalence is a map of degree ± 1.

The following theorem says that the transversal pre-image of an LFCS under a proper map is an LFCS.

Theorem 2.4.3 Let $f\colon \Sigma' \to \Sigma$ be a smooth proper map between two non-compact surfaces, and let $\mathscr{A}$ be an LFCS on Σ such that $f \pitchfork \mathscr{A}$. Then, for each component $\mathcal{C}$ of $\mathscr{A}$, either $f^{-1}(\mathcal{C})$ is empty or a pairwise disjoint finite collection of smoothly embedded circles on Σ'. Therefore, $f^{-1}(\mathscr{A})$ is an LFCS on Σ'.

Proof. By the definition of transversality, $f \pitchfork \mathscr{A}$ implies $f \pitchfork \mathcal{C}$ for each component $\mathcal{C}$ of $\mathscr{A}$. Thus, $f^{-1}(\mathcal{C})$ is either empty or is a compact (since f is proper) one-dimensional boundaryless smoothly embedded submanifold of Σ'. Now, by classification of closed one-dimensional manifolds, we complete the first part.

Next, if possible, let K' be a compact subset of Σ' such that K' intersects infinitely many components of $f^{-1}(\mathscr{A})$. By the first part, it means the compact set $f(K')$ intersects infinitely many components of $\mathscr{A}$, which contradicts the fact that $\mathscr{A}$ is a locally finite collection. □

2.5 Surgery on a pseudo proper homotopy equivalence in proper category

2.5.1 Disk removal

Previously, as observed, after a proper homotopy, the number of components in the collection of transversal pre-images of all decomposition circles can be infinite, and many components (possibly infinitely many) of this collection, maybe trivial circles. Here, at first, our goal is to group all these trivial circles in terms of the size of the disk bounded by a trivial circle and then remove all groups of trivial circles simultaneously by a proper homotopy.

At first, our intended grouping requires a technical lemma, which asserts that on a non-simply connected surface, an LFCS consisting of concentric trivial circles doesn't exist. Roughly, it means, on a non-simply connected surface, arbitrarily large disks bounded by components of an LFCS don't exist.

Lemma 2.5.1.1 Let Σ be a surface, and let $\mathscr{A} := \{\mathcal{C}_i : i \in \mathbb{N}\}$ be an LFCS on Σ such that for each i the circle $\mathcal{C}_i$ bounds a disk $\mathcal{D}_i \subset \Sigma$ with $\mathcal{C}_i \subset \text{int}(\mathcal{D}_{i+1})$. Then Σ is homeomorphic to $\mathbb{R}^2$.

Proof. At first, notice that Σ must be non-compact as $\mathscr{A}$ is a locally finite, pairwise disjoint, infinite collection of circles. Using inductive construction (see Theorem 1.3.1), we have a sequence $\{\mathbf{S}_j : j \in \mathbb{N}\}$ of compact bordered sub-surfaces of Σ such that $\cup_j \mathbf{S}_j = \Sigma$ and for each $j \in \mathbb{N}$, $\mathbf{S}_j \subset \text{int}(\mathbf{S}_{j+1})$. Consider any $p \in \Sigma$. So, a $j_0 \in \mathbb{N}$ exists such that $p \in \mathbf{S}_{j_0}$ and $\mathbf{S}_{j_0} \cap \big(\cup_i \mathcal{C}_i\big) \neq \varnothing$. Since $\mathscr{A}$ is a locally finite collection, only finitely many components of $\mathscr{A}$ intersect the compact set $\mathbf{S}_{j_0}$. Let $\mathcal{C}_{i_1}, ..., \mathcal{C}_{i_\ell}$ be the only components of $\mathscr{A}$ intersecting $\mathbf{S}_{j_0}$, where $i_1 < \cdots < i_\ell$. Pick an integer $i_0 > i_\ell$. Then $\mathcal{C}_{i_0} \cap \mathbf{S}_{j_0} = \varnothing$. Now, since $\mathcal{C}_{i_\ell} \subset \text{int}(\mathcal{D}_{i_0})$, $\mathbf{S}_{j_0}$ is connected, and Σ is locally Euclidean, we can say that $\mathbf{S}_{j_0} \subseteq \text{int}(\mathcal{D}_{i_0})$. Thus, every point $x \in \Sigma$ has an open neighbourhood $\mathcal{U}_x$ in Σ such that $\mathcal{U}_x \subseteq \mathcal{D}_i$ for some $i \in \mathbb{N}$. Therefore, every loop on Σ is contained in a disk $\mathcal{D}_i$ for some large $i \in \mathbb{N}$, i.e., Σ is simply-connected. By Theorem 2.3.9, $\Sigma \cong \mathbb{R}^2$. □

The following lemma is the primary tool for showing that a homotopy is proper. It tells how a proper map can be properly homotoped so that it changes on infinitely many pairwise disjoint compact sets.

Lemma 2.5.1.2 Let $f\colon \Sigma' \to \Sigma$ be a proper map between two non-compact surfaces, and let $\{\Sigma'_n : n \in \mathbb{N}\}$ be a pairwise disjoint collection of compact bordered sub-surfaces of Σ'. For each $n \in \mathbb{N}$, suppose $H_n\colon \Sigma'_n \times [0,1] \to \Sigma$ is a homotopy relative to $\partial\Sigma'_n$ such that $H_n(-,0) = f|\Sigma'_n$ and $\text{im}(H_n) \to \infty$. Then $\mathcal{H}\colon \Sigma' \times [0,1] \to \Sigma$ defined by

$$\mathcal{H}(p,t) := \begin{cases} H_n(p,t) & \text{if } p \in \Sigma'_n \text{ and } t \in [0,1], \\ f(p) & \text{if } p \in \Sigma' \setminus \big(\cup_{n\in\mathbb{N}} \Sigma'_n\big) \text{ and } t \in [0,1] \end{cases}$$

is a proper map.

Proof. Let $\mathcal{K}$ be a compact subset of Σ. By continuity of $\mathcal{H}$, $\mathcal{H}^{-1}(\mathcal{K})$ is closed in Σ'. Since $\text{im}(H_n) \to \infty$, there exists $n_0 \in \mathbb{N}$ such that $\text{im}(H_\ell) \cap \mathcal{K} = \varnothing$ for all integers $\ell \geq n_0 + 1$. Now, $f^{-1}(\mathcal{K})$ is compact as f is proper. Also, the domain of each H_n is compact. Hence, the closed subset $\mathcal{H}^{-1}(\mathcal{K})$ of Σ' is contained in the compact set $f^{-1}(\mathcal{K}) \cup \bigcup_{\ell=1}^{n_0} H_n^{-1}(\mathcal{K})$. So, we are done. □

To remove trivial components from the transversal pre-image of an LFCS with infinitely many components, we need to impose some conditions on this LFCS. One such preferred LFCS is given in Theorem 2.3.5. But for future use, not only this type of LFCS, we require other kinds of LFCS on the co-domain. So, here is the list of different preferred LFCS.

Definition 2.5.1.3 Let Σ be a non-compact surface such that $\Sigma \not\cong \mathbb{R}^2$. Suppose, $\mathscr{A}$ is a given LFCS on Σ. We say $\mathscr{A}$ is a *preferred* LFCS on Σ if either of the following happens: (i) $\mathscr{A}$ is a finite collection of primitive circles on Σ; (*ii*) $\mathscr{A}$ decomposes Σ into bordered sub-surfaces, and a complementary component of this decomposition is homeomorphic to either $S_{1,1}$, $S_{0,3}$, $S_{0,2}$, or $S_{0,1,1}$.

Remark 2.5.1.4 The only use of case (*i*) of Definition 2.5.1.3 is in Section 2.6, where we consider the process of removing unnecessary circles from the transversal pre-image of the boundary of an essential pair of pants or an essential punctured disk. It is worth noting that by a finite LFCS, one can't decompose an infinite-type surface into finite-type bordered surfaces.

In the theorem below, we construct a proper homotopy, which removes all trivial components, keeping a neighbourhood of each primitive component stationary from the transversal pre-image of a preferred LFCS. Recall that a homotopy $H\colon X \times [0,1] \to Y$ is said to be *stationary on a subset A of X* if $H(a,t) = H(a,0)$ for all $(a,t) \in A \times [0,1]$.

Theorem 2.5.1.5 Let $f\colon \Sigma' \to \Sigma$ be a smooth proper map between two non-compact surfaces, where $\Sigma' \ncong \mathbb{R}^2 \ncong \Sigma$; and let $\mathscr{A}$ be a preferred LFCS on Σ such that $f \pitchfork \mathscr{A}$. Then we can properly homotope f to remove all trivial components of the LFCS $f^{-1}(\mathscr{A})$ such that each primitive component of $f^{-1}(\mathscr{A})$ has an open neighbourhood on which this proper homotopy is stationary.

Proof. Since $\Sigma' \ncong \mathbb{R}^2$ and $f^{-1}(\mathscr{A})$ is an LFCS (see Theorem 2.4.3), by Lemma 2.5.1.1, there don't exist infinitely many components $\mathcal{C}'_1, \mathcal{C}'_2, \ldots$ of $f^{-1}(\mathscr{A})$ bounding the disks $\mathcal{D}'_1, \mathcal{D}'_2, \ldots$, respectively such that $\mathcal{C}'_n$ is contained in the interior of $\mathcal{D}'_{n+1}$ for each n. Thus, if $f^{-1}(\mathscr{A})$ has a trivial component, we can introduce the notion of an outermost disk bounded by a component of $f^{-1}(\mathscr{A})$ in the following way: A disk $\mathcal{D}' \subset \Sigma'$ bounded by a component of $f^{-1}(\mathscr{A})$ is called an outermost disk; if given another disk $\mathcal{D}'' \subset \Sigma$ bounded by a component of $f^{-1}(\mathscr{A})$, then either $\mathcal{D}'' \subseteq \mathcal{D}'$ or $\mathcal{D}' \cap \mathcal{D}'' = \varnothing$.

Let $\{\mathcal{D}'_n\}$ be the pairwise disjoint collection (which may be an infinite collection) of all outermost disks. Assume $\mathcal{C}_n$ represents that component of $\mathscr{A}$ for which $f(\partial\mathcal{D}'_n) \subseteq \mathcal{C}_n$. Note $\mathcal{C}_n$ may equal to $\mathcal{C}_m$ even if $m \neq n$.

Now, for each integer n, we will construct a compact bordered subsurface $\mathcal{Z}_n$ with $f(\mathcal{D}'_n) \subseteq \mathcal{Z}_n$ such that $\mathcal{Z}_n \to \infty$. Roughly, $\mathcal{Z}_n$ will be obtained from taking the union of all those complementary components of Σ (if a punctured disk appears, truncate it), which are hit by $f(\mathcal{D}'_n)$.

Fix an integer n. Let $\mathcal{X}'_{n,1}, \ldots, \mathcal{X}'_{n,k_n}$ be the all connected components of $\mathcal{D}'_n \setminus f^{-1}(\mathscr{A})$. By continuity of f, for each $\mathcal{X}'_{n,\ell}$, there exists a complementary component $\mathcal{Y}_{n,\ell}$ of Σ decomposed by $\mathscr{A}$ such that $f(\mathcal{X}'_{n,\ell}) \subseteq \mathcal{Y}_{n,\ell}$ and $\partial\mathcal{X}'_{n,\ell} \subseteq f^{-1}(\partial\mathcal{Y}_{n,\ell})$ (see Figure 2.5.1). For each ℓ, define a compact bordered subsurface $\mathcal{Z}_{n,\ell}$ of Σ as follows: If $\mathcal{Y}_{n,\ell}$ is homeomorphic to either $S_{1,1}$, $S_{0,3}$, or $S_{0,2}$; define $\mathcal{Z}_{n,\ell} := \mathcal{Y}_{n,\ell}$. On the other hand, if $\mathcal{Y}_{n,\ell}$ is homeomorphic to $S_{0,1,1}$, then let $\mathcal{Z}_{n,\ell}$ be an annulus in $\mathcal{Y}_{n,\ell}$ such that $\partial\mathcal{Z}_{n,\ell} \cap \partial\mathcal{Y}_{n,\ell} = \partial\mathcal{Y}_{n,\ell}$ and $f\left(\overline{\mathcal{X}'_{n,\ell}}\right) \subseteq \mathcal{Z}_{n,\ell}$. Finally, define $\mathcal{Z}_n := \mathcal{Z}_{n,1} \cup \cdots \cup \mathcal{Z}_{n,k_n}$.

Now, we show $\mathcal{Z}_n \to \infty$. So, consider a compact subset $\mathcal{K}$ of Σ. Let $\mathbf{S}_1, \ldots, \mathbf{S}_m$ be a collection of complementary components of Σ decomposed by $\mathscr{A}$ such that $\mathcal{K} \subseteq \operatorname{int}\left(\bigcup_{\ell=1}^{m} \mathbf{S}_\ell\right)$. Define

$\mathbf{S} := \bigcup_{\ell=1}^{m} \mathbf{S}_\ell$. Thus, for an integer n, $f(\mathcal{D}'_n) \cap \mathbf{S} \neq \varnothing$ if and only if $\mathcal{D}'_n$ contains at least one component of $\bigcup_{\ell=1}^{m} f^{-1}(\partial \mathbf{S}_\ell)$. This is due to the construction of each $\mathcal{Z}_n$; see Figure 2.5.1. For each component $\mathcal{C}$ of $\mathscr{A}$, Theorem 2.4.3 tells that $f^{-1}(\mathcal{C})$ has only finitely many components. So $\mathcal{D}'_n$ doesn't contain any component of $\bigcup_{\ell=1}^{m} f^{-1}(\partial \mathbf{S}_\ell)$ for all sufficiently large n, i.e., $f(\mathcal{D}'_n) \cap \mathbf{S} = \varnothing$ for all sufficiently large n. Since $\mathcal{K} \subseteq \operatorname{int}(\mathbf{S})$ and each $\mathcal{Z}_n$ is obtained from taking the union of all those complementary components of Σ (if a punctured disk appears, truncate it), which are hit by $f(\mathcal{D}'_n)$, we can say that $\mathcal{Z}_n \cap \mathcal{K} = \varnothing$ for all sufficiently large n. Therefore, $\mathcal{Z}_n \to \infty$ as $\mathcal{K}$ is an arbitrary compact subset of Σ.

For each n, adding a small external collar to one of the boundary components of $\mathcal{Z}_n$ (if needed), we can construct a compact bordered surface Σ_n with $\mathcal{C}_n \subseteq \operatorname{int}(\Sigma_n)$, $f(\mathcal{D}'_n) \subseteq \Sigma_n$ such that $\{\Sigma_n\}$ is a locally finite collection, i.e., $\Sigma_n \to \infty$ (see Figure 2.5.1).

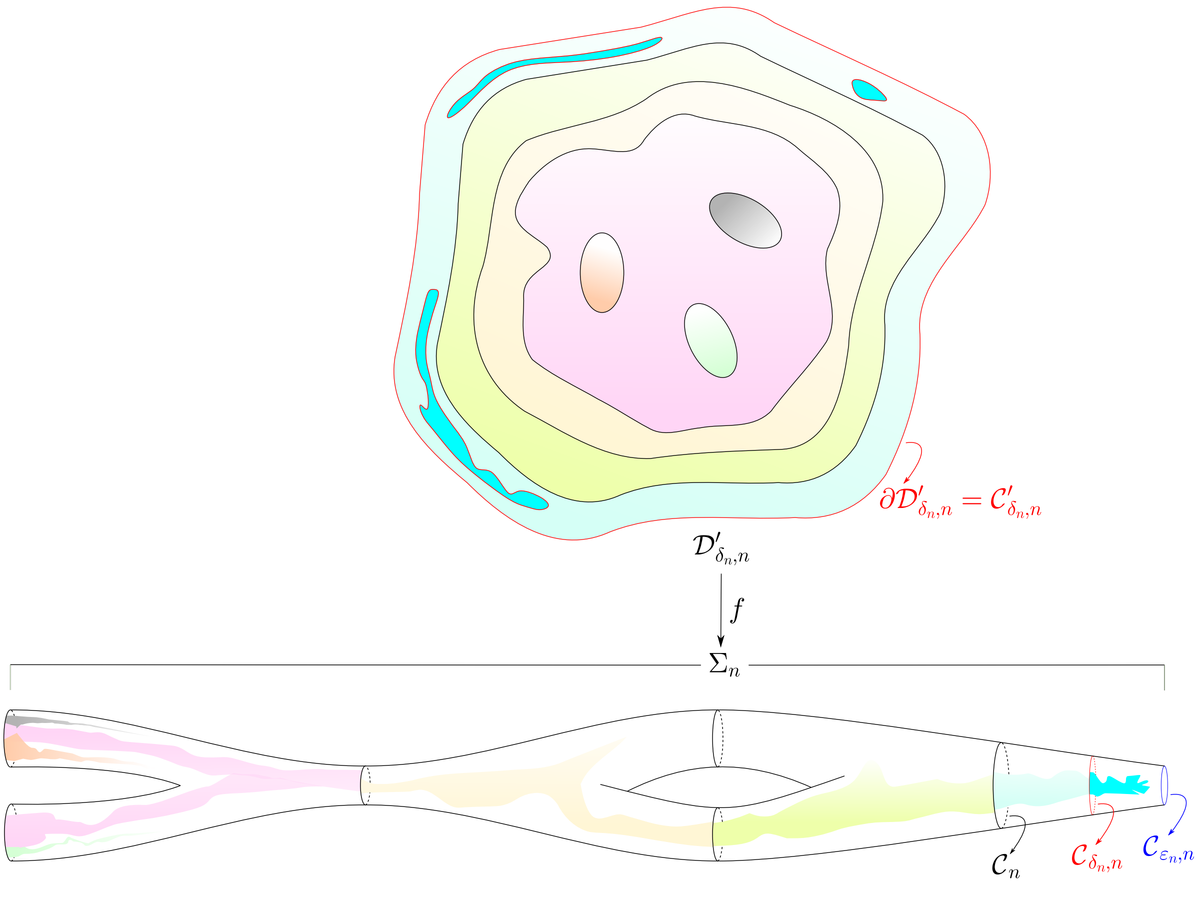

Fig. 2.5.1 Each component of $\mathcal{D}'_n \setminus f^{-1}(\mathscr{A})$ maps into a component of $\Sigma \setminus \mathscr{A}$. This fact, together with Theorem B.5, provides Σ_n. A black circle denotes a component of either $\mathscr{A}$ or a component of $f^{-1}(\mathscr{A})$.

For each n, write $\mathcal{C}'_n := \partial \mathcal{D}'_n$. Thus $f(\mathcal{C}'_n) \subseteq \mathcal{C}_n$. Since $\mathcal{C}_n \subseteq \operatorname{int}(\Sigma_n)$, using Theorem B.1, choose a one-sided tubular neighbourhood $\mathcal{C}_n \times [0, \varepsilon_n]$ of $\mathcal{C}_n$ in Σ with $\mathcal{C}_n \times 0 \equiv \mathcal{C}_n$ such that $f \,\overline{\pitchfork}\, (\mathcal{C}_n \times t_n)$ for each $t_n \in [0, \varepsilon_n]$ and $\mathcal{C}_n \times [0, \varepsilon_n] \subseteq \Sigma_n$. Without loss of generality, we may further assume that $f(x') \in \mathcal{C}_n \times [0, \varepsilon_n]$ for each $x' \in \Sigma' \setminus \mathcal{D}'_n$ sufficiently near to $\mathcal{C}'_n$. Next, since $f^{-1}(\mathscr{A})$ is an LFCS, for each n, Theorem B.3 gives a one-sided compact tubular neighbourhood $\mathcal{U}'_n$ of $\mathcal{C}'_n$

such that the following hold: $\mathcal{U}'_n \cap \mathcal{D}'_n = \mathcal{C}'_n = \mathcal{U}'_n \cap f^{-1}(\mathscr{A}), f(\mathcal{U}'_n) \subseteq \mathcal{C}_n \times [0, \varepsilon_n]$ for each n; and $\mathcal{U}'_n \cap \mathcal{U}'_m = \varnothing$ for $m \neq n$. Finally, Theorem B.5 gives $\delta_n \in (0, \varepsilon_n)$ and a component $\mathcal{C}'_{\delta_n,n}$ of $f^{-1}(\mathcal{C}_{\delta_n,n})$ such that $\mathcal{C}'_{\delta_n,n}$ bounds a disk $\mathcal{D}'_{\delta_n,n}$ in Σ' with $(\mathcal{U}'_n \cup \mathcal{D}'_n) \supseteq \mathcal{D}'_{\delta_n,n} \supset \operatorname{int}(\mathcal{D}'_{\delta_n,n}) \supset \mathcal{D}'_n$ (equivalently, $\mathcal{U}'_n$ contains the annulus co-bounded by $\mathcal{C}'_{\delta_n,n}$ and $\mathcal{C}'_n$) and $f\left(\mathcal{D}'_{\delta_n,n} \setminus \operatorname{int}(\mathcal{D}'_n)\right) \subseteq \mathcal{C}_n \times [0, \varepsilon_n]$. Thus, $\mathcal{D}'_{\delta_n,n} \cap f^{-1}(\mathscr{A}) = \mathcal{D}'_n \cap f^{-1}(\mathscr{A}), f(\mathcal{D}'_{\delta_n,n}) \subseteq \Sigma_n$ for each n; and $\mathcal{D}'_{\delta_n,n} \cap \mathcal{D}'_{\delta_m,m} = \varnothing$ when $m \neq n$.

Since $\mathcal{C}_{\delta_n,n}$ co-bounds an annulus with the primitive circle $\mathcal{C}_n$, the inclusion $\mathcal{C}_{\delta_n,n} \hookrightarrow \Sigma_n$ is π_1-injective (see Theorem 1.2.2). Also, Σ_n is homotopy equivalent to $\bigvee_{\text{finite}} \mathbb{S}^1$, which implies that the universal cover of Σ_n is contractible, and thus $\pi_2(\Sigma_n) = 0$. Therefore, exactness of

$$\cdots \longrightarrow \pi_2(\Sigma_n) \longrightarrow \pi_2(\Sigma_n, \mathcal{C}_{\delta_n,n}) \longrightarrow \pi_1(\mathcal{C}_{\delta_n,n}) \longrightarrow \pi_1(\Sigma_n) \longrightarrow \cdots$$

gives $\pi_2(\Sigma_n, \mathcal{C}_{\delta_n,n}) = 0$, i.e., we have a homotopy $H_n \colon \mathcal{D}'_{\delta_n,n} \times [0,1] \to \Sigma_n$ relative to $\mathcal{C}'_{\delta_n,n}$ from $f|(\mathcal{D}'_{\delta_n,n}, \mathcal{C}'_{\delta_n,n}) \to (\Sigma_n, \mathcal{C}_{\delta_n,n})$ to a map $\mathcal{D}'_{\delta_n,n} \to \mathcal{C}_{\delta_n,n}$ for each n; see [51, Lemma 4.6.]. Now, to conclude, apply Lemma 2.5.1.2 on $\{H_n\}$. □

Remark 2.5.1.6 In Theorem 2.5.1.5, the number of components of $\mathscr{A}$ can be infinite; thus, the number of trivial components of $f^{-1}(\mathscr{A})$ can be infinite. That's why we have removed all trivial components of $f^{-1}(\mathscr{A})$ by a single proper homotopy upon considering all outermost disks simultaneously. This process is in contrast to the finite-type surface theory, where the number of decomposition circles is finite, and therefore, all trivial circles in the collection of transversal pre-images of all decomposition circles can be removed one by one, considering the notion of an innermost disk.

2.5.2 Homotope a degree-one map between circles to a homeomorphism

Previously, we have removed all trivial components, keeping a neighbourhood of each primitive component stationary from the transversal pre-image $f^{-1}(\mathscr{A})$ of a preferred LFCS $\mathscr{A}$. In this section, we properly homotope our pseudo proper homotopy equivalence $f \colon \Sigma' \to \Sigma$ to send each component $\mathcal{C}'$ of $f^{-1}(\mathscr{A})$ homeomorphically onto a component $\mathcal{C}$ of $\mathscr{A}$ so that the restriction of f to a small one-sided tubular neighbourhood $\mathcal{C}' \times [1,2]$ of $\mathcal{C}'$ (on either side of $\mathcal{C}'$) can be described by the following homeomorphism:

$$\mathcal{C}' \times [1,2] \ni (z,t) \longmapsto \big(f(z), t\big) \in \mathcal{C} \times [1,2].$$

First, we fix a few notations. Define $\partial_\varepsilon := \mathbb{S}^1 \times \varepsilon$ for $\varepsilon \in \mathbb{R}$ and $\mathbf{I} := [0,1]$. Let $p \colon \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1$ be the projection. The following lemma roughly says that a self-map of $\mathbb{S}^1 \times [0,2]$ can be homotoped rel. $\mathbb{S}^1 \times 0$ to map $\mathbb{S}^1 \times [1,2]$ into itself by the product $\theta \times \mathrm{Id}_{[1,2]}$, where θ is a self-map of $\mathbb{S}^1$.

Lemma 2.5.2.1 Let Φ be a self-map of $A := \mathbb{S}^1 \times [0,2]$ such that $\Phi^{-1}(\partial_b) = \partial_b$ for each $b \in \{0,2\}$. If we are given a map $\varphi_2 \colon \partial_2 \to \partial_2$ and a homotopy $h_{(2)} \colon \partial_2 \times \mathbf{I} \to \partial_2$ from $\Phi|\partial_2 \to \partial_2$ to φ_2, then Φ can be homotoped relative to ∂_0 to map $\mathbb{S}^1 \times [0,1]$ into $\mathbb{S}^1 \times [0,1]$ and to satisfy $\Phi(-,r) = \big(p \circ \varphi_2(-,2), r\big)$ for each $r \in [1,2]$.

Remark 2.5.2.2 In Lemma 2.5.2.1, up to homotopy, φ_2 is either a constant map or a covering map.

Proof. Homotope Φ relative to $\partial_0 \cup \partial_2$ so that $\Phi(\mathbb{S}^1 \times [0,1]) \subseteq \mathbb{S}^1 \times [0,1]$ and $\Phi(z,r) = \big(p \circ \Phi(z,2), r\big)$ for all $(z,r) \in \mathbb{S}^1 \times [1,2]$. For each $r \in [1,2]$, $h_{(2)}$ provides a homotopy $h_{(r)} \colon \partial_r \times \mathbf{I} \to \partial_r$. Let $H \colon (\partial_0 \cup \partial_1) \times \mathbf{I} \to \partial_0 \cup \partial_1$ be the homotopy defined as follows: $H|\partial_1 \times \mathbf{I} = h_{(1)}$ and $H(-,t)|\partial_0 = \Phi|\partial_0$ for any $t \in [0,1]$. Homotopy extension theorem gives a homotopy $\widetilde{H} \colon \mathbb{S}^1 \times [0,1] \times \mathbf{I} \to \mathbb{S}^1 \times [0,1]$ such that $\widetilde{H}|(\partial_0 \cup \partial_1) \times \mathbf{I} = H$. Finally, paste $\widetilde{H}$ with the collection $h_{(r)},\ 1 \leq r \leq 2$. $\square$

The following theorem is the simple modification (in the proper category) of the analogue theorem for closed surfaces.

Theorem 2.5.2.3 Let $f \colon \Sigma' \to \Sigma$ be a smooth pseudo proper homotopy equivalence between two non-compact surfaces, where $\Sigma' \ncong \mathbb{R}^2 \ncong \Sigma$; and let $\mathscr{A}$ be a preferred LFCS on Σ such that $f \pitchfork \mathscr{A}$. Then f can be properly homotoped to remove all trivial components of the $f^{-1}(\mathscr{A})$ as well as to map each primitive component of $f^{-1}(\mathscr{A})$ homeomorphically onto a component of $\mathscr{A}$. Moreover, after this proper homotopy, near each component of $f^{-1}(\mathscr{A})$, the map f can be described as follows:

Let $\mathcal{C}'_{\mathbf{p}}$ (resp. $\mathcal{C}$) be a component of $f^{-1}(\mathscr{A})$ (resp. $\mathscr{A}$) such that $f|\mathcal{C}'_{\mathbf{p}} \to \mathcal{C}$ is a homeomorphism. Then $\mathcal{C}'_{\mathbf{p}}$ (resp. $\mathcal{C}$) has two one-sided tubular neighbourhoods $\mathcal{M}'$ and $\mathcal{N}'$ (resp. $\mathcal{M}$ and $\mathcal{N}$) with some specific identifications $(\mathcal{M}', \mathcal{C}'_{\mathbf{p}}) \cong (\mathcal{C}'_{\mathbf{p}} \times [1,2], \mathcal{C}'_{\mathbf{p}} \times 2) \cong (\mathcal{N}', \mathcal{C}'_{\mathbf{p}})$ (resp. $(\mathcal{M}, \mathcal{C}) \cong (\mathcal{C} \times [1,2], \mathcal{C} \times 2) \cong (\mathcal{N}, \mathcal{C})$) such that the following hold:

1. $\mathcal{M}' \cup \mathcal{N}'$ is a (two-sided) tubular neighbourhood of $\mathcal{C}'_{\mathbf{p}}$;
2. $f|\mathcal{M}' \to \mathcal{M}$ and $f|\mathcal{N}' \to \mathcal{N}$ are homeomorphisms described by $\mathcal{C}'_{\mathbf{p}} \times [1,2] \ni (z,t) \longmapsto \big(f(z), t\big) \in \mathcal{C} \times [1,2]$.

Remark 2.5.2.4 In Theorem 2.5.2.3, though $\mathcal{M}' \cup \mathcal{N}'$ is a (two-sided) tubular neighbourhood of $\mathcal{C}'_{\mathbf{p}}$, both $\mathcal{M}$ and $\mathcal{N}$ may lie on the same side of $\mathcal{C}$, i.e., $\mathcal{M} \cup \mathcal{N}$ may not be a two-sided tubular neighbourhood of $\mathcal{C}$.

Proof. Let $\{\mathcal{C}'_{\mathbf{p}n}\}$ be the collection of all primitive components of $f^{-1}(\mathscr{A})$. Assume $\mathcal{C}_n$ represents that component of $\mathscr{A}$ for which $f(\mathcal{C}'_{\mathbf{p}n}) \subseteq \mathcal{C}_n$. Note $\mathcal{C}_n$ may equal to $\mathcal{C}_m$ even if $m \neq n$.

Claim 2.5.2.4.1 There are one-sided compact tubular neighbourhoods $\mathcal{U}'_n, \mathcal{V}'_n (\subseteq \Sigma')$ of $\mathcal{C}'_{\mathbf{p}n}$, and there are one-sided compact tubular neighbourhoods $\mathcal{U}_n, \mathcal{V}_n (\subseteq \Sigma)$ of $\mathcal{C}_n$ such that after defining $\mathcal{T}'_n := \mathcal{U}'_n \cup \mathcal{V}'_n$, the following hold:

(1) $\widetilde{\mathscr{A}} := \mathscr{A} \cup \{(\partial \mathcal{U}_n \cup \partial \mathcal{V}_n) \setminus \mathcal{C}_n\}_n$ is an LFCS and $f \pitchfork \widetilde{\mathscr{A}}$;

(2) $\partial \mathcal{U}'_n \setminus \mathcal{C}'_{\mathbf{p}n}$ (resp. $\partial \mathcal{V}'_n \setminus \mathcal{C}'_{\mathbf{p}n}$) is the only component of $f^{-1}(\partial \mathcal{U}_n \setminus \mathcal{C}_n) \cap \mathcal{U}'_n$ (resp. $f^{-1}(\partial \mathcal{V}_n \setminus \mathcal{C}_n) \cap \mathcal{V}'_n$) that co-bounds an annulus with $\mathcal{C}'_{\mathbf{p}n}$ (see Figure 2.5.2);

(3) each point of $\operatorname{int}(\mathcal{U}'_n)$ (resp. $\operatorname{int}(\mathcal{V}'_n)$) that is sufficiently near to $\mathcal{C}'_{\mathbf{p}n}$ is mapped into $\operatorname{int}(\mathcal{U}_n)$ (resp. $\operatorname{int}(\mathcal{V}_n)$);

(4) $\mathcal{T}'_n$ is a two-sided tubular neighbourhood of $\mathcal{C}'_{\mathbf{p}n}$ with $f^{-1}(\mathscr{A}) \cap \mathcal{T}'_n = \mathcal{C}'_{\mathbf{p}n}$; and

(5) $\mathcal{T}'_n \cap \mathcal{T}'_m = \varnothing$ if $m \neq n$, and $(\mathcal{U}_n \cup \mathcal{V}_n) \to \infty$.

Proof of Claim 2.5.2.4.1. For any positive integer n_0, Theorem 2.4.3 says that the set $\{m \in \mathbb{N} : \mathcal{C}_m = \mathcal{C}_{n_0}\}$ is finite. Also, $\mathscr{A}$ is locally finite. Thus $\{\mathcal{C}_n : n \in \mathbb{N}\}$ is locally finite. So, for each n, there exists a two-sided tubular neighbourhood $\mathcal{C}_n \times [-\varepsilon_n, \varepsilon_n]$ of $\mathcal{C}_n$ with $\mathcal{C}_n \times 0 \equiv \mathcal{C}_n$ such that $\{\mathcal{C}_n \times [-\varepsilon_n, \varepsilon_n] : n \in \mathbb{N}\}$ is a locally finite collection. Further, for each $n \in \mathbb{N}$, we may assume that $f \pitchfork (\mathcal{C}_n \times t_n)$ whenever $t_n \in [-\varepsilon_n, \varepsilon_n]$ by Theorem B.1.

Now, since $f^{-1}(\mathscr{A})$ is a locally finite collection, for each n, there are one-sided compact tubular neighbourhoods $\mathcal{U}'_n, \mathcal{V}'_n$ of $\mathcal{C}'_{\mathbf{p}n}$ in Σ' such that after defining $\mathcal{T}'_n := \mathcal{U}'_n \cup \mathcal{V}'_n$, the following hold: $\mathcal{T}'_n$ is a two-sided tubular neighbourhood of $\mathcal{C}'_{\mathbf{p}n}$, $f^{-1}(\mathscr{A}) \cap \mathcal{T}'_n = \mathcal{C}'_{\mathbf{p}n}$, and $\mathcal{T}'_n \cap \mathcal{T}'_m = \varnothing$ if $m \neq n$. Moreover, using Theorem B.3, $f(\mathcal{T}'_n) \subseteq \mathcal{C}_n \times [-\varepsilon_n, \varepsilon_n]$ can also be assumed for each n.

Next, by Theorem B.5, we may further assume $\partial\mathcal{U}'_n \setminus \mathcal{C}'_{\mathbf{p}n}$ (resp. $\partial\mathcal{V}'_n \setminus \mathcal{C}'_{\mathbf{p}n}$) is a component of $f^{-1}(\mathcal{C}_n \times x_n)$ (resp. $f^{-1}(\mathcal{C}_n \times y_n)$) for some $x_n, y_n \in (-\varepsilon_n, 0) \cup (0, \varepsilon_n)$ such that after defining $\mathcal{U}_n$ (resp. $\mathcal{V}_n$) as the annulus in $\mathcal{C}_n \times [-\varepsilon_n, \varepsilon_n]$ co-bounded by $\mathcal{C}_n \times 0$ and $\mathcal{C}_n \times x_n$ (resp. $\mathcal{C}_n \times y_n$), both (2) and (3) of Claim 2.5.2.4.1 do hold. Finally, $\mathcal{C}_n \times [-\varepsilon_n, \varepsilon_n] \to \infty$ implies $(\mathcal{U}_n \cup \mathcal{V}_n) \to \infty$. □

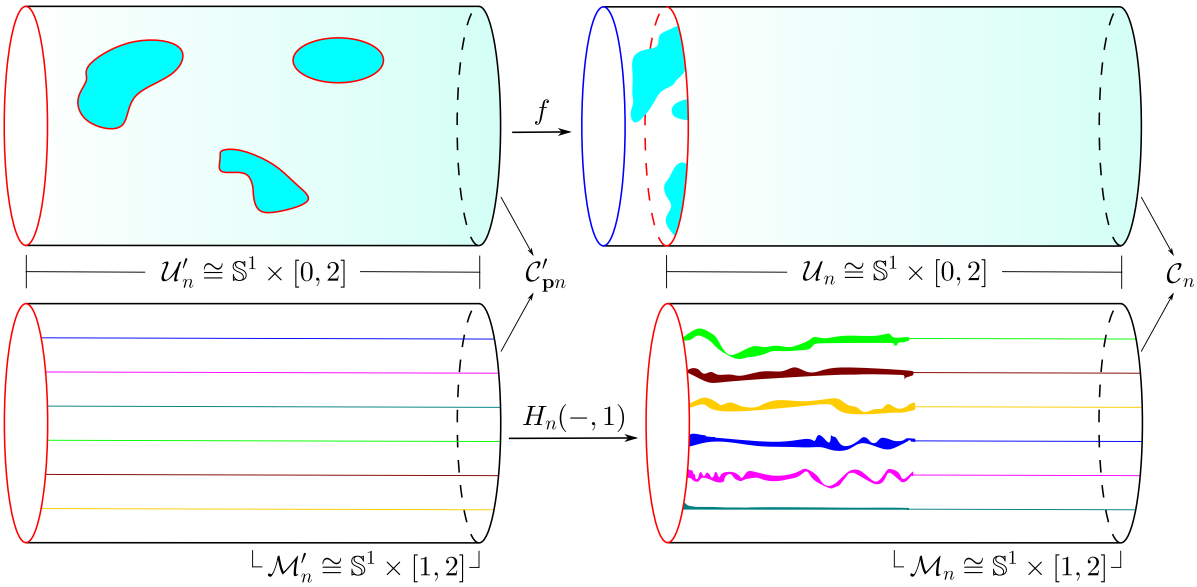

Fig. 2.5.2 On the top: description of $f|\mathcal{U}'_n \to \mathcal{U}_n$, using Theorem B.5. At the bottom: after removing all trivial components of $f^{-1}(\partial\mathcal{U}_n \setminus \mathcal{C}_n)$ from $\mathcal{U}'_n$ and then applying Lemma 2.5.2.1 to $f|\mathcal{U}'_n \to \mathcal{U}_n$, we obtain $H_n(-,1)|\mathcal{U}'_n \to \mathcal{U}_n$.

Using Theorem 2.5.1.5, keeping stationary a neighbourhood of each primitive component of $f^{-1}(\widetilde{\mathscr{A}})$, we can properly homotope f to remove all trivial components from $f^{-1}(\widetilde{\mathscr{A}})$. So, after this proper homotopy, (2) and (3) of Claim 2.5.2.4.1 imply that $f(\mathcal{U}'_n) \subseteq \mathcal{U}_n$, $f^{-1}(\partial\mathcal{U}_n) \cap \mathcal{U}'_n = \partial\mathcal{U}'_n$

and $f(\mathcal{V}'_n) \subseteq \mathcal{V}_n$, $f^{-1}(\partial\mathcal{V}_n) \cap \mathcal{V}'_n = \partial\mathcal{V}'_n$. Notice the abuse of notation; the initial and final maps of this proper homotopy are both denoted by f.

Now, let $h_n \colon \mathcal{C}'_{\mathbf{p}n} \times [0,1] \to \mathcal{C}_n$ be a homotopy from $f|\mathcal{C}'_{\mathbf{p}n} \to \mathcal{C}_n$ such that $h_n(-,1)$ is either a constant map or a covering map between two circles. Applying Lemma 2.5.2.1 on $f|\mathcal{U}'_n \to \mathcal{U}_n$ and $f|\mathcal{V}'_n \to \mathcal{V}_n$ separately upon considering h_n; a homotopy $H_n \colon \mathcal{T}'_n \times [0,1] \to \mathcal{U}_n \cup \mathcal{V}_n$ relative to $\partial\mathcal{T}'_n$ exists such that $H_n(-,0) = f|\mathcal{T}'_n$, $(H_n(-,1))^{-1}(\mathcal{C}_n) = \mathcal{C}'_{\mathbf{p}n}$, and $H_n(-,1)|\mathcal{C}'_{\mathbf{p}n} \to \mathcal{C}_n$ is the same as $h_n(-,1)$ (see Figure 2.5.2).

Next, (5) of Claim 2.5.2.4.1 tells that we can apply Lemma 2.5.1.2 on $\{H_n\}$ to obtain a proper homotopy $\mathcal{H} \colon \Sigma' \times [0,1] \to \Sigma$ starting from f. Next, being an isomorphism, $\pi_1(f) = \pi_1(\mathcal{H}(-,1))$ preserves primitiveness, i.e., $h_n(-,1) = \mathcal{H}(-,1)|\mathcal{C}'_{\mathbf{p}n} \to \mathcal{C}_n$ must be a homeomorphism. Thus, $\mathcal{H}$ is our ultimate required homotopy.

Finally, we need to describe f near each component of $f^{-1}(\mathscr{A})$ after the proper homotopy $\mathcal{H}$. Abusing notation, the final map of $\mathcal{H}$ will be denoted by f. Since Lemma 2.5.2.1 is being used, we have $\mathcal{M}'_n \subseteq \mathcal{U}'_n$ and $\mathcal{M}_n \subseteq \mathcal{U}_n$ with the identifications $(\mathcal{M}'_n, \mathcal{C}'_{\mathbf{p}n}) \cong (\mathcal{C}'_{\mathbf{p}n} \times [1,2], \mathcal{C}'_{\mathbf{p}n} \times 2)$, $(\mathcal{M}_n, \mathcal{C}_n) \cong (\mathcal{C}_n \times [1,2], \mathcal{C}_n \times 2)$ such that after the proper homotopy $\mathcal{H} \colon \Sigma' \times [0,1] \to \Sigma$, the map f sends $\mathcal{C}'_{\mathbf{p}n} \times r$ onto $\mathcal{C}_n \times r$ using the homeomorphism $f|\mathcal{C}'_{\mathbf{p}n} \to \mathcal{C}_n$ for all $r \in [1,2]$ (see Figure 2.5.2). Similar reasoning for $f|\mathcal{V}'_n \to \mathcal{V}_n$. $\square$

The following proposition tells what happens if we drop the phrase "homotopy equivalence" in the statement of Theorem 2.5.2.3. Its proof is almost the same.

Proposition 2.5.2.5 Let $f \colon \Sigma' \to \Sigma$ be a smooth proper map between two non-compact surfaces, where $\Sigma' \ncong \mathbb{R}^2 \ncong \Sigma$; and let $\mathscr{A}$ be a preferred LFCS on Σ such that $f \pitchfork \mathscr{A}$. Then f can be properly homotoped to remove all trivial components of the $f^{-1}(\mathscr{A})$ as well as to map each primitive component of $f^{-1}(\mathscr{A})$ into a component of $\mathscr{A}$ so that for any component $\mathcal{C}$ of $\mathscr{A}$ and any primitive component $\mathcal{C}'_{\mathbf{p}}$ of $f^{-1}(\mathcal{C})$ after this proper homotopy, $f|\mathcal{C}'_{\mathbf{p}} \to \mathcal{C}$ is either a constant map or a covering map.

2.5.3 Annulus removal

In the previous two sections, after removing all trivial components from the transversal pre-image of a decomposition circle, the remaining primitive circles have been mapped homeomorphically to that decomposition circle. This section aims to remove all these primitive circles except one from the inverse image of each decomposition circle using the following three steps: annulus bounding, then annulus compression, and finally, annulus pushing.

At first, annulus bounding: Consider the collection of inverse images of all decomposition circles. The following lemma says that any two circles in this collection co-bound an annulus in the domain if and only if their images are the same. In other words, in the domain, by pasting all small annuli, we get the outermost annulus corresponding to a decomposition circle.

Lemma 2.5.3.1 Let $f \colon \Sigma' \to \Sigma$ be a homotopy equivalence between two non-compact surfaces, and let $\mathscr{A}'$, $\mathscr{A}$ be two LFCS on Σ', Σ, respectively, such that f maps each component of $\mathscr{A}'$

homeomorphically onto a component of $\mathscr{A}$. Suppose each component of $\mathscr{A}$ is primitive, and any two distinct components of $\mathscr{A}$ don't co-bound an annulus in Σ. Let $\mathcal{C}_0', \mathcal{C}_1'$ be two distinct components of $\mathscr{A}'$. Then $\mathcal{C}_0', \mathcal{C}_1'$ co-bound an annulus in Σ' if and only if $f(\mathcal{C}_0') = f(\mathcal{C}_1')$.

Proof. To prove the only if part, let $\Phi\colon \mathbb{S}^1 \times [0,1] \hookrightarrow \Sigma'$ be an embedding such that $\Phi(\mathbb{S}^1, k) = \mathcal{C}_k'$ for $k = 0, 1$. Note that f maps each component of $\mathscr{A}'$ homeomorphically onto a component of $\mathscr{A}$, and each component of $\mathscr{A}$ is a primitive circle on Σ. Thus, the embeddings $f\Phi(-,0), f\Phi(-,1)\colon \mathbb{S}^1 \hookrightarrow \Sigma$ are freely homotopic; and hence $f\Phi(-,0), f\Phi(-,1)\colon \mathbb{S}^1 \hookrightarrow \Sigma$ represent the same non-trivial conjugacy class in $\pi_1(\Sigma, *)$. Since any two distinct components of $\mathscr{A}$ don't co-bound an annulus in Σ, by Theorem 1.2.3, $f(\mathcal{C}_0') = f(\mathcal{C}_1')$.

To prove the if part, let $g\colon \Sigma \to \Sigma'$ be a homotopy inverse of f, and let $\mathcal{C}$ be the component of $\mathscr{A}$ defined by $\mathcal{C} := f(\mathcal{C}_0') = f(\mathcal{C}_1')$. Now, $f|\mathcal{C}_k' \to f(\mathcal{C}_k')$ is a homeomorphism for $k = 0, 1$. Thus, for a homeomorphism $j\colon \mathbb{S}^1 \xrightarrow{\cong} \mathcal{C}$, there are homeomorphisms $\ell_0\colon \mathbb{S}^1 \xrightarrow{\cong} \mathcal{C}_0'$ and $\ell_1\colon \mathbb{S}^1 \xrightarrow{\cong} \mathcal{C}_1'$ such that $f\ell_0 = j = f\ell_1$. Since $\ell_0 \simeq gf\ell_0 = gj = gf\ell_1 \simeq \ell_1$, applying Theorem 1.2.3 to ℓ_0, ℓ_1, we are done. $\square$

The following theorem, which will be used to compress each annulus bounded by two primitive circles of the domain, roughly states that most homotopies of a circle embedded in a surface are trivial. We will provide a proof of it based on [95, Lemma 4.9.15.].

Theorem 2.5.3.2 Let $\mathbf{S}$ be a compact bordered surface other than the disk, and let Φ be a map from $\mathbf{A} := \mathbb{S}^1 \times [0,1]$ to $\mathbf{S}$ such that $\Phi\big(\mathrm{int}(\mathbf{A})\big) \subseteq \mathrm{int}(\mathbf{S})$ and there is a boundary component $\mathbf{C}$ of $\mathbf{S}$ for which $\Phi(-,0), \Phi(-,1)\colon \mathbb{S}^1 \xrightarrow{\cong} \mathbf{C}$ are the same homeomorphism. Then Φ can be homotoped relative to $\partial\mathbf{A}$ to map $\mathbf{A}$ onto $\mathbf{C}$.

Proof. By our hypothesis $\mathbf{S} = S_{g,b}$ for some $(g,b) \neq (0,1)$. Embed $S_{g,b}$ into $S_{g,0,b}$ so that each components of $\partial S_{g,b}$ bounds a puncture of $S_{g,0,b}$. So, $\Phi\colon \mathbf{A} \to S_{g,0,b}$ is π_1-injcetive. Now, we have a covering $p\colon \mathbb{R}^2 \backslash \mathbf{0} \to S_{g,0,b}$ corresponding to the subgroup $\pi_1(\mathbf{A}) \cong \mathbb{Z}$ of $\pi_1(S_{g,0,b})$. Let $\widetilde{\Phi}\colon \mathbf{A} \to \mathbb{R}^2\backslash\mathbf{0}$ be a lift of Φ, i.e., $p \circ \widetilde{\Phi} = \Phi$. By the hypothesis, $\widetilde{\Phi}$ sends $\mathbb{S}^1 \times k$ homeomorphically onto $\widetilde{C}_k := \widetilde{\Phi}(\mathbb{S}^1 \times k)$ and p sends $\widetilde{C}_k$ homeomorphically onto C, for each $k = 0, 1$. Since $\widetilde{\Phi}$ is π_1-injective, both $\widetilde{C}_0$ and $\widetilde{C}_1$ bound $\mathbf{0}$. Also, p is a local homeomorphism implies either $\widetilde{C}_0 \cap \widetilde{C}_1 = \varnothing$ or $\widetilde{C}_0 = \widetilde{C}_1$.

Claim 2.5.3.2.1 *$\widetilde{C}_0 \cap \widetilde{C}_1 = \varnothing$ is not possible.*

Proof of Claim 2.5.3.2.1. On the contrary, assume $\widetilde{C}_0 \cap \widetilde{C}_1 = \varnothing$. Let $\widetilde{\mathbf{A}}$ be the annulus in $\mathbb{R}^2\backslash\mathbf{0}$ bounded by $\widetilde{C}_0$ and $\widetilde{C}_1$. Denote the homeomorphism $p|\widetilde{C}_k \to C$ by p_k for each $k = 0, 1$. Orient $\widetilde{\mathbf{A}}$ and give the induced orientation on $\partial\widetilde{\mathbf{A}}$. Identifying each point $x \in \widetilde{C}_0$ with the point $p_1^{-1}p_0(x) \in \widetilde{C}_1$, we get a torus $\widetilde{T}$ from $\widetilde{\mathbf{A}}$ and a map $\widetilde{p}\colon \widetilde{T} \to S_{g,0,b}$ from $p|\widetilde{\mathbf{A}} \to S_{g,0,b}$. Let $\widetilde{C}$ be the circle on $\widetilde{T}$ obtained from $\widetilde{C}_0$ and $\widetilde{C}_1$.

Consider any $k \in \{0,1\}$. Fix an orientation for $S_{g,0,b}$, and give the pull-back orientation on $\mathbb{R}^2\backslash\mathbf{0}$ using the covering p. Also, orient $\widetilde{C}_k$ and C so that $p_k\colon \widetilde{C}_k \to C$ is an orientation-preserving homeomorphism. Any orientation-preserving embedding $C \times [-1,1] \hookrightarrow S_{g,0,b}$ with the identification $C\times 0 = C$ gives a lifted orientation-preserving embedding $\widetilde{C}_k\times[-1,1] \hookrightarrow \mathbb{R}^2\backslash\mathbf{0}$ with the identifications $\widetilde{C}_k\times 0 = \widetilde{C}_k$ such that p sends $(x,t) \in \widetilde{C}_k\times[-1,1]$ to $\big(p_k(x),t\big) \in C\times[-1,1]$. Thus, small neighbourhoods of $\widetilde{C}_0, \widetilde{C}_1$ in $\widetilde{\mathbf{A}}$ are mapped homeomorphically by p to opposite sides of C in $C\times[-1,1]$. Using the hypothesis $\Phi\big(\text{int}(\mathbf{A})\big) \subsetneq \text{int}(S_{g,b})$ and the previous paragraph, $[0,1] \ni t \longmapsto \widetilde{\Phi}(1,t) \in \widetilde{\mathbf{A}}$ gives a loop $\widetilde{\ell}$ on $\widetilde{T}$ such that the following hold:

(1) $[\widetilde{\ell}]$ and $[\widetilde{C}]$ generate $\pi_1(\widetilde{T}) \cong \mathbb{Z}\times\mathbb{Z}$.

(2) $\widetilde{p}\widetilde{\ell}(t) \in C$ if and only if $t = 0, 1$.

(3) $\widetilde{p}\widetilde{\ell}(t)$ maps $(0,\varepsilon]$ and $[1-\varepsilon, 1)$ to opposite sides of C in $C\times[-1,1]$ for small enough $\varepsilon > 0$.

Observations (1), (2), and (3) tell $\widetilde{p}\colon \widetilde{T} \to S_{g,0,b}$ is π_1-injective, which is impossible as a subgroup of a free group is free. This completes the proof of the claim. $\square$

By Claim 2.5.3.2.1, $\widetilde{C}_0 = \widetilde{C}_1$, in any case. If $\widetilde{H}$ a strong deformation retract of $\mathbb{R}^2\backslash\mathbf{0}$ onto $\widetilde{C}_0 = \widetilde{C}_1$, then our required homotopy is defined by $H(a,t) := p\widetilde{H}\big(\widetilde{\Phi}(a),t\big)$ for all $a \in \mathbf{A}, t\in[0,1]$. $\square$

Remark 2.5.3.3 To extend Theorem 2.5.3.2 to any (bordered) surface, we need to assume that $\Phi^{-1}(C) = \mathbb{S}^1\times\{0,1\}$ instead of $\Phi\big(\text{int}(\mathbf{A})\big) \subsetneq \text{int}(S_{g,b})$.

Remark 2.5.3.4 Note that for any $\theta\colon \mathbb{S}^1\times[0,1] \to \mathbb{S}^1\times[0,1]$ which sends $\mathbb{S}^1\times\{0,1\}$ into $\mathbb{S}^1\times 0$ can be homotoped rel. $\mathbb{S}^1\times\{0,1\}$ to send $\mathbb{S}^1\times[0,1]$ into $\mathbb{S}^1\times 0$. To see this, let $\theta_1\colon \mathbb{S}^1\times[0,1]\to\mathbb{S}^1$ and $\theta_2\colon \mathbb{S}^1\times[0,1]\to[0,1]$ be the components of θ. Then $G\colon \mathbb{S}^1\times[0,1]\times[0,1] \ni (z,s,t) \longmapsto \big(\theta_1(z,s), (1-t)\theta_2(z,s)\big) \in \mathbb{S}^1\times[0,1]$ is a homotopy rel. $\mathbb{S}^1\times\{0,1\}$ such that $G(-,0) = \theta$ and $G(-,1) \subseteq \mathbb{S}^1\times 0$.

The following theorem considers the last two steps - annulus compressing and annulus pushing. At first, by a proper homotopy, each outermost annulus will be mapped onto its decomposition circle; after that, by another proper homotopy, each outermost annulus will be pushed into a one-sided tubular neighbourhood of one of its boundary components.

Theorem 2.5.3.5 Let $f\colon \Sigma' \to \Sigma$ be a smooth pseudo proper homotopy equivalence between two non-compact surfaces, where $\Sigma' \ncong \mathbb{R}^2 \ncong \Sigma$; and let $\mathscr{A}$ be a preferred LFCS on Σ such that $f \pitchfork \mathscr{A}$. Suppose any two distinct components of $\mathscr{A}$ don't co-bound an annulus in Σ. In that case, f can be properly homotoped to a proper map g such that for each component $\mathcal{C}$ of $\mathscr{A}$, either $g^{-1}(\mathcal{C})$ is empty or $g^{-1}(\mathcal{C})$ is a component of $f^{-1}(\mathscr{A})$ that is mapped homeomorphically onto $\mathcal{C}$ by g.

Proof. Using Theorem 2.5.2.3, we may assume each component of $f^{-1}(\mathscr{A})$ is primitive and also mapped homeomorphically onto a component of $\mathscr{A}$. So, for simplicity, we may drop the subscript $\mathbf{p}$ to indicate a primitive component of $f^{-1}(\mathscr{A})$. Let $\{\mathcal{C}_n\}$ be the pairwise disjoint collection of all those components of $\mathscr{A}$ so that for each $n, f^{-1}(\mathcal{C}_n)$ has more than one component. By Lemma 2.5.3.1, for each n, an annulus $\mathcal{A}'_n$ (say the n-th outermost annulus) exists with the following properties: *(i)* $\partial\mathcal{A}'_n \subseteq f^{-1}(\mathcal{C}_n)$, *(ii)* $\mathcal{A}'_n$ is not contained in the interior of an annulus bounded by any two components of $f^{-1}(\mathscr{A})$. Thus $\mathcal{A}'_n \cap f^{-1}(\mathscr{A}) = f^{-1}(\mathcal{C}_n)$ and $\mathcal{A}'_n \cap \mathcal{A}'_m = \varnothing$ for $m \neq n$. Now, using Theorem 1.2.3, find a parametrization $\tau_n \colon \mathbb{S}^1 \times [0, k_n] \xrightarrow{\cong} \mathcal{A}'_n$ for some integer $k_n \geq 1$ so that $\tau_n(\mathbb{S}^1 \times \{0, \ldots, k_n\}) = f^{-1}(\mathcal{C}_n)$ and $f\tau_n(-, \ell) \colon \mathbb{S}^1 \xrightarrow{\cong} \mathcal{C}_n$ represents the same homeomorphism of $\mathcal{C}_n$ for each $\ell = 0, \ldots, k_n$. This is possible because a parametrization of $\mathcal{C}_n$ yields a collection of pairwise freely homotopic parametrizations, one for each component of $f^{-1}(\mathcal{C}_n)$, as we noticed in the proof of the if part of Lemma 2.5.3.1.

Claim 2.5.3.5.1 The proper map $f \colon \Sigma' \to \Sigma$ can be properly homotoped relative to $\Sigma' \setminus \bigcup_n \operatorname{int}(\mathcal{A}'_n)$ so that $f(\mathcal{A}'_n) = \mathcal{C}_n$ for each n.

Proof of Claim 2.5.3.5.1. For each integer n, we will construct a compact bordered sub-surface Σ_n of Σ with $f(\mathcal{A}'_n) \subseteq \operatorname{int}(\Sigma_n)$ such that $\Sigma_n \to \infty$. Roughly, Σ_n will be obtained from taking the union of all those complementary components of Σ (if a punctured disk appears, truncate it), which are hit by $f(\mathcal{A}'_n)$.

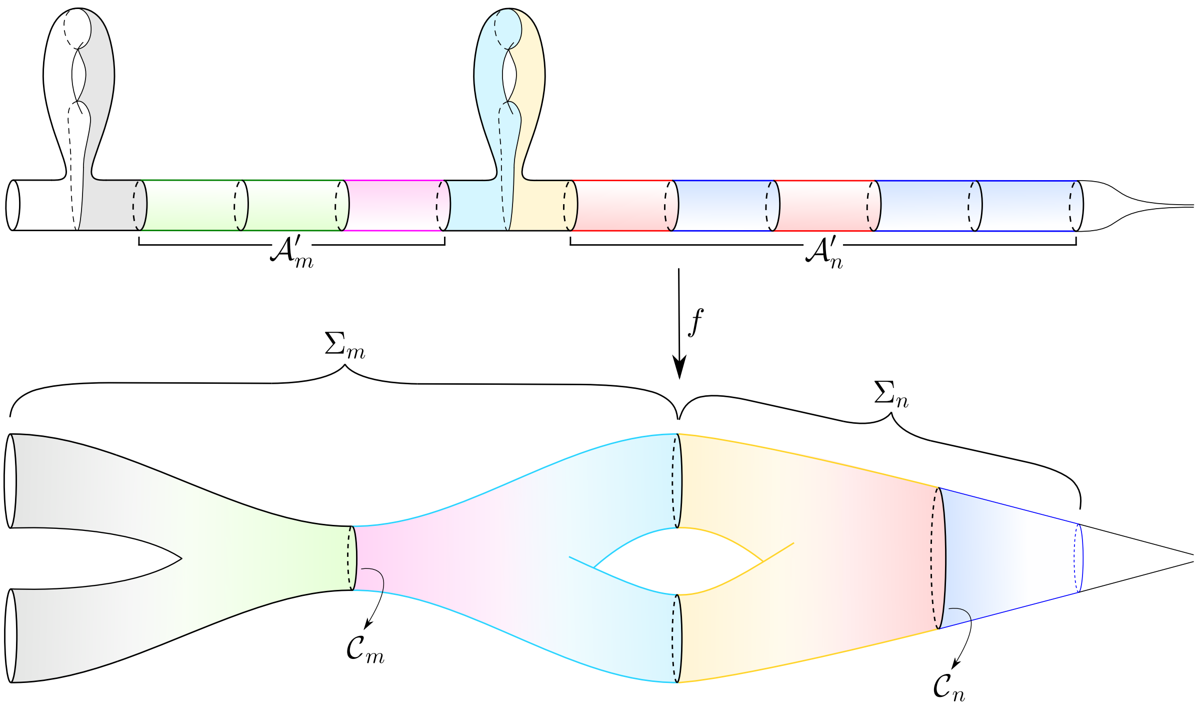

Fig. 2.5.3 Illustration of parts (1) and (3) of the definition of Σ_n given $\mathcal{C}_n$ in the proof of Claim 2.5.3.5.1. Only black circles denote a component of either $\mathscr{A}$ or a component of $f^{-1}(\mathscr{A})$.

Using continuity of $f|\Sigma' \setminus f^{-1}(\mathscr{A}) \to \Sigma \setminus \mathscr{A}$, we can say that $f(\mathcal{A}'_n) \subseteq \mathcal{X}_n \cup \mathcal{Y}_n$, where $\mathcal{X}_n$ and $\mathcal{Y}_n$ are complementary components of Σ decomposed by $\mathscr{A}$ such that $\mathcal{C}_n \subseteq \partial\mathcal{X}_n \cap \partial\mathcal{Y}_n$.

(1) We define Σ_n as $\Sigma_n := \mathcal{X}_n \cup \mathcal{Y}_n$ if either of the following happens: (*i*) $\mathcal{X}_n \cong S_{0,3} \cong \mathcal{Y}_n$; or (*ii*) $\mathcal{X}_n \cong S_{1,1}$ and $\mathcal{Y}_n \cong S_{0,3}$; or (*iii*) $\mathcal{Y}_n \cong S_{1,1}$ and $\mathcal{X}_n \cong S_{0,3}$ (see Figure 2.5.3).

(2) If $\mathcal{X}_n \cong S_{0,1,1} \cong \mathcal{Y}_n$ (in this case, Σ is homeomorphic to the punctured plane), then using compactness of $f(\mathcal{A}'_n)$, let Σ_n be an annulus in $\mathcal{X}_n \cup \mathcal{Y}_n$ so that $f(\mathcal{A}'_n) \subseteq \operatorname{int}(\Sigma_n)$.

(3) If $\mathcal{X}_n \cong S_{0,1,1}$, and $\mathcal{Y}_n$ is homeomorphic to either $S_{0,3}$ or $S_{1,1}$, then using compactness of $f(\mathcal{A}'_n)$, find an annulus $\mathcal{A}_n$ in $\mathcal{X}_n$ so that $f(\mathcal{A}'_n) \subseteq \text{int}(\mathcal{A}_n \cup \mathcal{Y}_n)$. Define $\Sigma_n := \mathcal{A}_n \cup \mathcal{Y}_n$ (see Figure 2.5.3).

(4) If $\mathcal{Y}_n \cong S_{0,1,1}$, and $\mathcal{X}_n$ is homeomorphic to either $S_{0,3}$ or $S_{1,1}$, define Σ_n similarly, as given in (3).

Thus, $f(\mathcal{A}'_n) \subseteq \text{int}(\Sigma_n)$ for each n. Now, we show $\Sigma_n \to \infty$. So, consider a compact subset $\mathcal{K}$ of Σ. Let $\mathbf{S}_1, \dots, \mathbf{S}_m$ be a collection of complementary components of Σ decomposed by $\mathscr{A}$ such that $\mathcal{K} \subseteq \text{int}\left(\bigcup_{\ell=1}^m \mathbf{S}_\ell\right)$. Define $\mathbf{S} := \bigcup_{\ell=1}^m \mathbf{S}_\ell$. Notice that for an integer n, $f(\mathcal{A}'_n) \cap \mathbf{S} \neq \varnothing$ if and only if $\mathcal{C}_n$ is a component of $\bigcup_{\ell=1}^m \partial\mathbf{S}_\ell$. This is due to the construction of each Σ_n; see Figure 2.5.3. Since $\mathcal{C}_n \to \infty$ and $\bigcup_{\ell=1}^m \partial\mathbf{S}_\ell$ is compact, we can say that $f(\mathcal{A}'_n) \cap \mathbf{S} = \varnothing$ for all sufficiently large n. Now, $\mathcal{K} \subseteq \text{int}(\mathbf{S})$ and each Σ_n is obtained from taking the union of all those complementary components of Σ (if a punctured disk appears, truncate it), which are hit by $f(\mathcal{A}'_n)$. Thus, $\Sigma_n \cap \mathcal{K} = \varnothing$ for all sufficiently large n. Therefore, $\Sigma_n \to \infty$, as $\mathcal{K}$ is an arbitrary compact subset of Σ.

Next, for each $\ell \in \{1, \dots, k_n\}$, applying Theorem 2.5.3.2 to each $f\tau_n|\mathbb{S}^1 \times [\ell - 1, \ell] \to \mathcal{Z}_n$, where $\mathcal{Z}_n$ can be either $\Sigma_n \cap \mathcal{X}_n$ or $\Sigma_n \cap \mathcal{Y}_n$, we have a homotopy $H_n\colon \mathcal{A}'_n \times [0,1] \to \Sigma_n$ relative to $\partial\mathcal{A}'_n$ such that $H_n(-,0) = f|\mathcal{A}'_n$ and $H_n(\mathcal{A}'_n, 1) = \mathcal{C}_n$. Finally, apply Lemma 2.5.1.2 on $\{H_n\}$ to complete the proof of Claim 2.5.3.5.1. □

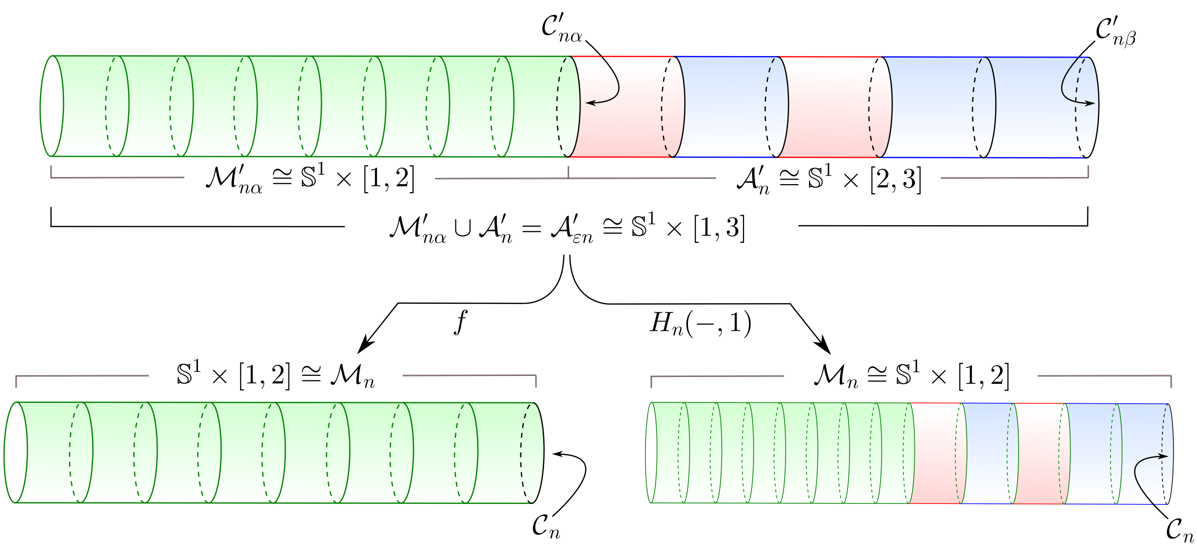

Fig. 2.5.4 Description of $f|\mathcal{A}'_{\varepsilon n} \to \mathcal{M}_n$ (resp. $H_n(-,1)\colon \mathcal{A}'_{\varepsilon n} \to \mathcal{M}_n$) using Theorem 2.5.2.3, Claim 2.5.3.5.1 (resp. Lemma 2.5.3.6). Only black circles denote a component of either $\mathscr{A}$ or a component of $f^{-1}(\mathscr{A})$.

Now, consider Figure 2.5.4, where $\mathcal{M}'_{n\alpha}, \mathcal{M}_n$ are provided by Theorem 2.5.2.3 such that after defining $\mathcal{A}'_{\varepsilon n}$ as $\mathcal{A}'_n \cup \mathcal{M}'_{n\alpha}$, we can think

$$(\mathcal{A}'_{\varepsilon n}, \mathcal{M}'_{n\alpha}, \mathcal{A}'_n) \cong (\mathbb{S}^1 \times [1,3], \mathbb{S}^1 \times [1,2], \mathbb{S}^1 \times [2,3]) \text{ and } (\mathcal{M}_n, \mathcal{C}_n) \cong (\mathbb{S}^1 \times [1,2], \mathbb{S}^1 \times 2)$$

resulting in the following description of f: If $\theta\colon \mathbb{S}^1 \to \mathbb{S}^1$ describes the homeomorphism $f|\mathcal{C}'_{n\alpha} \to \mathcal{C}_n$ under the above identification, then $f(z,t) = \big(\theta(z), t\big)$ for $z \in \mathbb{S}^1 \times [1,2]$ and $f(z,t) \in \mathbb{S}^1 \times 2$ for $(z,t) \in \mathbb{S}^1 \times [2,3]$. Consider Claim 2.5.3.5.1 to see why $f\big(\mathbb{S}^1 \times [2,3]\big) = \mathbb{S}^1 \times 2$.

Now, use Lemma 2.5.3.6 to construct a homotopy $H_n\colon \mathcal{A}'_{\varepsilon n} \times [0,1] \to \mathcal{M}_n$ relative to $\partial\mathcal{A}'_{\varepsilon n}$ from $f|\mathcal{A}'_{\varepsilon n} \to \mathcal{M}_n$ to the map $H_n(-,1)$ so that $\big(H_n(-,1)\big)^{-1}(\mathcal{C}_n) = \mathcal{C}'_{n\beta}$ and $H_n(-,1)|\mathcal{C}'_{n\beta} \to \mathcal{C}_n$ is a homeomorphism.

Notice that we are using the setup of proof of Theorem 2.5.2.3. By (4) and (5) of Claim 2.5.2.4.1 given in the proof of Theorem 2.5.2.3 show that $\mathcal{A}'_{\varepsilon n} \cap f^{-1}(\mathscr{A}) = f^{-1}(\mathcal{C}_n)$, $\mathcal{A}'_{\varepsilon n} \cap \mathcal{A}'_{\varepsilon m} = \varnothing$ if $m \neq n$, and $\mathcal{M}_n \to \infty$. Now, consider Lemma 2.5.1.2 with $\{H_n\}$ to obtain the desired homotopy. $\square$

Now, we prove the annulus-pushing lemma used in the proof of the previous theorem.

Lemma 2.5.3.6 Any map $\varphi\colon \mathbb{S}^1 \times [1,3] \to \mathbb{S}^1 \times [1,2]$ which sends $\mathbb{S}^1 \times r$ into $\mathbb{S}^1 \times r$ for $1 \leq r \leq 2$ and sends $\mathbb{S}^1 \times r$ into $\mathbb{S}^1 \times 2$ for $2 \leq r \leq 3$; can be homotoped relative to $\mathbb{S}^1 \times \{1,3\}$ to satisfy $\varphi^{-1}(\mathbb{S}^1 \times 2) = \mathbb{S}^1 \times 3$.

Proof. Let $\varphi_1\colon \mathbb{S}^1 \times [1,3] \to \mathbb{S}^1$ and $\varphi_2\colon \mathbb{S}^1 \times [1,3] \to [1,2]$ be the components of φ. Consider a homeomorphism $\ell\colon [1,3] \to [1,2]$ with $\ell(1) = 1$, $\ell(3) = 2$. Now, $H\colon \mathbb{S}^1 \times [1,3] \times [0,1] \to \mathbb{S}^1 \times [1,2]$ defined by

$$H\big((z,s),t\big) := \big(\varphi_1(z,s), (1-t)\varphi_2(z,s) + t\ell(s)\big) \text{ for } (z,s) \in \mathbb{S}^1 \times [1,3] \text{ and } t \in [0,1]$$

is our required homotopy. $\square$

Remark 2.5.3.7 In Theorem 2.5.3.5, the number of components of $\mathscr{A}$ can be infinite; thus, the number of outermost annuli (one outermost annulus for each component of $\mathscr{A}$, if any) can be infinite. That's why we have removed all outermost annuli simultaneously by a single proper homotopy, not one by one. Also, to prove the topological rigidity of closed surfaces, one may ignore the annulus removal process considering induction on the genus; see [26, Theorem 3.1.] or [95, Theorems 4.6.2 and 4.6.3]. But, since the genus of a non-compact surface can be infinite, we can't ignore the annulus removal process here.

2.6 The degree of a pseudo proper homotopy equivalence

Let $f\colon \Sigma' \to \Sigma$ be a pseudo proper homotopy equivalence between two non-compact oriented surfaces, where surfaces are homeomorphic to neither the plane nor the punctured plane. Our aim in this section is to properly homotope f to obtain a closed disk $\mathcal{D} \subseteq \Sigma$ so that $f|f^{-1}(\mathcal{D}) \to \mathcal{D}$ becomes a homeomorphism, and thus we show $\deg(f) = \pm 1$; see Theorem 1.7.1. Having got this and then using Theorem 2.6.3.1, it can be said that f is surjective, which further implies that after a proper homotopy for removing unnecessary components from the transversal pre-image of a decomposition circle C, a single circle will still be left that can be mapped onto C homeomorphically; see Theorem 2.6.3.5.

The argument for finding such a disk $\mathcal{D}$ is based on finding a finite-type bordered surface $\mathbf{S}$ in Σ such that for each component $\mathcal{C}$ of $\partial\mathbf{S}$, we have $f^{-1}(\mathcal{C}) \neq \varnothing$, even after any proper homotopy of f. Once we get $\mathbf{S}$, after a proper homotopy, we may assume that $f|f^{-1}(\partial\mathbf{S}) \to \partial\mathbf{S}$ is a homeomorphism; see Theorem 2.5.3.5. Now, since f is π_1-injective, by the topological rigidity of pair of pants together with the proper rigidity of the punctured disk, after a proper homotopy, one can show that $f|f^{-1}(\mathbf{S}) \to \mathbf{S}$ is a homeomorphism. Therefore, the required $\mathcal{D}$ can be any disk in int($\mathbf{S}$).

Now, to find such an $\mathbf{S}$, we consider two cases: If Σ is either an infinite-type surface or any $S_{g,0,p}$ with high complexity (to us, high complexity always means $g+p \geq 4$ or $p \geq 6$), then using π_1-surjectivity of f, we can choose $\mathbf{S}$ as a pair of pants in Σ so that $\Sigma \setminus \mathbf{S}$ has at least two components and every component of $\Sigma \setminus \mathbf{S}$ has a non-abelian fundamental group. On the other hand, if Σ is a finite-type surface, then we choose a punctured disk in Σ as $\mathbf{S}$ so that the puncture of $\mathbf{S}$ is an end $e \in \operatorname{im}(\operatorname{Ends}(f)) \subseteq \operatorname{Ends}(\Sigma)$.

The reason for excluding the plane and the punctured plane from consideration has already been mentioned in Page 12, in the paragraph immediately following Theorem I.

2.6.1 Essential pair of pants and the degree of a pseudo proper homotopy equivalence

Definition 2.6.1.1 A smoothly embedded pair of pants $\mathbf{P}$ in a surface Σ is said to be an *essential pair of pants* of Σ if $\Sigma \setminus \mathbf{P}$ has at least two components and every component of $\Sigma \setminus \mathbf{P}$ has a non-abelian fundamental group.

The process of finding an essential pair of pants in a non-compact surface will be divided into two cases: when the genus is at least two and when the space of ends has at least six elements.

Definition 2.6.1.2 Let $\mathbf{P}$ be a smoothly embedded copy of the pair of pants in $\mathcal{S}$, where $\mathcal{S}$ is a torus with two disks removed (i.e., $\mathcal{S}$ is a copy of $S_{1,2}$). We say $\mathbf{P}$ *is obtained from decomposing $\mathcal{S}$ into two copies of the pair of pants* if there exists another smoothly embedded copy $\widetilde{\mathbf{P}}$ of the pair of pants in $\mathcal{S}$ such that $\mathbf{P} \cup \widetilde{\mathbf{P}} = \mathcal{S}$ and $\mathbf{P} \cap \widetilde{\mathbf{P}} = \partial\mathbf{P} \cap \partial\widetilde{\mathbf{P}}$ is the union of two smoothly embedded disjoint circles in the interior of $\mathcal{S}$ (i.e., $\partial\mathbf{P}$ shares exactly two of its components with $\partial\widetilde{\mathbf{P}}$).

The following lemma says that every non-compact surface with a genus of at least two has an essential pair of pants with some additional properties.

Lemma 2.6.1.3 Let Σ be a non-compact surface of the genus of at least two. Then Σ has an essential pair of pants $\mathbf{P}$ with the following additional properties: (1) Σ contains a smoothly embedded copy $\mathcal{S}$ of $S_{1,2}$ such that $\Sigma \setminus \mathcal{S}$ has precisely two components and each component of $\Sigma \setminus \mathcal{S}$ has a non-abelian fundamental group, (2) $\mathbf{P}$ is a smoothly embedded copy of the pair of pants in $\mathcal{S}$ obtained by decomposing $\mathcal{S}$ into two copies of the pair of pants.

Proof. Consider an inductive construction of Σ; see Theorem 1.3.1. Since $g(\Sigma) \geq 2$, at least two smoothly embedded copies of $S_{1,2}$ are used in this inductive construction. By Lemma 2.3.8, without loss of generality, we may assume that two smoothly embedded copies of $S_{1,2}$ are used successively just after the initial disk; see Figure 2.6.1. Among these two copies of $S_{1,2}$, breaking the last one (i.e., that copy of $S_{1,2}$ which we just used to construct K_3 from K_2) into two copies of the pair of pants, as illustrated in Figure 2.6.1,

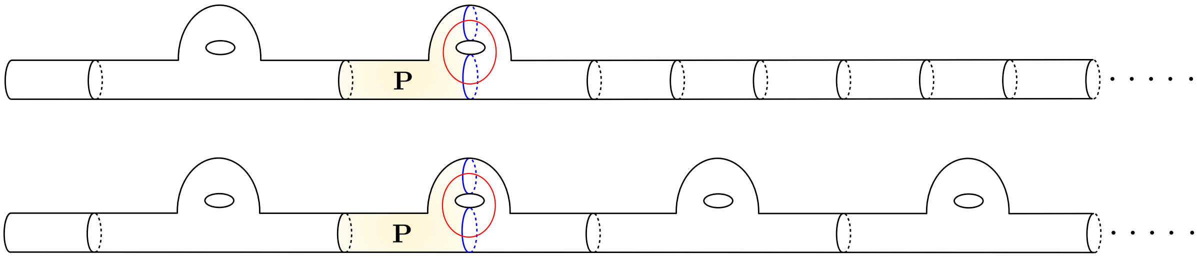

Fig. 2.6.1 Finding an essential pair of pants **P** in each of $S_{2,0,1}$ and Loch Ness Monster by decomposing a torus with two disks removed into two copies of the pair of pants.

we get the required essential pair of pants. $\square$

Lemma 2.6.1.4 Let $f\colon \Sigma' \to \Sigma$ be a π_1-surjective map between two non-compact surfaces, where Σ has the genus of at least two. Consider an essential pair of pants $\mathbf{P}$ in Σ given by Lemma 2.6.1.3. Then $f^{-1}(\text{int}\,\mathbf{P}) \neq \varnothing$ and $f^{-1}(c) \neq \varnothing$ for each component c of $\partial\mathbf{P}$.

Proof. Let $\mathcal{S}$ be a smoothly embedded copy of $S_{1,2}$ in Σ such that $\mathbf{P}$ is obtained from decomposing $\mathcal{S}$ into two copies of the pair of pants. If possible, let $f^{-1}(\text{int}\,\mathbf{P}) \neq \varnothing$. By continuity of f, the image of f is contained in precisely one of the two components of $\Sigma \setminus \text{int}(\mathbf{P})$. But each component of $\Sigma \setminus \text{int}(\mathbf{P})$ has a non-abelian fundamental group, i.e., $\pi_1(f)\colon \pi_1(\Sigma') \to \pi_1(\Sigma)$ is not surjective, a contradiction. Therefore, $f^{-1}(\text{int}\,\mathbf{P})$ must be non-empty.

To prove the second part, let c_1, c_2, and c_3 denote all three components of $\mathbf{P}$ such that both $\Sigma \setminus c_1$ and $\Sigma \setminus (c_2 \cup c_2)$ are disconnected, but neither $\Sigma \setminus c_2$ nor $\Sigma \setminus c_3$ is disconnected. In Figure 2.6.1, c_2 and c_3 are blue circles, whereas the color of the third component c_1 is black. Notice that we have a smoothly embedded primitive circle $\mathcal{C} \subseteq \text{int}(\mathcal{S})$ (in Figure 2.6.1, each red circle denotes $\mathcal{C}$) so that for each $k = 2, 3$, $c_k \cap \mathcal{C}$ is a single point, where c_k intersects $\mathcal{C}$ transversally. Therefore, for each $k = 2, 3$, using the bigon criterion [30, Proposition 1.7], any loop belonging to class $[\mathcal{C}] \in \pi_1(\Sigma)$ must intersect c_k. That is, if any of $f^{-1}(c_2)$ or $f^{-1}(c_3)$ were empty, then $[\mathcal{C}]$ would not belong to the image of $\pi_1(f)\colon \pi_1(\Sigma') \to \pi_1(\Sigma)$. But f is π_1-surjective. Thus $f^{-1}(c_2) \neq \varnothing \neq f^{-1}(c_3)$. On the other hand, $\Sigma \setminus c_1$ has precisely two components, and each component of $\Sigma \setminus c_1$ has a non-abelian fundamental group, i.e., by continuity and π_1-surjectivity of f, we can say that $f^{-1}(c_1) \neq \varnothing$. $\square$

Now, we consider the second case of finding an essential pair of pants in a non-compact surface, namely when the space of ends has at least six elements.

Lemma 2.6.1.5 Let Σ be a non-compact surface with at least six ends. Then Σ has an essential pair of pants $\mathbf{P}$ such that $\Sigma \setminus \mathbf{P}$ has precisely three components and each component of $\Sigma \setminus \mathbf{P}$ has a non-abelian fundamental group.

Proof. Consider an inductive construction of Σ; see Theorem 1.3.1. Since $|\mathrm{Ends}(\Sigma)| \geq 6$, at least five smoothly embedded copies of $S_{0,3}$ are used in this inductive construction. By Lemma 2.3.8, without loss of generality, we may assume that five smoothly embedded copies of $S_{0,3}$ are used successively just after the initial disk. Let $\mathbf{P}$ be the copy that shares all three boundary components with three other copies of this sequence of five copies of $S_{0,3}$; see Figure 2.6.2. Thus, $\Sigma \setminus \mathbf{P}$ has precisely three components, and each component of $\Sigma \setminus \mathbf{P}$ has a non-abelian fundamental group.

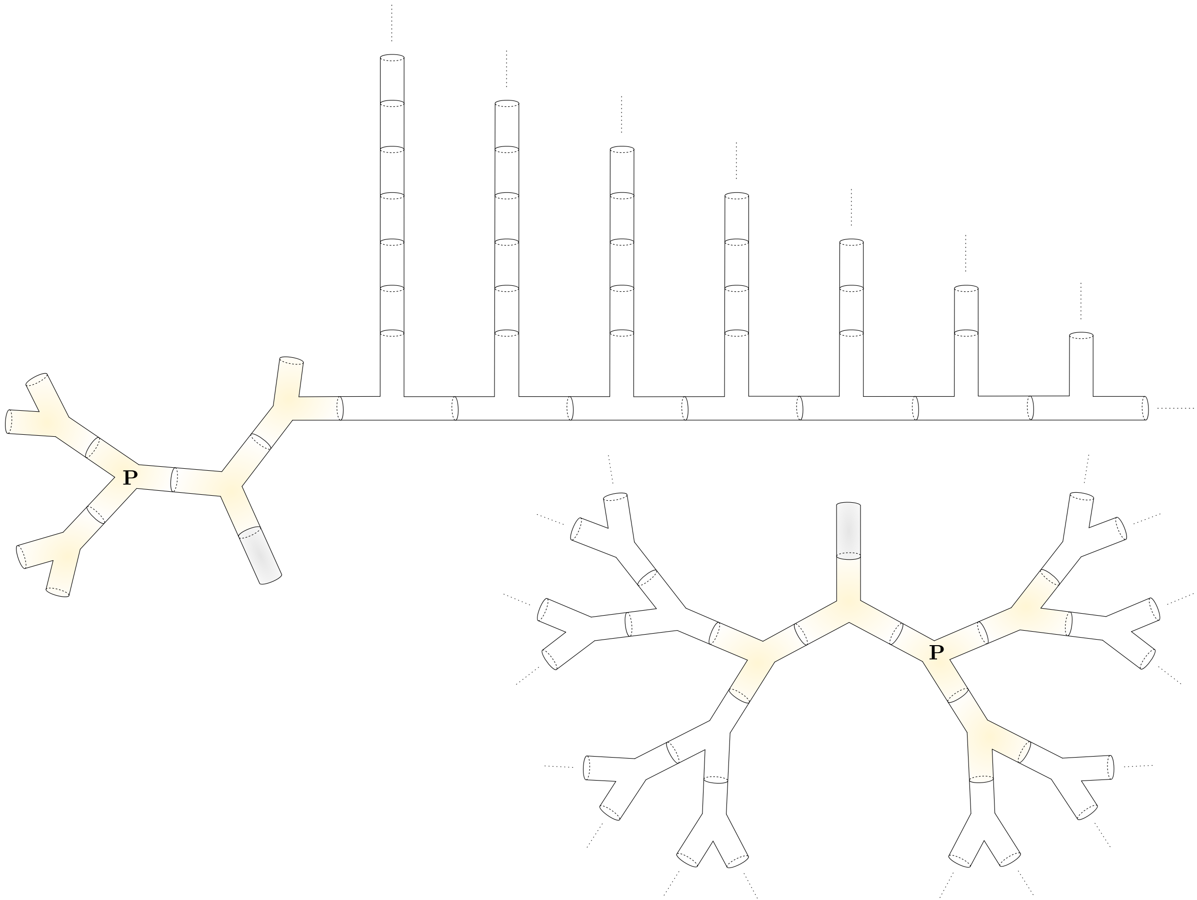

Fig. 2.6.2 Finding an essential pair of pants $\mathbf{P}$ in a non-compact surface with at least six ends.

In Figure 2.6.2, inductive constructions (up to a sufficient number of steps) of two surfaces have been given: the surface at the top contains a copy of $\{z \in \mathbb{C} : |z| \geq 1, z \notin \mathbb{N} \times 0\}$, and the bottom is the Cantor tree surface (i.e., the planar surface whose space of ends is homeomorphic to the Cantor set). In each surface, an essential pair of pants $\mathbf{P}$ is contained in the shaded compact bordered subsurface. □

We can prove the following Lemma by a similar argument given in the proof of Lemma 2.6.1.4.

Lemma 2.6.1.6 Let $f \colon \Sigma' \to \Sigma$ be a π_1-surjective proper map between two non-compact surfaces, where Σ has at least six ends. Consider an essential pair of pants $\mathbf{P}$ in Σ given by Lemma 2.6.1.5. Then $f^{-1}(\mathrm{int}\, \mathbf{P}) \neq \varnothing$ and $f^{-1}(c) \neq \varnothing$ for each component c of $\partial \mathbf{P}$.

The following theorem completes the whole process of finding an essential pair of pants, which will be used to find the degree of a pseudo-proper homotopy equivalence.

Theorem 2.6.1.7 Let $f\colon \Sigma' \to \Sigma$ be a π_1-surjective proper map between two non-compact surfaces. Suppose Σ is either an infinite-type surface or a finite-type surface $S_{g,0,p}$ with high complexity (i.e., $g+p \geq 4$ or $p \geq 6$). Then Σ contains an essential pair of pants $\mathbf{P}$ such that $f^{-1}(\text{int}\,\mathbf{P}) \neq \varnothing$ and $f^{-1}(c) \neq \varnothing$ for each component c of $\partial\mathbf{P}$.

Proof. If an infinite-type surface has a finite genus, then it must have infinitely many ends; see Proposition 2.3.10. Thus, by applying Lemma 2.6.1.3, Lemma 2.6.1.4, Lemma 2.6.1.5, and Lemma 2.6.1.6, the proof is complete in all cases, except when Σ is homeomorphic to either $S_{1,0,3}$ or $S_{1,0,4}$ or $S_{1,0,5}$. We consider the case when $\Sigma \cong S_{1,0,3}$; the other cases are similar.

Define an inductive construction of $S_{1,0,3}$ in the following way: Start with a copy of $S_{0,1}$, then consecutively add two copies of $S_{0,3}$, and after that, a copy of $S_{1,2}$; and finally, add three sequences of annuli to obtain three planar ends; see Figure 1.3.1. Therefore, in this inductive construction, K_4 is obtained from K_3, adding a copy $\mathcal{S}$ of $S_{1,2}$. Let $\mathbf{P}$ be a smoothly embedded copy of the pair of pants in $\mathcal{S}$ such that $\mathbf{P}$ is obtained from decomposing $\mathcal{S}$ into two copies of the pair of pants and $\mathbf{P} \cap K_3 \neq \varnothing$. Now, an argument similar to that given in Lemma 2.6.1.4 completes the proof. □

At this stage, we need a couple of lemmas. The first, Lemma 2.6.1.8, is well-known, and its proof is provided for the convenience of the reader.

Lemma 2.6.1.8 Let Σ be a surface, and let $\mathbf{S}$ be a smoothly embedded bordered sub-surface of Σ. Then the inclusion induced map $\pi_1(\mathbf{S}) \to \pi_1(\Sigma)$ is injective if either of the following satisfies:

(1) $\partial\mathbf{S}$ is a separating primitive circle on Σ and $\mathbf{S}$ is one of the two sides of $\partial\mathbf{S}$ in Σ;

(2) $\mathbf{S}$ is compact and each component of $\partial\mathbf{S}$ is a primitive circle on Σ.

Proof of part (1) of Lemma 2.6.1.8. Since $\pi_1(\Sigma) \cong \pi_1(\mathbf{S}) *_{\pi_1(\partial\mathbf{S})} \pi_1(\Sigma \setminus \text{int}\,\mathbf{S})$ (by Seifert-van Kampen theorem) and the inclusions $\partial\mathbf{S} \hookrightarrow \mathbf{S}, \Sigma \setminus \text{int}(\mathbf{S})$ are π_1-injective, we are done. □

Proof of part (2) of Lemma 2.6.1.8. It is enough to construct a sequence $\Sigma = \mathbf{S}_0 \supseteq \mathbf{S}_1 \supseteq \cdots \supseteq \mathbf{S}_n = \mathbf{S}$ of sub-surfaces of Σ, where n is the number of components of $\partial\mathbf{S}$, such that for each $k = 1, ..., n$, the following hold:

(i) $\mathbf{S}_k$ is a bordered sub-surface of $\mathbf{S}_{k-1}$ and the inclusion map $\mathbf{S}_k \hookrightarrow \mathbf{S}_{k-1}$ is π_1-injective;

(ii) $\partial\mathbf{S}_k \setminus \partial\mathbf{S}_{k-1}$ is either a component of $\partial\mathbf{S}_k$ or union of two components of $\partial\mathbf{S}_k$. In either case, $\partial\mathbf{S}_k \setminus \partial\mathbf{S}_{k-1}$ shares only one component with $\partial\mathbf{S}$.

We construct this sequence inductively as follows: To construct $\mathbf{S}_k$ from $\mathbf{S}_{k-1}$, pick a component $\mathbf{c}$ of $\partial\mathbf{S} \setminus \partial\mathbf{S}_{k-1}$. If $\mathbf{c}$ separates $\mathbf{S}_{k-1}$, define $\mathbf{S}_k$ as that side of $\mathbf{c}$ in $\mathbf{S}_{k-1}$, which contains $\mathbf{S}$; then consider an argument similar to the proof of part (1) of Lemma 2.6.1.8. Now, if $\mathbf{c}$ doesn't separate $\mathbf{S}_{k-1}$, pick a smoothly embedded annulus $\mathbf{A} \subset \operatorname{int}(\mathbf{S}_{k-1})$ such that $\mathbf{A} \cap \mathbf{S} = \mathbf{c}$. Define $\mathbf{S}_k := \mathbf{S}_{k-1} \setminus \operatorname{int}(\mathbf{A})$. Now, $\mathbf{S}_{k-1}$ is obtained from $\mathbf{S}_k$ identifying $\mathbf{c}$ with $\partial\mathbf{A} \setminus \mathbf{c}$ by an orientation-reversing diffeomorphism $\varphi\colon \mathbf{c} \to \partial\mathbf{A} \setminus \mathbf{c}$. By HNN-Seifert-van Kampen theorem, $\pi_1(\mathbf{S}_{k-1}) \cong \pi_1(\mathbf{S}_k)*_{\pi_1(\varphi)}$; where the map $\pi_1(\mathbf{S}_k) \to \pi_1(\mathbf{S}_{k-1})$ (which is inclusion induced) is injective due to Britton's lemma. This completes the proof. $\square$

The following lemma roughly says that the degree of a map between two compact bordered surfaces can be determined from the degree of its restriction on the boundaries.

Lemma 2.6.1.9 Let $\varphi\colon \mathbf{S}_{g_1,b_1} \to \mathbf{S}_{g_2,b_2}$ be a map between two compact bordered surfaces. If $\varphi|\partial\mathbf{S}_{g_1,b_1} \hookrightarrow \partial\mathbf{S}_{g_2,b_2}$ is an embedding, then $\varphi(\partial\mathbf{S}_{g_1,b_1}) = \partial\mathbf{S}_{g_2,b_2}$ and $\deg(\varphi) = \pm 1$.

Proof. Notice that φ maps each component of $\partial\mathbf{S}_{g_1,b_1}$ homeomorphically onto a component of $\partial\mathbf{S}_{g_2,b_2}$, and any two distinct components of $\partial\mathbf{S}_{g_1,b_1}$ have distinct φ-images. Now, naturality of homology long exact sequences of $(\mathbf{S}_{g_1,b_1}, \partial\mathbf{S}_{g_1,b_1})$ and $(\mathbf{S}_{g_2,b_2}, \partial\mathbf{S}_{g_2,b_2})$ give following commutative diagram:

$$\begin{tikzcd}
H_2\left(\mathbf{S}_{g_1,b_1}, \partial\mathbf{S}_{g_1,b_1}\right) \cong \mathbb{Z} \arrow[r, "1 \longmapsto \overset{b_1}{\oplus} 1"] \arrow[d, "\times \deg(\varphi)"'] & \overset{b_1}{\oplus}\mathbb{Z} \cong H_1\left(\partial\mathbf{S}_{g_1,b_1}\right) \arrow[d, "\overset{b_1}{\oplus} 1 \longmapsto \overset{b_1}{\oplus}(\pm 1) \bigoplus \overset{b_2-b_1}{\oplus} 0"] \\
H_2\left(\mathbf{S}_{g_2,b_2}, \partial\mathbf{S}_{g_2,b_2}\right) \cong \mathbb{Z} \arrow[r, "1 \longmapsto \overset{b_2}{\oplus} 1"] & \overset{b_2}{\oplus}\mathbb{Z} \cong H_1\left(\partial\mathbf{S}_{g_2,b_2}\right)
\end{tikzcd}$$

The horizontal maps are the connecting homomorphisms for homology long exact sequences, and for their description, see [51, Exercise 31 of Section 3.3]. Now, commutativity of this diagram gives $b_2 = b_1$ (the integer $\deg(\varphi)$ can't be simultaneously 0 as well as ± 1), and thus $\deg(\varphi) = \pm 1$. $\square$

The proof of Theorem 2.6.1.10 below can be found in [95, Theorem 4.6.2.]. It also follows from the much more general result, [26, Theorem 3.1.]. Since compact bordered surfaces are aspherical, an application of the Whitehead theorem [51, Theorem 4.5.] says that the assumption "$\varphi\colon \mathbf{S}' \to \mathbf{S}$ is a homotopy equivalence" in Theorem 2.6.1.10 is equivalent to the assumption "$\pi_1(\varphi)$ is an isomorphism".

Theorem 2.6.1.10 (Rigidity of compact bordered surfaces) Let $\varphi\colon \mathbf{S}' \to \mathbf{S}$ be a homotopy equivalence between two compact bordered surfaces such that $\varphi^{-1}(\partial\mathbf{S}) = \partial\mathbf{S}'$. If $\varphi|\partial\mathbf{S}' \to \partial\mathbf{S}$ is a homeomorphism, then φ is homotopic to a homeomorphism relative to $\partial\mathbf{S}'$.

The following lemma gives some sufficient conditions so that the pre-image of a compact bordered subsurface under a proper map becomes a compact bordered subsurface of the same homeomorphism type. Its usage is two-fold: firstly, in Theorem 2.6.1.12, to find the degree of a pseudo proper homotopy equivalence; and secondly, in the proof of Theorem I.

Lemma 2.6.1.11 Let $f\colon \Sigma' \to \Sigma$ be a π_1-injective proper map between two non-compact oriented surfaces, and let $\mathbf{S}$ be a smoothly embedded compact bordered subsurface of Σ with $f^{-1}(\text{int}\,\mathbf{S}) \neq \varnothing$. Suppose $f^{-1}(\partial\mathbf{S})$ is a pairwise disjoint collection of smoothly embedded primitive circles on Σ' such that f sends $f^{-1}(\partial\mathbf{S})$ homeomorphically onto $\partial\mathbf{S}$. Then $f^{-1}(\mathbf{S})$ is a copy $\mathbf{S}$ in Σ' with $\partial f^{-1}(\mathbf{S}) = f^{-1}(\partial\mathbf{S})$, and $\deg(f) = \pm 1$.

Proof. Since $f^{-1}(\text{int}\,\mathbf{S}) \neq \varnothing$ and f is proper, the continuity of $f|\Sigma' \setminus f^{-1}(\partial\mathbf{S}) \to \Sigma \setminus \partial\mathbf{S}$ tells that $\Sigma' \setminus f^{-1}(\partial\mathbf{S})$ is disconnected. Let $\mathbf{S}' \subset \Sigma'$ be a bordered sub-surface obtained as a complementary component of the decomposition of Σ' by $f^{-1}(\partial\mathbf{S})$ such that $f(\mathbf{S}') \subseteq \mathbf{S}$. That is, $\mathbf{S}'$ is the closure of one of the components of $\Sigma' \setminus f^{-1}(\partial\mathbf{S})$ and $\mathbf{S}'$ is contained in the compact set $f^{-1}(\mathbf{S})$. So, $\mathbf{S}'$ is a compact bordered subsurface of Σ', and each component of $\partial\mathbf{S}'$ is a component of $f^{-1}(\partial\mathbf{S})$. In the following few lines, we will show that each component of $f^{-1}(\partial\mathbf{S})$ is also a component of $\partial\mathbf{S}'$. Anyway, since $f|f^{-1}(\partial\mathbf{S}) \to \partial\mathbf{S}$ is a homeomorphism, we can say that $f|\partial\mathbf{S}' \hookrightarrow \partial\mathbf{S}$ is an embedding. Now, by Lemma 2.6.1.9, $\partial\mathbf{S}' = f^{-1}(\partial\mathbf{S})$ and $\deg\,(f|\mathbf{S}' \to \mathbf{S}) = \pm 1$. Next, by Theorem 1.7.3, $f|\mathbf{S}' \to \mathbf{S}$ is π_1-surjective. Since the inclusion $\mathbf{S}' \hookrightarrow \Sigma'$ and f are π_1-injective, $f|\mathbf{S}' \to \mathbf{S}$ is also so (see part (2) of Lemma 2.6.1.8). Thus, $f|\mathbf{S}' \to \mathbf{S}$ is π_1-bijective, and so Theorem 2.6.1.10 tells that $\mathbf{S}' \cong \mathbf{S}$. Finally, if $\mathbf{S}''$ is another bordered sub-surface obtained as a complementary component of decomposition of Σ' by $f^{-1}(\partial\mathbf{S})$ with $f(\mathbf{S}'') \subseteq \mathbf{S}$, then similarly, $\mathbf{S}'' \cong \mathbf{S}$. Since $f|f^{-1}(\partial\mathbf{S}) \to \partial\mathbf{S}$ is a homeomorphism and Σ' is connected, $\mathbf{S}'' = \mathbf{S}'$ (otherwise, Σ' would be the compact surface $\mathbf{S}' \cup_{\partial\mathbf{S}'=\partial\mathbf{S}''} \mathbf{S}''$). Therefore, $f^{-1}(\mathbf{S}) = \mathbf{S}' \cong \mathbf{S}$, and thus the proof of the first part is completed.

Now, we will prove that $\deg(f) = \pm 1$. Since $\deg(f)$ remains invariant after any proper homotopy of f, we can properly homotope f as we want. So, apply Theorem 2.6.1.10 to $f|\mathbf{S}' \to \mathbf{S}$. Thus, $f\colon \Sigma' \to \Sigma$ can be properly homotoped relative to $\Sigma' \setminus \text{int}(\mathbf{S}')$ to map $\mathbf{S}' = f^{-1}(\mathbf{S})$ is homeomorphically onto $\mathbf{S}$. Now, by Theorem 1.7.1, $\deg(f) = \pm 1$. □

We are now ready to prove that a pseudo proper homotopy equivalence is a map of degree ± 1 if the co-domain contains an essential pair of pants, as said before.

Theorem 2.6.1.12 Let $f\colon \Sigma' \to \Sigma$ be a pseudo proper homotopy equivalence between two non-compact oriented surfaces, where Σ is either an infinite-type surface or a finite-type non-compact surface $S_{g,0,p}$ with high complexity (to us, high complexity means $g + p \geq 4$ or $p \geq 6$). Then $\deg(f) = \pm 1$.

Proof. Since $\deg(f)$ remains invariant after any proper homotopy of f, we can properly homotope f as we want. Now, Theorem 2.6.1.7 gives an essential pair of pants $\mathbf{P}$ in Σ such that $f^{-1}(\text{int}(\mathbf{P})) \neq \varnothing$ and $f^{-1}(c) \neq \varnothing$ for each component c of $\partial\mathbf{P}$, even after any proper homotopy of f. Using Theorem 2.4.1 and then Theorem 2.5.3.5, after a proper homotopy, we may assume that $f^{-1}(\text{int}\,\mathbf{P}) \neq \varnothing$ and $f^{-1}(\partial\mathbf{P})$ is a pairwise-disjoint collection of three smoothly embedded circles on Σ such that $f|f^{-1}(\partial\mathbf{P}) \to \partial\mathbf{P}$ is a homeomorphism.

Now, if possible, let c' be a component of $f^{-1}(\partial\mathbf{P})$ such that there is an embedding $i'\colon \mathbb{D}^2 \hookrightarrow \Sigma'$ with $c' = i'(\mathbb{S}^1)$. Then the embedding $f \circ i'|\mathbb{S}^1 \hookrightarrow \Sigma$ is null-homotopic and $c := f \circ i'(\mathbb{S}^1)$ is a component of $\partial\mathbf{P}$. But $\mathbf{P}$ is an essential pair of pants in Σ implies each component of $\partial\mathbf{P}$ is a primitive circle on Σ. Now, Theorem 1.2.2 tells us we have reached a contradiction. Hence, each component of $f^{-1}(\partial\mathbf{P})$ is a primitive circle on Σ'. Finally, applying Lemma 2.6.1.11, we complete the proof. $\square$

2.6.2 An essential punctured disk of a proper map and the degree of a pseudo proper homotopy equivalence

We first build up notations for Section 2.6.2. Let Σ be a non-compact surface. Since the Ends(Σ) is independent of the choice of efficient exhaustion of Σ by compacta, we will use Goldman's inductive construction to define Ends(Σ); see Section 1.4. So, consider an inductive construction of Σ. For each $i \geq 1$, define K_i to be the compact bordered subsurface of Σ after the i-th step of the induction. Then $\{K_i\}_{i=1}^{\infty}$ is an efficient exhaustion of Σ by compacta. Also, notice that $\text{int}(K_1) \subseteq \text{int}(K_2) \subseteq \cdots$ is an increasing sequence of open subsets of Σ such that $\cup_{i=1}^{\infty}\text{int}(K_i) = \Sigma$; and thus every compact subset of Σ is contained in some $\text{int}(K_i)$.

Suppose Σ' is another non-compact surface and $f\colon \Sigma' \to \Sigma$ is a proper map. Let $(V_1, V_2, ...)$ be an end of Σ, i.e., V_i is a component of $\Sigma \setminus K_i$ and $V_1 \supseteq V_2 \supseteq \cdots$. With this setup, we are now ready to state a lemma that is more or less related to Proposition 1.6.4.

Theorem 2.6.2.1 Assume that $f^{-1}(V_i) \neq \varnothing$ for each $i \geq 1$. Then for every proper map $g\colon \Sigma' \to \Sigma$, which is properly homotopic to f, we have $g^{-1}(V_i) \neq \varnothing$ for each $i \geq 1$.

Proof. Let $g\colon \Sigma' \to \Sigma$ be a proper map, and let $\mathcal{H}\colon \Sigma' \times [0,1] \to \Sigma$ be a proper homotopy from f to g. Notice that $V_i \to \infty$: If $\mathcal{X}$ is a compact subset of Σ, then $\mathcal{X} \subseteq \text{int}(K_{i_0})$ for some positive integer i_0, i.e., $\mathcal{X} \cap V_i = \varnothing$ for all $i \geq i_0$. Therefore, $f^{-1}(V_i) \to \infty$: If $\mathcal{X}'$ is a compact subset of Σ', then $f(\mathcal{X}')$ is compact, so $f(\mathcal{X}') \cap V_i = \varnothing$ for all but finitely many i, i.e., $\mathcal{X}' \cap f^{-1}(V_i) = \varnothing$ for all but finitely many i.

Let n be any positive integer. Consider the compact subset $p(\mathcal{H}^{-1}(K_n))$ of Σ', where $p\colon \Sigma' \times [0,1] \to \Sigma'$ is the projection. Since $f^{-1}(V_i) \to \infty$, we have an integer $i_n > n$ such that $f^{-1}(V_{i_n}) \subseteq \Sigma' \setminus p(\mathcal{H}^{-1}(K_n))$. Now, consider any $x_{i_n} \in f^{-1}(V_{i_n})$. Then $\mathcal{H}(x_{i_n} \times [0,1]) \subseteq \Sigma \setminus K_n$, i.e., the connected set $\mathcal{H}(x_{i_n} \times [0,1])$ is contained in one of the components of $\Sigma \setminus K_n$. But $\mathcal{H}(x_{i_n}, 0) = f(x_{i_n}) \in V_{i_n} \subseteq V_n$, i.e., $\mathcal{H}(x_{i_n} \times [0,1]) \subseteq V_n$. In particular, this means $g(x_{i_n}) = \mathcal{H}(x_{i_n}, 1) \in V_n$. Since n is an arbitrary positive integer, we are done. $\square$

Definition 2.6.2.2 Let $e = (V_1, V_2, ...)$ be an end of Σ such that for some non-negative integer i_e, $\overline{V_i} \cong S_{0,1,1}$ for all $i \geq i_e$ (i.e., e is an isolated planar end of Σ). If $f^{-1}(V_i) \neq \varnothing$ for all $i \geq 1$, then for each integer $i \geq i_e$, we say $\overline{V_i}$ is an *essential punctured disk* of f.

Theorem 2.6.2.1 says that the notion of an essential punctured disk is invariant under the proper homotopy. In Theorem 2.6.2.5, we show that, after a proper homotopy, the pre-image of the boundary of an essential punctured disk under a pseudo proper homotopy equivalence bounds a planar end of the domain. But before moving into its proof, we need to prove the following lemma, which gives some sufficient conditions so that the pre-image of a punctured disk in the co-domain under a proper map becomes a punctured disk in the domain.

Lemma 2.6.2.3 Let $f\colon \Sigma' \to \Sigma$ be a π_1-injective proper map between two non-compact oriented surfaces, and let $\mathcal{C}$ be a smoothly embedded separating circle on Σ such that one of the two sides of $\mathcal{C}$ in Σ is a punctured disk $\mathcal{D}_*$. Also, let Σ' is homeomorphic to neither $\mathbb{S}^1 \times \mathbb{R}$ nor $\mathbb{R}^2$. If $f^{-1}(\mathcal{C})$ is a smoothly embedded primitive circle on Σ' so that $f|f^{-1}(\mathcal{C}) \to \mathcal{C}$ is a homeomorphism and $f^{-1}(\text{int}\, \mathcal{D}_*) \neq \varnothing$, then $f^{-1}(\mathcal{D}_*)$ is a copy the punctured disk in Σ' with $\partial f^{-1}(\mathcal{D}_*) = f^{-1}(\mathcal{C})$ and $\deg(f) = \pm 1$.

Proof. Notice that $\Sigma' \not\cong \mathbb{R}^2, \mathbb{S}^1 \times \mathbb{R}$, i.e., $\pi_1(\Sigma')$ is non-abelian by Theorem 2.3.9. Since $f^{-1}(\text{int}\, \mathcal{D}_*) \neq \varnothing$ and $\pi_1(f)(\pi_1(\Sigma'))$ is non-abelian, by continuity of $f|\Sigma' \setminus f^{-1}(\mathcal{C}) \to \Sigma \setminus \mathcal{C}$, we can say that $\Sigma' \setminus f^{-1}(\mathcal{C})$ is disconnected. Let $\mathbf{S}'$ be a side of $f^{-1}(\mathcal{C})$ in Σ' for which $f(\mathbf{S}') \subseteq \mathcal{D}_*$. Since f is π_1-injective, by part (1) of Lemma 2.6.1.8, $f|\mathbf{S}' \to \mathcal{D}_*$ is also so. Thus, $\pi_1(\mathbf{S}')$ is a subgroup of $\mathbb{Z}$. Now, $\text{int}(\mathbf{S}')$ is homotopy equivalent to $\mathbf{S}'$ and bounded by the primitive circle $f^{-1}(\mathcal{C})$ on Σ'; so, using Theorem 2.3.9, $\mathbf{S}' \cong S_{0,1,1}$. Next, if $\mathbf{S}''$ is another side of $f^{-1}(\mathcal{C})$ in Σ' for which $f(\mathbf{S}'') \subseteq \mathcal{D}_*$, then similarly, $\mathbf{S}'' \cong S_{0,1,1}$. Since $f|f^{-1}(\mathcal{C}) \to \mathcal{C}$ is homeomorphism and Σ' is connected, $\mathbf{S}'' = \mathbf{S}'$; otherwise, Σ' would be $\mathbf{S}' \cup_{f^{-1}(\mathcal{C})} \mathbf{S}'' \cong \mathbb{S}^1 \times \mathbb{R}$. Therefore, $f^{-1}(\mathcal{D}_*) = \mathbf{S}' \cong \mathcal{D}_*$, and thus the proof of the first part is completed.

Now, we will prove that $\deg(f) = \pm 1$. Since $\deg(f)$ remains invariant after any proper homotopy of f, we can properly homotope f as we want. So, apply Theorem 2.6.2.4 to $f|\mathbf{S}' \to \mathcal{D}_*$. Thus, $f\colon \Sigma' \to \Sigma$ can be properly homotoped relative to $\Sigma' \setminus \text{int}(\mathbf{S}')$ to map $\mathbf{S}' = f^{-1}(\mathcal{D}_*)$ is homeomorphically onto $\mathcal{D}_*$. Now, by Theorem 1.7.1, $\deg(f) = \pm 1$. □

In the previous lemma, we employed the following well-known theorem, for which a proof will be provided by modifying the proof of the Alexander trick [30, Lemma 2.1].

Theorem 2.6.2.4 (Proper rigidity of the punctured disk) Let $\mathbf{D}_*$ be a punctured disk, and let $\varphi\colon \mathbf{D}_* \to \mathbf{D}_*$ be a proper map such that $\varphi^{-1}(\partial \mathbf{D}_*) = \partial \mathbf{D}_*$ and $\varphi|\partial \mathbf{D}_* \to \partial \mathbf{D}_*$ is a homeomorphism. Then φ is properly homotopic to a homeomorphism $\mathbf{D}_* \to \mathbf{D}_*$ relative to the boundary $\partial \mathbf{D}_*$.

Proof. Without loss of generality, we may assume $\mathbf{D}_* = \{z \in \mathbb{C} : 0 < |z| \leq 1\}$. Define $\mathcal{H}\colon \mathbf{D}_* \times [0,1] \to \mathbf{D}_*$ by

$$\mathcal{H}(z,t) := \begin{cases} (1-t)\cdot \varphi\left(\frac{z}{1-t}\right) & \text{if } 0 < |z| \leq 1-t, \\ |z| \cdot \varphi\left(\frac{z}{|z|}\right) & \text{if } 1-t < |z| \leq 1. \end{cases}$$

Notice that $\varphi \simeq \mathcal{H}(-,1)$ relative to $\partial \mathbf{D}_*$, and $\mathcal{H}(-,1)\colon \mathbf{D}_* \to \mathbf{D}_*$ is a homeomorphism.

Now, we prove $\mathcal{H}$ is a proper map. So let $\{(z_n, t_n)\}$ is a sequence in $\mathbf{D}_* \times [0,1]$ with $z_n \to 0$. We need to show that $\mathcal{H}(z_n, t_n) \to 0$. Define $\mathscr{A} := \{n \in \mathbb{N} : 1 - t_n < |z_n|\}$ and $\mathscr{B} := \{n \in \mathbb{N} : |z_n| \leq 1 - t_n\}$. Then $\mathbb{N} = \mathscr{A} \cup \mathscr{B}$. Therefore, it is enough to show $\{\mathcal{H}(z_n, t_n) : n \in \mathscr{A}\} \to 0$ (resp. $\{\mathcal{H}(z_n, t_n) : n \in \mathscr{B}\} \to 0$) whenever $\mathscr{A}$ (resp. $\mathscr{B}$) is infinite.

If $\mathscr{A}$ is infinite, then $\{\mathcal{H}(z_n, t_n) : n \in \mathscr{A}\} \to 0$, since $|\mathcal{H}(z_n, t_n)| = |z_n| \cdot \left|\varphi\left(\frac{z_n}{|z_n|}\right)\right| \leq |z_n|$ for all $n \in \mathscr{A}$.

Next, assume $\mathscr{B}$ is infinite. We will prove that $\{\mathcal{H}(z_n, t_n) : n \in \mathscr{B}\} \to 0$. So consider any $\varepsilon > 0$. We need to show $|\mathcal{H}(z_n, t_n)| < \varepsilon$ for all but finitely many $n \in \mathscr{B}$. Let $\mathscr{B}'_\varepsilon := \{n \in \mathscr{B} : 1 - t_n < \varepsilon\}$. Therefore $|\mathcal{H}(z_n, t_n)| = (1 - t_n) \cdot \left|\varphi\left(\frac{z_n}{1-t_n}\right)\right| \le (1 - t_n) < \varepsilon$ for all $n \in \mathscr{B}'_\varepsilon$. Also, if $\mathscr{B} \setminus \mathscr{B}'_\varepsilon$ is infinite, then $\left\{\frac{z_n}{1-t_n} : n \in \mathscr{B} \setminus \mathscr{B}'_\varepsilon\right\} \to 0$, which implies $\left\{\varphi\left(\frac{z_n}{1-t_n}\right) : n \in \mathscr{B} \setminus \mathscr{B}'_\varepsilon\right\} \to 0$ (as φ is proper), and thus $|\mathcal{H}(z_n, t_n)| \le \left|\varphi\left(\frac{z_n}{1-t_n}\right)\right| < \varepsilon$ for all but finitely many $n \in \mathscr{B} \setminus \mathscr{B}'_\varepsilon$. Now, the previous two lines together imply that $|\mathcal{H}(z_n, t_n)| < \varepsilon$ for all but finitely many $n \in \mathscr{B}$. $\square$

Theorem 2.6.2.5 Let $f : \Sigma' \to \Sigma$ be a pseudo proper homotopy equivalence between two non-compact oriented surfaces. Suppose $\pi_1(\Sigma)$ is finitely-generated non-abelian group (equivalently $\Sigma \cong S_{g,0,p}$ for some $(g, p) \neq (0, 1), (0, 2)$). Then $\deg(f) = \pm 1$.

Proof. Since $\deg(f)$ remains invariant after any proper homotopy of f, we can properly homotope f as we want. Now, Σ is a finite-type non-compact surface implies each end of Σ is an isolated planar end, i.e., for every $e = (V_1, V_2, ...) \in \text{Ends}(\Sigma)$, we have an integer i_e such that $\overline{V_i}$ is homeomorphic to the punctured disk for each $i \ge i_e$. Next, f is proper implies there exists $\mathcal{E} = (\mathcal{W}_1, \mathcal{W}_2, ...) \in \text{Ends}(\Sigma)$ such that $f^{-1}(\mathcal{W}_i) \neq \varnothing$ for each $i \ge 1$. Notice that $\overline{\mathcal{W}_{i_\mathcal{E}}}$ is an essential punctured disk and $\mathcal{C}_{i_\mathcal{E}} := \partial\overline{\mathcal{W}_{i_\mathcal{E}}}$ is a smoothly embedded separating circle on Σ. Also, $\mathcal{C}_{i_\mathcal{E}}$ is a primitive circle on Σ as $\mathcal{C}_{i_\mathcal{E}}$ bound the punctured disk $\overline{\mathcal{W}_{i_\mathcal{E}}}$ on $\Sigma \ncong \mathbb{R}^2$.

We aim to use Lemma 2.6.2.3, but some observations are needed before that. Let $g : \Sigma' \to \Sigma$ be a proper map such that g is properly homotopic to f (note that f is properly homotopic to itself, i.e., g can be f). If possible, assume $g^{-1}(\mathcal{C}_{i_\mathcal{E}}) = \varnothing$. Then continuity of g implies $g(\Sigma')$ is contained in one of the two components of $\Sigma \setminus \mathcal{C}_{i_\mathcal{E}}$. By Theorem 2.6.2.1, $g(\Sigma')$ must be contained in $\mathcal{W}_{i_\mathcal{E}}$. But, then $\pi_1(f) = \pi_1(g)$ is non-surjective as $\pi_1(\Sigma \setminus \mathcal{W}_{i_\mathcal{E}}) = \pi_1(\Sigma)$ is non-abelian. Therefore, $g^{-1}(\mathcal{C}_{i_\mathcal{E}}) \neq \varnothing$. Also, by Theorem 2.6.2.1, $g^{-1}(\mathcal{W}_i) \neq \varnothing$ for each $i \ge 1$, and thus $g^{-1}(\mathcal{W}_{i_\mathcal{E}}) \neq \varnothing$.

Now, we are ready to apply Lemma 2.6.2.3 after the observation given in the previous paragraph. At first, notice that Σ' is homeomorphic to neither the plane nor the punctured plane as $\pi_1(\Sigma') = \pi_1(\Sigma)$ is non-abelian. After a proper homotopy of f, we may assume that $f \pitchfork \mathcal{C}_{i_\mathcal{E}}$; see Theorem 2.4.1. By the previous paragraph, $f^{-1}(\mathcal{C}_{i_\mathcal{E}})$ is a pairwise disjoint non-empty collection of finitely many smoothly embedded circles on Σ'. Now, by Theorem 2.5.3.5 and the previous paragraph, after a proper homotopy of f, we may further assume that $\mathcal{C}'_{i_\mathcal{E}} := f^{-1}(\mathcal{C}_{i_\mathcal{E}})$ is a (single) smoothly embedded circle on Σ' and $f|\mathcal{C}'_{i_\mathcal{E}} \to \mathcal{C}_{i_\mathcal{E}}$ is a homeomorphism. The previous paragraph also tells that after all these proper homotopies, $f^{-1}(\mathcal{W}_{i_\mathcal{E}})$ remains non-empty.

We show that $\mathcal{C}'_{i_\mathcal{E}}$ is a primitive circle on Σ'. On the contrary, let there be an embedding $i' : \mathbb{D}^2 \hookrightarrow \Sigma'$ with $\mathcal{C}'_{i_\mathcal{E}} = i'(\mathbb{S}^1)$. Then the embedding $f \circ i'|\mathbb{S}^1 \hookrightarrow \Sigma$ is null-homotopic and $\mathcal{C}_{i_\mathcal{E}} = f \circ i'(\mathbb{S}^1)$. But $\mathcal{C}_{i_\mathcal{E}}$ is a primitive circle on Σ. Now, Theorem 1.2.2 tells us we have reached a contradiction. Finally, applying Lemma 2.6.2.3, we can say that $\deg(f) = \pm 1$. $\square$

2.6.3 Most pseudo proper homotopy equivalences between non-compact surfaces are maps of degree ± 1

Consider a non-surjective map $\varphi : \mathscr{M} \to \mathscr{N}$ between two closed, oriented, connected n-manifolds. Then for any $p \in \mathscr{N} \setminus \text{im}(\varphi)$, the map $H^n(\varphi)$ factors through the inclusion-induced zero map $H^n(\mathscr{N}) \cong \mathbb{Z} \longrightarrow 0 \cong H^n(\mathscr{N} \setminus p)$ (recall that top integral singular cohomology of any connected, non-compact, boundaryless manifold is zero), i.e., $\deg(\varphi) = 0$. The theorem below generalizes this phenomenon in the proper category.

Theorem 2.6.3.1 Let $\Phi\colon M \to N$ be a proper map between two connected, oriented, boundaryless, smooth k-dimensional manifolds. If $\deg(\Phi) \neq 0$, then Φ is surjective.

Proof. Being a proper map between two manifolds, Φ is a closed map; see Theorem 1.6.2. Now, if possible, suppose Φ is non-surjective. Therefore, $N \setminus \Phi(M)$ is a non-empty open subset of N. Pick a point $y \in N \setminus \Phi(M)$. Since N is locally Euclidean, there is a smoothly embedded closed k-dimensional ball $B \subset N$ such that $B \subseteq N \setminus \Phi(M)$. Notice that $N \setminus \operatorname{int}(B)$ is a smoothly embedded co-dimension zero submanifold of N with $\partial(N \setminus \operatorname{int}(B)) = \partial B$. By Poincaré duality (see [51, Exercise 35 of Section 3.3]), $H^k_{\mathbf{c}}(N \setminus \operatorname{int}(B); \mathbb{Z}) \cong H_0(N \setminus \operatorname{int}(B), \partial B; \mathbb{Z})$. Also, $H_0(N \setminus \operatorname{int}(B), \partial B; \mathbb{Z}) = 0$ as N is path-connected; see [51, Exercise 16.(a) of Section 2.1]. Now, $\Phi\colon M \to N$ can be thought as the composition $M \xrightarrow{\Phi^\dagger} N \setminus \operatorname{int}(B) \overset{i}{\hookrightarrow} N$, where i is the inclusion map and $\Phi^\dagger(m) := \Phi(m)$ for all $m \in M$. Certainly, $\Phi^\dagger$ and i are both proper maps. Therefore, $H^k_{\mathbf{c}}(\Phi)$ is the composition

$$H^k_{\mathbf{c}}(N; \mathbb{Z}) \xrightarrow{H^k_{\mathbf{c}}(i)} H^k_{\mathbf{c}}(N \setminus \operatorname{int}(B); \mathbb{Z}) = 0 \xrightarrow{H^k_{\mathbf{c}}(\Phi^\dagger)} H^k_{\mathbf{c}}(M; \mathbb{Z}),$$

i.e., $H^k_{\mathbf{c}}(\Phi) = 0$, which contradicts $\deg(\Phi) \neq 0$. Thus, Φ must be a surjective map. $\square$

Corollary 2.6.3.2 Let $\Phi\colon M \to N$ be a proper map of non-zero degree between two connected, oriented, boundaryless, smooth k-dimensional manifolds, and let $\Psi\colon M \to N$ be a proper map such that Ψ is properly homotopic to Φ. Then Ψ is a surjective map.

A straightforward modification of Theorem 2.6.3.1 above yields the following result when considering the boundary case.

Proposition 2.6.3.3 A proper map between two connected, oriented, smooth k-dimensional manifolds sending the boundary of the domain into the boundary of the codomain is a surjective map if its degree is non-zero.

Theorem 2.6.3.4 Let $f\colon \Sigma' \to \Sigma$ be a pseudo proper homotopy equivalence between two non-compact oriented surfaces. If $\Sigma \not\cong \mathbb{S}^1 \times \mathbb{R}, \mathbb{R}^2$ (equivalently $\Sigma' \not\cong \mathbb{S}^1 \times \mathbb{R}, \mathbb{R}^2$), then $\deg(f) = \pm 1$ and f is surjective. Moreover, if $g\colon \Sigma' \to \Sigma$ is proper map, which is properly homotopic to f, then $\deg(g) = \pm 1$ and g is surjective.

Proof. By Theorem 2.6.1.12 and Theorem 2.6.2.5, $\deg(f) = \pm 1$. Now, Theorem 2.6.3.1 tells that f is surjective. Finally, if g is a proper map that is properly homotopic to f, then by definition, g is also a pseudo proper homotopy equivalence, so we're done by the previous two parts. $\square$

Here is the main application of the non-vanishing degree of a pseudo proper homotopy equivalence.

Theorem 2.6.3.5 Let $f\colon \Sigma' \to \Sigma$ be a smooth pseudo proper homotopy equivalence between two non-compact surfaces, where $\mathbb{S}^1 \times \mathbb{R} \not\cong \Sigma \not\cong \mathbb{R}^2$; and let $\mathscr{A}$ be a preferred LFCS on Σ such that $f \pitchfork \mathscr{A}$. Suppose any two distinct components of $\mathscr{A}$ don't co-bound an annulus in Σ. In that case, f can be properly homotoped to a proper map g such that for each component $\mathcal{C}$ of $\mathscr{A}$, $g^{-1}(\mathcal{C})$ is a component of $f^{-1}(\mathscr{A})$ that is mapped homeomorphically onto $\mathcal{C}$ by g.

Proof. Theorem 2.5.3.5 gives a proper map $g\colon \Sigma' \to \Sigma$ such that the following hold: (1) g is properly homotopic to f, and (2) for each component $\mathcal{C}$ of $\mathscr{A}$, if $g^{-1}(\mathcal{C}) \neq \varnothing$, then $g^{-1}(\mathcal{C})$ is a component of $f^{-1}(\mathscr{A})$ such that $g|g^{-1}(\mathcal{C}) \to \mathcal{C}$ is homeomorphism. But $\deg(f) = \pm 1$, by Theorem 2.6.3.4. Thus, the map g is surjective since it is properly homotopic to the non-zero degree map f; see Corollary 2.6.3.2. So, for each component $\mathcal{C}$ of $\mathscr{A}$, $g^{-1}(\mathcal{C})$ is a component of $f^{-1}(\mathscr{A})$ such that $g|g^{-1}(\mathcal{C}) \to \mathcal{C}$ is homeomorphism. $\square$

Remark 2.6.3.6 For closed surfaces, the analogue of Theorem 2.6.3.5 can be stated far before, exactly in the "annulus removal" section, as every homotopy equivalence between two closed manifolds has a homotopy inverse, hence is a map of degree ± 1 and hence is surjective. But, before Section 2.6, we didn't know the degree of a pseudo proper homotopy equivalence; even in this stage, we don't know whether a pseudo proper homotopy equivalence has a proper homotopy inverse or not.

We conclude this section by proving two more facts, although they will never be needed to prove anything. First, the following proposition states that if either of the integers 1 and -1 appears as the degree of pseudo proper homotopy equivalence between two non-compact oriented surfaces, then the other also appears.

Proposition 2.6.3.7 Let $f\colon \Sigma' \to \Sigma$ be a pseudo proper homotopy equivalence between two non-compact oriented surfaces. Then there exists another pseudo proper homotopy equivalence $\bar{f}\colon \Sigma' \to \Sigma$ such that $\deg(\bar{f}) = -\deg(f)$.

Proof. By Corollary 1.5.3, choose an orientation-reversing self-homeomorphism φ of Σ. Then, the degree of $\bar{f} := \varphi \circ f$ is $-\deg(f)$ as the degree is multiplicative; see Section 1.7. $\square$

The next proposition asserts that an infinite-type surface cannot be properly deformed onto any of its spines.

Proposition 2.6.3.8 Let Σ be a non-compact surface with a CW-complex structure such that Σ is homeomorphic to neither the plane nor the punctured plane. Suppose $\mathcal{G}$ is a one-dimensional subcomplex of Σ with a deformation retract $r\colon \Sigma \to \mathcal{G}$. Then r cannot be a proper map.

Proof. Since every sub-complex of Σ is a closed in Σ, the inclusion map $i\colon \mathcal{G} \hookrightarrow \Sigma$ is a proper map. If possible, suppose r is a proper map. Then $i \circ r\colon \Sigma \to \Sigma$ is a proper map homotopic to the identity map of Σ. Thus, $i \circ r$ is a pseudo proper homotopy equivalence, and hence $\deg(i \circ r) = \pm 1$. Therefore, $i \circ r$ is surjective by Theorem 2.6.3.1. But $\mathcal{G} \subsetneq \Sigma$ implies $i \circ r$ is non-surjective, a contradiction. $\square$

2.7 Finishing the proof of Theorem I

With the tools developed previously, we are now ready to prove Theorem I, which has been rewritten below for the reader's convenience.

Theorem 3.1.1 (Strong topological rigidity) Let $f\colon \Sigma' \to \Sigma$ be a pseudo proper homotopy equivalence between two non-compact surfaces. Then Σ' is homeomorphic to Σ. If we further assume that Σ is homeomorphic to neither the plane nor the punctured plane, then f is a proper homotopy equivalence, and there exists a homeomorphism $f_{\text{homeo}}\colon \Sigma \to \Sigma'$ as a proper homotopy inverse of f.

Proof. First, assume that Σ is homeomorphic to neither the plane nor the punctured plane. Consider an LFCS $\mathscr{C}$ on Σ provided by Theorem 2.3.5. Using Theorem 2.4.1, assume f is smooth as well as $f \pitchfork \mathscr{C}$. Thus $f^{-1}(\mathscr{C})$ is a non-empty LFCS on Σ'; see Theorem 2.6.3.4 and Theorem 2.4.3. By Theorem 2.6.3.5, f can be properly homotoped to a proper map g such that for each component $\mathcal{C}$ of $\mathscr{C}$, $g^{-1}(\mathcal{C})$ is a component of $f^{-1}(\mathscr{C})$ that is mapped homeomorphically onto $\mathcal{C}$ by g. Thus, $g^{-1}(\mathscr{C})$ decomposes Σ' into bordered sub-surfaces and each component of $\Sigma \setminus \mathscr{C}$ has non-empty pre-image; see Theorem 2.6.3.4. Let $\mathbf{S} \subset \Sigma$ be a bordered sub-surface obtained as a complementary component of the decomposition of Σ by $\mathscr{C}$. Now, $\mathbf{S} \cong g^{-1}(\mathbf{S})$; by Lemma 2.6.1.11 (see its proof also) and Lemma 2.6.2.3. Since g sends $\operatorname{int}(g^{-1}(\mathbf{S}))$ onto $\operatorname{int}(\mathbf{S})$ and $\partial g^{-1}(\mathbf{S})$ homeomorphically onto $\partial\mathbf{S}$, we can properly homotope $g|g^{-1}(\mathbf{S}) \to \mathbf{S}$ relative to $\partial g^{-1}(\mathbf{S})$ to a homeomorphism $g^{-1}(\mathbf{S}) \to \mathbf{S}$; see Theorem 2.6.1.10 and Theorem 2.6.2.4. Finally, vary $\mathbf{S}$ over different complementary components of Σ decomposed by $\mathscr{C}$ to collect these boundary-relative proper homotopies and then paste them to get a proper homotopy from g to a homeomorphism $g_{\text{homeo}}\colon \Sigma' \to \Sigma$. Since g is properly homotopic to f, the homeomorphism $g_{\text{homeo}}\colon \Sigma' \to \Sigma$ is properly homotopic to f. Therefore, if Σ is homeomorphic to neither the plane nor the punctured plane, the Σ' is homeomorphic to Σ, and f is a proper homotopy equivalence with the homeomorphism $f_{\text{homeo}} := g_{\text{homeo}}^{-1}$ as a proper homotopy inverse.

Now, assume Σ is homeomorphic to either the plane or the punctured plane. Since Σ' is a non-compact surface homotopy equivalent to Σ, by Theorem 2.3.9, Σ' is also homeomorphic to Σ. This completes the proof. □

The proof of Theorem I shows that we are using the non-zero degree assumption of the pseudo proper homotopy equivalence (which is gifted by Theorem 2.6.3.4) to ensure surjectivity after each proper homotopy. Thus, by a similar argument, we can prove the following.

Theorem 2.7.1 Let $f\colon \Sigma' \to \Sigma$ be a pseudo proper homotopy equivalence between two non-compact oriented surfaces. Suppose Σ is not homeomorphic to $\mathbb{R}^2$ and $\deg(f) \neq 0$. Then Σ' is homeomorphic to Σ, and f is properly homotopic to a homeomorphism.

The following proposition is a direct application of Theorem I.

Proposition 2.7.2 Let Σ be a non-compact surface such that Σ is homeomorphic to neither the plane nor the punctured plane. Suppose $f, g\colon \Sigma \to \Sigma$ are two pseudo proper homotopy equivalences. If f is homotopic to g, then f is properly homotopic to g.

Proof. By applying Theorem I up to proper homotopy, we may assume both f and g are homeomorphisms without loss of generality. Thus the homeomorphism $f^{-1}g$ is homotopic to Id_Σ. By [28, Theorem 6.4. (a)], there exists a level-preserving homeomorphism $\mathcal{H}\colon \Sigma \times [0,1] \to \Sigma \times [0,1]$ which agrees with $f^{-1}g$ on $\Sigma \times 0$ and with Id_Σ on $\Sigma \times 1$. The projection $\Sigma \times [0,1] \to \Sigma$ is proper implies $f^{-1}g$ is properly homotopic to Id_Σ. So we are done. □

2.8 Consequences and application of strong topological rigidity

2.8.1 Classification of π_1-injective proper maps

The main aim of this section is to classify π_1-injective proper maps between two non-compact surfaces. In 1927, Nielsen [82] proved that any π_1-injective map between two compact surfaces is homotopic to a covering map; see also [54, Theorem 13.1] and [107, Lemma 1.4.3.]. The analogue of Nielsen's theorem for non-compact surfaces is Theorem 2.8.1.1 below, which asserts that the majority of π_1-injective proper maps between surfaces become finite-sheeted covering maps through proper homotopy.

In dimension three, it is also possible to achieve an analogous classification of π_1-injective proper maps, subject to various constraints. For example, Waldhausen [107, Theorem 6.1.] proved that if $f\colon N \to M$ is π_1-injective map between two connected, closed, orientable, irreducible 3-manifolds, where N is non simply-connected and M is Haken, then f homotopic to a covering map. Brown and Tucker [12, Theorem 4.2] showed that if $\mathfrak{f}\colon \mathfrak{N} \to \mathfrak{M}$ is a π_1-injective proper map between two connected, non-compact, orientable, irreducible, end-irreducible, boundaryless 3-manifolds such that $\pi_1(\mathfrak{N})$ is not isomorphic to the fundamental group of any compact surface, then $\mathfrak{f}$ is properly homotopic to a finite-sheeted covering map.

Theorem 2.8.1.1 (Classification of π_1-injective proper maps) Let Σ' and Σ be two non-compact oriented surfaces, where Σ' is neither the plane nor the punctured plane. Suppose $f\colon \Sigma' \to \Sigma$ is a π_1-injective proper map. Then f is properly homotopic to a d-sheeted covering map $p\colon \Sigma' \to \Sigma$ for some positive integer d. Thus, $\deg(f) = \pm d (\neq 0)$.

Proof. Let $p\colon \widetilde{\Sigma} \to \Sigma$ be the covering corresponding to the subgroup $\operatorname{im} \pi_1(f)$ of $\pi_1(\Sigma)$, and let $\widetilde{f}\colon \Sigma' \to \widetilde{\Sigma}$ be a lift of f with respect to p, i.e., $p \circ \widetilde{f} = f$. Thus, $\operatorname{im} \pi_1(p) = \operatorname{im} \pi_1(f)$, and hence $\pi_1(\widetilde{f})\colon \pi_1(\Sigma') \to \pi_1(\widetilde{\Sigma})$ is an isomorphism because a covering map induces injection between fundamental groups. Recall that if a non-compact surface has an infinite cyclic (resp. trivial) fundamental group, then it is homeomorphic to the punctured plane (resp. plane); see Theorem 2.3.9. Also, the properness of f implies the properness of $\widetilde{f}$ by Lemma 2.8.1. Since non-compact surfaces are $K(\pi, 1)$ CW-complexes, by Whitehead theorem [51, Theorem 4.5.] $\widetilde{f}$ is a homotopy equivalence. Thus, by Theorem I, $\widetilde{f}$ is properly homotopic to a homeomorphism, which implies $\deg(\widetilde{f}) = \pm 1$. Therefore, using Lemma 2.8.1, we may assume that p is a d-sheeted covering for some positive integer d and $\widetilde{\Sigma}$ is orientable. Fix an orientation of $\widetilde{\Sigma}$. By Theorem 1.7.1, $\deg(p) = \pm d$. Since $\deg(f) = \deg(p\widetilde{f}) = (\pm d) \cdot \deg(\widetilde{f}) = (\pm d) \cdot (\pm 1)$, we can conclude that $\deg(f) = \pm d (\neq 0)$, and f is properly homotopic to the d-sheeted covering map p. □

Notice that we have used the Lemma 2.8.1 below to prove Theorem 2.8.1.1. We will give a proof of Lemma 2.8.1 drawing upon ideas from [29, Theorem 3.1]. To understand the various unlabeled maps and ι present in the four commutative diagrams of the proof of Lemma 2.8.1, let's first recall the definition of singular cohomology with compact support. Consider a closed subset A of a topological space X, and let n be a non-negative integer. Recall that the n-th singular cohomology with compact support $H^n_{\mathsf{c}}(X, A; \mathbb{Z})$ is equal to the direct limit $\varinjlim H^n(X, A \cup (X \setminus K); \mathbb{Z})$, where K is a compact subset of X and various maps to define this direct system are induced by inclusions. Hence, for a compact subset K of X, the definition of direct limit yields an *obvious map* $H^n(X, A \cup (X \setminus K); \mathbb{Z}) \to H^n_{\mathsf{c}}(X, A; \mathbb{Z})$.

Lemma 2.8.1 Let $f\colon M \to N$ be a proper map between two connected, oriented, topological manifolds of the same dimension n such that $f(\partial M) \subseteq \partial N$, and let $p\colon \widetilde{N} \to N$ be the covering corresponding to the subgroup $\operatorname{im}(\pi_1(f))$ of $\pi_1(N)$.

1. Then $\widetilde{N}$ is a connected, orientable topological manifold of dimension n, and the covering map p sends $\partial\widetilde{N}$ onto ∂N and int $(\widetilde{N})$ onto int(N).
2. If $\widetilde{f}\colon M \to \widetilde{N}$ is a lift of f, then $\widetilde{f}$ is a proper map and $\widetilde{f}(\partial M) \subseteq \partial\widetilde{N}$.
3. If $\deg(f) \neq 0$, then p is a proper map.
4. If $\widetilde{f}\colon M \to \widetilde{N}$ is a lift of f with $\deg(\widetilde{f}) \neq 0$, then p is a proper map.

Proof. Recall that every point of an n-dimensional topological manifold X has a neighbourhood homeomorphic to either $\mathbb{R}^n$ or $\mathbb{H}^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$ such that a point $x \in X$ belongs to int(X) (resp. ∂X) if and only if x admits a neighbourhood homeomorphic to $\mathbb{R}^n$ (resp. $\mathbb{H}^n$). Since $p\colon \widetilde{N} \to N$ is a local homeomorphism, $\widetilde{N}$ is a topological manifold of dimension n. Furthermore, by the invariance of domain, p sends $\partial\widetilde{N}$ onto ∂N and int $(\widetilde{N})$ onto int(N). The connectedness of $\widetilde{N}$ follows from covering space theory. Also, $\widetilde{N}$ is orientable because N is orientable, and p is a local homeomorphism [36, Lemma 125.17. (a)]. This completes the proof of 1.

To prove 2., note that $p \circ \widetilde{f} = f$, which implies that $\widetilde{f}(\partial M) \subseteq \partial\widetilde{N}$. Moreover, $\widetilde{f}$ is proper by Theorem 1.6.1 because f is proper. This completes the proof of 2.

Now, we are ready to prove 3. and 4. via contradiction. Let $\widetilde{f}\colon M \to \widetilde{N}$ be a lift of f. Suppose the covering map p is not proper, i.e., $p^{-1}(x)$ is infinite for some $x \in \operatorname{int}(N)$. Define $C := f^{-1}(x) \subseteq \operatorname{int}(M)$. So, the compact set $\widetilde{f}(C)$ is contained in the discrete space $p^{-1}(x)$, i.e., we can write $\widetilde{f}(C) = \{\widetilde{x}_1, \ldots, \widetilde{x}_k\}$ for points $\widetilde{x}_1, \ldots, \widetilde{x}_k \in p^{-1}(x) \subset \widetilde{N}$. Since $p^{-1}(x)$ is infinite, we have an $\widetilde{x} \in p^{-1}(x) \setminus \{\widetilde{x}_1, \ldots, \widetilde{x}_k\}$. Thus, $\widetilde{f}(M) \subseteq \widetilde{N} \setminus \widetilde{x}$.

Since $\widetilde{N}$ is orientable, $H^n\left(\widetilde{N}, \widetilde{N} \setminus \widetilde{y}\right) \to H^n_{\mathsf{c}}\left(\widetilde{N}, \partial\widetilde{N}\right)$ is an isomorphism for every $y \in \operatorname{int}(\widetilde{N})$. Now, the following commutative diagram shows that $\deg(\widetilde{f}) = 0$.

$$\begin{tikzcd}
H^n_{\mathsf{c}}\left(\widetilde{N}, \partial\widetilde{N}\right) \arrow[r, "\widetilde{f}^*"] & H^n_{\mathsf{c}}(M, \partial M) \\
H^n\left(\widetilde{N}, \widetilde{N} \setminus \widetilde{x}\right) \arrow[u, "\cong"] \arrow[r, "\widetilde{f}^*"'] & H^n(M, M) = 0 \arrow[u]
\end{tikzcd}$$

With the contradiction method, we conclude the proof of 3.

Now, to finalize the proof of 4. via contradiction, we show that $\deg(f) = 0$. Considering the outermost part of the following commutative diagram, notice that it suffices to prove that the composition of two blue arrows is zero, thereby implying $\deg(f) = 0$.

$$
\begin{tikzcd}
H^n_{\mathbf{c}}(N, \partial N) \arrow[r, "f^*"] & H^n_{\mathbf{c}}(M, \partial M) \\
H^n(N, N \setminus x) \arrow[u, "\cong"] \arrow[r, "f^*"] \arrow[d, "p^*"'] & H^n(M, M \setminus C) \arrow[u, "\iota"'] \\
H^n\left(\widetilde{N}, \widetilde{N} \setminus p^{-1}(x)\right) \arrow[ur, "\widetilde{f}^*"] \arrow[r] & H^n\left(\widetilde{N}, \left(\widetilde{N} \setminus p^{-1}(x)\right) \bigcup \{\widetilde{x}_1, \dots, \widetilde{x}_k\}\right) \bigoplus \oplus_{\ell=1}^{k} H^n\left(\widetilde{N}, \widetilde{N} \setminus \widetilde{x}_\ell\right) \arrow[u, "\widetilde{f}^*"']
\end{tikzcd}
$$

The following commutative diagram

$$
\begin{tikzcd}
H^n\left(\widetilde{N}, \left(\widetilde{N} \setminus p^{-1}(x)\right) \bigcup \{\widetilde{x}_1, \dots, \widetilde{x}_k\}\right) \arrow[r, "\widetilde{f}^*"] \arrow[dr, "\widetilde{f}^*"'] & H^n(M, M \setminus C) \\
& H^n(M, M) = 0 \arrow[u]
\end{tikzcd}
$$

tells that $H^n\left(\widetilde{N}, \left(\widetilde{N} \setminus p^{-1}(x)\right) \bigcup \{\widetilde{x}_1, \dots, \widetilde{x}_k\}\right) \xrightarrow{\widetilde{f}^*} H^n(M, M \setminus C) \xrightarrow{\iota} H^n_{\mathbf{c}}(M, \partial M)$ is zero map. Also, for each $\ell = 1, \dots, k$, if we consider the following commutative diagram

$$
\begin{tikzcd}
H^n\left(\widetilde{N}, \widetilde{N} \setminus \widetilde{x}_\ell\right) \arrow[r, "\widetilde{f}^*"] \arrow[d, "\cong"'] & H^n\left(M, M \setminus C\right) \arrow[d, "\iota"] \\
H^n_{\mathbf{c}}\left(\widetilde{N}, \partial \widetilde{N}\right) \arrow[r, "\widetilde{f}^*"] & H^n_{\mathbf{c}}\left(M, \partial M\right) \\
H^n\left(\widetilde{N}, \widetilde{N} \setminus \widetilde{x}\right) \arrow[u, "\cong"] \arrow[r, "\widetilde{f}^*"] & H^n(M, M) = 0 \arrow[u]
\end{tikzcd}
$$

then $\oplus_{\ell=1}^{k} H^n\left(\widetilde{N}, \widetilde{N} \setminus \widetilde{x}_\ell\right) \xrightarrow{f^*} H^n(M, M \setminus C) \xrightarrow{\iota} H^n_{\mathbf{c}}(M, \partial M)$ is zero map. This completes the proof of 4. $\square$

Now, we are ready to provide the proof of Theorem 1.7.3, as promised. For convenience, let's recall its statement once more.

Theorem 1.7.3 Let $f\colon M \to N$ be a proper map between two connected, oriented, topological manifolds of the same dimension such that $f(\partial M) \subseteq \partial N$. If $\deg(f) \neq 0$, then the index $[\pi_1(N) : \mathrm{im}\ \pi_1(f)]$ divides $\deg(f)$. In particular, if $\deg(f) = \pm 1$, then $\pi_1(f)$ is surjective.

Proof. Let $p\colon \widetilde{N} \to N$ be the covering corresponding to the subgroup im $\pi_1(f)$ of $\pi_1(N)$. Thus, im $\pi_1(p) = $ im $\pi_1(f)$. Consider a lift $\widetilde{f}\colon M \to \widetilde{N}$ of f with respect to the covering p, i.e., $f = p \circ \widetilde{f}$. By Lemma 2.8.1, $\widetilde{N}$ is a connected, orientable manifold such that $\dim \widetilde{N} = \dim N$, $p(\partial\widetilde{N}) = \partial N$, and $\widetilde{f}(\partial M) \subseteq \partial\widetilde{N}$. Furthermore, Lemma 2.8.1 tells that p and $\widetilde{f}$ both are proper maps because $\deg(f) \neq 0$. Therefore, p is a d-sheeted covering map for some positive integer d. By covering space theory, $d = [\pi_1(N) : \text{im}\, \pi_1(p)] = [\pi_1(N) : \text{im}\, \pi_1(f)]$. Fix an orientation for $\widetilde{N}$. Then, $\deg(p) = \pm d$. Therefore, $\deg(f) = \deg(p \circ \widetilde{f}) = \deg(p)\deg(\widetilde{f})$, i.e., $[\pi_1(N) : \text{im}\, \pi_1(f)]$ divides $\deg(f)$. □

Let's consider the surfaces excluded from Theorem 2.8.1.1, namely the plane and the punctured plane, and see what happens. First, we consider the case when the domain is $\mathbb{R}^2$, and the degree is zero.

Theorem 2.8.1.2 Suppose f is a proper map from $\mathbb{R}^2$ to a non-compact oriented surface Σ of degree zero. If K is a compact subset of Σ, then there exists a proper map g properly homotopic to f such that $\text{im}(g) \subseteq \Sigma \setminus K$.

Proof. First, assume $\Sigma = \mathbb{R}^2$. By Theorem 1.7.2, properly homotope f so that the image of f misses a point $a \in \mathbb{R}^2$. Since any translation map of $\mathbb{R}^2$ is properly homotopic to the identity map of $\mathbb{R}^2$, we may assume $a = 0$. Thus, there exists $r > 0$ such that $|f| \geq r$ because proper maps between manifolds are closed maps by Theorem 1.6.2. Consider a compact subset K of $\mathbb{R}^2$. Let n be a positive integer such that $K \subseteq \{z \in \mathbb{R}^2 : |z| \leq nr\}$. Since $1 + nt \geq 1$ for every $t \in [0,1]$, the map $\mathbb{R}^2 \times [0,1] \ni (z,t) \longmapsto (1+nt)\cdot f(z) \in \mathbb{R}^2$ is a proper homotopy from f to $g := (n+1)f$. Certainly, $\text{im}(g) \cap K = \varnothing$. So, we are done when $\Sigma = \mathbb{R}^2$.

Now, we may assume Σ is not homeomorphic to $\mathbb{R}^2$. Let K be a compact subset of Σ. Using Goldman's inductive procedure, find a compact bordered subsurface $\mathbf{S}$ of Σ such that $K \subseteq \text{int}(\mathbf{S})$ and each component of $\partial\mathbf{S}$ is a primitive circle on Σ. By Theorem A.1, we may assume f is smooth. Further, by Theorem A.2, we may assume $f \pitchfork \partial\mathbf{S}$. Therefore, $f^{-1}(\partial\mathbf{S})$ is a pairwise disjoint finite collection of smoothly embedded circles on $\mathbb{R}^2$. Since $f^{-1}(\partial\mathbf{S})$ has finitely many components, each of which bounds a disk in $\mathbb{R}^2$, by considering a procedure similar to that given in proof of Theorem 2.5.1.5, we can properly homotope f to a proper map $g\colon \mathbb{R}^2 \to \Sigma$ so that $g^{-1}(\partial\mathbf{S}) = \varnothing$. Using continuity of g, either $\text{im}(g) \subset \mathbf{S}$ or $\text{im}(g) \subseteq \Sigma \setminus \mathbf{S}$. Since g is proper, the former can't happen, i.e., $\text{im}(g) \subseteq \Sigma \setminus \mathbf{S}$, and hence $\text{im}(g) \cap K = \varnothing$. □

Notice that, unlike the first part, in the second part of the above proof, we haven't utilized the geometric realization of the fact that $\deg(f) = 0$. This is because if f is a proper map from $\mathbb{R}^2$ to a non-compact oriented surface Σ of non-zero degree, then Σ must be homeomorphic to $\mathbb{R}^2$, by Theorem 1.7.3 and Theorem 2.3.9. The following theorem tells us what all proper maps $\mathbb{R}^2 \to \mathbb{R}^2$ of non-zero degree are.

Theorem 2.8.1.3 Let $f\colon \mathbb{C} \to \mathbb{C}$ be a proper map of degree $n \neq 0$. Then f is properly homotopic to the branched covering $\mathbb{C} \ni z \longmapsto z^n \in \mathbb{C}$ if $n > 0$, and to the branched covering $\mathbb{C} \ni z \longmapsto \overline{z}^{-n} \in \mathbb{C}$ if $n < 0$.

Proof. At first, suppose $n > 0$. Let $\mathscr{A}$ be the collection of all circles in $\mathbb{C}$ centred at 0 with integer radii. Then $\mathscr{A}$ is an LFCS on $\mathbb{C}$. By Theorem 2.4.1, we may assume f is a smooth proper map of degree $n > 0$ such that $f \pitchfork \mathscr{A}$. Therefore, $f^{-1}(\mathscr{A})$ is an LFCS by Theorem 2.4.3. Now, Theorem 1.7.2 gives a proper map g properly homotopic to f with the following properties: the map g equals f outside a compact subset K of the form $\{z \in \mathbb{C} : |z| \leq r\}$ for some $r > 0$, and there exists a disk D in $\mathbb{C}$ such that $g^{-1}(D)$ is the union of pairwise disjoint disks $D_1, \ldots, D_n$ in $\mathbb{C}$ such that $g|D_j \to D$ is an orientation-preserving homeomorphism for each $j = 1, \ldots, n$. Without loss of generality, we may assume K contains $\cup_{j=1}^{n} D_j$. Consider the bordered surfaces $\mathbf{S}' := \mathbb{C} \setminus \cup_{j=1}^{n} \operatorname{int}(D_k)$ and $\mathbf{S} := \mathbb{C} \setminus \operatorname{int}(D)$. Notice that $g(\mathbf{S}') \subseteq \mathbf{S}$ and g sends each component of $\partial\mathbf{S}' = g^{-1}(\partial\mathbf{S})$ homeomorphically onto $\partial\mathbf{S}$. Denote the restriction map $g|\mathbf{S}' \to \mathbf{S}$ by g_{res}. Since $\mathscr{A}$ is an LFCS, there exists a component $\mathcal{C}$ of $\mathscr{A}$ such that $\mathcal{C} \cap \big(f(K) \cup g(K)\big) = \varnothing$ and the interior of $\mathcal{C}$ (in $\mathbb{C}$) contains D. Since $f = g$ on $\mathbb{C} \setminus K$, we can say that $f^{-1}(\mathcal{C}) = g^{-1}(\mathcal{C}) = g_{\text{res}}^{-1}(\mathcal{C})$ doesn't intersect with K, g_{res} is smooth outside the compact subset $\mathbf{K}' := \mathbf{S}' \cap K$ of $\mathbf{S}'$, and $g_{\text{res}} \pitchfork \mathcal{C}$. In particular, each component of $g_{\text{res}}^{-1}(\mathcal{C})$ is either a trivial circle or primitive circle on $\mathbf{S}'$ lying in $\mathbf{S}' \setminus \mathbf{K}' = \mathbb{C} \setminus K = \{z \in \mathbb{C} : |z| > r\}$.

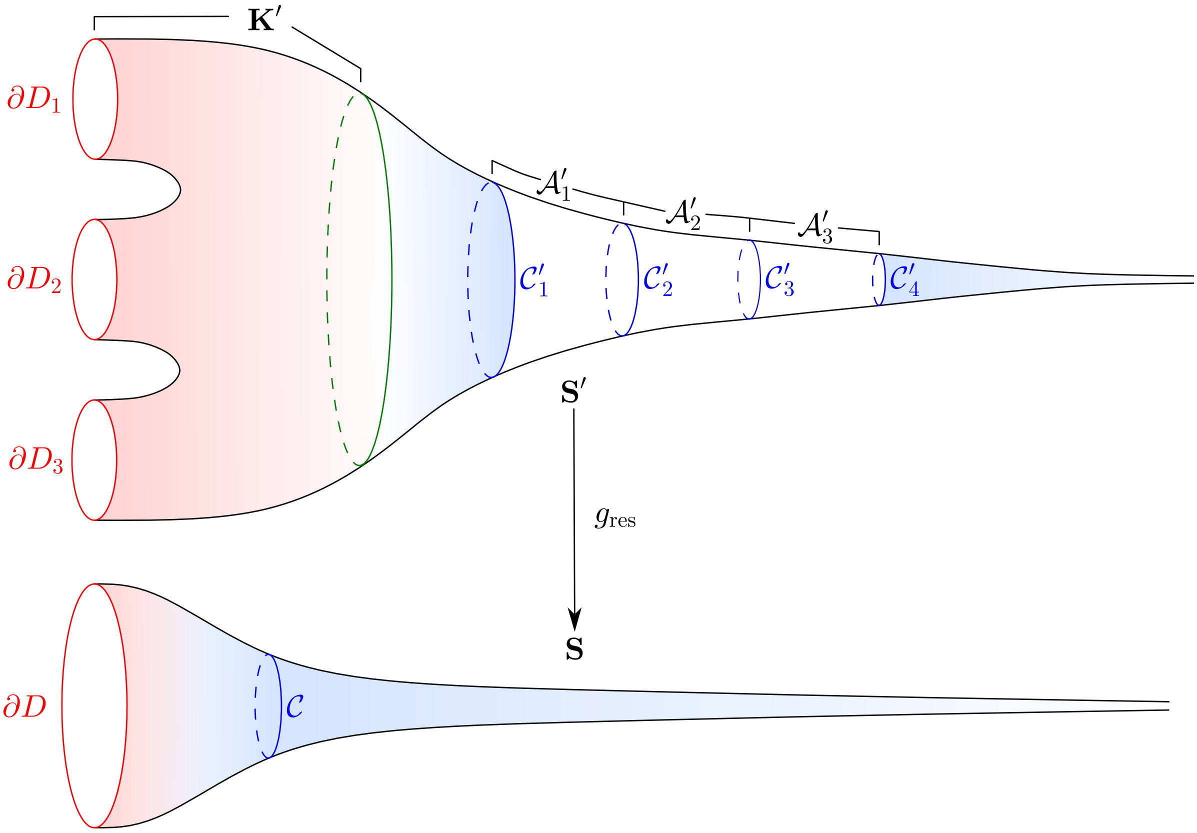

Fig. 2.8.1 The description of $g_{\text{res}} : \mathbf{S}' \to \mathbf{S}$ after removing trivial components from $g_{\text{res}}^{-1}(\mathcal{C})$.

Now, using the tricks mentioned in the proofs of Theorem 2.5.1.5, Theorem 2.5.2.3, and Theorem 2.5.3.5, we aim to properly homotope $g_{\text{res}} : \mathbf{S}' \to \mathbf{S}$ to a proper map $\widetilde{g_{\text{res}}} : \mathbf{S}' \to \mathbf{S}$ such that $\mathcal{C}' := \widetilde{g_{\text{res}}}^{-1}(\mathcal{C})$ is a single circle in $\operatorname{int}(\mathbf{S}')$ and $\widetilde{g_{\text{res}}}|\mathcal{C}' \to \mathcal{C}$ is an n-fold covering map, but here we want every proper homotopy to be relative to $\partial\mathbf{S}'$. So, here are the steps.

At first, applying Theorem 1.7.2 on g_{res}, and then using Theorem 1.7.1, we can tell $\deg(g_{\text{res}}) = \deg(g)$, i.e., in particular, $\deg(g_{\text{res}}) \neq 0$. Therefore, g_{res} remains surjective after any proper homotopy rel. $\partial\mathbf{S}'$; see Proposition 2.6.3.3.

Since $\mathcal{C}$ is a primitive circle on $\mathbf{S}$, using a trick similar to the one given in the proof of Theorem 2.5.1.5, after a proper homotopy $\mathcal{H}_1$, we may assume each component of the non-empty set $g_{\text{res}}^{-1}(\mathcal{C})$ is a primitive circle on $\mathbf{S}'$. Note that the homotopy that appears in the proof of Theorem 2.5.1.5 is, in fact, relative to the complement of every small enough neighbourhood of the union of all disks bounded by trivial circles. Thus, we may assume $\mathcal{H}_1$ is relative to $\partial\mathbf{S}'$.

Therefore, for some positive integer m, we can write $g_{\text{res}}^{-1}(\mathcal{C}) = \mathcal{C}_1' \sqcup \cdots \sqcup \mathcal{C}_m'$ (see Figure 2.8.1 for labeling of these circles), where each $\mathcal{C}_i'$ is a primitive circle on $\mathbf{S}'$. Thus $g_{\text{res}}^{-1}(\mathcal{C})$ decomposes $\mathbf{S}'$ into a copy of $S_{0,n+1}$, $(m-1)$-copies $\mathcal{A}_1', \ldots, \mathcal{A}_{m-1}'$ of $S_{0,2}$ (see Figure 2.8.1 for labelling of these annuli), and a copy of $S_{0,1,1}$. Similarly, $\mathcal{C}$ decomposes $\mathbf{S}$ into a copy of $S_{0,2}$ and a copy of $S_{0,1,1}$. Restricting g_{res} on those copies, we get a map $\xi\colon S_{0,n+1} \to S_{0,2}$ such that $\xi|\partial D_j \to \partial D$ is an orientation-preserving homeomorphism for each $j = 1, \ldots, n$ and ξ sends the other boundary component $\partial S_{0,n+1} \setminus \cup_{j=1}^{n} \partial D_j$ into $\mathcal{C}$. The naturality of the homology long exact sequence and [51, Exercise 31 of Section 3.3] give the following commutative diagram, where $\ell := \deg\left(\xi|\partial S_{0,n+1} \setminus \cup_{j=1}^{n}\partial D_j \to \mathcal{C}\right)$.

$$\begin{tikzcd}
H_2\left(S_{0,n+1}, \partial S_{0,n+1}\right) \cong \mathbb{Z} \arrow[r, "1\longmapsto \overset{n+1}{\oplus} 1"] \arrow[d, "\times \deg(\xi)"'] & \overset{n+1}{\oplus}\mathbb{Z} \cong H_1\left(\partial S_{0,n+1}\right) \arrow[d, "\overset{n+1}{\oplus} 1 \longmapsto n \oplus \ell"] \\
H_2\left(S_{0,2}, \partial S_{0,2}\right) \cong \mathbb{Z} \arrow[r, "1\longmapsto \overset{2}{\oplus} 1"'] & \overset{2}{\oplus}\mathbb{Z} \cong H_1\left(\partial S_{0,2}\right)
\end{tikzcd}$$

The commutativity of the diagram tells $\deg(\xi) = n$ and $\ell = n$. Since any two components of $g_{\text{res}}^{-1}(\mathcal{C})$ co-bound an annulus in $\mathbf{S}'$ and any two homotopic maps $\mathbb{S}^1 \to \mathbb{S}^1$ have the same degree, $\deg\left(g_{\text{res}}|\mathcal{C}_i' \to \mathcal{C}\right) = \pm n$ for each $i = 1, \ldots, m$ (the minus sign comes because two boundary components of an oriented annulus are oppositely oriented). Thus, using a trick similar to the one mentioned in the proof of Theorem 2.5.2.3, after a proper homotopy $\mathcal{H}_2$, we may assume that $g_{\text{res}}|\mathcal{C}_i' \to \mathcal{C}$ is an n-fold covering map for each $i = 1, \ldots, m$. Since the homotopy mentioned in the proof of Theorem 2.5.2.3 is relative to the complement of a small neighbourhood of $g_{\text{res}}^{-1}(\mathcal{C}) = \mathcal{C}_1' \sqcup \cdots \sqcup \mathcal{C}_m'$, one can ensure that $\mathcal{H}_2$ is relative to $\partial\mathbf{S}'$.

Now, we remove the components $\mathcal{C}_2', \ldots, \mathcal{C}_m'$ from $g_{\text{res}}^{-1}(\mathcal{C})$. First, consider the annulus $\mathcal{A}_1'$. By continuity, $g_{\text{res}}(\mathcal{A}_1')$ will be one of two sides of $\mathcal{C}$ in $\mathbf{S}$, i.e., the compact set $g_{\text{res}}(\mathcal{A}_1')$ is contained in a one-sided tubular neighbourhood of $\mathcal{C}$ in $\mathbf{S}$. Note that if $\mathbf{S}''$ is a compact bordered subsurface of $\mathbf{S}'$ such that $\mathbf{S}'' \cap \partial\mathbf{S}' = \varnothing$, then a homotopy of $g_{\text{res}}|\mathbf{S}''$ rel. $\partial\mathbf{S}''$ can be extended to a proper homotopy of g_{res} rel. $\mathbf{S}' \setminus \operatorname{int}(\mathbf{S}'')$ $(\supset \partial\mathbf{S}')$; see Lemma 2.5.1.2. Thus, after a homotopy of $g_{\text{res}}|\mathcal{A}_1'$ rel. $\partial\mathcal{A}_1'$ (see Remark 2.5.3.4), we may assume $g_{\text{res}}(\mathcal{A}_1') = \mathcal{C}$. Applying this argument to each of $\mathcal{A}_2', \ldots, \mathcal{A}_{m-1}'$, after a homotopy of $g_{\text{res}}|\mathcal{A}_1' \cup \cdots \cup \mathcal{A}_{m-1}'$ rel. $\partial\left(\mathcal{A}_1' \cup \cdots \cup \mathcal{A}_{m-1}'\right)$, we may assume $g_{\text{res}}\left(\mathcal{A}_1' \cup \cdots \cup \mathcal{A}_{m-1}'\right) = \mathcal{C}$. Further, the part 2. of Theorem 2.5.2.3 tells that there exists a one-sided tubular neighbourhood $\mathcal{C}_m' \times [1,2] \equiv \mathcal{V}' \subset S_{0,1,1}$ of $\mathcal{C}_m' \equiv \mathcal{C}_m' \times 2$ and a one-sided tubular neighbourhood $\mathcal{C} \times [1,2] \equiv \mathcal{V}$ of $\mathcal{C} \equiv \mathcal{C} \times 2$ such that $g_{\text{res}}(z,r) = g_{\text{res}}(z) \times r$ for all

$(z, r) \in \mathcal{C}'_m \times [1, 2]$. Applying Lemma 2.5.3.6, after a homotopy of $g_{\text{res}}|\,(\mathcal{A}'_1 \cup \cdots \cup \mathcal{A}'_{m-1}) \cup \mathcal{V}'$ rel. $\mathcal{C}'_1 \sqcup (\mathcal{C}'_m \times 1)$, we have $g_{\text{res}}^{-1}(\mathcal{C}) = \mathcal{C}'_1$ and $g_{\text{res}}|\mathcal{C}'_1 \to \mathcal{C}$ is an n-fold covering map.

Since every proper homotopy of g_{res} has been done rel. $\partial\mathbf{S}'$, pasting $g|g^{-1}(D) \to D$ with all those proper homotopies, we can say that $g\colon \mathbb{C} \to \mathbb{C}$ can be properly homotoped so that $g|g^{-1}(\mathcal{C}) \to \mathcal{C}$ becomes an orientation-preserving n-fold covering map from a (single) circle onto a circle. Since an n-fold orientation-preserving covering $\mathbb{S}^1 \to \mathbb{S}^1$ is of the form $\mathbb{S}^1 \ni z \longmapsto h(z^n) \in \mathbb{S}^1$ for some orientation-preserving self-homomorphism $h\colon \mathbb{S}^1 \to \mathbb{S}^1$, by an application of Theorem 2.8.1.4 and Theorem 1.6.3, after a proper homotopy, we may assume $g|g^{-1}(\mathbb{S}^1) = \mathbb{S}^1 \ni z \longmapsto z^n \in \mathbb{S}^1$. Now, applying Proposition 2.8.1.5 and Proposition 2.8.1.6, we can conclude that g, and consequently f as well, is properly homotopic to $\mathbb{C} \ni z \longmapsto z^n \in \mathbb{C}$. This completes the proof of the case when $\deg(f) = n > 0$.

Now, assume $\deg(f) = n < 0$. Since the complex conjugation is an orientation-reversing self-homeomorphism of $\mathbb{C}$, the map $\bar{f}\colon \mathbb{C} \ni z \longmapsto \overline{f(z)} \in \mathbb{C}$ is orientation-preserving of degree $-n$. Thus, $\bar{f}$ is properly homotopic to $\mathbb{C} \ni z \longmapsto z^{-n} \in \mathbb{C}$ by the previous case. Therefore, f is properly homotopic to $\mathbb{C} \ni z \longmapsto \bar{z}^{-n} \in \mathbb{C}$. $\square$

The famous Annulus Theorem says that if two orientation-preserving locally flat embeddings φ and ψ of the closed unit ball $\mathbb{B}^n := \{x \in \mathbb{R}^n : |x| \leq 1\}$ into $\mathbb{R}^n$ satisfies $\varphi(\mathbb{B}^n) \subset \psi(\operatorname{int}(\mathbb{B}^n))$, then $\psi(\mathbb{B}^n) \setminus \varphi(\operatorname{int}(\mathbb{B}^n))$ is homeomorphic to $\mathbb{S}^{n-1} \times [0, 1]$. The annulus theorem was proved in dimension two by Radó [90], in dimension three by Moise [78, Theorem 1], in dimension four by Quinn [89, page 506], and in dimensions at least five by Kirby [67, page 576]. It is well known that the annulus theorem establishes the following theorem, which is the main tool for showing that the connected sum operation of topological n-manifolds is independent of the choice of embeddings of the n-balls. The analogue of Theorem 2.8.1.4 for smooth manifolds is called the Cerf-Palais' disk theorem [85, Theorem 5.5.] [86, Theorem B].

Theorem 2.8.1.4 [37, Proof of Theorem 4.12.] [36, Theorem 153.10.] Let M be an oriented, connected n-dimensional topological manifold without boundary. If $\varphi, \psi\colon \mathbb{B}^n \hookrightarrow M$ are two locally flat orientation-preserving embeddings of the closed unit balls into M, then there exists a homotopy $\mathcal{H}\colon M \times [0, 1] \to M$ through homeomorphisms such that $\mathcal{H}(-, 0) = \mathrm{Id}_M$ and $\mathcal{H}(-, 1) \circ \varphi = \psi$.

In the proof of Theorem 2.8.1.3, we have also used the following well-known facts, for which proof will be provided by modifying the proof of the Alexander trick [30, Lemma 2.1].

Proposition 2.8.1.5 Let $f\colon \mathbb{B}^2 \to \mathbb{B}^2$ be a map such that $f^{-1}(\mathbb{S}^1) = \mathbb{S}^1$. Then f can be homotoped rel. $\mathbb{S}^1$ to a map $g\colon \mathbb{B}^2 \to \mathbb{B}^2$ such that $g(0) = 0$ and $g(z) = |z| \cdot f\left(\frac{z}{|z|}\right)$ if $z \neq 0$.

Proof. Consider $H\colon \mathbb{B}^2 \times [0, 1] \to \mathbb{B}^2$ defined as follows

$$H(z, t) := \begin{cases} (1-t) \cdot f\left(\frac{z}{1-t}\right) & \text{if } 0 \leq |z| < 1 - t, \\ |z| \cdot f\left(\frac{z}{|z|}\right) & \text{if } 1 - t \leq |z| \leq 1 \text{ and } (z, t) \neq (0, 1), \\ 0 & \text{if } (z, t) = (0, 1). \end{cases}$$

Define $g := H(-, 1)$. $\square$

Proposition 2.8.1.6 Let $\mathbb{B}^2_* := \{z \in \mathbb{C} : 0 < |z| \leq 1\}$. Suppose $f\colon \mathbb{B}^2_* \to \mathbb{B}^2_*$ is a proper map such that $f^{-1}(\mathbb{S}^1) = \mathbb{S}^1$. Then f can be properly homotoped rel. $\mathbb{S}^1$ to a proper map $g\colon \mathbb{B}^2_* \to \mathbb{B}^2_*$ such that $g(z) = |z| \cdot f\left(\frac{z}{|z|}\right)$ for each $z \in \mathbb{B}^2_*$.

Proof. Define $\mathcal{H}\colon \mathbb{B}^2_* \times [0,1] \to \mathbb{B}^2_*$ by

$$\mathcal{H}(z,t) := \begin{cases} (1-t)\cdot f\left(\frac{z}{1-t}\right) & \text{if } 0 < |z| \leq 1-t, \\ |z| \cdot f\left(\frac{z}{|z|}\right) & \text{if } 1-t < |z| \leq 1. \end{cases}$$

Define $g := \mathcal{H}(-,1)$. By an argument similar to what is given in the proof of Theorem 2.6.2.4, one can show that $\mathcal{H}$ is proper. □

Now, let's consider the surface $\mathbb{S}^1 \times \mathbb{R}$.

Theorem 2.8.1.7 Let f be a π_1-injective proper map from $\mathbb{S}^1 \times \mathbb{R}$ to a non-compact oriented surface Σ. Suppose $\deg(f) = 0$. Then there exists a π_1-injective, proper embedding $\iota\colon \mathbb{S}^1 \times [0,\infty) \hookrightarrow \Sigma$, along with a non-zero integer d, such that after a proper homotopy, f can be described by the proper map $\mathbb{S}^1 \times \mathbb{R} \ni (z,t) \longmapsto \iota\left(z^d, |t|\right) \in \Sigma$. Thus, Σ has an isolated planar end, and given any compact subset K of Σ, there exists a proper map g properly homotopic to f such that $\operatorname{im}(g) \subseteq \Sigma \setminus K$.

To prove Theorem 2.8.1.7, we need the following lemma.

Lemma 2.8.1.8 Let f be a π_1-injective proper map from $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ to a non-compact oriented surface Σ. Suppose $\deg(f) = 0$. Then Σ contains an essential punctured disk D_* such that f can be properly homotoped so that $f(\mathbb{C}^*) \subseteq \mathrm{D}_*$.

Proof. Notice that $\Sigma \not\cong \mathbb{R}^2$ because $\pi_1(f)$ is injective. So, there exists an LFCS $\mathscr{A}$ on Σ such that $\mathscr{A}$ decomposes Σ into bordered sub-surfaces, and a complementary component of this decomposition is homeomorphic to either $S_{1,1}$, $S_{0,3}$, or $S_{0,1,1}$. By Theorem A.1, we may assume f is smooth. Further, by Theorem A.2, we may assume $f \pitchfork \mathscr{A}$. Therefore, $f^{-1}(\mathscr{A})$ is an LFCS on $\mathbb{C}^*$ by Theorem 2.4.3. Since each component of $\mathscr{A}$ is a primitive circle on Σ, and f is π_1-injective, by Proposition 2.5.2.5, after a proper homotopy, we may further assume that each component of $f^{-1}(\mathscr{A})$, if any, is a primitive circle on $\mathbb{C}^*$, and f maps for every component of $f^{-1}(\mathscr{A})$ onto a component of $\mathscr{A}$ by a covering map. Moreover, any two primitive circles on $\mathbb{C}^*$ co-bound an annulus in $\mathbb{C}^*$, but no two distinct components of $\mathscr{A}$ co-bound an annulus in Σ together imply either $f^{-1}(\mathscr{A}) = \varnothing$ or there exists a component $\mathcal{C}$ of $\mathscr{A}$ such that $f^{-1}(\mathscr{A}) = f^{-1}(\mathcal{C}) \neq \varnothing$.

At first, suppose $f^{-1}(\mathscr{A}) = \varnothing$. Then $f(\mathbb{C}^*)$, being non-compact, must be contained in a punctured disk D_* that appears as a complementary component of the decomposition of Σ by $\mathscr{A}$. Hence, the proof for this case is complete.

From now onwards we will assume $f^{-1}(\mathscr{A}) = f^{-1}(\mathcal{C}) \neq \varnothing$. If $f^{-1}(\mathcal{C})$ has more than one component, then we want to properly homotope f so that $f^{-1}(\mathcal{C})$ becomes a single circle and $f|f^{-1}(\mathcal{C}) \to \mathcal{C}$ is a covering map. The main tool for all those proper homotopies is the observation that if S' is a compact bordered subsurface of $\mathbb{C}^*$, then a homotopy of $f|\mathrm{S}'$ rel. $\partial\mathrm{S}'$ can be extended to a proper homotopy of f.

Notice that any two distinct components of $f^{-1}(\mathcal{C})$, being primitive circles on $\mathbb{C}^*$, co-bound an annulus in $\mathbb{C}^*$. So, let $\mathcal{A}'$ be an annulus co-bounded by two components of $f^{-1}(\mathcal{C})$. The continuity of f tells that in the decomposition of Σ by $\mathscr{A}$, $f(\mathcal{A}')$ lies within one of the two complementary components that share $\mathcal{C}$ as a common boundary. Thus, we can always find a compact bordered subsurface S of Σ such that the inclusion map $\mathrm{S} \hookrightarrow \Sigma$ is π_1-injective and $f(\mathcal{A}') \subseteq \mathrm{S}$. Since $f|\mathcal{A}' \to \mathrm{S}$ is π_1-injective and $f|\partial\mathcal{A}' \to \mathcal{C}$ is a local homeomorphism, after a homotopy of $f|\mathcal{A}' \to \mathrm{S}$ rel. $\partial\mathcal{A}'$, we may assume $f(\mathcal{A}') \subseteq \mathcal{C}$ [107, Lemma 1.4.3]. Let $\mathcal{A}'_{\text{out}}$ denote the union of all annuli co-bounded by two components of $f^{-1}(\mathcal{C})$. Then, after a homotopy of $f|\mathcal{A}'_{\text{out}} \to \Sigma$ rel. $\partial\mathcal{A}'_{\text{out}}$, we may assume $f\left(\mathcal{A}'_{\text{out}}\right) \subseteq \mathcal{C}$. Denote the boundary components of $\mathcal{A}'_{\text{out}}$ by $\mathcal{C}'$ and $\mathcal{C}''$. The part 2. of Theorem 2.5.2.3 tells that there exists a one-sided tubular neighborhood $\mathcal{C}' \times [1,2] \equiv \mathcal{V}'$ of $\mathcal{C}' \equiv \mathcal{C}' \times 2$ and a one-sided tubular neighborhood $\mathcal{C} \times [1,2] \equiv \mathcal{V}$ of $\mathcal{C} \equiv \mathcal{C} \times 2$ such that $f(z,r) = \big(f(z), r\big)$ for all $(z,r) \in \mathcal{C}' \times [1,2]$. Applying Lemma 2.5.3.6, after a homotopy of $f|\mathcal{A}'_{\text{out}} \cup \mathcal{V}'$ rel. $\mathcal{C}'' \sqcup (\mathcal{C}' \times 1)$, we can assume that $f^{-1}(\mathcal{C}) = \mathcal{C}''$ and $f|\mathcal{C}'' \to \mathcal{C}$ is an n-sheeted covering map.

At this point, two cases arise depending on whether Σ is equal to $\mathbb{C}^*$ or not. Let's begin by assuming $\Sigma \not\cong \mathbb{C}^*$. Thus, if S_1 and S_2 are two distinct, complementary components of the decomposition of Σ by $\mathscr{A}$ such that S_1 is a punctured disk and $\partial\mathrm{S}_1 \cap \partial\mathrm{S}_2 \neq \varnothing$, then S_2 must be compact. Since f is proper, $f(\mathbb{C}^*)$ is non-compact. Now, upon considering the decomposition of Σ by $\mathscr{A}$, we can tell that $f(\mathbb{C}^*)$ is contained in one of the two complementary components that share $\mathcal{C}$ as a common boundary. Hence, there exists a punctured disk D_* appears as a complementary component of the decomposition of Σ by $\mathscr{A}$ such that $\partial\mathrm{D}_* = \mathcal{C}$ and $f(\mathbb{C}^*) \subseteq \mathrm{D}_*$.

Next, assume $\Sigma = \mathbb{C}^*$, and let $\mathrm{D}_*^{(1)}$ and $\mathrm{D}_*^{(2)}$ be the punctured disks appear as complementary components of the decomposition of $\mathbb{C}^*$ by $\mathcal{C}$. Note that in this case, $f(\mathbb{C}^*)$ cannot be equal to $\mathbb{C}^*$; otherwise, applying Proposition 2.8.1.6 on each $f|f^{-1}\left(\mathrm{D}_*^{(1)}\right) \to \mathrm{D}_*^{(1)}$ and $f|f^{-1}\left(\mathrm{D}_*^{(2)}\right) \to \mathrm{D}_*^{(2)}$, would yield $\deg(f) = \pm n \neq 0$. So, $f(\mathbb{C}^*)$ must contained in one of $\mathrm{D}_*^{(1)}$ or $\mathrm{D}_*^{(2)}$. So, we are done. □

Notice that, except in the last paragraph of the previous proof, we haven't utilized the fact that $\deg(f) = 0$. This is because of the following proposition.

Proposition 2.8.1.9 Let f be a π_1-injective proper map from $\mathbb{C}^*$ to a non-compact oriented surface Σ such that $\deg(f) \neq 0$. Then Σ must be homeomorphic to $\mathbb{C}^*$.

Proof. Since $\pi_1(f)$ is injective, $\Sigma \not\cong \mathbb{R}^2$. Now, note that the fundamental group of a non-compact surface other than the plane and the punctured plane is a free group of rank at least two; see Theorem 2.3.9. By Theorem 1.7.3, the index $[\pi_1(\Sigma) : \operatorname{im} \pi_1(f)]$ must be finite. Thus, $\pi_1(\Sigma)$ is finitely generated. So, if $\Sigma \neq \mathbb{C}^*$, then by the Nielsen-Schreier index formula [36, Proposition 74.4.(2)], $\operatorname{im} \pi_1(f)$ is a free group of rank at least 2, a contradiction. □

Now, we are ready to prove the first part of Theorem 2.8.1.7.

Proof of Theorem 2.8.1.7. Notice that $\Sigma \not\cong \mathbb{R}^2$ because f is π_1-injective. By Lemma 2.8.1.8, we may assume that $f(\mathbb{S}^1 \times \mathbb{R}) \subseteq \mathrm{D}_*$ for some essential punctured disk $\mathrm{D}_* \subset \Sigma$. Let e be the (isolated planar) end of Σ determined by D_*. Consider a locally-finite, pairwise-disjoint collection $\mathscr{A} := \{\mathcal{C}_i : i = 1, 2, ..\}$ of smoothly embedded primitive circles on Σ such that each $\mathcal{C}_i$ is contained in $\operatorname{int}(\mathrm{D}_*)$. Observe that if a proper map g is properly homotopic to f, then $g^{-1}(\mathscr{A}) \neq \varnothing$ because $\operatorname{Ends}(g) = \operatorname{Ends}(f)$ sends both elements of $\operatorname{Ends}(\mathbb{S}^1 \times \mathbb{R})$ to e. By Theorem A.1, we may assume f is smooth as well as transverse to $\mathscr{A}$. Since each component of $\mathscr{A}$ is a primitive circle on Σ, and f is π_1-injective, by Proposition 2.5.2.5, after a proper homotopy, we may further assume that each component of the non-empty LFCS $f^{-1}(\mathscr{A})$ is a primitive circle on $\mathbb{S}^1 \times \mathbb{R}$ and f maps for every component of $f^{-1}(\mathscr{A})$ onto a component of $\mathscr{A}$ by a covering map. Thus, there exists a positive integer n_0 such that $f^{-1}(\mathcal{C}_{n_0}) \neq \varnothing$ and f sends every component of $f^{-1}(\mathcal{C}_{n_0})$ onto $\mathcal{C}_{n_0}$ by a covering map. Now, by an argument similar to what is given in the proof of Lemma 2.8.1.8, after a proper homotopy, we may assume that $\mathcal{C}'_{n_0} := f^{-1}(\mathcal{C}_{n_0})$ is a single circle and $f|\mathcal{C}'_{n_0} \to \mathcal{C}_{n_0}$ is a finite-sheeted covering. Let $\mathcal{D}_*$ be the essential punctured disk contained in D_* such that $\partial\mathcal{D}_* = \mathcal{C}_{n_0}$. Notice that $f(\mathbb{S}^1 \times \mathbb{R}) = \mathcal{D}_*$.

Now, choose $r \neq 0$ so that $\mathbb{S}^1 \times r$ doesn't intersect $\mathcal{C}'_{n_0}$. Since $\mathbb{S}^1 \times r$ co-bounds an annulus with each of the circles $\mathbb{S}^1 \times 0$ and $\mathcal{C}'_{n_0}$, the isotopy extension theorem [30, Proposition 1.11] [57, Theorem 1.3.] gives two homotopies $\mathcal{H}_1, \mathcal{H}_2 \colon (\mathbb{S}^1 \times \mathbb{R}) \times [0,1] \to \mathbb{S}^1 \times \mathbb{R}$ through homeomorphisms such that $\mathcal{H}_1(-,0) = \mathrm{Id}_{\mathbb{S}^1 \times \mathbb{R}} = \mathcal{H}_2(-,0)$, $\mathcal{H}_1(\mathcal{C}'_{n_0}, 1) = \mathbb{S}^1 \times r$, and $\mathcal{H}_2(\mathbb{S}^1 \times r, 1) = \mathbb{S}^1 \times 0$. By Theorem 1.6.3, $\mathcal{H}_1$ and $\mathcal{H}_2$ are proper homotopies. Thus, there exists a homeomorphism of $\mathbb{S}^1 \times \mathbb{R}$ properly homotopic to the identity, which sends $\mathcal{C}'_{n_0}$ onto $\mathbb{S}^1 \times 0$. Therefore, after a proper homotopy, we may assume that $f(\mathbb{S}^1 \times \mathbb{R}) = \mathcal{D}_*$ and $f|\mathbb{S}^1 \times 0 \to \partial\mathcal{D}_* = \mathcal{C}_{n_0}$ is a finite-sheeted covering. Now, by the classification of finite-sheeted covering maps of the circle and together with an application of Proposition 2.8.1.6 on each of $f|\mathbb{S}^1 \times (-\infty, 0] \to \mathcal{D}_*$ and $f|\mathbb{S}^1 \times [0,\infty) \to \mathcal{D}_*$, we can conclude that there exists a π_1-injective, proper embedding $\iota \colon \mathbb{S}^1 \times [0,\infty) \hookrightarrow \Sigma$ with $\operatorname{im}(\iota) = \mathcal{D}_*$ such that after a proper homotopy, f can be described by the proper map $\mathbb{S}^1 \times \mathbb{R} \ni (z,t) \longmapsto \iota\left(z^d, |t|\right) \in \Sigma$ for some non-zero integer d. Moreover, by an argument similar to that given in the first part of the proof of Theorem 2.8.1.2, we can conclude that given any compact subset K of Σ, there exists a proper map g properly homotopic to f such that $\operatorname{im}(g) \subseteq \Sigma \setminus K$. □

An argument similar to the one given in the proof of Theorem 2.8.1.7 provides the following:

Proposition 2.8.1.10 Let $f \colon \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$ be a π_1-injective proper map of degree 0. Then f is properly homotopic to $\mathbb{S}^1 \times \mathbb{R} \ni (z,t) \longmapsto \left(z^d, |t|\right) \in \mathbb{S}^1 \times \mathbb{R}$ for some integer $d \neq 0$.

Now, we proceed to classify the non-zero degree π_1-injective maps from $\mathbb{S}^1 \times \mathbb{R}$ into surfaces.

Theorem 2.8.1.11 Let f be a π_1-injective proper map from $\mathbb{C}^*$ to a non-compact oriented surface Σ. Suppose $n := \deg(f)$ is non-zero. Then, $\Sigma \cong \mathbb{C}^*$ and f is properly homotopic to the covering $\mathbb{C}^* \ni z \longmapsto z^n \in \mathbb{C}^*$ if $n > 0$, and to the covering $\mathbb{C}^* \ni z \longmapsto \overline{z}^{-n} \in \mathbb{C}^*$ if $n < 0$

Proof. By Proposition 2.8.1.9, $\Sigma \cong \mathbb{C}^*$. Let $p \colon \widetilde{\Sigma} \to \mathbb{C}^*$ be the covering corresponding the subgroup $\operatorname{im} \pi_1(f)$ of $\pi_1(\mathbb{C}^*)$, and let $\widetilde{f} \colon \mathbb{C}^* \to \widetilde{\Sigma}$ be a lift of f w.r.t. p, i.e., $p \circ \widetilde{f} = f$. Thus,

$\operatorname{im} \pi_1(p) = \operatorname{im} \pi_1(f)$, and hence $\pi_1(\widetilde{f})$ is an isomorphism because a covering map induces injection between fundamental groups. In particular, the fundamental group of $\widetilde{\Sigma}$ is infinite cyclic. Hence, $\widetilde{\Sigma} \cong \mathbb{C}^*$ by Theorem 2.3.9. The properness of f implies the properness of $\widetilde{f}$ by Lemma 2.8.1. Since non-compact surfaces are $K(\pi, 1)$ CW-complexes, by Whitehead theorem [51, Theorem 4.5.] $\widetilde{f}$ is a homotopy equivalence. By Lemma 2.8.1, we may assume p is a d-sheeted covering for some positive integer d and $\widetilde{\Sigma}$ is orientable. Fix an orientation of $\widetilde{\Sigma}$. By Theorem 1.7.1, $\deg(p) = \pm d$. Now, $\deg(\widetilde{f}) \neq 0$ because $0 \neq n = \deg(f) = \deg(p\widetilde{f}) = (\pm d) \cdot \deg(\widetilde{f})$. Thus, $\widetilde{f} \colon \mathbb{C}^* \to \mathbb{C}^*$ is a homotopy equivalence and $\deg(\widetilde{f}) \neq 0$. So $\widetilde{f}$ is properly homotopic to a homeomorphism by Theorem 2.7.1, i.e., $\deg(\widetilde{f}) = \pm 1$. Hence, $n = \pm d$, and f is properly homotopic to a d-sheeted covering.

At first, suppose $n > 0$. By the previous paragraph, without loss of generality, we may assume $f \colon \mathbb{C}^* \to \mathbb{C}^*$ is an n-sheeted covering map. Now, covering space theory [51, Proposition 1.37.] gives a self-homeomorphism h of $\mathbb{C}^*$ such that $f(z) = h(z^n)$ for all $z \in \mathbb{C}^*$. Certainly, h is orientation-preserving. By Proposition 2.8.1.12 below, we may assume $f(1) = 1 = h(1)$. Now, [28, Theorem 5.7.] tells that there exists a level-preserving homeomorphism $\mathcal{H} \colon \mathbb{C}^* \times [0, 1] \to \mathbb{C}^* \times [0, 1]$ which agrees with h on $\mathbb{C}^* \times 0$ and with $\mathrm{Id}_{\mathbb{C}^*}$ on $\mathbb{C}^* \times 1$. The projection $\mathbb{C}^* \times [0, 1] \to \mathbb{C}^*$ is proper implies h is properly homotopic to $\mathrm{Id}_{\mathbb{C}^*}$. So we are done when $\deg(f) = n > 0$.

Now, assume $\deg(f) = n < 0$. Since the complex conjugation is an orientation-reversing self-homeomorphism of $\mathbb{C}$, the map $\bar{f} \colon \mathbb{C}^* \ni z \longmapsto \overline{f(z)} \in \mathbb{C}^*$ is orientation-preserving of degree $-n$. Thus, $\bar{f}$ is properly homotopic to $\mathbb{C}^* \ni z \longmapsto z^{-n} \in \mathbb{C}^*$ by the previous case. Therefore, f is properly homotopic to $\mathbb{C}^* \ni z \longmapsto \bar{z}^{-n} \in \mathbb{C}^*$. □

It is known that every connected, boundaryless manifold M is homogeneous, i.e., for any two points x and y of M, there exists a homeomorphism h of M sending x to y. Moreover, one can also show that the homeomorphism h can be chosen so that it is homotopic to the identity map of M [106, Lemma 6.4]. In case M is non-compact, Theorem 2.8.1.4 tells us that the homeomorphism h can be chosen so that it becomes properly homotopic to the identity of M.

Proposition 2.8.1.12 Let M be an orientable, connected topological manifold without boundary. If x and y are two points of M, then there exists a self-homeomorphism $\varphi \colon M \to M$ properly homotopic to the identity of M such that $\varphi(x) = y$.

Note that Theorem 2.8.1.1 tells π_1-injective degree one maps are homeomorphisms upto proper homotopy, provided the domain is neither the plane nor the punctured plane. Moreover, if $f \colon \mathbb{R}^2 \to \Sigma$ is a proper map from $\mathbb{R}^2$ to a non-compact oriented surface Σ of non-zero degree, then Σ must be homeomorphic to $\mathbb{R}^2$, by Theorem 1.7.3 and Theorem 2.3.9. Therefore, utilizing Theorem 2.8.1.3 and Theorem 2.8.1.11, we obtain the following classification theorem.

Theorem 2.8.1.13 (Classification of π_1-injective degree one maps) Let Σ, Σ' be any two non-compact oriented surfaces. Suppose there exists a π_1-injective proper map $f \colon \Sigma' \to \Sigma$ of degree ± 1. Then Σ is homeomorphic to Σ', and f is properly homotopic to a homeomorphism.

Another proof of Theorem 2.8.1.13. Since $\deg(f) = \pm 1$, by Theorem 1.7.3, $\pi_1(f)$ is surjective. Thus, $\pi_1(f)$ is bijective. Now, both Σ' and Σ are homotopy equivalent to $\bigvee_{\mathscr{I}} \mathbb{S}^1$ for some index set $\mathscr{I}$

with $|\mathscr{I}| \leq \aleph_0$, i.e., $\pi_k(\Sigma') = 0 = \pi_k(\Sigma)$ for all $k \geq 2$. So, by Whitehead theorem [51, Theorem 4.5.], f is a homotopy equivalence (note that each surface has a CW-complex structure due to its C^∞-smooth structure). Now, a simply-connected non-compact surface is homeomorphic to $\mathbb{R}^2$; see Theorem 2.3.9. So, combining Theorem 2.7.1 and Theorem 2.8.1.3, we are done. □

In 1927, Nielsen [82] proved that any π_1-injective map between two compact surfaces is homotopic to a covering map. Thus, combining Nielsen's result with Theorem 2.8.1.13, we obtain the following.

Theorem 2.8.1.14 Any π_1-injective proper map of degree ± 1 between two oriented surfaces is properly homotopic to a homeomorphism.

Our final theorem of this section asserts that the proper Borel Conjecture is true in dimension 2.

Theorem 2.8.1.15 (Proper rigidity) If $f\colon \Sigma' \to \Sigma$ is a proper homotopy equivalence between two non-compact surfaces, then Σ' is homeomorphic Σ and f is properly homotopic to a homeomorphism.

Proof. A proper homotopy equivalence is a π_1-injective map of degree ± 1. Now, apply Theorem 2.8.1.13. □

2.8.2 Characterizing the image of the Dehn-Nielsen-Baer map

In the 1980s, Goldman [47] introduced a Lie algebra structure on the $\mathbb{Z}$-module generated by the set of all free homotopy classes of directed loops on an oriented surface, while studying the Weil-Petersson symplectic form on the Teichmüller space. The Goldman Lie algebra provides enough information about how two closed curves intersect on a surface. In fact, its definition is based on both algebraic and geometric intersection numbers of two oriented closed curves on an oriented surface. This section aims to discuss an application of Theorem I in establishing a relationship between the big mapping class group (resp. big extended mapping class groups) and the Goldman Lie algebra (resp. geometric intersection number). This relationship asserts that each element of the image of the natural map $\mathrm{MCG}^+ \to \mathrm{Out}(\pi_1)$ (resp. the Dehn-Nielsen-Baer map $\mathrm{MCG}^\pm \to \mathrm{Out}(\pi_1)$) is induced by a homotopy equivalence that preserves the Goldman bracket (resp. geometric intersection number).

Let S be an oriented surface, possibly of infinite type. Define $\widehat{\pi}(\mathrm{S}) := [\mathbb{S}^1, \mathrm{S}]$, the set of free homotopy classes of curves in S. Note that if $p \in \mathrm{S}$, there is a bijection between the set of all conjugacy classes of $\pi_1(\mathrm{S}, p)$ and $\widehat{\pi}(\mathrm{S})$ [51, Problem 6 of Section 1.1]. For a closed curve γ, let $\widehat{\gamma}$ denote the free homotopy class of γ.

First, let's recall the definition of the geometric intersection number. Consider two elements $x, y \in \widehat{\pi}(\mathrm{S})$. The self-transversality theorem [91, Proposition 7.7 and Proposition 7.11] gives representatives $\varphi\colon \mathbb{S}^1 \to \mathrm{S}$ and $\psi\colon \mathbb{S}^1 \to \mathrm{S}$ of x and y, respectively, such that

1. $\varphi \sqcup \psi\colon \mathbb{S}^1 \sqcup \mathbb{S}^1 \to \mathrm{S}$ is an immersion,

2. each fiber of $\varphi \sqcup \psi \colon \mathbb{S}^1 \sqcup \mathbb{S}^1 \to \mathrm{S}$ has at most two elements, and

3. for two distinct points $a, b \in \mathbb{S}^1 \sqcup \mathbb{S}^1$ with $(\varphi \sqcup \psi)(a) = (\varphi \sqcup \psi)(b)$, we have

$$d(\varphi \sqcup \psi)\left(T_a\left(\mathbb{S}^1 \sqcup \mathbb{S}^1\right)\right) + d(\varphi \sqcup \psi)\left(T_b\left(\mathbb{S}^1 \sqcup \mathbb{S}^1\right)\right) = T_c(\mathrm{S}),$$

where $c = (\varphi \sqcup \psi)(a) = (\varphi \sqcup \psi)(b)$.

In this case, we say φ and ψ *intersects tranversally at double points*, and write $\varphi \underset{\text{double}}{\pitchfork} \psi$. Notice that 3. implies $(\varphi \sqcup \psi) \times (\varphi \sqcup \psi) \colon \left(\mathbb{S}^1 \sqcup \mathbb{S}^1\right) \times \left(\mathbb{S}^1 \sqcup \mathbb{S}^1\right) \to \mathrm{S} \times \mathrm{S}$ is transverse to the diagonal Δ_{S}, i.e., the set $\left\{c \in \mathrm{S} : \left|(\varphi \sqcup \psi)^{-1}(c)\right| = 2\right\}$ of double points of $\varphi \sqcup \psi$ is a finite set. The *geometric intersection number* $\mathrm{I}_{\mathrm{S}}(x, y)$ of x and y is defined as follows:

$$\mathrm{I}_{\mathrm{S}}(x, y) := \min\left\{|\mathrm{im}(\varphi) \cap \mathrm{im}(\psi)| : \varphi \in x, \psi \in y, \text{ and } \varphi \underset{\text{double}}{\pitchfork} \psi\right\}. \tag{2.8.2.1}$$

Now, we recall the definition of the Goldman Lie algebra $[\cdot,\cdot] \colon \mathbb{Z}[\widehat{\pi}(\mathrm{S})] \times \mathbb{Z}[\widehat{\pi}(\mathrm{S})] \to \mathbb{Z}[\widehat{\pi}(\mathrm{S})]$, where $\mathbb{Z}[\widehat{\pi}(\mathrm{S})]$ denotes the free $\mathbb{Z}$-module generated by the set $\widehat{\pi}(\mathrm{S})$. We will define it on $\widehat{\pi}(\mathrm{S})$, and then extend it on $\mathbb{Z}[\widehat{\pi}(\mathrm{S})]$ using bi-linearity. So, let $x, y \in \widehat{\pi}(\mathrm{S})$. Consider representatives $\varphi \colon \mathbb{S}^1 \to \mathrm{S}$ and $\psi \colon \mathbb{S}^1 \to \mathrm{S}$ of x and y, respectively, such that φ and ψ intersects tranversally at double points, and then define

$$[x, y] := \sum_{p \in \mathrm{im}(\varphi) \cap \mathrm{im}(\psi)} \varepsilon_p \cdot \widehat{\varphi *_p \psi}, \tag{2.8.2.2}$$

where for any $p \in \mathrm{im}(\varphi) \cap \mathrm{im}(\psi)$, the integer $\varepsilon_p \in \{\pm 1\}$ and the closed curve $\varphi *_p \psi$ are defined in the following way:

We view φ and ψ as curves in S based at p and define $\varphi *_p \psi$ as the product of based curves as in the definition of the fundamental group.

Since immersions are local embeddings, we have open neighbourhoods $U_p \subset \mathrm{im}(\varphi)$ and $V_p \subset \mathrm{im}(\psi)$ of p such that $\varphi^{-1}(U_p)$ and $\psi^{-1}(V_p)$ are open intervals of $\mathbb{S}^1$ which are mapped diffeomorphically onto U_p and V_p under φ and ψ, respectively. Let z, w be the unique elements of $\mathbb{S}^1$ satisfying $\varphi(z) = p = \psi(w)$. Pick vectors $\alpha \in T_z\mathbb{S}^1$ and $\beta \in T_w\mathbb{S}^1$ representing the standard orientations for $\mathbb{S}^1$ at z and w, respectively. If the ordered pair $\left(d\varphi_z(\alpha), d\psi_w(\beta)\right)$ represents the orientation for S ate p, then define $\varepsilon_p := +1$; otherwise, let $\varepsilon_p := -1$.

Goldman proved that $[\cdot,\cdot] \colon \mathbb{Z}[\widehat{\pi}(\mathrm{S})] \times \mathbb{Z}[\widehat{\pi}(\mathrm{S})] \to \mathbb{Z}[\widehat{\pi}(\mathrm{S})]$ is a well-defined bilinear map [47, Theorem 5.2.]. This map is skew-symmetric and satisfies the Jacobi identity [47, Theorem 5.3.], meaning that $[x, y] = -[y, x]$ for all $x, y \in \widehat{\pi}(\mathrm{S})$, and $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \widehat{\pi}(\mathrm{S})$. Furthermore, the Goldman bracket, in some situations, allows us to separate two closed curves. For example,

Theorem 2.8.2.1 [47, Theorem 5.17. (i)] Consider $x, y \in \widehat{\pi}(\mathrm{S})$. If x and y have disjoint representatives, then $[x, y] = 0$. Moreover, if $[x, y] = 0$, and one of x or y has a simple representative, then x and y have disjoint representatives.

As an application of Theorem I, one can prove the following theorem, the analogue version of which for compact bordered surfaces was proved in [42].

Theorem 2.8.2.2 [22] Let $f\colon \Sigma' \to \Sigma$ be a homotopy equivalence between non-compact oriented surfaces, which may not be a proper map. If Σ is homeomorphic to neither the plane nor the punctured plane, then

1. f is homotopic to a homeomorphism if and only if f preserves the geometric intersection number, i.e., $\mathrm{I}_{\Sigma'}(\alpha,\beta) = \mathrm{I}_{\Sigma}\big(f_*(\alpha), f_*(\beta)\big)$ for all $\alpha,\beta \in \widehat{\pi}(\Sigma')$.
2. f is homotopic to an orientation-preserving homeomorphism if and only if f preserves the Goldman bracket, i.e., $[\alpha,\beta] = \big[f_*(\alpha), f_*(\beta)\big]$ for all $\alpha,\beta \in \widehat{\pi}(\Sigma')$.

Here, f_* denotes the induced map on $\widehat{\pi}(\Sigma') \to \widehat{\pi}(\Sigma)$.

Here is a very brief outline of the proof of Theorem 2.8.2.2. The only if parts are easy compared to the if parts. For the if parts, we need to homotope the homotopy equivalence f to a proper map so that Theorem I can be used to conclude. The idea is to construct two "nice" enough exhaustions $\{K_i'\}$ of Σ' and $\{K_i\}$ of Σ by compact bordered sub-surfaces, and a filling $\{\gamma_{i,\ell} : \ell = 1, \ldots, n_i\}$ of K_i by simple closed curves such that $\cup_{\ell=1}^{n_i} g\left(\gamma_{i,\ell}\right) \subseteq K_i'$ for each i, where $g\colon \Sigma \to \Sigma'$ is a homotopy inverse of f. Thus, for any closed curve δ' in $\Sigma' \setminus \mathrm{int}(K_i')$, (using Theorem 2.8.2.1 for part 2. of Theorem 2.8.2.2) $f(\delta')$ can be freely homotoped to a loop disjoint from K_i since $\mathrm{I}_{\Sigma'}\big(\delta', g(\gamma_{i,\ell})\big) = 0 = \mathrm{I}_{\Sigma}\big(f(\delta'), \gamma_{i,\ell}\big)$ for each ℓ. This implies that for each component V' of $\overline{K_{i+1}' \setminus K_i'}$, there exists a component V of $\overline{\Sigma \setminus K_i}$ such that $f_*\pi_1(V') \subseteq \pi_1(V)$. Therefore, for each i, $f|K_{i+1}' \to \Sigma$ can be homotoped relative to $\partial K_i'$ to send $\overline{K_{i+1}' \setminus K_i'}$ into $\Sigma \setminus K_i$, provided f has already been homotoped so that $f(\partial K_i') \subseteq \Sigma \setminus K_i$. This tells us that f can be homotoped to a proper map.

Finally, we recall the definition of the Dehn-Nielsen-Baer map. Fix a point $p \in \mathrm{S}$. Denote the basepoint-preserving homotopy classes of maps $(\mathrm{S},p) \to (\mathrm{S},p)$ by $\langle \mathrm{S},\mathrm{S}\rangle$ and the free homotopy class of maps $\mathrm{S} \to \mathrm{S}$ by $[\mathrm{S},\mathrm{S}]$. The natural map $\Xi\colon \langle \mathrm{S},\mathrm{S}\rangle \to [\mathrm{S},\mathrm{S}]$ is surjective: if $\varphi\colon \mathrm{S} \to \mathrm{S}$ is any (un-based) map, then considering a path μ in S from p to $\varphi(p)$, the homotopy extension theorem gives a homotopy $H^\varphi\colon \mathrm{S} \times [0,1] \to \mathrm{S}$ from a based map $\varphi_b\colon (\mathrm{S},p) \to (\mathrm{S},p)$ to φ satisfying $H^\varphi(p,-) = \mu$. Note that the base-point change isomorphism $H^\varphi(p,-)_\sharp\colon \pi_1\big(\mathrm{S},\varphi(p)\big) \to \pi_1(\mathrm{S},p)$ induced by the path $H^\varphi(p,-)$ satisfies $\pi_1(\varphi_b) = H^\varphi(p,-)_\sharp \circ \pi_1(\varphi)$; see [51, Lemma 1.19. of Section 1.1].

Now, there is a right action of $\pi_1(\mathrm{S},p)$ on $\langle \mathrm{S},\mathrm{S}\rangle$ defined by setting $[f_0]\cdot[\gamma] = [f_1]$ whenever there exists a (free) homotopy $H\colon \mathrm{S} \times [0,1] \to \mathrm{S}$ from $f_0\colon (\mathrm{S},p) \to (\mathrm{S},p)$ to $f_1\colon (\mathrm{S},p) \to (\mathrm{S},p)$ such that the loops $H(p,-)$ and γ give the same element in $\pi_1(\mathrm{S},p)$ by [51, Proposition 4A.1.]. Moreover, Ξ induces a bijection of the orbit set $\langle \mathrm{S},\mathrm{S}\rangle/\pi_1(\mathrm{S},p)$ onto $[\mathrm{S},\mathrm{S}]$; see [51, Proposition 4A.2.].

From now on, assume S is aspherical also, i.e., S is not homeomorphic to $\mathbb{S}^2$. Then the map $\langle \mathrm{S},\mathrm{S}\rangle \ni [f] \longmapsto \pi_1(f) \in \hom\big(\pi_1(\mathrm{S},p), \pi_1(\mathrm{S},p)\big)$ is a bijection [108, Theorem 4.3 of Chapter V]. Consider two based maps $f_0, f_1\colon (\mathrm{S},p) \to (\mathrm{S},p)$. If $[f_0]\cdot\xi = [f_1]$ for some $\xi \in \pi_1(\mathrm{S},p)$, then $\pi_1(f_0)(x) = \xi\cdot\pi_1(f_1)(x)\cdot\xi^{-1}$ for all $x \in \pi_1(\mathrm{S},p)$ [51, Lemma 1.19. of Section 1.1], and in which case we say that $\pi_1(f_0)$ and $\pi_1(f_1)$ have the *same conjugacy class in* $\hom\big(\pi_1(\mathrm{S},p), \pi_1(\mathrm{S},p)\big)$. Conversely, if $\pi_1(f_0)$ and $\pi_1(f_1)$ have the same conjugacy class in $\hom\big(\pi_1(S,p),\pi_1(S,p)\big)$, then f_0 and f_1 are freely homotopic [108, Corollary 4.4 of Chapter V], and thus $[f_0]\cdot\xi=[f_1]$ for some $\xi\in\pi_1(S,p)$. Therefore, two maps $\varphi,\psi\colon S\to S$ are freely homotopic if and only if $\pi_1(\varphi_b)=H^{\varphi}(p,-)_\sharp\circ\pi_1(\varphi)$ and $\pi_1(\psi_b)=H^{\psi}(p,-)_\sharp\circ\pi_1(\psi)$ have the same conjugacy class in $\hom\big(\pi_1(S,p),\pi_1(S,p)\big)$.

Consider any homeomorphism $\varphi\colon S\to S$. Then $\pi_1(\varphi)\colon\pi_1(S,p)\to\pi_1\big(S,\varphi(p)\big)$ is an isomorphism [51, Proposition 1.18. of Section 1.1]. Note that for paths μ,ν in S from p to $\varphi(p)$, the base-point change isomorphisms $\mu_\sharp,\nu_\sharp\colon\pi_1\big(S,\varphi(p)\big)\to\pi_1(S,p)$ satisfies $\nu_\sharp\pi_1(\varphi)=(\nu*\overline{\mu})_\sharp\mu_\sharp\pi_1(\varphi)$, i.e., left cosets of the automorphisms $\nu_\sharp\pi_1(\varphi)$ and $\mu_\sharp\pi_1(\varphi)$ of $\pi_1(S,p)$ in quotient group $\mathrm{Out}\big(\pi_1(S,p)\big):=\mathrm{Aut}\big(\pi_1(S,p)\big)/\mathrm{Inn}\big(\pi_1(S,p)\big)$ are the same. Thus,

$$\sigma\colon \mathrm{Homeo}(S)\ni\varphi\longmapsto\mu_\sharp\pi_1(\varphi)\cdot\mathrm{Inn}\big(\pi_1(S,p)\big)\in\mathrm{Out}\big(\pi_1(S,p)\big),\qquad(2.8.2.3)$$

where μ is any path in S from p to $\varphi(p)$, is a well-defined map. A straightforward calculation shows σ is a group homomorphism. Further, our previous discussion tells two self-homeomorphisms φ and ψ of S are homotopic if and only if $\sigma(\varphi)=\sigma(\psi)$, i.e., σ is injective. A similar remark can be made for the group of all self homotopy equivalences of S.

We say two homeomorphisms $\varphi,\psi\colon S\to S$ are *isotopic* if there exists a homotopy $S\times[0,1]\to S$ through homeomorphisms from φ to ψ. The following theorem states that homotopic homeomorphisms of S are isotopic in almost all cases.

Theorem 2.8.2.3 Suppose $\varphi,\psi\colon S\to S$ are two homotopic homeomorphisms. If S is homeomorphic to neither the plane nor the punctured plane, then φ and ψ are isotopic.

Proof. Since the homeomorphism $\psi^{-1}\varphi$ is homotopic to Id_S and S is homeomorphic to neither the plane nor the punctured plane, by [28, Theorem 6.4. (a)], there exists a level-preserving homeomorphism $H\colon S\times[0,1]\to S\times[0,1]$ which agrees with $\psi^{-1}\varphi$ on $S\times 0$ and with Id_S on $S\times 1$. Then $\psi\circ p\circ H\colon S\times[0,1]\to S$ is a homotopy through homeomorphisms from φ to ψ, where $p\colon S\times[0,1]\to S$ is the projection. □

From now on, we will also assume that S is homeomorphic to neither the plane nor the punctured plane. Note that "being isotopic" is an equivalence relation on Homeo(S), and the set $\mathrm{MCG}^{\pm}(S)$ of equivalence classes becomes a group in an obvious way. We call $\mathrm{MCG}^{\pm}(S)$ the *extended mapping class group* of S. The above discussion tells

$$\boldsymbol{\sigma}\colon \mathrm{MCG}^{\pm}(S)\ni[\varphi]\longmapsto\mu_\sharp\pi_1(\varphi)\cdot\mathrm{Inn}\big(\pi_1(S,p)\big)\in\mathrm{Out}\big(\pi_1(S,p)\big),\qquad(2.8.2.4)$$

where μ is any path in S from p to $\varphi(p)$, is a well-defined injective group homomorphism, which is often called *the Dehn-Nielsen-Baer map* [25, 82, 6, 73].

Since S is aspherical, if $\tau\in\mathrm{Out}\big(\pi_1(S,p)\big)$, there exists a homotopy equivalence $h_\tau\colon(S,p)\to(S,p)$ that induces τ, meaning that $\tau=\pi_1(h_\tau)\cdot\mathrm{Inn}\big(\pi_1(S,p)\big)$; see [51, Proposition 1B.9. and Theorem 4.5. of Section 4.1]. Thus, an element $\tau\in\mathrm{Out}\big(\pi_1(S,p)\big)$ is in the image of $\boldsymbol{\sigma}$ if and only if τ is induced by a homotopy equivalence $h_\tau\colon S\to S$ that is homotopic to a self-homeomorphism of S. For example, if $S=S_{g,0}$ for some $g\geq 1$, then $\boldsymbol{\sigma}$ is surjective because $S_{g,0}$ is topologically rigid. In general, $\boldsymbol{\sigma}$ may fail to be surjective. For example, if $S=S_{g,0,\ell}$, where $2-2g-\ell<0$, then the image($\boldsymbol{\sigma}$) consists of elements that preserve the set of conjugacy classes of simple closed curves surrounding individual punctures [30, Theorem 8.8]. Now, Theorem 2.8.2.2 gives the following characterization of the image of $\boldsymbol{\sigma}$ for almost all non-compact surfaces in terms of preservation of the geometric intersection number.

Theorem 2.8.2.4 If Σ is a non-compact surface, possibly of infinite type, such that Σ is homeomorphic to neither the plane nor the punctured plane, then each element of the image of $\boldsymbol{\sigma}\colon \mathrm{MCG}^{\pm}(\Sigma) \to \mathrm{Out}\big(\pi_1(\Sigma, p)\big),\ p \in \Sigma$ is induced by a self homotopy equivalence of Σ that preserves the geometric intersection number of the free homotopy classes of any two closed curves on Σ.

Now, we want to consider the group $\mathrm{Homeo}^{+}(\mathrm{S})$ of orientation-preserving self homeomorphisms of S. Notice that if $\mathcal{H}\colon \mathrm{S}\times[0,1] \to \mathrm{S}$ is an isotopy between two elements of $\mathrm{Homeo}^{+}(\mathrm{S})$, then $\mathcal{H}(-,t) \in \mathrm{Homeo}^{+}(\mathrm{S})$ for each $t \in [0,1]$ because $\mathcal{H}$ is a proper homotopy by Theorem 1.6.3, and thus preserves the degree. Now "being isotopic" is an equivalence relation on $\mathrm{Homeo}^{+}(\mathrm{S})$, and the set MCG(S) of equivalence classes becomes a group in an obvious way. We call MCG(S) the *mapping class group* of S. Certainly, the map $\mathrm{MCG}(\mathrm{S}) \ni [\varphi] \mapsto [\varphi] \in \mathrm{MCG}^{\pm}(\mathrm{S})$ is an injective group homomorphism. Moreover, since S admits an orientation-reversing self-homeomorphism (see Corollary 1.5.3), the index of MCG(S) in $\mathrm{MCG}^{\pm}(\mathrm{S})$ is 2. So, the restriction of (2.8.2.4) on MCG(S) gives a well-defined injective group homomorphism $\boldsymbol{\sigma}|\mathrm{MCG}(\mathrm{S}) \to \mathrm{Out}\big(\pi_1(\mathrm{S}, p)\big)$, whose image can be characterized by the following theorem.

Theorem 2.8.2.5 If Σ is a non-compact surface, possibly of infinite type, such that Σ is homeomorphic to neither the plane nor the punctured plane, then each element of the image of $\boldsymbol{\sigma}|\mathrm{MCG}(\Sigma) \to \mathrm{Out}\big(\pi_1(\Sigma, p)\big),\ p \in \Sigma$ is induced by a self homotopy equivalence of Σ that preserves the Goldman bracket.

Chapter 3

Non-Hopfian property

3.1 Background and motivation

Around 1951, H. Hopf proposed the following famous problems and gave rise to the concept of Hopfian and non-Hopfian groups [80]:

H.1. If two closed, oriented topological manifolds M and N admit degree one maps $M \to N$ and $N \to M$, do they necessarily have isomorphic fundamental groups?

H.2. If two finitely generated groups can be mapped homomorphically onto each other, are they necessarily isomorphic?

A positive answer to the second problem would also lead to a positive answer to the first, due to Theorem 1.7.3. But the answer to the second problem turns out to be negative [80]. However, it introduces the notion of the Hopfian group: A group G is said to be *Hopfian* if every surjective group homomorphism $G \twoheadrightarrow G$ is an isomorphism. The simplest example of a Hopfian group is a finite group. On the other hand, $(\mathbb{R}, +)$ serves as an example of a non-Hopfian group; this is because any $\mathbb{Q}$-basis of $\mathbb{R}$, being an infinite set, admits a surjective map that is not injective. Nielsen [81] proved that any finitely generated free group is Hopfian. However, a free group generated by an infinite set S is not Hopfian, as a surjective function $S \to S$ that is not injective extends to a surjective homomorphism on the free group generated by S, which is not injective. H. Hopf [60, 61] proved that the fundamental group of any compact surface is Hopfian. For a purely algebraic proof of the Hopficity of the compact surface group, refer to [34].

The natural topological analogue of a surjective group homomorphism is a degree-one map, and that of an isomorphism is a homotopy equivalence. Following [23, page 162], we call a connected, oriented, topological boundaryless manifold M, possibly non-compact, *Hopfian* if every degree one map $M \to M$ is a homotopy equivalence. This chapter aims to prove that an oriented surface is Hopfian if and only if it is of finite type. Since the fundamental group of a compact surface is Hopfian, one-half of this characterization, namely, that any proper map of degree one from an oriented surface of finite type to itself is a homotopy equivalence, follows from Theorem 2.8.1.14. Now, the other part follows from the theorem below.

Theorem II (Infinite-type surfaces are non-Hopfian) Let Σ be any infinite-type oriented surface. Then there exists a proper map $f\colon \Sigma \to \Sigma$ of degree one such that $\pi_1(f)\colon \pi_1(\Sigma) \to \pi_1(\Sigma)$ is not injective. In particular, f is not a homotopy equivalence.

It is worth noting that finite generation and Hopficity don't imply each other; for example, $(\mathbb{Q}, +)$ is not finitely generated, but Hopfian, and on the other hand, the Baumslag-Solitar group $\mathrm{BS}(2,3) := \langle x, y | x^{-1}y^2x = y^3\rangle$ is finitely generated but non-Hopfian. More generally, the Baumslag-Solitar group $\mathrm{BS}(m,n) := \langle x, y | x^{-1}y^m x = y^n\rangle$, $mn \neq 0$, is known to be Hopfian if and only if m or n divides the other, or if m and n have precisely the same prime divisors [8, Theorem 1]. However, a theorem of Mal'cev [74] states that a finitely generated residually finite group is Hopfian. This provides the following interesting examples of Hopfian groups: (*i*) Any finitely generated abelian group is Hopfian; (*ii*) The mapping class group of any compact surface is Hopfian: residual finiteness follows from [48] and finite generation follows from the Dehn-Lickorish theorem [25, 71]; (*iii*) The fundamental group of any compact orientable manifold of dimension ≤ 3 is Hopfian: residual finiteness follows from [53], [55, Corollary 1.2.], and the Geometrization Conjecture (nowadays a theorem due to the epochal work of Perelman), while finite generation follows from the fact that any compact (connected) topological manifold is homotopy equivalent to a finite simplicial complex [66, Theorem III]. Note that, unlike the small mapping class groups, the big mapping class groups may fail to be Hopfian. For instance, if $\mathscr{E}$ is a closed subset of the Cantor set such that the set $\mathscr{E}'$ of accumulation points of $\mathscr{E}$ satisfies $\mathscr{E}' \subsetneq \mathscr{E}$ and $\mathscr{E}'$ is homeomorphic to $\mathscr{E}$, then $\mathrm{MCG}(\mathbb{S}^2 \setminus \mathscr{E})$ is non-Hopfian [4, § 12.5.1.1]. Also, note that the fundamental group of a manifold may not always be Hopfian. In fact, for $n \geq 4$, every finitely presentable group is isomorphic to the fundamental group of a closed, orientable, smooth n-manifold [17, Theorem 5.1.1]. However, the fundamental group of a closed, complete Riemannian manifold whose all sectional curvatures are negative is Hopfian as it is word-hyperbolic [43, 32.- Théorème] [38, Corollary 2.9].

The task of identifying all Hopfian manifolds in a fixed dimension has been a long-standing problem, and except in very few cases, the solution to this problem remains elusive. Notice that for every n, we have a Hopfian n-manifold, namely $\mathbb{S}^n$ by the Hopf degree theorem. Below, we will discuss some other examples of Hopfian manifolds.

A theorem of Hausmann [52, Proposition 1] states that if M is a closed, oriented, connected topological n-manifold such that either $n \leq 4$ or the integral group ring $\mathbb{Z}\pi_1(M)$ is Noetherian[1], then for any π_1-injective map $f\colon M \to M$ of degree ± 1, the induced map $f_*\colon H_k\big(M; \mathbb{Z}\pi_1(M)\big) \to H_k\big(M; \mathbb{Z}\pi_1(M)\big)$ on k-th homology of M with local coefficients in $\mathbb{Z}\pi_1(M)$ is an isomorphism for each $k \geq 0$. Now, recall that if $\varphi\colon X \to Y$ is a map between two connected CW-complexes such that $\pi_1(\varphi)\colon \pi_1(X) \to \pi_1(Y)$ is an isomorphism, and also $\varphi_*\colon H_k\big(X, \mathbb{Z}\pi_1(X)\big) \to H_k\big(Y, \mathbb{Z}\pi_1(Y)\big)$ is an isomorphism for each $k \geq 0$, then φ is a homotopy equivalence [51, Exercise 12 of Section 4.2 and Example 3H.2.]. Therefore, if M is a closed, oriented, connected, topological

[1]If the integral group ring of a group G is Noetherian then every subgroup of G must be finitely generated [68, Proposition 1. (b) of Appendix 2]. Moreover, if G is any polycyclic group (or more generally contains a polycyclic subgroup of finite index), then $\mathbb{Z}G$ is Noetherian [50, page 429]. Thus, if a group G is either of the following types, then $\mathbb{Z}G$ is Noetherian: (*i*) G is finite [64, Exercise 19.1.5.], (*ii*) G is a solvable group with no non-finitely generated subgroup (such as a finitely generated abelian) [64, Theorem 19.2.3.], or (*iii*) G is a supersolvable group (such as a finitely generated nilpotent group) [64, page 137].

n-manifold with Hopfian fundamental group, then M is a Hopfian manifold if either $n \leq 4$ or $\mathbb{Z}\pi_1(M)$ is Noetherian. In particular, every closed, oriented, connected 3-manifold is Hopfian. Hausmann [52, Corollary 2] also proved that if the fundamental group of a closed, oriented, connected topological n-manifold M contains a nilpotent subgroup of finite index, then $\mathbb{Z}\pi_1(M)$ is Noetherian and $\pi_1(M)$ is Hopfian, making M a Hopfian manifold.

An application of the Whitehead theorem [51, Theorem 4.5.] tells us that a closed, oriented, aspherical topological manifold with a Hopfian fundamental group is a Hopfian manifold. More generally, if M is a closed, oriented, connected topological n-manifold with Hopfian fundamental group such that $\pi_i(M) = 0$ for $1 < i < n-1$, then M is Hopfian. This is because the fact that if M and N are two closed, oriented, topological n-manifolds such that $\pi_i(M) = 0 = \pi_i(N)$ for $1 < i < n-1$, then any map $f\colon M \to N$ that induces an isomorphism between the fundamental groups is a homotopy equivalence if and only if $\deg(f) = \pm 1$ [101, Lemma 1.1.]. Therefore, a closed, oriented, complete, negatively curved Riemannian manifold M is Hopfian because $\pi_1(M)$ is Hopfian by the earlier mentioned argument, and M is aspherical by the Cartan-Hadamard theorem.

3.2 Proof of Theorem II

Let Σ be an oriented surface of infinite type. By Proposition 2.3.10, one of the following three cases may arise, and accordingly, the proof of Theorem II will be divided into three parts: (1) Σ has infinite genus, (2) Σ has a finite genus, and the set of isolated points $\mathscr{I}(\Sigma)$ of $\text{Ends}(\Sigma)$ is finite, and (3) Σ has a finite genus, and the set of isolated points $\mathscr{I}(\Sigma)$ of $\text{Ends}(\Sigma)$ is infinite. Our first result proves Theorem II in the case with infinite genus.

Theorem 3.2.1 Let Σ be an oriented surface of infinite genus. Then there exists a degree one map $f\colon \Sigma \to \Sigma$ which is not π_1-injective.

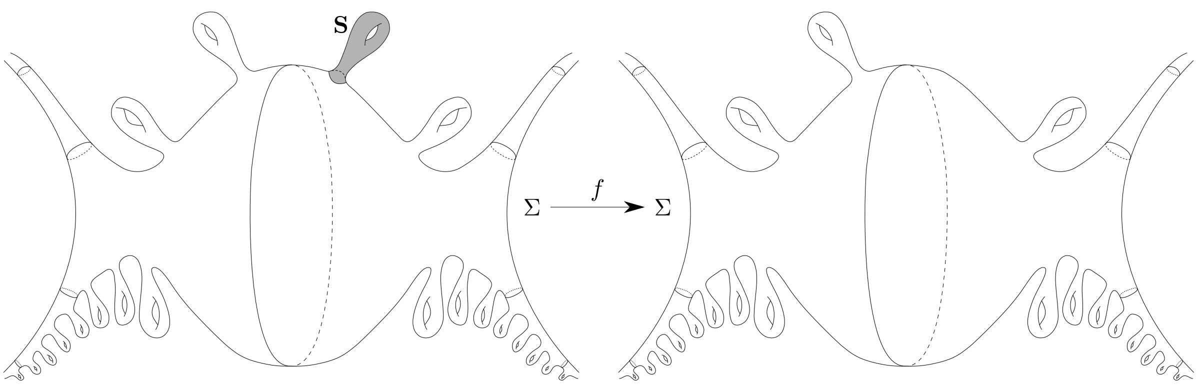

Fig. 3.2.1 Crushing the handle $\mathbf{S}$ of the infinite-genus surface Σ gives a non π_1-injective self-map $f\colon \Sigma \to \Sigma$ of degree ± 1.

Proof. Since Σ has infinite genus, there exists a compact bordered subsurface $\mathbf{S} \subset \Sigma$ such that $\mathbf{S}$ is homeomorphic to $S_{1,1}$. Define Σ' as $\Sigma' := \Sigma/\mathbf{S}$, i.e., Σ' is the quotient of Σ with $\mathbf{S}$ pinched to a point. Let $q\colon \Sigma \to \Sigma'$ be the quotient map. Thus, Σ' is also an infinite genus surface. Further, there are compact sets in $K \subset \Sigma$ and $K' \subset \Sigma'$ whose complements are homeomorphic, so the pair $(\mathscr{E}(\Sigma), \mathscr{E}_{\mathrm{np}}(\Sigma))$ is homeomorphic to the pair $\left(\mathscr{E}\left(\Sigma'\right), \mathscr{E}_{\mathrm{np}}\left(\Sigma'\right)\right)$. Hence, by Theorem 1.5.1, there is a homeomorphism $\varphi\colon \Sigma' \to \Sigma$.

Let $f\colon \Sigma \to \Sigma$ be the composition $f := \varphi \circ q$. By Theorem 1.7.1, the quotient map $q\colon \Sigma \to \Sigma'$ is of degree 1. Thus, $\deg(f) = \pm 1$ as homeomorphisms have degree ± 1. Notice that f sends $\partial \mathbf{S}$ to a point. But $\partial \mathbf{S}$ does not bound any disk in Σ, i.e., $\partial \mathbf{S}$ represents a non-trivial element of $\pi_1(\Sigma)$ by Theorem 1.2.2. Hence, f is not π_1-injective. If $\deg(f) = 1$, then we are done. Otherwise, we replace f by $f \circ f$ to get a map that has degree one and is not injective on π_1. $\square$

For the remaining two cases, we use a map from the sphere to the sphere, which has degree ± 1 but with some disks identified. We will replace these disks with appropriate surfaces to get Σ.

Lemma 3.2.2 There exist pairwise disjoint closed disks $\mathcal{D}_0, \mathcal{D}_1 \subseteq \mathbb{S}^2$ and a map $f\colon \mathbb{S}^2 \to \mathbb{S}^2$ such that the following hold:

- $f^{-1}(\mathcal{D}_0) = \mathcal{D}_0$ and $f|\mathcal{D}_0 \to \mathcal{D}_0$ is the identity map.
- $f^{-1}(\mathcal{D}_1)$ is the union of pairwise-disjoint closed disks $\mathcal{D}_{1,1}, \mathcal{D}_{1,2}$, and $\mathcal{D}_{1,3}$ in $\mathbb{S}^2$; and $f|\mathcal{D}_{1k} \to \mathcal{D}_1$ is a homeomorphism for each $k \in \{1, 2, 3\}$.

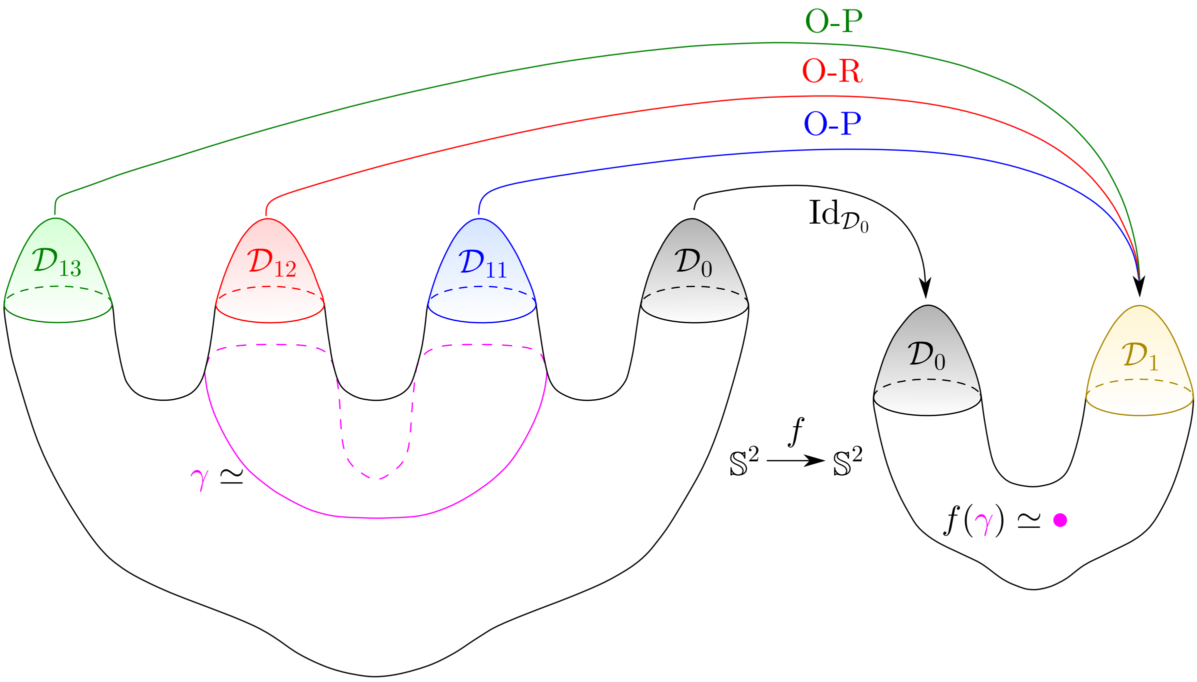

Fig. 3.2.2 Description of $f\colon \mathbb{S}^2 \to \mathbb{S}^2$. Here, O-P (resp. O-R) indicates that $f|\mathcal{D}_{1k} \to \mathcal{D}_1$ for some k is orientation-preserving (resp. orientation-reversing).

Further, there is a loop γ in $\mathbb{S}^2 \setminus \operatorname{int}(\mathcal{D}_0 \cup \mathcal{D}_{1,1} \cup \mathcal{D}_{1,2} \cup \mathcal{D}_{1,3})$ which is not homotopically trivial in $\mathbb{S}^2 \setminus \operatorname{int}(\mathcal{D}_0 \cup \mathcal{D}_{1,1} \cup \mathcal{D}_{1,2} \cup \mathcal{D}_{1,3})$, but such that $f(\gamma)$ is null-homotopic in $\mathbb{S}^2 \setminus \operatorname{int}(\mathcal{D}_0 \cup \mathcal{D}_1)$.

Proof. For each $k \in \{0,1,2,3\}$, choose $(a_k, b_k) \in \mathbb{R}^2$ such that if we define

$$\mathcal{B}_k := \left\{(x,y) \in \mathbb{R}^2 : (x-a_k)^2 + (y-b_k)^2 \le 1\right\},$$

then $\{\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3\}$ is a pairwise-disjoint collection of closed disks.

Define $X := \mathbb{S}^2 \setminus \bigcup_{i=0}^{3} \operatorname{int}(\mathcal{B}_i)$ and $Y := \mathbb{S}^2 \setminus \bigcup_{i=0}^{1} \operatorname{int}(\mathcal{B}_i)$. Next, define a map $f\colon \partial X \to Y$ as follows:

- $f|\partial\mathcal{B}_k\colon \partial\mathcal{B}_k \to \partial\mathcal{B}_k$ is the identity map for each $k \in \{0,1\}$;
- $f|\partial\mathcal{B}_2\colon \partial\mathcal{B}_2 \to \partial\mathcal{B}_1$ is defined as $f(x,y) := (-x + a_2 + a_1, y - b_2 + b_1)$ for all $(x,y) \in \partial\mathcal{B}_2$.
- $f|\partial\mathcal{B}_3\colon \partial\mathcal{B}_3 \to \partial\mathcal{B}_1$ is defined as $f(x,y) := (x - a_3 + a_1, y - b_3 + b_1)$ for all $(x,y) \in \partial\mathcal{B}_3$.

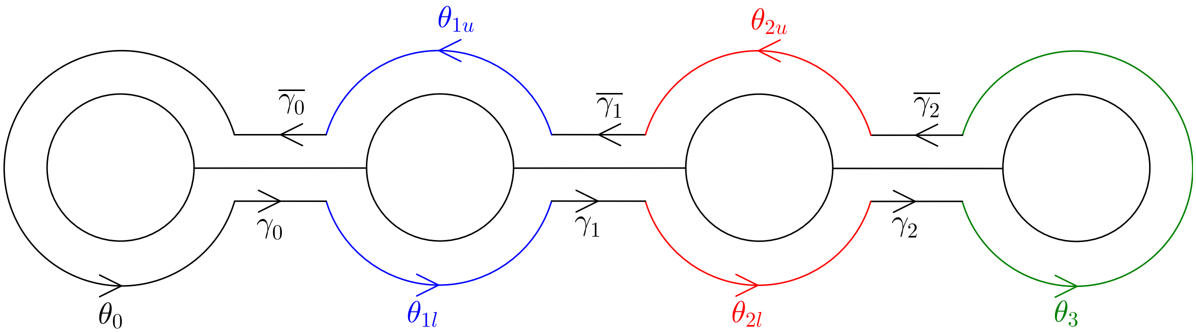

Fig. 3.2.3 The four-holed sphere X by attaching a 2-cell.

For each $k \in \{0,1,2\}$, let $\gamma_k\colon [0,1] \hookrightarrow X$ be an embedding such that $\operatorname{im}(\gamma_k) \cap \partial X$ consists of $\gamma_k(0) = (a_k + 1, b_k) \in \partial\mathcal{B}_k$ and $\gamma_k(1) = (a_{k+1} - 1, b_{k+1}) \in \partial\mathcal{B}_{k+1}$.

Define $\Gamma_0\colon [0,1] \to Y$ as $\Gamma_0(t) := \gamma_0(t)$ for all $t \in [0,1]$. Let $\Gamma_1, \Gamma_2\colon [0,1] \to Y$ be the constant loops based at the points $(a_1 + 1, b_1) \in \partial Y$ and $(a_1 - 1, b_1) \in \partial Y$, respectively.

Next, define $X^{(1)} := \partial X \cup \operatorname{im}(\gamma_0) \cup \operatorname{im}(\gamma_1) \cup \operatorname{im}(\gamma_2)$. Extend $f\colon \partial X \to Y$ to a map $X^{(1)} \to Y$, which we again denote by $f\colon X^{(1)} \to Y$, by mapping γ_0 onto Γ_0 by the identity, and, for each $k = 1,2$, mapping γ_k to the constant loop Γ_k.

Let θ_0 (resp. θ_3) be the simple loop that traverses $\partial\mathcal{B}_0$ (resp. $\partial\mathcal{B}_3$) in the counter-clockwise direction starting from $(a_0 + 1, b_0)$ (resp. $(a_3 - 1, b_3)$).

Let $\theta_{1,l}$ (resp. $\theta_{1,u}$) be the simple arc that traverses $\partial\mathcal{B}_1 \cap \{y \le b_1\}$ (resp. $\partial\mathcal{B}_1 \cap \{y \ge b_1\}$) counter-clockwise direction. Similarly, define $\theta_{2,l}$ and $\theta_{2,u}$.

Now, $X \cong X^{(1)} \cup_\varphi \mathbb{D}^2$, (see Figure 3.2.3) where the attaching map $\varphi\colon \mathbb{S}^1 \to X^{(1)}$ can be described as

$$\varphi := \theta_0 * \gamma_0 * \theta_{1,l} * \gamma_1 * \theta_{2,l} * \gamma_2 * \theta_3 * \overline{\gamma}_2 * \theta_{2,u} * \overline{\gamma}_1 * \theta_{1,u} * \overline{\gamma_0}.$$

Notice that $f(\gamma_1) = \Gamma_1$ and $f(\gamma_2) = \Gamma_2$ are constant loops. Also, as in Figure 3.2.4, $\overline{f \circ \theta_{1,l}} = f \circ \theta_{2,l}$ and $\overline{f \circ \theta_{1,u}} = f \circ \theta_{2,u}$. Thus, $f \circ \varphi$ is homotopic to $(f \circ \theta_0) * \Gamma_0 * (f \circ \theta_3) * \overline{\Gamma_0}$.

If $r\colon Y \cong \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1$ is the projection then $r \circ f \circ \theta_0$ and $r \circ f \circ \theta_3$ traverse $\mathbb{S}^1$ in opposite directions. Since r is a strong deformation retract, $(f \circ \theta_0) * \Gamma_0 * (f \circ \theta_3) * \overline{\Gamma_0}$, and hence $f \circ \varphi$ is null-homotopic. Now, the null-homotopic map $f \circ \varphi\colon \mathbb{S}^1 \to Y$ can be extended to a map $\mathbb{D}^2 \to Y$. Thus $f\colon X^{(1)} \to Y$ can be extended to a map $X \cong X^{(1)} \cup_\varphi \mathbb{D}^2 \to Y$, which will be again denoted by $f\colon X \to Y$. Note that every homeomorphism $\mathbb{S}^1 \to \mathbb{S}^1$ can be extended to a homeomorphism

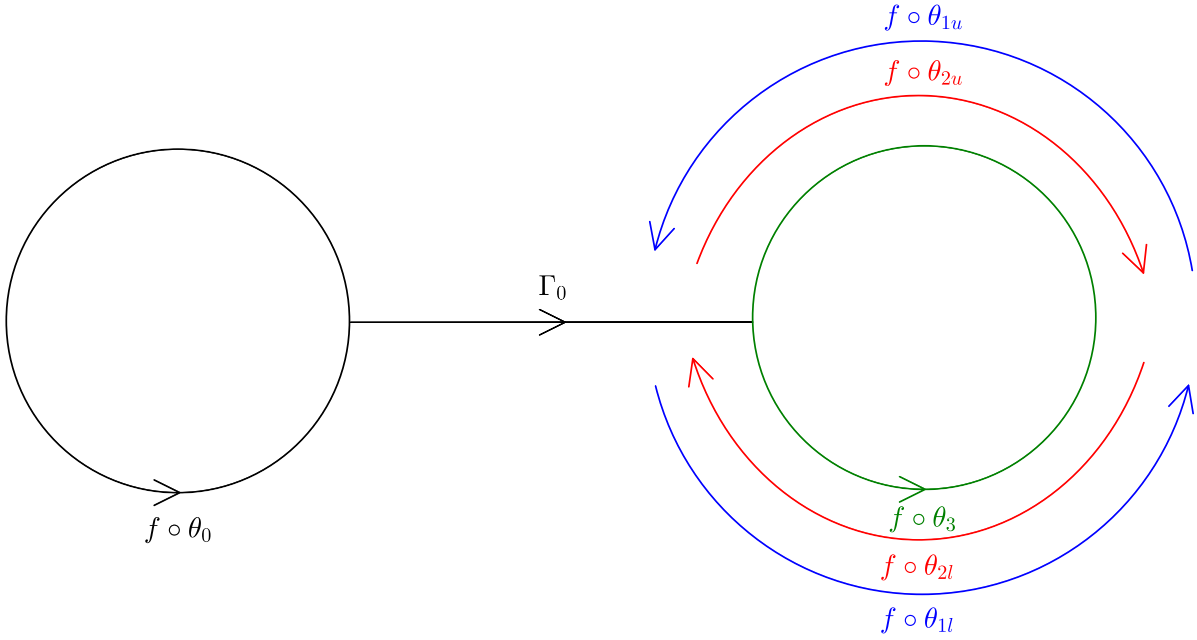

Fig. 3.2.4 The map on the $X^{(1)}$.

$\mathbb{D}^2 \to \mathbb{D}^2$ naturally. Thus, we can extend $f\colon X \to Y$ to a map $\mathbb{S}^2 \to \mathbb{S}^2$, which will be again denoted by $f\colon \mathbb{S}^2 \to \mathbb{S}^2$. Let $\mathcal{D}_0$ (resp. $\mathcal{D}_1$) be any closed disk, which is contained in $\operatorname{int}(\mathcal{B}_0)$ (resp. $\operatorname{int}(\mathcal{B}_1)$).

Finally, note that if $\gamma = \theta_{1u} * \theta_{1l} * \gamma_1 * \theta_{2l} * \theta_{2u} * \overline{\gamma_1}$, then γ is a loop in $\mathbb{S}^2 \setminus \operatorname{int}(\mathcal{D}_0 \cup \mathcal{D}_{1,1} \cup \mathcal{D}_{1,2} \cup \mathcal{D}_{1,3})$ which is not homotopically trivial in $\mathbb{S}^2 \setminus \operatorname{int}(\mathcal{D}_0 \cup \mathcal{D}_{1,1} \cup \mathcal{D}_{1,2} \cup \mathcal{D}_{1,3})$, but $f(\gamma)$ is null-homotopic in $\mathbb{S}^2 \setminus \operatorname{int}(\mathcal{D}_0 \cup \mathcal{D}_1)$, as claimed. $\square$

We now prove Theorem II in the two remaining cases, in both of which we have a finite genus surface. Note that for a finite genus surface, all ends are planar, so in applying Theorem 1.5.1, it suffices to consider the genus and the space of ends.

Theorem 3.2.3 Let Σ be a finite genus infinite-type surface such that $\operatorname{Ends}(\Sigma)$ has finitely many isolated points. Then there is a degree one map $f\colon \Sigma \to \Sigma$ which is not π_1-injective.

Proof. Let $\mathscr{I}(\Sigma)$ be the set of all isolated points of $\operatorname{Ends}(\Sigma)$, let $k \in \mathbb{N} \cup \{0\}$ be the cardinality of $\mathscr{I}(\Sigma)$, and let g be the genus of Σ. Then $\mathscr{C}(\Sigma) := \operatorname{Ends}(\Sigma) \setminus \mathscr{I}(\Sigma)$ is a non-empty, perfect, compact, totally-disconnected, metrizable space as it is infinite (by Proposition 2.3.10) and has no isolated points. Thus $\mathscr{C}(\Sigma)$ is a Cantor space (see [79, Theorem 8 of Chapter 12]).

Let $\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_{1,1}, \mathcal{D}_{1,2}, \mathcal{D}_{1,3} \subseteq \mathbb{S}^2, f\colon \mathbb{S}^2 \to \mathbb{S}^2$, and γ be as in the conclusion of Lemma 3.2.2. Also, let $C_1 \subset \operatorname{int}(\mathcal{D}_1)$ be a subset homeomorphic to the Cantor set, and let $\mathscr{I} \subset \operatorname{int}(\mathcal{D}_0)$ be a set consisting of k points (hence homeomorphic to $\mathscr{I}(\Sigma)$). Now, define $C_{1,j}$ as $C_{1,j} := f^{-1}(C_1) \cap \mathcal{D}_{1,j}$ for $j = 1, 2, 3$. Note that each $C_{1,j}$ is homeomorphic to the Cantor set (see Figure 3.2.5).

As $f^{-1}(\mathcal{D}_0) = \mathcal{D}_0$ and $f|\mathcal{D}_0 \to \mathcal{D}_0$ is the identity map, we can say that $f^{-1}(\mathscr{I}) = \mathscr{I}$. Let Σ_1 be the surface obtained from $\mathbb{S}^2 \setminus (\mathscr{I} \cup C_1)$ by attaching g handles along disjoint disks $\Delta_k \subset \operatorname{int}(\mathcal{D}_0) \setminus \mathscr{I}$, $1 \leq k \leq g$, and let Σ_2 be the surface obtained from $\mathbb{S}^2 \setminus (\mathscr{I} \cup C_{1,1} \cup C_{1,2} \cup C_{1,3})$ by attaching g handles along the (same) disks Δ_k, $1 \leq k \leq g$. Then f induces a proper map, which we also call f, from Σ_2 to Σ_1. By Theorem 1.7.1, $\deg(f) = \pm 1$.

Further, we claim that $f\colon\ \Sigma_2 \to \Sigma_1$ is not injective on π_1. Namely, $\pi_1(\Sigma_2)$ is the amalgamated free product of four groups, one of which is $\pi_1\left(\mathbb{S}^2 \setminus \operatorname{int}\left(\mathcal{D}_0 \cup \mathcal{D}_{1,1} \cup \mathcal{D}_{1,2} \cup \mathcal{D}_{1,3}\right)\right)$. As γ is not homotopic to the trivial loop in $\mathbb{S}^2 \setminus \operatorname{int}\left(\mathcal{D}_0 \cup \mathcal{D}_{1,1} \cup \mathcal{D}_{1,2} \cup \mathcal{D}_{1,3}\right)$, and components of an amalgamated free product inject, γ is not homotopic to the trivial loop in Σ_2. However, $f(\gamma)$ is homotopic to the trivial loop in $\mathbb{S}^2 \setminus \operatorname{int}\left(\mathcal{D}_0 \cup \mathcal{D}_1\right)$ and hence in Σ_1. Therefore, f is not injective on π_1.

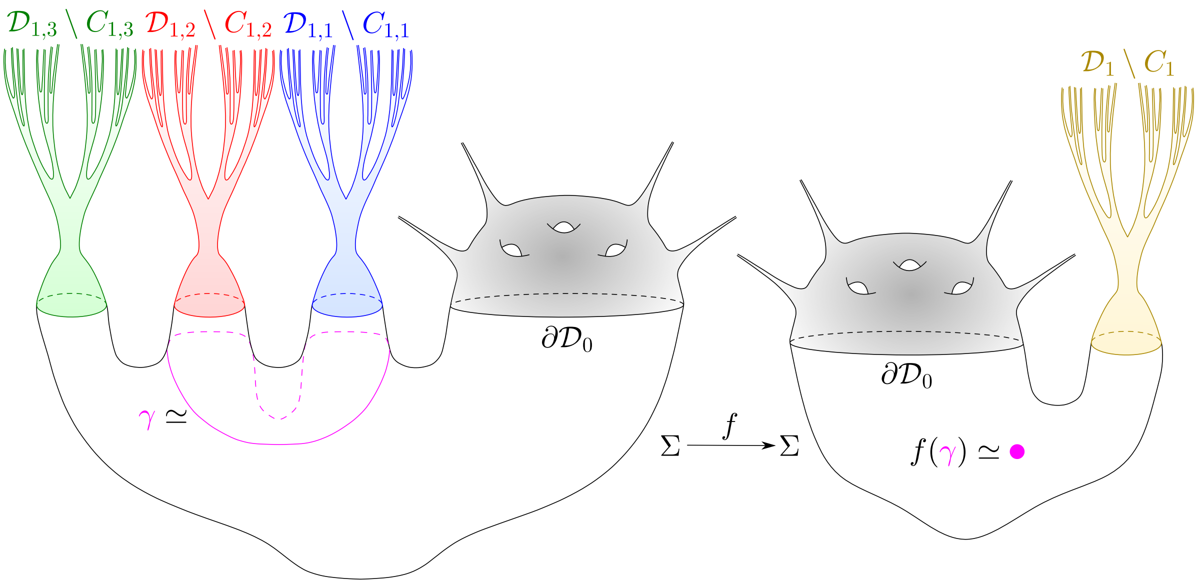

Fig. 3.2.5 A non π_1-injective degree ± 1 map $f\colon \Sigma \to \Sigma$, where $g = 3$ and $|\mathscr{I}| = 4$.

Both Σ_1 and Σ_2 have genus the same as Σ, and the space of ends homeomorphic to that of Σ (as a finite disjoint union of Cantor spaces is a Cantor space by the universality of the Cantor set) with all ends planar. Hence, by Theorem 1.5.1, both Σ_1 and Σ_2 are homeomorphic to Σ.

Identifying Σ_1 and Σ_2 with Σ by homeomorphisms, we get a proper map $f\colon\ \Sigma \to \Sigma$ which is not injective on π_1. As homeomorphisms have degree ± 1, it follows that $\deg(f) = \pm 1$. Replacing f by $f \circ f$ if necessary, we obtain a proper map of degree one that is not injective on π_1. □

Theorem 3.2.4 Let Σ be an oriented finite genus surface such that $\operatorname{Ends}(\Sigma)$ has infinitely many isolated points. Then there is a degree one map $f\colon \Sigma \to \Sigma$ which is not π_1-injective.

Proof. Let $\mathscr{I}(\Sigma)$ be the set of all isolated points of $\operatorname{Ends}(\Sigma)$, and let g be the genus of Σ. Also, let $\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_{1,1}, \mathcal{D}_{1,2}, \mathcal{D}_{1,3} \subseteq \mathbb{S}^2$, $f\colon\ \mathbb{S}^2 \to \mathbb{S}^2$, and γ be as in the conclusion of Lemma 3.2.2.

Now, consider a subset $\mathscr{E}$ of $\operatorname{int}(\mathcal{D}_0)$ such that $\mathscr{E}$ is homeomorphic to $\operatorname{Ends}(\Sigma)$. Also, consider points $p_1 \in \operatorname{int}(\mathcal{D}_1)$ and $p_{1,i} \in \operatorname{int}(\mathcal{D}_{1,i})$, $i = 1, 2, 3$ such that $f(p_{1,i}) = p_1$ for each $i = 1, 2, 3$ (see Figure 3.2.6).

Recall that $f^{-1}(\mathcal{D}_0) = \mathcal{D}_0$ and $f|\mathcal{D}_0 \to \mathcal{D}_0$ is the identity map. Thus $f^{-1}(\mathscr{E}) = \mathscr{E}$. Now, let Σ_1 be the surface obtained from $\mathbb{S}^2 \setminus (\mathscr{E} \cup \{p_1\})$ by attaching g handles along disjoint disks $\Delta_k \subset \operatorname{int}(\mathcal{D}_0) \setminus \mathscr{I}$, $1 \leq k \leq g$, and let Σ_2 be the surface obtained from $\mathbb{S}^2 \setminus (\mathscr{E} \cup \{p_{1,1}, p_{1,2}, p_{1,3}\})$ by attaching g handles along the same disks Δ_k, $1 \leq k \leq g$. Then f induces a proper map, which we also call f, from Σ_2 to Σ_1. By Theorem 1.7.1, $\deg(f) = \pm 1$.

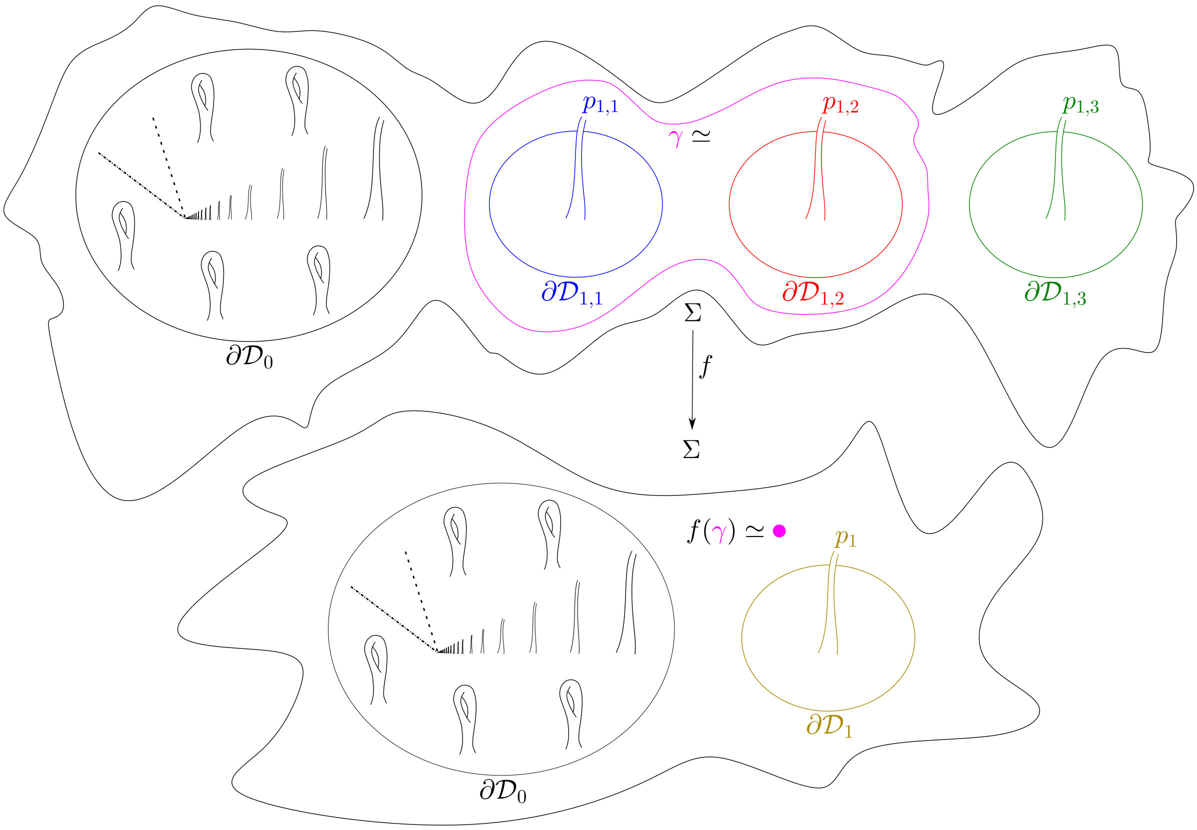

Fig. 3.2.6 A non π_1-injective degree ± 1 map $f\colon \Sigma \to \Sigma$, where $g = 5$ and $\mathscr{I}$ is an infinite set.

Further, we claim that $f\colon \Sigma_2 \to \Sigma_1$ is not injective on π_1. Namely, $\pi_1(\Sigma_2)$ is the amalgamated free product of four groups, one of which is $\pi_1\left(\mathbb{S}^2 \setminus \operatorname{int}\left(\mathcal{D}_0 \cup \mathcal{D}_{1,1} \cup \mathcal{D}_{1,2} \cup \mathcal{D}_{1,3}\right)\right)$. As γ is not homotopic to the trivial loop in $\mathbb{S}^2 \setminus \operatorname{int}\left(\mathcal{D}_0 \cup \mathcal{D}_{1,1} \cup \mathcal{D}_{1,2} \cup \mathcal{D}_{1,3}\right)$, and components of an amalgamated free product inject, γ is not homotopic to the trivial loop in Σ_2. However, $f(\gamma)$ is homotopic to the trivial loop in $\mathbb{S}^2 \setminus \operatorname{int}\left(\mathcal{D}_0 \cup \mathcal{D}_1\right)$ and hence in Σ_1. Therefore, f is not injective on π_1.

Both Σ_1 and Σ_2 have genus the same as Σ and, by Lemma 3.2.5 below, $\operatorname{Ends}(\Sigma_1)$ and $\operatorname{Ends}(\Sigma_2)$ are homeomorphic to $\operatorname{Ends}(\Sigma)$. Further, all ends of Σ, Σ_1 and Σ_2 are planar. Hence, by Theorem 1.5.1 both Σ_1 and Σ_2 are homeomorphic to Σ.

Identifying Σ_1 and Σ_2 with Σ by homeomorphisms, we get a proper map $f\colon \Sigma \to \Sigma$ which is not injective on π_1. As homeomorphisms have degree ± 1, it follows that $\deg(f) = \pm 1$.

Replacing f by $f \circ f$ if necessary, we obtain a proper map of degree one that is not injective on π_1. $\square$

Lemma 3.2.5 Let $\mathscr{E}$ be a closed totally disconnected subset of $\mathbb{S}^2$. Let $\mathscr{I}$ be the set of all isolated points of $\mathscr{E}$. Assume $\mathscr{I}$ is infinite. If $\mathscr{F}$ is a finite subset of $\mathbb{S}^2 \setminus \mathscr{E}$, then $\mathscr{E} \cup \mathscr{F}$ is homeomorphic to $\mathscr{E}$.

Proof. Let $\mathscr{A} := \{a_1, a_2, ...\}$ be a subset of $\mathscr{I}$ such that $a_n \to \ell \in \mathscr{E}$ ($\mathscr{A}$ exists as $\mathscr{E}$ is compact and infinite). Define $\mathscr{B} := \mathscr{A} \cup \mathscr{F}$. Write $\mathscr{B}$ as $\mathscr{B} = \{b_1, b_2, ...\}$. Then the map $g\colon \mathscr{E} \cup \mathscr{F} \to \mathscr{E}$ defined by

$$g(z) := \begin{cases} z & \text{if } z \in (\mathscr{E} \cup \mathscr{F}) \setminus \mathscr{B}, \\ a_n & \text{if } z = b_n \in \mathscr{B}, \end{cases}$$

is a homeomorphism. To prove this, note that g is a bijection from a compact space to a Hausdorff space, so it suffices to show that g is continuous. But observe that g restricted to the closed set $(\mathscr{E} \cup \mathscr{F}) \setminus \mathscr{B}$ is the identity, so g is continuous on $(\mathscr{E} \cup \mathscr{F}) \setminus \mathscr{B}$. Also g restricted to the closed set $\mathscr{B} \cup \{\ell\}$ is continuous as $b_n \to \ell$ and $g(b_n) = a_n \to \ell = g(\ell)$, and all other points of $\mathscr{B} \cup \{\ell\}$ are isolated. Thus, g is continuous, as required. $\square$

Since an oriented surface is amphichiral (see Corollary 1.5.3), Theorem II gives the following.

Theorem 3.2.6 Let Σ be any infinite-type oriented surface. Then there exists a proper map $g\colon \Sigma \to \Sigma$ of degree -1 such that $\pi_1(g)\colon \pi_1(\Sigma) \to \pi_1(\Sigma)$ is not injective. In particular, g is not a homotopy equivalence.

Chapter 4

Future directions

4.1 Strong topological rigidity for bordered surfaces

Up to this point, our focus has primarily been on surfaces. It is natural to wonder what the analogues of those theories should be for bordered surfaces. According to Remark 1.5.4, the classification of bordered surfaces, including those of infinite type, is fully understood, as proved in [14]. Nielsen showed that if $f\colon \mathbf{S}' \to \mathbf{S}$ is a homotopy equivalence between compact bordered surfaces, and if $f|\partial\mathbf{S}' \to \partial\mathbf{S}$ is a homeomorphism, then f is homotopic to a homeomorphism relative to $\partial\mathbf{S}$ [82]. Waldhausen [107, Theorem 6.1.] proved that if $f\colon (N, \partial N) \to (M, \partial M)$ is a homotopy equivalence between two connected, compact, orientable, irreducible, boundary-irreducible 3-manifolds, where N is non-simply-connected and M is Haken, and if $f|\partial N \to \partial M$ is a homeomorphism, then f is homotopic to a homeomorphism relative to ∂N. Brown and Tucker [12, Theorem 4.2] showed that if $\mathfrak{f}\colon (\mathfrak{N}, \partial\mathfrak{N}) \to (\mathfrak{M}, \partial\mathfrak{M})$ is a pseudo proper homotopy equivalence between two connected, non-compact, orientable, irreducible, end-irreducible, boundary-irreducible 3-manifolds such that components of $\partial\mathfrak{N}$ are compact, $\mathfrak{f}|\partial\mathfrak{N} \to \partial\mathfrak{M}$ is a homeomorphism, and $\pi_1(\mathfrak{N})$ is not isomorphic to the fundamental group of any compact surface; then $\mathfrak{f}$ is properly homotopic to a homeomorphism relative to $\partial\mathfrak{N}$. These theories inspire the following question.

Question 4.1.1 Let $\mathbf{S}'$ and $\mathbf{S}$ be two non-compact bordered surfaces whose fundamental groups are non-abelian. Consider a pseudo proper homotopy equivalence $f\colon \left(\mathbf{S}', \partial\mathbf{S}'\right) \to (\mathbf{S}, \partial\mathbf{S})$ such that $f|\partial\mathbf{S}' \to \partial\mathbf{S}$ is a homeomorphism. If each component of $\partial\mathbf{S}$ is compact, then is f properly homotopic to a homeomorphism relative to $\partial\mathbf{S}'$?

4.2 Simple loop conjecture with pinch

In 1985, Gabai proved the simple loop conjecture, which states that if $f\colon S \to T$ is a map between compact surfaces with $\ker\left(\pi_1(f)\right) \neq 1$, then f has geometric kernel, i.e., there is a simple loop in S representing a non-trivial element of $\ker\left(\pi_1(f)\right)$ [39, Theorem 2.1]. Before that, Edmonds proved the simple loop conjecture when $\chi(S) \leq |\deg(f)| \cdot (\chi(T) - 1)$ [26, Theorem 4.5]. Needless to say, both proofs are heavily based on the induction on the genus of the surfaces.

So, the modification of these theories for infinite-type surfaces is indeed tricky. In fact, there are examples of non π_1-injective proper maps of non-zero degree with no geometric kernel. For instance, consider the quotient map $q\colon \mathbb{R}^2 \setminus (\mathscr{P} \cup \mathscr{P}^-) \to \mathbb{R}^2 \setminus \mathscr{P}^-$ that identifies each (x, y) with $(-x, -y)$, where $\mathscr{P}$ is a closed totally disconnected subset of $[1, 2] \times \{0\}$ and $\mathscr{P}^-$ is the reflection of $\mathscr{P}$ w.r.t. **Y**-axis. That said, some restrictions on the induced map between the spaces of ends must exist to have a geometric kernel. A project to find such restrictions can begin by considering the following question, which is motivated by Theorem 3.2.1.

Question 4.2.1 (Degree one map pinches a handle) Consider an oriented surface Σ having finitely many ends. Let $f\colon \Sigma \to \Sigma$ be a proper map of degree ± 1. If $\pi_1(f)$ is not injective, does there exist an embedded copy $\mathbf{h}$ of $S_{1,1}$ in Σ such that f can be properly homotoped to send $\mathbf{h}$ to a point? In particular, does f admit a geometric kernel?

4.3 Domination relation

Given a positive integer d, we define the d-domination relation $\succcurlyeq_d$ on the set $\mathscr{C}_n$ of all homeomorphism classes of closed, oriented n-manifolds as follows: For $[M]$ and $[N] \in \mathscr{C}_n$, we say that $[M] \succcurlyeq_d [N]$ if and only if there exists a map $f\colon M \to N$ of degree $\pm d$. The domination relation $\succcurlyeq$, as suggested by Gromov, is defined as follows: For $[M]$ and $[N] \in \mathscr{C}_n$, we write $[M] \succcurlyeq [N]$ if $[M] \succcurlyeq_d [N]$ for some positive integer d. Notice that $\succcurlyeq_1$ is a partial order on $\mathscr{C}_n$ if and only if $n = 2$. The following three types of questions have attracted a lot of attention [24].

Existence: Given $[M], [N] \in \mathscr{C}_n$, decide whether $[M] \succcurlyeq_1 [N]$.

Finiteness: Given $[M] \in \mathscr{C}_n$, decide whether there are only finitely many $[N] \in \mathscr{C}_n$ with $[M] \succcurlyeq [N]$.

So far, our discussion has focused on degree-one maps whose domain and codomain are either the same infinite-type surfaces or turn out to be the same infinite-type surfaces after a proper homotopy. The following conjecture addresses the existence of a degree-one map between any two oriented infinite-type surfaces. Certainly, it also helps address the analogous problem related to finiteness.

Conjecture 4.3.1 (Degree one domination) Let Σ' and Σ be two oriented non-compact surfaces without boundary. Then there exists a proper map $\Sigma' \to \Sigma$ of degree ± 1 if and only if $g(\Sigma') \geq g(\Sigma)$ and there exists an embedding $\iota\colon \mathrm{Ends}(\Sigma) \hookrightarrow \mathrm{Ends}(\Sigma')$ with $\iota(\mathrm{Ends}_{\mathrm{np}}(\Sigma)) \subseteq \mathrm{Ends}_{\mathrm{np}}(\Sigma')$.

Appendix A

Approximation and transversality in the proper category

Throughout this chapter, M, N will denote two smooth boundaryless manifolds, possibly non-compact. Let $F\colon N \to M$ be a smooth map, and let X be a smoothly embedded boundaryless submanifold of M. We say F *is transverse to* X, and write $F \pitchfork X$, if for every $p \in F^{-1}(X)$, we have $T_{F(p)}X + dF_p(T_pN) = T_{F(p)}M$. If F is transverse to X and $F(N) \cap X \neq \varnothing$, then $F^{-1}(X)$ is a smoothly embedded boundaryless submanifold of N such that $\dim(N) - \dim\left(F^{-1}(X)\right) = \dim(M) - \dim(X)$; see [70, Theorem 6.30.(a)].

The Whitney approximation theorem [70, Theorem 6.26] says that any continuous map $N \to M$ is homotopic to a smooth map. The transversality homotopy theorem [70, Theorem 6.36] says that for any smooth map $F\colon N \to M$ and for any smoothly embedded boundaryless submanifold X of M, the smooth map F can be homotoped to another smooth map $\widetilde{F}\colon N \to M$ such that $\widetilde{F} \pitchfork X$. We modify these two theorems in the proper category. Our interest is in the properness of homotopies; the extra stuff not related to properness is in [70, Theorems 6.26, 6.36].

Theorem A.1 (Proper Whitney approximation theorem) Let $f\colon N \to M$ be a continuous proper map. Then f is properly homotopic to a smooth proper map.

Theorem A.2 (Proper transversality homotopy theorem) Let $f\colon N \to M$ be a smooth proper map, and let X be a smoothly embedded boundaryless submanifold of M. Then f is properly homotopic to a smooth proper map $g\colon N \to M$ which is transverse to X.

We start by summarizing key facts in and around the tubular neighbourhood theorem. Let $M \hookrightarrow \mathbb{R}^\ell$ be a smooth proper embedding; see [70, Theorems 6.15]. For each $x \in M$, define the *normal space* $\mathcal{N}_xM$ to M at x as $\mathcal{N}_xM := \{v \in \mathbb{R}^\ell : v \perp T_xM\}$. Then $\mathcal{N}M := \{(x, v) \in \mathbb{R}^\ell \times \mathbb{R}^\ell : x \in M, v \perp T_xM\}$ is a smoothly embedded ℓ-dimensional submanifold of $\mathbb{R}^\ell \times \mathbb{R}^\ell$ and $\pi\colon \mathcal{N}M \ni (x, v) \longmapsto x \in M$ is vector bundle of rank $\ell - \dim(M)$, called the *normal bundle* of M in $\mathbb{R}^\ell$; see [70, Corollary 10.36].

Consider the smooth map $E\colon \mathcal{N}M \ni (x, v) \longmapsto x + v \in \mathbb{R}^\ell$. One can show that $dE_{(x,0)}$ is bijective for each $x \in M$. Thus, for each $x \in M$, we have $\delta > 0$ such that E maps diffeomorphically $V_\delta(x) := \{(x', v') \in \mathcal{N}M : |x - x'| < \delta, |v'| < \delta\}$ onto an open neighbourhood of x in $\mathbb{R}^\ell$. Now, the map $\rho\colon M \to (0, 1]$ defined by

$$\rho(x) := \sup\left\{\delta \leq 1 : E \text{ maps } V_\delta(x) \text{ diffeomorphically onto an open neighbourhood of } x \text{ in } \mathbb{R}^\ell\right\}$$

is continuous. Further, $V := \left\{(x, v) \in \mathcal{N}M : |v| < \frac{1}{2}\rho(x)\right\}$ is an open subset of NM and E maps diffeomorphically V onto an open subset U of $\mathbb{R}^\ell$ with $M \subseteq U$, i.e., U is a tubular neighbourhood of M in $\mathbb{R}^\ell$; see [70, Theorem 6.24]. Note that the map $r\colon U \to M$ defined by $r := \pi \circ \big(E|V \to U\big)^{-1}$ is a retraction and submersion; see [70, Proposition 6.25.]. Denote $\{y \in \mathbb{R}^\ell : |y - x| < \varepsilon\}$ by $B_\varepsilon(x)$. By an argument similar to showing the continuity of ρ, one can prove that $\delta\colon M \to (0, 1]$ defined by $\delta(x) := \sup\{\varepsilon \leq 1 : B_\varepsilon(x) \subseteq U\}$ for any $x \in M$, is also continuous.

With this setup, we are now ready to state a crucial lemma, which in particular says that if two points are at the most unit distance, then the distance between their images under the tubular neighbourhood retraction can be, at most, 2.

Lemma A.3 Let $\varepsilon > 0$. If $y, y' \in U$ with $|y - y'| < \varepsilon$, then $|r(y) - r(y')| \leq \varepsilon + 1$.

Proof. Notice $|r(y) - r(y')| - |y - y'| \leq |y - r(y)| + |y' - r(y')| \leq \frac{1}{2}\rho \circ r(y) + \frac{1}{2}\rho \circ r(y')$ to conclude. □

Consider another smooth proper embedding $N \hookrightarrow \mathbb{R}^k$ for the proof of the following three facts. The following lemma says that a homotopy lying in a λ-neighbourhood (where λ is a fixed positive number) of a proper map is a proper homotopy.

Lemma A.4 Let $h\colon N \to M$ be a continuous proper map, and let $\mathcal{H}\colon N \times [0, 1] \to M$ be a homotopy. If there exists a constant λ so that $|\mathcal{H}(p, t) - h(p)| \leq \lambda$ for each $(p, t) \in N \times [0, 1]$, then $\mathcal{H}$ is proper.

Proof. Note that the embeddings $M \hookrightarrow \mathbb{R}^\ell$ and $N \hookrightarrow \mathbb{R}^k$ are closed maps as they are proper maps; see Theorem 1.6.2. Consider the induced metric d_M on M inherited from $\mathbb{R}^\ell$, i.e., $d_M(m, m') = |m - m'|$ for all $m, m' \in M$. Also, we have the induced metric $d_{N\times[0,1]}$ on $N \times [0, 1]$ inherited from $\mathbb{R}^k \times [0, 1]$, i.e., $d_{N\times[0,1]}\big((n, t), (n', t')\big) = |n - n'| + |t - t'|$ for all $(n, t), (n', t') \in N \times [0, 1]$. Thus, a subset of $N \times [0, 1]$ (respectively, M) is compact if and only if it is closed and bounded in $N \times [0, 1]$ (respectively, M).

Let C be a compact subset of M. Continuity of $\mathcal{H}$ implies $\mathcal{H}^{-1}(C)$ is closed in $N \times [0, 1]$. Also, if there were an unbounded sequence $\{(n_i, t_i)\} \subseteq \mathcal{H}^{-1}(C)$, then $\{n_i\}$, and hence $\{h(n_i)\}$ would be unbounded (as h is proper); and thus the unbounded set $\{h(n_i)\}$ would be inside the λ-neighbourhood of the bounded set C, a contradiction. Therefore, $\mathcal{H}^{-1}(C)$ is closed and bounded in $N \times [0, 1]$; and hence $\mathcal{H}^{-1}(C)$ is compact. Since C is an arbitrary compact subset of M, we are done. □

Now, we are ready to prove the analogues of the Whitney approximation theorem and transversality homotopy theorem in the proper category.

Proof of Theorem A.1. Whitney Approximation theorem gives a smooth function $\widetilde{f}\colon N \to \mathbb{R}^\ell$ such that $\left|\widetilde{f}(y) - f(y)\right| < \delta(f(y))$ for each $y \in N$; see [70, Theorem 6.21]. Now, define $\mathcal{H}\colon N \times [0,1] \to M$ as $\mathcal{H}(p,t) := r\left((1-t)f(p) + t\widetilde{f}(p)\right)$ for all $(p,t) \in N \times [0,1]$. If $(p,t) \in N \times [0,1]$, then

$$\left|(1-t)f(p) + t\widetilde{f}(p) - f(p)\right| \leq t \cdot \left|\widetilde{f}(p) - f(p)\right| \leq 1.$$

Therefore, $|\mathcal{H}(p,t) - f(p)| = |\mathcal{H}(p,t) - r \circ f(p)| \leq 2$ for all $(p,t) \in N \times [0,1]$ by Lemma A.3. Now, Lemma A.4 tells $\mathcal{H}$ is proper. Therefore, $\mathcal{H}(-,1) = r \circ \widetilde{f}$ is a smooth proper map that is properly homotopic to f (recall that r is a smooth retraction). So, we are done. □

Proof of Theorem A.2. Whitney Approximation theorem gives a smooth function $e\colon N \to (0,\infty)$ with $0 < e < \delta \circ f$; see [70, Corollary 6.22]. Let $\mathring{\mathbb{B}}^\ell := \{\mathbf{s} \in \mathbb{R}^\ell : |\mathbf{s}| < 1\}$. Define $F\colon N \times \mathring{\mathbb{B}}^\ell \to M$ as $F(p,\mathbf{s}) := r\big(f(p) + e(p)\mathbf{s}\big)$ for any $(p,\mathbf{s}) \in N \times \mathring{\mathbb{B}}^\ell$. If $p \in N$, the restriction of F to $\{p\} \times \mathring{\mathbb{B}}^\ell$ is the composition of the local diffeomorphism $\mathbf{s} \longmapsto f(p) + e(p)\mathbf{s}$ followed by the smooth submersion r, so F is a smooth submersion and hence transverse to X.

By parametric transversality theorem [70, Theorem 6.35], $F(-,\mathbf{s}_0)$ is transverse to X for some $\mathbf{s}_0 \in \mathring{\mathbb{B}}^\ell$. Now, define $\mathcal{H}\colon N \times [0,1] \to M$ as $\mathcal{H}(p,t) := r\big(f(p) + te(p)\mathbf{s}_0\big)$ for all $(p,t) \in N \times [0,1]$. If $(p,t) \in N \times [0,1]$, then

$$\left|\big(f(p) + te(p)\mathbf{s}_0\big) - f(p)\right| \leq te(p) \cdot |\mathbf{s}_0| < \delta\big(f(p)\big) \leq 1.$$

Therefore, $|\mathcal{H}(p,t) - f(p)| = |\mathcal{H}(p,t) - r \circ f(p)| \leq 2$ for all $(p,t) \in N \times [0,1]$ by Lemma A.3. Now, Lemma A.4 tells that $\mathcal{H}$ is proper. Define $g := \mathcal{H}(-,1)$, i.e., $g = r\big(f(-) + e(-)\mathbf{s}_0\big) = F(-,\mathbf{s}_0)$ is properly homotopic to f (recall that r is a smooth retraction) as well as transverse to X. □

Appendix B

Transversality of a proper map between two surfaces with respect to a circle

Here are a couple of notations that will be used throughout Appendix B. Let $f\colon \Sigma' \to \Sigma$ be a smooth *proper* map between two surfaces, and let $\mathcal{C}$ be a smoothly embedded circle on Σ such that f is transverse to $\mathcal{C}$. Also, let $\varphi\colon \mathcal{C} \times [-1,1] \hookrightarrow \Sigma$ be a smooth embedding with $\varphi(\mathcal{C},0) = \mathcal{C}$, i.e., $\operatorname{im}(\varphi)$ is a *two-sided (trivial) tubular neighbourhood* of $\mathcal{C}$. We call each of $\varphi(\mathcal{C} \times [-1,0])$ and $\varphi(\mathcal{C} \times [0,1])$ a *one-sided tubular neighbourhood of* $\mathcal{C}$ (in short, *a side of* $\mathcal{C}$). By scaling, we may replace $[-1,0]$ and $[0,1]$ with other closed intervals.

The following theorem says that f is transverse to all circles, which are parallel to $\mathcal{C}$ and sufficiently near to $\mathcal{C}$.

Theorem B.1 There exists $\varepsilon_0 \in (0,1)$ such that f is transverse to $\mathcal{C}_\varepsilon := \varphi(\mathcal{C},\varepsilon)$ for each $\varepsilon \in [-\varepsilon_0,\varepsilon_0]$. Thus for any $\varepsilon \in [-\varepsilon_0,\varepsilon_0]$, $f^{-1}(\mathcal{C}_\varepsilon)$ is either empty or a pairwise disjoint collection of finitely many smoothly embedded circles on Σ'.

At first, we need a lemma to prove Theorem B.1.

Lemma B.2 Let $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth map and $x_n \to x$ in $\mathbb{R}^2$ with $r_n := |g(x_n)| \to 1$. Write $S_r := \{z \in \mathbb{R}^2 : |z| = r\}$ and assume $\operatorname{im}(dg_{x_n}) = T_{g(x_n)}(S_{r_n})$ for all n. If $dg_x \neq 0$, then $\operatorname{im}(dg_x) = T_{g(x)}(S_1)$.

Proof. The derivative map $dg\colon \mathbb{R}^2 \to \mathrm{L}(\mathbb{R}^2,\mathbb{R}^2)$ is continuous implies $dg_{x_n} \to dg_x$, and this convergence can be thought as convergence of 2×2-matrices. In particular, if $\mathbf{i},\mathbf{j} \in \mathbb{R}^2$ are two perpendicular unit vectors, then $dg_{x_n}(\mathbf{i}) \to dg_x(\mathbf{i})$ and $dg_{x_n}(\mathbf{j}) \to dg_x(\mathbf{j})$.

Recall that the tangent space at any point of a circle is the vector space of all points perpendicular to this point. So, $\langle dg_{x_n}(\mathbf{i}), g(x_n)\rangle = 0 = \langle dg_{x_n}(\mathbf{j}), g(x_n)\rangle$ by hypothesis, and now $\langle dg_x(\mathbf{i}), g(x)\rangle = 0 = \langle dg_x(\mathbf{j}), g(x)\rangle$ by the convergence of inner-product. Hence, $\operatorname{im}(dg_x) \subseteq T_{g(x)}(S_1)$. Since $dg_x \neq 0$ and $\dim T_{g(x)}(S_1) = 1$, we are done. $\square$

Proof of Theorem B.1. Suppose not. So, a sequence $\varepsilon_n \to 0$ and points $x_n \in f^{-1}(\mathcal{C}_{\varepsilon_n})$ exist such that $\operatorname{im}(df_{x_n}) + T_{f(x_n)}\mathcal{C}_{\varepsilon_n} \subsetneqq T_{f(x_n)}\Sigma$ for all n. Hence, $\operatorname{im}(df_{x_n}) \subseteq T_{f(x_n)}\mathcal{C}_{\varepsilon_n}$ as $T_{f(x_n)}\mathcal{C}_{\varepsilon_n} \oplus \mathcal{N}_{f(x_n)}\mathcal{C}_{\varepsilon_n} = T_{f(x_n)}\Sigma$ for all n. Now, $\{x_n\}$ is contained in the compact set $f^{-1}(\operatorname{im}(\varphi))$ (recall that f is a proper map), i.e., passing to sub-sequence, if needed, assume $x_n \to x \in f^{-1}(\mathcal{C})$.

The continuity of the derivative map says $df_{x_n} \to df_x$. After discarding first few terms, we may assume $df_{x_n} \neq 0$ for all n (otherwise, we would have $df_x = 0$, i.e., $T_{f(x)}\mathcal{C} + \operatorname{im}(df_x) = T_{f(x)}\mathcal{C}$ wouldn't be equal to $T_{f(x)}\Sigma$, i.e., f wouldn't be transverse to $\mathcal{C}$). So, $\operatorname{im}(df_{x_n}) = T_{f(x_n)}(\mathcal{C}_{\varepsilon_n})$ for all n (a non-zero vector subspace of a one-dimensional vector space is equal to the whole space).

Now, restricting f to a coordinate ball containing x and then post composing with φ^{-1}, we can consider Lemma B.2 above, which gives $\operatorname{im}(df_x) = T_{f(x)}(\mathcal{C})$, a contradiction to the assumption $f \pitchfork \mathcal{C}$. $\square$

The previous theorem guarantees transversality near $\mathcal{C}$. In the rest part of Appendix B, we aim to prove that every small one-sided tubular neighbourhood of a component of $f^{-1}(\mathcal{C})$ maps into a small one-sided tubular neighbourhood of $\mathcal{C}$.

At first, we fix some notations. So, let $\mathcal{C}'$ be a component of $f^{-1}(\mathcal{C})$. Also, consider an $\varepsilon_0 \in (0, 1)$ such that $f \pitchfork \mathcal{C}_\varepsilon$ for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$; see Theorem B.1.

Theorem B.3 Let $\varepsilon \in (0, \varepsilon_0]$, and let $\mathcal{T}'$ be a two-sided compact tubular neighbourhood of $\mathcal{C}'$ in Σ'. Then there exist two one-sided compact tubular neighbourhoods $\mathcal{U}'_l, \mathcal{U}'_r$ of $\mathcal{C}'$ in Σ' such that $\mathcal{U}'_l \cup \mathcal{U}'_r$ is a two-sided tubular neighbourhood of $\mathcal{C}'$ with $\mathcal{U}'_l \cup \mathcal{U}'_r \subseteq \mathcal{T}'$, and for each $s \in \{l, r\}$ the following hold: (i) $f^{-1}(\mathcal{C}) \cap \mathcal{U}'_s = \mathcal{C}'$, (ii) either $f(\mathcal{U}'_s) \subseteq \varphi(\mathcal{C} \times [0, \varepsilon])$ or $f(\mathcal{U}'_s) \subseteq \varphi(\mathcal{C} \times [-\varepsilon, 0])$.

Proof. By Theorem B.1, $f^{-1}(\mathcal{C}_{-\varepsilon}) \cup f^{-1}(\mathcal{C}) \cup f^{-1}(\mathcal{C}_\varepsilon)$ is a pairwise disjoint collection of finitely many smoothly embedded circles on Σ'. Now, consider two one-sided compact tubular neighbourhoods $\mathcal{U}'_l, \mathcal{U}'_r$ of $\mathcal{C}'$ in Σ' such that $\mathcal{U}'_l \cup \mathcal{U}'_r$ is a two-sided tubular neighbourhood of $\mathcal{C}'$ with $\mathcal{U}'_l \cup \mathcal{U}'_r \subseteq \mathcal{T}'$, and for each $s \in \{l, r\}$ the following hold: (i) $f^{-1}(\mathcal{C}) \cap \mathcal{U}'_s = \mathcal{C}'$, (ii) $\mathcal{U}'_s \cap f^{-1}(\mathcal{C}_\varepsilon) = \varnothing = \mathcal{U}'_s \cap f^{-1}(\mathcal{C}_{-\varepsilon})$.

Now, fix $s \in \{l, r\}$. Since $\mathcal{U}'_s \setminus \mathcal{C}'$ is connected and f is continuous, $f(\mathcal{U}'_s \setminus \mathcal{C}')$ is contained in one of the components of $\Sigma \setminus (\mathcal{C}_{-\varepsilon} \cup \mathcal{C} \cup \mathcal{C}_\varepsilon)$. But $f(\mathcal{C}') \subseteq \mathcal{C}$ implies either $f(\mathcal{U}'_s) \subseteq \varphi(\mathcal{C} \times [0, \varepsilon])$ or $f(\mathcal{U}'_s) \subseteq \varphi(\mathcal{C} \times [-\varepsilon, 0])$. So, we are done. $\square$

Remark B.4 In Theorem B.3, it is possible that $f(\mathcal{U}'_l \cup \mathcal{U}'_r)$ is contained in either $\varphi(\mathcal{C} \times [0, \varepsilon])$ or $\varphi(\mathcal{C} \times [-\varepsilon, 0])$, i.e., f may map both sides of $\mathcal{C}'$ in one of the two sides of $\mathcal{C}$.

Consider the one-sided compact tubular neighbourhoods $\mathcal{U}'_l, \mathcal{U}'_r$ of $\mathcal{C}'$ in Σ' given by Theorem B.3. Notice that for some $s \in \{l, r\}$, it is possible that $f((\partial \mathcal{U}'_s) \setminus \mathcal{C}') \nsubseteq \varphi(\mathcal{C} \times t)$ for any $t \in [-\varepsilon, \varepsilon]$. A remedy for this is given in the following theorem.

Theorem B.5 Let $\varepsilon \in (0, \varepsilon_0]$, and let $\mathcal{U}'$ be a one-sided compact tubular neighbourhood of $\mathcal{C}'$ such that $f^{-1}(\mathcal{C}) \cap \mathcal{U}' = \mathcal{C}'$, and $f(\mathcal{U}') \subseteq \varphi(\mathcal{C} \times [0, \varepsilon])$. Then there is a $\delta \in (0, \varepsilon)$ and there is a component $\mathcal{C}'_\delta$ of $f^{-1}(\mathcal{C}_\delta)$ such that the following hold:

(1) $\mathcal{C}'_\delta$ together with $\mathcal{C}'$ co-bounds an annulus $\mathcal{A}' \subseteq \mathcal{U}'$ so that any other component of $f^{-1}(\mathcal{C}_\delta)$ in $\operatorname{int}(\mathcal{A}')$, if any, bounds a disk inside $\mathcal{A}'$.

(2) The map f sends $\mathcal{A}'$ into $\varphi(\mathcal{C} \times [0, \varepsilon])$. Also, after removing the interiors of all disks bounded by components of $f^{-1}(\mathcal{C}_\delta)$ from $\mathcal{A}'$, we can send it to $\varphi(\mathcal{C} \times [0, \delta])$ by f.

Proof of part (1) of Theorem B.5. Choose a $\delta \in (0, \varepsilon)$ such that $\varphi(\mathcal{C} \times [0, \delta]) \cap f((\partial\mathcal{U}') \setminus \mathcal{C}') = \varnothing$. Note that such a δ exists; otherwise, using the compactness of $(\partial\mathcal{U}') \setminus \mathcal{C}'$, we would have a sequence $\{x'_n\} \subseteq (\partial\mathcal{U}') \setminus \mathcal{C}'$ converging to some $x' \in (\partial\mathcal{U}') \setminus \mathcal{C}'$ such that $f(x'_n) \in \varphi\left(\mathcal{C} \times [0, 1/n]\right)$, i.e., $f(x')$ would belong to $\mathcal{C}$, a contradiction to the assumption $f^{-1}(\mathcal{C}) \cap \mathcal{U}' = \mathcal{C}'$. Define an open set $\mathcal{W}'$ as follows

$$\mathcal{W}' := \operatorname{int}(\mathcal{U}') \cap f^{-1}\big(\varphi(\mathcal{C} \times (0, \delta))\big).$$

Notice that no sequence in $\mathcal{W}'$ converges to some point of $(\partial\mathcal{U}') \setminus \mathcal{C}'$. Otherwise, if we assume $\mathcal{W}'_n \ni w'_n \longrightarrow x' \in (\partial\mathcal{U}') \setminus \mathcal{C}'$, then $\varphi(\mathcal{C} \times (0, \delta)) \ni f(w'_n) \longrightarrow f(x')$. Since $\varphi(\mathcal{C} \times [0, \delta])$ is a closed set containing the sequence $\{f(w'_n)\}$, we can say that $f(x') \in f\big((\partial\mathcal{U}') \setminus \mathcal{C}'\big) \cap \varphi\big(\mathcal{C} \times [0, \delta]\big)$, which is impossible by our choice of δ.

Therefore, $\overline{\mathcal{W}'} \subseteq \mathcal{U}'$ (as $\mathcal{U}'$ is compact) but $\big((\partial\mathcal{U}') \setminus \mathcal{C}'\big) \cap \overline{\mathcal{W}'} = \varnothing$. In particular, $\partial\mathcal{W}' \subseteq \mathcal{U}'$ but $\big((\partial\mathcal{U}') \setminus \mathcal{C}'\big) \cap \partial\mathcal{W}' = \varnothing$.

Claim B.6 $\partial\mathcal{W}' \subseteq \mathcal{C}' \cup f^{-1}(\mathcal{C}_\delta)$. Thus, $\partial\mathcal{W}'$ is contained in a finite union of pairwise disjoint circles.

Proof of Claim B.6. Let $y' \in \partial\mathcal{W}'$ and consider a sequence $\{y'_n\} \subseteq \mathcal{W}'$ converging to y'. Then $\varphi(\mathcal{C} \times (0, \delta)) \ni f(y'_n) \longrightarrow f(y') \in \varphi(\mathcal{C} \times [0, \delta])$. If $f(y') \in \varphi\big(\mathcal{C} \times \{0, \delta\}\big) = \mathcal{C} \cup \mathcal{C}_\delta$, then we are done since $f^{-1}(\mathcal{C}) \cap \mathcal{U}' = \mathcal{C}'$. On the other hand, if $f(y') \in \varphi\big(\mathcal{C} \times (0, \delta)\big)$, then the definition of $\mathcal{W}'$ and $\mathcal{W}' \cap \partial\mathcal{W}' = \varnothing$ (as $\mathcal{W}'$ is open) together imply $y' \in \mathcal{U}' \setminus \operatorname{int}(\mathcal{U}') = \partial\mathcal{U}'$, i.e., $y' \in \mathcal{C}'$ as $\big((\partial\mathcal{U}') \setminus \mathcal{C}'\big) \cap \partial\mathcal{W}' = \varnothing$. Since $y' \in \partial\mathcal{W}'$ is arbitrary, we are done. $\square$

The definition of $\mathcal{W}'$ tells that each point of $\operatorname{int}(\mathcal{U}')$ that is sufficiently near to $\mathcal{C}'$ must belong to $\mathcal{W}'$. Now, using Claim B.6, we can say that there is at least one component of $f^{-1}(\mathcal{C}_\delta)$, which co-bounds an annulus with $\mathcal{C}'$ inside $\mathcal{U}'$. Of all the $\mathcal{C}'$-parallel components of $f^{-1}(\mathcal{C}_\delta)$, we consider the closest to $\mathcal{C}'$ as $\mathcal{C}'_\delta$. $\square$

Proof of part (2) of Theorem B.5. Certainly, $f(\mathcal{A}') \subseteq f(\mathcal{U}') \subseteq \varphi\big(\mathcal{C} \times [0, \varepsilon]\big)$. Now, the rest follows, once we observe that removing the interiors of all disks bounded by components of $f^{-1}(\mathcal{C}_\delta)$ from $\mathcal{A}'$, $\mathcal{A}'$ remains connected, so by continuity of $f|\Sigma' \setminus f^{-1}(\mathcal{C} \cup \mathcal{C}_\delta) \to \Sigma \setminus (\mathcal{C} \cup \mathcal{C}_\delta)$, it maps into $\varphi\big(\mathcal{C} \times (0, \delta)\big)$. $\square$

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Colophon

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