MA232 Course of IISc Bangalore

Sumanta (Teaching Assistant)

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1 Introductory level problems

Theorem 1 *A map $f\colon \mathbb{S}^n \to Y$ is null-homotopic if and only if it can be extended to $\mathbb{D}^{n+1} := \{x \in \mathbb{R}^{n+1} : \|x\| \le 1\}$.*

Proof. We first prove the only if part. Let $f\colon \mathbb{S}^n \to Y$ be null-homotopic, say $F\colon \mathbb{S}^n \times [0,1] \to Y$ with $F(\bullet, 0) = f$ and $F(\bullet, 1) = c_{y_0}$, where c_{y_0} is the constant map based at $y_0 \in Y$. Then, $g\colon \mathbb{D}^{n+1} \to Y$ defined by

$$g(z) := \begin{cases} y_0 & \text{if } 0 \le \|z\| \le \frac{1}{2}, \\ F\left(\frac{z}{\|z\|}, 2 - 2\|z\|\right) & \text{if } \frac{1}{2} \le \|z\| \le 1. \end{cases}$$

is continuous by pasting lemma and $g(z) = F(z, 0) = f(z)$, i.e., g extends f.

Now, we prove the if part. Suppose $f\colon \mathbb{S}^n \to Y$ is a map and $g\colon \mathbb{D}^{n+1} \to Y$ extends f, i.e., $g|_{\mathbb{S}^n} = f$. Define, $F\colon \mathbb{S}^n \times [0,1] \to Y$ as

$$F(z,t) := g\big((1-t)z + tz_0\big), \text{ where } z_0 \in \mathbb{S}^n \text{ is a fixed point.}$$

Notice that $F(z,1) = g(z_0) = f(z_0)$ for all $z \in \mathbb{S}^n$. Hence, $F : f \simeq c_{f(z_0)}$. □

Theorem 2 *Let $x, y \in X$. Denote by $P(x,y)$ the set of equivalence classes of paths in X from x to y under the equivalence relation "homotopic relative to $\{0,1\}$". Then there is a one-to-one correspondence between $P(x,y)$ and $P(x,x)$ if and only if $P(x,y) \neq \varnothing$.*

Proof. Note that $P(x,x)$ is always non-empty (consider the constant loop based at x). So, existence of a one-to-one correspondence between $P(x,y)$ and $P(x,x)$ implies $P(x,y) \neq \varnothing$.

Next, let $P(x,y) \neq \varnothing$, and so choose a path $\alpha\colon [0,1] \to X$ with $\alpha(0) = x$ and $\alpha(1) = y$. Now, define

$$f\colon P(x,y) \ni [\gamma] \longmapsto [\gamma * \overline{\alpha}] \in P(x,x) \text{ and}$$

$$g\colon P(x,x) \ni [\ell] \longmapsto [\ell * \alpha] \in P(x,y).$$

Now, it is easy to check that f, g are well-defined maps and $f \circ g = \mathrm{Id}_{P(x,x)}$, $g \circ f = \mathrm{Id}_{P(x,y)}$. □

Theorem 3 *Let $p, q\colon I \to X$ be paths with $p(1) = q(0)$. For $0 < s < 1$ define $h_s\colon I \to X$ by*

$$h_s(t) := \begin{cases} p\left(\frac{t}{s}\right) & \text{if } 0 \le t < s, \\ q\left(\frac{t-s}{1-s}\right) & \text{if } s \le t \le 1. \end{cases}$$

Then, $h_s \simeq_{\text{rel }\{0,1\}} h_{\frac{1}{2}} := p * q$.

Proof. Consider $\mathcal{H}\colon I \times I \to X$ defined by

$$\mathcal{H}(t,\ell) := \begin{cases} p\left(\dfrac{t}{(1-\ell)s + \dfrac{\ell}{2}}\right) & \text{if } 0 \le t \le s, \ell \in [0,1]; \\ q\left(\dfrac{t-(1-\ell)s - \dfrac{\ell}{2}}{1-(1-\ell)s - \dfrac{\ell}{2}}\right) & \text{if } s \le t \le 1, \ell \in [0,1]. \end{cases}$$

Notice that $\mathcal{H}(-,0) = h_s$ and $\mathcal{H}(-,1) = p * q$, and $\mathcal{H}(0,-) = p(0), \mathcal{H}(1,-) = q(1)$. $\square$

Theorem 4 *Let $f, g\colon I \to X$ be continuous. Define $\overline{f}\colon I \to X$ as $\overline{f}(s) := f(1-s)$ for all $s \in I$. Then $f \simeq_{\text{rel }\{0,1\}} g$ if and only if $\overline{f} \simeq_{\text{rel }\{0,1\}} \overline{g}$.*

Proof. Let $H\colon I \times I \to X$ be a homotopy with $H(-,0) = f,\ H(-,1) = g$ and $H(0,t) = f(0),\ H(1,t) = f(1)$ for all $t \in I$. Define $\overline{H}\colon I \times I \to X$ by

$$\overline{H}(s,t) := H(1-s,t) \text{ for } (s,t) \in I \times I.$$

Then, $\overline{H}\colon \overline{f} \simeq_{\text{rel }\{0,1\}} \overline{g}$ as $\overline{H}(-,0) = \overline{f},\ \overline{H}(-,1) = \overline{g}$ and $\overline{H}(0,t) = H(1,t) = f(1),\ \overline{H}(1,t) = H(0,t) = f(0)$ for all $t \in I$. The reverse direction is similar. $\square$

Theorem 5 *Let $f_0, f_1\colon X \to Y$ and $g\colon Y \to Z$ be continuous and $A \subseteq X$. If $f_0 \simeq_{\text{rel }A} f_1$, then $g \circ f_0 \simeq_{\text{rel }A} g \circ f_1$.*

Proof. Let $H\colon X \times I \to Y$ be the relative homotopy from f_0 to f_1, i.e., $H(-,0) = f_0,\ H(-,1) = f_1$ and $H(a,t) = f_0(a)$ for $(a,t) \in A \times I$. Then the map $g \circ H\colon X \times I \to Z$ gives a homotopy from $g \circ f_0$ to $g \circ f_1$ relative to A. $\square$

Theorem 6 *Let $f_0, f_1\colon X \to Y$ and $g_0, g_1\colon Y \to Z$ be continuous. If $f_0 \simeq f_1$ and $g_0 \simeq g_1$ then $g_0 \circ f_0 \simeq g_1 \circ f_1$.*

Proof. Let $F\colon X \times I \to Y$ be a homotopy from f_0 to f_1 and $G\colon Y \times I \to Z$ be a homotopy from g_0 to g_1. Define $\mathscr{H}\colon X \times I \to Z$ by

$$\mathscr{H}(x,t) := G\big(F(x,t),t\big) \text{ for all } (x,t) \in X \times I.$$

Now for any $x \in X$ we have,

$$\mathscr{H}(x,0) = G\big(F(x,0),0\big) = G\big(f_0(x),0\big) = g_0 \circ f_0(x),$$
$$\mathscr{H}(x,1) = G\big(F(x,1),1\big) = G\big(f_1(x),1\big) = g_1 \circ f_1(x).$$

So, $\mathscr{H}\colon g_0 \circ f_0 \simeq g_1 \circ f_1$. $\square$

Theorem 7 *Let X, Y be topological spaces and let $\mathscr{F}(X,Y)$ be the set of continuous functions from X to Y with the compact-open topology.*

- *If $f \simeq g\colon X \to Y$ then there is a path from f to g in the space $\mathscr{F}(X,Y)$.*
- *Suppose that X is compact and Hausdorff; prove that there is a path from f to g in $\mathscr{F}(X,Y)$ if and only if $f \simeq g\colon X \to Y$.*

Proof. To prove the first part, let $H\colon X \times [0,1] \to Y$ be a homotopy from f to g. Now, consider $p\colon [0,1] \ni t \longmapsto H(-,t) \in \mathscr{F}(X,Y)$. The continuity of p follows from [Mun00, Theorem 46.11 on page 287].

Now, to prove the second part, let $p\colon [0,1] \to \mathscr{F}(X,Y)$ be a path from f, g. Then $H\colon X \times [0,1] \to Y$ defined by $H(x,t) := p(t)(x)$ for all $(x,t) \in X \times [0,1]$ is a continuous map as X is locally compact Hausdorff, see [Mun00, Theorem 46.11 on page 287]. Also, H is a homotopy from f to g. For the converse part, see the first paragraph. $\square$

Lemma 8 *Let X and Y be two topological spaces, and $\sim_X$ and $\sim_Y$ be two equivalence relations on X and Y, respectively. Let $F\colon X \times [0,1] \to Y$ be a continuous map such that $F(x,t) \sim_Y F(x',t)$ for all $t \in [0,1]$ whenever $x \sim_X x'$. Then, $\widetilde{F}\colon \frac{X}{\sim_X} \times [0,1] \to \frac{Y}{\sim_Y}$ defined by $([x],t) \longmapsto [F(x,t)]$ is continuous.*

Proof. Note that for quotient map $p\colon X \to X/\sim_X$ and $q\colon Y \to Y/\sim_Y$ we have $p \times \mathrm{Id}_{[0,1]}$ is quotient map as $[0,1]$ is locally compact Hausdorff space. Now, consider the commutative diagram below:

$$\begin{tikzcd}
& X \times [0,1] \arrow[dl, "p\times \mathrm{Id}_{[0,1]}"'] \arrow[dr, "q\circ F"] & \\
\frac{X}{\sim_X} \times [0,1] \arrow[rr, "\widetilde{F}"'] & & \frac{Y}{\sim_Y}
\end{tikzcd}$$

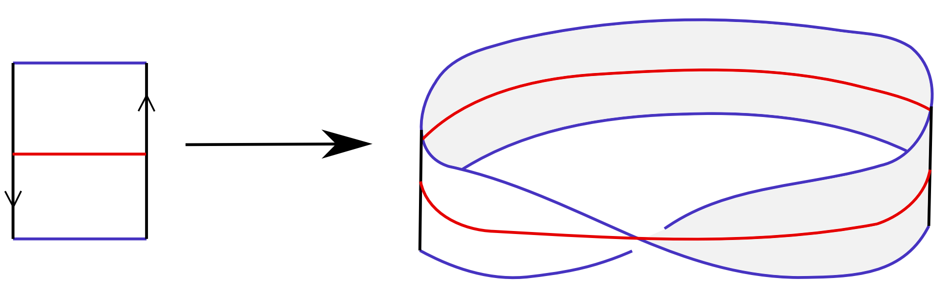
Note that $\widetilde{F} \circ \big(p \times \mathrm{Id}_{[0,1]}\big) = q \circ F$, so for any $U \subseteq_{\text{open}} Y/\sim_Y$ we have $(q \circ F)^{-1}(U)$ is open in $X \times [0,1]$, i.e., $\big(p \times \mathrm{Id}_{[0,1]}\big)^{-1}\big(\widetilde{F}^{-1}(U)\big) = (q \circ F)^{-1}(U)$ is open in $X \times [0,1]$, hence $\widetilde{F}^{-1}(U)$ is open in $\frac{X}{\sim_X} \times [0,1]$. $\square$

Theorem 9 *Möbius strip has a strong deformation retract onto a circle embedded in itself. Thus, the Möbius strip and cylinder are homotopy equivalent.*

Proof. Consider the Möbius strip $M := \dfrac{[-1,1] \times [-1,1]}{(-1,-y) \sim (1,y)}$. Then, there is a deformation retract of M onto its central circle $C := \{\, [x,0] : -1 \le x \le 1\}$. To prove this consider, $H\colon M \times [0,1] \to M$ defined by

$$H\colon \big([x,y],t\big) \longmapsto [x,(1-t)y] \ \text{ for } -1 \le x,y \le 1,\ t \in I.$$

Note that H is continuous by Lemma 8.



Möbius strip has a strong deformation retract onto the central circle.

For the second part, consider the Cylinder $\mathbb{S}^1 \times [0,1]$ and $H\colon \mathbb{S}^1 \times [0,1] \times [0,1] \ni (z,s,t) \longmapsto \big(z,(1-t)\cdot s\big) \in \mathbb{S}^1 \times [0,1]$. Notice that H gives a strong deformation retract of $\mathbb{S}^1 \times [0,1]$ onto $\mathbb{S}^1 \times 0$. Hence, a Cylinder is homotopy equivalent to a circle. Since being homotopy equivalent is an equivalence relation in the category of **Top**, we are done. $\square$

Theorem 10 *A space X is contractible if and only if Id_X is homotopic to a constant map.*

Proof. X is contractible means there are continuous maps $f\colon X \to \mathrm{pt}$ and $g\colon \mathrm{pt} \to X$ such that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_{\mathrm{pt}}$. Note that $g \circ f$ is a constant map. Hence, Id_X is homotopic to a constant map.

Conversely, let X be a space such that Id_X is homotopic to a constant map, say $H\colon \mathrm{Id}_X \simeq C_x$ where $C_x\colon X \to X$ is the constant map based at $x \in X$. Consider the inclusion map $\mathscr{I}_x\colon \{x\} \hookrightarrow X$. Let $\mathscr{C}_x\colon X \to \{x\}$ be the obvious map. Then $\mathscr{C}_x \circ \mathscr{I}_x = \mathrm{Id}_{\{x\}}$ and $H\colon \mathrm{Id}_X \simeq C_x = \mathscr{I}_x \circ \mathscr{C}_x$. □

Theorem 11 *The following two statements are equivalent:*

(a) There is a retract $r\colon \overline{\mathbb{B}^n(1)} \to \mathbb{S}^{n-1}$.

(b) $\mathbb{S}^{n-1}$ is contractible.

Proof. Suppose we have a retract $r\colon \overline{\mathbb{B}^n(1)} \to \mathbb{S}^{n-1}$. Consider a homotopy $H\colon \mathbb{S}^{n-1} \times I \to \mathbb{S}^{n-1}$ given by

$$H(z,t) := r\big((1-t)\cdot z\big) \text{ for } z \in \mathbb{S}^{n-1},\ t \in I.$$

Note that $H(-,0) = \mathrm{Id}_{\mathbb{S}^{n-1}}$ and $H(-,0)$ is a constant map. So, $(a) \implies (b)$.

To prove $(b) \implies (a)$ suppose $\mathbb{S}^{n-1}$ is contractible, so we have a homotopy $H\colon \mathbb{S}^{n-1} \times I \to \mathbb{S}^{n-1}$ from $\mathrm{Id}_{\mathbb{S}^{n-1}}$ to the constant map $H(-,1)$. Define $r\colon \overline{\mathbb{B}^n(1)} \to \mathbb{S}^{n-1}$ as

$$r(x) := \begin{cases} H(1,1) & \text{if } 0 \le \|x\| \le \frac{1}{2}, \\ H\left(\frac{x}{\|x\|}, 2 - 2\|x\|\right) & \text{if } \frac{1}{2} \le \|x\| \le 1. \end{cases}$$

□

Definition 12 *Let $\mathscr{C}(X)$ be the set of all connected components of X. If $f\colon X \to Y$ is continuous then define $\mathscr{C}(f)\colon \mathscr{C}(X) \to \mathscr{C}(Y)$ as*

$$\mathscr{C}(f)(\textit{connected component } C \textit{ of } X) := \textit{ the unique connected component of } Y \textit{ containing } f(C).$$

Remark 13 *For $X \xrightarrow{f} Y \xrightarrow{g} Z$ it is easy to show that $\mathscr{C}(\mathrm{Id}_X) = \mathrm{Id}_{\mathscr{C}(X)}$ and $\mathscr{C}(g \circ f) = \mathscr{C}(g) \circ \mathscr{C}(f)$. That is, $\mathscr{C}\colon$ **Top** $\to$ **Set** is a functor.*

Lemma 14 *If $f_1, f_2\colon X \to Y$ are homotopic, then $\mathscr{C}(f_1) = \mathscr{C}(f_2)$.*

Proof. Let $\Phi : f_1 \simeq f_2$ be a homotopy, then for any connected component C of X we have

$$f_1(C) = \Phi(C \times 0) \subseteq \Phi\left(C \times [0,1]\right) \text{ and } f_2(C) = \Phi(C \times 1) \subseteq \Phi\left(C \times [0,1]\right).$$

Now, $\Phi\left(C \times [0,1]\right)$ is contained in a unique connected component of Y. So, both $f_1(C)$ and $f_2(C)$ are contained in a unique connected component of Y. □

Lemma 15 *Spaces having the same homotopy type have the same number of connected components.*

Proof. Suppose $f\colon X \to Y$ and $g\colon Y \to X$ be such that $f \circ g \simeq \mathrm{Id}_Y$ and $g \circ f \simeq \mathrm{Id}_X$. Then, $\mathscr{C}(f) \circ \mathscr{C}(g) = \mathscr{C}(f \circ g) = \mathscr{C}(\mathrm{Id}_Y) = \mathrm{Id}_{\mathscr{C}(Y)}$, and similarly, $\mathscr{C}(g) \circ \mathscr{C}(f) = \mathscr{C}(g \circ f) = \mathscr{C}(\mathrm{Id}_X) = \mathrm{Id}_{\mathscr{C}(X)}$. That is, both $\mathscr{C}(f)$ and $\mathscr{C}(g)$ are bijections. □

Theorem 16 *Prove that if X is connected and has the same homotopy type as Y, then Y is also connected.*

Proof. This is a particular case of Lemma 15. □

Definition 17 *A subset $A \subseteq X$ is said to be a weak retract of X if there exists a continuous map $r\colon X \to A$ such that $r \circ i \simeq \mathrm{Id}\colon A \to A$ where $i\colon A \hookrightarrow X$ is the inclusion map.*

Theorem 18 *There exist spaces $A \subseteq X$ such that A is a weak retract of X but not a retract of X.*

Proof. Consider the comb space

$$A := \left\{ \left(\frac{1}{n}, t \right) : 0 \le t \le 1, n \in \mathbb{N} \right\} \bigcup \big(0 \times [0,1]\big) \bigcup \big([0,1] \times 0\big).$$

Consider the map $H\colon A \times [0,1] \to A$ given by

$$H\big((x,y),t\big) := \begin{cases} \big(x,(1-2t)y\big) & \text{for } 0 \le t \le \frac{1}{2}, \\ \big(2(1-t)x, 0\big) & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

Now, notice the following:

- A is contractible as $H\colon \mathrm{Id}_A \simeq c_{(0,0)}$.
- $H\big((0,0),t\big) = (0,0)$ for all $t \in [0,1]$, i.e., A is a deformation retract onto $\{(0,0)\}$.

Let $X := [0,1]^2$ and $r\colon X \to A$ be the constant map based at $(0,0)$. Therefore, $r \circ i = c_{(0,0)} \simeq \mathrm{Id}_A$, i.e., A is weak retract of X.

Now, we show that A is not a retract of X. On the contrary, let's assume $r\colon X \to A$ is a retract. Consider the point $q = \big(0, \frac{1}{2}\big)$. Let $V := A \cap B(q;\varepsilon)$, where $\varepsilon = \frac{1}{4}$. Now, $r(q) = q$ implies we must have an open ball $B(q;\delta)$ such that $U := X \cap B(q;\delta)$ is mapped into V by r. Since U is connected, $r(U)$ is contained in the set $\big\{(0,t) | \frac{1}{4} < t < \frac{3}{4}\big\}$, the component of V containing q. However, for n sufficiently large, we have the point $p_n = \big(\frac{1}{n}, \frac{1}{2}\big)$ in U, which is clearly moved by r . So, r cannot be a retraction. □

Theorem 19 *There exist spaces $A \subseteq X$ such that A is a deformation retract of X but not a strong deformation retract of X.*

Proof. Let

$$X := \left\{ \left(\frac{1}{n}, t \right) : 0 \le t \le 1, n \in \mathbb{N} \right\} \bigcup \big(0 \times [0,1]\big) \bigcup \big([0,1] \times 0\big),$$

and $A := 0 \times [0,1]$. Define $H\colon X \times [0,1] \to X$ as

$$H\big((x,y),t\big) := \begin{cases} \big(x,(1-3t)y\big) & \text{if } 0 \le t \le \frac{1}{3}; \\ ((2-3t)x, 0) & \text{if } \frac{1}{3} \le t \le \frac{2}{3}; \\ \big(0,(3t-2)y\big) & \text{if } \frac{2}{3} \le t \le 1. \end{cases}$$

Then H is a homotopy between Id_X and $i \circ r$, where $i\colon A \hookrightarrow X$ is the inclusion map and $r\colon (x,y) \longmapsto (0,y)$ is the retraction of X onto A . So, A is a deformation retract of X.

Now, we show A is not a strong deformation retract of X. On the contrary, that there is a homotopy $X \times [0,1] \to X$ such that $F(p,0) = p, F(p,1) \in A$ for all $p \in X$, and $F(q,t) = q$ for every $q \in A$ and all $t \in [0,1]$. Let q be a point of A other than the point $(0,0)$ and let $B(q;\varepsilon)$ be an open ball in $\mathbb{R}^2$, which does not meet the set $[0,1] \times \{0\} \subset X$. Then, $W = X \cap B(q;\varepsilon)$ is a nbd of q in X, and $\{q\} \times [0,1] \subset F^{-1}(W)$. By the Tube Lemma [Mun00, Lemma 26.8.], there is an open nbd U of q in X such that $U \times [0,1] \subseteq F^{-1}(W)$. So, for each $p \in U$, $F\big(\{p\} \times [0,1]\big)$ is contained in the component of W containing p, and this component is the intersection of $B(q;\varepsilon)$ with the tooth containing p. This contradicts the fact that $F(p,1) \in A$ for every $p \in X$, and hence our claim. $\square$

Definition 20 *A subset $A \subseteq X$ is said to be a weak deformation retract of X if the inclusion map $i\colon A \hookrightarrow X$ is a homotopy equivalence.*

Theorem 21 *There exist spaces $A \subseteq X$ such that A is a deformation retract of X but not a strong deformation retract of X.*

Proof. Consider the proof of Theorem 18. The inclusion map of the comb space into the unit square is a homotopy equivalence, as both spaces are contractible. But there is no retraction of the unit square onto the comb space. $\square$

Theorem 22 *Let $\varnothing \neq A \subseteq X,\ Y \neq \varnothing$. Then $A \times Y$ is a retract of $X \times Y$ if and only if A is retract of X.*

Proof. If $r\colon X \to A$ is a retract then $r \times \mathrm{Id}_Y\colon X \times Y \to A \times Y$ is a retraction. Conversely, for any retraction $R\colon X \times Y \to A \times Y$ and any $y_0 \in Y$ the map $r\colon X \to A$ defined by

$$r(x) := \pi_X \circ R(x, y_0) \text{ for } x \in X$$

is retraction of X onto A. $\square$

Theorem 23 *Let $A \subseteq B \subseteq C$. If A is a retract of B, and B is a retract of C, then A is a retract of C.*

Proof. Let $r_1\colon B \to A$ and $r_2\colon C \to B$ be retractions. Then $r_1 \circ r_2\colon C \to A$ is a retraction of C onto A. $\square$

Theorem 24 *Let $x_0 \in \mathbb{R}^2$. Then there exists a circle $\mathcal{C}$, which is a strong deformation retract of $\mathbb{R}^2 \setminus \{x_0\}$.*

Proof. Let $\mathcal{C} := \{z \in \mathbb{R}^2 : |z - x_0| = 1\}$. Define $H\colon \mathbb{R}^2 \setminus \{x_0\} \times [0,1] \to \mathbb{R}^2 \setminus \{x_0\}$ as

$$H(z,t) := (1-t)z + t\left(\frac{z - x_0}{|z - x_0|} + x_0\right) \text{ for all } (z,t) \in \mathbb{R}^2 \setminus \{x_0\} \times [0,1].$$

$\square$

Lemma 25 *Define $\mathbb{D}^n := \{z \in \mathbb{R}^n : \|z\| \leq 1\}$. Let $x, y \in \operatorname{int}(\mathbb{D}^n)$. Then, there is a homeomorphism $\varphi\colon \mathbb{D}^n \to \mathbb{D}^n$ such that $\varphi(w) = w$ for $\|w\| = 1$ and $\varphi(x) = y$.*

Proof. Consider the homeomorphism $\psi\colon \operatorname{int}(\mathbb{D}^n) \to \mathbb{R}^n$ given by

$$\psi(z) = \frac{z}{1 - \|z\|}.$$

Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be the translation given by $T(z) = z - \psi(x) + \psi(y)$. Now, we show that $\psi^{-1} \circ T \circ \psi\colon \operatorname{int}(\mathbb{D}^n) \to \operatorname{int}(\mathbb{D}^n)$ can be extended to a homeomorphism $\mathbb{D}^n \to \mathbb{D}^n$.

Note that for $\|z\| < 1$, write $z = rv$ for some $v \in \mathbb{S}^{n-1}$ and some $r \in [0, 1)$. Then, $\psi(z) = \frac{r}{1-r}v$ and for any $R \in [0, \infty)$ and any $w \in \mathbb{S}^{n-1}$ we have $\psi^{-1}(Rw) = \frac{R}{1+R}w$. So, for any $r \in [0, 1)$ and any $v \in \mathbb{S}^{n-1}$ we have

$$\psi^{-1} \circ T \circ \psi(rv) = \frac{\left|\left|\frac{r}{1-r}v - \psi(x) + \psi(y)\right|\right|}{1 + \left|\left|\frac{r}{1-r}v - \psi(x) + \psi(y)\right|\right|} = \frac{||rv - (1-r)\psi(x) + (1-r)\psi(y)||}{(1-r) + ||rv - (1-r)\psi(x) + (1-r)\psi(y)||}.$$

Therefore extension is possible. □

Theorem 26 *Every connected manifold is homogeneous, i.e., for a connected manifold M and any two points $a, b \in M$, there is a homeomorphism $\Phi\colon M \to M$ such that $\Phi(a) = b$.*

Proof. To prove this, consider the non-empty set

$$S := \{x \in M |\text{ there is a homeomorphism } f\colon M \to M \text{ with } f(x) = b\}.$$

Now, consider $y \in S$ with a homeomorphism $g\colon M \to M$ such that $g(y) = b$. Let $\psi\colon U\big(\subseteq_{\text{closed}} M\big) \to \mathbb{D}^n$ be a homeomorphism with $y \in \operatorname{int}(U)$. Now, for any $x \in \operatorname{int}(U)$, choose $\varphi\colon \mathbb{D}^n \to \mathbb{D}^n$ such that $\varphi(\psi(x)) = \psi(y)$ and $\varphi(w) = w$ for $\|w\| = 1$. So, define a homeomorphism $f\colon M \to M$ as

$$f(z) := \begin{cases} g\big(\psi^{-1} \circ \varphi \circ \psi(z)\big) & \text{if } z \in \operatorname{int}(U), \\ g(z) & \text{if } z \in M \setminus \operatorname{int}(U). \end{cases}$$

In other words, $\operatorname{int}(U) \subseteq S$. That is, S is open in M. Similarly, prove that $M \setminus S$ is open. Now, M is connected to imply the result. □

Theorem 27 *Every connected manifold is 2-homogeneous, i.e., given $\{a_1, a_2\} \cup \{b_1, b_2\} \subseteq M$, we have a homeomorphism $\psi\colon M \to M$ such that $\psi(a_k) = b_k$ for each $k = 1, 2$.*

Proof. Let

$$T := \{(x_1, x_2) \in M \times M |\text{ there is a homeomorphism } f\colon M \to M \text{ with } f(x_1) = b_1, f(x_2) = b_2\}.$$

Now, consider $(y_1, y_2) \in T$ with a homeomorphism $g\colon M \to M$ such that $g(y_1) = b, g(y_2) = b_2$. Let $\psi_k\colon U_k\big(\subseteq_{\text{closed}} M\big) \to \mathbb{D}^n$ be a homeomorphism with $y_k \in \operatorname{int}(U_k)$ for $k = 1, 2$ with $U_1 \cap U_2 = \varnothing$. Now, for any $x_k \in \operatorname{int}(U_k)$, choose $\varphi_k\colon \mathbb{D}^n \to \mathbb{D}^n$ such that $\varphi_k(\psi_k(x_k)) = \psi_k(y_k)$ and $\varphi_k(w) = w$ for $\|w\| = 1$ where $k = 1, 2$. So, define a homeomorphism $f\colon M \to M$ as

$$f(z) := \begin{cases} g\big(\psi_1^{-1} \circ \varphi_1 \circ \psi_1(z)\big) & \text{if } z \in \operatorname{int}(U_1), \\ g\big(\psi_2^{-1} \circ \varphi_2 \circ \psi_2(z)\big) & \text{if } z \in \operatorname{int}(U_2), \\ g(z) & \text{if } z \in M \setminus \big(\operatorname{int}(U_1) \cup \operatorname{int}(U_2)\big). \end{cases}$$

In other words, $\operatorname{int}(U_1) \times \operatorname{int}(U_2) \subseteq T$. That is, T is open in $M \times M$. Similarly, $\big(M \times M\big) \setminus T$ is open. Now, $M \times M$ is connected implies the result. □

Remark 28 *Similarly, one can show that every connected manifold is k-homogeneous for each integer $k \geq 1$.*

Theorem 29 *Torus minus a point is homotopy equivalent to figure eight.*

Proof. Note that Torus is the quotient space $\mathbb{T} := \dfrac{[-1,1]\times[-1,1]}{(-1,t)\sim(1,t) \text{ and } (s,-1)\sim(s,1)}$ and figure-eight is the space $\mathbb{S}^1 \vee \mathbb{S}^1 := \mathbb{S}^1 \times \{1\} \cup \{1\} \times \mathbb{S}^1$. Let $q\colon [-1,1]\times[-1,1] \to \mathbb{T}$ be the quotient map.

Note that for any two points $a, b \in \mathbb{T}$ we have a homeomorphism $\varphi\colon \mathbb{T} \to \mathbb{T}$ with $\varphi(a) = b$, see Theorem 26. In other words, $\mathbb{T} \setminus \{a\} \cong \mathbb{T} \setminus \{b\}$. So, without loss of generality, we may remove the point $q(0,0)$ from the Torus to solve this problem.

Now, note that there is a strong deformation retract $H\colon \big([-1,1]^2 \setminus \{(0,0)\}\big) \times I \to [-1,1]^2 \setminus \{(0,0)\}$ of $[-1,1]^2 \setminus \{(0,0)\}$ onto its boundary $\partial[-1,1]^2$ considering the radial projections starting from the origin. That is

$$H(x,y,t) := \left(\frac{t}{\max(|x|,|y|)} + 1 - t\right)(x,y) \text{ for } (x,y) \in [-1,1]^2 \setminus \{(0,0)\} \text{ and } t \in I.$$

Now, the map

$$\big(\mathbb{T} \setminus \{q(0,0)\}\big) \times I \ni \big(q(x,y), t\big) \longmapsto q \circ H(x,y,t) \in \mathbb{T}$$

is a strong deformation retract of $\mathbb{T} \setminus \{q(0,0)\}$ onto $q\big(\partial[-1,1]^2\big) \cong \mathbb{S}^1 \vee \mathbb{S}^1$. □

Theorem 30 *$\mathbb{S}^n$ is a strong deformation retract of $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$.*

Proof. Consider $\big(\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}\big) \times [0,1] \ni (\mathbf{z}, t) \longmapsto (1-t)\mathbf{z} + t\frac{\mathbf{z}}{|\mathbf{z}|} \in \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$. □

Theorem 31 *Let $X := \{x, y\}$ be the two-point Sierpinski space where the only open sets are $X, \varnothing, \{x\}$. Then, there is a strong deformation retract of X onto $\{x\}$.*

Proof. Consider the map $H\colon X \times [0,1] \to X$ defined by

$$H(z,t) := \begin{cases} z & \text{if } t = 0, \\ x & \text{if } t > 0. \end{cases}$$

Now, $H^{-1}\left(\{x\}\right) = \{(x,0)\} \cup X \times (0,1] = \big(\{x\} \times [0,1]\big) \cup \big(\{y\} \times (0,1]\big)$, which is an open subset of $X \times [0,1]$, as it's complement in $X \times [0,1]$ is $\{y\} \times \{0\} \subseteq_{\text{closed}} X \times [0,1]$.

This shows that H is continuous. Also, since $H(-,0) = \mathrm{Id}_X$ and $H(-,1) = c_x$ for all $z \in X$, then this shows that $H\colon \mathrm{Id}_X \simeq c_x$. □

Theorem 32 *If X is Hausdorff, and $r\colon X \to A$, then A is closed in X.*

Proof. If there were $x \in \overline{A} \setminus A$, then because $x \neq r(x)$, and X is Hausdorff, there would exist disjoint nbds $U \supset \{x\}$, and $V \supset \{(r(x)\}$ such that $r(U) \subset V$; however since $x \in \overline{A}$, there must be $a \in A$ in U, and since $a = r(a) \in V$, this contradicts the disjointness of U and V. □

Remark 33 *If X is not Hausdorff, then Theorem 32 may not be true; for example, consider Theorem 31.*

Theorem 34 *Let Y be a subspace of $\mathbb{R}^n$ and let $f, g\colon X \to Y$ be two continuous maps. Prove that if for each $x \in X$, $f(x)$ and $g(x)$ can be joined by a straight-line segment in Y, then $f \simeq g$. Deduce that any two maps $f, g\colon X \to \mathbb{R}^n$ must be homotopic.*

Proof. Consider $X \times [0,1] \ni (x,t) \longmapsto tf(x) + (1-t)g(x) \in Y$. $\square$

Theorem 35 *Let Y be contractible; then any $f, g\colon X \to Y$ are homotopic.*

Proof. Let $H\colon \mathrm{Id}_Y \simeq c_{y_0}$. Then consider,

$$G(x,t) := \begin{cases} H\left(f(x), 2t\right) & \text{if } 0 \le t \le \frac{1}{2}, \\ H\left(g(x), 2t-1\right) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

$\square$

Theorem 36 *Let X be any space and let $f, g\colon X \to \mathbb{S}^n$ be two continuous maps such that $f(x) \neq -g(x)$ for all $x \in \mathbb{S}^n$. Then f is homotopic to g.*

Proof. Consider

$$X \times [0,1] \ni (x,t) \longmapsto \frac{(1-t)f(x) + tg(x)}{|(1-t)f(x) + tg(x)|} \in \mathbb{S}^n.$$

$\square$

Theorem 37 *Any rotation on $\mathbb{S}^n$ is homotopic to the identity map of $\mathbb{S}^n$.*

Proof. Let $A \in \mathrm{SO}(n+1)$, then there is a invertible matrix P such that PAP^{-1} has the form $\left(\frac{n+1}{2}\right)$ many 2×2 matrices of the form $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ along the diagonal, 1 in the last diagonal place if n is even, and 0 elsewhere. Replacing θ by $t\theta$ gives a homotopy $H : \mathrm{Id}_{(n+1)\times(n+1)} \simeq PAP^{-1}$. So, the required homotopy is $P^{-1}HP$. Antipodal map on $\mathbb{S}^n$ homotopic to identity map if n is odd. $\square$

Theorem 38 *There is a deformation retract of $\mathrm{GL}(n, \mathbb{R})$ onto $\mathrm{O}(n)$.*

Proof. Here we show there is a deformation retract of $\mathrm{GL}(2, \mathbb{R})$ onto $\mathrm{O}(2)$. Let $A := [A_1 : A_2] \in \mathrm{GL}(2, \mathbb{R})$. Let $O := [O_1 : O_2]$ be the orthogonal matrix obtained from A by the Gram-Schmidt process. That is

$$O_1 = \frac{A_1}{\|A_1\|}, \; O_2 = \frac{A_2 - \frac{\langle A_2, A_1\rangle}{\|A_1\|^2} A_1}{\left\| A_2 - \frac{\langle A_2, A_1\rangle}{\|A_1\|^2} A_1 \right\|}.$$

That is we can write $O_1 = \lambda_{11} A_1$ and $O_2 = \lambda_{21} A_1 + \lambda_{22} A_2$ with $\lambda_{kk} > 0$ for $k = 1, 2$. So, consider the homotopy, $H\colon \mathrm{GL}(2, \mathbb{R}) \times [0,1] \to \mathrm{GL}(2, \mathbb{R})$ given by

$$H(A,t) := \left[(t\lambda_{11} + 1 - t)A_1 \;:\; t\lambda_{21} A_1 + (t\lambda_{22} + 1 - t)A_2\right].$$

$\square$

Theorem 39 *Let $n > m$ be positive integers. Write $\mathbb{S}^n = \{(z,w) \in \mathbb{R}^{m+1} \times \mathbb{R}^{n-m} : |z|^2 + |w|^2 = 1\}$ and let $\mathbf{S}^m := \{(z,w) \in \mathbb{S}^n : |w| = 0\}$. Then $\mathbb{S}^n \backslash \mathbf{S}^m \cong \mathbb{R}^{m+1} \times \mathbb{S}^{n-m-1}$.*

Proof. Consider

$$\Phi : \mathbb{S}^n \backslash \mathbf{S}^m \ni (z, w) \longmapsto \left(\frac{z}{|w|}, \frac{w}{|w|} \right) \in \mathbb{R}^{m+1} \times \mathbb{S}^{n-m-1}$$

with it's inverse

$$\Psi : \mathbb{R}^{m+1} \times \mathbb{S}^{n-m-1} \ni (a, b) \longmapsto \left(\frac{a}{\sqrt{|a|^2 + |b|^2}}, \frac{b}{\sqrt{|a|^2 + |b|^2}} \right) \in \mathbb{S}^n \backslash \mathbf{S}^m.$$

□

Theorem 40 *Let $n > m$ be positive integers. Let $\mathbf{V}$ be an m-dimensional vector subspace of $\mathbb{R}^n$, and $\mathbf{W}$ be it's complimentary subspace, i.e., $\mathbb{R}^n = \mathbf{V} \oplus \mathbf{W}$. Then, $\mathbb{R}^n \backslash \mathbf{V} \cong \mathbf{V} \times (\mathbf{W} \backslash \mathbf{0})$.*

Proof. Consider the homeomorphism

$$\Phi \colon \mathbb{R}^n \backslash \mathbf{V} \ni v \oplus w \longmapsto (v, w) \in \mathbf{V} \times (\mathbf{W} \backslash \mathbf{0}).$$

□

2 Problems related to fundamental groups

Theorem 41 *Let $(X_\alpha, x_\alpha)_{\alpha \in \mathcal{A}}$ be a collection of pointed topological spaces. Then*

$$\pi_1 \left(\prod_{\alpha \in \mathcal{A}} X_\alpha, \{x_\alpha\}_{\alpha \in \mathcal{A}} \right) \cong \prod_{\alpha \in \mathcal{A}} \pi_1(X_\alpha, x_\alpha).$$

Proof. For $\beta \in \mathcal{A}$, let $p_\beta \colon \prod_{\alpha \in \mathcal{A}} X_\alpha \to X_\beta$ be the projection. Define $\Phi \colon \pi_1 \left(\prod_{\alpha \in \mathcal{A}} X_\alpha, \{x_\alpha\}_{\alpha \in \mathcal{A}} \right) \to \prod_{\alpha \in \mathcal{A}} \pi_1(X_\alpha, x_\alpha)$ as

$$\Phi\big([f]\big) = \{[p_\alpha \circ f]\}_{\alpha \in \mathcal{A}},$$

for any loop $f \colon I \to \prod_{\alpha \in \mathcal{A}} X_\alpha$ based at the point $\{x_\alpha\}_{\alpha \in \mathcal{A}}$. Now, for any two loops $f, g \colon I \to \prod_{\alpha \in \mathcal{A}} X_\alpha$ based at the point $\{x_\alpha\}_{\alpha \in \mathcal{A}}$ we have

$$\begin{aligned} \Phi\big([f] \cdot [g]\big) &= \Phi\big([f * g]\big) = \big\{ \big[p_\alpha \circ (f * g)\big] \big\}_{\alpha \in \mathcal{A}} \\ &= \big\{ \big[(p_\alpha \circ f) * (p_\alpha \circ g)\big] \big\}_{\alpha \in \mathcal{A}} = \{[p_\alpha \circ f] \cdot [p_\alpha \circ g]\}_{\alpha \in \mathcal{A}} = \Phi\big([f]\big) \cdot \Phi\big([g]\big). \end{aligned}$$

So, Φ is a group homomorphism.

Now, $\Phi\big([f]\big) = \{[p_\alpha \circ f]\}_{\alpha \in \mathcal{A}}$ is trivial element implies for each $\alpha \in \mathcal{A}$ we have a path-homotopy $\mathcal{H}_\alpha \colon I \times I \to X_\alpha$ from the loop $p_\alpha \circ f$ to the constant loop c_{x_α} based at x_α. Define $\mathcal{H} \colon I \times I \to \prod_{\alpha \in \mathcal{A}} X_\alpha$ as $\mathcal{H} \coloneqq \prod_{\alpha \in \mathcal{A}} \mathcal{H}_\alpha$. Then $\mathcal{H}$ defines a path-homotopy from the loop $f = \prod_{\alpha \in \mathcal{A}} p_\alpha \circ f$ to the constant loop $\prod_{\alpha \in \mathcal{A}} c_{x_\alpha}$ based at $\{x_\alpha\}_{\alpha \in \mathcal{A}}$. So, Φ is a monomorphism.

Let f_α be a loop in X_α based at x_α for each $\alpha \in \mathcal{A}$. Consider the loop $f \coloneqq \prod_{\alpha \in \mathcal{A}} f_\alpha$ in $\prod_{\alpha \in \mathcal{A}} X_\alpha$ based at $\{x_\alpha\}_{\alpha \in \mathcal{A}}$. Then, $\Phi\big([f]\big) = \{[f_\alpha]\}_{\alpha \in \mathcal{A}}$, i.e., Φ is epimorphism. □

Theorem 42 *Let $\mathcal{C}$ be a circle and $x, y \in \mathcal{C}$ be two distinct points. Let $f_0, f_1 \colon [0, 1] \to \mathcal{C}$ be the paths defined by two distinct arcs of $\mathcal{C}$ starting at x and ending at y. Then f_0 is not homotopic to f_1 relative to $\{x, y\}$.*

Proof. On the contrary, let's assume $f_0 \simeq_{\text{rel }\{x,y\}} f_1$. Then $f_0 * \overline{f_1} \simeq_{\text{rel }\{x\}} f_1 * \overline{f_1}$ by [Kos80, Lemma 14.2] and $f_1 * \overline{f_1} \simeq_{\text{rel }\{x\}} \epsilon_x$ by [Kos80, Lemma 14.4], where ϵ_x is the constant loop based at x. Thus, $f_0 * \overline{f_1}$ is a loop that traverses the circle once and is homotopic to the ϵ_x relative to $\{x\}$, which is impossible by [Kos80, Theorem 16.7.]$\big($actually $f_0 * \overline{f_1}$ is a generator of $\pi_1(\mathcal{C}, x) \cong \mathbb{Z}\big)$. □

Theorem 43 *Prove that the subset $\mathbb{S}^1 \times \{x_0\}$ is a retract of $\mathbb{S}^1 \times \mathbb{S}^1$, but that it is not a strong deformation retract of $\mathbb{S}^1 \times \mathbb{S}^1$ for any point $x_0 \in \mathbb{S}^1$. Is it a deformation retract? Is it a weak deformation retract?*

Proof. Consider the retract

$$r\colon \mathbb{S}^1 \times \mathbb{S}^1 \ni (z, w) \longmapsto (z, x_0) \in \mathbb{S}^1 \times \{x_0\}.$$

Torus and circle are not homotopy equivalent as $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1) \cong \mathbb{Z} \times \mathbb{Z}$ and $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$. Hence, a circle can't be a weak retract of torus. In particular, a circle can not be (strong) deformation retract of a torus.

Here is an alternative way of showing that the circle $\mathbb{S}^1 \times \{x_0\}$ is not deformation retract of $\mathbb{S}^1 \times \mathbb{S}^1$ for any $x_0 \in \mathbb{S}^1$. On the contrary, let $r\colon \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \{x_0\}$ be a retraction and $H\colon \mathbb{S}^1 \times \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1 \times \mathbb{S}^1$ be a homotopy such that $i \circ r \simeq \mathrm{Id}_{\mathbb{S}^1 \times \mathbb{S}^1}$, where $i\colon \mathbb{S}^1 \times \{x_0\} \hookrightarrow \mathbb{S}^1 \times \mathbb{S}^1$ is the inclusion. Let $j\colon \{x_0\} \times \mathbb{S}^1 \hookrightarrow \mathbb{S}^1 \times \mathbb{S}^1$ be the inclusion map and $p_2\colon \mathbb{S}^1 \times \mathbb{S}^1 \ni (z, w) \longmapsto (x_0, w) \in \{x_0\} \times \mathbb{S}^1$. Now, $i \circ r \simeq \mathrm{Id}_{\mathbb{S}^1 \times \mathbb{S}^1} \implies p_2 \circ i \circ r \circ j \simeq p_2 \circ \mathrm{Id}_{\mathbb{S}^1 \times \mathbb{S}^1} \circ j = p_2 \circ j$. But, $p_2 \circ i$ is a constant map and $p_2 \circ j$ is the identity map of $\{x_0\} \times \mathbb{S}^1$. Thus, identity map of $\{x_0\} \times \mathbb{S}^1$ is null-homotopic, i.e., $\mathrm{Id}_{\mathbb{S}^1}$ is also null-homotopic. By Theorem 1, we have a retraction $\mathbb{D}^2 \to \mathbb{S}^1$, which is impossible, see [Cha03, Theorem 1] for a purely point set topological proof of no retraction theorem. □

Theorem 44 *Let X be a path-connected space having an abelian fundamental group. Let $x_1, x_2 \in X$. Now, for any two paths α, β from x_1 to x_2 we have $\alpha_\# = \beta_\# : \pi_1(X, x_1) \to \pi_1(X, x_2)$.*

Proof. To prove this, let $[f] \in \pi_1(X, x_1)$ then,

$$\alpha_\#\big([f]\big) = \big[\overline{\alpha} * f * \alpha\big] = \big[\overline{\alpha} * f * \beta * \overline{\beta} * \alpha\big] = \big[\overline{\alpha} * f * \beta\big]\big[\overline{\beta} * \alpha\big]$$

$$= \big[\overline{\beta} * \alpha\big]\big[\overline{\alpha} * f * \beta\big] = \big[\overline{\beta} * \alpha * \overline{\alpha} * f * \beta\big] = \big[\overline{\beta} * f * \beta\big] = \beta_\#\big([f]\big).$$

□

Theorem 45 *Let X be a path-connected space. Suppose for any two points $x_1, x_2 \in X$ and any two paths α, β in X from x_1 to x_2 we have $\alpha_\# = \beta_\#$. Then, the fundamental group of X is abelian.*

Proof. Let $[f] \in \pi_1(X, x_1)$ and α be a path from x_1 to x_2. Define $\beta := f * \alpha$. Now, for any $[g] \in \pi_1(X, x_1)$ we have $\beta_\#\big([g]\big) = \alpha_\#\big([g]\big)$ from the hypothesis. In other words,

$$\beta_\#\big([g]\big) = \big[\overline{\beta} * g * \beta\big] = \big[\overline{f * \alpha} * g * f * \alpha\big] = \big[\overline{\alpha} * \overline{f} * g * f * \alpha\big] \text{ is same as } \alpha_\#\big([g]\big) = \big[\overline{\alpha} * g * \alpha\big].$$

$$\implies [f]^{-1}[g][f] = \big[\overline{f} * g * f\big] = \overline{\alpha}_\#\big(\big[\overline{\alpha} * \overline{f} * g * f * \alpha\big]\big) = \overline{\alpha}_\#\big(\big[\overline{\alpha} * g * \alpha\big]\big) = [g].$$

□

Theorem 46 *All paths with the same endpoints are homotopic in a simply connected space X.*

Proof. Let α, β be two paths starting at $x \in X$ and end at $y \in X$. Then,

$$\alpha \simeq_{\text{rel }\{0,1\}} \alpha * c_y \simeq_{\text{rel }\{0,1\}} \alpha * (\overline{\beta} * \beta) \simeq_{\text{rel }\{0,1\}} (\alpha * \overline{\beta}) * \beta \simeq_{\text{rel }\{0,1\}} c_x * \beta \simeq_{\text{rel }\{0,1\}} \beta.$$

So, we are done. $\square$

Definition 47 *A pointed space (X, x_0) is called an $\mathbf{H}$-space if there is a map $m\colon X \times X \to X$ with $m(x_0, x_0) = x_0$ such that we have two homotopies*

$$\mathscr{H}_1 : m(x_0, -) \simeq_{rel\, x_0} \mathrm{Id}_X \text{ and } \mathscr{H}_2 : m(-, x_0) \simeq_{rel\, x_0} \mathrm{Id}_X.$$

Theorem 48 *The fundamental group of an $\mathbf{H}$-space (X, x_0) is abelian.*

Proof. Let α, β be any two loops based at x_0 and c_{x_0} be the constant loop based at x_0. Here, $m(\alpha, \beta)$ is a loop based at x_0 defined as $m(\alpha, \beta)(s) := m\big(\alpha(s), \beta(s)\big)$ for $0 \leq s \leq 1$. Similarly, $m(x_0, \beta)$ is a loop in X based at x_0 defined as $m(x_0, \beta)(s) := m\big(x_0, \beta(s)\big)$ for all $0 \leq s \leq 1$.

Notice that $\beta \simeq_{\text{rel }x_0} m(x_0, \beta)$. To prove this consider

$$\mathscr{F} : [0,1] \times [0,1] \ni (s,t) \longmapsto \mathscr{H}_1\big(\beta(s), 1-t\big) \in X.$$

Then $\mathscr{F}(-,0) = \mathscr{H}_1\big(\beta(-), 1\big) = \beta(-)$ and $\mathscr{F}(-,1) = \mathscr{H}_1\big(\beta(-), 0\big) = m\big(x_0, \beta(-)\big)$. Thus $\mathscr{F} : \beta \simeq_{\text{rel }x_0} m(x_0, \beta)$.

Similarly, $\alpha \simeq_{\text{rel }x_0} m(\alpha, x_0)$.

Now, if $\mathscr{G}_\alpha\colon c_{x_0} * \alpha \simeq_{\text{rel }x_0} \alpha$ and $\mathscr{G}_\beta\colon \beta * c_{x_0} \simeq_{\text{rel }x_0} \beta$, then

$$[0,1] \times [0,1] \ni (s,t) \longmapsto m\big(\mathscr{G}_\alpha(s,t), \mathscr{G}_\beta(s,t)\big) \in X$$

is a homotopy relative to $\{x_0\}$ from $m(c_{x_0} * \alpha, \beta * c_{x_0})$ to $m(\alpha, \beta)$.

Thus

$$\beta * \alpha \simeq_{\text{rel }x_0} m(x_0, \beta) * m(\alpha, x_0) = m(c_{x_0} * \alpha, \beta * c_{x_0}) \simeq_{\text{rel }x_0} m(\alpha, \beta).$$

Similarly,

$$m(\alpha, \beta) \simeq_{\text{rel }x_0} m(\alpha * c_{x_0}, c_{x_0} * \beta) = m(\alpha, x_0) * m(x_0, \beta) \simeq_{\text{rel }x_0} \alpha * \beta.$$

So, we are done. $\square$

Lemma 49 [Eckmann-Hilton Argument] Let X be a set with two binary operations, which we will write $\circ$ and $\otimes$, and suppose

1. there are elements $1_\circ, 1_\otimes \in X$ such that $1_\circ \circ a = a = a \circ 1_\circ$ and $1_\otimes \otimes a = a = a \otimes 1_\otimes$ for all $a \in X$.
2. $(a \otimes b) \circ (c \otimes d) = (a \circ c) \otimes (b \circ d)$ for all $a, b, c, d \in X$.

Then $\circ$ and $\otimes$ are the same and, in fact, commutative and associative.

Proof. First, observe that the units of the two operations coincide: $1_\circ = 1_\circ \circ 1_\circ = (1_\otimes \otimes 1_\circ) \circ (1_\circ \otimes 1_\otimes) = (1_\otimes \circ 1_\circ) \otimes (1_\circ \circ 1_\otimes) = 1_\otimes \otimes 1_\otimes = 1_\otimes$.
Now, let $a, b \in X$. Then, $a \circ b = (1 \otimes a) \circ (b \otimes 1) = (1 \circ b) \otimes (a \circ 1) = b \otimes a = (b \circ 1) \otimes (1 \circ a) = (b \otimes 1) \circ (1 \otimes a) = b \circ a$. This establishes that the two operations coincide and are commutative.
Now, For associativity, let $a, b, c \in X$. Then $(a \otimes b) \otimes c = (a \otimes b) \otimes (1 \otimes c) = (a \otimes 1) \otimes (b \otimes c) = a \otimes (b \otimes c)$.
So, we are done. $\square$

Theorem 50 *Let $(G, \bullet)$ be a topological group with identity element e. Then, $\pi_1(G, e)$ is abelian.*

Proof. To prove this, we apply the Eckmann-Hilton argument on the set $\big[(I, \partial I), (G, e)\big]$ of all relative-homotopy classes of loops based at e with two binary operations. So, for two loops α, β of G based at e consider two operations $\circ$ and $\otimes$ defined as follows:

$$\big(\alpha \circ \beta\big)(t) := \alpha(t)\bullet\beta(t) \text{ for } t \in [0,1] \text{ and } \big(\alpha \otimes \beta\big)(t) := \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \beta(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Now, these operations induce two operations on $\big[(I, \partial I), (G, e)\big]$. Note that the second operation gives the fundamental group based at e. $\square$

Theorem 51 *Give an example of an injective (surjective) continuous map $\varphi \colon X \to Y$ for which φ_* is not injective (surjective).*

Proof. Consider $\mathbb{S}^1 \hookrightarrow \mathbb{D}$ and $[0,1] \ni t \longmapsto e^{2\pi i t} \in \mathbb{S}^1$. $\square$

Theorem 52 [Hat02, Exercise 13 Chapter 1.1] *Given a space X, a path connected subspace A and $a_0 \in A$, show that the map $i_* \colon \pi_1(A, a_0) \to \pi_1(X, a_0)$ induced by the inclusion $i \colon A \hookrightarrow X$ is surjective if and only if every path in X with end points in A is homotopic to a path in A.*

Proof. Suppose, the inclusion induced map $i_* \colon \pi_1(A, a_0) \to \pi_1(X, a_0)$ is surjective and $\alpha \colon I \to X$ be a path with $\alpha(0), \alpha(1) \in A$. Since, A is path-connected we have a path $\beta \colon I \to A$ from $\alpha(1)$ to $\alpha(0)$. Therefore, $\alpha * \beta$ is a loop in X based at $\alpha(0)$. Since, $\pi_1(A, a_0) \cong \pi_1\big(A, \alpha(0)\big)$ and $\pi_1(X, a_0) \cong \pi_1\big(X, \alpha(0)\big)$, the inclusion induced map $\pi_1\big(A, \alpha(0)\big) \to \pi_1\big(X, \alpha(0)\big)$ is also surjective. So, there is a loop $\gamma \colon I \to A$ based at $\alpha(0)$ such that $\gamma \simeq_{\text{rel }\{0,1\}} \alpha * \beta$, this implies $\gamma * \overline{\beta} \simeq_{\text{rel }\{0,1\}} \alpha$. $\square$

Theorem 53 [Hat02, Exercise 7 Chapter 1.1] *Let* $f \colon \mathbb{S}^1 \times [0,1] \ni \big(e^{2\pi i\theta}, s\big) \longmapsto \big(e^{2\pi i(\theta+s)}, s\big) \in \mathbb{S}^1 \times [0,1]$. *Then f is homotopic to the identity by a homotopy that is stationary on one boundary circle but not by any homotopy that is stationary on both boundary circles.*

Proof. For the first part, consider

$$H \colon \mathbb{S}^1 \times [0,1] \times [0,1] \to \mathbb{S}^1 \times [0,1] \text{ given by} \big(e^{2\pi i\theta}, s, t\big) \longmapsto \big(e^{2\pi i(\theta+ts)}, s\big) \text{ for all } \theta \in \mathbb{R}.$$

Suppose there is a homotopy $F \colon \mathbb{S}^1 \times [0,1] \times [0,1] \to \mathbb{S}^1 \times [0,1]$ such that $F(-,-,0) = \mathrm{Id}_{\mathbb{S}^1 \times [0,1]}$ and $F(-,-,1) = f$ with

$$F\big(e^{2\pi i\theta}, 0, t\big) = \big(e^{2\pi i\theta}, 0\big) \text{ and } F\big(e^{2\pi i\theta}, 1, t\big) = \big(e^{2\pi i\theta}, 1\big) \text{ for all } \theta \in \mathbb{R}, \text{ and for all } t \in [0,1].$$

Consider $\alpha \colon [0,1] \to \mathbb{S}^1 \times [0,1]$ be given by $\alpha(s) := (1, s)$ for $0 \leq s \leq 1$. Let $\Pi \colon \mathbb{S}^1 \times [0,1] \ni (z, s) \longmapsto z \in \mathbb{S}^1$ be the projection map.

Now, $\Pi \circ \alpha \equiv 1$, and $\Pi \circ f \circ \alpha(s) = \Pi \circ f(1, s) = \Pi\big(e^{2\pi i s}, s\big) = e^{2\pi i s}$ for $0 \leq s \leq 1$, i.e., $\Pi \circ f \circ \alpha \colon [0,1] \to \mathbb{S}^1$ be a generator of the fundamental group of $\mathbb{S}^1$. But, we have a homotopy $G \colon [0,1]^2 \to \mathbb{S}^1$ given by $G(s,t) := \Pi \circ F\big(\alpha(s), t\big)$ such that $G : 1 \simeq_{\text{rel }\{0,1\}} \Pi \circ f \circ \alpha$, contradiction. $\square$

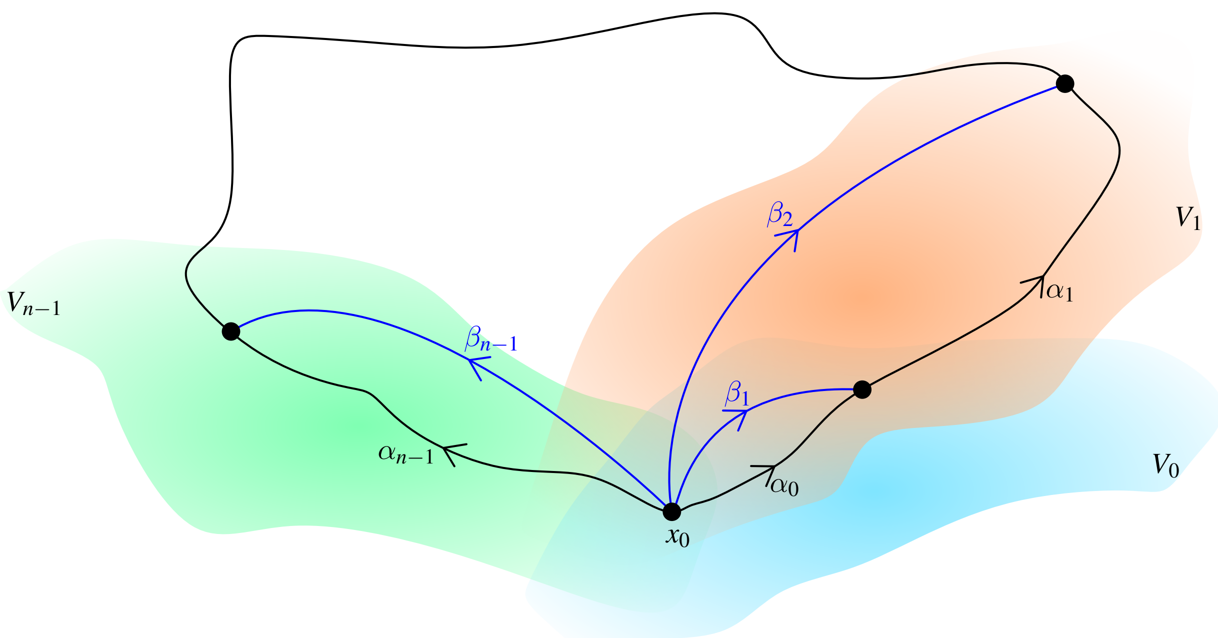
Theorem 54 *Let $\mathscr{U} = \{V_i : i \in \Lambda\}$ be an open covering of X by simply-connected open subsets V_i of X. Suppose, $\bigcap_{i \in \Lambda} V_i \neq \varnothing$ and for each $i, j \in \Lambda$ the space $V_i \cap V_j$ is path-connected. Then, X is simply connected.*

Proof. Since X is path-connected, it is enough to show that $\pi_1(X, x_0) = 0$ for some $x_0 \in \bigcap_{i\in\Lambda} V_i$. For a loop $\alpha\colon \big([0,1], \{0,1\}\big) \to (X, x_0)$, considering the cover $\{\alpha^{-1}(V_i) : i \in \Lambda\}$ of $[0,1]$, we have a partition $0 = t_0 < \ldots < t_n = 1$ of $[0,1]$ such that $\alpha\big([t_j, t_{j+1}]\big) \subseteq V_j \in \mathscr{U}$ for $j = 0, \ldots, (n-1)$. Define $\alpha_j(s) := \alpha\big((1-s)t_j + st_{j+1}\big)$ for $0 \le s \le 1$ and $j = 0, \ldots, (n-1)$. Then,

$$\alpha \underset{\text{rel } x_0}{\simeq} \alpha_0 * \alpha_1 * \alpha_2 * \cdots * \alpha_{n-1}$$

$$\underset{\text{rel } x_0}{\simeq} \alpha_0 * \big(\overline{\beta}_1 * \beta_1\big) * \alpha_1 * \big(\overline{\beta}_2 * \beta_2\big) * \alpha_2 * \cdots * \big(\overline{\beta}_{n-1} * \beta_{n-1}\big) * \alpha_{n-1}$$

$$\underset{\text{rel } x_0}{\simeq} \big(\alpha_0 * \overline{\beta}_1\big) * \big(\beta_1 * \alpha_1 * \overline{\beta}_2\big) * \big(\beta_2 * \alpha_2 * \overline{\beta}_3\big) * \cdots * \big(\beta_{n-2} * \alpha_{n-2} * \overline{\beta}_{n-1}\big) * (\beta_{n-1} * \alpha_{n-1}).$$



Here, $\beta_1\colon [0,1] \to V_0 \cap V_1$ is a path from x_0 to $\alpha_0(1) = \alpha(t_1) = \alpha_1(0)$, hence $\alpha_0 * \overline{\beta}_1$ is a loop based at x_0 in the simply-connected space V_0.
$\beta_2\colon [0,1] \to V_1 \cap V_2$ is a path from x_0 to $\alpha_1(1) = \alpha(t_2) = \alpha_2(0)$, hence $\beta_1 * \alpha_1 * \overline{\beta}_2$ is a loop based at x_0 in the simply-connected space V_1.
$\beta_3\colon [0,1] \to V_2 \cap V_3$ is a path from x_0 to $\alpha_2(1) = \alpha(t_3) = \alpha_3(0)$, hence $\beta_2 * \alpha_2 * \overline{\beta}_3$ is a loop based at x_0 in the simply-connected space V_2.

$$\vdots$$

$\beta_{n-1}\colon [0,1] \to V_{n-1} \cap V_n$ is a path from x_0 to $\alpha_{n-2}(1) = \alpha(t_{n-1}) = \alpha_{n-1}(0)$, hence $\beta_{n-1} * \alpha_{n-1}$ is a loop based at x_0 in the simply-connected space V_{n-1}. Therefore,

$$\alpha \underset{\text{rel } x_0}{\simeq} \big(\alpha_0 * \overline{\beta}_1\big) * \big(\beta_1 * \alpha_1 * \overline{\beta}_2\big) * \big(\beta_2 * \alpha_2 * \overline{\beta}_3\big) * \cdots * \big(\beta_{n-2} * \alpha_{n-2} * \overline{\beta}_{n-1}\big) * (\beta_{n-1} * \alpha_{n-1})$$

$$\underset{\text{rel } x_0}{\simeq} c_{x_0} * c_{x_0} * c_{x_0} * \cdots c_{x_0} * c_{x_0} = c_{x_0}$$

$\square$

Definition 55 *For a topological space X, the cone CX is defined as*

$$CX := \frac{X \times [0,1]}{(x,0) \sim (x',0)}$$

Theorem 56 *For a topological space X, the cone CX is contractible.*

Proof. Consider $H\colon CX \times [0,1] \ni ([x,t],s) \longmapsto [x, t(1-s)] \in CX$. $\square$

Theorem 57 *The cone $C\mathbb{S}^n$ is homeomorphic to $\mathbb{D}^{n+1}$.*

Proof. Consider the surjective map $g\colon \mathbb{S}^n \times [0,1] \ni (x,t) \longmapsto tx \in \mathbb{D}^{n+1}$, it sends $\mathbb{S}^n \times 0$ to $0 \in \mathbb{D}^{n+1}$. So, we have a continuous bijective map $f\colon C\mathbb{S}^n \to \mathbb{D}^{n+1}$. Since, $C\mathbb{S}^n$ is compact and $\mathbb{D}^{n+1}$ is Hausdorff, f is a homeomorphism. $\square$

Theorem 58 *A map $f\colon X \to Y$ is null-homotopic if and only if it can be extended to a map $\widetilde{f}\colon CX \to Y$.*

Proof. To prove only if direction, let $H\colon f \simeq c_y$ for some $y \in Y$. Then consider

$$\begin{tikzcd} X \times [0,1] \arrow[d, "q"'] \arrow[dr, "H"] & \\ CX = \frac{X\times[0,1]}{(x,1)\sim(x',1)} \arrow[r, dashed, "\widetilde{f}"'] & Y \end{tikzcd}$$

Now, to prove if direction notice that the composition $i\colon X \ni x \longmapsto [x,1] \in CX$ and $\widetilde{f}\colon CX \to Y$ is f, and CX is contractible. $\square$

Definition 59 *The suspension ΣX of a space X is defined as*

$$\Sigma X := \frac{X \times [0,1]}{(x,0) \sim (x',0) \text{ and } (y,1) \sim (y',1)}.$$

Theorem 60 $\Sigma\mathbb{S}^n \cong \mathbb{S}^{n+1}$.

Proof. Consider the map $g\colon \mathbb{S}^n \times [0,1] \to \mathbb{S}^{n+1}$ defined by $g(x,t) := (x\sin\pi t, \cos\pi t)$ to show $\Sigma\mathbb{S}^n$ is homeomorphic to $\mathbb{S}^{n+1}$. $\square$

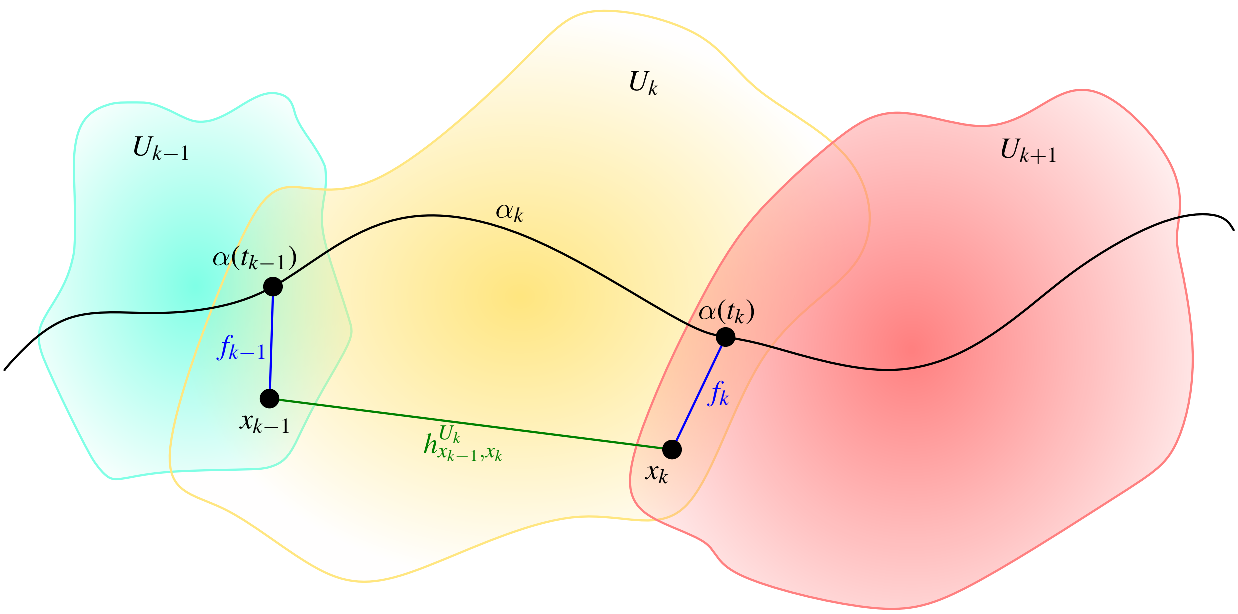
Remark 61 *If X is path-connected, then by Theorem 54, ΣX is simply connected as the cone over any space is contractible. In particular, $\mathbb{S}^n, n \geq 2$ is simply-connected. Note that $\mathbb{S}^1 = \Sigma\{\pm 1\}$ is not simply-connected.*

Theorem 62 [Hat02, Exercise 2 Chapter 1.2] *Let X be the union of convex open sets $X_1, ..., X_n$ such that $X_i \cap X_j \cap X_k \neq \varnothing$ for all i, j, k. Then, X is simply connected.*

Proof. For $n = 2$, consider the Theorem 54. Now, note that for every $1 \leq m \leq n-1$, the space $\big(X_1 \cup ... \cup X_m\big) \cap X_{m+1} = (X_1 \cap X_{m+1}) \cup ... \cup (X_m \cap X_{m+1})$ is path connected. So, we are done by induction. $\square$

Theorem 63 [Lee11, Theorem 7.21.] *The fundamental group of a topological manifold M is at most countable.*

Proof. Consider a cover $\mathscr{U}$ of M by countable many open sets, each of which is homeomorphic to $\mathbb{R}^n$. Now, the intersection of any two such open sets has at most countably many components, so picking up a point from each component of each intersection, we have an at most countable set $\mathscr{C}$. Next, for any such open set $U \in \mathscr{U}$ and $x, x' \in \mathscr{C} \cap U$, consider a path $h^U_{x,x'}$ in U from x to x'. Fix, $p \in \mathscr{C}$. There are at most countably many loops based at p which are finite concatenation of paths of the form $h^U_{x,x'}$.
Next, let α be any loop based at p. By Lebesgue Number Lemma we have a partition $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ of $[0,1]$ such that each $\alpha_k := \alpha|[t_{k-1}, t_k]$ has image contained in one of the element U_k of $\mathscr{U}$.



Find a point $x_k \in \mathscr{C}$ such that $\alpha(t_k)$ and x_k lie in same component of $U_k \cap U_{k+1}$ and choose a path f_k from x_k to $\alpha(t_k)$. We also take, $x_k = p$ for $k = 0, n$ and f_k to be constant path based at x_k for $k = 0, n$. Now,

$$\alpha \simeq_{\text{rel } p} \alpha_1 * \alpha_2 * \ldots * \alpha_{n-1} * \alpha_n \simeq_{\text{rel } p} \left(f_0 * \alpha_1 * \bar{f}_1\right) * \left(f_1 * \alpha_2 * \bar{f}_2\right) * \ldots * \left(f_{n-2} * \alpha_{n-1} * \bar{f}_{n-1}\right) * \left(f_{n-1} * \alpha_n * \bar{f}_n\right)$$

$$\simeq_{\text{rel } p} h^{U_1}_{x_0,x_1} * h^{U_2}_{x_1,x_2} * \ldots * h^{U_{n-1}}_{x_{n-2},x_{n-1}} * h^{U_n}_{x_{n-1},x_n} \text{ as each } U_k \text{ is simply-connected.}$$

So, we are done.

□

Theorem 64 *Consider the action of $\mathbb{Z}$ on $\mathbb{R}^m \backslash \mathbf{0}$ given by $n \cdot x := 2^n x$. Then, $(\mathbb{R}^m \backslash \mathbf{0})/\mathbb{Z} \cong \mathbb{S}^{m-1} \times \mathbb{S}^1$.*

Proof. Define $f \colon (\mathbb{R}^m \backslash \mathbf{0})/\mathbb{Z} \to \mathbb{S}^{m-1} \times \mathbb{S}^1$ by

$$[x] \longmapsto \left(\frac{x}{|x|}, \exp\left(2\pi i \log_2 |x|\right)\right)$$

with inverse $g \colon \mathbb{S}^{m-1} \times \mathbb{S}^1 \to (\mathbb{R}^m \backslash \mathbf{0})/\mathbb{Z}$ given by

$$g(z, e^{2\pi i t}) := \begin{cases} \left[2^t z\right] & \text{if } 0 \leq t < 1, \\ [z] & \text{if } t = 1. \end{cases}$$

□

Corollary 65 *If $m \geq 3$, then $\mathbb{R}^m \backslash \mathbf{0}$ is simply-connected, hence homeomorphic to the universal cover of $\mathbb{S}^{m-1} \times \mathbb{S}^1$.*

Definition 66 *Let X be a space. Two maps $f_0, f_1 \colon \mathbb{S}^1 \to X$ are said to be freely homotopic if there is a map $H \colon \mathbb{S}^1 \times [0,1] \to X$ such that $H(-,0) = f_0$ and $H(-,1) = f_1$. Note that being freely homotopic is an equivalence relation on the set of all maps from $\mathbb{S}^1$ to X. The set of all equivalence classes will be denoted by $[\mathbb{S}^1, X]$.*

Lemma 67 *Let X be a path-connected space and $x_0 \in X$. Consider a map $\alpha \colon \mathbb{S}^1 \to X$. Then there is a map $\beta \colon (\mathbb{S}^1, 1) \to (X, x_0)$ such that α, β are freely homotopic.*

Proof. Consider a path $f \colon [0,1] \to X$ with $f(0) = x_0$ and $f(1) = \alpha(1)$. Define $\mathcal{H} \colon \mathbb{S}^1 \times [0,1] \to X$ as follows:

$$\mathcal{H}(e^{2\pi i s}, t) := \begin{cases} f(t+3s) & \text{if } 0 \leq s \leq \frac{1-t}{3}, \\ \alpha \circ \exp\left(2\pi i \cdot \dfrac{3}{1+2t}\left(s - \dfrac{1-t}{3}\right)\right) & \text{if } \frac{1-t}{3} \leq s \leq \frac{2+t}{3}, \\ f\left(1 - 3\left(s - \dfrac{2+t}{3}\right)\right) & \text{if } \frac{2+t}{3} \leq s \leq 1. \end{cases}$$

Notice that $\mathcal{H}$ is well-defined continuous map. Note that as defined, $e^{2\pi i s} = 1$ if and only if $s = 0, 1$. Now, define $\beta := \mathcal{H}(-,0)$. Also, $\mathcal{H}(-,1) = \alpha$. So, we are done. □

Lemma 68 *Let X be a space and $x_0 \in X$. Suppose $[\beta] = [\gamma] \cdot [\alpha] \cdot [\gamma]^{-1}$ in $\pi_1(X, x_0)$. Then α, β are freely homotopic.*

Proof. Define $\mathcal{H} \colon \mathbb{S}^1 \times [0,1] \to X$ as follows:

$$\mathcal{H}(e^{2\pi i s}, t) := \begin{cases} \gamma \circ \exp\big(2\pi i \cdot (t+3s)\big) & \text{if } 0 \leq s \leq \frac{1-t}{3}, \\ \alpha \circ \exp\left(2\pi i \cdot \dfrac{3}{1+2t}\left(s - \dfrac{1-t}{3}\right)\right) & \text{if } \frac{1-t}{3} \leq s \leq \frac{2+t}{3}, \\ \gamma \circ \exp\left(2\pi i \cdot \left(1 - 3\left(s - \dfrac{2+t}{3}\right)\right)\right) & \text{if } \frac{2+t}{3} \leq s \leq 1. \end{cases}$$

Notice that $\mathcal{H}$ is well-defined continuous map. Note that as defined, $e^{2\pi i s} = 1$ if and only if $s = 0, 1$. Now, $\beta = \mathcal{H}(-,0)$ and $\mathcal{H}(-,1) = \alpha$. So, we are done. □

Lemma 69 *Let X be a space and x_0. Suppose $\alpha, \beta \colon (\mathbb{S}^1, 1) \to (X, x_0)$ are two freely homotopic maps. Then, there is map $\gamma \colon (\mathbb{S}^1, 1) \to (X, x_0)$ such that*

$$[\beta] = [\gamma] \cdot [\alpha] \cdot [\gamma]^{-1} \text{ in } \pi_1(X, x_0).$$

Proof. Consider a free homotopy $\mathcal{H} \colon \mathbb{S}^1 \times [0,1] \to X$ from $\beta = \mathcal{H}(-,0)$ and $\alpha = \mathcal{H}(-,1)$. Notice that $\mathcal{H}(\mathbf{1}, 0) = \beta(\mathbf{1}) = x_0 = \alpha(\mathbf{1}) = \mathcal{H}(\mathbf{1}, 1)$. Define a map $\gamma \colon (\mathbb{S}^1, 1) \to (X, x_0)$ as

$$\gamma(e^{2\pi i t}) := \mathcal{H}(1, t) \text{ for } 0 \leq s \leq 1.$$

Now, consider the map $\overline{\mathcal{H}} \colon \mathbb{S}^1 \times [0,1] \to X$ defined by

$$\overline{\mathcal{H}}(e^{2\pi i s}, t) := \begin{cases} \mathcal{H}(\mathbf{1}, 3s) & \text{if } 0 \leq s \leq \frac{t}{3}, \\ \mathcal{H}\left(\exp\left(2\pi i \cdot \dfrac{3s-t}{3-2t}\right), t\right) & \text{if } \frac{t}{3} \leq s \leq \frac{3-t}{3}, \\ \mathcal{H}\big(\mathbf{1}, 3-3s\big) & \text{if } \frac{3-t}{3} \leq s \leq 1. \end{cases}$$

Notice that $\overline{\mathcal{H}}$ is a well-defined continuous map with $\overline{\mathcal{H}}(-,0) = \beta$ and $\overline{\mathcal{H}}(-,1) = \gamma * \alpha * \overline{\gamma}$ such that $\overline{\mathcal{H}}(\mathbf{1}, t) = x_0$ for all $t \in [0,1]$. $\square$

Theorem 70 [Hat02, Exercise 6 Chapter 1.1] *Let X be a path-connected space and $x_0 \in X$. Define conjugacy equivalence relation on $\pi_1(X, x_0)$ as follows: Two elements $\alpha, \beta \in \pi_1(X, x_0)$ are said to be equivalent if and only if there is an element $\gamma \in \pi_1(X, x_0)$ such that $\alpha = \gamma\beta\gamma^{-1}$. We will denote this conjugacy equivalence relation by $\sim$. Then there is a bijection*

$$\frac{\pi_1(X, x_0)}{\sim} \longrightarrow [\mathbb{S}^1, X]$$

Proof. Define $\Phi \colon \pi_1(X, x_0) \to [\mathbb{S}^1, X]$ as

$$\Phi\big([f]\big) = \operatorname{cls}(f) \text{ for } [f] \in \pi_1(X, x_0).$$

That is, for a map $f \colon (\mathbb{S}^1, 1) \to (X, x_0)$ we are sending the loop-homotopy class $[f] \in \pi_1(X, x_0)$ of f to the free-homotopy class of f, i.e., we are ignoring the base-points to define Φ. Clearly, Φ is well-defined.

Now, Lemma 67 says that Φ is surjective. Also, Lemma 68 says that Φ induces a map $\frac{\pi_1(X, x_0)}{\sim} \longrightarrow [\mathbb{S}^1, X]$. Finally, Lemma 69 gives that this induced map is an injection. $\square$

Theorem 71 *Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Consider the homeomorphism $\varphi \colon \mathbb{C}^* \ni z \longmapsto -\overline{z} \in \mathbb{C}^*$. Then the orbit space $\mathbb{C}^*/\{\varphi, \mathrm{Id}\} \cong \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}$.*

Proof. Consider the map $f \colon \mathbb{C}^*/\{\varphi, \mathrm{Id}\} \longrightarrow \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}$ defined by

$$f\big([z]\big) := \begin{cases} z & \text{if } \mathrm{Re}(z) \geq 0, \\ -\overline{z} & \text{otherwise.} \end{cases}$$

with inverse

$$g \colon \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\} \ni z \longmapsto [z] \in \mathbb{C}^*/\{\varphi, \mathrm{Id}\}$$

$\square$

Theorem 72 *Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Consider the homeomorphism $\varphi \colon \mathbb{C}^* \ni z \longmapsto 2\overline{z} \in \mathbb{C}^*$. Then the orbit space $\mathbb{C}^*/\langle\varphi\rangle$ is Klein bottle.*

Proof. Notice that $\mathbb{C}^* \ni z \longmapsto \left(\frac{z}{|z|}, |z|\right) \in \mathbb{S}^1 \times (0, \infty)$ is a homeomorphism, and φ is equivalent to $\Phi \colon \mathbb{S}^1 \times (0,\infty) \ni (e^{i\theta}, r) \longmapsto (e^{-i\theta}, 2r) \in \mathbb{S}^1 \times (0, \infty)$. Now,

$$\frac{\mathbb{C}^*}{\langle\varphi\rangle} \cong \frac{\mathbb{S}^1 \times (0,\infty)}{\langle\Phi\rangle} \cong \frac{\mathbb{S}^1 \times [1,2]}{(z,1) \sim (\overline{z}, 2)}$$

is the Klein bottle. $\square$

3 Problems related to covering spaces

Theorem 73 *The exponential map* $\exp\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$ *is a covering map.*

Proof. Consider the homeomorphism $\varphi\colon \mathbb{R}^2\setminus\{0\} \ni (r\cos\theta, r\sin\theta) \longmapsto (e^r, e^{i\theta}) \in (0,\infty)\times\mathbb{S}^1$. Now, the covering map is the following composition

$$\mathbb{C} = \mathbb{R}^2 \ni (x,y) \xrightarrow{\text{homeomorphism} \times \text{covering map}} (e^x, e^{iy}) \xrightarrow{\varphi^{-1}} \exp(x+iy).$$

Note that homeomorphism is a covering map, and the product of any two covering maps is a covering map. $\square$

Theorem 74 (Uniformization Theorem) *Any simply-connected surface is homeomorphic to either* $\mathbb{S}^2$ *or* $\mathbb{R}^2$*. That is, the universal cover of any surface is either* $\mathbb{S}^2$ *or* $\mathbb{R}^2$*.*

Remark 75 *The only surfaces covered by* $\mathbb{S}^2$ *are* $\mathbb{S}^2$ *and* $\mathbb{R}P^2$*. Therefore, any surface other than* $\mathbb{S}^2, \mathbb{R}P^2$*; is a* $K(G,1)$ *space for some group* G*.*

Theorem 76 [GH81, Theorem 22.14] *Any connected non-orientable manifold has a connected orientable two-fold cover. In other words, the fundamental group of a connected non-orientable manifold has an index two subgroup.*

Remark 77 Below are some illustrations of Theorem 76.

- Covering $\mathbb{S}^{2n} \ni x \longmapsto \{x, -x\} \in \mathbb{R}P^{2n}$. Note that $\mathbb{R}P^m$ is orientable if and only if m is even.
- Covering from annulus to Möbius strip
$$\frac{\mathbb{R}\times[-1,1]}{(x,y)\sim(x+1,y)} \ni [x,y] \longmapsto [x,y] \in \frac{\mathbb{R}\times[-1,1]}{(x,y)\sim\left(x+\frac{1}{2},-y\right)}.$$
- Covering from Torus to Klein bottle
$$\left(e^{2\pi is}, e^{2\pi it}\right) \longmapsto \begin{cases} [2s,t]_K & \text{if } 0\le s\le \frac{1}{2}, \\ [2s-1, 1-t]_K & \text{if } \frac{1}{2}\le t\le 1. \end{cases}$$

Theorem 78 *There is no retraction from the Möbius strip to its boundary circle.*

Proof. Let
$$Q\colon [-1,1]\times[-1,1] \longrightarrow M = \frac{[-1,1]\times[-1,1]}{(-1,-y)\sim(1,y)}$$
be the quotient map. Then, $\partial M = Q\big([-1,1]\times\{\pm1\}\big)$. Let $C := \{Q(x,0) : -1\le x\le 1\}$ be the central circle. Consider inclusion maps $i\colon C \hookrightarrow M$ and $j\colon \partial M \hookrightarrow M$. Let $r_0\colon M \ni Q(x,y) \longmapsto Q(x,0) \in C$. Note that $r_0\circ j\colon \partial M \to C$ is a two-fold covering map, i.e., after suitable parameterization, we can say $r_0\circ j\colon \partial M \cong \mathbb{S}^1 \ni z \longmapsto z^2 \in \mathbb{S}^1 \cong C$. Hence, $(r_0\circ j)_*\colon \pi_1(\partial M) \to \pi_1(C)$ is multiplication by 2.

If possible assume there is a retract $r\colon M \to \partial M$, then $r\circ j = \mathrm{Id}_{\partial M}$. Now, using Theorem 9

$$r\circ(i\circ r_0)\circ j \simeq r\circ \mathrm{Id}_M\circ j = \mathrm{Id}_{\partial M}$$
$$\implies \big(r\circ(i\circ r_0)\circ j\big)_*\colon \pi_1\big(\partial M\big) \cong \mathbb{Z} \xrightarrow{\times(\pm1)} \mathbb{Z} \cong \pi_1(\partial M).$$

Also, $\big(r\circ(i\circ r_0)\circ j\big)_* = (r\circ i)_*\circ(r_0\circ j)_*$ that is composition of these two maps $(r_0\circ j)_*\colon \pi_1(\partial M) \cong \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \cong \pi_1(C)$ and $(r\circ i)_*\colon \pi_1(C) \cong \mathbb{Z} \xrightarrow{\times 1} \mathbb{Z} \cong \pi_1(M)$, contradicts the fact that $\big(r\circ(i\circ r_0)\circ j\big)_*$ is multiplication by ±1 in $\mathbb{Z}$. $\square$

Theorem 79 *Let $p\colon \mathbb{S}^2 \to \mathbb{R}P^2$ be the quotient map and Σ be a simple closed curve in $\mathbb{R}P^2$. Then $p^{-1}(\Sigma)$ is either a simple closed curve or is a union of two disjoint simple closed curves in $\mathbb{S}^2$.*

Proof. Let $f\colon [0,1] \to \mathbb{R}P^2$ be a map such that $f(s) \neq f(t)$ if $0 < s,t < 1$ and $f(0) = f(1)$, and $\operatorname{im}(f) = \Sigma$. Write $[x] := f(0) = f(1)$ for some $x \in \mathbb{S}^2$.

Now, let $f^+, f^-\colon [0,1] \to \mathbb{S}^2$ be the lifts of f with $f^+(0) = x$ and $f^-(0) = -x$. Note that $p^{-1}\big([a]\big) = \{a, -a\}$ for every $a \in \mathbb{S}^2$. By uniqueness of lifting, $f^+ = -f^-$.

Observation. *If $0 < s,t < 1$, then $f(s) \neq f(t)$, thus $f^+(s) \neq f^+(t)$ and $f^-(s) \neq f^-(t)$, and $f^+(s) \neq f^-(t)$ as $pf^+ = f = pf^-$.*

Now, we have to consider two cases, namely $f^+(1) = -x$ and $f^+(1) = x$.

(1) Since $f^+(1) = -x = f^-(0)$, define $g := f^+ * f^-$. By the above Observation, $g(s) \neq g(t)$ if $0 < s,t < 1$. Thus g is a simple loop based at x with $\operatorname{im}(g) = \operatorname{im}(f^+) \cup \operatorname{im}(f^-) = p^{-1}(\Sigma)$.

(2) Now, consider the case when $f^+(1) = x$. Thus, by the above Observation, we can say that $f^-(s) \neq f^+(t)$ if $0 \leq s,t \leq 1$, i.e., both f^+ and f^- gives disjoint simple loops. Since $\operatorname{im}(f^+) \cup \operatorname{im}(f^-) = p^{-1}(\Sigma)$, we are done.

□

Definition 80 *Here we define Pullback in the category **Top**. Define $A \times_B Y := \{(a,y) \in A \times Y \mid f(a) = g(y)\}$ and let $\overline{g}, \overline{f}$ be the restrictions of the projections on first and second components, respectively.*

$$\begin{tikzcd}
Z \arrow[drr, bend left, "h"] \arrow[dr, dashed, "\exists!\,\ell"] \arrow[ddr, bend right, "k"'] & & \\
& A \times_B Y \arrow[r, "\overline{f}"] \arrow[d, "\overline{g}"'] & Y \arrow[d, "g"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}$$

Then given any $h\colon Z \to Y$ and $k\colon Z \to A$ with $g \circ h = f \circ k$, we have a unique map $\ell\colon Z \to A \times_B Y$ such that following diagram commutes. Actually, $\ell\colon Z \ni z \longmapsto \big(k(z), h(z)\big) \in A \times_B Y$ does the job.

Lemma 81 *Let $g\colon Y \to B$ be a homeomorphism and $f\colon A \to B$ be a map. Then $\overline{g}\colon A \times_B Y \to A$ is a homeomorphism.*

Proof. Take $Z = A$, $k = \mathrm{Id}_A$, $h = g^{-1} \circ f$. Then, $\ell = (\mathrm{Id}_A, g^{-1} \circ f)$ and $\overline{g} \circ \ell = \mathrm{Id}_A$. Now, $\ell \circ \overline{g}(a,y) = \ell(a) = \big(a, g^{-1} \circ f(a)\big) = (a,y)$, i.e., $\ell \circ \overline{g} = \mathrm{Id}_{A \times_B Y}$. □

Theorem 82 *Let $g\colon Y \to B$ be a covering map. Then $\overline{g}\colon A \times_B Y \to A$ is also a covering.*

Proof. Take an admissible open subset U of B, and write $g^{-1}(U) = \bigsqcup_i V_i$ with $g|V_i \xrightarrow{\cong} U$. Then,

$$\bar{g}^{-1}\big(f^{-1}(U)\big) = \bar{f}^{-1}\big(g^{-1}(U)\big) = \bigsqcup_i \bar{f}^{-1}(V_i).$$

Now, $\bar{g}|\bar{f}^{-1}(V_i) \xrightarrow{\cong} f^{-1}(U)$ is the pull-back of the homeomorphism $g|V_i \xrightarrow{\cong} U$, so we are done by Lemma 81. $\square$

Theorem 83 *Let $g\colon Y \to B$ and $f\colon A \to B$ are covering maps. Then $\bar{f}\colon A \times_B Y \to Y$ is also a covering.*

Proof. Take an open subset U of B such that $g^{-1}(U) = \bigsqcup_i V_i$ and $f^{-1}(U) = \bigsqcup_j W_j$ with $g|V_i \xrightarrow{\cong} U$ and $f|W_j \xrightarrow{\cong} U$ (note an open subset of an admissible set is also admissible). Then,

$$\bigsqcup_j \bar{g}^{-1}(W_j) = \bar{g}^{-1}\big(f^{-1}(U)\big) = \bar{f}^{-1}\big(g^{-1}(U)\big) = \bigsqcup_i \bar{f}^{-1}(V_i)$$

$$\implies \bar{f}^{-1}\big(g^{-1}(U)\big) = \bigsqcup_{i,j} \bar{f}^{-1}(V_i) \cap \bar{g}^{-1}(W_j)$$

Now, $\bar{g}|\bar{f}^{-1}(V_i) \xrightarrow{\cong} f^{-1}(U)$ is a homeomorphism as it is the pull-back of the homeomorphism $g|V_i \xrightarrow{\cong} U$, see Lemma 81. Therefore, $\bar{g}|\bar{f}^{-1}(V_i) \cap \bar{g}^{-1}(W_j) \xrightarrow{\cong} f^{-1}(U) \cap W_j = W_j$ is a homeomorphism for each i, j. Also, $g|V_i \xrightarrow{\cong} U$ and $f|W_j \xrightarrow{\cong} U$ are homeomorphisms, i.e., we have the following commutative diagram, where three arrows are homeomorphisms:

$$\begin{tikzcd}
\bar{f}^{-1}(V_i) \cap \bar{g}^{-1}(W_j) \arrow[r, "\bar{f}"] \arrow[d, "\bar{g}"'] & V_i \arrow[d, "g"] \\
W_j \arrow[r, "f"'] & U
\end{tikzcd}$$

Thus $f|\bar{f}^{-1}(V_i) \cap \bar{g}^{-1}(W_j) \to V_i$ is also a homeomorphism, i.e., $g^{-1}(U)$ is an admissible open set of the covering $\bar{f}$ with the sheets $\left\{\bar{f}^{-1}(V_i) \cap \bar{g}^{-1}(W_j) : i, j\right\}$. $\square$

Theorem 84 *Let $p\colon E \to X$ be a covering map and X is connected, then $\#p^{-1}(x) = \#p^{-1}(x')$ for all $x, x' \in X$.*

Proof. Let $x_0 \in X$ and consider the set $S := \{x \in X | \#p^{-1}(x) = \#p^{-1}(x_0)\}$. Then S is non-empty. Now, for $x \in S$ and and an evenly covered open nbd V of x write $q^{-1}(V) = \bigsqcup_{i \in \Lambda} U_i$ where each U_i is open in E with $q|U_i \to V$ is a homeomorphism for each $i \in \Lambda$. Now, for each $y \in V$ we have $q^{-1}(y) \cap U_i$ is singleton, i.e., $\#q^{-1}(y) = \#\lambda = \#q^{-1}(x)$, hence, $V \subseteq S$. So, S is open in X. Similarly, $X \setminus S$ is open in X. Since, X is connected, $S = X$. $\square$

Theorem 85 *Let $p\colon E \to X$ be a covering and $f, g\colon Y \to E$ be such that $p \circ f = p \circ g$. Then the set of all points of Y where f and g agree, is a clopen subset of Y.*

Theorem 86 *Let $(X, \bullet)$ be a topological group with identity element x_0, and $p\colon (E, e_0) \to (X, x_0)$ be a covering map such that E connected, locally path-connected. Then there is a unique structure of topological group on E for which e_0 is the identity element, and p is a group-homomorphism.*

Proof. Let $m\colon X \times X \ni (x_1, x_2) \longmapsto x_1 \bullet x_2^{-1} \in X$. We wish to lift $m \circ (p \times p)$.

$$\begin{tikzcd}
 & & (E, e_0) \arrow[d, "p"] \\
\big(E \times E, (e_0, e_0)\big) \arrow[r, "p \times p"] \arrow[urr, dashed, bend left, "\exists!\ m'"] & \big(X \times X, (x_0, x_0)\big) \arrow[r, "m"'] & (X, x_0)
\end{tikzcd}$$

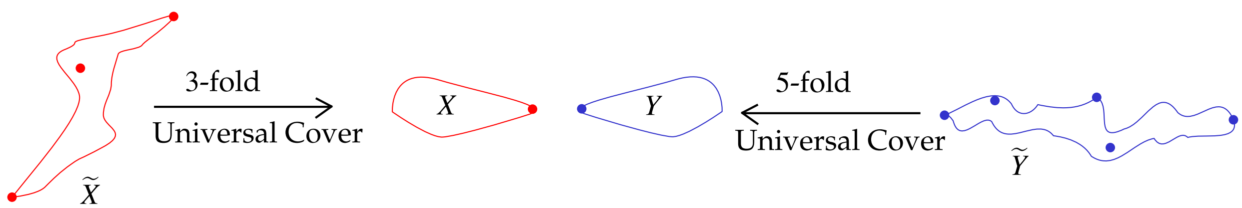
The criterion of existence of m' is

$$m_*(p \times p)_* \pi_1\big(E \times E, (e_0, e_0)\big) \subseteq p_* \pi_1(E, e_0).$$

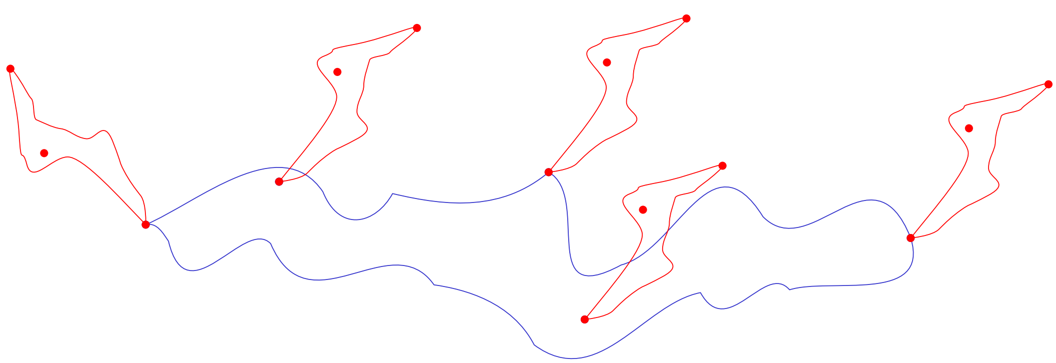
This is equivalent to say that for any two loops σ, τ in E based at e_0, the loop $\Gamma\colon [0,1] \ni t \longmapsto \big(p \circ \sigma(t)\big) \bullet \big(p \circ \tau(t)\big)^{-1} \in X$ based at x_0 is relatively homotopic to $p \circ \gamma$ for some loop γ in E based at e_0, i.e., $\Gamma \simeq_{\text{rel } x_0} p \circ \gamma$. But we know that $\Gamma \simeq_{\text{rel } x_0} (p \circ \sigma) * \overline{p \circ \tau} = p \circ (\sigma * \overline{\tau})$. So, our required $\gamma = \sigma * \overline{\tau}$. □

3.1 Schematic construction of the universal cover of wedge

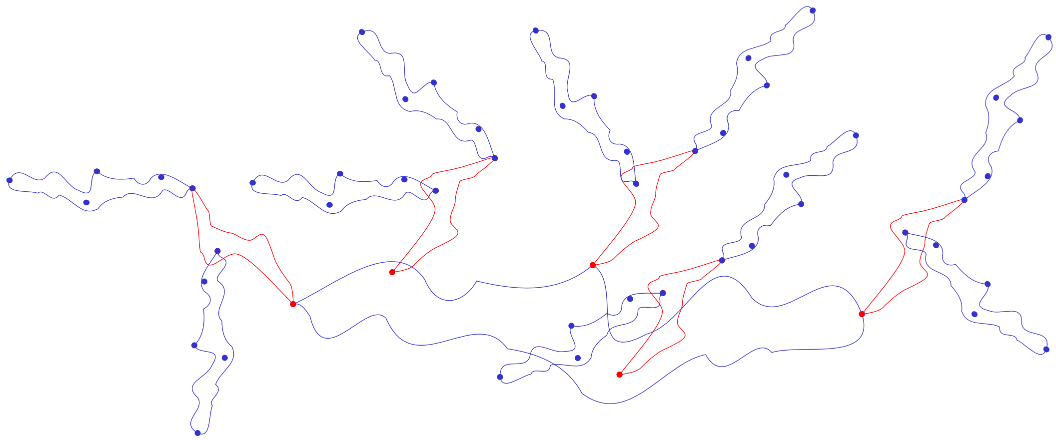
We follow the argument given in [Kup]. Let X and Y be two good spaces, e.g., CW-complexes, topological manifolds, etc. We are interested in finding the universal cover of $X \vee Y$. We will describe the drawing of the universal covering for a particular case, namely when the universal cover $p\colon \widetilde{X} \to X$ is a 3-fold cover and $q\colon \widetilde{Y} \to Y$ is a 5-fold cover. The general case is analogous.



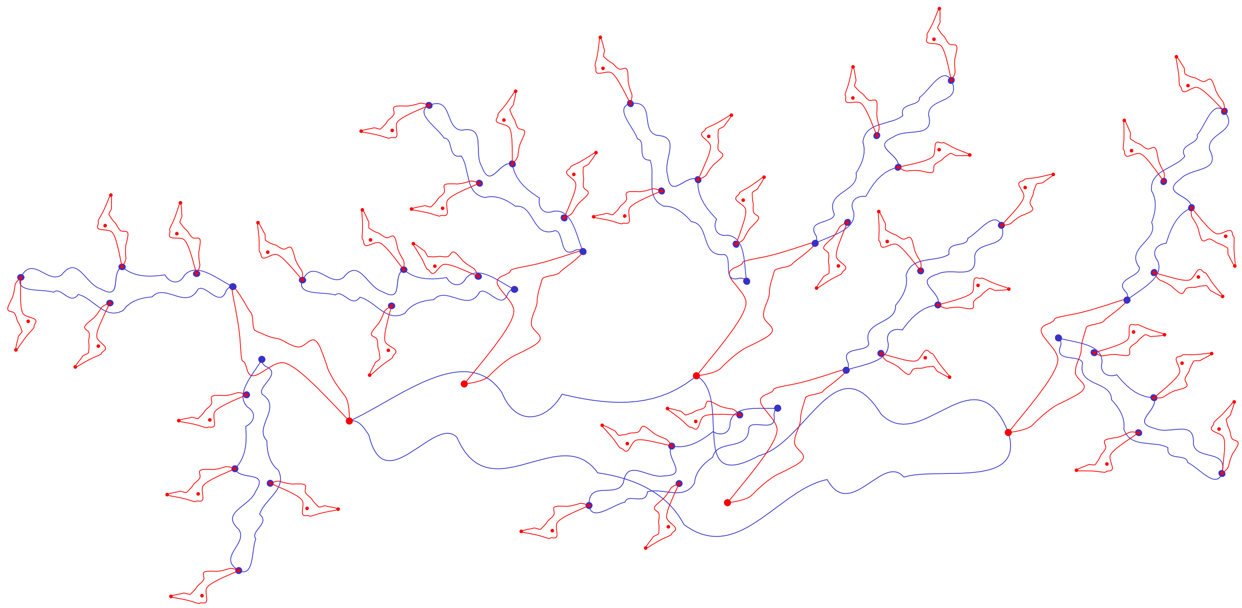
Step 0: Here, $X \vee Y$ is the space obtained from $X \sqcup Y$ identifying red base-point of X with blue base-point of Y. In the figure the fibers $p^{-1}(\bullet)$ and $q^{-1}(\bullet)$ are illustrated.



Step 1: Take a copy of $\widetilde{Y}$ and at each point $q^{-1}(\bullet)$ add a copy of $\widetilde{X}$ using a point of $p^{-1}(\bullet)$. Call this space as A_1. So, we now have a total of 10 red free vertices in the space A_1.



Step 2: At each red free-vertex of A_1 attach a copy of $\widetilde{Y}$ using a point of $q^{-1}(\bullet)$. Call this space as A_2. So, we now have a total of 40 blue free vertices in the space A_2.



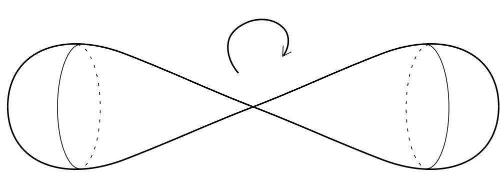
Step 3: At each blue free-vertex of A_2 attach a copy of $\widetilde{X}$ using a point of $p^{-1}(\textcolor{red}{\bullet})$. Call this space A_3.

Ultimate Step: If we continue (continue until no red/blue free-vertex remains) this way, the final space $\widetilde{X \vee Y}$ will be the universal cover of $X \vee Y$. Now, applying p on each copy of $\widetilde{X}$ inside $\widetilde{X \vee Y}$ and applying q on each copy of $\widetilde{Y}$ inside $\widetilde{X \vee Y}$, we have the universal covering map $\widetilde{X \vee Y} \to X \vee Y$.

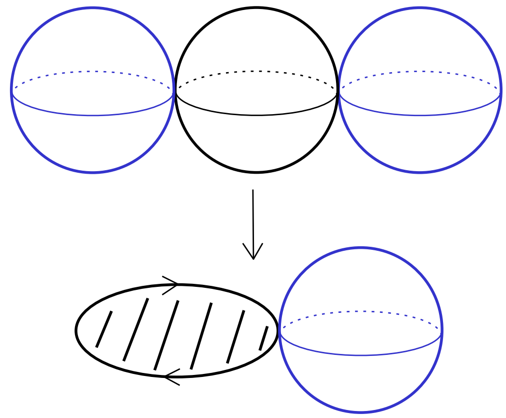
3.2 Examples to illustrate schematic construction

How to Draw, some tips.

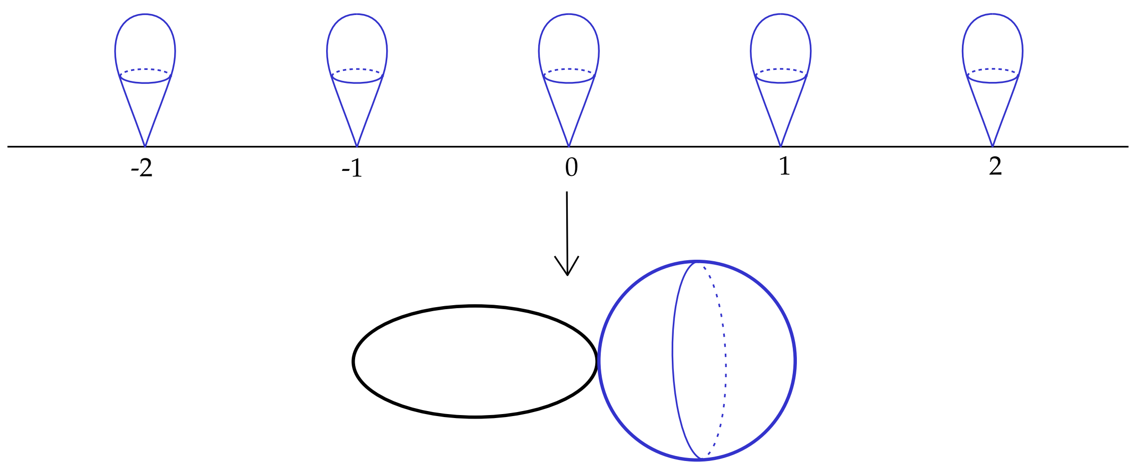
- Covering map is a local homeomorphism. So, the local nature of the connected base space is repeated in the cover, and the number of repetitions is the same as the number of folds of the covering. In particular, the cardinality of any two fibers will be the same.
- There will be no non-trivial loop in the universal cover.
- Let X be a connected, locally path-connected, semi-locally simply-connected space, and $p\colon (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ be its universal cover. Now, there is a bijection between $\pi_1(X, x_0) \cong \dfrac{\pi_1(X, x_0)}{p_* \pi_1(\widetilde{X}, \widetilde{x_0})}$ and $p^{-1}(x_0)$. In other words, if the fundamental group is an infinite group, then fiber is infinite.



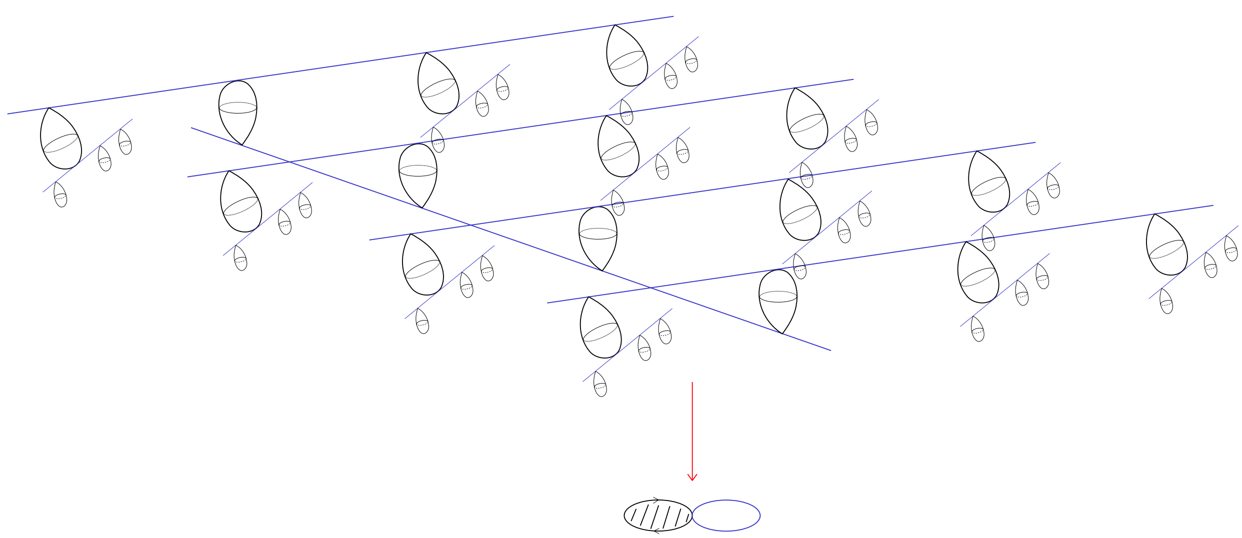
(1) $\mathrm{Id}\colon \mathbb{S}^2 \vee \mathbb{S}^2 \to \mathbb{S}^2 \vee \mathbb{S}^2$ is the universal cover



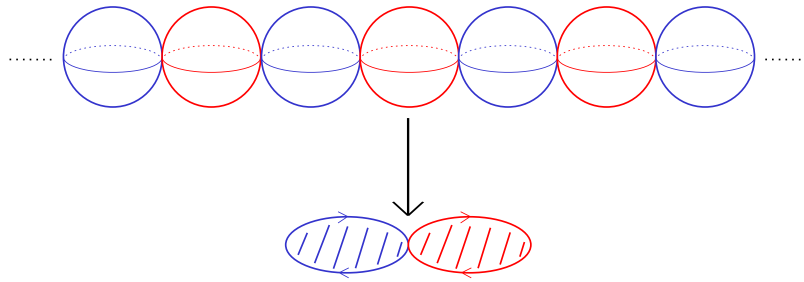
(2) The above map is the universal cover of $\mathbb{R}P^2 \vee \mathbb{S}^2$



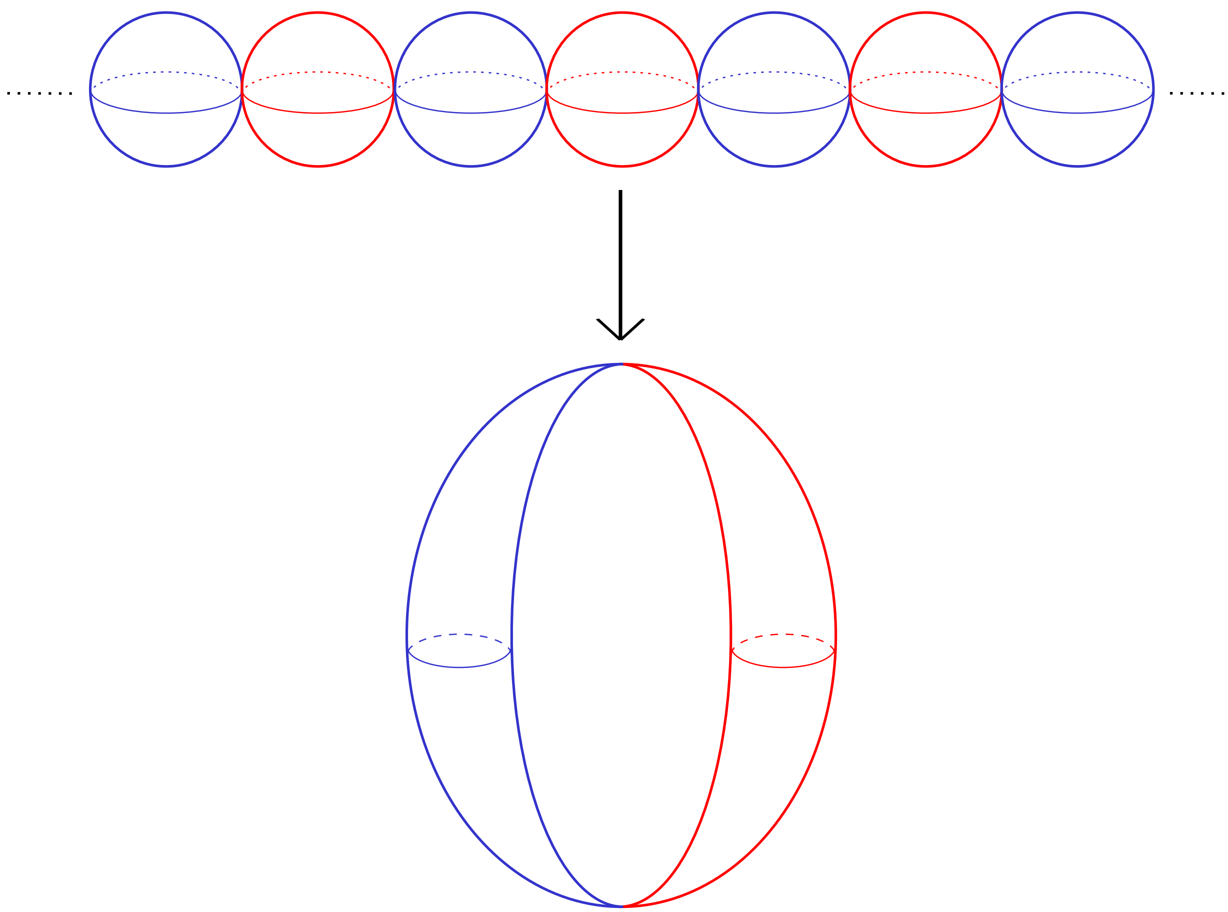
(3) The above map is the universal cover of $\mathbb{S}^1 \vee \mathbb{S}^2$



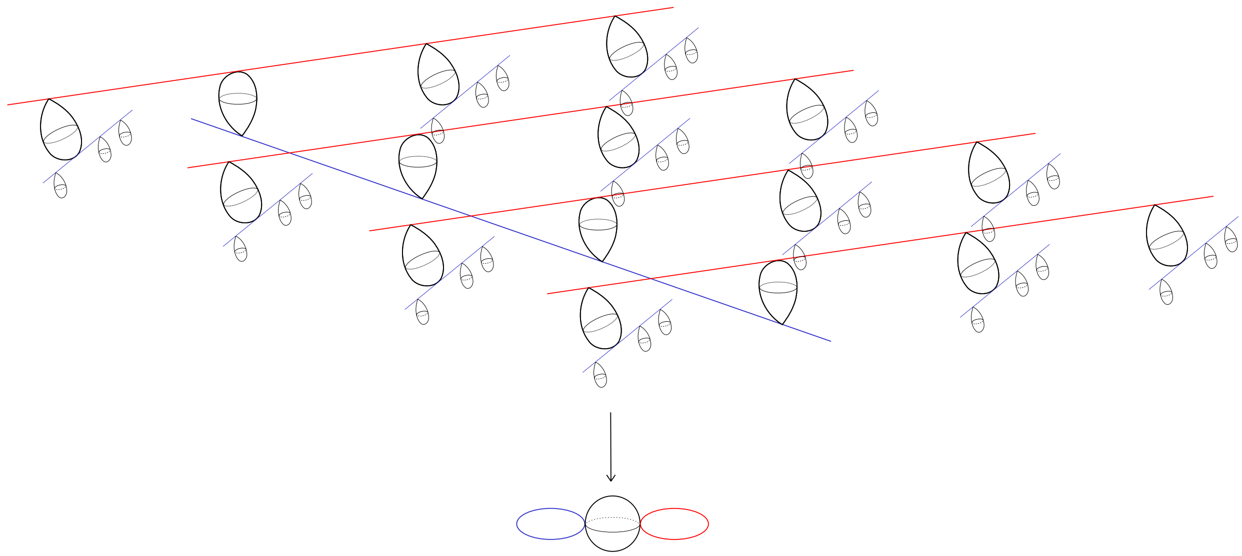
(4) The above map is the universal cover of $\mathbb{R}P^2 \vee \mathbb{S}^1$



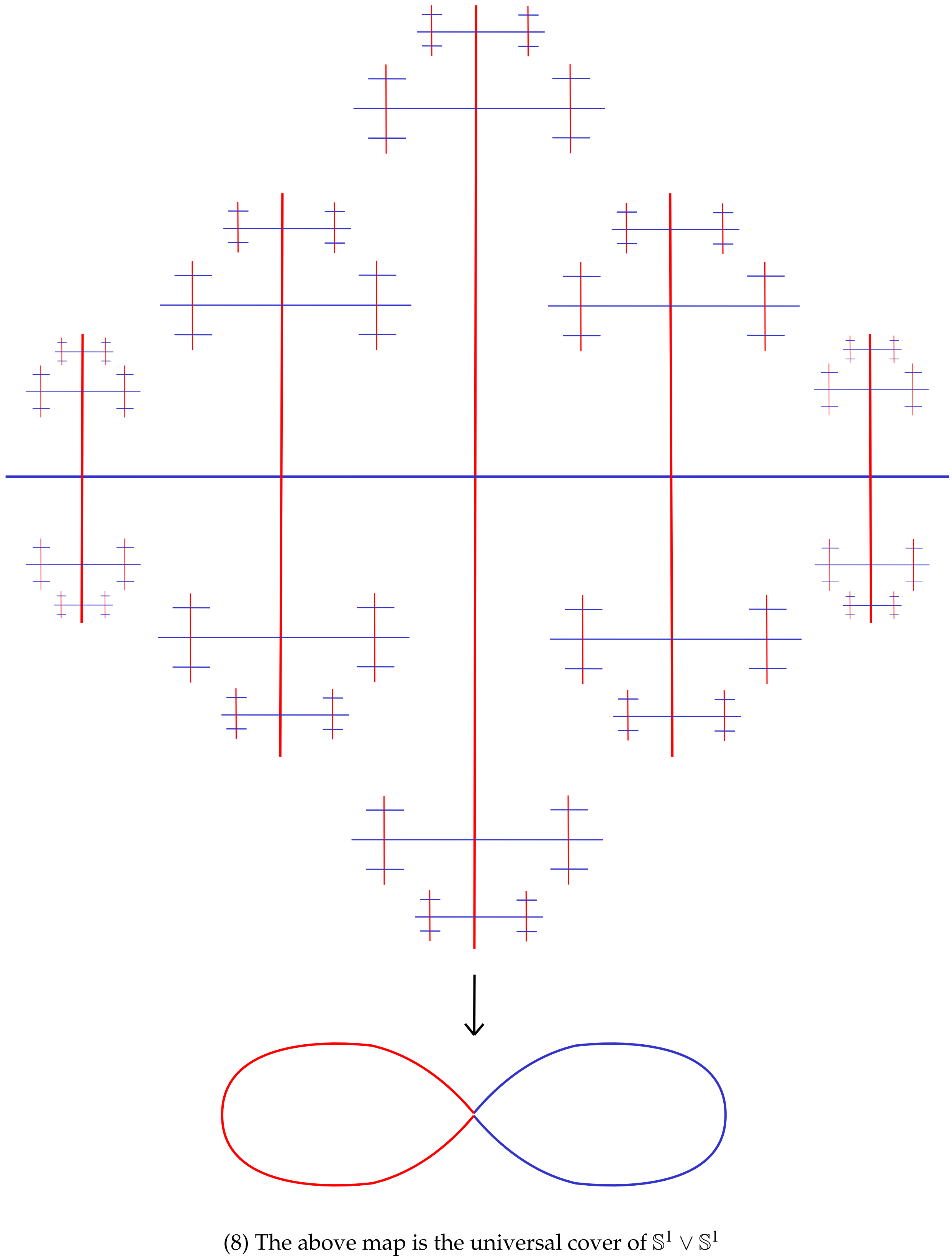
(5) The above map is the universal cover of $\mathbb{R}P^2 \vee \mathbb{R}P^2$



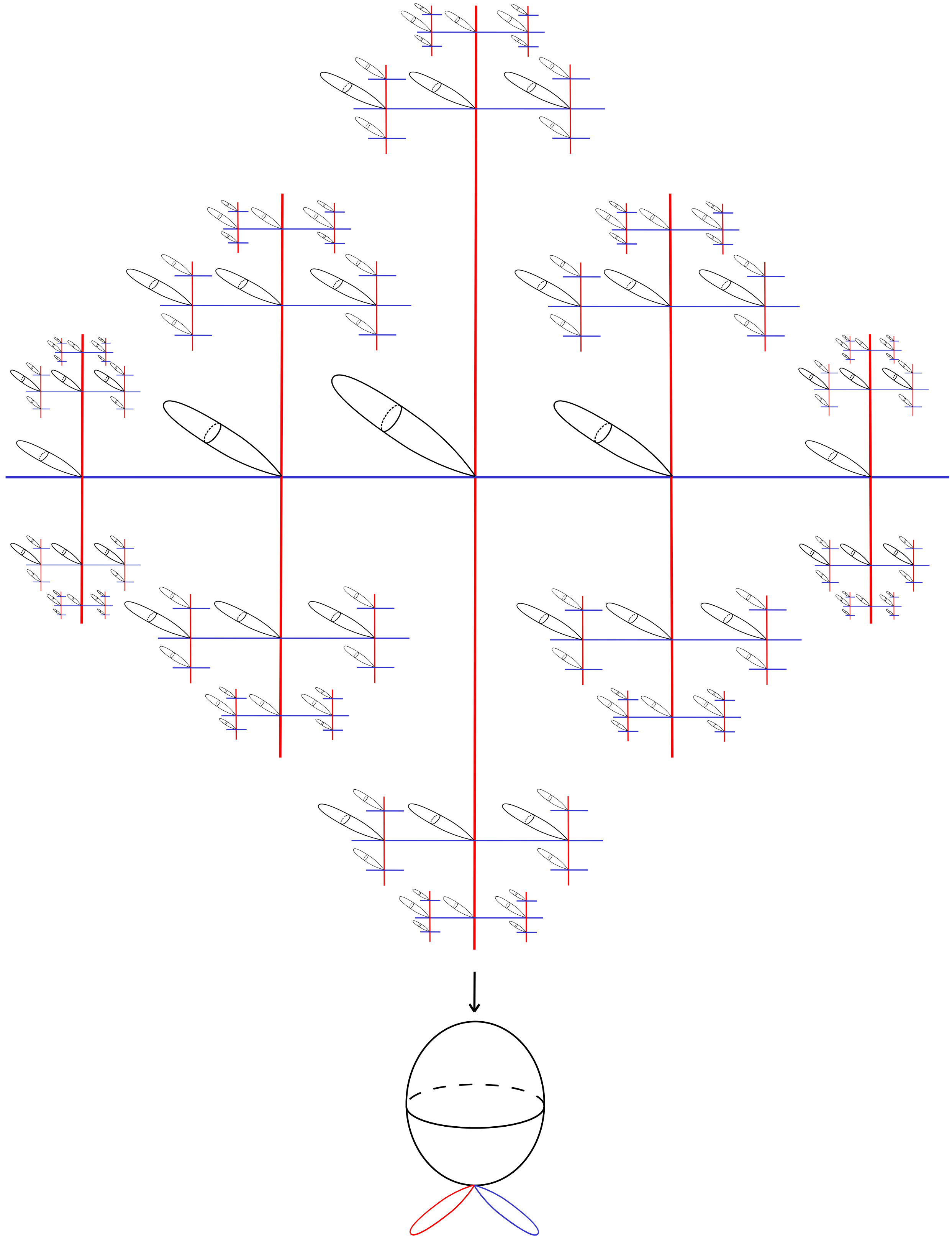
(6) The above map is the universal cover of two $\mathbb{S}^2$ having two points in common



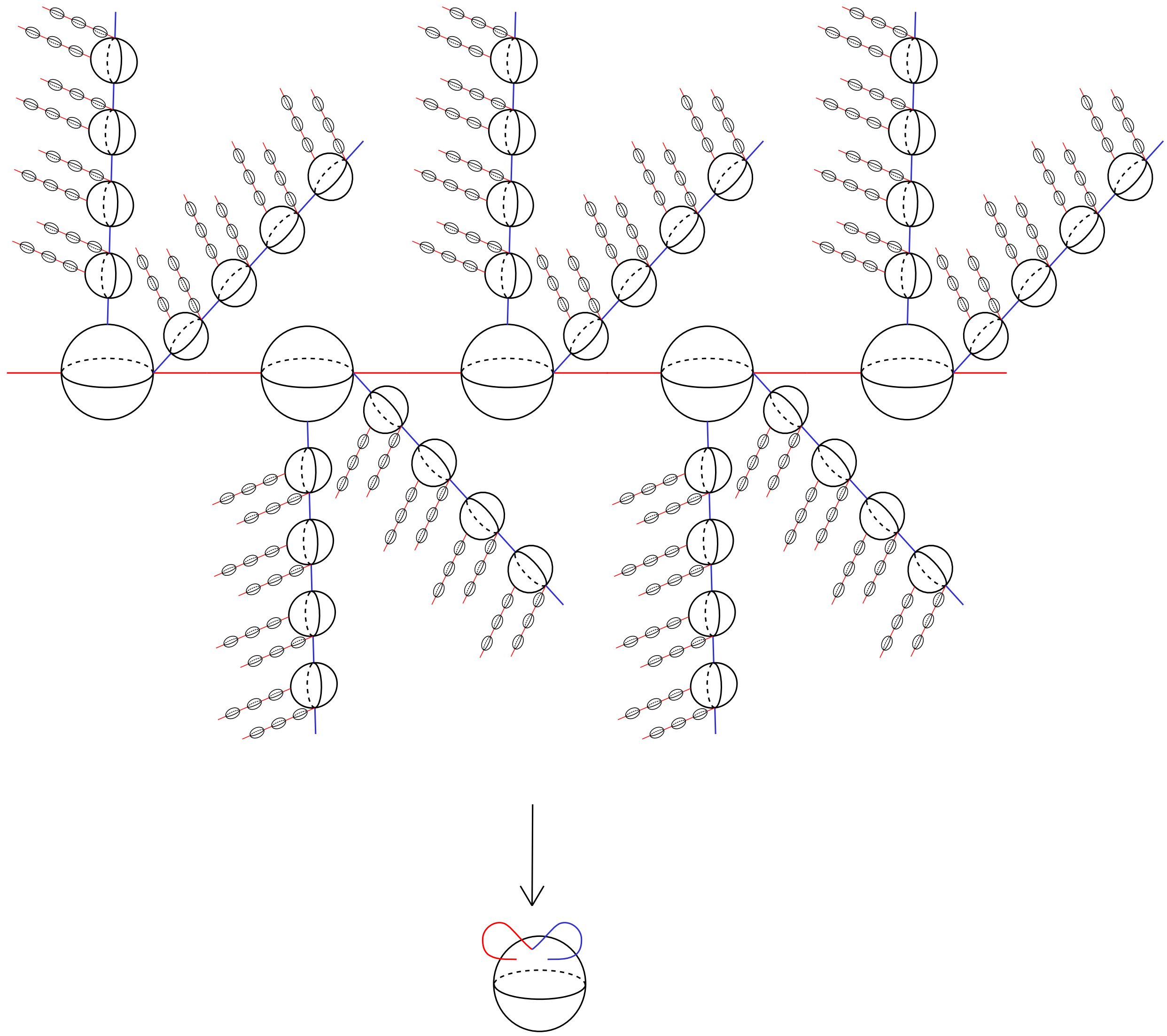
(7) The above map is the universal cover



(8) The above map is the universal cover of $\mathbb{S}^1 \vee \mathbb{S}^1$



(9) The above map is the universal cover of $\mathbb{S}^1 \vee \mathbb{S}^2 \vee \mathbb{S}^1$



(10) The above map is the universal cover

3.3 Covering spaces of graphs

Definition 87 [Hat02, 1.A Graphs and Free Groups] *A Hausdorff space X is called a graph or one-dimensional CW-complex if the following hold:*

- $$X = \bigsqcup_i e_i \sqcup \bigsqcup_j v_j$$

  where each e_i is a subspace of X and each v_j is a point of X. Each e_i is called an 1-cell or an edge, and each point v_j is called a 0-cell or a vertex.

- *For each i we have a continuous map $\varphi_i \colon [0,1] \to \overline{e_i}$ such that $\varphi_i|(0,1) \xrightarrow{\cong} e_i$ and $\varphi_i(0), \varphi_i(1) \in \bigsqcup_j v_j$.*

- *For a subset A of X we have $A \subseteq_{\text{closed}} X$ if and only if $A \cap \overline{e_i} \subseteq_{\text{closed}} \overline{e_i}$.*

Remark 88 *Each $\{v_j\}$ is closed in X as X is Hausdorff.*

Lemma 89 *Let X be a graph. Any subset of the set of all vertices is a discrete closed subspace of X.*

Proof. Let A be any subset of $\bigsqcup_j v_j$, then $A \cap \overline{e_i}$ is either empty or two vertices or one vertex. In other words, $A \cap \overline{e_i}$ is closed in X, hence in $\overline{e_i}$ also. Therefore, A is closed in X. Since A is an arbitrary subset of $\bigsqcup_j v_j$, we are done. □

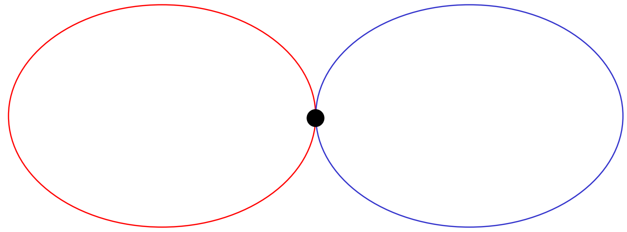
Remark 90 *Each edge e_i is open in X as $X \backslash e_i$ is closed in X.*

Example/Non-example $\bigvee_i \mathbb{S}^1 := \dfrac{\bigsqcup_i \mathbb{S}^1}{\bigsqcup_i \{1\}}$ *is a connected graph, where i varies over any non-empty index set. Under the subspace topology of $\mathbb{R}^2$, the Hawaiian Earring is **not** a CW-complex, as any compact CW-complex has only finitely many cells.*

Theorem 91 [Hat02, Lemma 1.A.3] [Rot88, Theorem 10.43.] *Let X be a connected graph and $p \colon \widetilde{X} \to X$ be an n-fold covering map, where n is a positive integer or infinity. Write $X = \bigsqcup_i e_i \sqcup \bigsqcup_j v_j$ as in the above definition. Then, one can give a CW-complex structure on $\widetilde{X}$ such that for each vertex v_j, we have exactly n-many vertices in $\widetilde{X}$, and for each edge e_i we have exactly n-many 1-cells in $\widetilde{X}$. Roughly,*

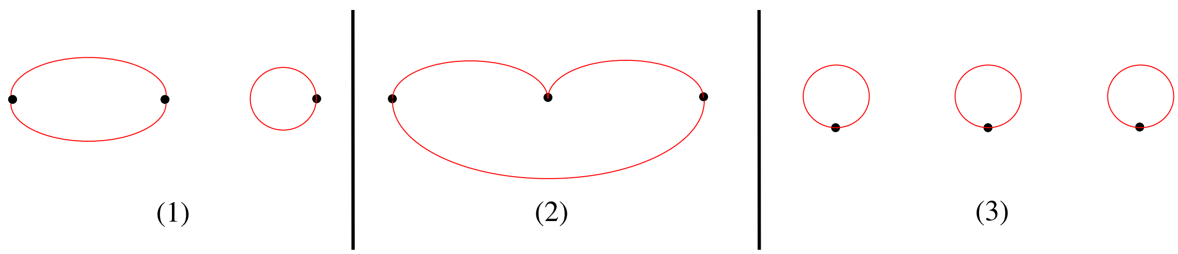
$$\widetilde{X} = \bigsqcup_i n \cdot e_i \sqcup \bigsqcup_j n \cdot v_j.$$

Theorem 92 [Hat02, Exercise 10 Chapter 1.3] *Let $\mathcal{C}$ be a collection of connected 3-fold covering maps of $\mathbb{S}^1 \vee \mathbb{S}^1$ such that no distinct two elements of $\mathcal{C}$ are homeomorphic via a Deck-transformation and given any 3-fold covering map $X \to \mathbb{S}^1 \vee \mathbb{S}^1$; there is an element $X' \to \mathbb{S}^1 \vee \mathbb{S}^1$ of $\mathcal{C}$ such that X is homeomorphic to X' via some homeomorphism. Then $|\mathcal{C}| = 7$.*



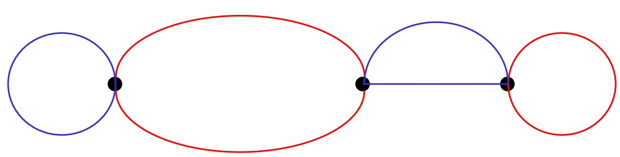
Roughly, we want to find all connected 3-fold covers of $\mathbb{S}^1 \vee \mathbb{S}^1$. Note that $\mathbb{S}^1 \vee \mathbb{S}^1$ has a CW structure with one vertex and two edges.

Step 1: So, every 3-fold cover has a CW-structure with three vertices

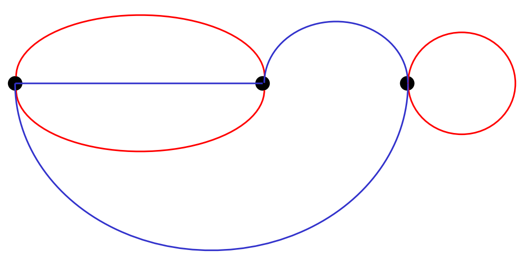


(1) (2) (3)

Step 2: Also, every 3-fold cover has a CW-structure with three red edges. Since the end(s) of an edge is either a single vertex or two distinct vertices, (1), (2), (3) are the only possibilities of attaching three red edges to three black vertices. Note that a small nbd of the wedge point of $\mathbb{S}^1 \vee \mathbb{S}^1$ is a wedge of four small arcs, out of which two are red arcs, and two are blue arcs.



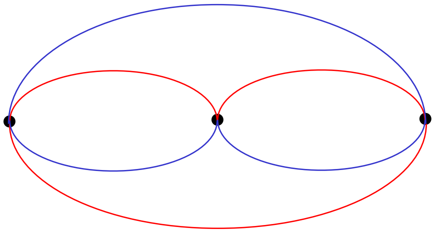
(1.1.)



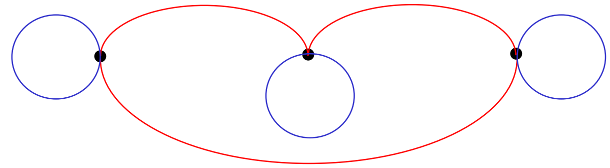
(1.2.)

Step 3: Like red edges, every 3-fold cover has a CW-structure with three blue edges. Considering only (1) from step 2, we have two possibilities of attaching three blue edges. Again, note that a small nbd of the wedge point of $\mathbb{S}^1 \vee \mathbb{S}^1$ is a wedge of four small arcs, out of which two are red arcs, and two are blue arcs. Notice that this pattern is repeated thrice near each black wedge point of the covers (1.1.) and (1.2.)

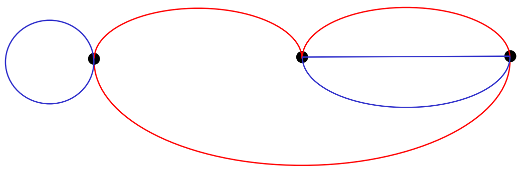
Proof.



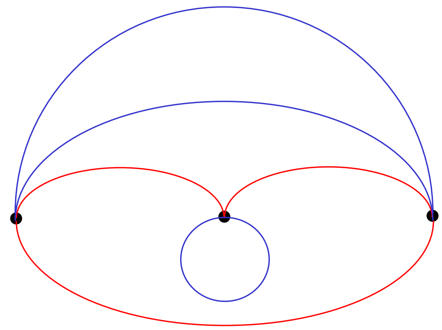
(2.1.)



(2.2.)

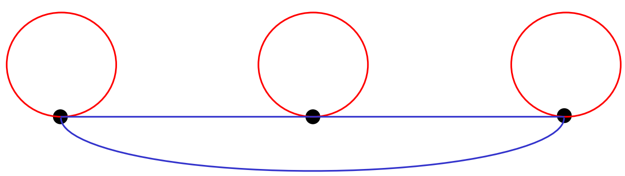


(2.3.)



(2.4.)

Step 4: Like red edges, every 3-fold cover has a CW-structure with three blue edges. Considering only (2) from step 2, we have four possibilities of attaching three blue edges. Again, note that a small nbd of the wedge point of $\mathbb{S}^1 \vee \mathbb{S}^1$ is a wedge of four small arcs, out of which two are red arcs, and two are blue arcs. Notice that this pattern is repeated thrice near each black wedge point of the covers (2.1.), (2.2.), (2.3.) and (2.4.)



Step 5: Like red edges, every 3-fold cover has a CW-structure with three blue edges. Considering only (3) from step 2, we have exactly one possibility of attaching three blue edges. Again, note that a small nbd of the wedge point of $\mathbb{S}^1 \vee \mathbb{S}^1$ is a wedge of four small arcs, out of which two are red arcs, and two are blue arcs. Notice that this pattern is repeated thrice near each black wedge point of this cover.

This shows $\mathcal{C}$ can have at least 7 elements (one needs to check that the above covers are pairwise non-isomorphic; note that the last cover is homeomorphic to cover of 2.2. but not via a Deck-transformation, see Remark 96). Notice that at each step, we have considered all possibilities of attaching an edge to a vertex or two vertices. Thus, $\mathcal{C}$ can have at most 7 elements. $\square$

Theorem 93 [Hat02, Exercise 10 Chapter 1.3] *Let $\mathcal{C}$ be a collection of connected 2-fold covering maps of $\mathbb{S}^1 \vee \mathbb{S}^1$ such that no distinct two elements of $\mathcal{C}$ are homeomorphic via a Deck-transformation and given any 2-fold covering map $X \to \mathbb{S}^1 \vee \mathbb{S}^1$; there is an element $X' \to \mathbb{S}^1 \vee \mathbb{S}^1$ of $\mathcal{C}$ such that X is homeomorphic to X' via some homeomorphism. Then $|\mathcal{C}| = 3$.*

Theorem 94 *Let $p\colon (X, x_0) \to (B, b_0)$ be a finite-covering space such that X is path-connected. The following are equivalent:*

- *The covering is regular: for every $x, x' \in p^{-1}(b_0)$ we have a deck transformation (a homeomorphism, $h\colon X \to X$ with $p \circ h = p$) such that $h(x) = x'$.*

- *$p_* \pi_1(X, x_0)$ is a normal subgroup of $\pi_1(B, b_0)$.*

- *The number of deck transformations is same as $\#p^{-1}(b_0)$.*

Remark 95 All 2-fold path-connected covers are regular as index-two subgroups are normal.

Remark 96 Consider any finite-fold cover of $\mathbb{S}^1 \vee \mathbb{S}^1$. A deck transformation d sends

- a vertex to a vertex,

- blue (resp. red) edges to a blue (reps. red) edges,

- an edge e with end-points P, Q to the edge $d(e)$ with end-points $d(p)$ and $d(Q)$,

- a loop l to the loop $d(l)$.

The 3-fold covers (1.1.) and (1.2.) are irregular as each of these contains only one red loop. Similarly, the 3-fold covers (2.3.) and (2.4.) are irregular as each of these contains only one blue loop. But, the covers (2.1.), (2.2.), and (3.1.) are regular.

Definition 97 *For a finite graph X, i.e., the number of 1-cells and the number of 0-cells both are finite; define Euler characteristic as*

$$\chi(X) := \text{number of vertices} - \text{number of edges.}$$

Remark 98 *If $p\colon \widetilde{X} \to X$ is an n-fold covering then $\chi(\widetilde{X}) = n \cdot \chi(X)$*

Definition 99 *Let X be a graph, a subspace Y of X is said to be a sub-graph of X if Y can be written as a union of edges and vertices of X such that if the edge $e_i \subseteq Y$ then $\overline{e_i} \subseteq Y$. Note that a sub-graph itself is a graph.*

Definition 100 *A simply connected graph is called a tree. One can show a tree is contractible. Roughly, a tree doesn't contain a non-trivial loop.*

Definition 101 *A sub-graph T of X is called a maximal tree if the graph T is a tree and T contains all the vertices of X.*

Theorem 102 [Hat02, Proposition 1.A.1] *Using the axiom of choice, one can show that every connected graph contains a maximal tree.*

Theorem 103 [Hat02, Proposition 1.A.2] *Let X be a connected graph with a base-vertex v_0. Fix an orientation on each edge. Let T be a maximal tree of X. Then each edge $e_i \subseteq X \backslash T$ determines a loop ℓ_i in X based at v_0 as follows: starting from v_0 go to a vertex of e_i by a path in T, then cross e_i following its orientation, then back to v_0 by a path in T. Then,*

$$\pi_1(X, v_0) = \langle [\ell_i] : \text{ where } \ell_i \text{ is the loop corresponding to the oriented edge } e_i \subseteq X \backslash T \rangle.$$

4 Lifting Problems

Theorem 104 *Let $p\colon (E,e_0)\to(X,x_0)$ be a covering map and and $H\colon [0,1]^2\to X$ be a map such that $H(0,0)=x_0$. Then there exists a map $\widetilde{H}\colon [0,1]^2\to E$ with $\widetilde{H}(0,0)=e_0$ such that the following diagram commutes.*

$$\begin{tikzcd} & (E,e_0) \arrow[d, "p"] \\ \big([0,1]\times[0,1],(0,0)\big) \arrow[ur, dashed, "\exists!\,\widetilde{H}"] \arrow[r, "H"'] & (X,x_0) \end{tikzcd}$$

Corollary 105 Suppose H in Theorem 104 is a path homotopy, i.e., $H(0,-)=a$ and $H(1,-)=b$. Thus $\widetilde{H}(0,-)$ maps into the discrete space $p^{-1}(a)$ and $\widetilde{H}(1,-)$ maps into the discrete space $p^{-1}(b)$. Therefore, $\widetilde{H}$ is also a path homotopy, i.e., $\widetilde{H}(0,-)$ and $\widetilde{H}(1,-)$ are constant maps.

Theorem 106 (Monodromy Theorem) *Let $q\colon (E,e)\to(B,b)$ be a covering map. Suppose f and g are paths in B with the same initial point and the same terminal point, and $\widetilde{f}_e,\widetilde{g}_e$ are their unique lifts with the same initial point $e\in E$. Then,*

$$\widetilde{f}_e\simeq_{\mathrm{rel}\ \{0,1\}}\widetilde{g}_e \iff f\simeq_{\mathrm{rel}\ \{0,1\}} g$$

So, in particular, $f\simeq_{\mathrm{rel}\ \{0,1\}} g$ implies $\widetilde{f}_e(1)=\widetilde{g}_e(1)$.

Proof. For $H\colon f\simeq_{\mathrm{rel}\ \{0,1\}} g$ and corresponding unique lift $\widetilde{H}\colon \big([0,1]^2,(0,0)\big)\to(E,e)$ we can say $\widetilde{f}_e=\widetilde{H}(-,0)$ and $\widetilde{g}_e=\widetilde{H}(-,1)$ by the uniqueness of lifting. Note that $\widetilde{H}(0,1)=\widetilde{g}_e(0)$ as $\widetilde{H}\big(0,-\big)$ is constants by previous statements. $\square$

Theorem 107 *Let $p\colon (E,e_0)\to(B,b_0)$ be a covering such that E is path-connected. Define,*

$$\Phi\colon \frac{\pi_1(B,b_0)}{p_*\pi_1(E,e_0)}\ni \mathrm{cls}\big([f]\big)\longmapsto \widetilde{f}(1)\in p^{-1}(b_0)$$

$$\textit{where } \widetilde{f}\colon \big([0,1],0\big)\to(E,e_0) \textit{ is the unique lift of } f.$$

Then Φ is a bijection.

Proof. Φ is well-defined: Suppose, $\mathrm{cls}\big([f]\big)=\mathrm{cls}\big([g]\big)$ in $\frac{\pi_1(B,b_0)}{p_*\pi_1(E,e_0)}$, so write $[f]=\big[(p\circ\widetilde{h})*g\big]$, where $\widetilde{h}:\big([0,1],\{0,1\}\big)\to(E,e_0)$. Let $\widetilde{f},\widetilde{g}:\big([0,1],0\big)\to(E,e_0)$ be the unique lifts of f and g, respectively. Then, $\widetilde{h}*\widetilde{g}$ is well-defined and

$$p\circ(\widetilde{h}*\widetilde{g})=(p\circ\widetilde{h})*g.$$

Since, $\widetilde{f},\widetilde{h}*\widetilde{g}:\big([0,1],0\big)\to(E,e_0)$ and $f\simeq_{\mathrm{rel}\ \{0,1\}}(p\circ\widetilde{h})*g$ we have $\widetilde{f}\simeq_{\mathrm{rel}\ \{0,1\}}\widetilde{h}*\widetilde{g}$.

Φ is injective: Suppose, for $\mathrm{cls}\big([f]\big)$ and $\mathrm{cls}\big([g]\big)$ in $\frac{\pi_1(B,b_0)}{p_*\pi_1(E,e_0)}$ we have $\widetilde{f}(1)=\widetilde{g}(1)$ where $\widetilde{f},\widetilde{g}:\big([0,1],0\big)\to(E,e_0)$ be the unique lifts of f and g, respectively. Now,

$$\begin{aligned}
&(\widetilde{f}*\overline{\widetilde{g}})*\widetilde{g}\simeq_{\mathrm{rel}\ \{0,1\}}\widetilde{f}\\
\implies\ & p\circ\big((\widetilde{f}*\overline{\widetilde{g}})*\widetilde{g}\big)\simeq_{\mathrm{rel}\ \{0,1\}} p\circ\widetilde{f}\\
\implies\ & \big(p\circ(\widetilde{f}*\overline{\widetilde{g}})\big)*g\simeq_{\mathrm{rel}\ \{0,1\}} f\\
\implies\ & \big(p_*[\widetilde{f}*\overline{\widetilde{g}}\,]\big)\cdot[g]=[f].
\end{aligned}$$

Φ is surjective: For any $e_1\in p^{-1}(b_0)$ choose a path $\gamma:[0,1]\to E$ with $\gamma(0)=e_0$ and $\gamma(1)=e_1$. Consider $f:=p\circ\gamma$. Then, Φ sends $\mathrm{cls}\big([f]\big)$ to $\gamma(1)=e_1$. $\square$

Corollary 108 *Using the injectivity of* Φ *one can show that for* $f\colon \big([0,1],\{0,1\}\big) \to (B,b_0)$ *we have* $[f] \in p_*\pi_1(E,e_0) \iff f$ *lifts to a loop in* E *based at* e_0*.*

Corollary 109 *Let* $n \geq 2$*. Considering the* 2*-fold covering* $\mathbb{S}^n \ni x \longmapsto [x] \in \mathbb{R}P^n$ *and* $\mathbb{S}^n$ *is simply-connected we have* $\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$*. Note that* $\mathbb{R}P^1 \ni [z] \longmapsto z^2 \in \mathbb{S}^1$ *is a homeomorphism.*

Corollary 110 *In Theorem 107,* $\#p^{-1}(b_0) = n$ *implies index of* $p_*\pi_1(E,e_0)$ *in* $\pi_1(B,b_0)$ *is* n*.*

Theorem 111 *Let* X *be a path-connected and locally path-connected space such that* $\pi_1(X)$ *is finite. Then every* $f\colon X \to \mathbb{S}^1$ *is null-homotopic. In particular,* $\pi_n(\mathbb{S}^1) = \big[(\mathbb{S}^n,*),(\mathbb{S}^1,1)\big]$ *is a trivial group for* $n \geq 2$*.*

Proof. Consider the lifting given below; lifting exists as $f_*\pi_1(X,x_0)$ is a finite subgroup of $\mathbb{Z}$, so the algebraic condition of lifting is satisfied as any finite subgroup of $\mathbb{Z}$ is the trivial group.

$$\begin{tikzcd}
 & (\mathbb{R},0) \arrow[d, "t\longmapsto\exp(2\pi i t)"] \\
(X,x_0) \arrow[r, "f"'] \arrow[ur, dashed, "\exists!\,\widetilde{f}"] & (\mathbb{S}^1,1)
\end{tikzcd}$$

Now, consider the homotopy $H\colon X \times [0,1] \to X$ given by

$$H(x,t) := \exp\big(2\pi i \cdot (1-t)\widetilde{f}(x)\big) \text{ for } x \in X, t \in [0,1].$$

□

Remark 112 Recall that we defined degree of a map $f\colon (\mathbb{S}^1,1) \to (\mathbb{S}^1,*)$ as follows: Consider the map $f_1\colon [0,1] \ni t \longmapsto f\big(e^{2\pi i t}\big) \in \mathbb{S}^1$. Then $f_1(0) = *$. Let $e^{2\pi i \bullet} = *$ for some $\bullet \in \mathbb{R}$. Now, consider the unique lift $\widetilde{f_1}\colon \big([0,1],0\big) \to (\mathbb{R},\bullet)$, i.e., $e^{2\pi i \widetilde{f_1}(-)} = f_1$. Define,

$$\deg(f) := \widetilde{f_1}(1) - \widetilde{f_1}(0).$$

Theorem 113 *Every odd map* $\mathbb{S}^1 \to \mathbb{S}^1$ *has an odd degree.*

Proof. Now, let $f\colon (\mathbb{S}^1,1) \to (\mathbb{S}^1,*)$ be a odd map. Then,

$$e^{2i\pi\widetilde{f_1}\left(t+\frac{1}{2}\right)} = f(-e^{2i\pi t}) = -f(e^{2i\pi t}) = -e^{2i\pi\widetilde{f_1}(t)} = e^{2i\pi\left(\widetilde{f_1}(t)+\frac{1}{2}\right)} \text{ for all } t \in [0,1].$$

So there exists $m \in \mathbb{Z}$ such that

$$\widetilde{f_1}\left(t+\frac{1}{2}\right) = \widetilde{f_1}(t) + \frac{1}{2} + m$$

Finally,

$$\widetilde{f_1}(t+1) - \widetilde{f_1}(t) = \left[\widetilde{f_1}(t+1) - \widetilde{f_1}\left(t+\frac{1}{2}\right)\right] + \left[\widetilde{f_1}\left(t+\frac{1}{2}\right) - \widetilde{f_1}(t)\right] = 2m+1$$

and the degree of f is odd. □

Theorem 114 *Let* $n \geq 2$*. Then there does not exist* $\varphi\colon \mathbb{S}^n \to \mathbb{S}^1$ *such that* $\varphi(-x) = -\varphi(x)$ *for all* $x \in \mathbb{S}^n$*.*

Proof. Suppose, such a $\varphi\colon \mathbb{S}^n \to \mathbb{S}^1$ exists. Then φ null-homotopic by Theorem 111. Consider the inclusion $i\colon \mathbb{S}^1 \hookrightarrow \mathbb{S}^n$. Then $\varphi \circ i$ is also null-homotopic. In particular, $\deg(\varphi \circ i) = 0$. Also, $\varphi \circ i$ is an odd map, i.e., $\deg(\varphi \circ i)$ is an odd integer by Theorem 113, a contradiction. □

Theorem 115 (Borsuk-Ulam Theorem) *If $f\colon \mathbb{S}^n \to \mathbb{R}^n$ is a map then there exists $x \in \mathbb{S}^n$ such that $f(-x) = f(x)$.*

Theorem 116 *Let G be a group with identity element e, which may or may not have topology, acting on a simply-connected topological space X such that for each $g \in G$ the map $X \ni x \mapsto g \cdot x \in X$ is continuous. That's G acts on X continuously.*
Suppose also that G acts discretely, i.e., for each $x \in X$ there is an open neighborhood U of x such that

$$U \cap gU = \varnothing \text{ for all } g \in G \backslash \{e\}.$$

Consider the orbit space $X/G := \frac{X}{x \sim g \cdot x}$ with quotient topology obtained from quotient map $q\colon X \to X/G$. Choose $x_0 \in X$ and define $\Phi\colon G \to \pi_1(X/G, [x_0])$ by

$$\Phi\colon g \longmapsto \big[q \circ (\text{ path in } X \text{ from } x_0 \text{ to } g \cdot x_0)\big].$$

Then, Φ is a group isomorphism.

Proof. The quotient map $q : X \to X/G$ is a covering map: For any $x \in X$ consider a nbd U of x as above. Then,

$$q^{-1}\big(q(U)\big) = \bigsqcup_{g \in G} g \cdot U \subseteq_{\text{open}} X.$$

Φ is well-defined:Given $g \in G$, let α be a path in X from x_0 to $g \cdot x_0$, we have $q \circ \alpha$ is a loop in X/G based at $[x_0] \in X/G$. For any other path β in X from x_0 to $g \cdot x_0$, we have a homotopy $H\colon \alpha \simeq_{\text{rel } \{0,1\}} \beta$ as X is simply connected, so $q \circ H\colon q \circ \alpha \simeq_{\text{rel } [x_0]} q \circ \beta$.
Φ is a group homomorphism: For $g_1, g_2 \in G$, consider a path γ in X from x_0 to $g_1 \cdot x_0$ and another path δ in X from x_0 to $g_2 \cdot x_0$. Then, $\gamma * (g_1 \cdot \delta)$ is a path from x_0 to $(g_1 g_2) \cdot x_0$.
Φ is surjective: For a loop $\Gamma\colon [0,1] \to X/G$ based at $[x_0]$. Then, for any lifting $\widetilde{\Gamma}\colon [0,1] \to X$ of Γ we have $\widetilde{\Gamma}(0) = x_0$ and $\widetilde{\Gamma}(1) = g \cdot x_0$ for some $g \in G$. So, $\Phi(g) = [\Gamma]$.
Φ is injective: Suppose for some $g \in G$ and some $\alpha\colon [0,1] \to X$ from x_0 to $g \cdot x_0$ we have

$$p \circ \alpha \ \simeq_{\text{rel } [x_0]} \ C_{[x_0]}$$

Now, $\alpha(0) = x_0 = C_{x_0}(0)$. So, Theorem 106 gives that $\alpha(1) = C_{x_0}(1)$ so that $g \cdot x_0 = x_0 \implies g = e$ as group action is discrete. □

Examples

- Consider the group action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ by addition. The orbit space is $\big(\mathbb{S}^1\big)^n$.
- Let $n \geq 2$ and consider the $\{\pm 1\}$ action on $\mathbb{S}^n$ as $(\varepsilon, x) \mapsto \varepsilon \cdot x$. The orbit space is $\mathbb{R}P^n$.
- Consider the $\mathbb{Z}$ action on $\mathbb{R} \times [-1, 1]$ given by $\big(n, (x, y)\big) \longmapsto \big(x + \frac{1}{2}n, (-1)^n y\big)$. The orbit space is the Möbius strip.
- Let G be the of self-homeomorphisms of $\mathbb{R}^2$ generated by $A\colon \mathbb{R}^2 \ni (x, y) \longmapsto (x+1, 1-y) \in \mathbb{R}^2$ and $B\colon \mathbb{R}^2 \ni (x, y) \longmapsto (x, y+1) \in \mathbb{R}^2$. Then $\mathbb{R}^2/G$ =Klein bottle. Note that G is non-abelian.

- Let $p, q \in \mathbb{N}$ with $\gcd(p, q) = 1$, p and q no need to be primes. Consider the group action $\mathbb{Z}_p \times \mathbb{S}^3 \to \mathbb{S}^3$ given by
$$\left([k]_p, (z_1, z_2)\right) \longmapsto \left(e^{\frac{2\pi ik}{p}} z_1, e^{\frac{2\pi ikq}{p}} z_2\right)$$
for all $(z_1, z_2) \in \mathbb{C}^2$ with $|z_1|^2 + |z_2|^2 = 1$. The orbit space is an orientable 3-manifold, called lens space, and denoted by $L(p, q)$.

Theorem 117 *Let $Y := \mathbb{C}^*/K$ where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and K is the group of homeomorphisms $\{\varphi^n : n \in \mathbb{Z}\}$ with $\varphi(z) = 4z$. Then, the fundamental group of Y is $\mathbb{Z} \times \mathbb{Z}$.*

Proof. □

Theorem 118 (Primary decomposition of finitely generated abelian group) *Every finitely generated abelian group G is isomorphic to a group of the form $\mathbb{Z}^n \oplus \mathbb{Z}_{p_1^{\ell_1}} \oplus \cdots \oplus \mathbb{Z}_{p_t^{\ell_t}}$ for some integers $n, t \geq 0$; and primes (not necessarily distinct) $p_1, ..., p_t$; and non-negative integers (not necessarily distinct) $\ell_1, ..., \ell_t$.*

Theorem 119 *Given any finitely generated abelian group G, we have a path-connected manifold M such that $\pi_1(M) = G$.*

Proof. Consider Theorem 118 with the manifold $M := \left(\mathbb{S}^1\right)^n \times L\left(p_1^{\ell_1}, 1\right) \times \cdots \times L\left(p_t^{\ell_t}, 1\right)$ □

Theorem 120 [Hat02, Page 30] [Rot88, Theorem 10.5] *Let $p\colon \widetilde{X} \to X$ be a covering space and Y be a connected space. Then, the commutative square on the left has a unique solution, which is the commutative square on the right.*

$$\begin{tikzcd} Y \arrow[r, "\widetilde{f}"] \arrow[d, "y\mapsto(y,0)"'] & \widetilde{X} \arrow[d, "p"] \\ Y \times [0,1] \arrow[r, "F"'] & X \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "\widetilde{f}"] \arrow[d, "y\mapsto(y,0)"'] & \widetilde{X} \arrow[d, "p"] \\ Y \times [0,1] \arrow[r, "F"'] \arrow[ur, dashed, "\exists!\widetilde{F}"] & X \end{tikzcd}$$

Theorem 121 [Hat02, Exercise 8 Chapter 1.3] *Let $\overline{X}, \overline{Y}$ be simply connected covering spaces of path-connected, locally path-connected spaces X and Y, respectively. Then $X \simeq Y$ implies $\overline{X} \simeq \overline{Y}$.*

Proof. Let $f\colon (X, x_0) \to (Y, y_0)$ be a homotopy-equivalence with $h\colon (Y, y_0) \to (X, x_1)$ as a homotopy inverse, i.e., $f \circ h \simeq \mathrm{Id}_Y$ and $h \circ f \simeq \mathrm{Id}_X$. Let $p\colon (\overline{X}, \overline{x}_0, \overline{x}_1) \to (X, x_0, x_1)$ and $q\colon (\overline{Y}, \overline{y}_0) \to (Y, y_0)$ be the universal covering maps. Consider the two lifts below.

$$\begin{tikzcd} & (\overline{Y}, \overline{y}_0) \arrow[d, "q"] \\ (\overline{X}, \overline{x}_0) \arrow[r, "fp"'] \arrow[ur, dashed, "\overline{fp}"] & (Y, y_0) \end{tikzcd} \qquad \begin{tikzcd} & (\overline{X}, \overline{x}_1) \arrow[d, "p"] \\ (\overline{Y}, \overline{y}_0) \arrow[r, "hq"'] \arrow[ur, dashed, "\overline{hq}"] & (X, x_1) \end{tikzcd}$$

Now, $\overline{fp} \circ \overline{hq}$ is a lift of fhq, and $F\colon fhq \simeq q$. To see these notice that

$$q \circ \left(\overline{fp} \circ \overline{hq}\right) = \left(q \circ \overline{fp}\right) \circ \overline{hq} = fp \circ \overline{hq} = f \circ \left(p \circ \overline{hq}\right) = f \circ hq \text{ and}$$

$$f \circ hq = fh \circ q \simeq \mathrm{Id}_Y \circ q = q.$$

Applying Theorem 120 w.r.t the covering $q\colon \overline{Y} \to Y$ and homotopy $F\colon fhq \simeq q$, we have homotopy lifting $\overline{F}\colon \left(\overline{fp} \circ \overline{hq}\right) \simeq \overline{F}(-,1)$, where $q \circ \overline{F} = F$. In particular, $q \circ \overline{F}(-,1) = q$. Then, $\overline{F}(-,1)\colon \overline{Y} \to \overline{Y}$ is a homeomorphism, see [Rot88, Corollary 10.15.]. Hence,

$$\left(\overline{fp} \circ \overline{hq}\right) \simeq \overline{F}(-,1) \implies \left(\overline{F}(-,1)\right)^{-1} \circ \left(\overline{fp} \circ \overline{hq}\right) \simeq \mathrm{Id}_{\overline{Y}} \implies \overline{hq} \text{ has a homotopy left-inverse.}$$

Similarly, $\overline{hq}$ has a right-inverse: $p \circ \left(\overline{hq} \circ \overline{fp}\right) = hq \circ \overline{fp} = h \circ fp \simeq \mathrm{Id}_X \circ p = p$. Applying Theorem 120 w.r.t. the covering $p\colon \overline{X} \to X$ and homotopy $G\colon hfp \simeq p$ we have homotopy lifting $\overline{G}\colon \left(\overline{hq} \circ \overline{fp}\right) \simeq \overline{G}(-,1)$ with $p \circ \overline{G}(-,1) = p$. Then $\overline{G}(-,1)\colon \overline{X} \to \overline{X}$ is a homeomorphism. So, $\overline{hq} \circ \overline{fp} \circ \left(\overline{G}(-,1)\right)^{-1} \simeq \mathrm{Id}_{\overline{X}}$. Therefore, $\overline{hq}\colon \overline{Y} \to \overline{X}$ is a homotopy-equivalence. □

Remark 122 *If a square matrix has left and right inverses, then the matrix has THE inverse.*

Theorem 123 [Hat02, Exercise 17 Chapter 1.1] *There are infinitely many non-homotopic retractions* $\mathbb{S}^1 \vee \mathbb{S}^1 \longrightarrow \mathbb{S}^1$.

Proof. Note that $\mathbb{S}^1 \vee \mathbb{S}^1 = \left(\mathbb{S}^1 \times \{1\}\right) \bigcup \left(\{1\} \times \mathbb{S}^1\right)$. For $k \in \mathbb{N}$, define retraction $r_k\colon \mathbb{S}^1 \vee \mathbb{S}^1 \to \mathbb{S}^1 \times \{1\}$ by

$$r_k\big(a,1\big) = (a,1) \text{ for all } a \in \mathbb{S}^1,$$

$$r_k(1,b) = (b^k,1) \text{ for all } b \in \mathbb{S}^1.$$

Hence $r_k|_{\mathbb{S}^1 \times \{1\}} = \mathrm{Id}_{\mathbb{S}^1 \times \{1\}}$ and $r_k|_{\{1\} \times \mathbb{S}^1}\colon \{1\} \times \mathbb{S}^1 \to \mathbb{S}^1 \times \{1\}$ is k-fold covering map. So, $r_k \not\simeq r_l$ if $k \neq l$. □

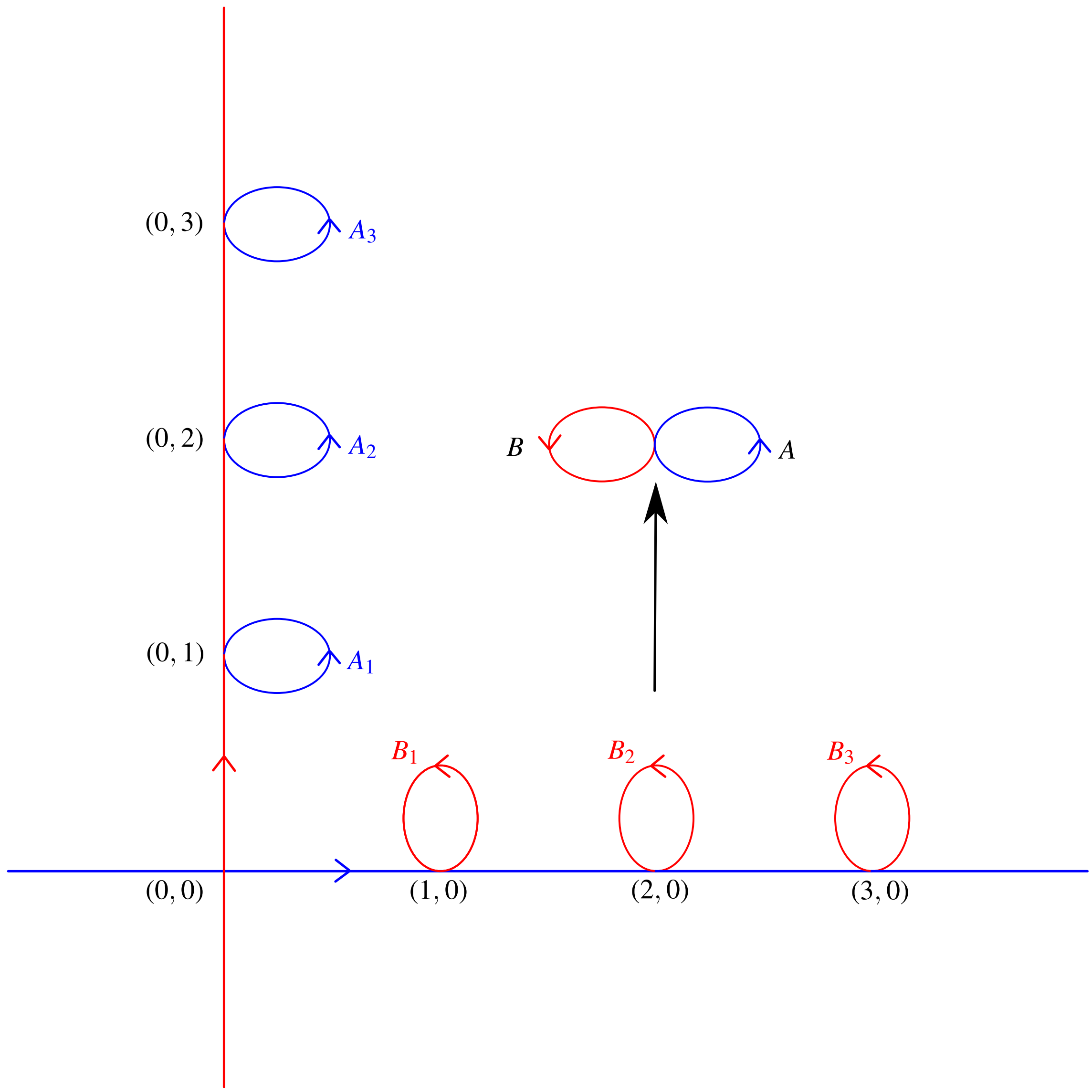
Theorem 124 [Hat02, Exercise 16.(e) Chapter 1.1] *If X is $\mathbb{D}^2$ with two points on its boundary identified and A is its boundary $\mathbb{S}^1 \vee \mathbb{S}^1$, then there is no retraction from $X \to A$.*

Proof. Let $q\colon \mathbb{D}^2 \to \frac{\mathbb{D}^2}{\{i,-i\}} \cong X$ be the quotient map. Now, $H\colon \mathbb{D}^2 \times [0,1] \ni (x,y,t) \longmapsto \big((1-t)x, y\big) \to \mathbb{D}^2$ is a strong deformation retract of $\mathbb{D}^2$ onto the line-segment $\ell := \{0\} \times [0,1]$. Thus

$$\widetilde{H}\colon X \times [0,1] \ni \big([x,y],t\big) \longmapsto \big[(1-t)x, y\big] \in X$$

is a strong deformation retract of X onto $q(\ell)$. Now, $q(\ell) \cong \mathbb{S}^1$ implies $\pi_1(X) \cong \mathbb{Z}$. So if there were a retraction $r\colon X \to A$, the inclusion induced map $\mathbb{Z} * \mathbb{Z} \cong \pi_1(A) \to \pi_1(X) \cong \mathbb{Z}$ would be injective, which is impossible as $\mathbb{Z} * \mathbb{Z}$ is non-abelian. □

Theorem 125 *The fundamental group of $\mathbb{S}^1 \vee \mathbb{S}^1$ is non-abelian.*



An infinite fold cover of $\mathbb{S}^1 \vee \mathbb{S}^1$

Proof. Consider the above cover of $\mathbb{S}^1 \vee \mathbb{S}^1$. Now, define a path $\widetilde{A * B}$ as follows: $\widetilde{A * B}$ starts at $(0,0)$, then traverses the blue horizontal straight line segment until it reaches $(1,0)$, and finally traverses the B_1 loop anti-clockwise once. Similarly, define a path $\widetilde{B * A}$ as follows: $\widetilde{B * A}$ starts at $(0,0)$, then traverses the red straight vertical straight line segment until it reaches $(0,1)$, and finally traverses the A_1 loop anti-clockwise once. Note that $\widetilde{A * B}$ and $\widetilde{B * A}$ are lifts of $A * B$ and $B * A$, respectively starting at $(0,0)$.

Now, if possible let $[A][B] = [B][A]$ in $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1, p)$, where p is the wedge point. Thus $A * B \simeq_{\text{rel } p} B * A$, and by Theorem 106, the endpoint of $\widetilde{A * B}$is the same as the endpoint of $\widetilde{B * A}$, i.e., $(1,0) = (0,1)$, which is impossible. So $[A][B] \neq [B][A]$, and we are done. $\square$

5 Calculating the fundamental groups of manifolds and CW-complexes

5.1 Seifert-Van Kampen theorem

Definition 126 Let G be and H be two groups. We consider the set $G * H$ of all finite sequences $(x_1, ..., x_m)$ such that the following conditions are satisfied:

- each x_i lies in one of the groups G or H,
- no x_j is the neutral element of G or of H,
- any two consecutive x_j's lie in two different groups.

Here we also allow the empty sequence (). Such sequences are sometimes called reduced words in G and H.

Now, define a group structure on $G * H$. Given two sequences $(x_1, ..., x_m)$ and $(y_1, ..., y_n)$, we stack them together $(x_1, ..., x_m, y_1, ..., y_n)$ and then we delete any occurrence of a subsequence of the form a, a^{-1} for $a \in G$ or $a \in H$ and if a subsequence is of the form a, b with $a, b \in G$ or $a, b \in H$, then we replace it by ab.

We henceforth refer to $G * H$ together with this product structure as the free product of G and H.

Definition 127 For a set S we refer to

$$\langle S \rangle := \text{free product of the infinite cyclic groups generated by } s \in S$$

as the free group on the (generating) set S.

Definition 128 Let G be a group and let A be a subset of G. Define

$$\text{subgroup of } G \text{ normally generated by } A := \overline{A} := \bigcap_{A \subseteq H \trianglelefteq G} H$$

Definition 129 Let G be a group. A presentation of G is a collection: A set X, a subset R of the free group $\langle X \rangle$, and an isomorphism $G \to \langle X | R \rangle := \dfrac{\langle X \rangle}{\overline{R}}$. If X and R both are finite sets, then we say G is finitely presented.

Definition 130 Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. We define the amalgamated product $A *_G B$ of A and B with amalgam G as

$$A *_G B := \frac{A * B}{\overline{\{\alpha(g)\beta(g)^{-1} | g \in G\}}}.$$

Definition 131 *Let G be a group and let $A := \{xyx^{-1}y^{-1} | x, y \in G\}$. Define the abelianization G_{ab} of G as $G_{\text{ab}} := \dfrac{G}{\overline{A}}$.*

Theorem 132 Let $\alpha\colon G \to A$, $\beta\colon G \to B$, $\widetilde{\beta}\colon H \to B$, and $\gamma\colon H \to C$ be group homomorphisms. Now, we have the following:

(1) If $G = \{e\}$, then $A *_G B = A * B$.

(2) If $B = \{e\}$, then $A *_G B = \dfrac{A}{\alpha(G)}$.

(3) If β is an isomorphism (resp. epimorphism), then the obvious map $A \to A *_G B$ is also an isomorphism (resp. epimorphism).

(4) $(A *_G B) *_H C \cong A *_G (B *_H C)$.

(5) Let $\varphi\colon H \to G$ be an epimorphism. Using $\alpha \circ \varphi\colon H \to A$ and $\beta\colon \varphi\colon H \to B$, we can talk about $A *_H B$. Now, the natural map $A *_G B \to A *_H B$ is an isomorphism.

(6) If α and β are both monomorphisms, then the natural homomorphisms $A \to A *_G B$ and $B \to A *_G B$ are also both monomorphisms.

(7) $(G * H)_{\text{ab}} \cong G_{\text{ab}} \times H_{\text{ab}}$. In particular, $\langle x_1, ..., x_n \rangle_{\text{ab}} \cong \mathbb{Z}^n$.

Theorem 133 (Seifert-Van Kampen theorem) Let X be a topological space and let $X = U \cup V$ be a decomposition of X in two open subsets U and V such that $U \cap V$ is non-empty and path-connected. Let $x_0 \in U \cap V$. Then there exists an isomorphism $\Phi\colon \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0) \to \pi_1(X, x_0)$ such that the following diagram commutes:

$$\begin{tikzcd}
\pi_1(U \cap V, x_0) \arrow[r] \arrow[d] & \pi_1(U, x_0) \arrow[d] \arrow[ddr, bend left] & \\
\pi_1(V, x_0) \arrow[r] \arrow[drr, bend right] & \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0) \arrow[dr, dashed, "\Phi"] & \\
& & \pi_1(x, x_0)
\end{tikzcd}$$

Here all the undercoated maps are the obvious inclusion-induced homomorphisms.

Remark 134 In Theorem 133, the inclusion induced maps $\pi_1(U, x_0) \to \pi_1(X, x_0)$ and $\pi_1(V, x_0) \to \pi_1(X, x_0)$ gives a surjection $\pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$. In particular, if U and V are simply-connected, then X is also so.

Remark 135 By Remark 134, the suspension of path-connected space is simply-connected.

Remark 136 Consider Theorem 133 again. If $U \cap V$ is simply-connected, then the inclusion induced maps $\pi_1(U, x_0) \to \pi_1(X, x_0)$ and $\pi_1(V, x_0) \to \pi_1(X, x_0)$ gives a bijection $\pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$.

Definition 137 We say a point x in a topological space X is good, if $\{x\}$ is a closed subset of X and there exists an open neighborhood U of x such that x is a deformation retract of U.

Remark 138 Every point of a topological manifold or a CW-complex is a good point.

Theorem 139 *Let A_1 and A_2 be two path-connected topological spaces, and let $a_1 \in A$ and $a_2 \in A$ be good points. Then, the inclusion maps induce an isomorphism $\pi_1(A_1, a_1) * \pi_1(A_2, a_2) \xrightarrow{\cong} \pi_1(A_1 \vee A_2, a_1 = a_2)$.*

Proof. We pick an open neighborhood W_1 in A_1 of that deformation retracts to a_1. Furthermore, we pick an open neighborhood W_2 of a_2 in A_2 that deformation retracts to a_2. We consider $U := A_1 \vee W_2 \subseteq_{\text{open}} A_1 \vee A_2$ and $V := W_1 \vee A_2 \subseteq_{\text{open}} A_1 \vee A_2$. Note that A_1 (resp. A_2) is a deformation retract onto U (resp. V) and $U \cap V$ has a deformation retract onto $x_0 := \{a_1, a_2\} \in A_1 \vee A_2$. Therefore, the inclusion induced maps $\pi_1(A_1, a_1) \to \pi_1(U, x_0)$, $\pi_1(A_2, a_2) \to \pi_1(V, x_0)$ are isomorphisms and $\pi_1(U \cap V, x_0)$ is trivial. Thus the inclusion induced map $\pi_1(A_1, a_1) * \pi_1(A_2, a_2) \to \pi_1(U, x_0) * \pi_1(V, x_0)$ is an isomorphism. Now, by Remark 136, we are done. $\square$

Remark 140 Let n be a positive integer. An induction on n together with Theorem 139, says that $\pi_1\left(\bigvee^n \mathbb{S}^1\right) \cong \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n\text{-times}}$.

Theorem 141 Let M be a topological manifold of dimension $n \geq 3$. Let p be a point and let $x_0 \in M \setminus \{p\}$ be a base point. Then, the inclusion induced map $\pi_1\big(M \setminus \{p\}, x_0\big) \to \pi_1(M, x_0)$ is an isomorphism.

Proof. Since M is locally $\mathbb{R}^n$, pick an open ball $B(p, r) \subseteq M$ of radius r centered at p. Define $U := B(p, r)$ and $V := M \setminus \{p\}$. Now, $U \cap V = B(p, r) \setminus \{p\} \cong \mathbb{S}^{n-1} \times (0, 1) \simeq \mathbb{S}^{n-1}$ is simply-connected as $n \geq 3$. Now, by Remark 136, we are done. $\square$

Lemma 142 *Let X be a topological space. Furthermore, let A and B be two subsets with $X = A \cup B$. If $A \cap B$ is a deformation retract of B and if A and B are closed subsets of X, then A is a deformation retract of X.*

Proof. We pick a deformation retraction $F\colon B \times [0, 1] \to A \cap B$ of B onto $A \cap B$. Now, define $G\colon X \times [0, 1] \to X$ as $G(x, t) = x$ if $(x, t) \in A \times [0, 1]$ and $G(x, t) = F(x, t)$ if $(x, t) \in B \times [0, 1]$. $\square$

Theorem 143 (Topological Collar Neighborhood Theorem) Given a topological manifold M, there is an embedding $\varphi\colon \partial M \times [0, 1] \hookrightarrow M$ such that $\varphi(p, 0) = p$ for $p \in \partial M$, $\operatorname{im}(\varphi) \subseteq_{\text{closed}} M$, and $\varphi\big(\partial M \times [0, 1)\big) \subseteq_{\text{open}} M$.

Theorem 144 Let M be an m-dimensional topological manifold and let $R, S \subseteq_{\text{closed}} M$ be two m-dimensional submanifolds such that $R \cup S = M$ and $R \cap S$ is a component of ∂R as well as a component of ∂S. Then for any base point $x_0 \in R \cap S$, the inclusion induced maps $\pi_1(R, x_0) *_{\pi_1(R \cap S, x_0)} \pi_1(S, x_0) \to \pi_1(M, x_0)$ is an isomorphism.

Proof. By Theorem 143, let $f\colon \partial R \times [0, 1] \hookrightarrow R$ and $g\colon \partial S \times [0, 1] \hookrightarrow S$ be two collars. Define

$$U := R \cup g\big([0, 1) \times (R \cap S)\big),\ V := S \cup f\big([0, 1) \times (R \cap S)\big).$$

Now, $R \cap U = R \subseteq_{\text{open}} R$ and $S \cap U = g\big([0, 1) \times (R \cap S)\big) \subseteq_{\text{open}} S$ as $g\big([0, 1) \times (R \cap S)\big)$ is open in M. Since R and S are closed subsets of M with $M = R \cup S$, the set U is open in M. Similarly, V is open in M.

Also, $R = R \cap U \subseteq_{\text{closed}} U$ as R is closed in M and $g\big([0, 1) \times (R \cap S)\big) = S \cap U \subseteq_{\text{closed}} U$ as U is closed in M. Thus by Lemma 142, R is a deformation retract of U. Similarly, S is a deformation retract of V.

One can also show that $R \cap S$ is a deformation retract of $U \cap V$. Thus the inclusion induced map $\pi_1(R \cap S, x_0) \to \pi_1(U \cap V, x_0)$, $\pi_1(R, x_0) \to \pi_1(U, x_0)$, and $\pi_1(S, x_0) \to \pi_1(V, x_0)$ are isomorphisms. Now, we are done, as Theorem 133 tells that the the inclusion induced maps give an isomorphism $\pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0) \to \pi_1(M, x_0)$. $\square$

5.2 Connected sum of two closed smooth manifolds

Theorem 145 (Palais disk theorem) Let M be a closed smooth n-manifold, where $n \geq 2$. If M is orientable, then we pick an orientation for M. Let $\varphi_1, \varphi_2 \colon \overline{\mathbb{B}^n} \hookrightarrow M$ be two smooth embeddings (if M is oriented, we demand that either φ_1, φ_2 both are orientation-preserving or φ_1, φ_2 both are orientation-reversing). Then there is smooth homotopy $H \colon M \times [0,1] \to M$ through diffeomorphisms starting from Id_M such that $H(-,1) \circ \varphi_1 = \varphi_2$.

Let M, N be two connected closed smooth n-manifolds, where $n \geq 2$. Let $\varphi_1, \varphi_2 \colon \overline{\mathbb{B}^n} \hookrightarrow M$ and $\psi_1, \psi_2 \colon \overline{\mathbb{B}^n} \hookrightarrow N$ be smooth embeddings. We will write $\varphi_1 \equiv \varphi_2$ (resp. $\psi_1 \equiv \psi_2$) if there is a diffeomorphism $f \colon M \to M$ homotopic to Id_M (resp. $g \colon N \to N$ homotopic to Id_N) with $f\varphi_1 = \varphi_2$ (resp. $g\psi_1 = \psi_2$). Define two smooth manifolds (smoothness checking is technical!)

$$M \sharp_{(\varphi_k, \psi_k)} N := \frac{\big(M \setminus \varphi_k(\mathbb{B}^n)\big) \sqcup \big(N \setminus \psi_k(\mathbb{B}^n)\big)}{\varphi_k(p) \sim \psi_k(p),\ p \in \mathbb{S}^{n-1}} \text{ for } k = 1, 2.$$

Remark 146 *Suppose $\varphi_1 \equiv \varphi_2$ and $\psi_1 \equiv \psi_2$. Choose diffeomorphisms $f \colon M \to M$ and $g \colon N \to N$ with $f\varphi_1 = \varphi_2$ and $g\psi_1 = \psi_2$. Now, the map $M\sharp_{(\varphi_1,\psi_1)}N \to M\sharp_{(\varphi_2,\psi_2)}N$ defined by*

$$x \longmapsto \begin{cases} f(x) & \text{if } x \in M \setminus \varphi_1(\mathbb{B}^n), \\ g(x) & \text{if } x \in N \setminus \psi_1(\mathbb{B}^n). \end{cases}$$

is a diffeomorphism (checking smoothness of this map is technical!).

Case 1: Both M, N are non-orientable. By Theorem 145, $\varphi_1 \equiv \varphi_2$ and $\psi_1 \equiv \psi_2$. Thus $M\sharp_{(\varphi_1,\psi_1)}N \cong M\sharp_{(\varphi_2,\psi_2)}N$ by Remark 146.

Case 2: Next, suppose M is oriented and N is non-orientable. Thus $\psi_1 \equiv \psi_2$. Now, we consider two sub-cases, namely $\varphi_1 \equiv \varphi_2$ and $\varphi_1 \not\equiv \varphi_2$.

- If former happens, then $M\sharp_{(\varphi_1,\psi_1)}N \cong M\sharp_{(\varphi_2,\psi_2)}N$ by Remark 146.
- For the later, $\varphi_1\tau \equiv \varphi_2$ for any orientation-reversing diffeomorphism $\tau \colon \overline{\mathbb{B}^n} \to \overline{\mathbb{B}^n}$. Thus $M\sharp_{(\varphi_1\tau,\psi_1\tau)}N \cong M\sharp_{(\varphi_2,\psi_2)}N$ by Remark 146 since $\varphi_1\tau \equiv \varphi_2$ and $\psi_1\tau \equiv \psi_2$. Now, $M\sharp_{(\varphi_1\tau,\psi_1\tau)}N = M\sharp_{(\varphi_1,\psi_1)}N$ by definition.

Therefore, $M\sharp_{(\varphi_1,\psi_1)}N \cong M\sharp_{(\varphi_2,\psi_2)}N$ in any case.

Case 3: If N is oriented and M is non-orientable, the same argument as in Case 2 tells that $M\sharp_{(\varphi_1,\psi_1)}N \cong M\sharp_{(\varphi_2,\psi_2)}N$.

Case 4: Finally, consider both M, N are oriented.

(a) If $\varphi_1 \equiv \varphi_2$ and $\psi_1 \equiv \psi_2$, then we can show that $M\sharp_{(\varphi_1,\psi_1)}N \cong M\sharp_{(\varphi_2,\psi_2)}N$ by Remark 146.

(b) If $\varphi_1 \not\equiv \varphi_2$ and $\psi_1 \not\equiv \psi_2$, then for any orientation-reversing diffeomorphism $\tau \colon \overline{\mathbb{B}^n} \to \overline{\mathbb{B}^n}$, we have $\varphi_1\tau \equiv \varphi_2$ and $\psi_1\tau \equiv \psi_2$, i.e., again using Remark 146, we can show that $M\sharp_{(\varphi_1,\psi_1)}N = M\sharp_{(\varphi_1\tau,\psi_1\tau)}N \cong M\sharp_{(\varphi_2,\psi_2)}N$

(c) If $\varphi_1 \not\equiv \varphi_2$ but $\psi_1 \equiv \psi_2$ (or $\varphi_1 \equiv \varphi_2$ but $\psi_1 \not\equiv \psi_2$), then $M\sharp_{(\varphi_1,\psi_1)}N$ may not be diffeomorphic to $M\sharp_{(\varphi_2,\psi_2)}N$.

(d) If $\varphi_1 \equiv \varphi_2$ but $\psi_1 \not\equiv \psi_2$, then $M\sharp_{(\varphi_1,\psi_1)}N$ may not be diffeomorphic to $M\sharp_{(\varphi_2,\psi_2)}N$.

Remark 147 Consider the Case 4 above. Note that for two other embeddings $\Phi\colon \overline{\mathbb{B}^n} \hookrightarrow M$ and $\Psi\colon \overline{\mathbb{B}^n} \hookrightarrow N$, either $M\natural_{(\Phi,\Psi)}N \cong M\natural_{(\varphi_1,\psi_1)}N$ or $M\natural_{(\Phi,\Psi)}N \cong M\natural_{(\varphi_2,\psi_2)}N$.

Remark 148 Consider the Case 4 (c) above. Suppose, $\theta\colon M \to M$ is an orientation-reversing diffeomorphism (for example, M can be any closed orientable surface), then $\theta\varphi_1 \equiv \varphi_2$ by Theorem 145. Choose diffeomorphism $f\colon M \to M$ with $f(\theta\varphi_1) = \varphi_2$, i.e., $f\theta$ is a diffeomorphism taking φ_1 to φ_2, i.e., $\varphi_1 \equiv \varphi_2$. By Remark 146, $M\natural_{(\varphi_1,\psi_1)}N \cong M\natural_{(\varphi_2,\psi_2)}N$. Thus if either of M or N has an orientation-reversing diffeomorphism $M\natural_{(\varphi_1,\psi_1)}N \cong M\natural_{(\varphi_2,\psi_2)}N$, in any case.

Remark 149 If M, N are orientable, then $M\natural_{(\varphi_1,\psi_1)}N$ and $M\natural_{(\varphi_2,\psi_2)}N$ are also so.

Theorem 150 *Let M, N be two connected closed smooth n-manifolds, where $n \geq 3$. Let $\varphi\colon \overline{\mathbb{B}^n} \hookrightarrow M$ and $\psi\colon \overline{\mathbb{B}^n} \hookrightarrow N$ be smooth embeddings. Then*

$$\pi_1\left(M\natural_{(\varphi,\psi)}N\right) \cong \pi_1(M) * \pi_1(N).$$

Proof. Let $X := M \setminus \varphi(\mathbb{B}^n) \subseteq M\natural_{(\varphi,\psi)}N$ and $Y := N \setminus \psi(\mathbb{B}^n) \subseteq M\natural_{(\varphi,\psi)}N$. Thus $X \cup Y = M\natural_{(\varphi,\psi)}N$ and $X \cap Y \cong \mathbb{S}^{n-1}$ is simply-connected. By Theorem 144, the inclusion induced maps $\pi_1(X) \to \pi_1\left(M\natural_{(\varphi,\psi)}N\right)$ and $\pi_1(Y) \to \pi_1\left(M\natural_{(\varphi,\psi)}N\right)$ gives the isomorphism $\pi_1(X) * \pi_1 * (Y) \cong \pi_1\left(M\natural_{(\varphi,\psi)}N\right)$. Also, $M = X \cup \varphi\left(\overline{\mathbb{B}^n}\right)$ and $X \cap \varphi\left(\overline{\mathbb{B}^n}\right) \cong \mathbb{S}^{n-1}$ is simply-connected. Again by Theorem 144, the inclusion induced map $\pi_1(X) \to \pi_1(M)$ is an isomorphism. Similarly, the inclusion induced map $\pi_1(Y) \to \pi_1(N)$ is an isomorphism. So, we are done. $\square$

5.3 Classification of closed surfaces

Consider the bordered surface obtained from deleting the interior of a closed disk inside the Klein bottle (resp. torus). In Figure 1, their equivalent planar representations are given.

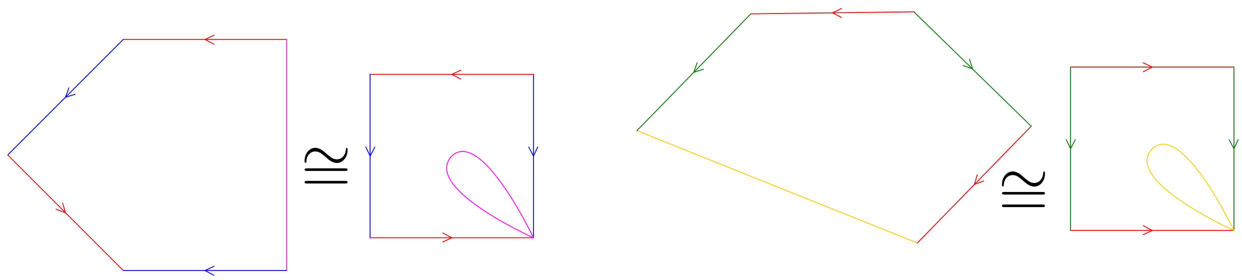


Fig. 1: One-holed Klein bottle and one-holed torus

Definition 151 Let g be a positive integer. Consider a regular $4g$-gon E_{4g} and label each edge of E_{4g} by the symbols $a_1, b_1, ..., a_g, b_g$ (each a_i or b_j appears twice) so that after orientating each edge of E_{4g}, the boundary $\partial E_{4g} \cong \mathbb{S}^1$ can be described by the word $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$.

Now, we identify the $(4j+1)$-st edge with the $(4j+3)$-rd edge and the $(4j+2)$-nd edge with the $(4j+4)$-th edge following the orientations provided. Denote the quotient space by Σ_g.

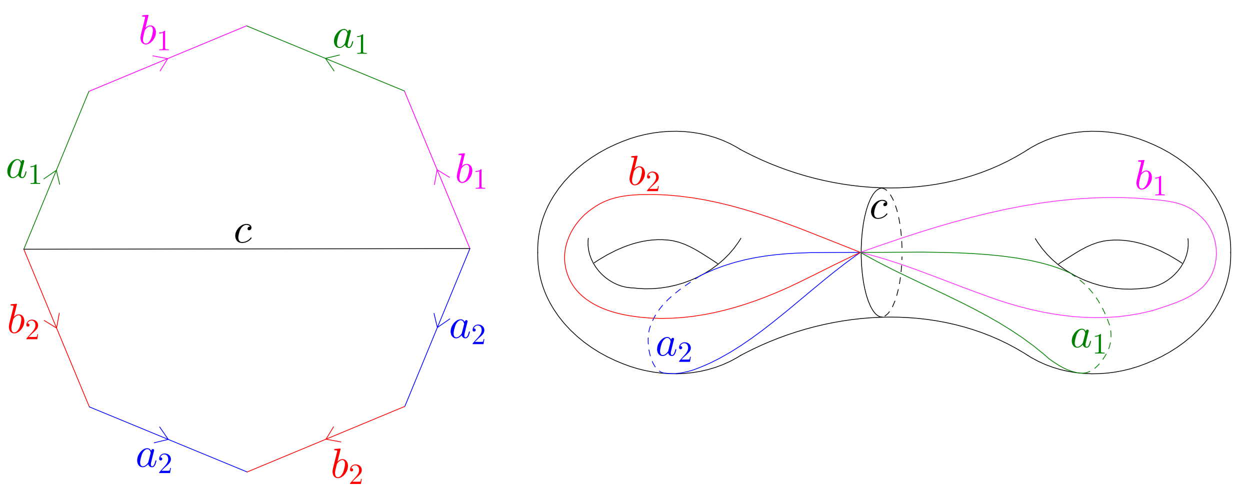


Fig. 2: Σ_2 is homeomorphic to $(\mathbb{S}^1 \times \mathbb{S}^1)\sharp(\mathbb{S}^1 \times \mathbb{S}^1)$

Definition 152 Let $h \geq 2$ be an integer. Consider a regular $2h$-gon E_{2h} and label each edge of E_{2h} by the symbols $a_1, ..., a_h$ (each a_i appears twice) so that after orientating each edge of E_{2h}, the boundary $\partial E_{2h} \cong \mathbb{S}^1$ can be described by the word $a_1^2 \cdots a_h^2$. Now, for any i, we identify two a_i-th edges following the orientation provided. Denote the quotient space by N_g.

Theorem 153 *The quotient space N_2 is homeomorphic to Klein Bottle.*

Proof. See Figure 3. $\square$

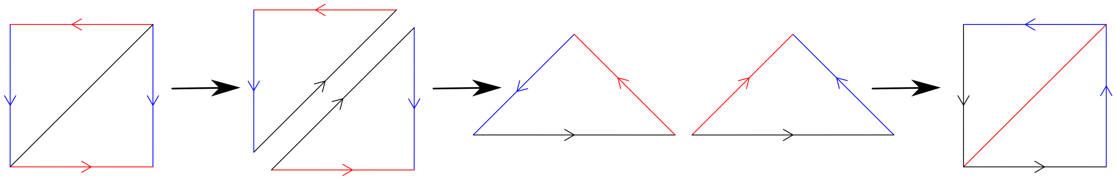


Fig. 3: Transformation of Klein Bottle to N_2

Theorem 154 *Let $\mathcal{D}$ be a closed disk embedded in $\mathbb{R}P^2$. Then $\mathbb{R}P^2 \setminus \operatorname{int}(\mathcal{D})$ is homeomorphic to the Möbius strip.*

Proof. Observe that $\mathbb{R}P^2$ is obtained from closed unit disk $\mathbb{D}^2$ with the identification $z \sim -z$, where $z \in \mathbb{S}^1$. Now, consider Figure 4, where we consider our favorite disk. For a general disk, consider Theorem 145.

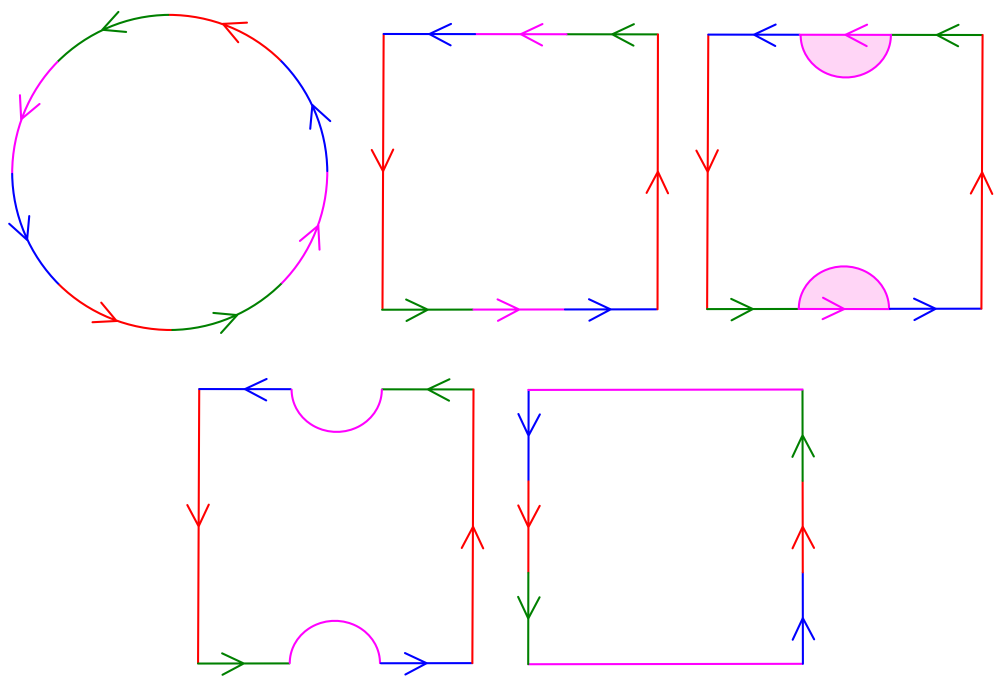


Fig. 4: The second row shows that $\mathbb{R}P^2$ minus a interior of small disk is the Möbius strip

□

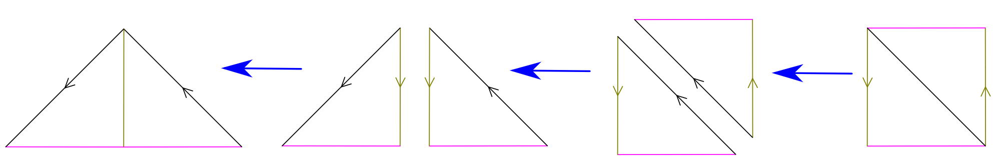


Fig. 5: An alternative presentation of Möbius strip, i.e., one-holed $\mathbb{R}P^2$

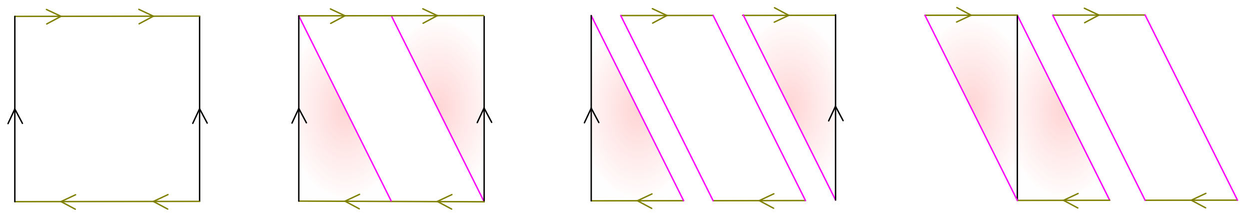


Fig. 6: Klein bottle is the double of Möbius strip

Theorem 155 *Klein Bottle is homeomorphic to* $\mathbb{R}P^2 \sharp \mathbb{R}P^2$.

Proof. The Figure 6 shows that the Klein bottle is the double of the Möbius strip. Now, applying Theorem 154, we are done.

□

Theorem 156 $(\mathbb{S}^1 \times \mathbb{S}^1) \sharp \mathbb{R}P^2$ *is homeomorphic to* $\mathbb{R}P^2 \sharp \mathbb{R}P^2 \sharp \mathbb{R}P^2$.

Proof. At first, observe Figure 1 and Figure 5. Now, consider Figure 7 below.

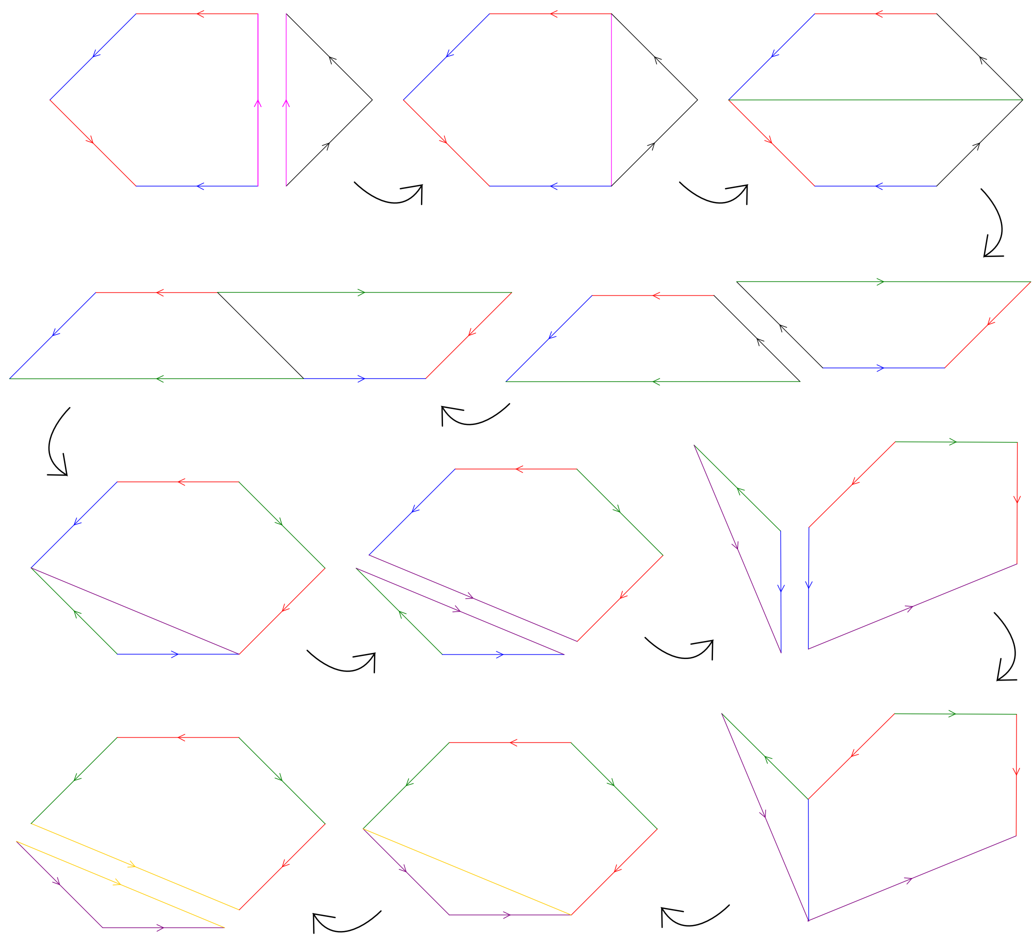


Fig. 7: Transformation of Klein bottle$\sharp\mathbb{R}P^2$ to $(\mathbb{S}^1 \times \mathbb{S}^1)\sharp\mathbb{R}P^2$

□

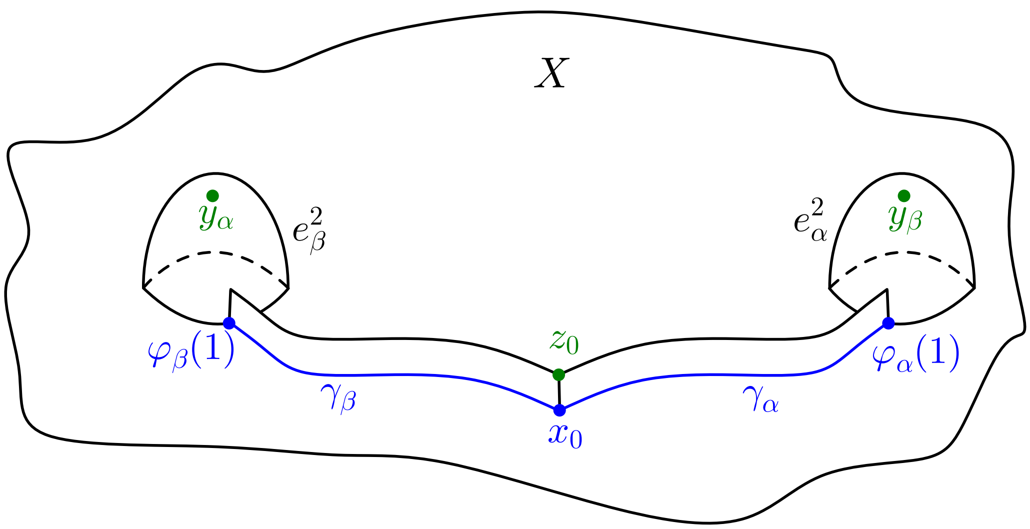
Theorem 157 [Lee11, Theorem 6.15.] Every nonempty, compact, connected 2-manifold is homeomorphic to one of the following:

- The sphere $\mathbb{S}^2$;
- A connected sum of one or more copies of torus $\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$, i.e., $\sharp_n \mathbb{T}^2 := \underbrace{\mathbb{T}^2 \sharp \cdots \sharp \mathbb{T}^2}_{n\text{-copies of } \mathbb{T}^2}$;
- A connected sum one or more copies of the real projective plane, i.e., $\sharp_n \mathbb{R}P^2 := \underbrace{\mathbb{R}P^2 \sharp \cdots \sharp \mathbb{R}P^2}_{n\text{-copies of } \mathbb{R}P^2}$.

Remark 158 Let $g, h \geq 2$ be integers. Now, planar representations of Σ_g and N_h tell that $\Sigma_g \cong \Sigma_{g-1} \sharp \mathbb{T}^2$ and $N_h \cong N_{h-1} \sharp \mathbb{R}P^2$. Thus, $\sharp_g \mathbb{T}^2 \cong \Sigma_g$ for all $g \geq 1$ and $\sharp_h \mathbb{R}P^2 \cong N_h$ for all $h \geq 2$.

Theorem 159 Suppose we attach a collection $\{e^2_\alpha\}_\alpha$ of 2-cells to a path-connected space X via maps $\varphi_\alpha \colon \mathbb{S}^1 \to X$, producing $Y := X \coprod_{\varphi_\alpha} \overline{\mathbb{B}^2_\alpha}$. Let $x_0 \in X$ and γ_α be a path from x_0 to $\varphi_\alpha(1)$ for each α. Consider the normal subgroup N of $\pi_1(X, x_0)$ generated by all $[\gamma_\alpha \varphi_\alpha \overline{\gamma_\alpha}]$ for varying α. Then the kernel of the inclusion induced map $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ is N.

Proof. Define a space Z as follows: The space Z is obtained from Y by attaching rectangular strips $S_\alpha := [0,1] \times [0,1]$, with the lower edge $[0,1] \times 0$ attached along γ_α, the right edge $1 \times [0,1]$ attached along an arc that starts at $\varphi_\alpha(1)$ and goes radially into e^2_α, and all the left edges $0 \times [0,1]$ of the different strips identified together. The top edges of the strips are not attached to anything, allowing us to deformation retract Z onto Y.



In each cell, e^2_α, choose a point y_α not in the arc along which S_α is attached. Define $A := Z \setminus \bigcup_\alpha \{y_\alpha\}$ and $B := Z \setminus X$. Then, A deformation retracts onto X, and B is contractible. Choose a base point z_0 near x_0 on the segment where all the strips S_α intersect. Let h be the line segment connecting z_0 to x_0 in the intersection of the S_α's. Consider the base change isomorphism $\beta_h \colon \pi_1(A, x_0) \ni [\ell] \longmapsto \left[h \ell \overline{h}\right] \in \pi_1(A, z_0)$. In particular, β_h sends $[\gamma_\alpha \varphi_\alpha \overline{\gamma_\alpha}]$ sends to $\left[h \gamma_\alpha \varphi_\alpha \overline{\gamma_\alpha} \overline{h}\right]$. Let δ_α be loop in $A \cap B$ based at z_0 such that $\delta_\alpha \simeq_{\text{rel } z_0} h \gamma_\alpha \varphi_\alpha \overline{\gamma_\alpha} \overline{h}$. Thus, if τ_α is the top edge of S_α and ℓ_α is non-trivial simple loop in $e^2_\alpha \setminus \{y_\alpha\}$ based at the point $\tau_\alpha \cap e^2_\alpha$, then δ_α is homotopic rel. $\{z_0\}$ to either $\tau_\alpha \ell_\alpha \overline{\tau_\alpha}$ or $\tau_\alpha \overline{\ell_\alpha} \overline{\tau_\alpha}$

We claim that $\pi_1(A\cap B, z_0)$ is a free group generated by $[\delta_\alpha]$ for varying α. To prove this, cover $A\cap B$ by the open sets $A_\alpha := (A\cap B)\setminus \bigcup_{\beta\neq\alpha} e^2_\beta$. Since A_α deformation retracts onto a circle in $e^2_\alpha\setminus\{y_\alpha\}$, we have $\pi_1(A_\alpha, z_0)\simeq\mathbb{Z}$.

Now, Theorem 133 to together with (2) of Theorem 132, tells that the kernel of inclusion induced map is $\pi_1(A, z_0)\to\pi_1(Z, z_0)$ is $\pi_1(A\cap B, z_0)$. Under the base change isomorphisms $\beta_{\overline{h}}\colon \pi_1(A, z_0)\to \pi_1(A, x_0)$ and $\beta_{\overline{h}}\colon \pi_1(Z, z_0)\to\pi_1(Z, x_0)$, the group $\pi_1(A\cap B, z_0)$ correspondence to N, i.e., kernel of inclusion induced map is $\pi_1(A, x_0)\to\pi_1(Z, x_0)$ is N. Finally, X (resp. Y) is a deformation retract of A (resp. Z), i.e., we have the following commutative diagram of the inclusion-induced maps:

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r] \arrow[d, "\cong"'] & \pi_1(Y, y_0) \arrow[d, "\cong"] \\
\pi_1(A, x_0) \arrow[r] & \pi_1(Z, x_0)
\end{tikzcd}$$

Thus the kernel of the inclusion induced map $\pi_1(X, x_0)\to\pi_1(Y, y_0)$ is N. □

Theorem 160 Let $n\geq 3$ be an integer. Suppose we attach a collection $\{e^n_\alpha\}_\alpha$ of n-cells to a path-connected space X via maps $\varphi_\alpha\colon \mathbb{S}^1\to X$, producing $Y := X\coprod_{\varphi_\alpha}\overline{\mathbb{B}^n_\alpha}$. Let $x_0\in X$. Then the inclusion induced map $\pi_1(X, x_0)\to\pi_1(Y, x_0)$ is an isomorphism.

Proof. In the proof of Theorem 159, now each $A_\alpha\simeq\mathbb{S}^{n-1}$, a simply connected space, i.e., $\pi_1(A\cap B, z_0)$ is a trivial group. □

Remark 161 The n-fold dunce cap D_n is the attaching a 2-cell to $\mathbb{S}^1$ via $\mathbb{S}^1\ni z\longmapsto z^n\in\mathbb{S}^1$. By Theorem 159, $\pi_1(D_n)\cong\langle x\rangle/\langle x^n\rangle\cong\mathbb{Z}_n$. For $n=2$, we have $D_2\cong\mathbb{R}P^2$. If $n\geq 3$, the space D_n is not a manifold.

Theorem 162 Let $g, h\geq 1$ be integers. By Seifert-Van Kampen theorem

$$\pi_1(\sharp_g\mathbb{T}^2) = \left\langle a_1, b_1, \cdots, a_n, b_n | a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}\right\rangle$$

and

$$\pi_1(\sharp_h\mathbb{R}P^2) = \left\langle a_1, .., a_h | a_1^2\cdots a_h^2\right\rangle.$$

Proof. The space $\sharp_g\mathbb{T}^2\cong\Sigma_g$ (resp. $\sharp_h\mathbb{R}P^2\cong N_h$) is the obtained from attaching a 2-cell to $\bigvee_{i=1}^{2g}\mathbb{S}^1$ (resp. $\bigvee_{i=1}^{h}\mathbb{S}^1$) via the attaching map $\mathbb{S}^1\to\bigvee_{i=1}^{2g}\mathbb{S}^1$ (resp. $\mathbb{S}^1\to\bigvee_{i=1}^{h}\mathbb{S}^1$) described the words $a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}$ (resp. $a_1^2\cdots a_g^2$). □

Theorem 163 *For every group G there is a 2-dimensional CW-complex X_G with $\pi_1(X_G)\cong G$.*

Theorem 164 (HNN-Seifert-van Kampen Theorem) Let A, B be two disjoint path-connected open subsets of a path-connected space X and $f\colon A\to B$ be a homeomorphism. Let $\alpha\colon\pi_1(A)\to\pi_1(X)$ be the inclusion induced map; and $\beta\colon\pi_1(A)\to\pi_1(X)$ be the composition of $\pi_1(f)\colon\pi_1(A)\to\pi_1(B)$ and the inclusion induced map $\pi_1(B)\to\pi_1(X)$. Let $q\colon X\to X(f) := \frac{X}{a\sim f(a)}$. Then there exists an isomorphism

$$\Phi\colon \pi_1\big(X(f)\big)\longrightarrow\frac{\pi_1(X)*\langle t\rangle}{N},$$

where N is the intersection of all normal subgroups of $\pi_1(X)*\langle t\rangle$ containing $\{\alpha(g)t\beta(g)^{-1}t^{-1} : g\in\pi_1(A)\}$ such that

$$\Phi\circ\pi_1(q)\colon \pi_1(X)\longrightarrow\frac{\pi_1(X)*\langle t\rangle}{N}$$

is the natural group homomorphism, which is injective by Britton's Lemma.

Corollary 165 Let $\mathscr{T}$ be a topological space. We suppose that it can be written as a union $\mathscr{T} = Y \cup Z$ such that the following conditions are satisfied: (1) Y and Z are open subsets of $\mathscr{T}$, (2) Y is path-connected, (3) Z is simply connected, (4) $Y \cap Z$ consists of two simply connected path components A and B, each of which is open in $\mathscr{T}$. Then $\pi_1(Y) * \langle t \rangle \cong \pi_1(\mathscr{T})$.

Proof. Let $Z' := Z \times 0$ and $W := \frac{Y \sqcup Z'}{b \sim b \times 0 \text{ for } b \in B}$ and $f\colon A \times 0 \to A$ be the obvious map. Now,

$$\pi_1\left(\mathscr{T} \cong \frac{W}{a \sim f(a \times 0)}\right) \cong \pi_1(W) * \langle t \rangle \cong \left(\pi_1(Y) *_{\pi_1(B)} \pi_1(Z')\right) * \langle t \rangle \cong \pi_1(Y) * \langle t \rangle.$$

□

6 Simplicial complex, triangulation, and simplicial homology

Definition 166 Given a set $\{a_0, ..., a_n\}$ of points of $\mathbb{R}^N$, this set is said to be geometrically independent if for any (real) scalars t_i, the equations

$$\sum_{i=0}^{n} t_i = 0 \text{ and } \sum_{i=0}^{n} t_i a_i = \mathbf{0}$$

imply that $t_0 = t_1 = \cdots = t_n = 0$. In other words, $\{a_0, ..., a_n\}$ is geometrically independent if and only if the vectors $a_1 - a_0, ..., a_n - a_0$ are linearly independent.

Definition 167 Given a geometrically independent set of points $\{a_0, ..., a_n\}$, we define the n-plane P spanned by these points to consist of all points x of $\mathbb{R}^N$ such that

$$x = \sum_{i=0}^{n} t_i a_i,$$

for some scalars t_i with $\sum_{i=0}^{n} t_i = 1$.

Definition 168 Let $\{a_0, ..., a_n\}$ be a geometrically independent set in $\mathbb{R}^N$. We define the n-simplex a spanned by $a_0, ., a_n$ to be the set of all points x of $\mathbb{R}^N$ such that

$$x = \sum_{i=0}^{n} t_i a_i \text{ where } \sum_{i=0}^{n} t_i = 1$$

and $t_i \geq 0$ for all i. The numbers t_i are uniquely determined by x; they are called the barycentric coordinates of the point x of a with respect to $a_0, ..., a_n$.

Remark 169 Let σ be the n-simplex spanned by the geometrically independent set $\{a_0, ..., a_n\}$. If $x \in \sigma$, let $\{t_i(x)\}$ be the barycentric coordinates of x; they are determined uniquely by the conditions

$$x = \sum_{i=0}^{n} t_i a_i \text{ where } \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i.$$

Now, we have the following observations:

- The barycentric coordinates $t_i(x)$ of x with respect to $a_0, ..., a_n$ are continuous functions of x.

- σ equals the union of all line segments joining a_0 to points of the simplex s spanned by $a_1, ..., a_n$. Two such line segments intersect only in the point a_0.

- σ is a compact, convex set in $\mathbb{R}^N$, which equals the intersection of all convex sets in $\mathbb{R}^N$ containing $a_0, ..., a_n$.

- There is one and only one geometrically independent set of points spanning σ.

Definition 170 The points $a_0, ..., a_n$, that span σ are called the vertices of σ; the number n is called the dimension of σ.

Any simplex spanned by a subset of $a_0, ..., a_n$ is called a face of σ. In particular, the face of a spanned by $a_1, ..., a_n$ is called the face opposite a_0.

The faces different from σ itself are called the proper faces of σ; their union is called the boundary of σ and denoted $\mathrm{Bd}(\sigma)$.

The interior of σ is defined by the equation $\mathrm{Int}(\sigma) = \sigma \setminus \mathrm{Bd}(\sigma)$; the set $\mathrm{Int}(\sigma)$ is sometimes called an open simplex.

Remark 171 Now, we have the following observations:

- $\mathrm{Bd}(\sigma)$ consists of all points x of σ such that at least one of the barycentric coordinates $t_i(x)$ is zero. $\mathrm{Int}(\sigma)$ consists of those points of σ for which $t_i(x) > 0$ for all i.

- Given $x \in \sigma$, there is exactly one face s of σ such that $x \in \mathrm{Int}(s)$, for s must be the face of σ spanned by those a_i; for which $t_i(x)$ is positive.

- $\mathrm{Int}(\sigma)$ is convex and is open in the plane P spanned by $\{a_0, ..., a_n\}$; its closure is σ. Furthermore, $\mathrm{Int}(\sigma)$ equals the union of all open line segments joining a_0 to points of $\mathrm{Int}(s)$, where s is the face of σ opposite a_0.

Definition 172 A simplicial complex K in $\mathbb{R}^N$ is a collection of simplices in $\mathbb{R}^N$ such that:

(1) Every face of a simplex of K is in K.

(2) The intersection of any two simplexes of K is a face of each of them.

In other words, the condition (2) is equivalent to

(2′) Every pair of distinct simplices of K have disjoint interiors.

Definition 173 If L is a sub-collection of K that contains all faces of its elements, then L is a simplicial complex in its own right; it is called a sub-complex of K. One sub-complex of K is the collection of all simplices of K of dimension at most p; it is called the p-skeleton of K and is denoted $K^{(p)}$. The points of the collection $K^{(0)}$ are called the vertices of K.

Definition 174 Let $|K|$ be the subset of $\mathbb{R}^N$ that is the union of the simplices of K. Giving each simplex its natural topology as a subspace of $\mathbb{R}^N$, we then topologize $|K|$ by declaring a subset A of $|K|$ to be closed in $|K|$ if and only if $A \cap \sigma$ is closed in σ, for each σ in K. It is easy to see that this defines a topology on $|K|$, for this collection of sets is closed under finite unions and arbitrary intersections. The space $|K|$ is called the underlying space of K, or the polytope of K.

A space that is the polytope of a simplicial complex will be called a polyhedron.

Remark 175 In general, the topology of $|K|$ is finer (larger) than the topology $|K|$ inherits as a subspace of $\mathbb{R}^N$: If A is closed in $|K|$ in the subspace topology, then $A = B \cap |K|$ for some closed set B in $\mathbb{R}^N$. Then $B \cap \sigma$ is closed in σ for each σ, so $B \cap |K| = A$ is closed in the topology of in the topology of $|K|$, by definition.

However, if K is finite, these two topologies are the same. For suppose K is finite and A is closed in $|K|$. Then $A \cap \sigma$ is closed in σ and hence closed in $\mathbb{R}^N$. Because A is the union of finitely many sets $A \cap \sigma$, the set A also is closed in $\mathbb{R}^N$.

Remark 176 If L is a sub-complex of K, then $|L|$ is a closed subspace of $|K|$. In particular, if $\sigma \in K$, then σ is a closed subspace of $|K|$.

Remark 177 A map $f : |K| \to X$ is continuous if and only if $f|\sigma$ is continuous for each $\sigma \in K$.

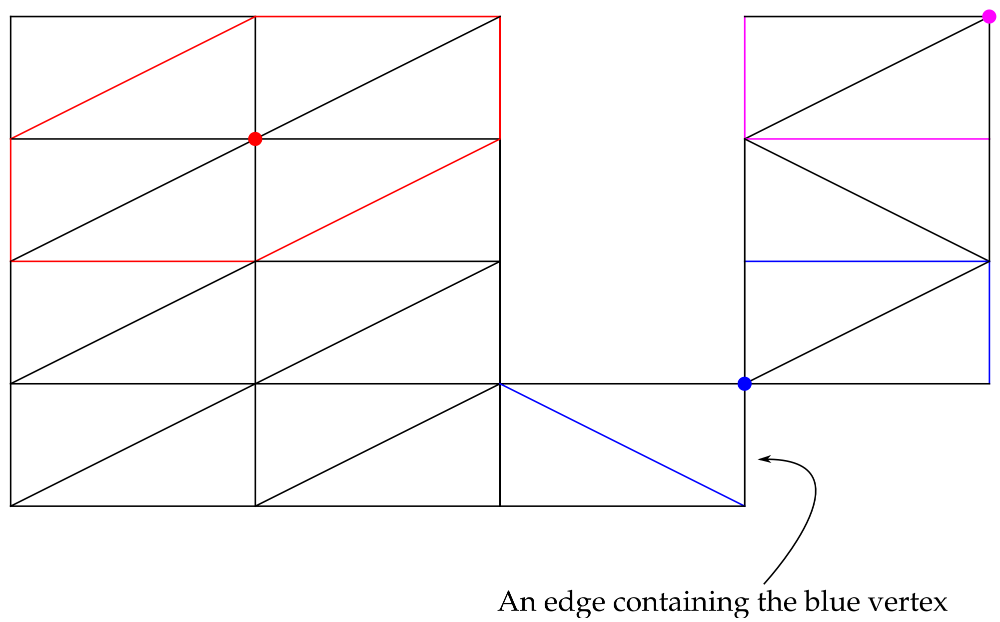


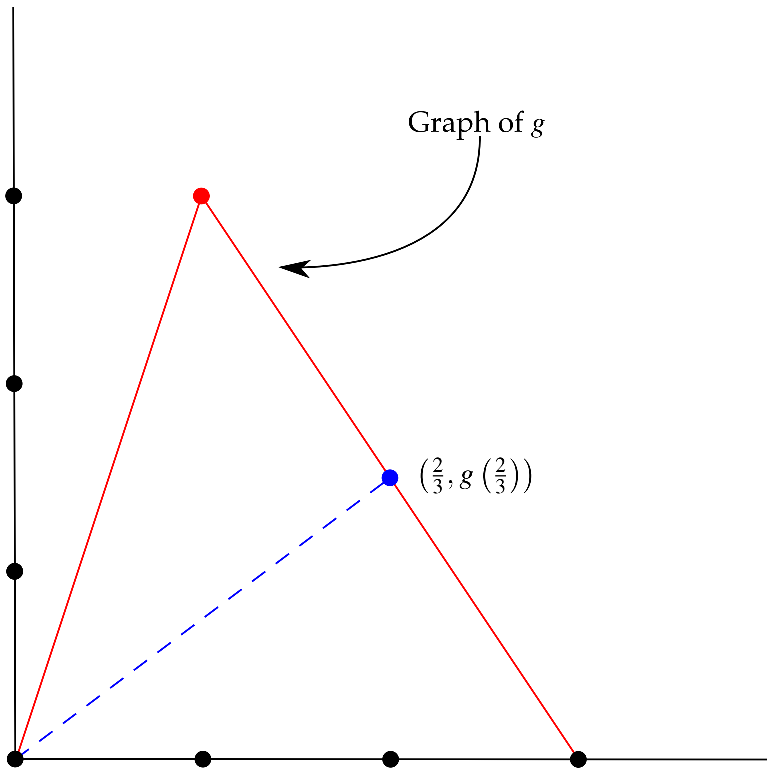
Fig. 8: Star and link

Definition 178 If v is a vertex of K, the star of v in K, denoted by $\mathrm{St}(v)$, is the union of the interiors of those simplices of K that have v as a vertex. Its closure, denoted $\overline{\mathrm{St}}(v)$, is called the closed star of v in K. It is the union of all simplices of K having v as a vertex and is the polytope of a sub-complex of K. The set $\overline{\mathrm{St}}(v) \setminus \mathrm{St}(v)$ is called the link of v in K and is denoted $\mathrm{Lk}(v)$. Figure 8 shows the link of three colored vertices.

Theorem 179 Let K and L be complexes, and let $f : K^{(0)} \to L^{(0)}$ be a map. Suppose that whenever the vertices $v_0, ..., v_n$ of K span a simplex of K, the points $f(v_0), ..., f(v_n)$ are vertices of a simplex of L. Then f can be extended to a continuous map $g : |K| \to |L|$ such that

$$x = \sum_{i=0}^{n} t_i v_i \implies g(x) = \sum_{i=0}^{n} t_i f(v_i).$$

We call g the (linear) simplicial map induced by the vertex map f.



Theorem 180 Let

$$K := \left\{0, \frac{1}{3}, 1, \left[0, \frac{1}{3}\right], \left[\frac{1}{3}, 1\right]\right\} \text{ and } L := \{0, 1, [0, 1]\}$$

be simplicial complexes in $\mathbb{R}$. Using Theorem 179, let $g\colon |K| \to |L|$ be a map defined by $g(0) = 0,\ g\left(\frac{1}{3}\right) = 1,\ g(1) = 0$. Consider the subdivision

$$K' := \left\{0, \frac{1}{3}, \frac{2}{3}, 1, \left[0, \frac{1}{3}\right], \left[\frac{1}{3}, \frac{2}{3}\right], \left[\frac{1}{3}, 1\right]\right\}$$

of K, i.e., each simplex of K' is contained in a simplex of K and and each simplex of K equals the union of finitely many simplices of K'. Show that there is **no** subdivision L' of L such that $g\colon |K'| = |K| \longrightarrow |L| = |L'|$ is a simplicial map induced by some vertex map $(K')^{(0)} \to (L')^{(0)}$.

Proof. On the contrary, assume there is a vertex map $f\colon (K')^{(0)} \to (L')^{(0)}$ which induces the map $g\colon |K'| \to |L'|$. Then in particular, $f(0) = 0 = f(1),\ f\left(\frac{1}{3}\right) = 1$ and $f\left(\frac{2}{3}\right) = g\left(\frac{2}{3}\right)$. Since, g is induced by f we have

$$g\left(t \cdot \frac{2}{3}\right) = g\left((1-t) \cdot 0 + t \cdot \frac{2}{3}\right) = (1-t) \cdot g(0) + t \cdot g\left(\frac{2}{3}\right) = t \cdot g\left(\frac{2}{3}\right) \text{ for } 0 \leq t \leq 1.$$

□

Definition 181 A space is triangulable if there is a simplicial complex whose geometric carrier is homeomorphic to the space.

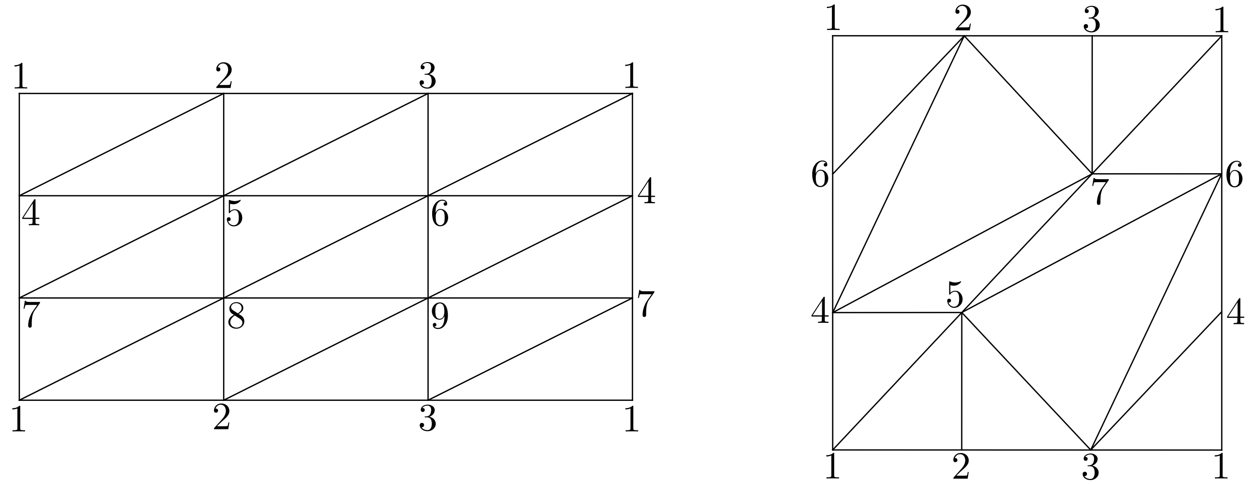


Fig. 9: Two different triangulations of torus. On the right side: A vertex-minimal triangulation.

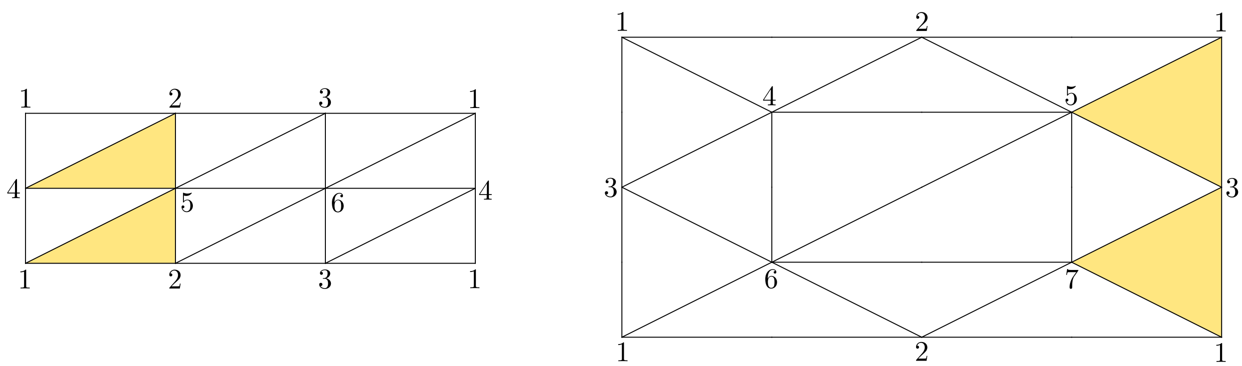


Fig. 10: These are **not** triangulations of the torus.

Remark 182 In a simplicial complex, the intersection of two simplices is either empty or a single common face of them. That is, if the intersection of two simplices is a union (possibly disjoint union) of common faces of them, then the union must be a common face also. See Figure 10.

Remark 183 Given a list of k vertices, one can tell whether there is no $(k-1)$-cell with those vertices or there is exactly one such $(k-1)$-cell and it's that one.

Remark 184 In order to get a triangulation of a compact surface $\mathcal{S}$, we first split up its planner representation $\mathcal{P}$, which is a polygon, into finitely many triangles, i.e., we construct a finite simplicial complex $\mathcal{K}$ whose geometric carrier is $\mathcal{P}$. Now, suppose we split up "correctly" the polygon $\mathcal{P}$. In that case, the restriction of the quotient map $q\colon \mathcal{P} \to \mathcal{S}$ on each triangle will be an embedding, and considering images of all elements of $\mathcal{K}$, we get a simplicial complex, denoted by $q(\mathcal{K})$ so that the surface $\mathcal{S}$ is homeomorphic to the geometric carrier of $q(\mathcal{K})$.

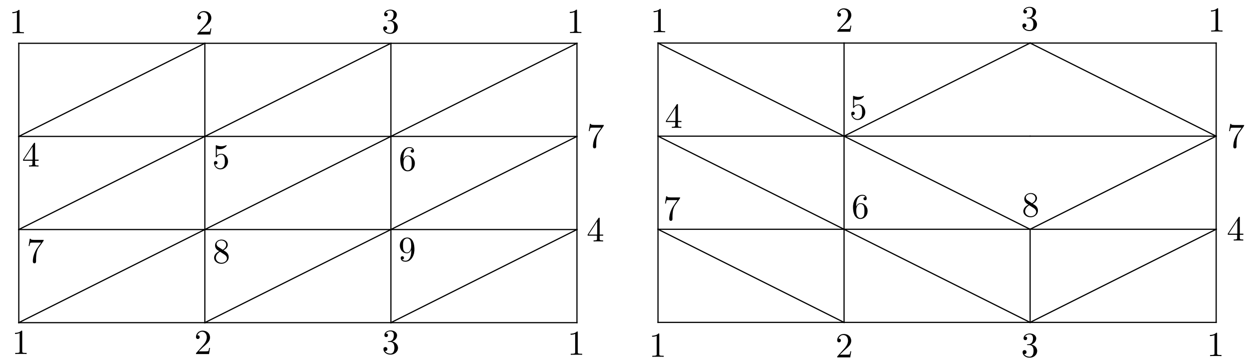


Fig. 11: Two different triangulations of Klein Bottle. Right side: A vertex-minimal triangulation.

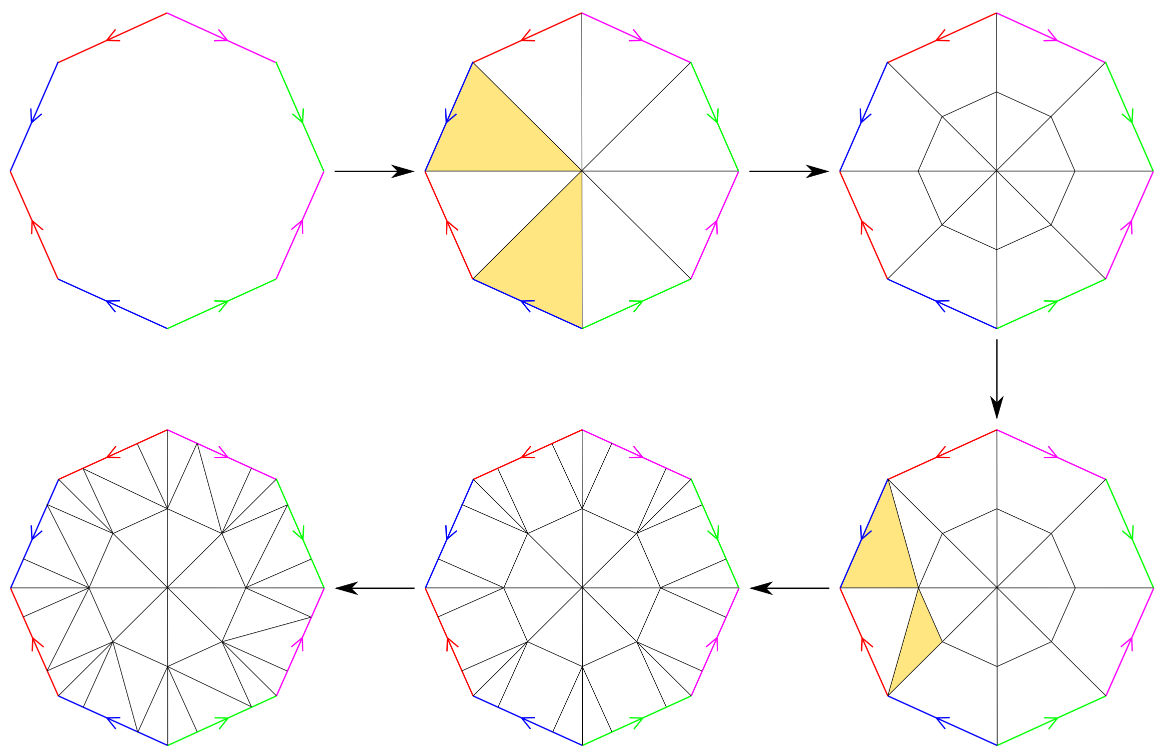


Fig. 12: A triangulation of Σ_2

Remark 185 Note that if a compact surface $\mathcal{S}$ has a triangulation having v vertices, e edges and t triangles then $v \geq \frac{1}{2}\left(7+\sqrt{49-24\cdot\chi(\mathcal{S})}\right)$, $e = 3\big(v-\chi(\mathcal{S})\big)$, and $3t = 2e$. Here, $\chi(\mathcal{S})$ is the Euler characteristic, i.e., $\chi(\mathcal{S}) = \sum_{n=0}^{2}(-1)^n \cdot \operatorname{rank}\big(H_n(\mathcal{S})\big)$. Recall that $\chi(\mathbb{S}^2) = 2$, $\chi\big(\sharp_g\mathbb{T}^2\big) = 2-2g$, and $\chi(\sharp_g\mathbb{R}P^2) = 2-g$.

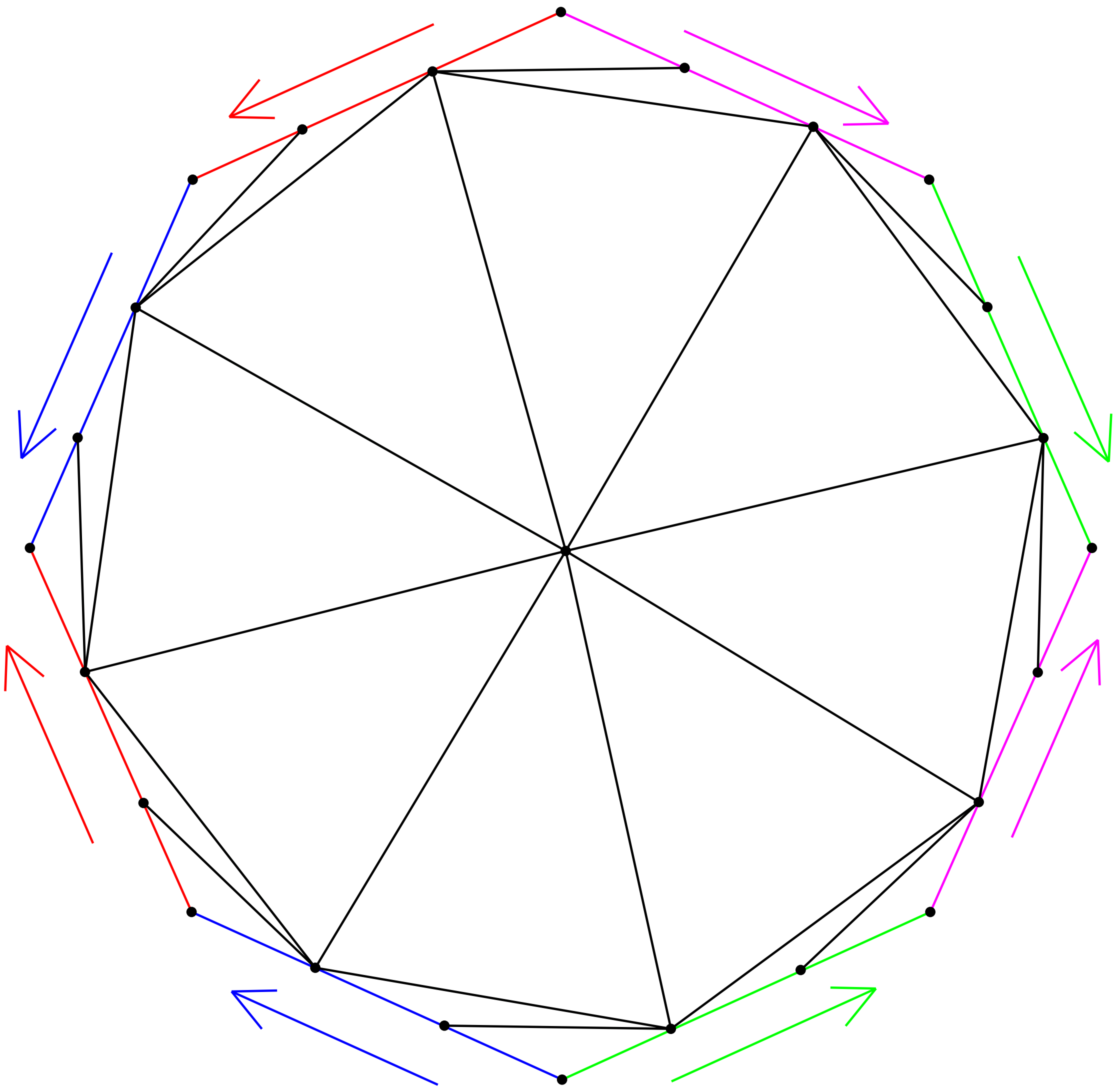


Fig. 13: A minimal triangulation of Σ_2. Note that for any triangulation of Σ_2 we have $v \geq 8.4244$. One can show (difficult!) that a triangulation by 9 vertices, 33 edges, and 22 triangles of Σ_2 is not possible.

Definition 186 Let Σ be a simplex. Define two orderings of its vertex set to be equivalent if they differ from one another by an even permutation. If $\dim \sigma > 0$, the orderings of the vertices of σ then fall into two equivalence classes. Each of these classes is called an orientation of σ. If σ is a 0-simplex, then there is only one class and hence only one orientation of σ. An oriented simplex is a simplex σ together with an orientation of σ.

A simplicial complex is said to be oriented if each of its simplexes is assigned an orientation.

Definition 187 Let K be an oriented simplicial complex and σ^p, σ^{p+1} be two simplexes whose dimensions differ by 1. Define the incidence number, denoted by $\left[\sigma^{p+1}, \sigma^p\right]$, as follows: If σ^p is not a face of σ^{p+1}, we put $\left[\sigma^{p+1}, \sigma^p\right] = 0$. Suppose σ^p is a face of σ^{p+1}, v is the additional vertex of σ^{p+1}, and $v_0 < v_1 < \cdots < v_p$ gives the orientation of σ^p. Then

$$\left[\sigma^{p+1}, \sigma^p\right] = \begin{cases} +1 \text{ if } v < v_0 < v_1 < \cdots < v_p \text{ gives the orientation of } \sigma^{p+1}, \\ -1 \text{ if } v < v_0 < v_1 < \cdots < v_p \text{ gives the opposite orientation of } \sigma^{p+1}. \end{cases}$$

Definition 188 Let K be an oriented simplicial complex. Thus, each simplex of K comes with a fixed orientation. For each integer n, let $C_n(K)$ be the free abelian group generated by all oriented n-simplices of K, called the group of n-chains of K. Define $\partial_n \colon C_n(K) \to C_{n-1}(K)$ as follows:

$$\partial_n(\sigma^n) := \sum_{\sigma^{n-1} \text{ is an oriented } n\text{-simplex of } K} \left[\sigma^n, \sigma^{n-1}\right] \sigma^{n-1}.$$

One can show that $\partial_{n-1} \circ \partial_n = 0$ for every integer n. Define

$$H_n(K;\mathbb{Z}) := \frac{\ker \partial_n}{\operatorname{im}\left(\partial_{n+1}\right)}$$

Theorem 189 [Cro78, Theorem 2.3.] Let K be a simplicial complex with two orientations, and let K_1 and K_2 denote the resulting oriented simplicial complexes. Then the homology groups $H_n(K_1;\mathbb{Z})$ and $H_n(K_2;\mathbb{Z})$ are isomorphic for each dimension n.

Theorem 190 [Mun84, Theorem 7.1.] Let K be an oriented simplicial complex. Then the group $H_0(K;\mathbb{Z})$ is free abelian. If $\{v_\alpha\}$ is a collection consisting of one vertex from each component of $|K|$, then the homology classes of the chains v_α form a basis for $H_0(K;\mathbb{Z})$.

Definition 191 An n-pseudomanifold is a simplicial complex K with the following properties:

- Each simplex of K is a face of some n-simplex of K.
- Each $(n-1)$-simplex is a face of exactly two n-simplexes of K.
- Given a pair σ_1^n, σ_2^n of two n-simplexes of K, there is a sequence of n-simplexes beginning with a σ_1^n and ending with σ_2^n such that any two successive terms of the sequence have a common $(n-1)$-face.

Definition 192 Let K be an n-pseudomanifold. For each $(n-1)$-simplex σ^{n-1} of K, let σ_1^n and σ_2^n denote the two n-simplexes of which σ^{n-1} is a face. An orientation for K having the property

$$\left[\sigma_1^n, \sigma^{n-1}\right] = -\left[\sigma_2^n, \sigma^{n-1}\right]$$

for each $(n-1)$-simplex σ^{n-1} of K is a coherent orientation. An n-pseudomanifold is orientable if it can be assigned a coherent orientation. Otherwise it is non-orientable.

6.1 Homology Calculation of Klein Bottle

Consider the following triangulation of Klein bottle K. Orient the 1-simplices of K randomly, keeping in mind the identification on the boundary of the square. Look at the two different orientations of e_{13} edges in the square in order to get oriented e_{13} edge in K. That is, a random orientation of 1-simplices of the square may not give an orientation of 1-simplices of Klein bottle, and this is due to identification on the boundary of the square. For example, if e, e' are edges of the square giving a single edge in quotient space (Klein bottle), then choosing one orientation for e, we have exactly one and only one way to orient the other edge e', so that after identification, they give an oriented edge in quotient space.

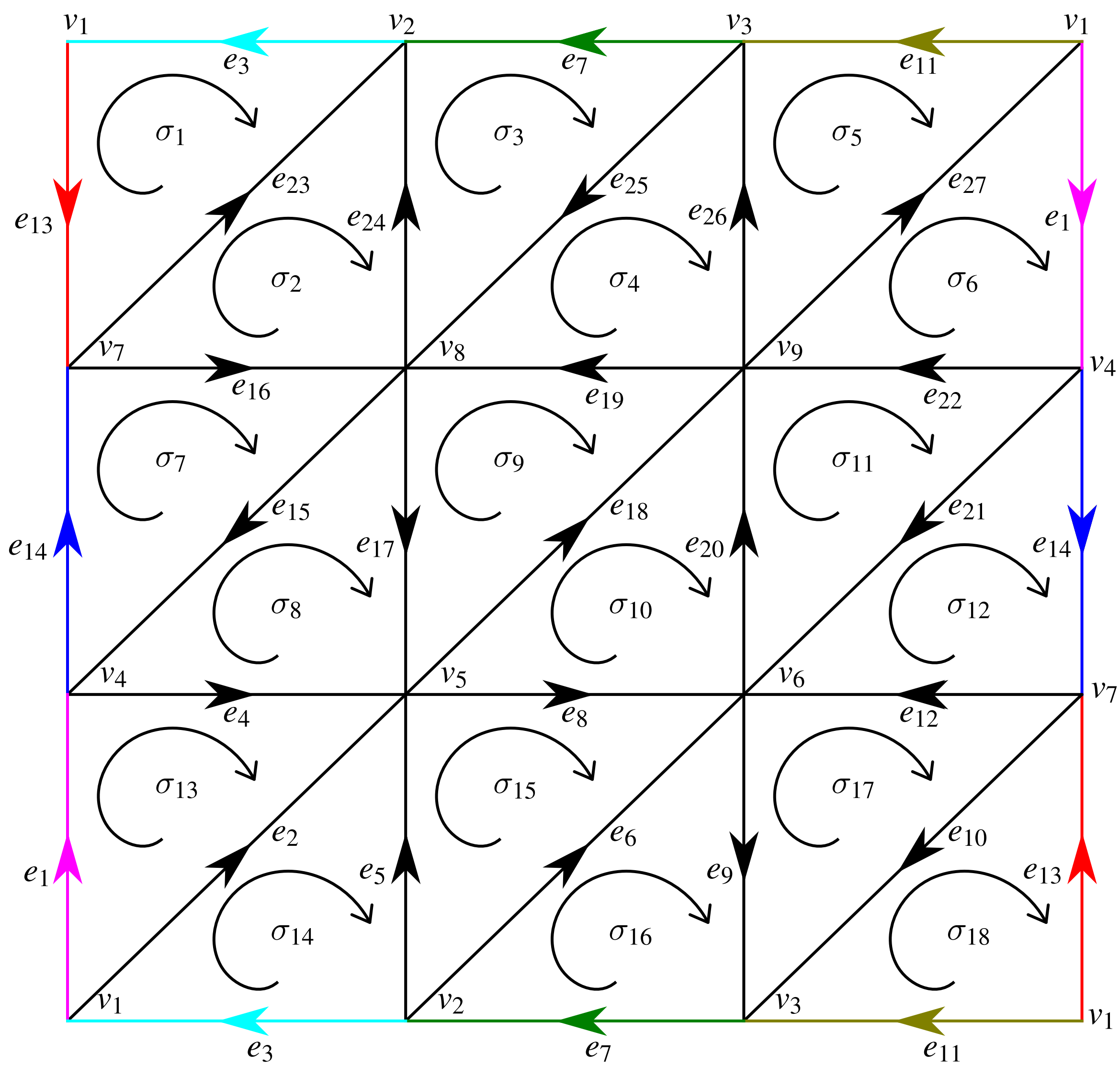


Fig. 14: Homology calculation of Klein bottle

Now, a priori, not knowing whether Klein bottle is orientable or not, let's try to orient all triangles coherently, if possible. In this case, if we choose an orientation for σ_1, say clockwise manner, then one has to orient σ_2 in the clockwise manner due to the edge e_{23}. Similarly, for edge e_{24}, we need to orient σ_3 in the clockwise manner also, and so on. In other words, due to the edges inside the square, once if we choose an orientation for a triangle, and the coherent orientation of the other triangles comes automatically. Now, σ_1 and σ_{18} share the edge e_{13} in the quotient space with the same incidence number, i.e., it is impossible to give a global coherent orientation of the triangles of Klein bottle.

Let $\alpha := \sum_{\ell=1}^{18} n_\ell \cdot \sigma_\ell \in C_2(K)$. Now,

$$\partial_2(\alpha) = \sum_{\ell=1}^{18} n_\ell \cdot \partial_2(\sigma_\ell)$$

$$= n_1 \cdot \big(- e_{13} - e_{23} - e_3 \big) + n_2 \cdot \big(- e_{16} - e_{24} + e_{23} \big) + n_3 \cdot \big(+ e_{24} + e_{25} - e_7 \big) + n_4 \cdot \big(- e_{25} - e_{26} + e_{19} \big)$$

$$+ n_5 \cdot \big(+ e_{26} - e_{11} - e_{27} \big) + n_6 \cdot \big(+ e_{27} + e_1 + e_{22} \big) + n_7 \cdot (+e_{14} + e_{16} + e_{15}) + n_8 \cdot \big(- e_{15} - e_4 + e_{17} \big)$$

$$+ n_9 \cdot (-e_{17} - e_{18} - e_{19}\big) + n_{10} \cdot \big(+ e_{18} - e_{20} - e_8 \big) + n_{11} \cdot \big(+ e_{20} - e_{22} + e_{21} \big) + n_{12} \cdot \big(- e_{21} + e_{14} + e_{12} \big)$$

$$+ n_{13} \cdot \big(+ e_1 + e_4 - e_2 \big) + n_{14} \cdot \big(+ e_2 - e_5 + e_3 \big) + n_{15} \cdot \big(+ e_5 + e_8 - e_6 \big) + n_{16} \cdot \big(+ e_6 + e_9 + e_7 \big)$$

$$+ n_{17} \cdot \big(- e_9 - e_{12} + e_{10} \big) + n_{18} \cdot \big(- e_{10} - e_{13} + e_{11} \big).$$

Now, using some basic calculations, one can show that $\partial_2(\alpha) = 0$ if and only if $n_1 = \cdots = n_{18} = 0$. Also,

$$\partial_2 \left(\sum_{\ell=1}^{18} \sigma_\ell \right) = 2 \cdot (e_1 + e_{14} - e_{13}). \tag{1}$$

Push-off Trick: Given a 1-chain β, to get a simple looking 1-chain β' such that β is homologous to β, we need to push β off 1-simplices that are in the interior of the polygon as many as possible.

Let $\beta^{(0)} = \sum_{\ell=1}^{27} m_\ell^{(0)} \cdot e_\ell \in C_1(K)$. We want to push $\beta^{(0)}$ off e_2 using σ_{13}, so consider $\beta^{(1)} := \beta^{(0)} + m_2^{(0)} \partial_2(\sigma_{13})$. Now, if we write $\beta^{(1)} = \sum_{\ell=1}^{27} m_\ell^{(1)} \cdot e_\ell$, then $m_2^{(1)} = 0$ and $\beta^{(1)}$ homologous to $\beta^{(0)}$. Look at the first two squares Figure 15.

Next, we want to push $\beta^{(1)}$ off e_5 using σ_{15}, so consider $\beta^{(2)} := \beta^{(1)} - m_{15}^{(1)} \partial_2(\sigma_{15})$. Now, if we write $\beta^{(2)} = \sum_{\ell=1}^{27} m_\ell^{(2)} \cdot e_\ell$, then $m_5^{(2)} = 0$, $m_2^{(2)} = 0$ and $\beta^{(2)}$ homologous to $\beta^{(1)}$. So, $\beta^{(2)}$ is homologous to $\beta^{(0)}$. Look at the second and third squares in the picture above.

Continue this fashion. At the end, we have $\beta^{(16)} = \sum_{\ell=1}^{27} m_\ell^{(16)} \cdot e_\ell$ with $m_\ell^{(16)} = 0$ for all ℓ except $\ell = 1, 3, 7, 11, 13, 14, 17, 20, 24, 26$, and $\beta^{(16)}$ is homologous to $\beta^{(0)}$.

Therefore,

$$\begin{aligned} \partial_1(\beta^{(0)}) = \quad & \partial_1(\beta^{(16)}) \\ = \quad & m_1^{(16)}(v_4 - v_1) + m_3^{(16)}(v_1 - v_2) + m_7^{(16)}(v_2 - v_3) + m_{11}^{(16)}(v_3 - v_1) + m_{13}^{(16)}(v_7 - v_1) \\ & + m_{14}^{(16)}(v_7 - v_4) + m_{17}^{(16)}(v_5 - v_8) + m_{20}^{(16)}(v_9 - v_6) + m_{24}^{(16)}(v_2 - v_8) + m_{26}^{(16)}(v_3 - v_9) \end{aligned}$$

Thus using some basic calculations, $\partial_1(\beta^{(0)}) = 0$ if and only if $\beta^{(0)}$ is homologous to $m \cdot (e_1 + e_{14} - e_{13}) + n \cdot (e_{11} + e_7 + e_3)$ for some integers m and n.

Next, suppose $m \cdot (e_1 + e_{14} - e_{13}) + n \cdot (e_{11} + e_7 + e_3) = \partial_2(\alpha)$ for some $\alpha \in C_2(K)$ and for some integers m, n. Write $\alpha := \sum_{\ell=1}^{18} n_\ell \cdot \sigma_\ell$. Then any 1-simplex e inside the polygon is a side of exactly two triangles σ_i, σ_j such that $[\sigma_i, e] = -[\sigma_j, e]$. Therefore, considering $\partial_2(\alpha)$, we can say that $n_1 = \cdots = n_{18}$.

Thus using Equation (1), we can say that $m \cdot (e_1 + e_{14} - e_{13}) + n \cdot (e_{11} + e_7 + e_3) \in \operatorname{im}(\partial_2)$ if and only if m is even and $n = 0$.

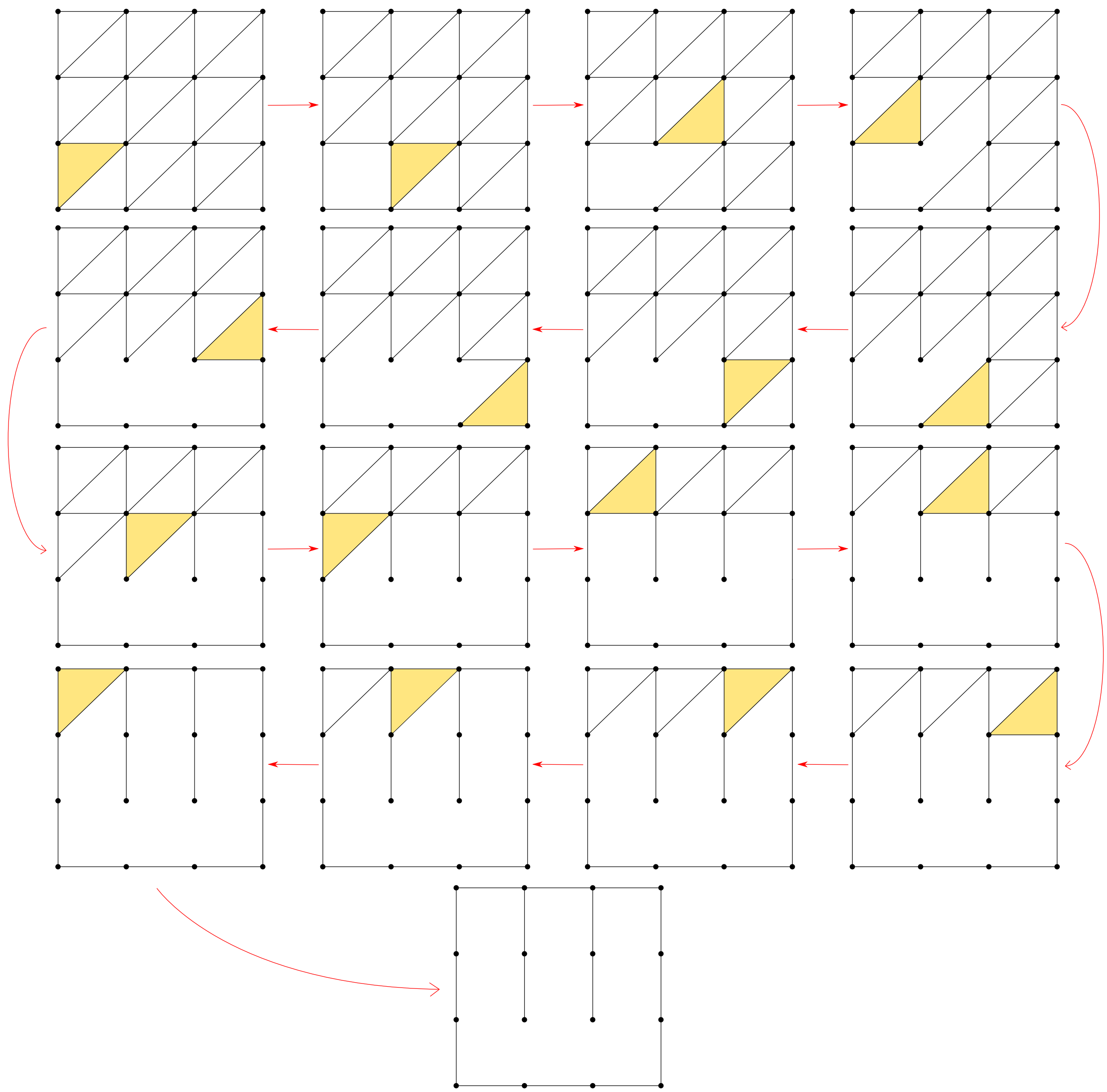


Fig. 15: Push-off trick

Finally, consider

$$0 \to C_2(K) \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} 0$$

Then

$$H_2(K;\mathbb{Z}) = \ker \partial_2 = 0.$$

Now, the map

$$H_1(K;\mathbb{Z}) = \frac{\ker \partial_1}{\mathrm{im}\, \partial_2} \ni \big[m \cdot (e_1 + e_{14} - e_{13}) + n \cdot (e_{11} + e_7 + e_3)\big] \longmapsto \big([m]_2, n\big) \in \mathbb{Z}_2 \oplus \mathbb{Z}$$

is an isomorphism, i.e., $H_1(K;\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}$. Also, $H_0(K;\mathbb{Z}) = \mathbb{Z}$ as K is path-connected.

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DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF SCIENCE, BANGALORE 560012, INDIA

✉ sumantadas@iisc.ac.in 🌍 https://sumanta0.github.io/