Surfaces of infinite-type are non-Hopfian

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Abstract: We show that an orientable surface Σ is of finite-type if and only if every proper map $f : \Sigma \to \Sigma$ of degree one is homotopic to a homeomorphism.

Hopfian groups

Definition A group G is said to be Hopfian if every epimorphism $G \to G$ is an isomorphism.

Theorem [Hem72; Thu82; Hem87; AFW15] The fundamental group of any compact orientable manifold of dim ≤ 3 is Hopfian.

Theorem 3 [BS62; Hem83] The Baumslag-Solitar group $\mathrm{BS}(m,n) := \langle x, y \big| x^{-1}y^m x = y^n \rangle$, $mn \neq 0$ is Hopfian if and only if m or n divides the other or if m and n have precisely the same prime divisors.

The degree of a proper map

Let $f\colon M \to N$ be a proper map between connected, oriented, boundaryless smooth n-manifolds.

Definition The integer $\deg(f)$ that describes $H^n_{\mathbf{c}}(f)\colon H^n_{\mathbf{c}}(N;\mathbb{Z}) \cong \mathbb{Z} \longrightarrow \mathbb{Z} \cong H^n_{\mathbf{c}}(M;\mathbb{Z})$, up to sign.

Theorem [Eps66, Theorems 3.1 and 4.1] If $f|f^{-1}(D) \to D$ is a homeomorphism between two smoothly embedded closed n-balls, then $\deg(f) = \pm 1$.

Olum's Theorem [Eps66, Corollary 3.4] If $\deg(f) = \pm 1$, then $\pi_1(f)\colon \pi_1(M) \longrightarrow \pi_1(N)$ is surjective.

The Hopf conjecture

Conjecture's Statement [Hau87] If the degree of $f\colon M \to M$ is ± 1, then f is a homotopy equivalence.

Theorem [Dav05; DIK00; Hau87; Swa74] Suppose M is closed and $f\colon M \to M$ is a π_1-injective map of deg ± 1. Then f is a homotopy equivalence if either of the following happens:

- $\pi_1(M)$ is finite,
- $\pi_i(M) = 0$ for $1 < i < \dim(M) - 1$,
- $\dim(M) \leq 4$, or
- $\pi_1(M)$ has a nilpotent subgroup of the finite index.

Finite-type surfaces are Hopfian

Well-Known Theorem [Das21, Theorem 2] *Let $S_{g,0,p}$ be a finite-type surface. Then any degree ± 1 map $f: S_{g,0,p} \to S_{g,0,p}$ is properly homotopic to a homeomorphism.*

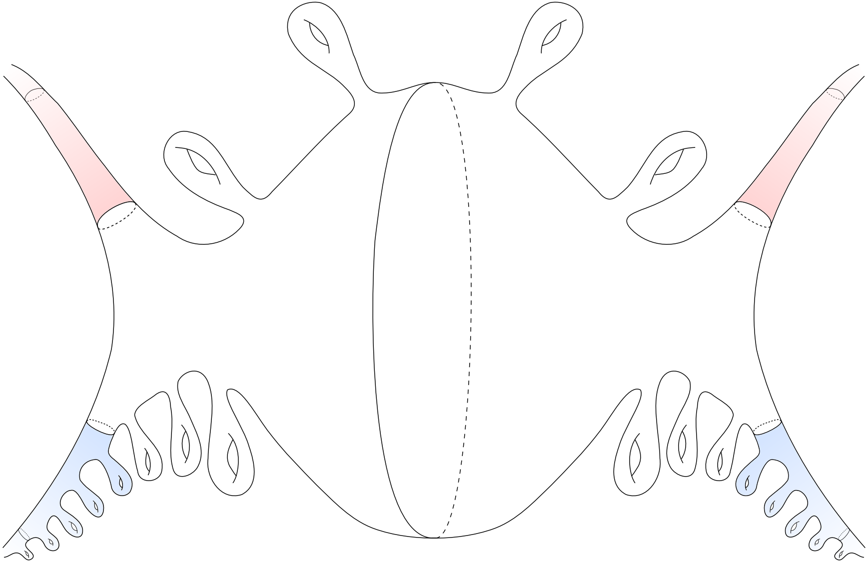
Infinite-type surfaces are non-Hopfian

Main Result [DG22, Theorem 1.1.] *Let Σ be any infinite-type surface. Then there exists a degree ± 1 map $f: \Sigma \to \Sigma$ such that $\pi_1(f): \pi_1(\Sigma) \to \pi_1(\Sigma)$ is not injective.*

Representation and classification of non-compact surfaces

Richrads' Representation Theorem [Ric63, Theorem 3] Any non-compact surface Σ is homeomorphic to a surface obtained by adding at most countably many handles to $\mathbb{S}^2 \setminus \mathscr{E}(\Sigma)$; where $\mathscr{E}(\Sigma)$ is a non-empty, closed, totally disconnected subset of $\mathbb{S}^2$.

The surface with two planar ends and two non-planar ends:



Let $\mathscr{E}_{\text{np}}(\Sigma)$ be the set of all those points of $\mathscr{E}(\Sigma)$, which are accumulated by handles.

Kerékjártó's classification Theorem [Ric63, Theorem 1] Two non-compact surfaces Σ_1 and Σ_2 are homeomorphic if and only if

- the number of handles of Σ_1 is the same as the number of handles of Σ_2, and
- there is a homeomorphism $\varphi\colon \mathscr{E}(\Sigma_1) \to \mathscr{E}(\Sigma_2)$ with $\varphi\big(\mathscr{E}_{\text{np}}(\Sigma_1)\big) = \mathscr{E}_{\text{np}}(\Sigma_2)$.

A pinching map of degree ± 1

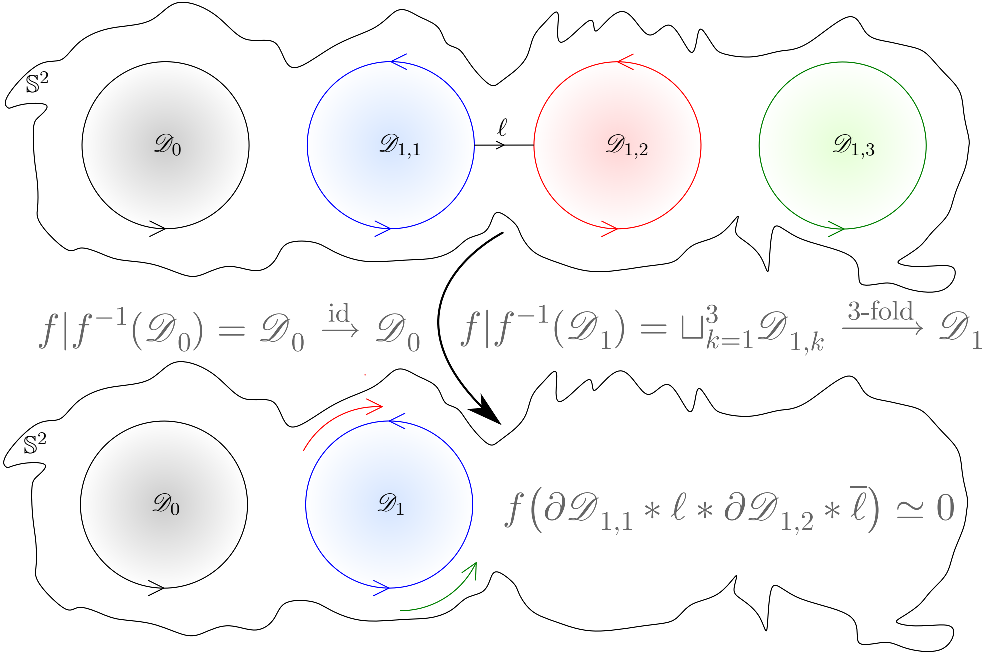
Infinite genus surfaces are non-Hopfian

[DG22, Theorem 3.3.] Let Σ be any infinite genus surface. Then the composition

$$\Sigma \xrightarrow{\text{quotient map}} \frac{\Sigma}{S_{1,1,0}} \xrightarrow{\text{homeomorphism}} \Sigma$$

is a non π_1-injective map of deg ± 1.

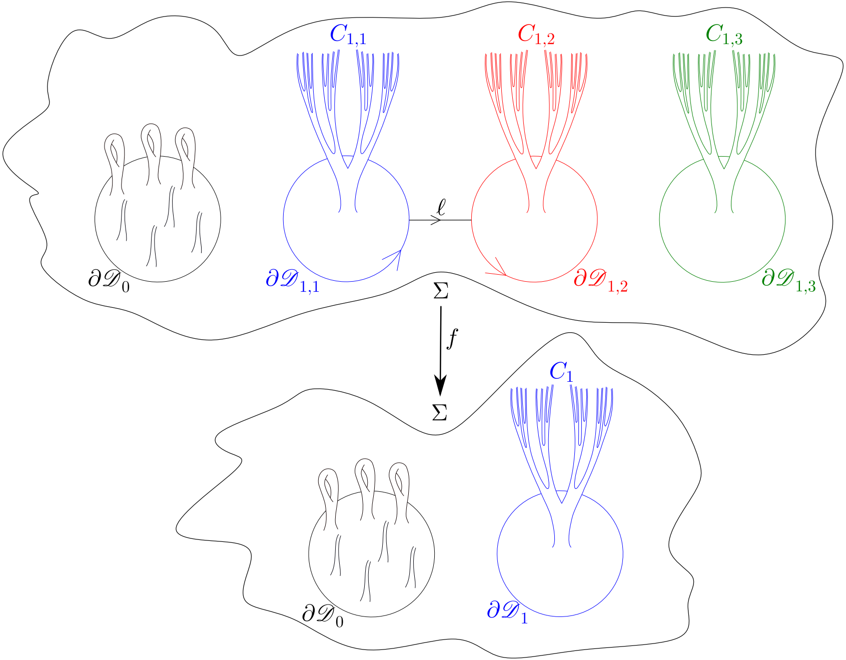
A folding map of degree ± 1 [DG22, Lemma 3.4.]



$\mathbb{S}^2$
$\mathscr{D}_0$
$\mathscr{D}_{1,1}$
ℓ
$\mathscr{D}_{1,2}$
$\mathscr{D}_{1,3}$
$f|f^{-1}(\mathscr{D}_0) = \mathscr{D}_0 \xrightarrow{\text{id}} \mathscr{D}_0$
$f|f^{-1}(\mathscr{D}_1) = \sqcup_{k=1}^{3} \mathscr{D}_{1,k} \xrightarrow{\text{3-fold}} \mathscr{D}_1$
$\mathbb{S}^2$
$\mathscr{D}_0$
$\mathscr{D}_1$
$f\left(\partial\mathscr{D}_{1,1} * \ell * \partial\mathscr{D}_{1,2} * \overline{\ell}\right) \simeq 0$

Finite genus infinite-type surfaces with finitely many isolated ends are non-Hopfian

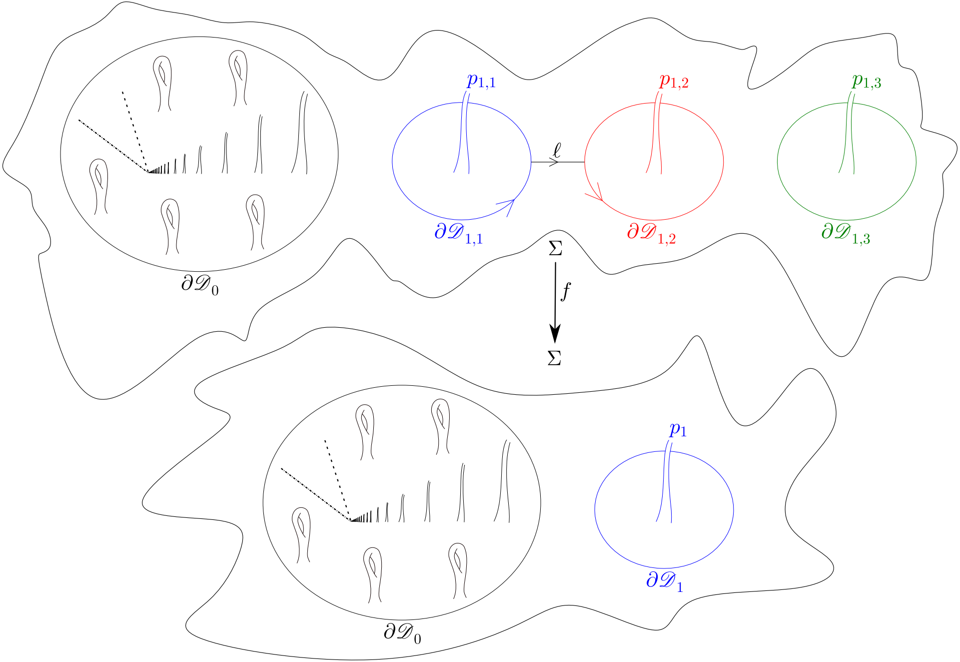
[DG22, Theorem 3.5.] Let Σ be a finite genus infinite-type surface such that $\mathscr{E}(\Sigma)$ has finitely many isolated points. Then there is a degree ± 1 map $f\colon \Sigma \to \Sigma$ which is not π_1-injective.



$C_{1,1}$
$C_{1,2}$
$C_{1,3}$
ℓ
$\partial\mathscr{D}_0$
$\partial\mathscr{D}_{1,1}$
$\partial\mathscr{D}_{1,2}$
$\partial\mathscr{D}_{1,3}$
Σ
f
Σ
C_1
$\partial\mathscr{D}_0$
$\partial\mathscr{D}_1$

Finite genus surfaces with infinitely many isolated ends are non-Hopfian

[DG22, Theorem 3.7.] Let Σ be a finite genus surface such that $\mathscr{E}(\Sigma)$ has infinitely many isolated points. Then there is a degree one map $f\colon \Sigma \to \Sigma$ which is not π_1-injective.



$p_{1,1}$
$p_{1,2}$
$p_{1,3}$
ℓ
$\partial\mathscr{D}_{1,1}$
$\partial\mathscr{D}_{1,2}$
$\partial\mathscr{D}_{1,3}$
$\partial\mathscr{D}_0$
Σ
f
Σ
p_1
$\partial\mathscr{D}_1$
$\partial\mathscr{D}_0$

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