MA 403-2025-1 | Real Analysis

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Homework 1

Theorem 1. *If* $n \in \mathbb{N}$ *is not a perfect square, then* $\sqrt{n}$ *is irrational.*

Proof. Suppose, for contradiction, that $\sqrt{n}$ is rational. Then we can write

$$\sqrt{n} = \frac{m}{d},$$

where $m, d \in \mathbb{Z}$, $d \neq 0$, and $\gcd(m, d) = 1$.
Squaring both sides gives

$$m^2 = nd^2.$$

Let

$$n = \prod_{i=1}^{k} {p_i}^{a_i}, \quad m^2 = \prod_{i=1}^{k} {p_i}^{2b_i}, \quad d^2 = \prod_{i=1}^{k} {p_i}^{2c_i}$$

be the prime factorizations of n, m^2, and d^2.
From $m^2 = nd^2$, we get

$$\prod_{i=1}^{k} {p_i}^{2b_i} = \left(\prod_{i=1}^{k} {p_i}^{a_i}\right)\left(\prod_{i=1}^{k} {p_i}^{2c_i}\right) = \prod_{i=1}^{k} {p_i}^{a_i+2c_i}.$$

Comparing exponents gives

$$2b_i = a_i + 2c_i \implies a_i = 2(b_i - c_i)$$

for each i.

Hence each a_i is even, which means $n = \prod_{i=1}^{k} {p_i}^{a_i}$ is a perfect square.
But this contradicts the assumption that n is not a perfect square.
Therefore, our assumption that $\sqrt{n}$ is rational is false, and $\sqrt{n}$ is irrational. □

Theorem 2. *The number $\sqrt{2} + \sqrt{3}$ is irrational.*

Proof. Suppose, for the sake of contradiction, that $\sqrt{2} + \sqrt{3}$ is rational. Then there exists $r \in \mathbb{Q}$ such that

$$\sqrt{2} + \sqrt{3} = r.$$

Rewriting, we have

$$\sqrt{3} = r - \sqrt{2}.$$

Squaring both sides gives

$$3 = (r - \sqrt{2})^2 = r^2 - 2r\sqrt{2} + 2.$$

Simplifying, we get

$$1 - r^2 = -2r\sqrt{2} \implies \sqrt{2} = \frac{r^2 - 1}{2r}.$$

But the right-hand side is rational, which contradicts the fact that $\sqrt{2}$ is irrational; see Theorem 1. Hence, our assumption is false. □

Theorem 3. *Let $r \in \mathbb{Q}$, $r \neq 0$, and $x \notin \mathbb{Q}$. Then $r + x$ and rx are irrational.*

Proof. (i) Suppose $r + x$ is rational, say $r + x = s$ with $s \in \mathbb{Q}$. Then

$$x = s - r \in \mathbb{Q},$$

contradicting x being irrational. Hence $r + x$ is irrational.

(ii) Suppose rx is rational, say $rx = t$ with $t \in \mathbb{Q}$. Then

$$x = \frac{t}{r} \in \mathbb{Q},$$

contradicting x being irrational. Hence rx is irrational. □

Theorem 4. *Given any real number $x > 0$, there exists an irrational number in $(0, x)$.*

Proof. We consider two cases depending on whether x is rational or irrational.

Case 1: x is rational. Let $x = r \in \mathbb{Q}$. Consider

$$z = \frac{r}{\sqrt{2}}.$$

Since $r \neq 0$ and $\sqrt{2}$ is irrational, z is irrational. Moreover,

$$0 < z = \frac{r}{\sqrt{2}} < r = x.$$

Hence z is an irrational number in $(0, x)$.

Case 2: x is irrational. Then $x/2$ is positive and irrational. Clearly,

$$0 < \frac{x}{2} < x,$$

so $x/2$ is an irrational number in $(0, x)$.

In either case, there exists an irrational number in $(0, x)$. $\square$

Theorem 5. *Suppose $x, y \in \mathbb{R}$ and for each $\varepsilon > 0$, $|x - y| \leqslant \varepsilon$. Then $x = y$.*

Proof. Assume $x \neq y$. Take $\varepsilon = \frac{|x-y|}{2} > 0$. Then

$$|x - y| \leqslant \varepsilon = \frac{|x - y|}{2},$$

which is impossible. Hence $x = y$. $\square$

Example 6. Consider the set

$$S = (0, 1] = \{x \in \mathbb{R} : 0 < x \leqslant 1\}.$$

Notice that S is bounded above and below. We have

$$\sup S = 1 \in S, \quad \text{however,} \quad \inf S = 0 \notin S.$$

Theorem 7. *Suppose $A, B \subset \mathbb{R}$ such that A is bounded above and B is bounded below. Then the intersection $A \cap B$ is bounded both above and below.*

Proof. Since A is bounded above, there exists $M \in \mathbb{R}$ such that

$$a \leqslant M \quad \forall a \in A.$$

For any $x \in A \cap B$, we have $x \in A$, hence

$$x \leqslant M.$$

Thus M is an upper bound for $A \cap B$.

Since B is bounded below, there exists $m \in \mathbb{R}$ such that

$$b \geqslant m \quad \forall b \in B.$$

For any $x \in A \cap B$, we have $x \in B$, hence

$$x \geqslant m.$$

Thus m is a lower bound for $A \cap B$.

Therefore, $A \cap B$ is bounded both above and below. □

Theorem 8. *Let $S \subset \mathbb{R}$ be a nonempty set such that* $\sup S$ *and* $\inf S$ *exist. Then* $\sup S$ *and* $\inf S$ *are uniquely determined.*

Proof. Supremum uniqueness: Suppose u_1 and u_2 are both suprema of S. We want to show $u_1 = u_2$.

By definition of supremum, for any $\varepsilon > 0$, there exist $s_1, s_2 \in S$ such that

$$u_1 - \varepsilon < s_1 \leqslant u_1 \quad \text{and} \quad u_2 - \varepsilon < s_2 \leqslant u_2.$$

Take $\varepsilon = |u_1 - u_2|/2$. Without loss of generality, assume $u_1 < u_2$. Then

$$u_2 - \varepsilon = u_2 - \frac{u_2 - u_1}{2} = \frac{u_1 + u_2}{2} > u_1.$$

But there exists $s_2 \in S$ such that $s_2 > u_2 - \varepsilon > u_1$, contradicting that u_1 is an upper bound of S. Hence $u_1 = u_2$.

Infimum uniqueness: Suppose l_1 and l_2 are both infima of S. For any $\varepsilon > 0$, there exist $s_1, s_2 \in S$ such that

$$l_1 \leqslant s_1 < l_1 + \varepsilon \quad \text{and} \quad l_2 \leqslant s_2 < l_2 + \varepsilon.$$

Take $\varepsilon = |l_1 - l_2|/2$. Without loss of generality, assume $l_1 < l_2$. Then

$$l_1 + \varepsilon = l_1 + \frac{l_2 - l_1}{2} = \frac{l_1 + l_2}{2} < l_2.$$

But there exists $s_1 \in S$ such that $s_1 < l_1 + \varepsilon < l_2$, contradicting that l_2 is a lower bound of S. Hence $l_1 = l_2$. □

Theorem 9. *Let A and B be sets of positive numbers which are bounded above. Let*

$$a = \sup A, \quad b = \sup B,$$

and define

$$C = \{xy : x \in A,\, y \in B\}.$$

Then

$$\sup C = ab.$$

Proof. Let $c \in C$. Then $c = xy$ for some $x \in A$ and $y \in B$. Since $x \leqslant a$ and $y \leqslant b$, we have

$$c = xy \leqslant ab.$$

Hence ab is an upper bound for C.

Let $\varepsilon > 0$ be arbitrary. Since $a = \sup A$, there exists $x_\varepsilon \in A$ such that

$$a - \frac{\varepsilon}{2b} < x_\varepsilon \leqslant a.$$

Similarly, since $b = \sup B$, there exists $y_\varepsilon \in B$ such that

$$b - \frac{\varepsilon}{2a} < y_\varepsilon \leqslant b.$$

Consider $c_\varepsilon = x_\varepsilon y_\varepsilon \in C$. Then

$$\begin{aligned} ab - c_\varepsilon &= ab - x_\varepsilon y_\varepsilon \\ &= ab - a y_\varepsilon + a y_\varepsilon - x_\varepsilon y_\varepsilon \\ &= a(b - y_\varepsilon) + y_\varepsilon(a - x_\varepsilon) \\ &< a \cdot \frac{\varepsilon}{2a} + b \cdot \frac{\varepsilon}{2b} = \varepsilon. \end{aligned}$$

Hence, for any $\varepsilon > 0$, there exists $c_\varepsilon \in C$ such that

$$ab - \varepsilon < c_\varepsilon \leqslant ab.$$

Since ab is an upper bound of C and for every $\varepsilon > 0$ there exists $c_\varepsilon \in C$ with $ab - \varepsilon < c_\varepsilon$, it follows that

$$\sup C = ab. \qquad \square$$

Homework 2

Theorem 10. *Let* $S = \{x \in \mathbb{R} : 3x^2 - 10x + 3 < 0\}$. *Then* $\inf S = \frac{1}{3}$ *and* $\sup S = 3$.

Proof. We first consider the general case.

Let

$$q(x) = ax^2 + bx + c, \quad a \neq 0, \quad \Delta = b^2 - 4ac.$$

Then

$$q(x) = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a}.$$

Define $S := \{x \in \mathbb{R} : q(x) < 0\}$. We consider three cases.

Case A: $\Delta < 0$

- If $a > 0$: $-\frac{\Delta}{4a} > 0$, and $a(x + b/2a)^2 \geqslant 0$, so $q(x) > 0$ for all x. Hence $S = \varnothing$.

- If $a < 0$: $-\frac{\Delta}{4a} < 0$, and $a(x + b/2a)^2 \leqslant 0$, so $q(x) < 0$ for all x. Hence $S = \mathbb{R}$.

Case B: $\Delta = 0$, root $r = -b/(2a)$

- If $a > 0$: $q(x) = a(x - r)^2 \geqslant 0$, equality at $x = r$. So $S = \varnothing$.

- If $a < 0$: $q(x) = a(x - r)^2 \leqslant 0$, equality at $x = r$. So $S = \mathbb{R}\backslash\{r\}$.

Case C: $\Delta > 0$, distinct roots $r_1 = \frac{-b-\sqrt{\Delta}}{2a}, r_2 = \frac{-b+\sqrt{\Delta}}{2a}$, with $\alpha = \min(r_1, r_2), \beta = \max(r_1, r_2)$

$$q(x) = a(x - r_1)(x - r_2) = a(x - \alpha)(x - \beta).$$

- If $a > 0$: $(x - \alpha)(x - \beta) < 0$ for $\alpha < x < \beta$, so $S = (\alpha, \beta)$.
- If $a < 0$: $(x - \alpha)(x - \beta) < 0$ for $x < \alpha$ or $x > \beta$, so $S = (-\infty, \alpha) \cup (\beta, \infty)$.

$\inf S$ and $\sup S$:

- $\Delta < 0$:
  - $a > 0$: $S = \varnothing$, $\inf S = +\infty$, $\sup S = -\infty$.
  - $a < 0$: $S = \mathbb{R}$, $\inf S = -\infty$, $\sup S = +\infty$.
- $\Delta = 0$:
  - $a > 0$: $S = \varnothing$, $\inf S = +\infty$, $\sup S = -\infty$.
  - $a < 0$: $S = \mathbb{R} \backslash \{r\}$, $\inf S = -\infty$, $\sup S = +\infty$.
- $\Delta > 0$, roots $\alpha < \beta$:
  - $a > 0$: $S = (\alpha, \beta)$, $\inf S = \alpha$, $\sup S = \beta$.
  - $a < 0$: $S = (-\infty, \alpha) \cup (\beta, \infty)$, $\inf S = -\infty$, $\sup S = +\infty$.

If $q(x) = 3x^2 - 10x + 3$, then $S = \left(\frac{1}{3}, 3\right)$. Hence, $\inf S = \frac{1}{3}$ and $\sup S = 3$. $\square$

Theorem 11 (Lagrange's Identity). *For all real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, we have*

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum_{1 \leqslant k < j \leqslant n} (a_k b_j - a_j b_k)^2.$$

Proof. Notice that

$$\left(\sum_{i=1}^{n} x_i\right)^2 = \sum_{i=1}^{n} x_i^2 + 2 \sum_{1 \leqslant i < j \leqslant n} x_i x_j.$$

Now let

$$A := \sum_{i=1}^{n} a_i^2, \quad B := \sum_{i=1}^{n} b_i^2, \quad C := \sum_{i=1}^{n} a_i b_i.$$

Then

$$AB = \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i^2 b_j^2 = \sum_{i=1}^{n} a_i^2 b_i^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{n} a_i^2 b_j^2.$$

Using the expansion with $x_i = a_i b_i$, we obtain

$$C^2 = \left(\sum_{i=1}^{n} a_i b_i\right)^2 = \sum_{i=1}^{n} a_i^2 b_i^2 + 2\sum_{1\leqslant i<j\leqslant n} a_i a_j b_i b_j.$$

Subtracting,

$$\begin{aligned} AB - C^2 &= \Big[\sum_{i=1}^{n} a_i^2 b_i^2 + \sum_{i\neq j} a_i^2 b_j^2\Big] - \Big[\sum_{i=1}^{n} a_i^2 b_i^2 + 2\sum_{i<j} a_i a_j b_i b_j\Big], \\ &= \sum_{i\neq j} a_i^2 b_j^2 - 2\sum_{i<j} a_i a_j b_i b_j. \end{aligned}$$

Grouping the $i \neq j$ terms:

$$\sum_{i\neq j} a_i^2 b_j^2 = \sum_{i<j} a_i^2 b_j^2 + \sum_{j<i} a_i^2 b_j^2 = \sum_{i<j} \left(a_i^2 b_j^2 + a_j^2 b_i^2\right).$$

Hence,

$$AB - C^2 = \sum_{i<j} \left(a_i^2 b_j^2 + a_j^2 b_i^2 - 2a_i a_j b_i b_j\right) = \sum_{i<j} (a_i b_j - a_j b_i)^2. \qquad \square$$

Corollary 12 (Cauchy–Schwarz Inequality)**.** For all real numbers a_1, $a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, we have

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geqslant \left(\sum_{i=1}^{n} a_i b_i\right)^2.$$

Theorem 13. *Let $f\colon S \to T$ be a function. The following statements are equivalent:*

(a) f is one-to-one on S.

(b) $f^{-1}(f(A)) = A$ for every subset A of S.

(c) For all subsets $A, B \subseteq S$ with $B \subseteq A$, we have

$$f(A \backslash B) = f(A) \backslash f(B).$$

Proof. (a) $\Rightarrow$ (b): Assume f is one-to-one on S. Let $A \subseteq S$. If $a \in A$, then $f(a) \in f(A)$, so $a \in f^{-1}(f(A))$. Hence $A \subseteq f^{-1}(f(A))$.

Conversely, let $x \in f^{-1}(f(A))$. Then $f(x) \in f(A)$, so there exists $a \in A$ such that $f(x) = f(a)$. Since f is injective, $x = a \in A$. Thus $f^{-1}(f(A)) \subseteq A$, and we conclude $f^{-1}(f(A)) = A$.

(b) $\Rightarrow$ (c): Assume $f^{-1}(f(X)) = X$ for every $X \subseteq S$. Let $A, B \subseteq S$ with $B \subseteq A$.

First, if $y \in f(A \backslash B)$, then $y = f(x)$ for some $x \in A \backslash B$. Clearly $y \in f(A)$. If $y \in f(B)$, then $f(x) = f(b)$ for some $b \in B$, implying $x \in f^{-1}(f(B)) = B$, a contradiction. Hence $y \notin f(B)$, and $y \in f(A) \backslash f(B)$. Thus $f(A \backslash B) \subseteq f(A) \backslash f(B)$.

Conversely, if $y \in f(A) \backslash f(B)$, then $y = f(a)$ for some $a \in A$ but $y \notin f(B)$. If $a \in B$, then $f(a) \in f(B)$, contradiction. Thus $a \in A \backslash B$, and $y \in f(A - B)$. Hence $f(A) \backslash f(B) \subseteq f(A \backslash B)$, giving equality.

(c) $\Rightarrow$ (a): Assume (c) holds. Suppose f is not one-to-one. Then there exist distinct $x, y \in S$ with $f(x) = f(y)$. Let $A = \{x, y\}$ and $B = \{y\}$, so $B \subseteq A$. Then (c) gives

$$f(A \backslash B) = f(A) \backslash f(B).$$

Now $A \backslash B = \{x\}$, so $f(A \backslash B) = \{f(x)\}$. Also $f(A) = \{f(x)\}$ and $f(B) = \{f(y)\} = \{f(x)\}$, hence $f(A) \backslash f(B) = \varnothing$. Thus $\{f(x)\} = \varnothing$, impossible. Therefore, f must be one-to-one.

Since (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), and (c) $\Rightarrow$ (a), the three statements are equivalent. □

Problem 14. *Let $S \subseteq \mathbb{R} \times \mathbb{R}$ be the relation defined in each case below.*

(a) $S = \{(x, y) \in \mathbb{R}^2 : x \leqslant y\}$.

(b) $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

For each case determine whether S is reflexive, symmetric, and/or transitive.

Solution. (a) $S = \{(x, y) : x \leqslant y\}$.

Reflexive. For every $x \in \mathbb{R}$ we have $x \leqslant x$, so $(x, x) \in S$. Thus S is reflexive.

Symmetric. If $(x, y) \in S$ then $x \leqslant y$. This does not imply $y \leqslant x$ in general (take $x = 0, y = 1$), so S is not symmetric.

Transitive. If $(x, y) \in S$ and $(y, z) \in S$ then $x \leqslant y$ and $y \leqslant z$, hence $x \leqslant z$, so $(x, z) \in S$. Thus S is transitive.

(b) $S = \{(x, y) : x^2 + y^2 = 1\}$.

Reflexive. Reflexivity would require $(x, x) \in S$ for every x, i.e. $2x^2 = 1$ for all x, which is false (for example $(0, 0) \notin S$). Hence S is not reflexive.

Symmetric. The defining equation is symmetric in x and y: if $x^2 + y^2 = 1$ then $y^2 + x^2 = 1$, so $(y, x) \in S$. Thus S is symmetric.

Transitive. Transitivity fails. For example $(1, 0) \in S$ and $(0, 1) \in S$, but $(1, 1) \notin S$ since $1^2 + 1^2 = 2 \neq 1$. Therefore S is not transitive. □

Theorem 15. *The set of all circles in $\mathbb{R}^2$ whose centers have rational coordinates and whose radii are rational (positive) numbers is countable.*

Proof. A circle in the plane is determined uniquely by its center and its radius. Let

$$\mathcal{C} = \{\, C((p, q), r) : (p, q) \in \mathbb{Q}^2,\ r \in \mathbb{Q}_{>0} \,\},$$

where $C((p, q), r)$ denotes the circle with center (p, q) and radius r. Consider the map

$$\varphi\colon \mathbb{Q}^2 \times \mathbb{Q}_{>0} \longrightarrow \mathcal{C}, \qquad \varphi((p, q), r) = C((p, q), r).$$

This map is surjective by definition and injective because distinct triples $((p, q), r)$ determine distinct circles. Hence $\mathcal{C}$ is in bijection with the set $\mathbb{Q}^2 \times \mathbb{Q}_{>0}$.

Since $\mathbb{Q}$ is countable and any finite Cartesian product of countable sets is countable, the set $\mathbb{Q}^2 \times \mathbb{Q}_{>0}$ is countable. Therefore, $\mathcal{C}$ is countable. This completes the proof. □

Theorem 16. *Any collection $\mathcal{I}$ of pairwise disjoint intervals in $\mathbb{R}$, each of positive length, is at most countable (i.e., finite or countably infinite).*

Proof. Let $\mathcal{I}$ be such a collection. For each interval $I \in \mathcal{I}$ its length $\ell(I) > 0$, so I contains more than one point. Since the rationals $\mathbb{Q}$ are dense in $\mathbb{R}$, every nondegenerate interval I contains at least one rational number. Choose and fix, for each $I \in \mathcal{I}$, a rational number $q_I \in I$.

We claim the map $I \mapsto q_I$ is injective. Indeed, if $I \neq J$ are two distinct intervals in $\mathcal{I}$ then, because the intervals are pairwise disjoint, $I \cap J = \varnothing$. Hence $q_I \in I$ and $q_J \in J$ cannot be equal. Thus distinct intervals are assigned distinct rationals.

Therefore the set $\{q_I : I \in \mathcal{I}\}$ is an injective image of $\mathcal{I}$ and is a subset of $\mathbb{Q}$. Since $\mathbb{Q}$ is countable, every subset of $\mathbb{Q}$ is at most countable. It follows that $\mathcal{I}$ is at most countable. □

Theorem 17. *The set of real numbers $\mathbb{R}$ is uncountable.*

Proof. We show that the set of real numbers $\mathbb{R}$ is uncountable using the Cantor's diagonal argument.
Recall that a *decimal expansion* of a real number $x \in \mathbb{R}$ is a representation of the form

$$x = d_0.d_1d_2d_3 \ldots := d_0 + \sum_{i=1}^{\infty} d_i 10^{-i},$$

where d_0 is the integer part of x, and each $d_i \in \{0, 1, 2, \ldots, 9\}$ is a decimal digit. For numbers in $[0, 1)$, the expansion is of the form $x = 0.d_1d_2d_3 \ldots$. Some numbers have two decimal expansions (e.g., $0.5 = 0.5000 \ldots = 0.4999 \ldots$).

It suffices to prove that the interval $[0, 1) \subset \mathbb{R}$ is uncountable. Assume, for contradiction, that $[0, 1)$ is countable. Suppose that all numbers in $[0, 1)$ can be listed in a sequence:

$$x_1, x_2, x_3, \ldots$$

To avoid ambiguity from numbers with two expansions, we adopt the convention: choose the decimal expansion *not ending with infinitely many 9's*. Under this rule, every number in $[0, 1)$ has a unique decimal expansion. Using this convention, write the sequence as:

$$
\begin{array}{ccccccc}
x_1 & = & 0. & d_{11} & d_{12} & d_{13} & \ldots, \\
x_2 & = & 0. & d_{21} & d_{22} & d_{23} & \ldots, \\
 & & & \vdots & & &
\end{array}
$$

Define a number

$$
y \quad = \quad 0. \quad a_1 \quad a_2 \quad a_3 \quad \ldots
$$

by choosing each digit a_i as

$$
a_i \neq d_{ii}, \quad a_i \in \{1, 2, \ldots, 8\}.
$$

This ensures that y differs from x_i in the i-th decimal place. Since we avoided 0 and 9, y does not create ambiguity with decimal expansions.

By construction, $y \in [0, 1)$. However, $y \neq x_i$ for all $i \in \mathbb{N}$, so y is *not* in the list. This contradicts the assumption that all numbers in $[0, 1)$ were listed. Therefore, $[0, 1)$ is uncountable. Consequently, $\mathbb{R}$ is uncountable. $\square$

Homework 3

Theorem 18. *Let $S \subset \mathbb{R}^n$. Then* $\operatorname{int} S$ *(the interior of S) is an open set.*

Proof. Recall that $x \in \operatorname{int} S$ iff there exists $\varepsilon > 0$ such that the open ball $B_\varepsilon(x) = \{y \in \mathbb{R}^n : \|y - x\| < \varepsilon\}$ is contained in S.

Let $x \in \operatorname{int} S$. By definition, choose $\varepsilon > 0$ with $B_\varepsilon(x) \subset S$. We claim $B_\varepsilon(x) \subset \operatorname{int} S$, which will show that $\operatorname{int} S$ is a neighborhood of each of its points and hence open.

Indeed, let $y \in B_\varepsilon(x)$. Then $\|y - x\| < \varepsilon$. Put $\delta = \varepsilon - \|y - x\| > 0$. For any $z \in \mathbb{R}^n$ with $\|z - y\| < \delta$ we have

$$
\|z - x\| \leqslant \|z - y\| + \|y - x\| < \delta + \|y - x\| = \varepsilon,
$$

so $z \in B_\varepsilon(x) \subset S$. Thus $B_\delta(y) \subset S$, hence $y \in \operatorname{int} S$. This proves $B_\varepsilon(x) \subset \operatorname{int} S$.

Since every $x \in \operatorname{int} S$ has an open ball around it contained in $\operatorname{int} S$, the set $\operatorname{int} S$ is open. $\square$

Theorem 19. *The set $\mathbb{Z}$ has no accumulation points. Thus, $\mathbb{Z}$ is closed. However, $\mathbb{Z}$ is not open.*

Proof. Let $x \in \mathbb{R}$.

Case 1. If $x = k \in \mathbb{Z}$, choose $\varepsilon = \frac{1}{2}$. Then

$$(k - \tfrac{1}{2}, k + \tfrac{1}{2}) \cap \mathbb{Z} = \{k\}.$$

Hence the punctured neighborhood $(k-\varepsilon, k+\varepsilon)\backslash\{k\}$ contains no point of $\mathbb{Z}$; thus k is not an accumulation point.

Case 2. If $x \notin \mathbb{Z}$, let $d = \inf\{|x - n| : n \in \mathbb{Z}\} > 0$ be the distance from x to the nearest integer. Take $\varepsilon = \frac{d}{2}$. Then $(x - \varepsilon, x + \varepsilon)$ contains no integer, so it contains no point of $\mathbb{Z}$. Hence x is not an accumulation point.

Therefore $\mathbb{Z}$ has no accumulation points.

Now, $\mathbb{Z}$ is not open (no nonempty interval lies entirely inside $\mathbb{Z}$) and closed, since it contains all of its accumulation points (vacuously, because there are none). □

Theorem 20. *Every real number is an accumulation point of $\mathbb{Q}$.*

Proof. Let $x \in \mathbb{R}$ and $\varepsilon > 0$ be arbitrary. Choose an integer N such that $\frac{1}{N} < \varepsilon$. There exists an integer k with

$$\frac{k}{N} \leqslant x < \frac{k+1}{N}.$$

Then $\frac{k}{N}$ is rational and lies in $[x - \frac{1}{N}, x] \subset (x - \varepsilon, x + \varepsilon)$. If $\frac{k}{N} \neq x$, we are done. If $\frac{k}{N} = x$, then

$$0 < \frac{k+1}{N} - x < \frac{1}{N} < \varepsilon,$$

so $\frac{k+1}{N} \in (x-\varepsilon, x+\varepsilon)$ and $\frac{k+1}{N} \neq x$. Thus every punctured neighborhood of x contains a rational distinct from x, and hence x is an accumulation point of $\mathbb{Q}$. □

Remark 21. $\mathbb{Q}$ is not open (every interval contains irrationals) and not closed (irrationals are accumulation points not in $\mathbb{Q}$).

Theorem 22. *Let*

$$S = \left\{ \frac{1}{n} + \frac{1}{m} : m, n \in \mathbb{Z}_+ \right\}.$$

Then the accumulation points of S are precisely

$$\{0\} \cup \left\{ \frac{1}{k} : k \in \mathbb{Z}_+ \right\}.$$

Moreover, S is neither open nor closed.

Proof. For $m, n \in \mathbb{Z}_+$, define $s_{n,m} := \frac{1}{n} + \frac{1}{m}$.

(1) 0 is an accumulation point: Let $\varepsilon > 0$. Choose N such that $\frac{2}{N} < \varepsilon$. Then for all $m, n \geqslant N$,

$$0 < s_{n,m} \leqslant \frac{1}{N} + \frac{1}{N} = \frac{2}{N} < \varepsilon.$$

Hence $s_{n,m} \in (0 - \varepsilon, 0 + \varepsilon)$ and $s_{n,m} \neq 0$. Thus every punctured neighborhood of 0 contains a point of S, so 0 is an accumulation point.

(2) Each $\frac{1}{k}$ is an accumulation point: Fix $k \in \mathbb{Z}_+$ and let $\varepsilon > 0$. Choose M such that $\frac{1}{M} < \varepsilon$. Then

$$s_{k,M} = \frac{1}{k} + \frac{1}{M} \in \left(\tfrac{1}{k} - \varepsilon, \tfrac{1}{k} + \varepsilon\right),$$

and $s_{k,M} \neq \frac{1}{k}$. Hence each punctured neighborhood of $\frac{1}{k}$ contains a point of S, so $\frac{1}{k}$ is an accumulation point.

(3) No other accumulation points exist: Let $y \in \mathbb{R}$ and suppose y is an accumulation point of S. We will show that $y = 0$ or $y = \frac{1}{k}$ for some $k \in \mathbb{Z}_+$.

First observe that $S \subset (0, 2]$, so any accumulation point y must satisfy $0 \leqslant y \leqslant 2$. If $y = 0$, we are done. Assume $y > 0$.

Because y is an accumulation point, for every $\varepsilon > 0$, the punctured neighborhood $(y - \varepsilon, y + \varepsilon) \backslash \{y\}$ contains some $s_{n,m} \neq y$. Consider the set of index pairs

$$P(\varepsilon) = \{(n, m) \in \mathbb{Z}_+^2 : s_{n,m} \in (y - \varepsilon, y + \varepsilon)\}.$$

Suppose, for contradiction, that both coordinates n and m are bounded on $P(\varepsilon_0)$ for some sufficiently small $\varepsilon_0 > 0$. That is, there exist integers N_0, M_0 such that whenever $(n, m) \in P(\varepsilon_0)$, we have $n \leqslant N_0$ and $m \leqslant M_0$. Then the set of possible values

$$F = \{s_{n,m} : 1 \leqslant n \leqslant N_0,\ 1 \leqslant m \leqslant M_0\}$$

is finite.

If $y \notin F$, let

$$\delta = \min\{|y - f| : f \in F\} > 0,$$

and choose $\varepsilon < \frac{\delta}{2}$. Then $(y-\varepsilon, y+\varepsilon) \cap F = \varnothing$, contradicting $P(\varepsilon_0) \neq \varnothing$.

If $y \in F$, let

$$\delta = \min\{|y - f| : f \in F,\ f \neq y\} > 0,$$

and take $\varepsilon < \frac{\delta}{2}$. Then the punctured neighborhood $(y - \varepsilon, y + \varepsilon)\backslash\{y\}$ contains no element of F, again contradicting $P(\varepsilon) \neq \varnothing$.

Therefore, it is impossible that both coordinates are bounded for arbitrarily small ε. Therefore, at least one coordinate is unbounded among pairs (n, m) whose sums $s_{n,m}$ lie arbitrarily close to y.

Case A: both coordinates can be made arbitrarily large.

Then for any $\varepsilon > 0$ we can find n, m so large that

$$s_{n,m} = \frac{1}{n} + \frac{1}{m} < \varepsilon.$$

(Choose N with $\frac{2}{N} < \varepsilon$ and take $n, m \geqslant N$.) Hence, we must have $y = 0$. But we assumed $y > 0$, so this case cannot occur for $y > 0$.

Case B: exactly one coordinate is unbounded while the other takes only finitely many values.

Then there exists some fixed $k \in \mathbb{Z}_+$ and arbitrarily large integers m (or vice versa) such that $s_{k,m}$ lies within any given ε-neighborhood of y. But for every $\varepsilon > 0$ there exists M with $|s_{k,m} - \frac{1}{k}| < \varepsilon$ for all $m \geqslant M$. By the punctured-neighborhood definition, this forces $y = \frac{1}{k}$.

Combining the impossibility of Case A for $y > 0$ and the conclusion of Case B, we find that any positive accumulation point y must be equal to some $\frac{1}{k}$.

Thus the only accumulation points are 0 and the numbers $\frac{1}{k}$ for $k \in \mathbb{Z}_+$.

Now, S is not open (its points are isolated in the sense that for a fixed (n, m) we can choose ε small enough to exclude all other $s_{n',m'}$), and not closed because 0 (and the points $\frac{1}{k}$) are accumulation points not in S. $\square$

Theorem 23. *The set of accumulation points of* $S = \{(x, y) \in \mathbb{R}^2 : x > 0\} \subset \mathbb{R}^2$ *is* $\{(x, y) \in \mathbb{R}^2 : x \geqslant 0\}$. *Moreover, S is open but not closed.*

Proof. Let $p = (x, y) \in \mathbb{R}^2$.

(i) If $x > 0$. Fix $\varepsilon > 0$. Take $q = (x', y')$ with $x' = x + \min\{\varepsilon/2,\ x/2\} > 0$ and $y' = y$. Then $\|q - p\| = |x' - x| < \varepsilon$ and $q \in S$, $q \neq p$. Thus every punctured neighborhood of p meets S; so p is an accumulation point.

(ii) If $x = 0$. Fix $\varepsilon > 0$. Let $q = (\varepsilon/2, y)$. Then $\|q - p\| = \varepsilon/2 < \varepsilon$ and $q \in S$. Hence $(0, y)$ is an accumulation point (though $(0, y) \notin S$).

(iii) If $x < 0$. Put $\varepsilon = -x/2 > 0$. If $\|q - p\| < \varepsilon$ then the first coordinate x' of q satisfies $|x' - x| < \varepsilon$, so $x' \leqslant x + \varepsilon = x/2 < 0$. Thus no point of S lies in $B_\varepsilon(p)$. Hence p is not an accumulation point.

Combining (i)–(iii) gives that the accumulation points are exactly those with $x \geqslant 0$.

Now, we show S is open. Let $p = (x, y) \in S$. Then $x > 0$. Take $\varepsilon = \frac{x}{2} > 0$. If $q = (x', y')$ satisfies $\|q - p\| < \varepsilon$, then $|x' - x| < \varepsilon$, so

$$x' > x - \varepsilon = x/2 > 0.$$

Hence $q \in S$. Therefore every neighborhood of p lies in S, so S is open.

Since points with $x = 0$ are accumulation points not in S, S is not closed. $\square$

Theorem 24. *The set of accumulation points of* $S = \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 < 1\}$ *is* $\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 \leqslant 1\}$. *Moreover, S is open but not closed.*

Proof. Let $p = (x, y) \in \mathbb{R}^2$. Define $g(x, y) = x^2 - y^2$.

(i) If $g(x,y) < 1$. Set $\Delta := 1 - g(x,y) > 0$. Choose

$$\delta = \min\left\{1,\ \frac{\Delta}{4(|x|+|y|+1)}\right\} > 0.$$

If $\|(x',y') - (x,y)\| < \delta$ then in particular $|x'-x| < \delta$ and $|y'-y| < \delta$. Now

$$|x'^2 - x^2| = |x'-x|\,|x'+x| \leqslant \delta(2|x|+\delta) \leqslant \delta(2|x|+1),$$

and similarly

$$|y'^2 - y^2| \leqslant \delta(2|y|+1).$$

Hence

$$\begin{aligned} |g(x',y') - g(x,y)| &\leqslant |x'^2 - x^2| + |y'^2 - y^2| \\ &\leqslant \delta\big(2(|x|+|y|)+2\big) \\ &\leqslant 2\delta(|x|+|y|+1). \end{aligned}$$

By the choice of δ we have $2\delta(|x|+|y|+1) \leqslant \Delta/2$, so $|g(x',y') - g(x,y)| < \Delta/2$. Therefore

$$g(x',y') < g(x,y) + \Delta/2 = 1 - \Delta/2 < 1.$$

Thus every punctured neighborhood of p contains points of S; so p is an accumulation point (and an interior point).

(ii) If $g(x,y) = 1$. Note $x \neq 0$ (otherwise $-y^2 = 1$ impossible). Fix $\varepsilon > 0$. Choose $\delta > 0$ with $\delta|x| < \varepsilon$, for example $\delta = \min\{\varepsilon/(2|x|), 1/2\}$. Let $x' = (1-\delta)x,\ y' = y$. Then

$$\|(x',y') - (x,y)\| = |x - x'| = \delta|x| < \varepsilon,$$

and

$$g(x',y') = (1-\delta)^2x^2 - y^2 = x^2 - y^2 - 2\delta x^2 + \delta^2 x^2 = 1 - 2\delta x^2 + \delta^2 x^2 < 1.$$

Thus every punctured neighborhood of a boundary point (x,y) meets S, so every boundary point is an accumulation point (but not in S).

(iii) If $g(x,y) > 1$. Put $\Gamma := g(x,y) - 1 > 0$. Choose

$$\delta = \min\left\{1,\ \frac{\Gamma}{4(|x|+|y|+1)}\right\} > 0.$$

Arguing as in (i) we obtain

$$|g(x',y') - g(x,y)| \leqslant 2\delta(|x|+|y|+1) \leqslant \Gamma/2,$$

whenever $\|(x',y') - (x,y)\| < \delta$. Hence for such (x',y'),

$$g(x',y') > g(x,y) - \Gamma/2 = 1 + \Gamma/2 > 1,$$

so no point of S lies in $B_\delta(p)$. Thus p is not an accumulation point.

Combining (i)–(iii) shows the accumulation points are exactly those with $x^2 - y^2 \leqslant 1$.

Now, we show S is open. Let $p = (x,y) \in S$ and define $\Delta = 1 - (x^2 - y^2)$. Then If $\Delta > 0$. Choose

$$\delta = \min\left\{1, \frac{\Delta}{4(|x|+|y|+1)}\right\} > 0.$$

If $\|(x',y') - (x,y)\| < \delta$, then $|x' - x| < \delta$, $|y' - y| < \delta$, so

$$|x'^2 - x^2| \leqslant \delta(2|x|+1), \quad |y'^2 - y^2| \leqslant \delta(2|y|+1).$$

Hence

$$|(x'^2 - y'^2) - (x^2 - y^2)| \leqslant 2\delta(|x|+|y|+1) \leqslant \Delta/2,$$

so $x'^2 - y'^2 < 1$. Thus $B_\delta(p) \subset S$, and S is open.

Boundary points (where $x^2 - y^2 = 1$) are accumulation points not in S, so S is not closed. $\square$

Theorem 25. *Every point of* $\mathbb{R}^n$ *is an accumulation point of* $\mathbb{Q}^n$*. Moreover,* $\mathbb{Q}^n$ *is neither open nor closed.*

Proof. Fix $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\varepsilon > 0$. Choose a positive integer N with

$$\frac{1}{N} < \frac{\varepsilon}{\sqrt{n}}.$$

For each coordinate x_i choose an integer k_i with

$$\frac{k_i}{N} \leqslant x_i < \frac{k_i + 1}{N}.$$

Set $q_i = \frac{k_i}{N}$ for $i = 1, \ldots, n$ and $q = (q_1, \ldots, q_n)$. Then each $q_i \in \mathbb{Q}$ and

$$|x_i - q_i| < \frac{1}{N} < \frac{\varepsilon}{\sqrt{n}}.$$

Therefore

$$\|x - q\| = \sqrt{\sum_{i=1}^{n} (x_i - q_i)^2} < \sqrt{n \cdot \frac{\varepsilon^2}{n}} = \varepsilon.$$

If $q \neq x$ we are done. If $q = x$ (this can only occur when $x \in \mathbb{Q}^n$), then modify one coordinate slightly: replace q_1 by $q_1 + \frac{1}{N}$ (which is rational and still satisfies $|x_1 - (q_1 + 1/N)| \leqslant 1/N < \varepsilon/\sqrt{n}$), so the modified rational vector $q' \in \mathbb{Q}^n$ satisfies $\|x - q'\| < \varepsilon$ and $q' \neq x$. Hence every punctured ball around x contains a rational point distinct from x, proving the claim.

But $\mathbb{Q}^n$ has no interior points, since every ball contains irrationals. Therefore $\mathbb{Q}^n$ is neither open nor closed. □

Theorem 26. *Let*

$$S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \left\{ 1 + \frac{1}{n} : n \in \mathbb{N} \right\} \cup \left\{ 2 + \frac{1}{n} : n \in \mathbb{N} \right\}.$$

Then the set of accumulation points of S is exactly $\{0, 1, 2\}$.

Proof. Let $a \in \{0, 1, 2\}$ and let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Then the point $a + \frac{1}{N} \in S$ (for $a = 0$ we interpret this as $\frac{1}{N} \in S$) satisfies

$$0 < |a + \tfrac{1}{N} - a| = \tfrac{1}{N} < \varepsilon.$$

Hence every punctured neighborhood $(a - \varepsilon, a + \varepsilon) \setminus \{a\}$ contains points of S. Thus a is an accumulation point of S.

Let $y \in \mathbb{R}\setminus\{0,1,2\}$. Define

$$d = \min\{|y-0|, |y-1|, |y-2|\} > 0, \qquad r = \frac{d}{2}.$$

Let $F = \{s \in S : |s-y| < r\}$. Suppose for contradiction that F is infinite. Then there exists $i \in \{0,1,2\}$ and an infinite subset $\mathscr{A} \subseteq \mathbb{N}$ such that

$$\left|i + \frac{1}{n} - y\right| < r \qquad \text{for all } n \in \mathscr{A}.$$

Fix $n \in \mathscr{A}$ with $n > \frac{2}{d}$ (such an n exists because $\mathscr{A}$ is infinite). Then $\frac{1}{n} < \frac{d}{2}$ and

$$|y-i| \leqslant \left|y - \left(i + \frac{1}{n}\right)\right| + \frac{1}{n} < r + \frac{1}{n} = \frac{d}{2} + \frac{1}{n} < \frac{d}{2} + \frac{d}{2} = d,$$

which contradicts the definition of d (since $|y-i| \geqslant d$). Hence F must be finite.

If $F = \varnothing$, then $B_r(y)$ contains no point of S and we are done. Otherwise, set

$$\varepsilon = \min\left\{r, \frac{1}{2}\min_{s\in F}|s-y|\right\} > 0.$$

Then no point of S (other than possibly y itself, but $y \notin S$) lies in $(y-\varepsilon, y+\varepsilon)$. Hence the punctured neighborhood $(y-\varepsilon, y+\varepsilon)\setminus\{y\}$ contains no point of S, so y is not an accumulation point.

Therefore, the set of accumulation points of S is exactly {0,1,2}. □

Theorem 27. *Let $S \subset \mathbb{R}^n$. The closure $\overline{S}$ is the intersection of all closed subsets of $\mathbb{R}^n$ that contain S, i.e.*

$$\overline{S} = \bigcap\{\, F \subset \mathbb{R}^n : F \text{ is closed and } S \subset F \,\}.$$

Proof. Let $\mathcal{F} = \{F \subset \mathbb{R}^n : F$ is closed and $S \subset F\}$ and set

$$K := \bigcap_{F\in\mathcal{F}} F.$$

We will show $\overline{S} = K$.

(1) $\overline{S} \subset K$. By definition $\overline{S}$ is a closed set containing S. Since K is the intersection of *all* closed sets that contain S, every such closed set in particular contains $\overline{S}$. Hence $\overline{S} \subset F$ for every $F \in \mathcal{F}$, and therefore $\overline{S} \subset K$.

(2) $K \subset \overline{S}$. Suppose $x \notin \overline{S}$. By the definition of closure there exists $\varepsilon > 0$ such that the open ball $B_\varepsilon(x)$ satisfies

$$B_\varepsilon(x) \cap S = \varnothing.$$

Equivalently, $S \subset \mathbb{R}^n \backslash B_\varepsilon(x)$. The complement $\mathbb{R}^n \backslash B_\varepsilon(x)$ is closed and contains S, but it does not contain x. Thus $\mathbb{R}^n \backslash B_\varepsilon(x) \in \mathcal{F}$ and $x \notin \bigcap_{F \in \mathcal{F}} F = K$. Hence every $x \notin \overline{S}$ is also not in K, so $K \subset \overline{S}$.

Combining (1) and (2) yields $\overline{S} = K$, which proves the claim. □

Theorem 28. *Let*

$$\mathcal{F} = \left\{ \left(\frac{1}{n}, \frac{2}{n} \right) : n \in \mathbb{Z}_+ \right\}.$$

Then $\mathcal{F}$ is an open cover of $(0, 1)$, but no finite subcollection of $\mathcal{F}$ covers $(0, 1)$.

Proof. Each set $(1/n, 2/n)$ is open, so $\mathcal{F}$ is a collection of open sets. Let $x \in (0, 1)$ be arbitrary. Then $1/x > 1$, hence

$$\frac{2}{x} - \frac{1}{x} = \frac{1}{x} > 1,$$

so the open interval $(1/x, 2/x)$ has length $1/x > 1$ and therefore contains at least one integer. Thus there exists $n \in \mathbb{Z}_+$ with

$$\frac{1}{x} < n < \frac{2}{x}.$$

Rewriting the inequalities gives

$$\frac{1}{n} < x < \frac{2}{n},$$

so $x \in (1/n, 2/n) \in \mathcal{F}$. Since x was arbitrary, $\bigcup \mathcal{F} = (0, 1)$, i.e., $\mathcal{F}$ is an open cover of $(0, 1)$.

We show that no finite subcollection of $\mathcal{F}$ covers $(0,1)$. Suppose, for contradiction, that a finite subcollection $\{(1/n_i, 2/n_i) : i = 1, \ldots, k\} \subset \mathcal{F}$ covers $(0,1)$. Let $N = \max\{n_1, \ldots, n_k\}$. Consider the point

$$x = \frac{1}{N+1} \in (0,1).$$

For any chosen index i we have $n_i \leqslant N$, hence

$$\frac{1}{n_i} \geqslant \frac{1}{N} > \frac{1}{N+1} = x,$$

so $x \notin (1/n_i, 2/n_i)$. Thus x is not contained in any of the finitely many chosen intervals, contradicting the assumption that the finite subcollection covers $(0,1)$. Therefore no finite subcollection of $\mathcal{F}$ can cover $(0,1)$.

This completes the proof. $\square$

Theorem 29. *Let*

$$\mathcal{B} = \{\, B((q,q),q) : q \in \mathbb{Q}_{>0} \,\},$$

where $B((q,q),q) = \{(u,v) \in \mathbb{R}^2 : \sqrt{(u-q)^2 + (v-q)^2} < q\}$*. Then* $\mathcal{B}$ *is a countable collection and*

$$\bigcup_{q \in \mathbb{Q}_{>0}} B((q,q),q) = \{(x,y) \in \mathbb{R}^2 : x > 0,\ y > 0\}.$$

In particular $\mathcal{B}$ *is a countable cover of the open first quadrant.*

Proof. The set $\mathbb{Q}_{>0}$ of positive rationals is countable, hence the indexed family $\mathcal{B}$ is countable.

Let (a,b) be an arbitrary point with $a > 0$ and $b > 0$. Define the function

$$F(r) = (a-r)^2 + (b-r)^2 - r^2.$$

A point (a,b) lies in $B((r,r),r)$ precisely when $F(r) < 0$. Expand and simplify:

$$F(r) = (a^2 + b^2) - 2(a+b)r + r^2.$$

Thus $F(r) < 0$ is equivalent to

$$r^2 - 2(a+b)r + (a^2+b^2) < 0.$$

The quadratic on the left has discriminant

$$\Delta = 4(a+b)^2 - 4(a^2+b^2) = 8ab > 0,$$

so the inequality holds exactly for r lying between the two real roots

$$r_\pm = (a+b) \pm \sqrt{2ab}.$$

Hence

$$F(r) < 0 \quad \Longleftrightarrow \quad r \in \big(r_-, r_+\big).$$

Note that $r_- > 0$ because $(a+b)^2 - 2ab = a^2 + b^2 > 0$, so the open interval (r_-, r_+) is a nonempty interval contained in $(0, \infty)$.

By density of the rationals there exists some $q \in \mathbb{Q}_{>0} \cap (r_-, r_+)$. For such a rational q we have $F(q) < 0$, i.e.

$$\sqrt{(a-q)^2 + (b-q)^2} < q,$$

so $(a, b) \in B((q, q), q)$. Since (a, b) was an arbitrary point of the first quadrant, every such point is contained in some ball from $\mathcal{B}$.

Combining the two parts, $\mathcal{B}$ is a countable cover of $\{(x, y) : x > 0, y > 0\}$. $\square$

Theorem 30. *Let $\mathcal{U}$ be a collection of pairwise disjoint nonempty open subsets of $\mathbb{R}^n$. Then $\mathcal{U}$ is at most countable.*

Proof. The set $\mathbb{Q}^n$ of points with rational coordinates is countable. Enumerate $\mathbb{Q}^n = \{q_1, q_2, q_3, \ldots\}$.

By Theorem 25, for each $U \in \mathcal{U}$ the intersection $U \cap \mathbb{Q}^n$ is nonempty. Define an assignment $f : \mathcal{U} \to \mathbb{Q}^n$ by letting $f(U)$ be the first rational q_i (with smallest index i) that lies in U. This is well defined because each U contains at least one rational and our enumeration gives a least index.

We claim f is injective. Indeed, if $U \neq V$ are two distinct sets in $\mathcal{U}$ then $U \cap V = \varnothing$ by hypothesis; hence no rational point can lie in both

U and V. Therefore the first rational in U cannot equal the first rational in V, so $f(U) \neq f(V)$.

Since f injects $\mathcal{U}$ into the countable set $\mathbb{Q}^n$, the collection $\mathcal{U}$ must itself be at most countable. $\square$

Remark 31. The hypothesis "nonempty" is essential: the empty set is open and many copies of it would be pairwise disjoint but not interesting.

Example 32. The family of singletons

$$\mathcal{C} = \{\{x\} : x \in [0,1]\}$$

is an uncountable collection of pairwise disjoint closed subsets of $\mathbb{R}$. Each $\{x\}$ is closed in $\mathbb{R}$, distinct singletons are disjoint, and the indexing set $[0,1]$ is uncountable, so $\mathcal{C}$ is uncountable.

Midterm

Theorem 33. *There is no continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(\mathbb{R}) = \mathbb{Q}$.*

Proof. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is continuous with $f(\mathbb{R}) = \mathbb{Q}$. Then f is not constant, so pick x_1, x_2 with $a := f(x_1) < b := f(x_2)$ (both rational). Choose any irrational $s \in (a,b)$ (every nonempty open interval contains irrationals), for example, we may take $s := a + \frac{b-a}{\sqrt{2}}$. By the Intermediate Value Theorem there exists $c \in (x_1, x_2)$ with $f(c) = s$, contradicting $f(\mathbb{R}) = \mathbb{Q}$. Thus no such continuous f exists. $\square$

Theorem 34. *There is a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f((0,1)) = (0,1]$.*

Proof. Consider $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 0, & x \leqslant 0, \\ 2x, & 0 < x \leqslant \frac{1}{2}, \\ 2(1-x), & \frac{1}{2} < x < 1, \\ 0, & x \geqslant 1. \end{cases}$$

Then f is continuous and $f((0,1)) = (0,1]$. $\square$

Theorem 35. *Consider the function $d\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by*

$$d(x,y) = |2x - y|.$$

Then d is not a metric on $\mathbb{R}$.

Proof. To be a metric, d must satisfy the following for all $x, y, z \in \mathbb{R}$:

(a) $d(x,y) \geqslant 0$

(b) $d(x,y) = 0$ if and only if $x = y$.

(c) $d(x,y) = d(y,x)$.

(d) $d(x,y) \leqslant d(x,z) + d(z,y)$

Notice that a holds. However, b does not hold in general; for instance, $d(1,2) = |2-2| = 0$, but $1 \neq 2$. Similarly, c also does not hold: $d(1,2) = |2-2| = 0$, but $d(2,1) = |4-1| = 3 \neq 0$. Hence, d is not a metric on $\mathbb{R}$. $\square$

Theorem 36. *The set $\mathbb{Z} \subset \mathbb{R}$ has no accumulation point.*

Proof. Let $x \in \mathbb{R}$. We consider two cases:

Case 1: $x \in \mathbb{Z}$.
Take $\varepsilon = \frac{1}{4}$. Then the interval $(x-\varepsilon, x+\varepsilon)$ contains no integer other than x itself. By the definition of an accumulation point, x would need to have an integer in every interval around it different from x. Since $(x-\varepsilon, x+\varepsilon)$ contains no such point, x is not an accumulation point.

Case 2: $x \notin \mathbb{Z}$.
Let $n = \lfloor x \rfloor$ be the greatest integer less than x, and let $d := \min\{x - n,\ (n+1) - x\} > 0$ be the distance from x to the nearest integer. Take $\varepsilon = \frac{d}{2}$. Then the interval $(x-\varepsilon, x+\varepsilon)$ contains no integers at all. Hence, by the definition, x is not an accumulation point.

Since $x \in \mathbb{R}$ was arbitrary, no point of $\mathbb{R}$ is an accumulation point of $\mathbb{Z}$. Therefore, $\mathbb{Z}$ has no accumulation points. $\square$

Theorem 37. *Let $S \subset \mathbb{R}$ be nonempty with $b = \sup S$. Then for every $\varepsilon > 0$ there exists $x \in S$ satisfying $x \leqslant b < x + \varepsilon$.*

Proof. Let $\varepsilon > 0$ be given. By the definition of supremum, b is the least upper bound of S, so $b - \varepsilon < b$ is not an upper bound of S. Hence, there exists $x \in S$ such that $b - \varepsilon < x \leqslant b$. Adding ε to the left inequality, we get $x \leqslant b < x + \varepsilon$.

This proves that for every $\varepsilon > 0$, there exists $x \in S$ satisfying $x \leqslant b < x + \varepsilon$. □

Theorem 38. *Let $\mathcal{F} := \{I_\alpha : \alpha \in A\}$ be a family of non-empty open intervals in $\mathbb{R}$ which are pairwise disjoint, i.e., $I_\alpha \cap I_\beta = \varnothing$ whenever $\alpha \neq \beta$. Then A is a countable set.*

Proof. Since each $I_\alpha = (a_\alpha, b_\alpha)$ is nonempty and open, by the density of rationals in $\mathbb{R}$, there exists a rational number $q_\alpha \in I_\alpha$.

Because the intervals are pairwise disjoint, $q_\alpha \neq q_\beta$ whenever $\alpha \neq \beta$. Thus the map

$$\alpha \mapsto q_\alpha$$

is injective from A into $\mathbb{Q}$.

Since $\mathbb{Q}$ is countable, it follows that A is at most countable. □

Theorem 39. *Let $S \subset \mathbb{R}^n$ be open and $x_0 \in \mathbb{R}^n$ be fixed. Define*

$$T := \{x_0 + y : y \in S\}.$$

Then T is an open set.

Proof. Take any $t \in T$. Then there exists $y \in S$ such that $t = x_0 + y$. Since S is open, there exists $\varepsilon > 0$ such that

$$B(y, \varepsilon) := \{w \in \mathbb{R}^n : \|w - y\| < \varepsilon\} \subset S.$$

Now consider

$$B(t, \varepsilon) := \{z \in \mathbb{R}^n : \|z - t\| < \varepsilon\}.$$

For any $z \in B(t, \varepsilon)$, let $w := z - x_0$. Then

$$\|w - y\| = \|(z - x_0) - y\| = \|z - t\| < \varepsilon,$$

so $w \in B(y, \varepsilon) \subset S$. Hence $z = x_0 + w \in T$.

This shows $B(t, \varepsilon) \subset T$. Since $t \in T$ was arbitrary, T is open. □

Theorem 40. *Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Then f is not uniformly continuous on $\mathbb{R}$.*

Proof. Suppose, for contradiction, that $f(x) = x^2$ is uniformly continuous on $\mathbb{R}$. Consider any $\varepsilon > 0$. Then, there exists $\delta > 0$ such that for all $x, y \in \mathbb{R}$,

$$|x - y| < \delta \implies |x^2 - y^2| < \varepsilon.$$

Let N be a positive integer. Now take $x = \delta N$ and $y = x + \frac{\delta}{2}$. Then $|x - y| = \frac{\delta}{2} < \delta$ and

$$|x^2 - y^2| = |x - y|\,|x + y| = \frac{\delta}{2}\left(2\delta N + \frac{\delta}{2}\right) = \delta^2 N + \frac{\delta^2}{4}.$$

By taking N large enough, for instance,

$$N := \left\lfloor \frac{|\varepsilon - \frac{\delta^2}{4}|}{\delta^2} \right\rfloor + 1,$$

we can make $|x^2 - y^2| > \varepsilon$, contradicting the uniform continuity condition.

Hence, $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$. □

Theorem 41 (Heine–Borel)**.** *A subset $K \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded. Equivalently, every open cover of K has a finite sub-cover.*

Remark 42. Let $X = \mathbb{R}$ with the discrete metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

Consider $S = [-1, 1] \subset X$.
Then S is bounded, since $d(x, y) \leqslant 1$ for all $x, y \in S$, and S is closed (all subsets of a discrete metric space are closed).
However, S is not compact. Consider the open cover

$$\{\{x\} : x \in [-1, 1]\}.$$

No finite subcollection covers S, so S is not compact.

Hence, in this metric space, a set can be closed and bounded but not compact. Therefore, the Heine–Borel theorem does not hold in general metric spaces.

Theorem 43. *Let $a \in \mathbb{R}^n$ and $r > 0$. Then $\overline{B}(a;r) := \{x \in \mathbb{R}^n : \|x - a\| \leqslant r\}$ is a closed set.*

Proof. Consider the complement

$$\mathbb{R}^n \backslash \overline{B}(a;r) = \{x \in \mathbb{R}^n : \|x - a\| > r\}.$$

Take any $x \in \mathbb{R}^n \backslash \overline{B}(a;r)$. Then $\|x - a\| > r$, and let

$$\varepsilon := \|x - a\| - r > 0.$$

For any $y \in \mathbb{R}^n$ with $\|y - x\| < \varepsilon$, the triangle inequality gives

$$\|y - a\| \geqslant \|x - a\| - \|y - x\| > \|x - a\| - \varepsilon = r.$$

Hence $y \in \mathbb{R}^n \backslash \overline{B}(a;r)$, showing that the complement is open. Since the complement of $\overline{B}(a;r)$ is open, $\overline{B}(a;r)$ is closed. $\square$

Theorem 44. *Let S be a bounded subset of $\mathbb{R}^n$. Let $\varepsilon > 0$. Then S can be covered by a finite number of balls of radius ε.*

Proof. Let $S \subset \mathbb{R}^n$ be bounded. Then there exists $a \in \mathbb{R}^n$ and $r > 0$ such that

$$S \subset \overline{B}(a, r) := \{x \in \mathbb{R}^n : \|x - a\| \leqslant r\}.$$

The closure $\overline{S}$ of S satisfies

$$\overline{S} \subseteq \overline{B}(a, r),$$

so $\overline{S}$ is bounded. By definition, $\overline{S}$ is also closed.

By the Heine–Borel theorem, a set in $\mathbb{R}^n$ is compact if and only if it is closed and bounded. Hence $\overline{S}$ is compact.

Let $\varepsilon > 0$. Consider the open cover

$$\{B(x, \varepsilon) : x \in \overline{S}\}.$$

By compactness, there exists a finite subcollection of balls that covers $\overline{S}$. These balls also cover $S \subset \overline{S}$.

Therefore, S can be covered by finitely many balls of radius ε. $\square$

Theorem 45. *Let $S \subset \mathbb{R}^n$ be bounded. Then for every $\varepsilon > 0$ there exist finitely many points $x_1, x_2, \ldots, x_m \in S$ such that*

$$S \subset \bigcup_{i=1}^{m} B(x_i, \varepsilon).$$

In other words, every bounded subset of $\mathbb{R}^n$ is totally bounded, and the covering balls of fixed radius ε may be chosen with centers in S.

Proof. Suppose, for contradiction, that $S \subset \mathbb{R}^n$ is bounded but not totally bounded. Then there exists some $\varepsilon > 0$ such that no finite collection of ε-balls centered at points of S covers S.
Pick any $x_1 \in S$. Since $\{B(x_1, \varepsilon)\}$ does not cover S, we may choose $x_2 \in S \backslash B(x_1, \varepsilon)$. Inductively, having chosen $x_1, \ldots, x_k \in S$, the finite union $\bigcup_{i=1}^{k} B(x_i, \varepsilon)$ does not cover S, so we may pick

$$x_{k+1} \in S \backslash \bigcup_{i=1}^{k} B(x_i, \varepsilon).$$

This produces an infinite sequence $(x_m)_{m \geqslant 1} \subset S$ with the property that

$$\|x_i - x_j\| > \varepsilon \quad \text{for all } i \neq j.$$

Since S is bounded, the sequence (x_m) is bounded. By the Bolzano–Weierstrass theorem, there exists a subsequence (x_{m_k}) converging to some limit $x \in \mathbb{R}^n$. Choose K such that for all $k \geqslant K$,

$$\|x_{m_k} - x\| < \frac{\varepsilon}{2}.$$

Then for $k, \ell \geqslant K$ we have

$$\|x_{m_k} - x_{m_\ell}\| \leqslant \|x_{m_k} - x\| + \|x_{m_\ell} - x\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

contradicting the fact that all pairwise distances exceed ε. Therefore, our assumption was false, and S must be totally bounded. $\square$

Definition 46. Let (M, d) be a metric space. A sequence $\{x_n\}_{n \geqslant 1}$ in M

is said to *converge* to a point $p \in M$ if for every $\varepsilon > 0$, there exists an integer $N \geqslant 1$ such that

$$d(x_n, p) < \varepsilon \quad \text{for all } n \geqslant N.$$

In symbols, we write

$$x_n \to p \quad \text{as } n \to \infty.$$

Theorem 47. *Let $x \in \mathbb{R}$. Let $\{x_n\}_{n \geqslant 1}$ be a sequence of real numbers such that $x_n \to x$. Consider the sequence of arithmetic means $\{s_n\}_{n \geqslant 1}$, defined by*

$$s_n := \frac{1}{n} \sum_{k=1}^{n} x_k.$$

Then $\{s_n\}_{n \geqslant 1}$ also converges to x.

Proof. Fix $\varepsilon > 0$. Since $x_n \to x$, there exists a positive integer n_0 such that

$$|x_n - x| < \frac{\varepsilon}{2} \quad \text{for all } n \geqslant n_0.$$

Then for $n > n_0$, we can write

$$s_n - x = \frac{1}{n} \sum_{k=1}^{n} (x_k - x) = \frac{1}{n} \sum_{k=1}^{n_0 - 1} (x_k - x) + \frac{1}{n} \sum_{k=n_0}^{n} (x_k - x).$$

For the first sum,

$$\left| \frac{1}{n} \sum_{k=1}^{n_0 - 1} (x_k - x) \right| \leqslant \frac{1}{n} \sum_{k=1}^{n_0 - 1} |x_k - x| = \frac{C}{n},$$

where $C := \sum_{k=1}^{n_0 - 1} |x_k - x|$. Note that C does not depend on n.

For the second sum,

$$\left| \frac{1}{n} \sum_{k=n_0}^{n} (x_k - x) \right| \leqslant \frac{1}{n} \sum_{k=n_0}^{n} |x_k - x| \leqslant \frac{n - n_0 + 1}{n} \cdot \frac{\varepsilon}{2} \leqslant \frac{\varepsilon}{2}.$$

Hence,

$$|s_n - x| \leqslant \frac{C}{n} + \frac{\varepsilon}{2} \leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{2} < \varepsilon \quad \text{as } n \geqslant \max\left\{ n_0, \frac{4C}{\varepsilon} \right\}.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $s_n \to x$. $\square$

Consider the metric on $\mathbb{R}^n$ given by

$$d(x, y) := \max_{1 \leqslant i \leqslant n} |x_i - y_i|,$$

and let

$$\|x\| := \sqrt{x_1^2 + \cdots + x_n^2}$$

denote the Euclidean norm on $\mathbb{R}^n$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are two points in $\mathbb{R}^n$.

Write $B_d(a; r)$ for an open ball in the metric space $(\mathbb{R}^n, d)$, i.e.,

$$B_d(a; r) := \{x \in \mathbb{R}^n : d(a, x) < r\},$$

and write $B(a; r)$ for an open ball in $\mathbb{R}^n$ with the Euclidean norm, i.e.,

$$B(a; r) := \{x \in \mathbb{R}^n : \|x - a\| < r\}.$$

Theorem 48. *Let $x = (x_1, \ldots, x_n)$ and and $y = (y_1, \ldots, y_n)$ be two points in $\mathbb{R}^n$. Then*

$$d(x, y) \leqslant \|x - y\| \leqslant \sqrt{n}\, d(x, y).$$

Proof. There exists $k \in \{1, ..., n\}$ such that $d(x, y) = |x_k - y_k|$. Then

$$\|x - y\|^2 = \sum_{i=1}^{n} |x_i - y_i|^2 \geqslant |x_k - y_k|^2 = d(x, y)^2.$$

Furthermore,

$$\|x - y\|^2 = \sum_{i=1}^{n} |x_i - y_i|^2 \leqslant \sum_{i=1}^{n} d(x, y)^2 = n\, d(x, y)^2.$$

Taking square roots gives the desired inequalities. □

Theorem 49. *Let $a \in \mathbb{R}^n$ and $r > 0$. Then*

$$B_d(a; r) \subset B(a; \sqrt{n}\, r) \quad \textit{and} \quad B(a; r) \subset B_d(a; r).$$

Proof. If $x \in B_d(a; r)$, then $d(a, x) < r$. By Theorem 48,

$$\|x - a\| \leqslant \sqrt{n}\, d(a, x) < \sqrt{n}\, r,$$

so $x \in B(a; \sqrt{n}r)$.

If $x \in B(a; r)$, then $\|x - a\| < r$. By Theorem 48,

$$d(a, x) \leqslant \|x - a\| < r,$$

so $x \in B_d(a; r)$. □

Theorem 50. *Let $S \subset \mathbb{R}^n$. Then S is open in $\mathbb{R}^n$ with respect to the Euclidean norm if and only if S is open in the metric space $(\mathbb{R}^n, d)$.*

Proof. Suppose S is open in the Euclidean norm. For any $x \in S$, there exists $r > 0$ such that $B(x; r) \subset S$. By Theorem 49, $B_d(x; r) \subset B(x; r) \subset S$. Hence S is open in d.

Conversely, suppose S is open in d. For $x \in S$, there exists $r > 0$ such that $B_d(x; r) \subset S$. By Theorem 49, $B(x; r) \subset B_d(x; r) \subset S$. Hence S is open in the Euclidean norm. □

Homework 6

Theorem 51. *Let S be a non-empty closed subset of $\mathbb{R}$, and let $f\colon S \to \mathbb{R}$ be continuous. Define*

$$A := \{x \in S : f(x) = 0\}.$$

Then A is a closed subset of $\mathbb{R}$.

Proof. Consider the complement

$$\mathbb{R} \backslash A = (\mathbb{R} \backslash S) \cup \{x \in S : f(x) \neq 0\}.$$

Since S is closed, $\mathbb{R} \backslash S$ is open. Let

$$B := \{x \in S : f(x) \neq 0\}.$$

Take any $x \in B$. Since f is continuous at x and $f(x) \neq 0$, there exists $\varepsilon > 0$ such that

$$|f(y) - f(x)| < |f(x)| \quad \text{for all } y \in S \text{ with } |y - x| < \varepsilon.$$

Then

$$|f(y)| \geqslant |f(x)| - |f(y) - f(x)| > 0,$$

so $y \in B$. Therefore, B is open in $\mathbb{R}$.

Hence,

$$\mathbb{R}\backslash A = (\mathbb{R}\backslash S) \cup B$$

is a union of open sets, and thus open. Therefore, A is closed in $\mathbb{R}$. $\square$

Theorem 52. *Let $f\colon [a,b] \to \mathbb{R}$ be continuous and suppose $x_1, x_2 \in [a,b]$ with $x_1 < x_2$ are local maxima of f. Then there exists $c \in (x_1, x_2)$ such that $f(c)$ is a local minimum.*

Proof. Consider the interval $[x_1, x_2]$. By the Extreme Value Theorem, f attains a minimum on $[x_1, x_2]$, say

$$f(c) = \inf_{x \in [x_1, x_2]} f(x)$$

for some $c \in [x_1, x_2]$.

Since x_1 and x_2 are local maxima, this minimum cannot occur at the endpoints x_1 or x_2. Hence $c \in (x_1, x_2)$.

By definition of the minimum on $[x_1, x_2]$, there exists $\delta > 0$ such that

$$f(c) \leqslant f(x) \quad \text{for all } x \in (c - \delta, c + \delta) \subset (x_1, x_2),$$

so f has a local minimum at c. $\square$

Theorem 53. *There is a continuous function from $(0,1)$ onto $(0,1]$.*

Proof. Consider $f\colon (0,1) \to (0,1]$ defined by

$$f(x) := \begin{cases} 2x, & 0 < x \leqslant \frac{1}{2}, \\ 2(1-x), & \frac{1}{2} < x < 1. \end{cases}$$

Then f is continuous and $f((0,1)) = (0,1]$. $\square$

Theorem 54. *There is no continuous function from $(0,1)$ onto $(0,1) \cup (1,2)$.*

Proof. The domain $(0,1)$ is connected, but the range is disconnected. The continuous image of a connected set must be connected. $\square$

Theorem 55. *There is no continuous function from $\mathbb{R}$ onto $\mathbb{Q}$.*

Proof. See Theorem 33. ☐

Theorem 56. *There is no continuous function from* $[0,1] \times [0,1]$ *onto* $\mathbb{R}^2$.

Proof. The domain $[0,1]^2$ is compact, and the continuous image of a compact set is compact, but $\mathbb{R}^2$ is not compact. ☐

Theorem 57. *There is a continuous function from* $(0,1) \times (0,1)$ *onto* $\mathbb{R}^2$.

Proof. Define

$$f\colon (0,1) \to \mathbb{R}, \quad f(x) := \tan\big(\pi(x - 1/2)\big).$$

- f is continuous on $(0,1)$ because $\tan$ is continuous on $(-\pi/2, \pi/2)$.
- $\lim_{x \to 0^+} f(x) = -\infty$, $\lim_{x \to 1^-} f(x) = +\infty$.
- Therefore, $f((0,1)) = \mathbb{R}$, i.e., f is surjective.

Similarly, for a continuous surjection $g\colon (0,1)^2 \to \mathbb{R}^2$, define

$$g(x,y) := \big(\tan(\pi(x - 1/2)), \tan(\pi(y - 1/2))\big).$$

Then g is continuous and $g((0,1)^2) = \mathbb{R}^2$. ☐

Theorem 58. *Let* $f\colon (S, d_S) \to (T, d_T)$ *be a function between metric spaces. Then*

$$f \text{ is continuous on } S \iff f(\overline{A}) \subseteq \overline{f(A)} \text{ for all } A \subseteq S.$$

Proof. ($\Rightarrow$) Suppose f is continuous and let $x \in \overline{A}$. Then there exists a sequence $(x_n) \subset A$ with $x_n \to x$. By continuity, $f(x_n) \to f(x)$. Since each $f(x_n) \in f(A)$ and $\overline{f(A)}$ is closed, it follows that $f(x) \in \overline{f(A)}$. Hence $f(\overline{A}) \subseteq \overline{f(A)}$.

($\Leftarrow$) Suppose $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subseteq S$. Assume, for contradiction, that f is not continuous at some $x_0 \in S$. Then there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$ there exists $x \in S$ with $d_S(x, x_0) < \delta$ but $d_T(f(x), f(x_0)) \geqslant \varepsilon_0$.

Construct a sequence $(x_n) \subset S$ such that $d_S(x_n, x_0) < 1/n$ and $d_T(f(x_n), f(x_0)) \geqslant \varepsilon_0$. Let $A = \{x_n : n \geqslant 1\}$. Then $x_0 \in \overline{A}$, so $f(x_0) \in f(\overline{A}) \subset \overline{f(A)}$.

By definition of closure, there exists a subsequence $(f(x_{n_k})) \subset f(A)$ such that $f(x_{n_k}) \to f(x_0)$. This is impossible, because by construction $d_T(f(x_n), f(x_0)) \geqslant \varepsilon_0$ for all n, so no subsequence can converge to $f(x_0)$.

This contradiction shows that f must be continuous at x_0. Since x_0 was arbitrary, f is continuous on S. $\square$

Alternative Proof. ($\Rightarrow$) Suppose f is continuous. Let $y \in f(\overline{A})$, so $y = f(x)$ with $x \in \overline{A}$. For any open neighborhood V of y in T, $f^{-1}(V)$ is open in S and contains x. Since $x \in \overline{A}$, we have $f^{-1}(V) \cap A \neq \varnothing$, i.e., $V \cap f(A) \neq \varnothing$. Hence $y \in \overline{f(A)}$. Therefore $f(\overline{A}) \subseteq \overline{f(A)}$.
($\Leftarrow$) Suppose $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subset S$. Let $U \subset T$ be open. Set $A = S \backslash f^{-1}(U)$. Then $f(A) \subset T \backslash U$, which is closed, so $\overline{f(A)} \subset T \backslash U$. By assumption, $f(\overline{A}) \subseteq \overline{f(A)} \subset T \backslash U$, hence $\overline{A} \subset S \backslash f^{-1}(U)$, so $S \backslash f^{-1}(U)$ is closed. Thus $f^{-1}(U)$ is open. Since U was arbitrary, f is continuous. $\square$

Theorem 59. *Let (S, d) be a metric space. Then S is connected if and only if the only subsets of S which are both open and closed (clopen) are $\varnothing$ and S.*

Proof. ($\Rightarrow$) Suppose S is connected. Assume for contradiction that there exists $A \subset S$ with $A \neq \varnothing$, $A \neq S$, and A both open and closed. Then $S \backslash A$ is also nonempty and open. Thus $S = A \cup (S \backslash A)$ is a union of two nonempty disjoint open sets, which is a separation of S. This contradicts the connectedness of S. Hence, the only clopen sets are $\varnothing$ and S.

($\Leftarrow$) Suppose the only clopen subsets of S are $\varnothing$ and S. Assume for contradiction that S is not connected. Then there exists a separation $S = U \cup V$ with U, V nonempty, disjoint, and open. Then U is open and $S \backslash U = V$ is also open, so U is clopen. This is a nonempty proper clopen subset, contradicting the assumption. Hence S must be connected. $\square$

Theorem 60. *Let S be a connected subset of a metric space and let T satisfy $S \subseteq T \subseteq \overline{S}$. Then T is connected. In particular, the closure $\overline{S}$ of a connected set S is connected.*

Theorem 61. *Let S be a connected subset of a metric space (X, d), and let T satisfy*

$$S \subset T \subset \overline{S}.$$

Then T is connected. In particular, the closure $\overline{S}$ of a connected set is connected.

Proof. Suppose, for contradiction, that T is not connected. Then there exists a separation $T = U \cup V$ where U and V are nonempty, disjoint, and open in the subspace topology of T. Define

$$U_S := U \cap S, \quad V_S := V \cap S.$$

Then U_S and V_S are open in the subspace topology of S, disjoint, and

$$U_S \cup V_S = (U \cup V) \cap S = T \cap S = S.$$

We need to show that U_S and V_S are nonempty. Suppose, for contradiction, that $U_S = \varnothing$. Then $U \subset T \backslash S \subset \overline{S} \backslash S$. But U is open in T, so there exists $u \in U$ and $\varepsilon > 0$ such that $B_\varepsilon(u) \cap T \subset U$. Since $u \in T \subset \overline{S}$, any neighborhood of u intersects S, so $B_\varepsilon(u) \cap T \cap S \neq \varnothing$. This contradicts $U_S = \varnothing$. Similarly, $V_S \neq \varnothing$.

Thus U_S and V_S are nonempty, disjoint, open in S, and cover S. This is a separation of S, contradicting its connectedness. Therefore, T must be connected.

In particular, taking $T = \overline{S}$, we conclude that the closure of a connected set is connected. □

Theorem 62. *Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Then f is not uniformly continuous on $\mathbb{R}$.*

Proof. See Theorem 40. □

Theorem 63. *Let $f : (S, d_S) \to (T, d_T)$ be uniformly continuous on S. If $\{x_n\} \subset S$ is a Cauchy sequence, then $\{f(x_n)\} \subset T$ is also a Cauchy sequence.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in S. We need to show that $\{f(x_n)\}$ is a Cauchy sequence in T. Let $\varepsilon > 0$ be given. By uniform continuity of f, there exists $\delta > 0$ such that

$$d_S(x, y) < \delta \implies d_T(f(x), f(y)) < \varepsilon \quad \text{for all } x, y \in S.$$

Since $\{x_n\}$ be a Cauchy sequence in S, there exists $N \in \mathbb{N}$ such that for all $m, n \geqslant N$,

$$d_S(x_m, x_n) < \delta.$$

Then, for all $m, n \geqslant N$,

$$d_T(f(x_m), f(x_n)) < \varepsilon.$$

Hence $\{f(x_n)\}$ is a Cauchy sequence in T. $\square$

Theorem 64. *The connected subsets of $\mathbb{R}$ are exactly the empty set, singletons, and intervals (open, closed, half-open, or infinite).*

Proof. The empty set $\varnothing$ and singletons $\{x_0\}$ are trivially connected.

Let $I \subset \mathbb{R}$ be an interval. Suppose, for contradiction, that I is not connected. Then there exists a separation $I = U \cup V$, where U and V are nonempty, disjoint, and open in the subspace topology of I. Pick $u \in U$ and $v \in V$ with $u < v$, and define

$$S := \{x \in [u, v] \cap I : [u, x] \subset U\}.$$

Then S is nonempty since $u \in S$. Let $s = \sup S$. If $s \in U$, then by openness of U in I, there exists $\varepsilon > 0$ such that $[s, s + \varepsilon) \cap I \subset U$, contradicting the definition of s as a supremum. If $s \in V$, then $s \in [u, v] \cap I$ but $s \notin U$, also contradicting the definition of s. In both cases we get a contradiction. Therefore, I cannot be separated, and hence I is connected.

Finally, let $S \subset \mathbb{R}$ be any connected subset. If $|S| \leqslant 1$, then S is either empty or a singleton. Suppose $|S| \geqslant 2$ and pick $x, y \in S$ with $x < y$. If there exists $z \in (x, y)$ with $z \notin S$, then

$$U := S \cap (-\infty, z), \quad V := S \cap (z, \infty)$$

are nonempty, disjoint, open subsets of S, and $S = U \cup V$, which is a separation of S. This contradicts the connectedness of S. Therefore, S contains all points between any two of its points, and hence S is an interval.

Combining all cases, the connected subsets of $\mathbb{R}$ are exactly the empty set, singletons, and intervals. $\square$

Homework 7

Theorem 65. *Let $f\colon \mathbb{R} \to \mathbb{R}$, and suppose that*

$$|f(x) - f(y)| \leqslant (x-y)^2 \quad \text{for all } x, y \in \mathbb{R}.$$

Then f is constant.

Proof. Fix $a, b \in \mathbb{R}$, and for an integer $n \geqslant 1$ partition the interval from a to b into n equal sub-intervals:

$$x_k = a + k\frac{b-a}{n}, \qquad k = 0, 1, \ldots, n.$$

By the triangle inequality and the given hypothesis, we have

$$\begin{aligned}
|f(b) - f(a)| &= \left|\sum_{k=0}^{n-1} (f(x_{k+1}) - f(x_k))\right| \\
&\leqslant \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \\
&\leqslant \sum_{k=0}^{n-1} (x_{k+1} - x_k)^2 \\
&= n\left(\frac{b-a}{n}\right)^2 \\
&= \frac{(b-a)^2}{n}.
\end{aligned}$$

Since this holds for every n, letting $n \to \infty$ gives

$$|f(b) - f(a)| \leqslant 0 \qquad \Longrightarrow \qquad f(b) = f(a).$$

Thus f is constant on $\mathbb{R}$. □

Lemma 66. Let $m \in \mathbb{N} \cup \{0\}$. Then $\lim_{x\to 0} |x|^{-m} e^{-1/x^2} = 0$.

Proof. For $t \geqslant 0$ the exponential series gives

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \geqslant \frac{t^{k+1}}{(k+1)!} \qquad (k \in \mathbb{N} \cup \{0\}).$$

Hence for $t > 0$

$$\frac{t^k}{e^t} \leqslant \frac{(k+1)!}{t} \xrightarrow[t \to \infty]{} 0,$$

so $\lim\limits_{t \to \infty} \dfrac{t^k}{e^t} = 0$.

Now let $m \geqslant 0$ be an integer and put $t = 1/x^2$ for $x \neq 0$. Then for $t \geqslant 1$, we have

$$\frac{e^{-1/x^2}}{|x|^m} = t^{m/2} e^{-t} \leqslant t^{\lceil m/2 \rceil} e^{-t} \xrightarrow[t \to \infty]{} 0,$$

which shows e^{-1/x^2} tends to 0 faster than any power of $|x|$ as $x \to 0$. $\square$

Theorem 67. *Define $f \colon \mathbb{R} \to \mathbb{R}$ by*

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then

(a) f is continuous for all $x \in \mathbb{R}$.

(b) For every $n \geqslant 1$, the derivative $f^{(n)}$ exists and is continuous on $\mathbb{R}$, and $f^{(n)}(0) = 0$.

Proof of (a). If $x \neq 0$, then f is the composition of the continuous functions $\mathbb{R}\backslash\{0\} \ni x \mapsto -1/x^2 \in \mathbb{R}\backslash\{0\}$ and $\mathbb{R}\backslash\{0\} \ni t \mapsto e^t \in \mathbb{R}\backslash\{0\}$, so f is continuous at every nonzero point. It remains to check continuity at 0. By Lemma 66, $\lim_{x \to 0} e^{-1/x^2} = 0$. Hence f is continuous at 0. Combining this with continuity away from 0 gives continuity on $\mathbb{R}$. $\square$

Lemma 68. Let $f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$ be a polynomial of degree m. Then

$$|f(x)| \leqslant |x|^m \left(|a_m| + |a_{m-1}| + \cdots + |a_0|\right)$$

for $|x| \geqslant 1$.

Proof. Let x be a real number such that $|x| \geqslant 1$. Then

$$\begin{aligned}
|f(x)| &= \left|a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0\right| \\
&= |x^m| \left|a_m + a_{m-1}\frac{1}{x} + \cdots + a_1 \frac{1}{x^{m-1}} + a_0 \frac{1}{x^m}\right| \\
&\leqslant |x|^m \left(|a_m| + |a_{m-1}|\frac{1}{|x|} + \cdots + |a_1|\frac{1}{|x|^{m-1}} + |a_0|\frac{1}{|x|^m}\right) \\
&\leqslant |x|^m \left(|a_m| + |a_{m-1}| + \cdots + |a_0|\right).
\end{aligned}$$

□

Proof of (b). We first prove by induction that for each $n \geqslant 1$ there exists a polynomial P_n (with real coefficients) such that for every $x \neq 0$

$$f^{(n)}(x) = P_n\big(1/x\big)\, e^{-1/x^2}. \tag{1}$$

For $n = 0$ take $P_0 \equiv 1$. Suppose (1) holds for some n. Differentiate (for $x \neq 0$):

$$f^{(n+1)}(x) = \big(P_n(1/x)\big)'\, e^{-1/x^2} + P_n(1/x)\, \big(e^{-1/x^2}\big)'.$$

Since $\big(e^{-1/x^2}\big)' = \frac{2}{x^3} e^{-1/x^2}$ and $\big(P_n(1/x)\big)'$ is again a rational function which can be written as a polynomial in $1/x$ (times a power of x^{-1}) , we see that $f^{(n+1)}(x)$ can be written in the form

$$f^{(n+1)}(x) = P_{n+1}(1/x)\, e^{-1/x^2}$$

for some polynomial P_{n+1}. This completes the induction.

Now fix $n \geqslant 0$. From (1) we have for $x \neq 0$

$$\left|f^{(n)}(x)\right| = \left|P_n(1/x)\right| e^{-1/x^2}.$$

The polynomial $\left|P_n(1/x)\right|$ grows at most like a fixed power of $|x|^{-1}$; hence, by Lemma 68, there exist constants $C > 0$ and $m \geqslant 0$ such that

$$\left|f^{(n)}(x)\right| \leqslant C\,|x|^{-m} e^{-1/x^2} \quad \text{for } |x| \leqslant 1.$$

As in part (a), with $t = 1/x^2$ we get

$$|x|^{-m} e^{-1/x^2} = t^{m/2} e^{-t} \to 0 \quad \text{as } x \to 0.$$

Thus $\lim_{x\to 0} f^{(n)}(x) = 0$. Define $f^{(n)}(0) := 0$. The preceding limit shows that this value agrees with the limit of $f^{(n)}(x)$ as $x \to 0$, so $f^{(n)}$ is continuous at 0. Together with smoothness on $\mathbb{R}\backslash\{0\}$, this proves $f^{(n)}$ exists and is continuous on all of $\mathbb{R}$, and $f^{(n)}(0) = 0$.

Finally, to see explicitly that the derivatives at 0 computed via the difference quotient agree with 0, one can check by induction that

$$\frac{d^n f}{dx^n}(0) = \lim_{x\to 0} \frac{f^{(n-1)}(x) - f^{(n-1)}(0)}{x - 0} = \lim_{x\to 0} \frac{f^{(n-1)}(x)}{x} = 0,$$

using the fact already established that $f^{(n-1)}(x)$ tends to 0 faster than any power of x. This gives another direct verification that all derivatives at 0 are 0. □

Theorem 69. *Let*

$$f_n(x) = \begin{cases} x^n \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then f_1 is continuous but not differentiable at 0. Also, f_2 is differentiable but not of class C^1. In general, $f_n \in C^k$ at 0 if and only if $n \geqslant k+1$.

Proof. For $n = 1$, we have

$$f_1(x) = \begin{cases} x\sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then

$$\lim_{x\to 0} f_1(x) = \lim_{x\to 0} x\sin(1/x).$$

Since $|\sin(1/x)| \leqslant 1$, we have $|x\sin(1/x)| \leqslant |x| \to 0$ as $x \to 0$. Hence f_1 is continuous at 0. Now,

$$\lim_{x\to 0} \frac{f_1(x) - f_1(0)}{x - 0} = \lim_{x\to 0} \frac{x\sin(1/x)}{x} = \lim_{x\to 0} \sin(1/x),$$

which does not exist due to oscillation. Therefore f_1 is not differentiable at 0.

Next, for $n = 2$, we have

$$f_2(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then

$$f_2'(0) = \lim_{x\to 0} \frac{x^2\sin(1/x) - 0}{x} = \lim_{x\to 0} x\sin(1/x) = 0.$$

Hence f_2 is differentiable at 0. Moreover, for $x \neq 0$, we have

$$f_2'(x) = \frac{d}{dx}\left(x^2\sin(1/x)\right) = 2x\sin(1/x) - \cos(1/x).$$

Now,

$$\lim_{x\to 0} f_2'(x) = \lim_{x\to 0} \left(2x\sin(1/x) - \cos(1/x)\right)$$

does not exist because $\cos(1/x)$ oscillates. Hence, f_2' is not continuous at 0, so $f_2 \notin C^1$.

Finally, we assume $n \geqslant 3$. For $x \neq 0$,

$$f_n'(x) = nx^{n-1}\sin(1/x) - x^{n-2}\cos(1/x).$$

To have $f_n'(0)$ exist, the term $x^{n-2}\cos(1/x)$ must vanish as $x \to 0$. This requires $n - 2 > 0 \implies n \geqslant 3$.

Hence the general pattern:

- f_n is continuous at 0 for all $n \geqslant 1$.
- f_n is differentiable at 0 if $n \geqslant 2$.
- $f_n \in C^1$ (i.e., derivative continuous at 0) if $n \geqslant 3$.

Now, we show that $f_n \in C^k$ at 0 if and only if $n \geqslant k+1$. For $x \neq 0$,

$$f_n'(x) = nx^{n-1}\sin(1/x) - x^{n-2}\cos(1/x).$$

The first term $nx^{n-1}\sin(1/x)$ vanishes as $x \to 0$ if $n-1 > 0$. The second term $-x^{n-2}\cos(1/x)$ vanishes as $x \to 0$ if $n-2 > 0$. Hence the term with the smallest power of x dominates the behavior near 0.

After taking k derivatives, the most singular term behaves like

$$x^{n-k} \cdot (\sin(1/x) \text{ or } \cos(1/x)).$$

This term determines whether $f_n^{(k)}(x)$ can extend continuously to 0.

For $f_n^{(k)}$ to be continuous at 0, we require

$$\lim_{x\to 0} x^{n-k}(\sin(1/x) \text{ or } \cos(1/x)) = 0,$$

which holds if and only if

$$n-k > 0 \quad \Longrightarrow \quad n \geqslant k+1.$$

Then we define $f_n^{(k)}(0) = 0$ to make it continuous. □

Theorem 70. *Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on $[a,b]$.*

Proof. Take any $x, y \in [a,b]$ with $x < y$. By the Mean Value Theorem there exists $c \in (x,y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}.$$

Since $f'(c) = 0$ by hypothesis, it follows that $f(y) - f(x) = 0$, so $f(y) = f(x)$. Because x, y were arbitrary points of $[a,b]$, the function f is constant on $[a,b]$. □

Theorem 71. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous, $f'(x)$ exists for all $x \neq 0$, and*

$$\lim_{x\to 0} f'(x) = 3.$$

Then $f'(0)$ exists and $f'(0) = 3$.

Proof. For $x \neq 0$, apply the Mean Value Theorem on $[0, x]$ (if $x > 0$) or $[x, 0]$ (if $x < 0$). There exists c_x between 0 and x such that

$$f(x) - f(0) = f'(c_x)\, x.$$

Dividing by x gives

$$\frac{f(x) - f(0)}{x} = f'(c_x).$$

As $x \to 0$, the point c_x lies between 0 and x, so $c_x \to 0$. By hypothesis,

$$\lim_{x \to 0} f'(x) = 3.$$

Hence

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 3,$$

so $f'(0)$ exists and $f'(0) = 3$. □

Theorem 72 (Banach Fixed-Point)**.** *Let (X, d) be a complete metric space, and let $T : X \to X$ satisfy*

$$d(T(x), T(y)) \leqslant \alpha\, d(x, y) \quad \textit{for all } x, y \in X,$$

for some $0 \leqslant \alpha < 1$. Then T has a unique fixed point $x^* \in X$. Moreover, for any $x_0 \in X$, the sequence defined by $x_{n+1} = T(x_n)$ converges to x^*.

Proof. Let $x_0 \in X$ and define $x_{n+1} = T(x_n)$ for $n \geqslant 0$. For $n \geqslant 1$,

$$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) \leqslant \alpha\, d(x_n, x_{n-1}).$$

By induction,

$$d(x_{n+1}, x_n) \leqslant \alpha^n d(x_1, x_0).$$

For $m > n$, by the triangle inequality,

$$\begin{aligned} d(x_m, x_n) &\leqslant \sum_{k=n}^{m-1} d(x_{k+1}, x_k) \\ &\leqslant d(x_1, x_0) \sum_{k=n}^{m-1} \alpha^k \\ &\leqslant \frac{\alpha^n}{1-\alpha} d(x_1, x_0) \to 0 \quad (n \to \infty). \end{aligned}$$

Hence (x_n) is Cauchy.
Since X is complete, there exists $x^* \in X$ with $x_n \to x^*$. By continuity of T,

$$T(x^*) = T\Big(\lim_{n\to\infty} x_n\Big) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x^*.$$

If $y^* \in X$ is another fixed point, then

$$d(x^*, y^*) = d(T(x^*), T(y^*)) \leqslant \alpha\, d(x^*, y^*) \implies d(x^*, y^*) = 0.$$

Thus $x^* = y^*$. □

Theorem 73. *Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), with*

$$a \leqslant f(x) \leqslant b \quad \text{for all } x \in [a,b],$$

and

$$|f'(x)| \leqslant \alpha < 1 \quad \text{for all } x \in (a,b).$$

Then f has a unique fixed point in $[a,b]$.

Proof. We first show that f is a contraction. For any $x, y \in [a,b]$, $x \neq y$, by the Mean Value Theorem there exists c between x and y such that

$$f(x) - f(y) = f'(c)(x-y),$$

so

$$|f(x) - f(y)| = |f'(c)||x-y| \leqslant \alpha|x-y|.$$

Hence f is a contraction with constant $\alpha < 1$.

Since $[a, b]$ is a closed interval in $\mathbb{R}$ (a complete metric space), the Banach fixed-point theorem guarantees that f has a unique fixed point $x^* \in [a, b]$. $\square$

Homework 8

Theorem 74. *Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} x + 2x^2 \sin \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

(a) Then $f'(0) = 1$ and $f'(x) = 1 - 2\cos(1/x) + 4x \sin(1/x)$ for $x \neq 0$.

(b) There exists a sequence of points $\{x_n\}$ with $x_n \neq 0$, $x_n \to 0$, and $f'(x_n) < 0$.

Proof. **(a)** For $x \neq 0$, we have

$$f(x) = x + 2x^2 \sin \frac{1}{x}, \qquad f(0) = 0.$$

Then

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{h + 2h^2 \sin(1/h)}{h} \\ &= \lim_{h \to 0} \big(1 + 2h \sin(1/h)\big) \\ &= 1. \end{aligned}$$

Hence $f'(0) = 1 > 0$.
For $x \neq 0$, differentiating directly gives

$$\begin{aligned} \frac{d}{dx}\big(2x^2 \sin(1/x)\big) &= 4x \sin(1/x) + 2x^2 \cos(1/x) \left(-\frac{1}{x^2}\right) \\ &= 4x \sin(1/x) - 2\cos(1/x). \end{aligned}$$

Therefore,

$$f'(x) = 1 - 2\cos(1/x) + 4x\sin(1/x), \quad x \neq 0.$$

(b) We seek a sequence $\{x_n\}$ with $x_n \to 0$ and $f'(x_n) < 0$. Choose x_n such that $\cos(1/x_n) = 1$ and $\sin(1/x_n) = 0$, for example,

$$x_n = \frac{1}{2\pi n}, \qquad n = 1, 2, 3, \ldots$$

Then $1/x_n = 2\pi n$, so $\cos(1/x_n) = 1$ and $\sin(1/x_n) = 0$. Substituting into the formula for $f'(x)$,

$$f'(x_n) = 1 - 2 \cdot 1 + 4x_n \cdot 0 = -1 < 0.$$

Hence $x_n \neq 0$, $x_n \to 0$, and $f'(x_n) = -1 < 0$.

Although $f'(0) = 1 > 0$, there are points arbitrarily close to 0 where $f'(x) < 0$. Thus, there is no open interval around 0 on which f is increasing. □

Theorem 75. *Suppose $f\colon (a,b) \to \mathbb{R}$ is r-th order differentiable at x. If $P(h)$ and $Q(h)$ are two polynomials of degree $\leqslant r$ such that*

$$\lim_{h \to 0} \frac{f(x+h) - P(h)}{h^r} = 0 = \lim_{h \to 0} \frac{f(x+h) - Q(h)}{h^r},$$

then $Q = P$.

Proof. Set $S(h) := P(h) - Q(h)$. Then

$$\lim_{h \to 0} \frac{S(h)}{h^r} = 0.$$

Suppose S is not the zero polynomial. Then we can write

$$\frac{S(h)}{h^r} = h^{m-r}\left(d_m + d_{m+1}h + \cdots + d_r h^{r-m}\right)$$

for some $m \leqslant r$ and some $d_m \neq 0$. Let $\varphi(h) := d_m + d_{m+1}h + \cdots + d_r h^{r-m}$. Then $\lim_{h \to 0} \varphi(h) = d_m$. Therefore, if $m < r$, then $|h^{m-r}| \to \infty$ as $h \to 0$, contradicting that the limit above equals 0. On the other hand, if $m = r$, then $\dfrac{S(h)}{h^r} \to d_m$ as $h \to 0$, so the limit is $d_m \neq 0$, again a contradiction. Hence no such m exists and all $d_k = 0$; therefore $S \equiv 0$ and $P(h) = Q(h)$. □

Theorem 76 (Peano form of the Taylor approximation)**.** *Let $f\colon (a,b) \to \mathbb{R}$ be r-times differentiable at x. Define the r-th order Taylor polynomial of f at x by*

$$P_r(h) := f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \cdots + \frac{f^{(r)}(x)}{r!}h^r.$$

Then the remainder

$$R(h) := f(x+h) - P_r(h)$$

satisfies

$$\frac{R(h)}{h^r} \longrightarrow 0 \quad \text{as } h \to 0,$$

i.e., $R(h)$ is r-th order flat at 0.

Proof. By the definition of the Taylor polynomial, $P_r(h)$ matches the first r derivatives of f at x. Therefore, for the remainder $R(h) = f(x+h) - P_r(h)$,

$$R(0) = R'(0) = \cdots = R^{(r)}(0) = 0.$$

By the Mean Value Theorem, there exists $\theta_1 \in (0,h)$ such that

$$R(h) - R(0) = R'(\theta_1)h \implies R(h) = R'(\theta_1)h.$$

Apply the Mean Value Theorem to $R'(\theta_1) - R'(0)$: there exists $\theta_2 \in (0,\theta_1)$ such that

$$R'(\theta_1) - R'(0) = R''(\theta_2)\theta_1 \implies R'(\theta_1) = R''(\theta_2)\theta_1.$$

Substituting back gives

$$R(h) = R''(\theta_2)\theta_1 h.$$

Repeating this process $(r-1)$ times, we obtain

$$R(h) = R^{(r-1)}(\theta_{r-1})\theta_{r-2}\cdots\theta_1 h,$$

where

$$0 < \theta_{r-1} < \cdots < \theta_1 < h.$$

Thus, when $0 < h < 1$,

$$\left|\frac{R(h)}{h^r}\right| = \left|\frac{R^{(r-1)}(\theta_{r-1})\theta_{r-2}\cdots\theta_1 h}{h^r}\right| \leqslant \left|\frac{R^{(r-1)}(\theta_{r-1}) - 0}{\theta_{r-1}}\right| \to 0.$$

as $h \to 0+$. Hence,

$$\frac{R(h)}{h^r} \longrightarrow 0 \quad \text{as } h \to 0+.$$

If $-1 < h < 0$, the same is true with

$$h < \theta_1 < \theta_2 < \cdots < \theta_{r-1} < 0.$$

Therefore, $R(h)$ is r-th order flat at 0. □

Theorem 77. *Suppose f is defined in an open interval containing a, and suppose $f''(a)$ exists. Then*

$$f''(a) = \lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}.$$

Proof. Since $f''(a)$ exists, we can write the Taylor expansions for small h:

$$f(a+h) = f(a) + f'(a)h + \frac{1}{2}f''(a)h^2 + o(h^2),$$

$$f(a-h) = f(a) - f'(a)h + \frac{1}{2}f''(a)h^2 + o(h^2),$$

where $o(h^2)$ denotes a term such that $\frac{o(h^2)}{h^2} \to 0$ as $h \to 0$.
Form the symmetric difference quotient:

$$f(a+h) - 2f(a) + f(a-h) = f''(a)h^2 + o(h^2).$$

Divide both sides by h^2:

$$\frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = f''(a) + \frac{o(h^2)}{h^2}.$$

Taking the limit as $h \to 0$, we get

$$\lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = f''(a). \qquad \square$$

Remark 78. Here is an example where the limit exists but $f''(a)$ does not. Consider

$$f(x) = x|x|, \quad a = 0.$$

The symmetric difference quotient is

$$\frac{f(h) - 2f(0) + f(-h)}{h^2} = \frac{h|h| + (-h)| - h|}{h^2} = \frac{h^2 - h^2}{h^2} = 0.$$

Therefore, the limit exists and equals 0:

$$\lim_{h\to 0} \frac{f(h) - 2f(0) + f(-h)}{h^2} = 0.$$

However, the second derivative $f''(0)$ does not exist, because

$$f''(x) = \begin{cases} 2 & x > 0, \\ -2 & x < 0, \end{cases}$$

so the left and right second derivatives at 0 are different.
Hence this function satisfies the required conditions.

Theorem 79 (Taylor's theorem (degree n with Lagrange remainder))**.** *If g is C^{n+1} on an interval containing 0 and t, then there exists ξ between 0 and t such that*

$$g(t) = g(0) + g'(0)t + \frac{g''(0)}{2!}t^2 + \cdots + \frac{g^{(n)}(0)}{n!}t^n + \frac{g^{(n+1)}(\xi)}{(n+1)!}t^{n+1}.$$

Theorem 80. *Let*

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then
$$\lim_{x\to 0}\frac{\sin x}{x}=1,$$
and the point $x=0$ is a removable discontinuity of f (since $f(0)=0\neq 1$). Re-defining $f(0):=1$ makes f continuous at 0.

Proof. We use Taylor's theorem with the Lagrange form of the remainder for the function $g(t)=\sin t$ about $t=0$.

Since $g(0)=0$, $g'(0)=1$, and $g''(u)=-\sin u$, for each x there exists ξ between 0 and x with
$$\sin x=0+1\cdot x+\frac{-\sin\xi}{2}\,x^2=x-\frac{\sin\xi}{2}\,x^2.$$
For $x\neq 0$ divide both sides by x to obtain
$$\frac{\sin x}{x}=1-\frac{\sin\xi}{2}\,x,$$
where ξ lies between 0 and x.

Since $|\sin\xi|\leqslant 1$ for all real ξ, we have the estimate
$$\left|\frac{\sin x}{x}-1\right|=\left|\frac{\sin\xi}{2}\,x\right|\leqslant\frac{|x|}{2}.$$
As $x\to 0$ the right-hand side $\frac{|x|}{2}\to 0$, therefore
$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$

The two-sided limit $\lim_{x\to 0}\dfrac{\sin x}{x}$ exists and equals 1, while the function value given is $f(0)=0$. Hence the limit and the value differ: the discontinuity at 0 is *removable*. If we redefine
$$\tilde f(x)=\begin{cases}\dfrac{\sin x}{x}, & x\neq 0,\\ 1, & x=0,\end{cases}$$
then $\tilde f$ is continuous at 0. $\square$

Theorem 81. *Let*

$$f(x) = \begin{cases} e^{1/x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then

$$\lim_{x \to 0^+} f(x) = +\infty, \qquad \lim_{x \to 0^-} f(x) = 0,$$

and the discontinuity of f at $x = 0$ is essential (equivalently: an infinite/non-removable discontinuity).

Proof. We shall use Taylor's theorem (Lagrange form of the remainder) for the function $g(t) = e^t$ about $t = 0$, for which $g^{(k)}(u) = e^u$ for all k and u.

(i) We claim that $\lim\limits_{x \to 0^+} e^{1/x} = +\infty$.

Put $t = \dfrac{1}{x}$. When $x \to 0^+$ we have $t \to +\infty$. Apply Taylor's theorem with $n = 1$ to $g(t) = e^t$ at 0: for each $t > 0$ there exists $\xi \in (0, t)$ such that

$$e^t = g(0) + g'(0)t + \frac{g''(\xi)}{2} t^2 = 1 + t + \frac{e^\xi}{2} t^2.$$

Since $e^\xi > 0$, the remainder term $\dfrac{e^\xi}{2} t^2$ is positive, so for every $t > 0$

$$e^t = 1 + t + \frac{e^\xi}{2} t^2 > 1 + t > t.$$

Now let $M > 0$ be arbitrary. Choose $T > M$. For $t > T$ we have $e^t > t > T > M$. Translating back to x: choose $\delta = \dfrac{1}{T}$. Then if $0 < x < \delta$ we get $t = \dfrac{1}{x} > T$ and hence $e^{1/x} > M$. Since M was arbitrary this proves $\lim_{x \to 0^+} e^{1/x} = +\infty$.

(ii) We claim that $\lim\limits_{x \to 0^-} e^{1/x} = 0$.

For $x \to 0^-$ set $t = \dfrac{1}{x}$; then $t \to -\infty$. Write $t = -s$ with $s \to +\infty$. Then

$$e^{1/x} = e^t = e^{-s} = \frac{1}{e^s}.$$

It suffices to show $e^s \to +\infty$ as $s \to +\infty$. Apply Taylor's theorem with $n = 2$ to $g(s) = e^s$ at 0: for each $s > 0$ there exists $\eta \in (0, s)$ such that

$$e^s = 1 + s + \frac{s^2}{2} e^{\eta}.$$

Since $e^{\eta} \geqslant 1$ for $\eta \geqslant 0$, we have

$$e^s \geqslant 1 + s + \frac{s^2}{2}.$$

The right-hand side tends to $+\infty$ as $s \to +\infty$, hence $e^s \to +\infty$. Therefore

$$e^{1/x} = e^{-s} = \frac{1}{e^s} \longrightarrow 0 \quad \text{as } s \to +\infty,$$

i.e. $\lim_{x\to 0^-} e^{1/x} = 0$.

We have $\lim_{x\to 0^-} f(x) = 0 = f(0)$, while $\lim_{x\to 0^+} f(x) = +\infty$. Thus the two one-sided limits are not both finite and equal (indeed the right-hand limit diverges to $+\infty$). Consequently the two-sided limit $\lim_{x\to 0} f(x)$ does not exist as a finite real number, and the point $x = 0$ is not removable. Because one one-sided limit is infinite, the usual real-analysis terminology classifies this as an *essential* (or *infinite / non-removable*) discontinuity at $x = 0$. □

Theorem 82. *Let f be an increasing function on $[a, b]$, and let $x_1, \ldots, x_n \in (a, b)$ with*

$$a < x_1 < x_2 < \cdots < x_n < b.$$

1. *Then*

$$\sum_{k=1}^{n} \left[f(x_k^+) - f(x_k^-) \right] \leqslant f(b) - f(a).$$

2. *For each $m \in \mathbb{Z}^+$, let*

$$S_m = \{x \in [a, b] : f(x^+) - f(x^-) > 1/m\}.$$

Then S_m is finite.

3. *Thus, the set of discontinuities of f is countable.*

Proof. Since f is increasing, the total change from a to b can be written as the sum of the continuous increases between the points and the jumps at the points:

$$\begin{aligned} f(b) - f(a) = & \left[f(x_1^-) - f(a)\right] + \left[f(x_1^+) - f(x_1^-)\right] \\ & + \left[f(x_2^-) - f(x_1^+)\right] + \left[f(x_2^+) - f(x_2^-)\right] \\ & + \cdots \\ & + \left[f(x_n^-) - f(x_{n-1}^+)\right] + \left[f(x_n^+) - f(x_n^-)\right] \\ & + \left[f(b) - f(x_n^+)\right]. \end{aligned}$$

By considering jumps at x_k, we immediately get:

$$\sum_{k=1}^{n} \left[f(x_k^+) - f(x_k^-)\right] \leqslant f(b) - f(a),$$

as required. This completes the proof of 1.

Suppose, for some $m \in \mathbb{Z}^+$, that S_m has infinitely many points. Let $l \in \mathbb{N}$ be such that $\frac{l}{m} > f(b) - f(a)$, and choose $x_1, ..., x_l$ distinct points from S. Then

$$\sum_{k=1}^{l} [f(x_k^+) - f(x_k^-)] > \#S_m \cdot \frac{l}{m} > f(b) - f(a),$$

which contradicts part 1. Therefore, S_m must be finite. This completes the proof of 2.

Let D be the set of discontinuities of f in $[a, b]$. Each discontinuity corresponds to a jump, so for each $x \in D$, there exists some $m \in \mathbb{Z}^+$ such that the jump at x is greater than $1/m$. Therefore, we can write

$$D = \bigcup_{m=1}^{\infty} S_m,$$

where each S_m is finite by part 2. A countable union of finite sets is countable. Hence, the set of discontinuities D is countable. $\square$

Homework 9

Definition 83. A function $f\colon [a, b] \to \mathbb{R}$ is said to satisfy a *uniform Lipschitz condition of order* $\alpha > 0$ on $[a, b]$ if there exists a constant $M > 0$ such that

$$|f(x) - f(y)| \leqslant M|x - y|^{\alpha}, \quad \forall x, y \in [a, b].$$

Theorem 84. *Let $f\colon [a, b] \to \mathbb{R}$ be a function that satisfy a uniform Lipschitz condition of order $\alpha > 0$ on $[a, b]$.*

1. *If $\alpha > 1$, then f is constant on $[a, b]$.*
2. *If $\alpha = 1$, then f is of bounded variation on $[a, b]$.*

Proof of 1. For $x \neq y$,

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leqslant M|x - y|^{\alpha - 1}.$$

Since $\alpha - 1 > 0$,

$$\lim_{y \to x} \left| \frac{f(x) - f(y)}{x - y} \right| \leqslant \lim_{y \to x} M|x - y|^{\alpha - 1} = 0.$$

Therefore,

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x} = 0 \quad \forall x \in [a, b].$$

Since $f'(x) = 0$ for all $x \in [a, b]$, the Mean Value Theorem implies that f is constant on $[a, b]$. $\square$

Proof of 2. For any partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$,

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leqslant \sum_{i=1}^{n} M|x_i - x_{i-1}| = M \sum_{i=1}^{n} (x_i - x_{i-1}) = M(b - a).$$

Since this bound holds for any partition P, we have

$$V_a^b(f) \leqslant M(b - a) < \infty,$$

so f is of bounded variation on $[a, b]$. $\square$

Theorem 85. *Let*

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then f is Lipschitz continuous on $[0, 1]$ with Lipschitz constant $L = 3$.

Proof. We need to show that there exists a constant $L > 0$ such that for all $x, y \in [0, 1]$,

$$|f(x) - f(y)| \leqslant L|x - y|.$$

First suppose $x, y \neq 0$ By the Mean Value Theorem, there exists c between x and y such that

$$f(x) - f(y) = f'(c)(x - y).$$

Hence,

$$\begin{aligned} |f(x) - f(y)| &= |f'(c)|\,|x - y| \\ &= |2c \sin(1/c) - \cos(1/c)|\,|x - y| \\ &\leqslant (2|c| + |\cos(1/c)|)\,|x - y| \\ &\leqslant 3\,|x - y|. \end{aligned}$$

Now, suppose one of the points is 0. Without loss of generality, let $x = 0$ and $y \neq 0$. Then

$$|f(y) - f(0)| = |y^2 \sin(1/y) - 0| \leqslant y^2 \leqslant |y - 0|.$$

The same estimate holds if $y = 0$ and $x \neq 0$.

Combining both cases, we obtain for all $x, y \in [0, 1]$:

$$|f(x) - f(y)| \leqslant 3|x - y|.$$

Therefore, f is Lipschitz continuous on $[0, 1]$ with Lipschitz constant $L = 3$. □

Theorem 86. *Let*

$$f(x) = \begin{cases} \sqrt{x}\, \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then f is not of bounded variation on $[0, 1]$.

Proof. Consider the sequence

$$x_n = \frac{1}{n\pi + \pi/2}, \quad n = 0, 1, 2, \dots$$

Then

$$f(x_n) = \sqrt{x_n}\,\sin(1/x_n) = (-1)^n\sqrt{x_n}.$$

Let

$$P_N = \{0, x_N, x_{N-1}, \dots, x_1, x_0, 1\}.$$

This is an increasing sequence from left to right (toward 0). The total variation along P_N is

$$\begin{aligned}
V(f, P_N) &= |f(0) - f(x_N)| + |f(x_0) - f(1)| + \sum_{n=1}^{N} |f(x_n) - f(x_{n-1})| \\
&\geqslant \sum_{n=1}^{N} |f(x_n) - f(x_{n-1})| \\
&= \sum_{n=1}^{N} |(-1)^n\sqrt{x_n} - (-1)^{n-1}\sqrt{x_{n-1}}| \\
&= \sum_{n=1}^{N} \left(\sqrt{x_n} + \sqrt{x_{n-1}}\right) \\
&\geqslant \sum_{n=1}^{N} \sqrt{x_n},
\end{aligned}$$

which goes to ∞ as $N \to \infty$

Since there exists a sequence of partitions $\{P_N\}$ with total variation tending to ∞, the function f is not of bounded variation on $[0, 1]$. $\square$

Definition 87. A function $f\colon [a, b] \to \mathbb{R}$ is said to be *absolutely continuous* if: For every $\epsilon > 0$, there exists $\delta > 0$ such that for any finite collection of pairwise disjoint open sub-intervals $(a_k, b_k) \subset [a, b]$, $k = 1, 2, \dots, n$, with

$$\sum_{k=1}^{n} (b_k - a_k) < \delta,$$

we have

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon.$$

Theorem 88. *Let $f\colon [a,b] \to \mathbb{R}$ is an absolutely continuous function. Then f is continuous on $[a,b]$.*

Proof. Fix $\epsilon > 0$. By absolute continuity, there exists $\delta > 0$ such that for any finite collection of disjoint intervals with total length less than δ, the sum of the function differences is less than ϵ. In particular, consider a single interval (x, y) with $|y - x| < \delta$. Then,

$$|f(y) - f(x)| < \epsilon.$$

This is exactly the definition of continuity at every point $x \in [a,b]$. $\square$

Proposition 89. The function

$$f(x) = \begin{cases} x\sin(1/x), & x \neq 0, \\ 0, & x = 0 \end{cases}$$

is continuous on $[0,1]$ but not absolutely continuous.

Proof. Clearly, f is continuous on $[0,1]$.
Define

$$x_n := \frac{1}{n\pi + \pi}, \quad y_n := \frac{1}{n\pi + \pi/2}, \quad n = 1, 2, 3, \ldots$$

The intervals $[x_n, y_n]$ are disjoint because

$$y_n = \frac{1}{n\pi + \pi/2} < \frac{1}{(n-1)\pi + \pi} = x_{n-1}$$

for $n \geqslant 2$. Moreover, we have

$$y_n - x_n = \frac{1}{n\pi + \pi/2} - \frac{1}{n\pi + \pi} = \frac{\pi/2}{(n\pi + \pi/2)(n\pi + \pi)} < \frac{1}{2n^2}.$$

Hence, for large enough N, the total length

$$\sum_{n=N}^{\infty}(y_n - x_n) < \delta$$

for any given $\delta > 0$.
On each interval $[x_n, y_n]$,

$$|f(y_n) - f(x_n)| = |y_n \cdot 1 - 0| = \frac{1}{n\pi + \pi/2}.$$

Thus, for $n \geqslant N$,

$$\sum_{n=N}^{\infty}|f(y_n) - f(x_n)| \geqslant \sum_{n=N}^{\infty}\frac{1}{2n\pi} = \infty.$$

Let $\varepsilon = 1$ and choose any $\delta > 0$. Then, as above, we can select large N such that the sum of interval lengths $\sum_{n=N}^{\infty}(y_n - x_n) < \delta$. However, the total change in f over these intervals is infinite, which exceeds ε. This contradicts the definition of absolute continuity.
Therefore, f is continuous but not absolutely continuous. □

Theorem 90. *Let $f\colon [a,b] \to \mathbb{R}$ is an absolutely continuous function. Then f is a bounded variation on $[a,b]$.*

Proof. Fix $\epsilon = 1$. Since f is absolutely continuous, there exists $\delta > 0$ such that for any finite collection of pairwise disjoint sub-intervals $(x_1, y_1), \ldots, (x_m, y_m)$ of $[a,b]$ with $\sum_{k=1}^{m}(y_k - x_k) < \delta$, we have

$$\sum_{k=1}^{m}|f(y_k) - f(x_k)| < \epsilon = 1.$$

Next, divide $[a,b]$ into sub-intervals of length at most $\delta/2$ by defining the partition

$$P^* = \{a_0 = a < a_1 < \cdots < a_N = b\}, \quad a_i - a_{i-1} \leqslant \frac{\delta}{2}.$$

Then the number of sub-intervals satisfies

$$N \leqslant \frac{2(b-a)}{\delta} + 1.$$

Now, take any partition $P = \{a = x_0 < x_1 < \cdots < x_s = b\}$ of $[a, b]$ and consider its refinement

$$P' = P \cup P^* = \{a = z_0 < z_1 < \cdots < z_m = b\}.$$

For each $i = 1, \ldots, N$, let $a_{i-1} = y_{i,1} < y_{i,2} < \cdots < y_{i,k_i} = a_i$ denote all the points of $P' \cap [a_{i-1}, a_i]$.

By construction, each sub-interval $[a_{i-1}, a_i]$ has length $\leqslant \delta/2 < \delta$. Therefore, applying absolute continuity to the points in $P' \cap [a_{i-1}, a_i]$ gives

$$\sum_{l=1}^{k_i - 1} |f(y_{i,l}) - f(y_{i,l+1})| < 1.$$

Summing over all $i = 1, \ldots, n$, we obtain

$$\begin{aligned}
V(P, f) &= \sum_{j=1}^{s} |f(c_j) - f(c_{j-1})| \\
&\leqslant \sum_{i=1}^{m} |f(z_i) - f(z_{i-1})| && \text{as } P \subseteq P' \\
&= \sum_{i=1}^{N} \sum_{l=1}^{k_i - 1} |f(y_{i,l}) - f(y_{i,l+1})| \\
&\leqslant N.
\end{aligned}$$

Since n is finite and depends only on $b - a$ and δ, we conclude that

$$V_a^b(f) := \sup_P \sum_{j=1}^{|P|} |f(c_j) - f(c_{j-1})| \leqslant N < \infty.$$

Thus, f is of bounded variation on $[a, b]$ $\square$

Remark 91. The Cantor function $c\colon [0,1] \to [0,1]$ is a continuous, non-decreasing function which is not absolutely continuous. In particular, the Cantor function is of bounded variation on $[0,1]$.

Theorem 92. *Let $f\colon [a,b] \to \mathbb{R}$ be integrable and let $c \in \mathbb{R}$. Then cf is integrable and*

$$\int_a^b cf = c\int_a^b f.$$

Proof. Let $\epsilon > 0$. Since f is integrable, there exists a partition P of $[a,b]$ such that

$$U(P,f) - L(P,f) < \begin{cases} \epsilon/|c|, & \text{if } c \neq 0, \\ \epsilon, & \text{if } c = 0. \end{cases}$$

If $c = 0$, then $cf = 0$ is constant and hence integrable, with $\int_a^b 0 = 0$. So suppose $c \neq 0$.

Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$. For each i define

$$M_i = \sup_{[x_{i-1},x_i]} f, \quad m_i = \inf_{[x_{i-1},x_i]} f.$$

Then for cf,

$$\sup_{[x_{i-1},x_i]} cf = \begin{cases} cM_i, & \text{if } c > 0, \\ cm_i, & \text{if } c < 0, \end{cases} \qquad \inf_{[x_{i-1},x_i]} cf = \begin{cases} cm_i, & \text{if } c > 0, \\ cM_i, & \text{if } c < 0. \end{cases}$$

Hence,

$$U(P,cf) - L(P,cf) = |c|\big(U(P,f) - L(P,f)\big) < |c| \cdot \frac{\epsilon}{|c|} = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, cf is integrable.

Finally, for $c > 0$, $L(P,cf) = cL(P,f)$ and $U(P,cf) = cU(P,f)$, while for $c < 0$, $L(P,cf) = cU(P,f)$ and $U(P,cf) = cL(P,f)$. Using $I_f = \int_a^b f = \sup_P L(P,f) = \inf_P U(P,f)$, we obtain

$$\int_a^b cf = \sup_P L(P,cf) = \begin{cases} c\sup_P U(P,f) = c\int_a^b f & \text{if } c > 0 \\ c\inf_P L(P,f) = c\int_a^b f & \text{if } c < 0. \end{cases}$$

$\square$

Theorem 93. *Let $f, g\colon [a,b] \to \mathbb{R}$ be integrable functions. Then $f+g$ is integrable and*

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

Proof. Let $\epsilon > 0$. Since f and g are integrable, there exist partitions P_f and P_g of $[a,b]$ such that

$$U(P_f, f) - L(P_f, f) < \frac{\epsilon}{2}, \quad U(P_g, g) - L(P_g, g) < \frac{\epsilon}{2}.$$

Let $P_0 = P_f \cup P_g$ be the common refinement. By the refinement property,

$$U(P_0, f) - L(P_0, f) < \frac{\epsilon}{2}, \quad U(P_0, g) - L(P_0, g) < \frac{\epsilon}{2}.$$

Write P_0 as $\{a = x_0 < \cdots < x_n = b\}$ and let

$$M_i^f = \sup_{[x_{i-1}, x_i]} f, \quad m_i^f = \inf_{[x_{i-1}, x_i]} f,$$

$$M_i^g = \sup_{[x_{i-1}, x_i]} g, \quad m_i^g = \inf_{[x_{i-1}, x_i]} g.$$

Then for each i,

$$\sup_{[x_{i-1}, x_i]} (f+g) \leqslant M_i^f + M_i^g, \qquad \inf_{[x_{i-1}, x_i]} (f+g) \geqslant m_i^f + m_i^g.$$

Hence the upper and lower sums satisfy

$$\begin{aligned} L(P_0, f) + L(P_0, g) &\leqslant L(P_0, f+g) \\ &\leqslant U(P_0, f+g) \\ &\leqslant U(P_0, f) + U(P_0, g), \end{aligned}$$

which implies

$$\begin{aligned} U(P_0, f+g) - L(P_0, f+g) &\leqslant (U(P_0, f) - L(P_0, f)) \\ &\quad + (U(P_0, g) - L(P_0, g)) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $f+g$ is integrable.

Let $I_f = \int_a^b f$ and $I_g = \int_a^b g$. Then,

$$I_f = \sup_P L(P,f) = \inf_P U(P,f)$$

and

$$I_g = \sup_P L(P,g) = \inf_P U(P,g).$$

Therefore,

$$\begin{aligned} I_f - \frac{\epsilon}{2} + I_g - \frac{\epsilon}{2} &\leqslant U(P_0,f) - \frac{\epsilon}{2} + U(P_0,g) - \frac{\epsilon}{2} \\ &< L(P_0,f) + L(P_0,g) \\ &\leqslant L(P_0,f+g) \\ &\leqslant U(P_0,f+g) \\ &\leqslant U(P_0,f) + U(P_0,g) \\ &< L(P_0,f) + \frac{\epsilon}{2} + L(P_0,g) + \frac{\epsilon}{2} \\ &\leqslant I_f + \frac{\epsilon}{2} + I_g + \frac{\epsilon}{2}. \end{aligned}$$

Thus,

$$\int_a^b (f+g) = \inf_P U(P,f+g) \leqslant U(P_0,f+g) \leqslant I_f + I_g + \epsilon$$

and

$$\int_a^b (f+g) = \sup_P L(P,f+g) \geqslant L(P_0,f+g) \geqslant I_f + I_g - \epsilon.$$

Since ϵ is arbitrary,

$$\int_a^b (f+g) = I_f + I_g = \int_a^b f + \int_a^b g. \qquad \square$$

Theorem 94. *Let $f,g\colon [a,b] \to \mathbb{R}$ be integrable functions such that $f(x) \geqslant g(x)$ for all $x \in [a,b]$. Then*

$$\int_a^b f \geqslant \int_a^b g.$$

Proof. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$, with $\Delta x_i = x_i - x_{i-1}$. Define the upper and lower sums for f:

$$U(f, P) = \sum_{i=1}^{n} M_i^f \Delta x_i, \quad M_i^f = \sup_{x \in [x_{i-1}, x_i]} f(x),$$

$$L(f, P) = \sum_{i=1}^{n} m_i^f \Delta x_i, \quad m_i^f = \inf_{x \in [x_{i-1}, x_i]} f(x),$$

and similarly for g:

$$U(g, P) = \sum_{i=1}^{n} M_i^g \Delta x_i, \quad L(g, P) = \sum_{i=1}^{n} m_i^g \Delta x_i.$$

Since $f(x) \geqslant g(x)$ for all x, we have for each interval $[x_{i-1}, x_i]$:

$$m_i^f \geqslant m_i^g \quad \text{and} \quad M_i^f \geqslant M_i^g.$$

Hence, for any partition P,

$$L(f, P) \geqslant L(g, P) \quad \text{and} \quad U(f, P) \geqslant U(g, P).$$

Taking the supremum of lower sums (or infimum of upper sums) over all partitions, and using Riemann integrability of f and g, we get

$$\int_a^b f = \sup_P L(f, P) \geqslant \sup_P L(g, P) = \int_a^b g. \qquad \square$$

Theorem 95. *Let $f \colon [a, b] \to \mathbb{R}$ be continuous and non-negative ($f(x) \geqslant 0$ for all $x \in [a, b]$). If*

$$\int_a^b f = 0,$$

then $f(x) = 0$ for all $x \in [a, b]$.

Proof. Suppose, for contradiction, that f is not identically zero. Then there exists $x_0 \in [a, b]$ such that

$$f(x_0) > 0.$$

Since f is continuous at x_0, for $\varepsilon = \frac{f(x_0)}{2}$, there exists $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon \quad \text{for all } x \in I := [x_0 - \delta, x_0 + \delta] \cap [a, b].$$

That is,

$$f(x_0) - f(x) = |f(x_0)| - |f(x)| \leqslant |f(x_0) - f(x)| < \frac{f(x_0)}{2} \quad \text{for all } x \in I.$$

Equivalently,

$$\frac{f(x_0)}{2} < f(x) \quad \text{for all } x \in I.$$

By the properties of the integral over sub-intervals:

$$\int_a^b f \geqslant \int_I f \geqslant \int_I \frac{f(x_0)}{2} = \frac{f(x_0)}{2} \cdot \text{length}(I) > 0.$$

This contradicts the assumption that $\int_a^b f = 0$. Hence no such x_0 exists, and we must have

$$f(x) = 0 \quad \text{for all } x \in [a, b].$$

□

Homework 10

Theorem 96. *Let $f\colon [a, b] \to \mathbb{R}$ be continuous and suppose that*

$$\int_a^b f = 0.$$

Then there exists a point $c \in [a, b]$ such that $f(c) = 0$.

Proof. Since f is continuous on the compact interval $[a, b]$, it attains both a minimum and a maximum on $[a, b]$.

Suppose, for contradiction, that $f(x) \neq 0$ for every $x \in [a, b]$. By continuity, f cannot change sign without vanishing, so it must have a constant sign on $[a, b]$. Hence either

1. $f(x) > 0$ for all $x \in [a, b]$, or

2. $f(x) < 0$ for all $x \in [a, b]$.

In the first case, let $m = \min_{[a,b]} f > 0$. Then

$$\int_a^b f \geqslant \int_a^b m = m(b-a) > 0,$$

contradicting the hypothesis $\int_a^b f = 0$. In the second case, let $M = \max_{[a,b]} f < 0$. Then

$$\int_a^b f \leqslant \int_a^b M = M(b-a) < 0,$$

again a contradiction.

Therefore our assumption was false, and there exists $c \in [a, b]$ such that $f(c) = 0$. □

Theorem 97 (Mean Value Theorem for Integrals)**.** *Let $f\colon [a, b] \to \mathbb{R}$ be continuous. Then there exists $c \in [a, b]$ such that*

$$\int_a^b f = (b-a)f(c).$$

Proof. If $a = b$ the identity is trivial (take $c = a$). Assume $a < b$. By continuity on the compact interval $[a, b]$, f attains a minimum m and a maximum M on $[a, b]$, so

$$m \leqslant f(x) \leqslant M \qquad \text{for all } x \in [a, b].$$

By Theorem 94,

$$m(b-a) \leqslant \int_a^b f \leqslant M(b-a).$$

Dividing by $b - a > 0$ yields

$$m \leqslant \frac{1}{b-a} \int_a^b f \leqslant M.$$

Since f attains every value between m and M (Intermediate Value Theorem), there exists $c \in [a, b]$ with

$$f(c) = \frac{1}{b-a} \int_a^b f,$$

and multiplying by $b - a$ gives the result. □

Definition 98. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of the interval $[a, b]$. A *sub-interval of* P is a closed interval $[x_{i-1}, x_i]$ for some $i = 1, \ldots, n$.

Theorem 99. *Let $f, g\colon [a, b] \to \mathbb{R}$ be bounded functions that are equal except at finitely many points. Then f is Riemann integrable if and only if g is Riemann integrable, and in that case*

$$\int_a^b f = \int_a^b g.$$

Proof. Set $h := f - g$. By hypothesis, there exists a finite subset $\mathscr{F} \subset [a, b]$ such that $h(x) = 0$ for all $x \in [a, b] \backslash \mathscr{F}$. Define

$$M := \max_{x \in \mathscr{F}} |h(x)|,$$

which is finite.

For any integer $n \geqslant 1$, let P_n be the partition of $[a, b]$ into n equal sub-intervals, each of length $(b - a)/n$. Denote by $\mathscr{I}_n$ the set of all sub-intervals of P_n, and let

$$\mathscr{A} := \{I \in \mathscr{I}_n : I \cap \mathscr{F} \neq \varnothing\}.$$

Then we have the following:

- $|\mathscr{A}| \leqslant 2|\mathscr{F}|$.
- If $I \in \mathscr{A}$, then $-M \leqslant \inf_I h \leqslant \sup_I h \leqslant M$.
- If $I \in \mathscr{I}_n \backslash \mathscr{A}$, then $\inf_I h = 0 = \sup_I h$.

Hence,

$$U(h, P_n) = \sum_{I \in \mathscr{I}_n} \frac{b - a}{n} \sup_I h = \sum_{I \in \mathscr{A}} \frac{b - a}{n} \sup_I h \leqslant 2|\mathscr{F}| \cdot \frac{b - a}{n} \cdot M,$$

and

$$L(h, P_n) = \sum_{I \in \mathscr{I}_n} \frac{b - a}{n} \inf_I h = \sum_{I \in \mathscr{A}} \frac{b - a}{n} \inf_I h \geqslant 2|\mathscr{F}| \cdot \frac{b - a}{n} \cdot -M.$$

Therefore, for every n,

$$-2|\mathscr{F}|\frac{b-a}{n}M \leqslant L(h, P_n) \leqslant \underline{\int_a^b} h \leqslant \overline{\int_a^b} h \leqslant U(h, P_n) \leqslant 2|\mathscr{F}|\frac{b-a}{n}M.$$

Letting $n \to \infty$ gives

$$0 \leqslant \underline{\int_a^b} h \leqslant \overline{\int_a^b} h \leqslant 0,$$

so the upper and lower integrals coincide and equal 0. Thus h is Riemann integrable and

$$\int_a^b h = 0.$$

The final statements follow immediately: if one of f, g is integrable then the other is (since they differ by the integrable function h; see Theorem 93), and

$$\int_a^b f = \int_a^b (g+h) = \int_a^b g + \int_a^b h = \int_a^b g.$$

This completes the proof. $\square$

Theorem 100. *Let $f\colon [0,1] \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} 0, & x = 0 \textit{ or } x \textit{ irrational}, \\ \dfrac{1}{q}, & x = \dfrac{p}{q} \in \mathbb{Q}\backslash\{0\} \textit{ written in lowest terms, } q > 0. \end{cases}$$

Then f is Riemann integrable on $[0,1]$ and

$$\int_0^1 f = 0.$$

Proof. First note that every subinterval of $[0,1]$ contains irrational points; hence on any subinterval the infimum of f is 0. Therefore every lower sum is 0, so the lower integral satisfies

$$\underline{\int_0^1} f = 0.$$

It remains to show that the upper integral is also 0.

Let $\varepsilon > 0$. Choose an positive integer N with

$$\frac{1}{N} < \frac{\varepsilon}{2}.$$

If x is a element of $\mathbb{Q} \cap (0, 1]$ such that $x = \frac{p}{q}$ for some positive integers p and q with $\gcd(p, q) = 1$, then the following are equivalent:

- $q \geqslant N + 1$.
- $f(x) < \varepsilon/2$

Let $\mathscr{F}$ denote the following set

$$\left\{ x \in (0,1] : x = \frac{p}{q} \text{ for some } p, q \in \mathbb{N} \text{ with } \gcd(p, q) = 1 \text{ and } q \leqslant N \right\}.$$

Then $\mathscr{F}$ is a finite set.

Choose a partition $P = \{0 = x_0 < x_1 < \cdots < x_k = 1\}$ such that

$$\max_i (x_i - x_{i-1}) < \frac{\varepsilon}{4|\mathscr{F}|}.$$

Denote by $\mathscr{I}$ the set of all sub-intervals of P, and let

$$\mathscr{A} := \{I \in \mathscr{I} : I \cap \mathscr{F} \neq \varnothing\}.$$

Then we have the following:

- $|\mathscr{A}| \leqslant 2|\mathscr{F}|$.
- If $I \in \mathscr{A}$, then $\sup_I f \leqslant 1$.
- If $I \in \mathscr{I} \backslash \mathscr{A}$, then $\sup_I f < \frac{\varepsilon}{2}$.
- $\sum_{I \in \mathscr{F} \backslash \mathscr{A}} |I| \leqslant |[0, 1]| = 1$, since the elements of $\mathscr{F} \backslash \mathscr{A}$ are sub-intervals of $[0, 1]$ with pairwise disjoint interiors.

Therefore,

$$\begin{aligned}
U(P, f) &= \sum_{I \in \mathscr{I}}^{k} |I| \sup_I f \\
&= \sum_{I \in \mathscr{A}} |I| \sup_I f + \sum_{I \in \mathscr{I} \setminus \mathscr{A}} |I| \sup_I f \\
&< \sum_{I \in \mathscr{A}} \frac{\varepsilon}{4|\mathscr{F}|} \cdot 1 + \sum_{I \in \mathscr{F} \setminus \mathscr{A}} |I| \cdot \frac{\varepsilon}{2} \\
&= |\mathscr{A}| \frac{\varepsilon}{4|\mathscr{F}|} + \frac{\varepsilon}{2} \sum_{I \in \mathscr{F} \setminus \mathscr{A}} |I| \\
&\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the infimum of the upper sums is 0:

$$\overline{\int_0^1} f = 0.$$

Combining the lower and upper integrals gives

$$\underline{\int_0^1} f = \overline{\int_0^1} f = 0,$$

so f is Riemann integrable on $[0, 1]$ and $\displaystyle\int_0^1 f = 0.$ □

Theorem 101. *Every monotone function $f \colon [a, b] \to \mathbb{R}$ is integrable.*

Proof. Suppose f is monotone increasing on $[a, b]$. (The argument for decreasing f is similar.)

Let $\epsilon > 0$. Choose $P = \{x_0 = a < x_1 < \cdots < x_n = b\}$ a partition of

$[a, b]$ such that $\|P\| := \max_{i=1}^{n}(x_i - x_{i-1}) < \frac{\epsilon}{f(b)-f(a)}$. Then

$$\begin{aligned}
U(P, f) - L(P, f) &= \sum_{i=1}^{n} \left(\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f \right) \cdot (x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1}) \big) \cdot (x_i - x_{i-1}) \\
&\leqslant \|P\| \cdot \sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1}) \big) \\
&= \|P\| \cdot \big(f(b) - f(a) \big) \\
&< \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, f is integrable.

For a monotone decreasing function, the same argument applies with $f(x_{i-1})$ and $f(x_i)$ interchanged. Hence, every monotone function on $[a, b]$ is integrable. □

Theorem 102. *Every piecewise-monotone function $f \colon [a, b] \to \mathbb{R}$ is integrable.*

Proof. By definition, f is piecewise-monotone if there exists a partition

$$P = \{a = x_0 < x_1 < \cdots < x_N = b\}$$

such that on each sub-interval of P, f is either increasing or decreasing.

On each sub-interval $[x_{i-1}, x_i]$, f is monotone. Every monotone function on a closed interval is integrable; see Theorem 101. That is, for any $\epsilon > 0$, there exists a partition Q_i of $[x_{i-1}, x_i]$ such that

$$U(Q_i, f) - L(Q_i, f) < \frac{\epsilon}{N}.$$

Let $Q = \bigcup_{i=1}^{N} Q_i$ be the union of all refinements. Then Q is a partition of $[a, b]$, and

$$U(Q, f) - L(Q, f) = \sum_{i=1}^{N} \big(U(Q_i, f) - L(Q_i, f) \big) < \sum_{i=1}^{N} \frac{\epsilon}{N} = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, f is integrable on $[a, b]$. □

Example 103. There exists a bounded function $f\colon [a,b] \to \mathbb{R}$ such that $|f|$ is integrable but $\displaystyle\int_a^b f$ does not exist.

Define

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [a,b], \\ -1 & \text{if } x \in (\mathbb{R} \backslash \mathbb{Q}) \cap [a,b]. \end{cases}$$

Every sub-interval of $[a,b]$ contains both rational and irrational numbers, so f is well defined and bounded with $|f(x)| \leqslant 1$ for all $x \in [a,b]$.

For every $x \in [a,b]$, $|f(x)| = 1$. Thus $|f|$ is the constant function 1, which is integrable, and

$$\int_a^b |f| = \int_a^b 1 = b - a.$$

Now, we show that f is not integrable.

Let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[a,b]$. On each sub-interval $[x_{i-1}, x_i]$, since the rationals and irrationals are both dense in $\mathbb{R}$, we have

$$\sup_{[x_{i-1},x_i]} f = 1, \qquad \inf_{[x_{i-1},x_i]} f = -1.$$

Hence,

$$U(P,f) = \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot 1 = b - a$$

and

$$L(P,f) = \sum_{i=1}^{n} (x_i - x_{i-1}) \cdot (-1) = -(b-a).$$

Therefore,

$$U(P,f) - L(P,f) = 2(b-a) > 0$$

for every partition P. Consequently,

$$\sup_P L(P,f) \neq \inf_P U(P,f),$$

and f is not integrable.

Thus, $|f|$ is integrable but $\displaystyle\int_a^b f$ does not exist.

Homework 11

Theorem 104. *Let $f : (0,1] \to \mathbb{R}$ be a function such that f is integrable on $[c,1]$ for each $c > 0$, and define the improper integral*

$$\int_0^1 f := \lim_{c \to 0^+} \int_c^1 f,$$

if the limit exists and is finite. Then:

(a) If f is integrable on $[0,1]$, then this definition agrees with the usual Riemann integral.

(b) There exists a function f for which the above improper integral exists, but the integral of $|f|$ does not exist.

Proof of (a). Suppose f is integrable on $[0,1]$. For any $c > 0$, by additivity of the integral we have

$$\int_0^1 f = \int_0^c f + \int_c^1 f.$$

Rewriting, we get

$$\int_c^1 f = \int_0^1 f - \int_0^c f.$$

Now, since f is integrable on $[0,1]$, it is bounded, say $|f(x)| \leqslant M$ for all $x \in [0,1]$. Hence, for any $c > 0$,

$$\left| \int_0^c f \right| \leqslant \int_0^c |f| \leqslant \int_0^c M = M \cdot c.$$

As $c \to 0^+$, we have $M \cdot c \to 0$. Therefore,

$$\lim_{c \to 0^+} \int_0^c f = 0.$$

Substituting this into the previous equality gives

$$\lim_{c \to 0^+} \int_c^1 f = \int_0^1 f,$$

which shows that the improper integral definition agrees with the usual integral.

□

Theorem 105 (Alternating Series Test (Leibniz)). *Let (a_n) be a sequence of positive real numbers such that*

1. $a_{n+1} \leqslant a_n$ *for all sufficiently large* n*, and*

2. $\lim_{n\to\infty} a_n = 0$.

Then the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges.

Proof. Let $S_n = a_1 - a_2 + a_3 - \cdots + (-1)^{n+1} a_n$. Then

$$S_{2k+1} - S_{2k-1} = a_{2k} - a_{2k+1} \geqslant 0,$$

$$S_{2k+2} - S_{2k} = -a_{2k+1} + a_{2k+2} \leqslant 0.$$

Hence the sequence (S_{2k}) is increasing, and (S_{2k+1}) is decreasing. Since $S_{2k} \leqslant S_{2k+1}$ for all k, both are bounded and monotone, so each converges. Moreover,

$$S_{2k+1} - S_{2k} = a_{2k+1} \to 0,$$

so both converge to the same limit. Thus S_n converges, and the alternating series converges. $\square$

Theorem 106 (p-series test). *For $p > 0$ the series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$.

Proof. Split the positive integers into dyadic blocks

$$B_k = \{2^k + 1, 2^k + 2, \ldots, 2^{k+1}\} \qquad (k = 0, 1, 2, \ldots).$$

Each block B_k contains exactly 2^k integers.

(1) If $p > 1$ then the series converges.
For $n \in B_k$ we have $n > 2^k$, hence

$$\frac{1}{n^p} \leqslant \frac{1}{(2^k)^p}.$$

Summing over the 2^k members of B_k,

$$\sum_{n \in B_k} \frac{1}{n^p} \leqslant 2^k \cdot \frac{1}{(2^k)^p} = 2^{k(1-p)}.$$

Thus

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{k=0}^{\infty} \sum_{n \in B_k} \frac{1}{n^p} \leqslant \sum_{k=0}^{\infty} 2^{k(1-p)}.$$

The right-hand side is a geometric series with ratio $2^{1-p} < 1$ (since $p > 1$), so it converges. Hence the p-series converges.

(2) If $0 < p \leqslant 1$ then the series diverges.
For $n \in B_k$ we have $n \leqslant 2^{k+1}$, hence

$$\frac{1}{n^p} \geqslant \frac{1}{(2^{k+1})^p}.$$

Summing over the 2^k members of B_k,

$$\sum_{n \in B_k} \frac{1}{n^p} \geqslant 2^k \cdot \frac{1}{(2^{k+1})^p} = 2^{-p}\, 2^{k(1-p)}.$$

If $0 < p < 1$ then $1 - p > 0$, so $2^{k(1-p)} \to \infty$ and the lower bounds on block-sums form a divergent geometric-type sequence; summing over blocks shows the whole series diverges. If $p = 1$ we get the constant lower bound

$$\sum_{n \in B_k} \frac{1}{n} \geqslant 2^k \cdot \frac{1}{2^{k+1}} = \frac{1}{2}$$

for every k, so the series certainly diverges (its partial sums grow by at least $1/2$ in each block).

Combining (1) and (2) completes the proof. $\square$

Proof of (b). Define $f\colon (0,1] \to \mathbb{R}$ by

$$f(x) = (-1)^n n \quad \text{if } \frac{1}{n+1} < x \leqslant \frac{1}{n},\ n \in \mathbb{N}.$$

For $c \in (1/(N+1), 1/N]$, we have

$$\begin{aligned}
\int_c^1 f &= \int_{1/2}^1 f + \int_{1/3}^{1/2} f + \cdots + \int_{1/(N-1)}^{1/N} f + \int_{1/N}^c f \\
&= \int_{1/2}^1 (-1)^1 \cdot 1 + \int_{1/3}^{1/2} (-1)^2 \cdot 2 + \cdots \\
&\quad + \int_{1/N}^{1/(N-1)} (-1)^{N-1} \cdot (N-1) + \int_c^{1/N} (-1)^N \cdot N \\
&= \sum_{n=1}^{N-1} (-1)^n n \left(\frac{1}{n} - \frac{1}{n+1} \right) + (-1)^N N \left(\frac{1}{N} - c \right) \\
&= \sum_{n=1}^{N-1} \frac{(-1)^n}{n+1} + (-1)^N N \left(\frac{1}{N} - c \right).
\end{aligned}$$

Now, $1 - \frac{1}{N+1} = \frac{N}{N+1} \leqslant Nc \leqslant 1$. Hence, taking the limit as $c \to 0^+$ (equivalently, $N \to \infty$) gives

$$\int_0^1 f = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1},$$

which converges by Theorem 105.

However,

$$\int_0^1 |f| = \sum_{n=1}^{\infty} n \left(\frac{1}{n} - \frac{1}{n+1} \right) = \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty.$$

Thus, the improper integral of f exists, but the integral of $|f|$ diverges. $\square$

Theorem 107. *Let $\gamma_1\colon [a,b] \to \mathbb{R}^k$ be a path, and let $\phi\colon [c,d] \to [a,b]$ be a continuous, 1-1, onto map such that $\phi(c) = a$. Define the reparametrized curve*

$$\gamma_2(s) := \gamma_1(\phi(s)), \quad s \in [c,d].$$

Then:

(a) γ_2 *is rectifiable if and only if* γ_1 *is rectifiable.*

(b) *If the curves are rectifiable, they have the same length, i.e.,*

$$L(\gamma_2) = L(\gamma_1).$$

Proof (a). Let $P = \{c = s_0 < s_1 < \cdots < s_n = d\}$ be a partition of $[c, d]$. Consider the corresponding points in $[a, b]$:

$$t_i := \phi(s_i), \quad i = 0, \ldots, n.$$

Since ϕ is 1-1 and onto, $Q = \{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition of $[a, b]$.

The polygonal sum for γ_2 is

$$\sum_{i=1}^{n} \|\gamma_2(s_i) - \gamma_2(s_{i-1})\| = \sum_{i=1}^{n} \|\gamma_1(t_i) - \gamma_1(t_{i-1})\|.$$

Every partition of $[c, d]$ corresponds to a partition of $[a, b]$, and conversely, since ϕ is onto. Taking the supremum over all partitions gives

$$\sup_{P \subset [c,d]} \sum_{i=1}^{n} \|\gamma_2(s_i) - \gamma_2(s_{i-1})\| = \sup_{Q \subset [a,b]} \sum_{i=1}^{n} \|\gamma_1(t_i) - \gamma_1(t_{i-1})\|.$$

Hence, γ_2 is rectifiable if and only if γ_1 is rectifiable. □

Proof of (b). By the calculation above, the polygonal sums of γ_2 and γ_1 are identical for corresponding partitions. Therefore, taking the supremum over all partitions,

$$\begin{aligned} L(\gamma_2) &= \sup_{P \subset [c,d]} \sum_{i=1}^{n} \|\gamma_2(s_i) - \gamma_2(s_{i-1})\| \\ &= \sup_{Q \subset [a,b]} \sum_{i=1}^{n} \|\gamma_1(t_i) - \gamma_1(t_{i-1})\| \\ &= L(\gamma_1). \end{aligned}$$

This shows that reparametrization via a continuous, 1-1, onto map preserves rectifiability and length. □

Theorem 108. *Let $\{a_n\}$ and $\{b_n\}$ be two real sequences which are bounded below. Then*

$$\limsup_{n\to\infty}(a_n + b_n) \leqslant \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Proof. Recall that for a real sequence $\{x_n\}$, the *lim sup* is defined as

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \sup_{k\geqslant n} x_k.$$

Since $\{a_n\}$ and $\{b_n\}$ are bounded below, their suprema over tails are finite.

Define for each $n \in \mathbb{N}$:

$$A_n := \sup_{k\geqslant n} a_k, \quad B_n := \sup_{k\geqslant n} b_k, \quad S_n := \sup_{k\geqslant n}(a_k + b_k).$$

For each fixed n, and for all $k \geqslant n$,

$$a_k + b_k \leqslant \sup_{j\geqslant n} a_j + \sup_{j\geqslant n} b_j = A_n + B_n.$$

Taking the supremum over $k \geqslant n$ on the left-hand side gives

$$S_n = \sup_{k\geqslant n}(a_k + b_k) \leqslant A_n + B_n.$$

The sequences $\{A_n\}$ and $\{B_n\}$ are non-increasing and bounded below, so the limits exist:

$$\lim_{n\to\infty} A_n = \limsup_{n\to\infty} a_n, \quad \lim_{n\to\infty} B_n = \limsup_{n\to\infty} b_n.$$

From the inequality $S_n \leqslant A_n + B_n$, we get

$$\lim_{n\to\infty} S_n \leqslant \lim_{n\to\infty}(A_n + B_n) = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

By the definition of lim sup,

$$\limsup_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} S_n \leqslant \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

This completes the proof. ∎

Theorem 109. *Let $\{a_n\}$ be a sequence of real numbers. Then:*

(a) $\liminf_{n\to\infty} a_n \leqslant \limsup_{n\to\infty} a_n$.

(b) The sequence $\{a_n\}$ converges if and only if $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ are both finite and equal. In this case,

$$\lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n.$$

Proof of (a). For each $n \in \mathbb{N}$, define

$$A_n := \sup_{k\geqslant n} a_k, \quad B_n := \inf_{k\geqslant n} a_k.$$

Then $B_n \leqslant A_n$ for all n, and the sequences $\{A_n\}$ and $\{B_n\}$ are non-increasing and non-decreasing respectively. Taking limits gives

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} B_n \leqslant \lim_{n\to\infty} A_n = \limsup_{n\to\infty} a_n. \qquad \square$$

Proof of (b). Suppose $\{a_n\}$ converges to $L \in \mathbb{R}$. Then for any $\varepsilon > 0$, there exists N such that for all $n \geqslant N$, $L - \varepsilon < a_n < L + \varepsilon$. This implies

$$\inf_{k\geqslant n} a_k \geqslant L - \varepsilon, \quad \sup_{k\geqslant n} a_k \leqslant L + \varepsilon \quad \forall n \geqslant N.$$

Taking limits as $n \to \infty$, we obtain

$$\liminf_{n\to\infty} a_n \geqslant L - \varepsilon \quad \text{and} \quad \limsup_{n\to\infty} a_n \leqslant L + \varepsilon.$$

By (a),

$$L - \varepsilon \leqslant \liminf_{n\to\infty} a_n \leqslant \limsup_{n\to\infty} a_n \leqslant L + \varepsilon.$$

Since ε is arbitrary positive,

$$\liminf_{n\to\infty} a_n = L = \limsup_{n\to\infty} a_n$$

Next, suppose $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = L$ (finite). Let $\varepsilon > 0$. There exists N_1 such that for all $n \geqslant N_1$, $\inf_{k\geqslant n} a_k > L - \varepsilon$, and

N_2 such that for all $n \geqslant N_2$, $\sup_{k \geqslant n} a_k < L + \varepsilon$. Let $N = \max(N_1, N_2)$. Then for all $n \geqslant N$,

$$L - \varepsilon < a_n < L + \varepsilon \implies |a_n - L| < \varepsilon.$$

Hence $\{a_n\}$ converges to L, which equals both the lim sup and lim inf.

□

Theorem 110. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers such that*

$$a_n \leqslant b_n \quad \textit{for all } n \in \mathbb{N}.$$

Then

$$\limsup_{n\to\infty} a_n \leqslant \limsup_{n\to\infty} b_n \quad \textit{and} \quad \liminf_{n\to\infty} a_n \leqslant \liminf_{n\to\infty} b_n.$$

Proof. Define for each $n \in \mathbb{N}$:

$$A_n := \sup_{k \geqslant n} a_k, \quad B_n := \sup_{k \geqslant n} b_k.$$

Since $a_k \leqslant b_k$ for all $k \geqslant n$, we have $A_n \leqslant B_n$ for all n.

Taking the limit as $n \to \infty$, we obtain

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} A_n \leqslant \lim_{n\to\infty} B_n = \limsup_{n\to\infty} b_n.$$

Similarly, define

$$C_n := \inf_{k \geqslant n} a_k, \quad D_n := \inf_{k \geqslant n} b_k.$$

Then $C_n \leqslant D_n$ for all n, and taking limits gives

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} C_n \leqslant \lim_{n\to\infty} D_n = \liminf_{n\to\infty} b_n.$$

This proves the theorem. □

Theorem 111 (Comparison Test)**.** *Let $\sum a_n$ and $\sum b_n$ be series with $a_n, b_n \geqslant 0$ for all n. If $a_n \leqslant b_n$ for all sufficiently large n, and $\sum b_n$ converges, then $\sum a_n$ also converges.*

Proof. Assume $a_n \leqslant b_n$ and $\sum b_n$ converges. Let $A_N = \sum_{n=1}^{N} a_n$ and $B_N = \sum_{n=1}^{N} b_n$. Then $A_N \leqslant B_N$ for every N. Since (B_N) converges, it is bounded above. Hence (A_N) is increasing and bounded above, so it also converges. Thus $\sum a_n$ converges. □

Remark 112. The converse also holds: if $a_n \leqslant b_n$ for all sufficiently large n, and $\sum a_n$ diverges, then $\sum b_n$ also diverges.

Theorem 113. *The geometric series*

$$\sum_{n=0}^{\infty} r^n$$

converges if and only if $|r| < 1$. *In that case,*

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Proof. Let $S_N = 1 + r + r^2 + \cdots + r^N$. Multiplying both sides by r gives

$$rS_N = r + r^2 + \cdots + r^{N+1}.$$

Subtracting, we obtain

$$S_N - rS_N = 1 - r^{N+1},$$

so

$$S_N = \frac{1 - r^{N+1}}{1-r}.$$

If $|r| < 1$, then $r^{N+1} \to 0$ as $N \to \infty$, giving

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Now, if $|r| \geqslant 1$, then r^n does not tend to zero as $n \to \infty$, so the series diverges. □

Theorem 114. *The series* $\sum_{n=1}^{\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$ *diverges.*

Proof. Using the identity

$$\sqrt{n+1}-\sqrt{n}=\frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}}=\frac{1}{\sqrt{n+1}+\sqrt{n}},$$

we have, for all $n \geqslant 1$,

$$\frac{1}{2\sqrt{n+1}} \leqslant \frac{1}{\sqrt{n+1}+\sqrt{n}} \leqslant \frac{1}{2\sqrt{n}}.$$

Since $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges (see Theorem 106), by Theorem 111,

$$\sum_{n=1}^{\infty}\left(\sqrt{n+1}-\sqrt{n}\right) \text{ diverges.}$$ □

Theorem 115. *The series* $\sum_{n=1}^{\infty}\dfrac{\sqrt{n+1}-\sqrt{n}}{n}$ *converges.*

Proof. Let $a_n=\dfrac{\sqrt{n+1}-\sqrt{n}}{n}$. Then

$$a_n=\frac{1}{n(\sqrt{n+1}+\sqrt{n})}.$$

Then for all $n \geqslant 1$,

$$\frac{1}{2n\sqrt{n+1}} \leqslant a_n \leqslant \frac{1}{2n\sqrt{n}}.$$

Compare with the p-series $\sum \frac{1}{n^{3/2}}$, which converges since $p=3/2>1$; see Theorem 106. Hence, by Theorem 111,

$$\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{n} \text{ converges.}$$ □

Theorem 116. *The series*

$$\sum_{n=1}^{\infty}\left(\sqrt[n]{n}-1\right)^{n}$$

converges.

Proof. Let $a_n = \sqrt[n]{n} - 1$. Then $(1 + a_n)^n = n$, and since $a_n > 0$,

$$n = (1 + a_n)^n = 1 + na_n + \frac{n(n-1)}{2} a_n^2 + \cdots \geqslant 1 + na_n + \frac{n(n-1)}{2} a_n^2.$$

Hence

$$1 + na_n + \frac{n(n-1)}{2} a_n^2 \leqslant n,$$

so

$$na_n + \frac{n(n-1)}{2} a_n^2 \leqslant n - 1.$$

Dropping the nonnegative term na_n gives

$$\frac{n(n-1)}{2} a_n^2 \leqslant n - 1,$$

and thus, for all $n \geqslant 2$,

$$a_n \leqslant \sqrt{\frac{2}{n}}.$$

Therefore

$$a_n^n \leqslant \left(\frac{2}{n}\right)^{n/2}.$$

For $n \geqslant 8$, we have $\frac{2}{n} \leqslant \frac{1}{2}$, so

$$a_n^n \leqslant \left(\frac{1}{2}\right)^{n/2}.$$

Hence

$$\sum_{n=8}^{\infty} a_n^n \leqslant \sum_{n=8}^{\infty} \left(\frac{1}{2}\right)^{n/2},$$

and the right-hand side is a convergent series; see Theorem 113. By Theorem 111,

$$\sum_{n=1}^{\infty} \left(\sqrt[n]{n} - 1\right)^n$$

converges. □

Homework 12

Problem 117. *Find the radius of convergence of each of the following power series using the root test only:*

$$\text{(a)} \ \sum_{n=0}^{\infty} 3^n x^n, \qquad \text{(b)} \ \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n.$$

Solution. We use the formula for the radius of convergence of a power series

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}.$$

(a) Here $a_n = 3^n$. Then

$$|a_n|^{1/n} = (3^n)^{1/n} = 3.$$

Thus

$$\limsup_{n\to\infty} |a_n|^{1/n} = 3,$$

and hence

$$R = \frac{1}{3}.$$

(b) Here $a_n = \dfrac{2^n}{n!}$. Then

$$|a_n|^{1/n} = \left(\frac{2^n}{n!}\right)^{1/n} = \frac{2}{(n!)^{1/n}}.$$

We first show that $(n!)^{1/n} \to \infty$.

If $n = 2k$ for some $k \in \mathbb{N}$, then

$$\begin{aligned}
(n!)^{1/n} &= (1 \cdot 2 \cdots (k-1) \cdot k \cdot (k+1) \cdots (2k))^{1/n} \\
&\geqslant (k \cdot (k+1) \cdots (2k))^{1/n} \\
&\geqslant (k \cdot k \cdots k)^{1/n} \\
&= k^{\frac{k+1}{2k}} \\
&\geqslant k^{1/2} \\
&= \sqrt{\frac{n}{2}},
\end{aligned}$$

which diverges to ∞ as $n \to \infty$.

On the other hand, if $n = 2k + 1$ for some $k \in \mathbb{N}$, then

$$
\begin{aligned}
(n!)^{1/n} &= (1 \cdot 2 \cdots (k-1) \cdot k \cdot (k+1) \cdots (2k+1))^{1/n} \\
&\geqslant ((k+1) \cdot (k+2) \cdots (2k+1))^{1/n} \\
&\geqslant ((k+1) \cdot (k+1) \cdots (k+1))^{1/n} \\
&= (k+1)^{\frac{k+1}{2k+1}} \\
&\geqslant (k+1)^{1/2} \\
&\geqslant \sqrt{\frac{n}{2}},
\end{aligned}
$$

which diverges to ∞ as $n \to \infty$.

Thus, in either case, $(n!)^{1/n} \to \infty$. Hence,

$$
\lim_{n\to\infty} \left(\frac{2^n}{n!}\right)^{1/n} = 0.
$$

Therefore,

$$
\limsup_{n\to\infty} |a_n|^{1/n} = 0, \quad \text{and} \quad R = \infty. \qquad \square
$$

Lemma 118. Let $(x_n)_{n\geqslant 1}$ be a sequence of non-negative reals with finite $L := \limsup_{n\to\infty} x_n$. Let $f\colon [0,\infty) \to \mathbb{R}$ be a continuous non-decreasing function. Then

$$
\limsup_{n\to\infty} f(x_n) \;=\; f\big(\limsup_{n\to\infty} x_n\big) = f(L).
$$

Proof. For $N \geqslant 1$ set $S_N := \sup_{n\geqslant N} x_n$. The sequence $(S_N)_{N\geqslant 1}$ is non-increasing and $\lim_{N\to\infty} S_N = \inf_{N\geqslant 1} S_N = L$. Fix N. Since f is non-decreasing, for every $n \geqslant N$ we have $f(x_n) \leqslant f(S_N)$, hence

$$
\sup_{n\geqslant N} f(x_n) \leqslant f(S_N).
$$

Conversely, by definition of supremum for every $\varepsilon > 0$ there exists some $n \geqslant N$ with $x_n > S_N - \varepsilon$. By monotonicity, $f(x_n) \geqslant$

$f(S_N - \varepsilon)$, so

$$\sup_{n \geqslant N} f(x_n) \geqslant f(S_N - \varepsilon).$$

Letting $\varepsilon \downarrow 0$ and using continuity of f at S_N gives $\sup_{n \geqslant N} f(x_n) \geqslant f(S_N)$. Thus

$$\sup_{n \geqslant N} f(x_n) = f(S_N) \qquad \text{for every } N \geqslant 1.$$

Taking infimum over N on both sides yields

$$\inf_{N \geqslant 1} \sup_{n \geqslant N} f(x_n) = \inf_{N \geqslant 1} f(S_N).$$

The left-hand side is $\limsup_{n \to \infty} f(x_n)$. Since $S_{N+1} \leqslant S_N$ for all N and f is non-decreasing, we have

$$f(S_{N+1}) \leqslant f(S_N),$$

so the sequence $(f(S_N))$ is non-increasing. Therefore,

$$\inf_N f(S_N) = \lim_{N \to \infty} f(S_N).$$

By continuity of f at L, we have $\lim_{N \to \infty} f(S_N) = f(L)$. Combining these equalities gives the desired identity

$$\limsup_{n \to \infty} f(x_n) = \inf_{N \geqslant 1} \sup_{n \geqslant N} f(x_n) = \inf_{N \geqslant 1} f(S_N) = f(L).$$

This completes the proof. $\square$

Theorem 119. *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R = 2$. Fix an integer $k \geqslant 1$ and consider the power series*

$$\sum_{n=0}^{\infty} a_n^k x^n.$$

Then the radius of convergence of this new series is $R' = 2^k$.

Proof. Put
$$L := \limsup_{n\to\infty} |a_n|^{1/n}.$$
By the root-test formula for radii of convergence we have $R = 1/L$. Since $R = 2$ we get $L = \frac{1}{2}$.

Define $b_n := a_n^k$. To find the radius R' of $\sum b_n x^n$ apply the root test:
$$\begin{aligned}
R' &= \frac{1}{\limsup_{n\to\infty} |b_n|^{1/n}} \\
&= \frac{1}{\limsup_{n\to\infty} |a_n|^{k/n}} \\
&= \frac{1}{\limsup_{n\to\infty} f\left(|a_n|^{1/n}\right)} && \text{where } f(t) = t^k \\
&= \frac{1}{f\left(\limsup_{n\to\infty} |a_n|^{1/n}\right)} && \text{by Lemma 118} \\
&= \frac{1}{f\left(\frac{1}{2}\right)} \\
&= 2^k.
\end{aligned}$$
□

Lemma 120. Let (a_n) be a real sequence such that
$$\rho = \limsup_{n\to\infty} |a_n|^{1/n}$$
is a positive real number. Define a new sequence (c_m) by
$$c_m = \begin{cases} a_n & \text{if } m = n^2, \\ 0 & \text{otherwise.} \end{cases}$$
Then
$$\limsup_{m\to\infty} |c_m|^{1/m} = 1.$$

Proof. For $m = n^2$ we have
$$|c_{n^2}|^{1/n^2} = |a_n|^{1/n^2} = \left(|a_n|^{1/n}\right)^{1/n}.$$

For m not a perfect square, $c_m = 0$, so $|c_m|^{1/m} = 0$.
Hence the values of $|c_m|^{1/m}$ are either 0 or $\left(|a_n|^{1/n}\right)^{1/n}$. Since

$$\inf_{N\in\mathbb{N}} \sup_{n\geqslant N} |a_n|^{1/n} = \limsup_{n\to\infty} |a_n|^{1/n}$$

is a positive real, the sequence $\left(|a_n|^{1/n}\right)$ is bounded. Thus, there exists $M > 0$ such that

$$|a_n|^{1/n} \leqslant M \quad \text{for all } n.$$

Let $\varepsilon > 0$ be arbitrary. Since $M^{1/n} \to 1$ as $n \to \infty$, there exists n_0 such that

$$M^{1/n} < 1 + \varepsilon \quad \text{for all } n \geqslant n_0.$$

Now for any $m \geqslant n_0^2$, either

$$|c_m|^{1/m} = 0 < 1 + \varepsilon,$$

or $m = n^2$ with $n \geqslant n_0$, in which case

$$|c_m|^{1/m} = (|a_n|^{1/n})^{1/n} \leqslant M^{1/n} < 1 + \varepsilon.$$

Therefore, for all $m \geqslant n_0^2$, we have $|c_m|^{1/m} < 1 + \varepsilon$. Hence

$$\limsup_{m\to\infty} |c_m|^{1/m} = \inf_{M\in\mathbb{N}} \sup_{m\geqslant M} |c_m|^{1/m} \leqslant \sup_{m\geqslant n_0^2} |c_m|^{1/m} \leqslant 1 + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$\limsup_{m\to\infty} |c_m|^{1/m} \leqslant 1.$$

Now, we show that $\limsup_{m\to\infty} |c_m|^{1/m} \geqslant 1$. By assumption,

$$\limsup_{n\to\infty} |a_n|^{1/n} = \inf_{N\in\mathbb{N}} \sup_{n\geqslant N} |a_n|^{1/n} = \rho$$

is a positive real. Then by the definition of $\limsup$, for every $\varepsilon > 0$ there exist infinitely many n such that

$$|a_n|^{1/n} > \rho - \varepsilon.$$

Fix an $\varepsilon \in (0, \rho)$ and pick a corresponding strictly increasing subsequence (n_k) with this property. Then for each k,

$$|c_{n_k^2}|^{1/n_k^2} = (|a_{n_k}|^{1/n_k})^{1/n_k} > (\rho - \varepsilon)^{1/n_k}.$$

Since $(\rho - \varepsilon)^{1/n_k} \to 1$ as $k \to \infty$, there exists K such that

$$(\rho - \varepsilon)^{1/n_k} > 1 - \varepsilon \quad \text{for all } k \geqslant K.$$

Hence for all large k,

$$|c_{n_k^2}|^{1/n_k^2} > 1 - \varepsilon.$$

Therefore, for every $\varepsilon > 0$ and every large enough index, there exist infinitely many $m = n_k^2$ satisfying $|c_m|^{1/m} > 1 - \varepsilon$. This means

$$\limsup_{m \to \infty} |c_m|^{1/m} = \inf_{M \in \mathbb{N}} \sup_{m \geqslant M} |c_m|^{1/m} \geqslant 1 - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude

$$\limsup_{m \to \infty} |c_m|^{1/m} \geqslant 1.$$

Combining the two inequalities gives

$$\limsup_{m \to \infty} |c_m|^{1/m} = 1. \qquad \square$$

Theorem 121. *Let the power series* $\sum_{n=0}^{\infty} a_n x^n$ *have radius of convergence* $R = 2$. *Then the series*

$$\sum_{n=0}^{\infty} a_n x^{n^2}$$

has radius of convergence $R' = 1$.

Proof. Follows from Lemma 120. $\square$