HypoElastic Models

Sumanta Mukherjee

July 24, 2015

1 The differential equation for a Tubular Structure

Hypoelastic Materials are important in everyday life. Below we create the mathematical framework for hypoelastic material which is an adhesive between two linear elastic materials.

1.1 The basic equations

$$\nabla \cdot \sigma = 0 \tag{1}$$

$$\epsilon = \frac{1}{2}((\nabla u) + (\nabla u)^T) \tag{2}$$

$$\epsilon_{ij} = \alpha_1 \sigma_{ij} - \alpha_2 \sigma_{kk} \delta_{ij} + \alpha_3 S_{ij} (\frac{\sigma_{eq}}{\sigma_0})^{(n-1)} \tag{3}$$

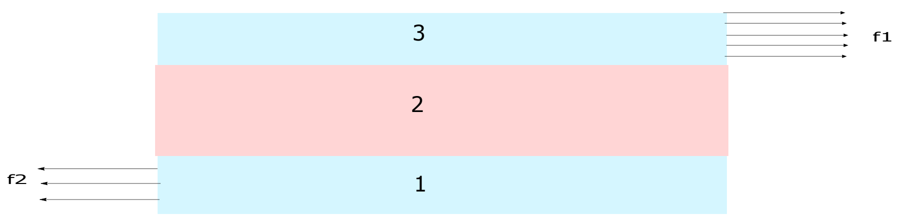

Figure 1

The three layers are marked 1,2, 3. The blue layers are made up of linear elastic material while the intermediate adhesive layer is made up of hypoelastic material.

The expanded equations for one of the tubular structure in r, θ, z direction is given as follows.

$$
\begin{aligned}
\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} &= 0 \\
\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} &= 0 \\
\frac{\partial \sigma_{rz}}{\partial r} + \frac{\sigma_{rz}}{r} + \frac{1}{r}\frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} &= 0
\end{aligned}
\tag{4}
$$

The expanded equations for the cylindrical geometry of the multilayer tubular structure are subject to the following assumptions.

- The radial stresses are directly proportiional to the hoop stresses in all the three domains
- Axisymmetric condition implies that only the shear stress σ_{rz} is non zero in all the three domains.
- variations with respect to θ is also insignificant

The equations therefore reduce to the following form.

$$
\begin{aligned}
\frac{1}{r}\frac{\partial (r\sigma_{rr}^{(i)})}{\partial r} + \frac{-\sigma_{\theta\theta}}{r} + \frac{\partial \sigma_{rz}}{\partial z} &= 0 \\
\frac{\partial \sigma_{rz}}{\partial r} + \frac{\sigma_{rz}}{r} + \frac{\partial \sigma_{zz}}{\partial z} &= 0
\end{aligned}
\tag{5}
$$

There are 3 unknowns from two equations and so a relation needs to be found out between the stresses $\sigma_{\theta\theta}, \sigma_{rz}, \sigma_{zz}$.

We describe the force balance in each of the layers

Layer i = 1,3: (Linear Elastic Material Layer)

$$
\begin{aligned}
\frac{1}{r}\frac{\partial (r\sigma_{rr}^{(i)})}{\partial r} + \frac{-\sigma_{\theta\theta}^{(i)}}{r} + \frac{\partial \sigma_{rz}^{(i)}}{\partial z} &= 0 \\
\frac{\partial \sigma_{rz}^{(i)}}{\partial r} + \frac{\sigma_{rz}^{(i)}}{r} + \frac{\partial \sigma_{zz}^{(i)}}{\partial z} &= 0
\end{aligned}
\tag{6}
$$

Layer 2: (Adhesive Hypo Elastic Material Layer)

$$
\begin{aligned}
\frac{1}{r}\frac{\partial (r\sigma_{rr}^{(2)})}{\partial r} + \frac{-\sigma_{\theta\theta}^{(2)}}{r} + \frac{\partial \sigma_{rz}^{(2)}}{\partial z} &= 0 \\
\frac{\partial \sigma_{rz}^{(2)}}{\partial r} + \frac{\sigma_{rz}^{(2)}}{r} + \frac{\partial \sigma_{zz}^{(2)}}{\partial z} &= 0
\end{aligned}
\tag{7}
$$

Solving the equations and using the notation $\frac{\partial \sigma_{zz}}{\partial z} = \sigma^{'}$ with the boundary condition $\sigma_{rz}(a) = 0$ we come to the following solution of the equation

$$
\frac{1}{r}\frac{\partial (r\sigma_{rz})}{\partial r} + \sigma^{'} = 0 \tag{8}
$$

as

$$\frac{\partial(r\sigma_{rz})}{\partial r} = -\sigma^{'} r$$
$$r\sigma_{rz} = -0.5\sigma^{'} r^2 + const \tag{9}$$

Applying the Boundary Condition in the first region

$$0.5\sigma^{'} a^2 = const \tag{10}$$

Applying the Boundary Condition in the second region

$$0.5\sigma^{'(1)}(a^2 - b^2) + 0.5\sigma^{'(2)}(b^2) = const \tag{11}$$

Applying the Boundary Condition in the third region

$$0.5\sigma^{'(1)}(a^2 - b^2) + 0.5\sigma^{'(2)}(b^2 - c^2) + 0.5\sigma^{'(3)}(c^2) = const \tag{12}$$

Substituting the solution into the first of the two equations for each of the regions and making the assumption that $\sigma_{\theta\theta} = \sigma_{rr}$

$$\boxed{\begin{aligned} \sigma_{rz}^{(1)} &= \sigma^{'(1)}\frac{(a^2 - r^2)}{2r} \\ \sigma_{rz}^{(2)} &= \sigma^{'(2)}\frac{(b^2 - r^2)}{2r} + \sigma^{'(1)}\frac{(a^2 - b^2)}{2r} \\ \sigma_{rz}^{(3)} &= \sigma^{'(3)}\frac{(c^2 - r^2)}{2r} + \sigma^{'(2)}\frac{(b^2 - c^2)}{2r} + \sigma^{'(1)}\frac{(a^2 - b^2)}{2r} \end{aligned}} \tag{13}$$

Given that the stress on the outside is also zero we get the following condition.

$$\boxed{\sigma^{'(3)}(c^2 - d^2) + \sigma^{'(2)}(b^2 - c^2) + \sigma^{'(1)}(a^2 - b^2) = 0} \tag{14}$$

The first equation is therefore with $\sigma_{\theta\theta} = \sigma_{rr}$ in each of the layers

$$\frac{1}{r}\frac{\partial(r\sigma_{rr})}{\partial r} + \frac{-\sigma_{rr}}{r} + \frac{\partial\sigma_{rz}}{\partial z} = 0 \tag{15}$$

This is equal to the following after substituting for the value of σ_{rz} and application of the boundary conditions in each of the regions

$$\boxed{\begin{aligned} \sigma_{rr}^{(1)} &= \sigma^{''(1)}\left(\frac{a^2 ln(r/a)}{2} - \frac{(a^2 - r^2)}{2.2}\right) \\ \sigma_{rr}^{(2)} &= \sigma^{''(2)}\left(\frac{b^2 ln(r/b)}{2} - \frac{(b^2 - r^2)}{2.2}\right) + \left(\sigma^{''(1)}\frac{(a^2 - b^2)}{2} ln(r/b)\right) \\ \sigma_{rr}^{(3)} &= \sigma^{''(3)}\left(\frac{c^2 ln(r/c)}{2} - \frac{(c^2 - r^2)}{2.2}\right) + \left(\sigma^{''(2)}\frac{(b^2 - c^2)}{2} ln(r/c)\right) + \left(\sigma^{''(1)}\frac{(a^2 - b^2)}{2} ln(r/c)\right) \end{aligned}} \tag{16}$$

Now for the model of $\sigma'^{(i)}$ in the different layers. Making a force balance along the z direction we get the following

$$\begin{aligned} f_1(b^2-a^2) = f_2(d^2-c^2) = \sigma_{zz}^{(1)}(b^2-a^2)+ \\ \sigma_{zz}^{(2)}(c^2-b^2)+\sigma_{zz}^{(3)}(d^2-c^2) \\ \sigma_{zz}^{(3)} = f_2 + \sigma_{zz}^{(1)}\frac{(b^2-a^2)}{(c^2-d^2)} + \sigma_{zz}^{(2)}\frac{(c^2-b^2)}{(c^2-d^2)} \end{aligned} \tag{17}$$

Thus substituting the values we get the following two parameter model based on $\sigma'^{(1)}$ and $\sigma'^{(2)}$

$$\boxed{\begin{aligned} \sigma_{rr}^{(1)} &= \sigma''^{(1)}\left(\frac{a^2 ln(r/a)}{2} - \frac{(a^2-r^2)}{2.2}\right) \\ \sigma_{rr}^{(2)} &= \sigma''^{(2)}\left(\frac{b^2 ln(r/b)}{2} - \frac{(b^2-r^2)}{2.2}\right) + \sigma''^{(1)}\left(\frac{(a^2-b^2)}{2} ln(r/b)\right) \\ \sigma_{rr}^{(3)} &= \sigma''^{(3)}\left(\frac{c^2 ln(r/c)}{2} - \frac{(c^2-r^2)}{2.2}\right) + \sigma''^{(2)}\left(\frac{(b^2-a^2)}{2} ln(r/c)\right) + \sigma''^{(1)}\left(\frac{(a^2-b^2)}{2} ln(r/b)\right) \\ &= \left(\sigma''^{(1)}\frac{(b^2-a^2)}{(c^2-d^2)} + \sigma''^{(2)}\frac{(c^2-b^2)}{(c^2-d^2)}\right)\left(\frac{c^2 ln(r/c)}{2} - \frac{(c^2-r^2)}{2.2}\right) + \sigma''^{(2)}\left(\frac{(b^2-a^2)}{2} ln(r/c)\right) + \sigma''^{(1)}\left(\frac{(c\ldots}{}\right. \end{aligned}} \tag{18}$$

Now substituting the values for the $\sigma_{rz}^{(i)}$ we can see the following

$$\boxed{\begin{aligned} \sigma_{rz}^{(1)} &= \sigma'^{(1)}\frac{(a^2-r^2)}{2r} \\ \sigma_{rz}^{(2)} &= \sigma'^{(2)}\frac{(b^2-r^2)}{2r} + \sigma'^{(1)}\frac{(a^2-b^2)}{2r} \\ \sigma_{rz}^{(3)} &= \sigma'^{(3)}\frac{(c^2-r^2)}{2r} + \sigma'^{(2)}\frac{(b^2-c^2)}{2r} + \sigma'^{(1)}\frac{(a^2-b^2)}{2r} \\ &= \left(\sigma'^{(1)}\frac{(b^2-a^2)}{(c^2-d^2)} + \sigma'^{(2)}\frac{(c^2-b^2)}{(c^2-d^2)}\right)\left(\frac{(c^2-r^2)}{2r}\right) + \sigma'^{(2)}\frac{(b^2-c^2)}{2r} + \sigma'^{(1)}\frac{(a^2-b^2)}{2r} \end{aligned}} \tag{19}$$

The energy of the system that is there needs to be minimized in this case and therefore the overall energy density in the three layers are given for each of the layers

$$E = \int_0^r \int_0^L \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{zr} & \sigma_{z\theta} & \sigma_{zz} \end{pmatrix} : \begin{pmatrix} \epsilon_{rr} & \epsilon_{r\theta} & \epsilon_{rz} \\ \epsilon_{\theta r} & \epsilon_{\theta\theta} & \epsilon_{\theta z} \\ \epsilon_{zr} & \epsilon_{z\theta} & \epsilon_{zz} \end{pmatrix} rdrdz$$

The above is an example of Hadamard product. Expanding and writing it shows the following. $E = \int_0^r \int_0^L \left(\sigma_{rr}\epsilon_{rr} + \sigma_{r\theta}\epsilon_{r\theta} + \sigma_{rz}\epsilon_{rz} + \sigma_{\theta r}\epsilon_{\theta r} + \sigma_{\theta\theta}\epsilon_{\theta\theta} + \sigma_{\theta z}\epsilon_{\theta z} + \sigma_{zr}\epsilon_{zr} + \sigma_{z\theta}\epsilon_{z\theta} + \sigma_{zz}\epsilon_{zz}\right) rdrdz$

Substituting the value of the various stresses and strains in each of the three regions we get the following.

For region 1.

For region 2.

For region 3.