Linköping University

Fluid-structure interaction problems with high-performance computing

Project course
TMPM10

Group 3
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Nomenclature

- $\vec{u_s}$ - Deformation of a solid
- $\vec{u_f}$ - Velocity of a fluid
- Ω - Domain of a fluid/solid
- Γ - Boundary of a fluid/solid
- $\vec{n}$ - Normal vector of a fluid/solid
- p - Pressure
- ρ - Optimization variable (1 if solid and 0 if fluid)
- $\vec{t}$ - Traction vector
- $\vec{F}$ - Mechanical load vector
- $\vec{F}_{traction}$ - Load vector as a result of a prescribed traction
- $\vec{F}_{FSI}$ - The FSI load vector
- div(G) = $\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial G}{\partial z}$, where G is an arbitrary function
- $K_{pp}(\sigma)$ - Pressure drop penalty
- $\vec{v}$ - Test function
- V_s - Test space (such that $\forall \vec{v} \ \epsilon \ V_s$)
- x_{max}, y_{max}, z_{max} - The size of the domain
- $\vec{D}$ - Constitutive matrix
- $\vec{d}$ - Global displacement vector
- $\vec{B}_e$ - Strain-displacement matrix
- $\vec{N}_e$ - Shape function
- $\vec{C}_e$ - Connectivity matrix
- $\vec{E}$ - Elasticity matrix

Abstract

This report focuses on the phenomenon called FSI (Fluid Solid Interactions). The phenomenon itself is used in relation to MMA structural optimization to improve on a hydraulic valve's design. Prior to computing FSI data and applying optimization to the valve, the theory is broken down into a strictly fluid and solid analysis and optimization. These separate steps are done first on the simple case of a wall and then later on a geometrically simplified hydraulic valve. The two separate load cases are later superpositioned to a combined state and then finally the complete FSI problem was solved for both the geometries. The result shows that the optimization has created feasible solutions for all of the load cases for the hydraulic valve.

Contents

1 Introduction

This project will focus on designing and optimizing a hydraulic valve using topology optimization. You can read more about the theory, assumptions, and governing equations for FSI in (2) Theory. The model for this project is inspired by EPIROC, who provided the data for the valve and hydraulic oil shown in Table 1 below. A figure of the valve's application, provided by EPIROC, can be seen in Figure 1 to provide an idea of how the valve looks and works. Within, the fluid (hydraulic oil) flow direction is shown by the arrows, and the illustration shows a piston and valve combination that transforms the energy of a pressurized fluid into linear motion.

Table 1: The data of the valve and hydraulic oil provided by EPIROC.

Given data	
Height, h	50 mm
Outer diameter, d_o	42 mm
Inner diameter, d_i	35 mm
Viscosity, μ	0.032 Ns/m^2

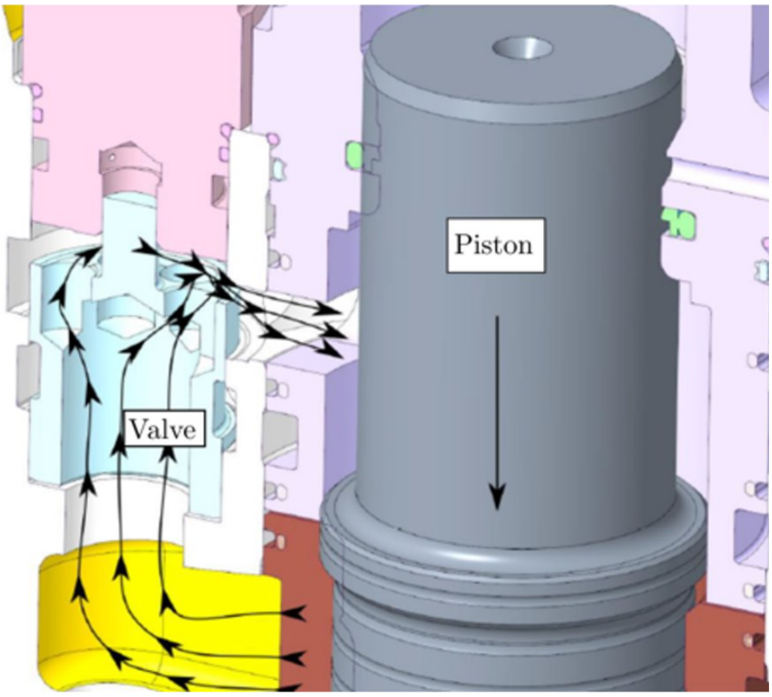

Figure 1: Shows the model from EPIROC that inspired the geometry used in the project

Fluid/structure interactions is needed in problems where a moving fluid interacting with a deformable solid, meaning that the pressure of the fluid will impact the deformation of the solid. Without FSI you would not get the coupling between the fluid and the solid, and such problems as the one in this project would not be able to be solved as well.

EPIROC has provided two objectives

- Minimize pressure drop in the hydraulic valve.
- Minimize/limit the deformation in the radial direction of the valve (y- direction in Figure 1)

This makes this a Multi-Objective Optimization problem. In order to solve this properly and meet the two objectives, the deformation demand will be rewritten as a constraint such that the deformation must be less than or equal to a given value rather than trying to minimize it. This will allow for the valve to not deform more than the allotted amount while also minimizing the pressure drop.

To begin, the group was given a framework of code programmed in C++ to be used as an frame of reference and was intended to be rewritten and complemented with their own code. Since this framework already was a working edition of performing FEA using object-oriented programming [1]. C++ is used since the code can be run using multiple cores and it is compatible with the library PETSc.
Within the code, a fluid dynamics code is used first to solve for pressure and velocity in the valve. This pressure is then used as the load for a linear elastic finite element model of the valve's wall. Once a connection was made between the fluid and solid using those two codes, the optimization code was then able to be set up and used.

1.1 Limitations

This is quite a complex problem so some simplifying assumptions was made.

- Slow stationary flow
- Small deformations, linear elastic solid material
- Linear viscous incompressible fluid
- The velocity of the fluid $\vec{u_f}$ is zero on the fluid-solid boundary Γ_{fs}
- Shear forces on Γ_{fs} are negligible

2 Theory

Fluid-Solid Interactions (FSI) is a multidisciplinary field that explores the complex interplay between fluid flow and deformable solid structures. The interaction between them can be stable or oscillatory. Neglecting the impact of oscillatory interactions can have disastrous results, particularly in the structures made of materials prone to fatigue. This interaction occurs in a wide range of natural and engineered systems, such as blood flow in arteries, the behavior of flexible structures in ocean currents, and the dynamics of bridges under wind loads. This segment delvers into the theoretical foundations of Fluid-Solid Interactions, highlighting key concepts and approaches that underpin the understanding of this intricate phenomenon.

The foundation of Fluid-Solid Interactions lie within the fundamental equations governing fluid dynamics (Navier-Stokes equations) and solid mechanics (linear elasticity or nonlinear elastodynamics). The simultaneous solution of these equations, referred to as the coupling term, forms the basis for studying how deformations in solids affect the surrounding fluid flow and, reciprocally, how fluid forces influence solid deformations.

The coupling between fluid and solid involves the iterative solution of the governing equations for both domains. Several methods exist to handle this coupling, ranging from monolithic approaches (which is the focus of this project) that solve fluid and solid equations simultaneously to partitioned approaches that alternate between solving fluid and solid problems. Understanding the strengths and limitations of each approach is crucial for accurate representation of FSI phenomena.

FSI modeling involves representing the behavior of flexible or deformable solids impacted by fluid forces. This includes capturing the dynamics of moving boundaries, such as the deformation of blood vessels or the motion of a flexible wing in an airflow. The choice of an appropriate model is highly dependent on the specific characteristics of the problem at hand. However, in this case, there will be no moving boundaries.

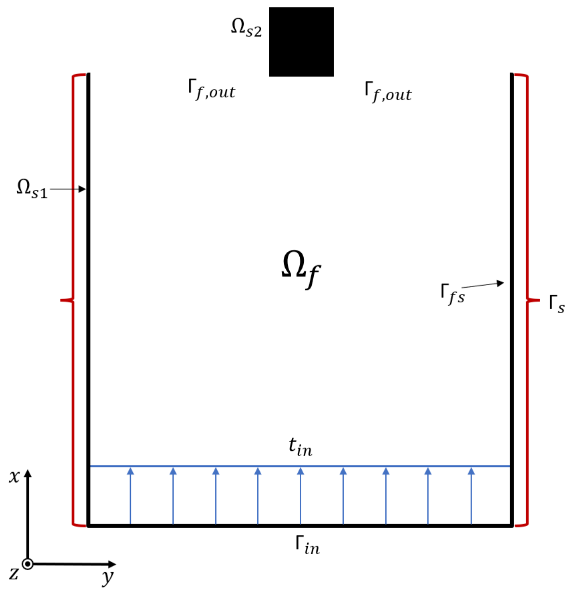

Figure 2: Shows the model for the simplified hydraulic valve

2.1 Boundary Conditions

Here the boundary conditions for the simplified valve model (see Figure 2) will be stated.

$$\vec{u}_s = \begin{bmatrix} u_s(x,y,z) \\ v_s(x,y,z) \\ w_s(x,y,z) \end{bmatrix} \tag{1}$$

$$\vec{u}_f = \begin{bmatrix} u_f(x,y,z) \\ v_f(x,y,z) \\ w_f(x,y,z) \end{bmatrix} \tag{2}$$

The boundary conditions are as follows:

- $\vec{u_f} = 0$ on Γ_{fs}
- $\vec{\sigma_f}\vec{n_f} = \vec{t}_{in}$ on Γ_{in}
- $\vec{\sigma_f}\vec{n_f} = 0$ on Γ_{out}
- $\vec{u_{s_y}} = [free\ free\ 0]^T$ on Γ_s, on the surfaces with normal in the y-direction.
- $\vec{u_{s_z}} = [free\ 0\ free]^T$ on Γ_s, on the surfaces with normal in the z-direction.
- $\vec{u_s} = \vec{0}$ in one arbitrary point on Γ_{in}, in order to avoid rigid body motion
- $\vec{u}_{s_{hat}} = [free\ 0\ 0]^T$ in Ω_{s2}

2.2 Boundary value problems (BVP)

Boundary value problems are differential equations with a set of additional constraints known as boundary conditions. For the Hydraulic valve problem the BVP could be used to describe the fluid dynamics within the valve. In Figure 2 above you can see the simplified model of the valve with all of the boundaries: Γ_{in}, Γ_{out}, Γ_s and Γ_{fs} and then the fluid domain, Ω_f The solid domain is the material between the boundaries Γ_{fs} and Γ_s and it is denoted as Ω_{s1}. On the inlet (Γ_{in}) you can see the boundary condition in the form of prescribed traction.

As it looks in the figure, Ω_{s2}, which is the domain of the solid that is going to be referred as the "hat" in the simplified valve model. The function of the hat is to steer the piston through the valve. It looks like the hat is floating freely, which it in fact does. The idea is the optimization is going to place material freely between the hat and the walls of the valve. The boundary condition placed on the hat (locked in y,z- direction and free in x- direction) was to ensure that the solid is centered in the valve, but can still move in the flow direction.

The boundary value problems (BVPs) for the solid and the fluid is stated in Equations (3) and (4) below.

$$Solid\ BVP = \begin{cases} div(\bar{E}\vec{\epsilon}(\vec{u}_s)) & in \quad \Omega_s \\ \vec{u}_{s_y} = (free\ \ free\ \ 0)^T & on \quad \Gamma_s \\ \vec{u}_{s_z} = (free\ \ 0\ \ free)^T & on \quad \Gamma_s \\ \vec{u}_{s_{hat}} = (free\ \ 0\ \ 0)^T & in \quad \Omega_{s2} \\ \vec{\sigma}_s\vec{n}_s = -p\vec{n}_s & on \quad \Gamma_{fs} \end{cases} \tag{3}$$

$$Fluid\ BVP = \begin{cases} 2\mu div(\bar{D}(\vec{u}_f)) - \frac{1}{\rho}\nabla p = 0 & in \quad \Omega_f \\ div\vec{u}_f = 0 & in \quad \Omega_f \\ \vec{u}_f = 0 & on \quad \Gamma_{fs} \\ \vec{\sigma_f}\vec{n_f} = \vec{t_{in}} & on \quad \Gamma_{in} \\ \vec{\sigma}_f\vec{n}_f = 0 & on \quad \Gamma_{out} \end{cases} \tag{4}$$

Where $\vec{u_{s_y}}$ and $\vec{u_{s_z}}$ are the displacements on the boundary on the surfaces with the y- respectively z-coordinate as normal. $\vec{u}_{s_{hat}}$ is the displacement on the so-called hat. The $div(\bar{E}\vec{\epsilon}(\vec{u}_s))$ is product of divergence of strain tensor and displacement vector of solid, which is the equation describes the linear elasticity in solid domain. The last term in Equation (3) is the coupling term. That means that the velocity of the fluid impacts the deformation of the solid. Lets look at the BVP for the fluid, the first term can be expressed as

$$2\mu div(\vec{D}(\vec{u_f})) - \frac{1}{\rho}\nabla p = 2\mu\nabla(\nabla\vec{u_f} + (\nabla\vec{u_f})^T) - \frac{1}{\rho}\nabla p \tag{5}$$

The Navier-Stokes equation for an incompressible Newtonian fluid is represented by the equation (5) in fluid mechanics. Where $\nabla(\nabla\vec{u_f} + (\nabla\vec{u_f})^T)$ is the divergence of the symmetric part of the velocity gradient. It represents the rate of change of the volume of fluid elements and is related to the spatial variation of velocity gradients. Now lets look at the whole term in Equation (5)

- The factor 2 is included for mathematical convenience in certain derivations.
- 2μ scales the viscous term, where μ is the kinematic viscosity of the fluid, which measures the fluid resistance and also ρ indicates the fluid density.
- The term $2\mu\nabla(\nabla\vec{u_f} + (\nabla\vec{u_f})^T)$ describes the viscous stresses in the fluid, accounting for the internal friction and DoF (Degrees of Freedom) within the fluid. It is often referred to as the viscous stress tensor.
- The entire term is subtracted from the pressure gradient term in the Navier-Stokes equations to account for the viscous forces resisting the fluid motion.

The second term in Equation (4) ensures that the fluid is incompressible. It states that the divergence of the velocity field is zero.

2.3 FSI Topology optimization

Topology optimization is a method that optimizes the material layout inside the given design space for a particular combination of loads, constraints, and boundary conditions to maximize the efficiency of the system. To conduct the optimization, one needs to first determine the main objective function, such as achieving a specific performance metric or decreasing weight. This objective function is meant to be either minimized or maximized according to the application. Next, the design variables which are allowed to be altered to satisfy the optimization requirement must be determined. These could be the distribution of materials or any geometric parameter's. Lastly, the problem state variables must be established. These are any variables that represent the response of the system when acted upon by loads, such as displacements and strains. The main task of this project was to provide an optimized geometry of the pressure valve based on the two objectives mentioned in the introduction. For this, in order to get better performance, it optimizes the shape and the distribution of the materials in both fluid and solid domains. Below, the equations for the topology optimization will be shown. You can look at the optimization of this project as a problem and a solution part.

- **Problem** - Ω_s and Ω_f will change during the optimization, where Ω_s and Ω_f are the solid and fluid domain. Thus, during the optimization, solid can be added or removed in order to optimize the objective function/functions.
- **Solution** - The *monolithic approach* to FSI will be used in this project, this means that the displacement, velocity and pressure exists in all of $\Omega = \Omega_s \cup \Omega_f$, but they are penalized so that displacement ≈ 0 in Ω_f and velocity ≈ 0 in Ω_s.

The design variable $\rho = \rho(x, y, z)$ is introduced, as a way for the optimization to show where there should be a solid and non-solid (void or fluid). The variable ρ can be introduced into the solid and fluid as Equations (6) - (10) shows. First let the stress $\vec{\sigma_s}$ and strain ϵ be related via

$$\vec{\sigma_s}(\vec{u_s}) = \rho\bar{E}\vec{\epsilon}(\vec{u_s}) \tag{6}$$

Which is just Hooke's law, but if you set $\rho = 0$ it means that it is a fluid. However, if you set $\rho = 1$ you will end up with the regular Hooke's law ($\sigma = E\epsilon$), which means that it is a solid. This gives the result to Equation (6).

$$\begin{cases} \vec{\sigma_s} = \vec{0} & in \quad \Omega_f \\ \vec{\sigma_s} = \bar{E}\vec{\epsilon}(\vec{u_s}) & in \quad \Omega_s \end{cases} \tag{7}$$

and consider the problem

$$\int_\Omega \rho\bar{\epsilon}(\bar{u}_s)^T\bar{E}\vec{\epsilon}(\vec{v})dv = \int_{\Gamma_{fs(\rho)}} -p\vec{n_s}^T\vec{v}dA \qquad \forall\ \vec{v}\ in\ V_s \tag{8}$$

Where V_s is the so called test space.

$$\int_\Omega \alpha(\rho)\vec{u_f}^T\vec{v}dV + \int_\Omega \vec{\epsilon}(\bar{u}_f)\bar{C}\vec{\epsilon}(\vec{v})dV - \int_\Omega pdiv\,\vec{v}dV = 0 \tag{9}$$

Where

$$\int_\Omega \alpha(\rho)\vec{u_f}^T\vec{v}dV \tag{10}$$

is called "Brikman term" which gives large resistance to flow in Ω_s. The term α is a parameter than can take form in various numerical value ($0 \leq \alpha < \infty$), such that if it is a fluid then $\alpha \to 0$ and if it is a solid then $\alpha \to \infty$. This basically means that if it is a solid α is so high that the resistance of the fluid goes to infinity, simulating a solid.

To clean up the result of the optimization, a linear filter (S) is used as a restriction method, such that $S(\rho)$ is a moving average of ρ. That basically means that the value of $S(\rho)$ at an arbitrary point in the domain is a weighted average of ρ- values in nearby points. The finer mesh, the more nearby points will be included in the average.

2.3.1 Method of Moving Asymptotes (MMA)

The algorithm used to solve the optimization problem is called **MMA** (Method of Moving Asymptotes). The theory behind MMA will be briefly described below.

The MMA is an optimization algorithm commonly used for solving engineering design problems, particularly in structural and mechanical design. The main objective of MMA is to efficiently handle optimization problems with nonlinear constraints, where the objective is to minimize or maximize a certain function while satisfying a set of constraints. Equation (11) below shows a MMA subproblem at iteration k.

$$(P)_{nf}^{M,k} = \begin{cases} min\ g_0^{M,k}(\vec{x}) \\ s.t. \begin{cases} g_i^k(\vec{x}) \leq 0\ \ i = 1, ..., m \\ x_j^{min} \leq x_j \leq x_j^{max}\ \ j = 1, ..., n \end{cases} \end{cases} \tag{11}$$

Where "M,k" stands for "MMA, k:th iteration". You can also see that the approximated objective function is denoted as "g_0", the constraints are denoted as g_i where $i = 1, ..., m$ and the lower and upper bounds for the box constraints are x_j^{min}, x_j^{max} where $j = 1, ..., n$.

Basic idea of MMA

- **Approximating the constraints** - MMA works by approximating the non-linear constraints $g_j(\vec{x})$ using a set of linear constraints. These linear approximations are updated iteratively.
- **Asymptotes** - The term "Moving Asymptotes" refers to the linear approximations, which are constructed as a set of asymptotes moving through the design space. These asymptotes are adapted during the optimization process.
- **Optimization variables** - MMA introduces additional variables, known as the "slack variables," to handle the constraints. These slack variables represent the distance between the current design point and the constraint boundaries.

Equation (11) is solved using an iterative process. Which is described below.

- **Update of Asymptotes** - The algorithm starts by defining initial asymptotes that intersect the current design point. These asymptotes are then updated iteratively based on the changes in the objective function and constraints.
- **Optimization step** - MMA performs an optimization step to update the design variables, seeking to minimize the objective function while satisfying the constraints. This is typically done using an optimization algorithm, such as a quasi-Newton method.
- **Update of Asymptotes and Slack Variables** - The asymptotes are updated based on the changes in the design variables, and the slack variables are adjusted to ensure that the constraints are satisfied.
- **Convergence** - The process repeats until convergence is achieved, meaning that the design variables and asymptotes no longer change significantly, and the constraints are satisfied within acceptable tolerances.

2.3.2 SIMP penalization

Because some of the values on the optimization variable ρ can take on values that are between 0 and 1, a penalization has to be introduced. What this means is that if $\rho = 1$ it is a solid while if $\rho = 0$ it is a fluid, so the intermediate values is hard to physically interpret and not wanted. For this reason, a method known as SIMP penalization is used. This penalization, represented as q, is introduced as follows:

$$\vec{E} = E_{min} + \tilde{\rho}_e(\vec{\rho})^q (E - E_{min}) \tag{12}$$

Such that $q > 1$. This basically means that the stiffness of the structure is penalized for intermediate ρ values and therefore less efficient to use, resulting in driving the optimization to choose values on ρ closer to 0 or 1. Where the E_{min} is a scalar such that $E_{min} << E$ to avoid singularity in the stiffness matrix when $\rho \to 0$.

2.3.3 Sensitivity analysis

Sensitivity analysis relies on derivatives and gradients. In the context of MMA, sensitivity information is crucial for updating the asymptotes effectively. The optimization algorithm uses sensitivity information to adjust the asymptotes, ensuring accurate linear approximations of the constraints. Sensitivity analysis enhances the understanding of the local and global characteristics of the optimization problem, guides the optimization algorithm, and helps conclude informed decisions/designs during the iterative optimization process. Proper handling of sensitivities contributes to higher efficiency and effectiveness of optimization algorithms like MMA.

These measures quantify how the objective function and constraints change concerning variations in the design variables. For example, the sensitivity of the objective function with respect to a particular variable x_j is denoted as $\frac{\partial g_0}{\partial x_j}$, and the sensitivity of a constraint g_i to x_j is denoted as $\frac{\partial g_i}{\partial x_j}$

In optimization algorithms like MMA, gradients are often used to determine the direction and magnitude of the search step in the design space. The gradient of the objective function guides the search for the minimum or maximum, and the gradients of the constraints help ensure feasibility.

To perform the derivatives of the constraint, the constraint needs to be defined.

$$g_i = \vec{F}_{FSI}^T(\rho, u_f(\rho))\vec{u_s}(\rho) - C_1 \tag{13}$$

The final expression for the sensitivity analysis for the prescribed traction is:

$$\frac{\partial g_i}{\partial \rho} = 2\frac{\partial \vec{F_{FSI}^T}}{\partial \rho}\vec{u_s} - \vec{u_s}^T\frac{\partial \vec{K_s}}{\partial \rho}\vec{u_s} + \vec{\lambda}^T\frac{\partial \vec{K_f}}{\partial \rho}\vec{u_f} \tag{14}$$

Where $\vec{\lambda}$ comes from solving the adjoint system

$$\vec{K_f}\vec{\lambda} = -2\frac{\partial \vec{F_{FSI}}}{\partial \vec{u_f}}\vec{u_s} \tag{15}$$

However, every time an optimization iteration is made, $\vec{\lambda}$ has to be solved again. This comes with a downside as solving for $\vec{\lambda}$ takes the same computational power as one whole fluid analysis. The full derivation of the sensitivity can be found in Appendix 7.

2.4 FE discretisation

The FE discretisation is displayed below and the variable ρ is the variable from the topology optimization. The elements used are Hex8.

The solid problem is as follows

$$\bar{K}_s(\vec{\rho})\vec{d} = \vec{f}(\vec{p},\vec{\rho}) \tag{16}$$

where $\vec{d}$ is the global displacement vector

$$\bar{K}_s(\vec{\rho}) = \sum_{e=1}^{m} \bar{C_e}^T \tilde{\rho}_e(\vec{\rho})^q \int_{\Omega_e} \bar{B_e}^T (E_{min} + \tilde{\rho}_e(\vec{\rho})^q (E - E_{min}))\vec{D}\bar{B}_e dV \bar{C}_e \tag{17}$$

Where $\tilde{\rho}_e$ is the filtered design and q is the SIMP variable. Which is a constant (like q = 2 or q = 3) that penalizes intermediate ρ values (like $\rho = \frac{1}{2}$), in order to make the optimization more inclined to choose ρ values that are closer to 0 and 1. The SIMP variable is spoken about in more detail in section 2.3.2.

and

$$\vec{f}(\vec{p},\vec{\rho}) = \sum_{e=1}^{m} \bar{C_e}^T \frac{1}{2} \int_{\Gamma_e \cap \Gamma_{fs}(\vec{\rho})} -p_e \bar{N_e}^T \vec{n_s} dA \tag{18}$$

Where $\vec{N_e}$ is the element shape function and $\Gamma_e \cap \Gamma_{fs}(\vec{\rho})$ is the intersecting boundary between the element and the fluid/solid boundary, i.e. the boundary where the two elements have different ρ values.

The fluid problem is

$$\begin{bmatrix} \bar{M}(\bar{\rho}) & \bar{G} \\ \bar{G}^T & \bar{K}_{pp}(\delta) \end{bmatrix} \begin{bmatrix} \vec{u_f} \\ \vec{p} \end{bmatrix} = \begin{bmatrix} \vec{F} \\ \vec{0} \end{bmatrix} \tag{19}$$

The full discretization of the solid and fluid problem can be found in the Appendix.

In (19), $\vec{K_{pp}}(\delta)$ is a pressure jump penalty and $\vec{F}$ is the load vector. Because Equation (19) is not enough to determine the pressure uniquely the pressure jump term is necessary to "smooth" out the pressure over all the elements. This is done by penalizing the pressure jump and leads to avoiding instabilities.

3 Method

The generalized steps taken in order to achieve the final results are visualized in the form of flow chart shown as Figure 3.

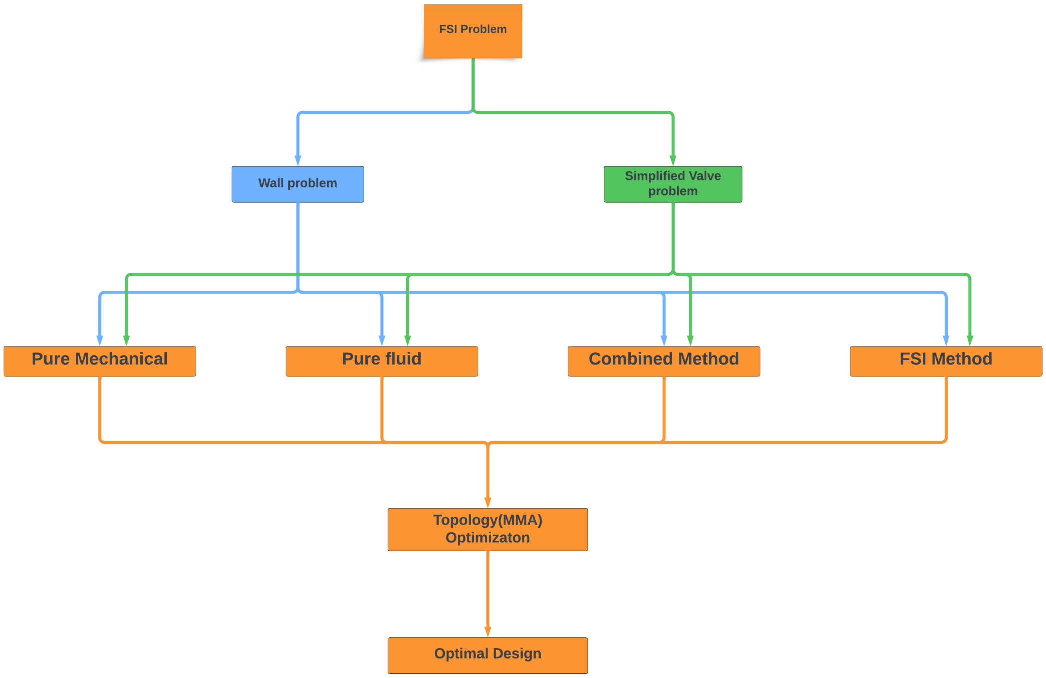

Figure 3: The flow chart of the generalized steps used for the project

3.1 Optimization formulations

Here the optimization formulations will be stated for both the analyzed cases(wall and simplified valve). The approach to the two problems was to solve them in increasingly difficult steps. Directly below, the four separate steps taken are very briefly explained prior to the deeper explanations.

- **Pure mechanical** - Replacing the FSI load vector with a purely mechanical load to see if the deformation and optimization look reasonable.
- **Pure fluid** - The wall problem was solved for *both* a prescribed traction and velocity profile while for the simplified valve model, only the prescribed traction. Both were used to see if the pressure field and optimization look reasonable.
- **Combined** - Replacing the FSI load vector with a combination of the pure mechanical load and the prescribed traction to see if the deformation and optimized version look reasonable.
- **FSI** - Using the FSI load vector to solve the two problems to get the final results on the deformation, pressure drop and then optimization

The optimization formulations for the **pure mechanical**, **pure fluid** and **combined** are the same for the wall and valve aside from a minor change in volume allowence.

3.1.1 Pure mechanical

The optimization formulation for the pure mechanical case is.

$$\begin{cases} \min \vec{F}^T \vec{u_s}(\rho), \;\; with\, respect\, to\, \rho \\ s.t \begin{cases} 0 \leq \rho \leq 1 \\ V_{max}(\rho) \leq C_0 V_{tot} \end{cases} \end{cases} \tag{20}$$

Where "s.t." stands for "subject to", which is how the constraints are displayed in the optimization formulation. In (20), F is the applied mechanical load and V_{max}, V_{tot} are the maximum and total volumes where C_0 is a constant which follows $0 \leq C_0 \leq 1$ and changes value depending on whether the wall or simplified valve problem is being solved.

Problem (20) states that the compliance is minimized (i.e. minimizing the displacement in the load direction), with a constraint that material can only occupy a given percentage of the total volume of the domain.

3.1.2 Pure fluid

The optimization formulation for the pure fluid states.

$$\begin{cases} \max \vec{F}^T_{traction} \vec{u_f}(\rho), \;\; with\, respect\, to\, \rho \\ s.t \begin{cases} 0 \leq \rho \leq 1 \\ V_{max}(\rho) \geq C_1 V_{tot} \end{cases} \end{cases} \tag{21}$$

Where C_1 is a constant that works in the same way as C_0

It might seem strange that the constraint states that the allowed volume should be greater or equal to the constant times the total volume. The reason for this is if you would have had the $V_{max}(\rho) \leq C_1 V_{tot}$, then the trivial solution would just be to remove all the material. This means that the optimization has to build **at least** some material rather than none at all.

3.1.3 Combined

The optimization for the combined solution states

$$\begin{cases} \max \vec{F}^T_{traction}\vec{u_f}(\rho), \;\; with\, respect\, to\, \rho \\ s.t \begin{cases} 0 \leq \rho \leq 1 \\ \vec{F}^T\vec{u_s}(\rho) \leq C_2 \end{cases} \end{cases} \tag{22}$$

Where C_2 is a constant such that it changes value depending on whether the wall or simplified valve problem is being solved. It is the value of the allowed compliance in the solid.

3.1.4 Wall problem

The FSI optimization formulation for the displacement of the solid in the direction of the force (also called the compliance). This compliance, made of the FSI force vector and the deformation of the solid, is the objective function and as such is minimized. Within this optimization, two constraints are used. ρ is used in an inequality to ensure it lies within the desired values of 1 and 0.

$$\begin{cases} \min \vec{F}^T_{FSI}(\rho, u_f(\rho))\vec{u_s}(\rho), \;\; with\, respect\, to\, \rho \\ s.t \begin{cases} 0 \leq \rho \leq 1 \\ V_{max}(\rho) \leq C_3 V_{tot} \end{cases} \end{cases} \tag{23}$$

Where V_{tot} is the total volume of the domain and C_3 is a constant such that $0 \leq C_3 \leq 1$. This means that the allowed volume that the solid can take up is limited to the volume of the domain.

3.1.5 Simplified valve problem

The FSI optimization formulation for the valve is stated as follows

$$\begin{cases} \max \vec{F}^T_{traction}\vec{u_f}(\rho), \;\; with\, respect\, to\, \rho \\ s.t \begin{cases} 0 \leq \rho \leq 1 \\ \vec{F}^T_{FSI}(\rho, u_f(\rho))\vec{u_s}(\rho) \leq C_4 \end{cases} \end{cases} \tag{24}$$

Where it basically minimizes the pressure drop in the valve. Here C_4 is a constant value given for the maximum allowed mechanical compliance.

3.1.6 Optimization variable (ρ)

You might have noticed that the variable ρ is allowed to be $0 \leq \rho \leq 1$. This may come as a surprise as it is said that ρ should either be 0 or 1 for topology optimization.

If one insists that ρ can only be 0 or 1 at each point, a highly non-linear integer problem arises which leads the optimization to have a non-linear function. Such problems are very difficult to solve, so the requirement that ρ can only be 0 or 1 is relaxed. Instead, all values in between are allowed, but penalties (like SIMP penalization coefficient q) are imposed on such intermediate values. This approach aims to obtain a solution where ρ is reasonably close to being 0 or 1 everywhere.

3.2 Wall problem

A staple in FSI is the wall problem. It is the problem formulation used to describe the FSI formulation. In this project, the wall problem will be solved for two different kind of boundary conditions on the inlet. For both the problems that will be presented, the solution path will be the same. With use of the wall problem, the group is able to move towards the end goal of the project with greater confidence in the code written.

3.2.1 Prescribed velocity profile

The model for the wall problem can be seen in Figure 4 below.

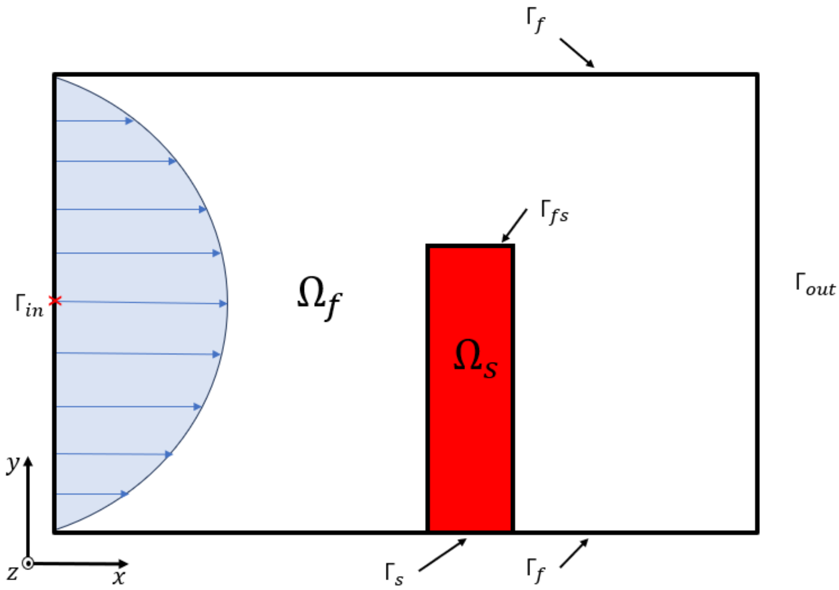

Figure 4: The wall problem for a prescribed velocity profile

Where in the figure the boundaries for the solid and the fluid are specified and where Γ_{fs} as before is the boundary where the solid meets the fluid.

The boundary conditions for the problem is as follows:

- $\vec{u_f} = \vec{0}$ velocity of the fluid on Fluid domain (Γ_f)
- $\vec{u_f} = \vec{0}$ velocity of the fluid on Fluid-solid domain (Γ_{fs})
- $\vec{u_s} = \vec{0}$ displacement on solid domain (Γ_s)
- $\vec{u_f} = \vec{u_{f0}}$ velocity of fluid at inlet (Γ_{in})
- $\vec{\sigma_f}\vec{n_f} = 0$ on Γ_{out}

3.2.2 Prescribed traction

For the problem at hand in this project, the hydraulic valve, a prescribed pressure is used instead of a prescribed velocity profile. This is due to the fact that adding material at the inlet would disrupt the flow when the prescribed velocity was applied while adding material at the pressure prescribed would allow for the addition of material. Therefore, the wall problem will be solved for the same prescribed pressure as the valve will. The model for that problem can be seen in Figure 5 below.

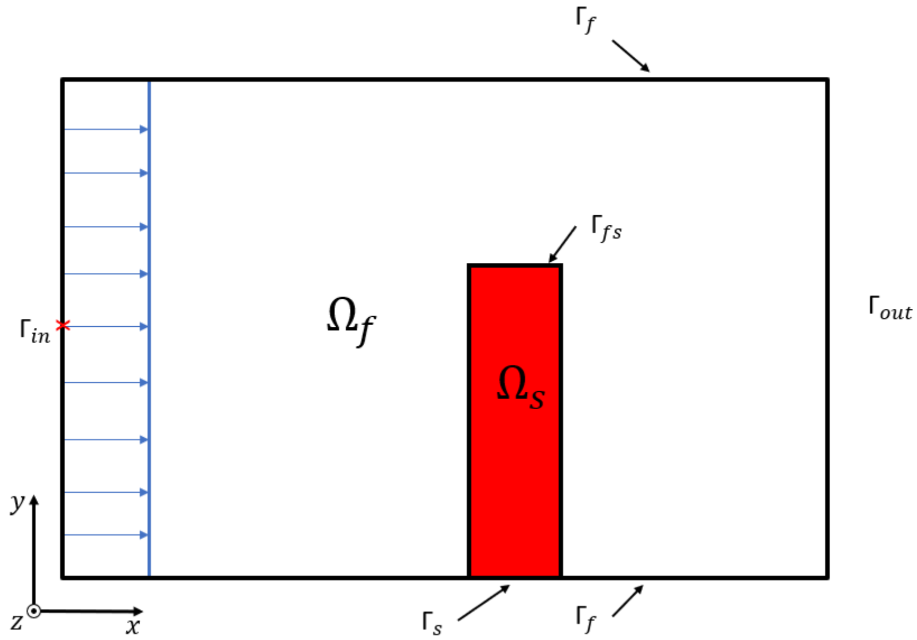

Figure 5: Shows the model of the basic wall problem for a prescribed pressure

Where the boundary conditions are the same as for Figure 4 except on Γ_{in} where now $\sigma_f n_f = t_{in}$ on Γ_{in}

3.3 Simplified valve model

Using a simplified model of the valve, saves on optimization time and lets the project focus more on the FSI phenomenon rather than the actual geometry of the valve.

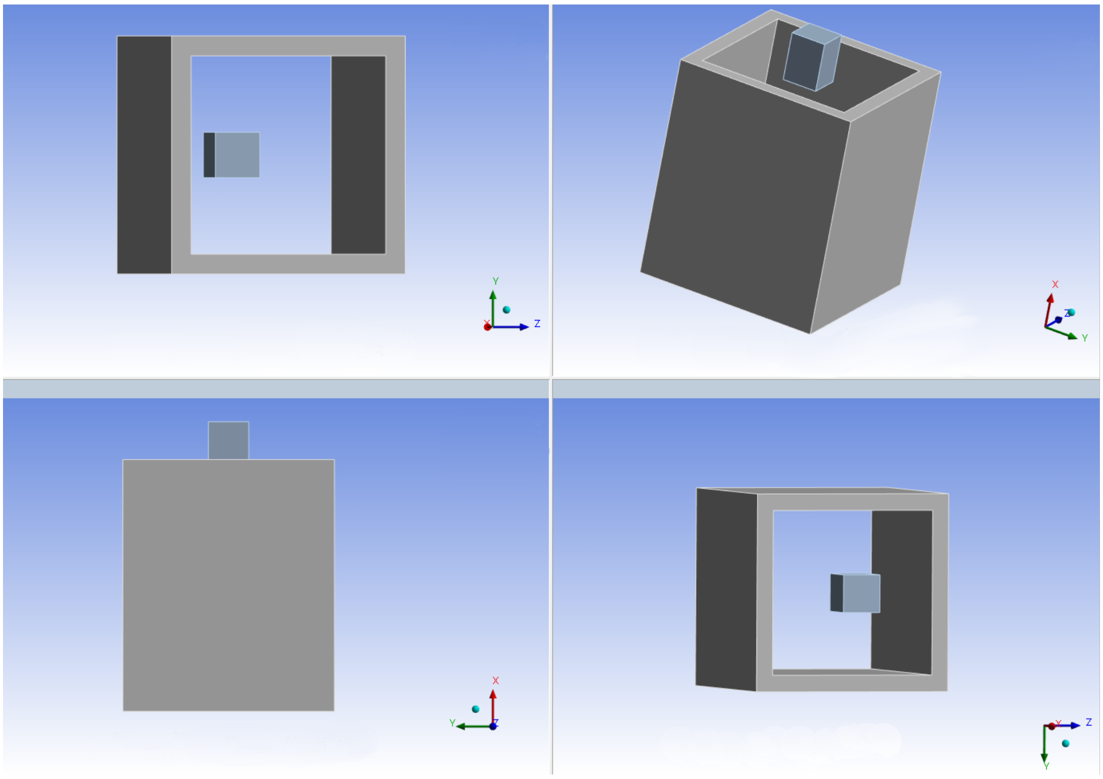

Figure 6: A simplified model of the valve.

Where the little floating solid, Ω_{s2} in Figure 2 and also referred to as the hat, has the function of guiding the piston through the hydraulic valve correctly.

The boundary conditions for the problem is as follows

- $\vec{u_f} = \vec{0}$ on the inside surface area of the surrounding walls.
- $\vec{u_{s_y}} = [free\ \ free\ \ 0]^T$ on the walls with normal in the Y-direction.
- $\vec{u_{s_z}} = [free\ \ 0\ \ free]^T$ on the walls with normal in the Z-direction.
- $\vec{\sigma_f}\vec{n_f} = \vec{t_{in}}$ in the inlet, the inlet is shown in the top left picture.
- $\vec{\sigma_f}\vec{n_f} = 0$ on the outlet, it is shown on the bottom right figure.
- $\vec{u_s} = \vec{0}$ on one arbitrary point on the inlet, in order to avoid RBM (rigid body motion)
- $\vec{u}_{s_{hat}} = [free\ \ 0\ \ 0]^T$ on the solid in the outlet

3.4 The implementation of the method within the code

In this section, the code will be explained in words in order to provide more understanding of the process. The code was written using C++.

- This project is based upon the library called PETSc [2], which stands for: *Portable, Extensible Toolkit for Scientific Computation*

- To begin with a bunch of codes were collected from [1] which was the outline of the analysis

- A library that was capable of performing matrix and vector operations was chosen, namely: *Eigen* [3].

- The mesh consist of Hex8 elements that are created using the *TopOpt* class, which also sets ρ-values and pressure to all elements inside the domain.

- The classes *LinearElasitcity* as well as *MixedStokes* are used to calculate the solid and the fluid parts which are coupled with the optimization object.

3.4.1 Implementation of FSI-traction

- After the elements and nodes were found, the Gauss quadrature points with corresponding weights were extracted from a given table ([4]). Since Hex8 elements were used, the double integral on each face is calculated on a rectangular and flat surface and is evaluated using four Gauss quadrature points.

- A loop over all the elements in the mesh was created, followed by a loop for all the faces to determine whether the face is internal or not (internal meaning that the element shares a face with the neighbouring element).

- Now that the internal faces has been found, the normal vectors to all the internal faces were set.

- After the internal faces with normal vectors has been found the corresponding shape functions (51) were set up and calculated from the Gauss quadrature points and put into one Matrix per Gauss point where each of them has the dimension 3x24.

- The contribution from each internal face was calculated by summing the local force contributions from its four Gauss points as: (q denotes - Gauss point and i the current face number of the current element - e, A_i - area of the current face, w - weight for the Gauss points).

$$\vec{f_{e,i}} = \sum_{q=1}^{4} -P(e)\vec{N}_q^T \vec{n}_i(\rho(e) - \rho(e-1))\frac{A_i ww}{8} \tag{25}$$

- Now, it was time to initiate the assembly of the global load vector. Each element and face had to couple the local Degrees of Freedom (DoFs) with the global ones, ensuring the entry of the local load vector into the correct positions of the global load vector.
- After the assembly of the global load vector, it was time to start solving the fluid problem to retrieve the pressure. From the pressure calculate the traction on the solid as a result of the fluid interacting with the solid.
- The next step was to insert the traction into the FE-solver for the solid. In both the solid and fluid solver the boundary conditions have been implemented.

3.4.2 Main code

In order to get an overview of the calculations inside of the main file a flow chart was constructed. There were no significant differences in the main script between the wall and the valve, but note that this section emphasizes on the valve. The flow chart is presented in Figure 7 and effectively demonstrates how the key components in the code interact with each other.

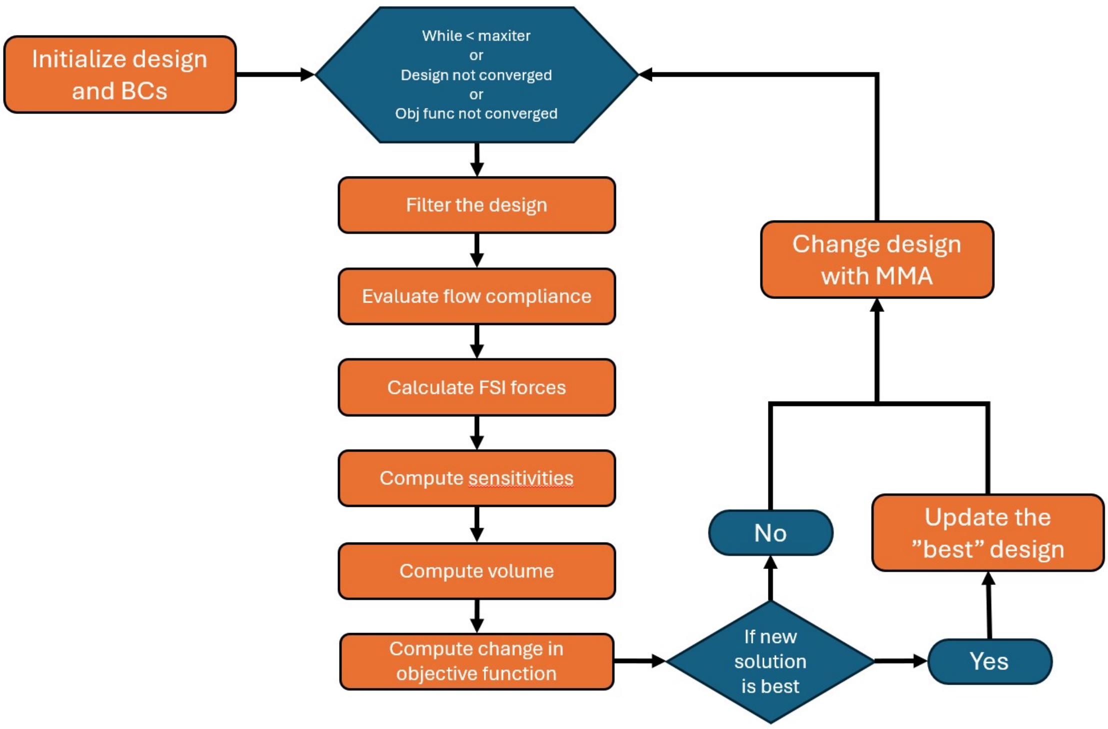

Figure 7: Flowchart over main code

- The code initiates by establishing the design and boundary conditions in alignment with Figure 2 for the valve. Placement of solid elements are defined, displacements are locked, no-slip walls are set and initial loads are prescribed.
- After that, the main optimization loop is initialized which runs until either the set maximum number of iterations has been reached, or if the objective function or design has converged.
- The first step inside of the main loop is where the design variables get filtered by a SIMP-filter.
- The flow compliance gets evaluated which also happens to be the objective function for the valve in accord with equation 24.
- The FSI forces are calculated in accord with Section 3.4.
- *Compute sensitivities* assembles and evaluates the equations used for the sensitivity analysis according to equation 14.
- *Compute volume* determines the amount of the domain's volume occupied by the new current design. As the number of solid elements within the domain increases, so does the overall volume.
- The check for a change in the objective function value verifies if the objective function has converged, which is one of the stopping criteria within the while loop.
- The script consistently monitors the current best design based on the objective function value and ensures that the design complies with the constraints. Should a new solution emerge with an improved objective function value while still meeting the constraints, it becomes the new "best" design.
- Afterwards a new possible design is evaluated. This continues as mentioned above until one of the convergence criteria has been met or the maximum number of iterations is reached.

4 Results

The results are divided into two separate cases, as discussed in the method section.

- The wall problem, which is a staple in displaying the FSI problem, though it is more of a ”toy” problem. It is simply used to understand the concept and implementation of FSI and to increase the knowledge of the phenomenon so that more complex problems (such as the valve) could be more easily understood and analyzed.

- The simplified valve problem, which was a simplified geometry of the valve model provided by EPIROC.

All of the figures where visualized in the program Paraview [5].

The threshold for ρ was chosen to be $0.6 \leq \rho \leq 1$ for both the wall and simplified valve problem for *all* load cases in order to find the optimal design.

4.1 Wall problem

For the wall problem, the domain and the wall within it were set up in the code. The model can be seen in Figure 8 below. The code is set up in a way that the domain size has been specified by the user and the coordinates of the wall have been based on fractions of the domain size.

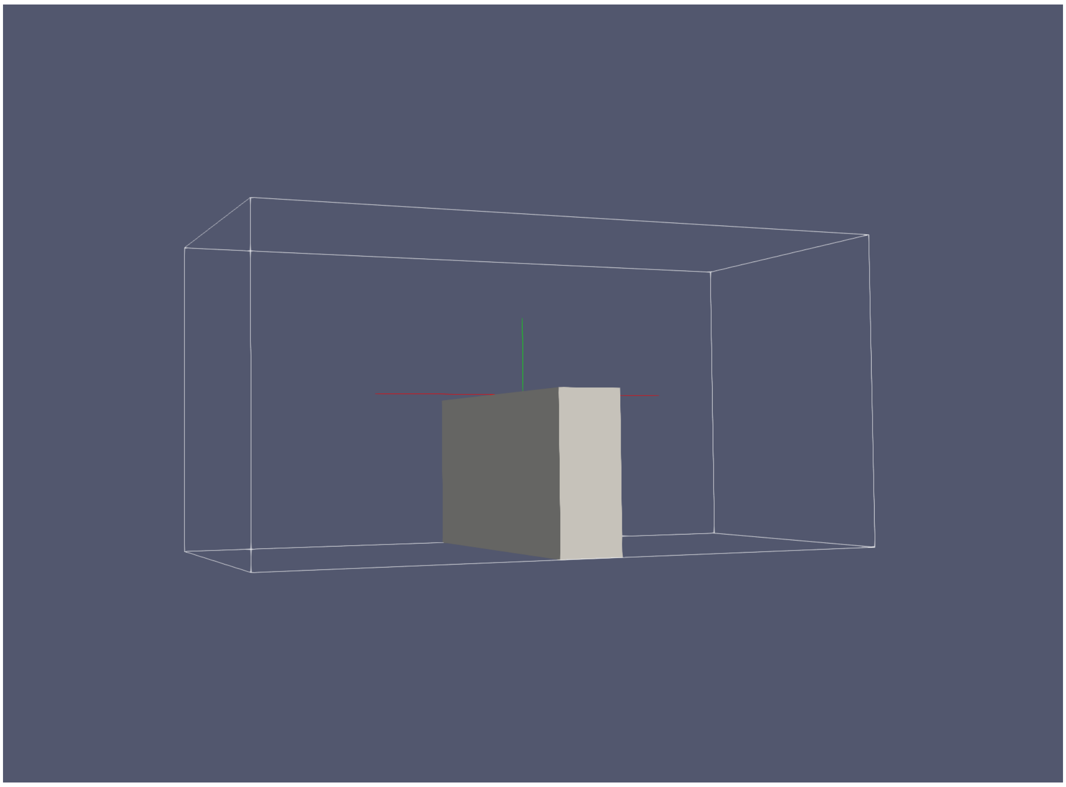

Figure 8: Shows the model of the wall set up in the code.

4.1.1 Pure Mechanical solution

The constant C_0 in Equation (20) is $C_0 = 0.1$, which means that the solid material is allowed to occupy 10% of the domain.

The solution for the pure Mechanical problem can be seen in Figure 9 below. This illustrates the deformation of the optimized wall under mechanical load, while Figure 10 below shows the geometry of the optimized model of the wall.

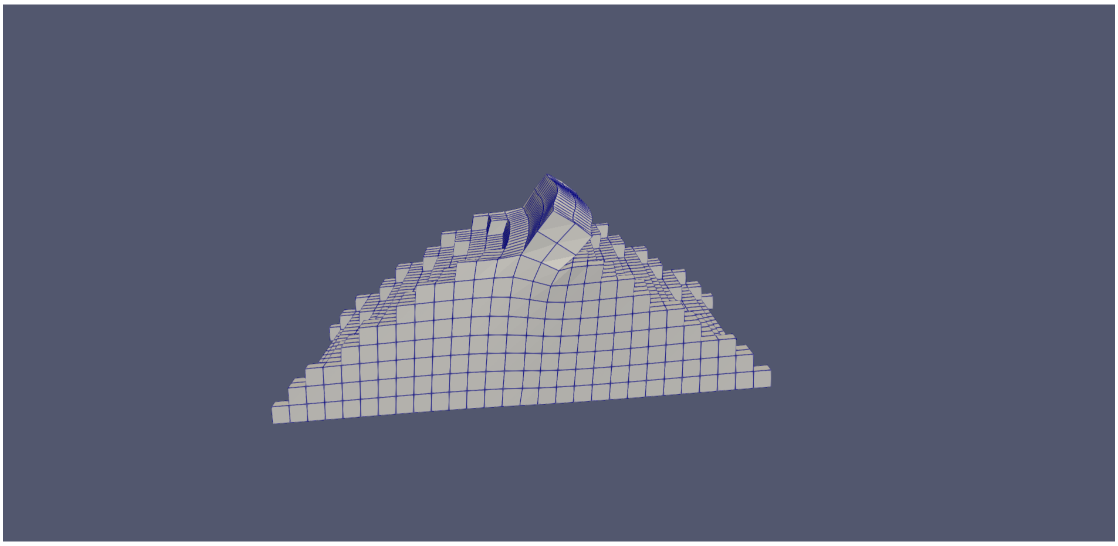

Figure 9: Shows the deformation of the wall **for the pure mechanical case**

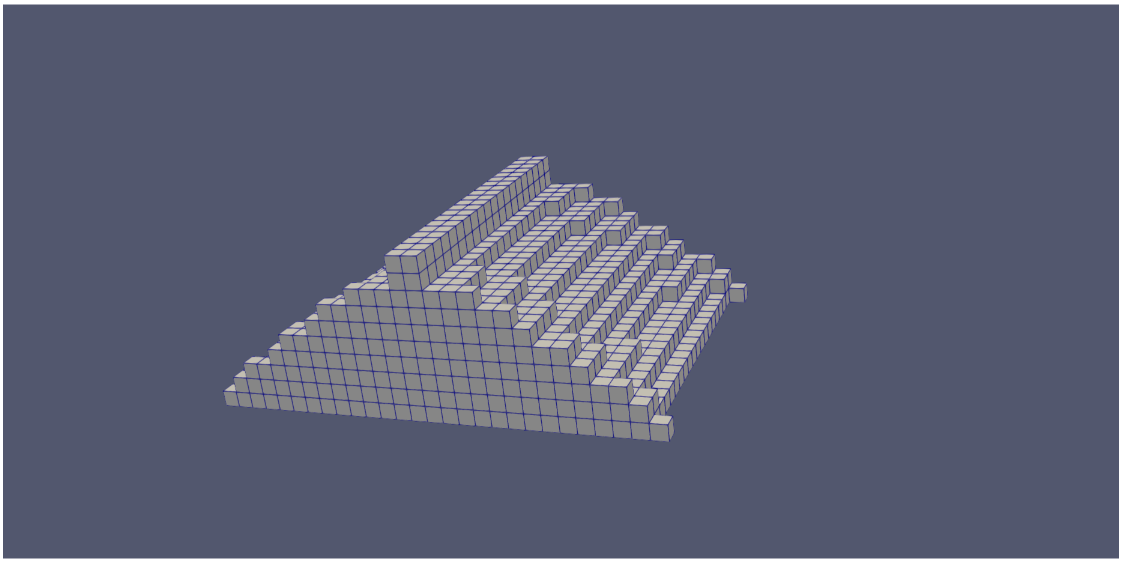

Figure 10: Shows the optimized model of the wall **for the pure mechanical case**

You can see that the optimization had placed materials in a staircase shape on both sides of the wall. This is the final optimal structure meant to minimize the deformation of the wall under a purely mechanical load.

4.1.2 Pure fluid solution

Where the constant C_1 has been set to $C_1 = 0.1$, which means that the optimization has to put material in at least 10% of the domain.

Prescribed velocity profile - The result for the pure fluid solution (for prescribed velocity profile) is shown in Figures 11 and 12 below.

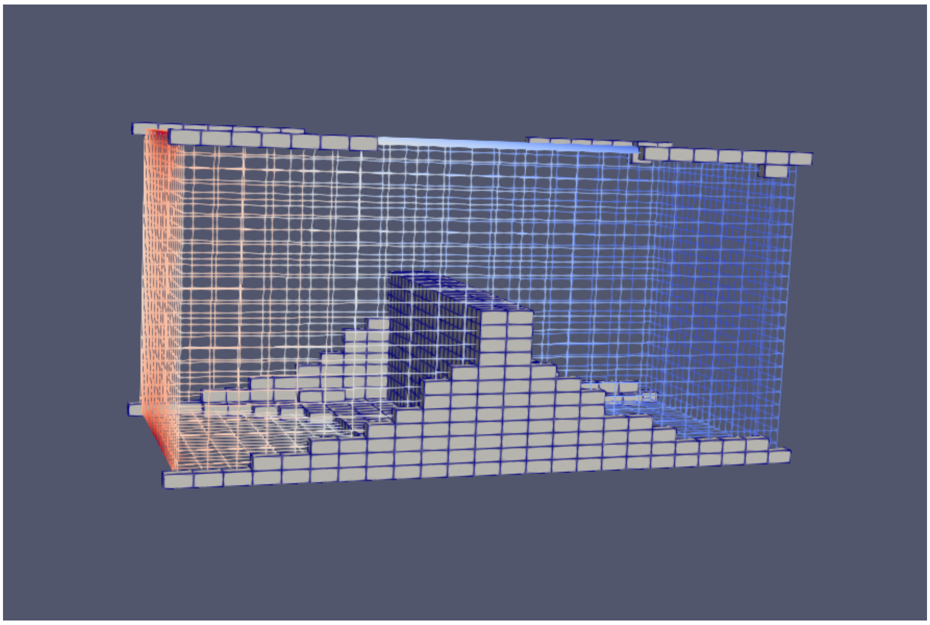

Figure 11: Shows the pressure field **for the pure fluid case with prescribed velocity profile**

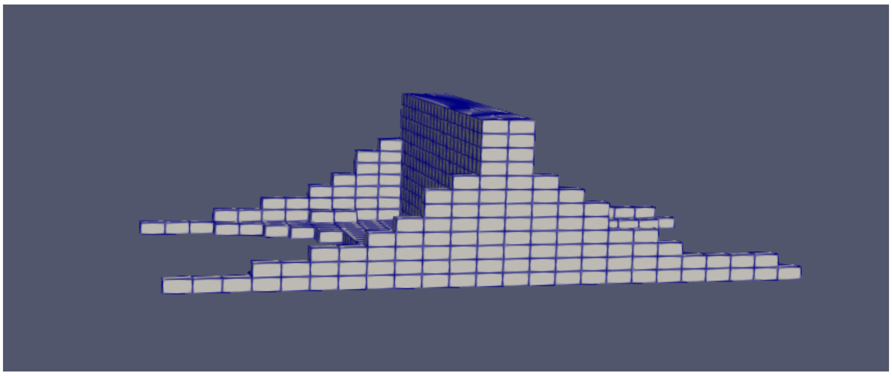

Figure 12: Shows the optimization **for the pure fluid case with prescribed velocity profile**

Prescribed traction - The result for the pure fluid solution (for prescribed traction) is shown in Figures 13 and 14 below.

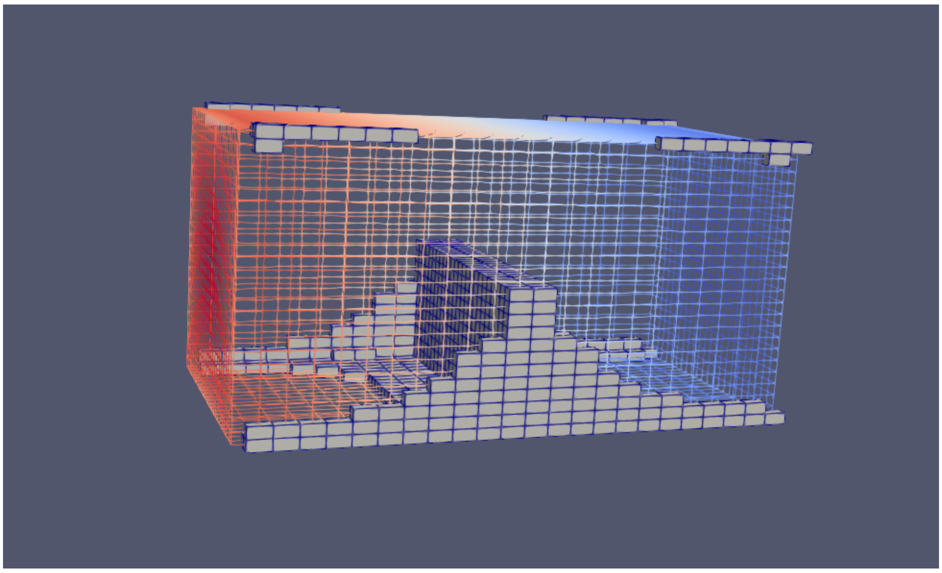

Figure 13: Shows the pressure field **for the pure fluid case with prescribed traction**

By looking at the pressure you can see that it is higher on the inlet- side and lower on the outlet- side, again, as it should be. Comparing Figure 13 to 11, you can see that the models are nearly identical with only a slight difference in the design.

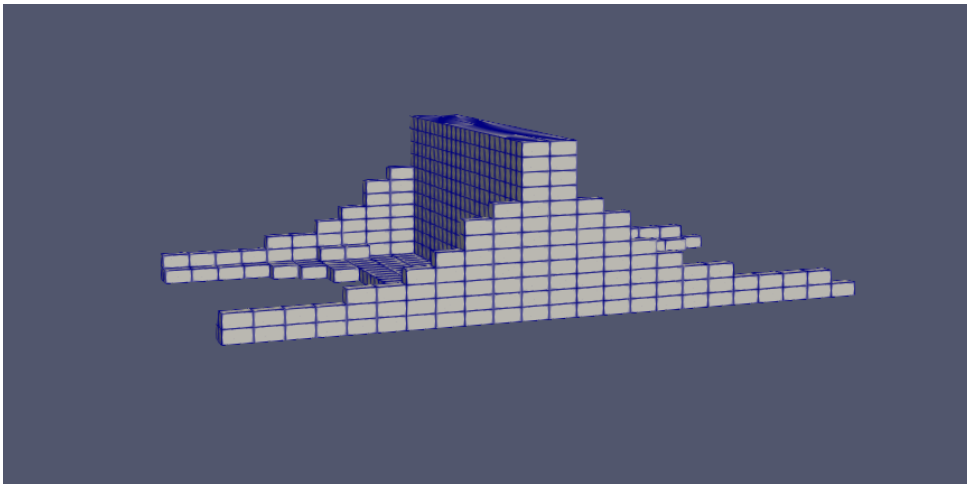

Figure 14: Shows the optimization **for the pure fluid case with prescribed traction**

4.1.3 Combined solution

The constant C_2 in Equation (22) is set to $C_2 = 10^6[Nm]$, which is the set value for the compliance that the valve has to fulfill.

The solution for the combined solution is displayed in Figures 15, 16 and 17 below. There you can see the deformation on the wall for the combined case and the pressure field.

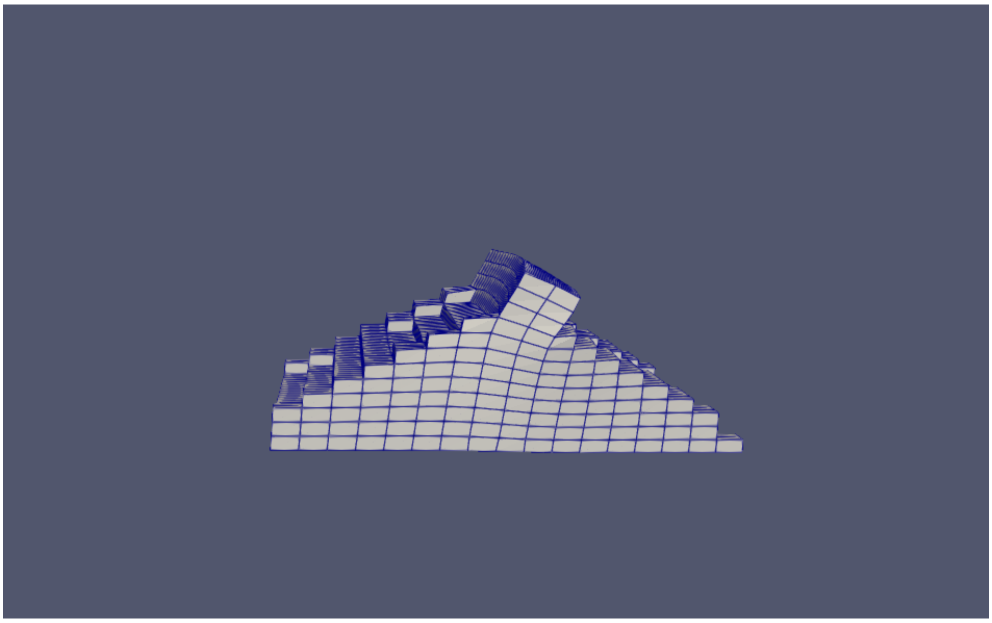

Figure 15: Shows the deformation **for the combined case**

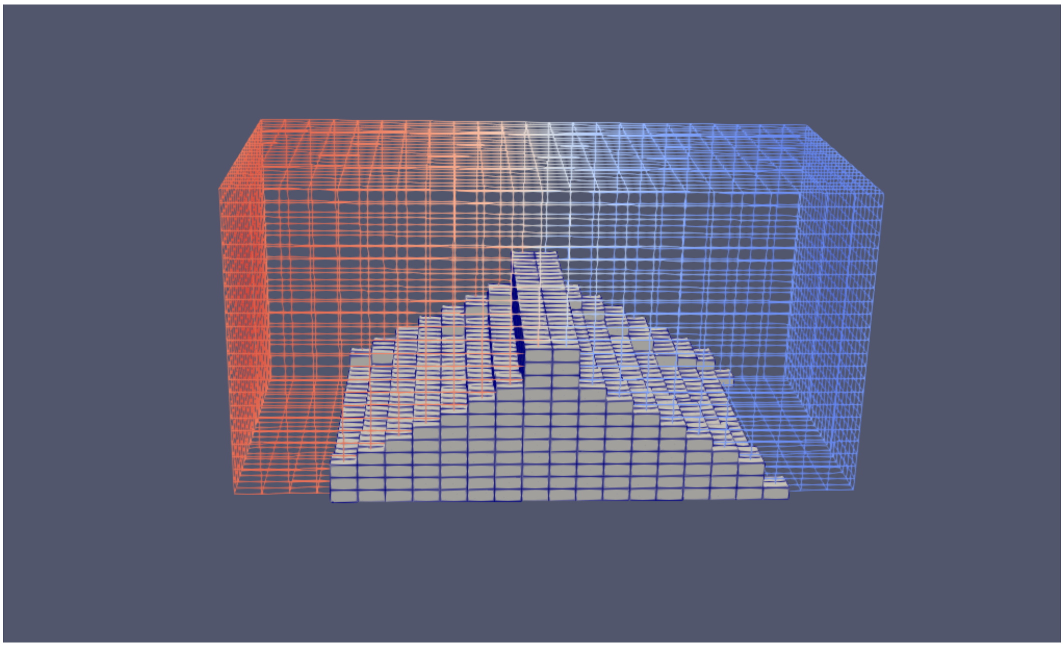

Figure 16: Shows the pressure field **for the combined case**

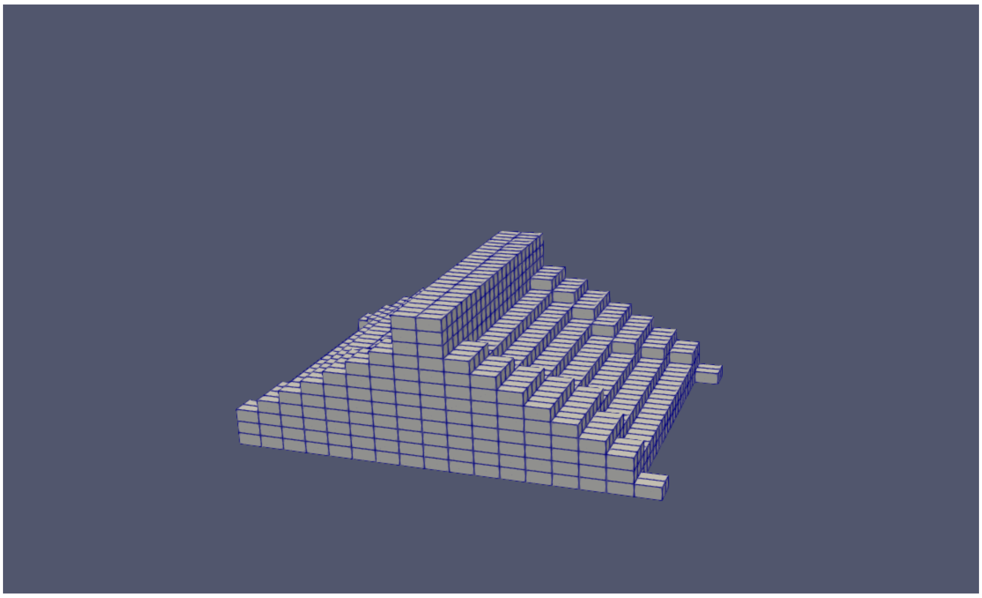

Figure 17: Shows the optimization **for the combined case**

Above you can see the optimized version of the wall for the combined load case. Note that the only difference between the combined case and the FSI solution is that in the combined case a mechanical force vector is acting on the wall, while in the FSI solution, it is the FSI load vector that is acting on the wall.

4.1.4 FSI solution

The wall for the FSI solution was first solved with a prescribed traction. The pressure field and deformation can be seen in Figures 18 - 20 below. Where the constant C_3 in Equation (23) is $C_3 = 0.3$, which means that the solid material is allowed to occupy 30% of the domain.

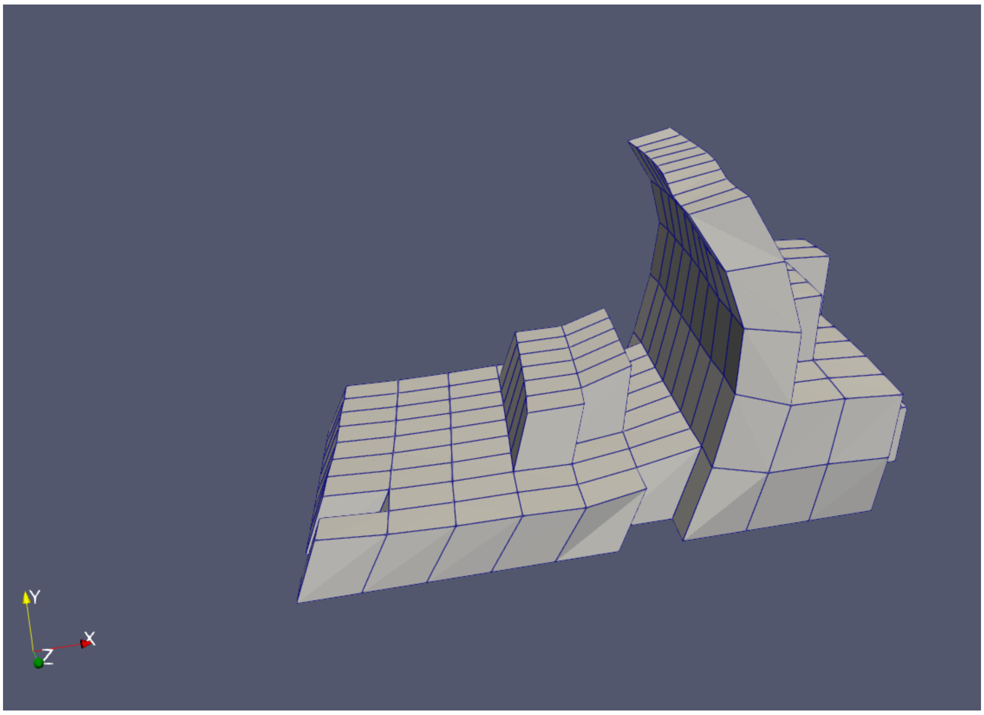

Figure 18: Shows the deformation **for the FSI case**

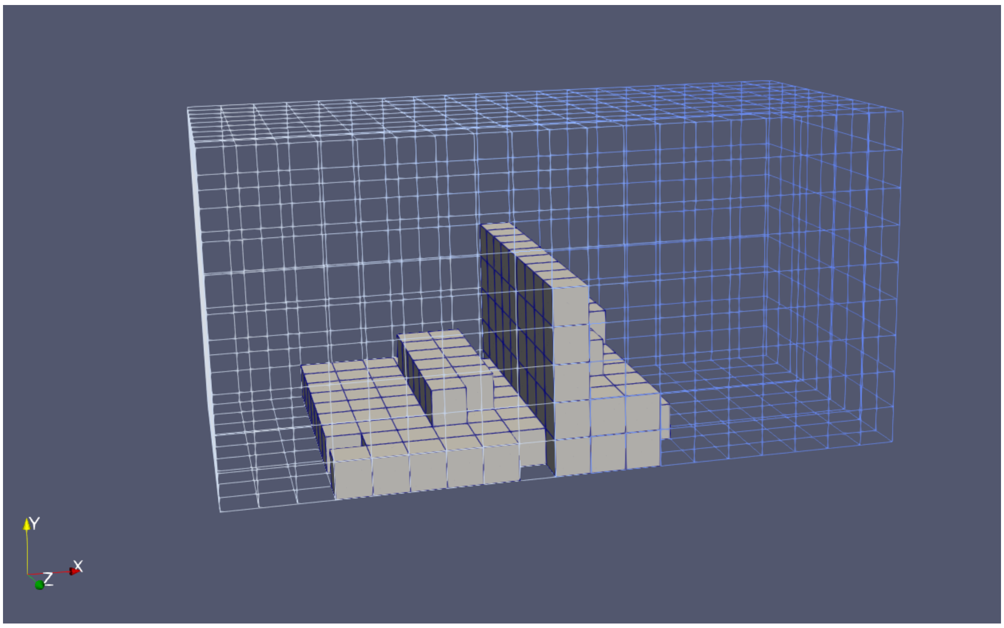

Figure 19: Shows the pressure field **for the FSI case**

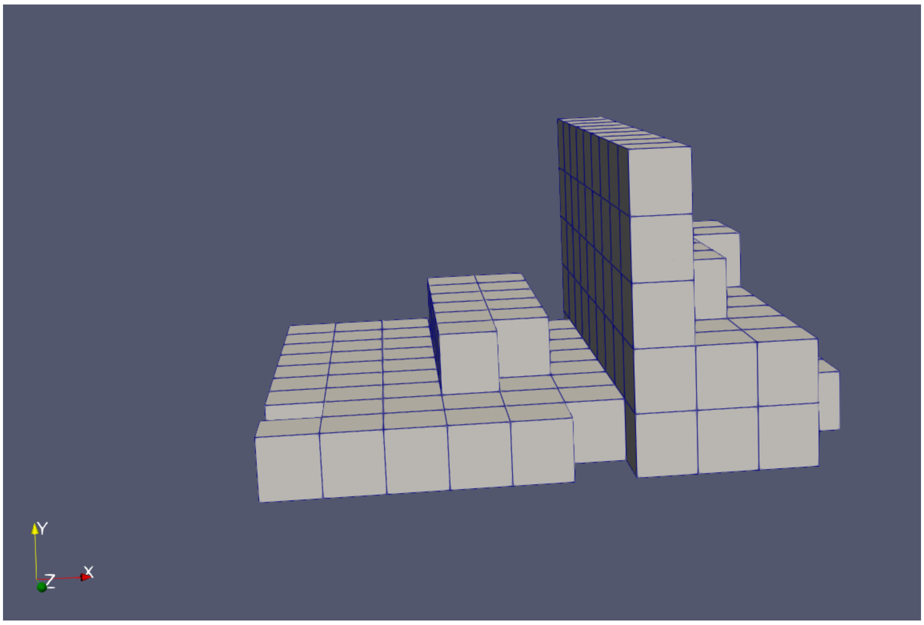

Figure 20: Shows the optimization **for the FSI case**

If you look at Figure 19, you can see that there is still higher pressure at the inlet than in the outlet as it should be.

4.2 Simplified valve model

The valve was built directly in the C++ code. It was created in the middle of the domain which was made to be the same size as the outer dimensions of the valve. The model was created using four separate solids for the walls and one other solid which was the hat. This was done by setting the solid's (x,y,z) coordinates. The domain and the valve were coded in such a way that allows for the valve's sizing to change proportionally with a change in the domain size. The valve can be seen in Figure 21 below.

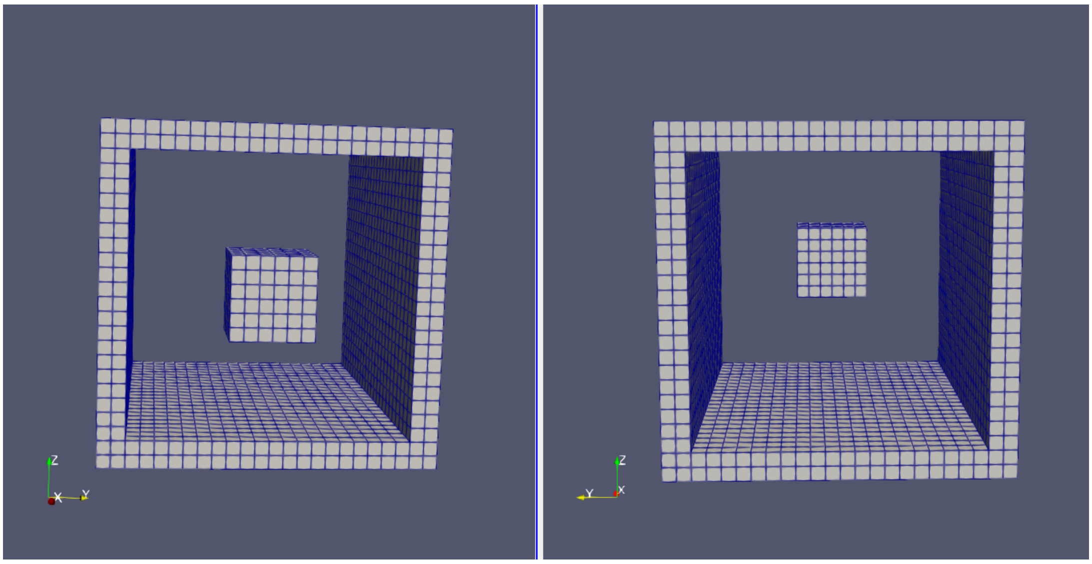

Figure 21: Shows the model of the valve that was set up directly in the code.

High-performance computing was used to optimize the pure mechanical and pure fluid case, meaning those two models were solved on the cluster *Sigma*. Furthermore, they were both solved with a mesh containing 100x100x100 Hex8 elements, with 50 max iterations and a filter radius of 0.01. During the analyses, it was found that the pure mechanical solution runs faster than the pure fluid solution runs. This was because the linear elasticity class is easier to solve than the Navier-Stokes class.

The combined and FSI case were solved on the standard computer with a mesh containing 25x25x25 Hex8 elements with 30 max iterations and a filter radius of 0.01. This was done because the script did not work for more than 1 core for these cases.

4.2.1 Pure Mechanical

The constant C_0 in Equation (20) is $C_0 = 0.7$, which means that the solid material is allowed to occupy 70% of the domain. The reason for this is that the valve is already taking up a major part of the volume from the start, so by setting for example $C_0 = 0.5$, it would look the same as the original valve as the volume constraint would already be fulfilled.

For this case, arbitrary forces were applied on the four walls of the simplified valve model. That means that the internal pressure of the fluid had been translated into external forces pulling on the walls. The numerical value of the forces has not been taken into consideration, it was just the behavior of the deformation that was of interest. The force vectors acting on the walls are perpendicular to the respective walls. The deformation as a result of the forces can be seen in Figure (22) below.

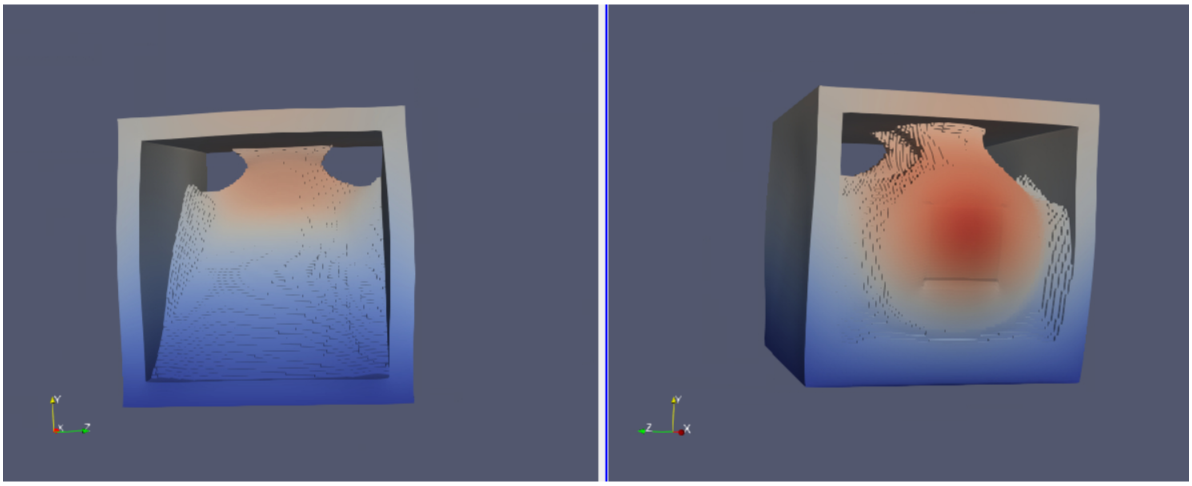

Figure 22: Shows the deformation **for the pure mechanical case**

The loading for the valve is stated below.

$$\vec{F_y} = \begin{bmatrix} 0 \\ 10F_0 \\ 0 \end{bmatrix} \vec{F_z} = \begin{bmatrix} 0 \\ 0 \\ F_0 \end{bmatrix} \vec{F_{hat}} = \begin{bmatrix} 100F_0 \\ 0 \\ 0 \end{bmatrix} \tag{26}$$

Where F_0 [N] is an arbitrary value of the force and the subscripts y, z, and hat are meant to indicate if the force is applied on the sides with the Y-coordinate as normal, Z-coordinate as normal, or on the hat. This load case means that the force is asymmetrical on the walls and then very large on the hat.

You can see that the optimization has placed material to make it stronger in the Y-direction, which makes sense since the loading is bigger in the Y-direction.

The optimized version of the valve can be seen below (Figure 23).

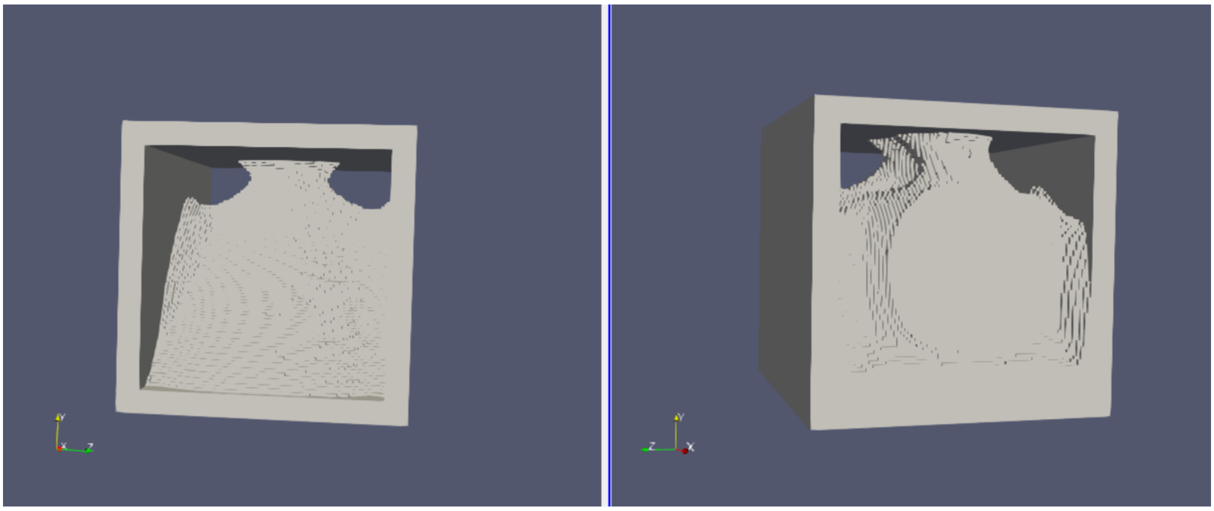

Figure 23: Shows the optimization **for the pure mechanical case**

This is a reasonable optimization since it was only the compliance being minimized, which is minimizing the displacement in the direction of the load. An "isosurface" was created in Paraview in order to make the design look more realistic.

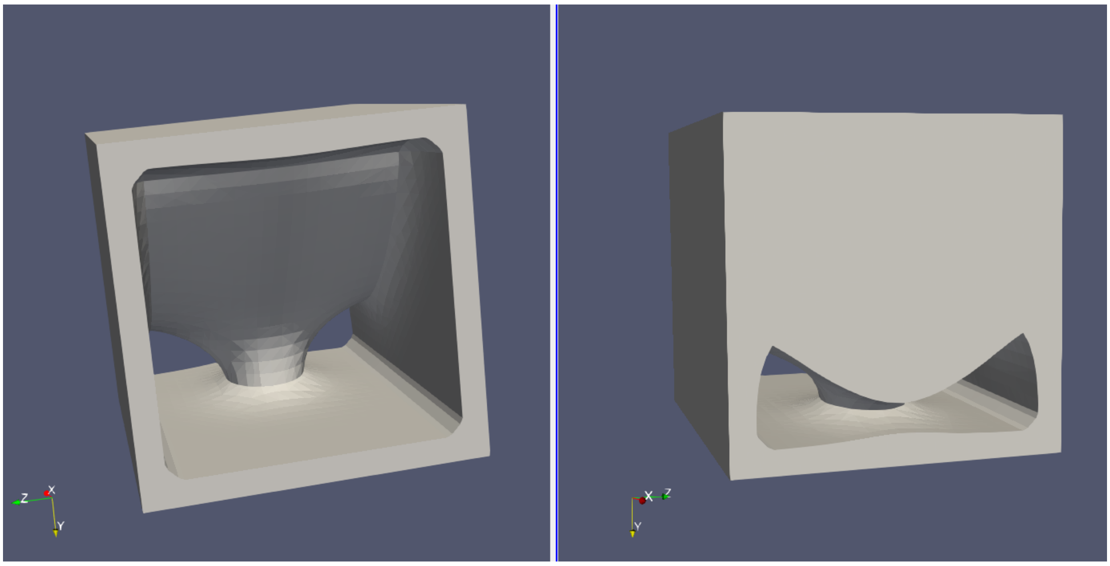

Figure 24: Shows the more smooth design of the pure mechanical case

4.2.2 Pure fluid

Where the constant C_1 in Equation (21) has been set to $C_1 = 0.7$, which means that the optimization has to put material in at least 70% of the domain.

Note, that the pure fluid case for the simplified valve has been analyzed with a prescribed traction and not with a prescribed velocity profile. The solution for the pure fluid case can be seen in Figures 25 and 26 below.

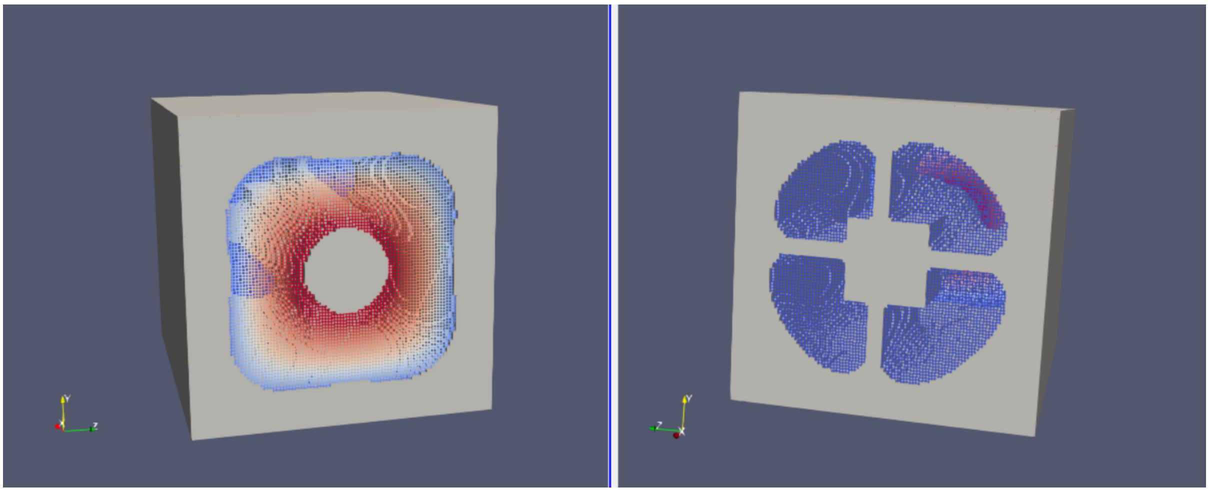

Figure 25: Shows the pressure drop **for the pure fluid case**

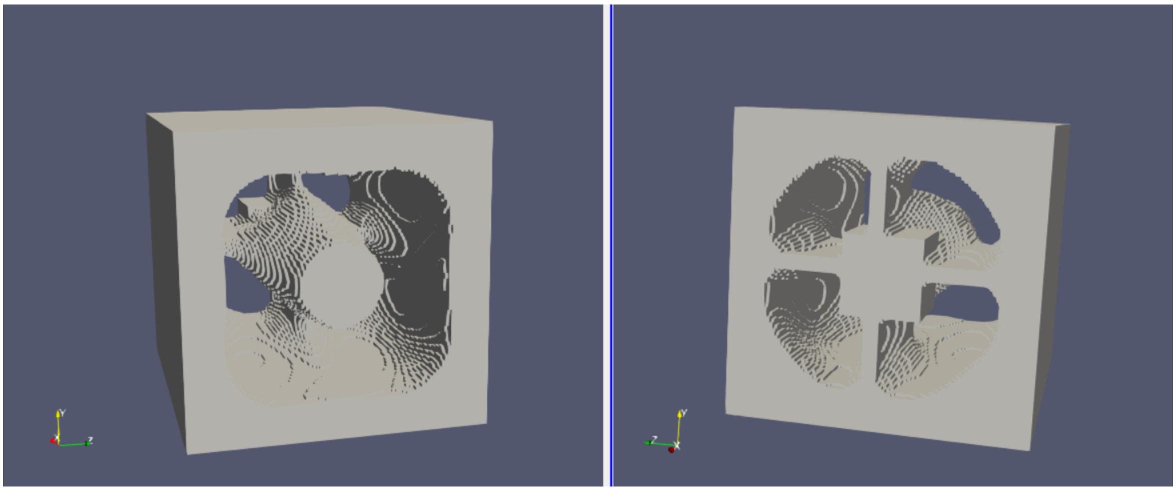

Figure 26: Shows the optimized valve **for the pure fluid case**

If you compare Figure 26 to Figure 23 you can see that not only has it created some kind of cone under the hat but it has also increased the thickness of the walls. You can also see that the optimization is symmetrical, that is because the loading for this case is symmetrical.

An "isosurface" was created in Paraview in order to make the design look more realistic.

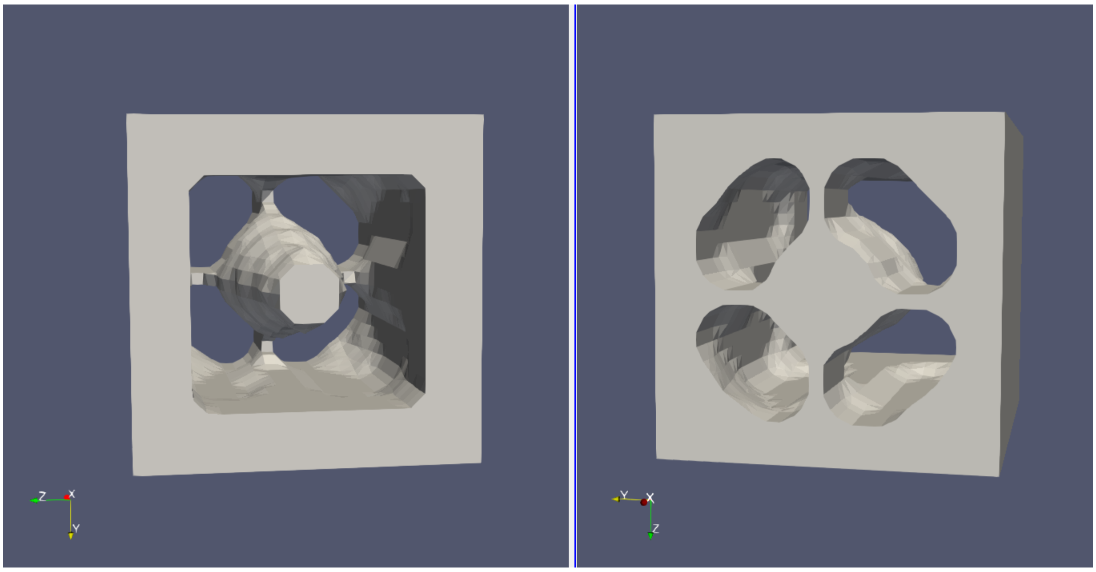

Figure 27: Shows the more smooth design for the pure fluid case

4.2.3 Combined solution

The constant C_2 in Equation (22) is set to $C_2 = 7 \cdot 10^5 [Nm]$, which is the set value for the compliance that the valve has to fulfill.

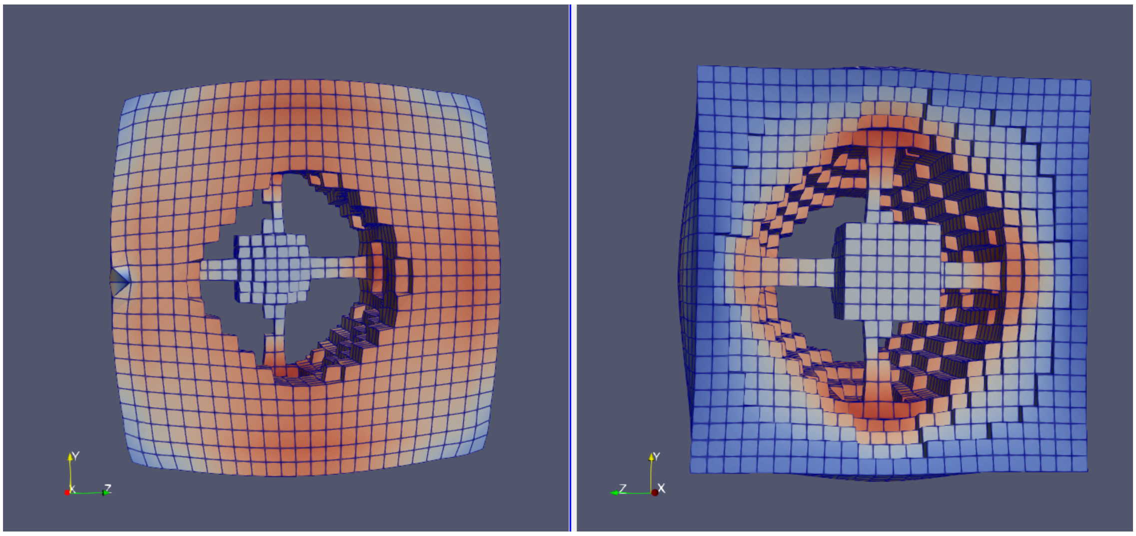

Figure 28: Shows the deformation **for the combined case**

Above the deformation of the valve can be seen and below the pressure field is shown. The pressure field shows high pressure at the inlet and low pressure at the outlet, which is still reasonable.

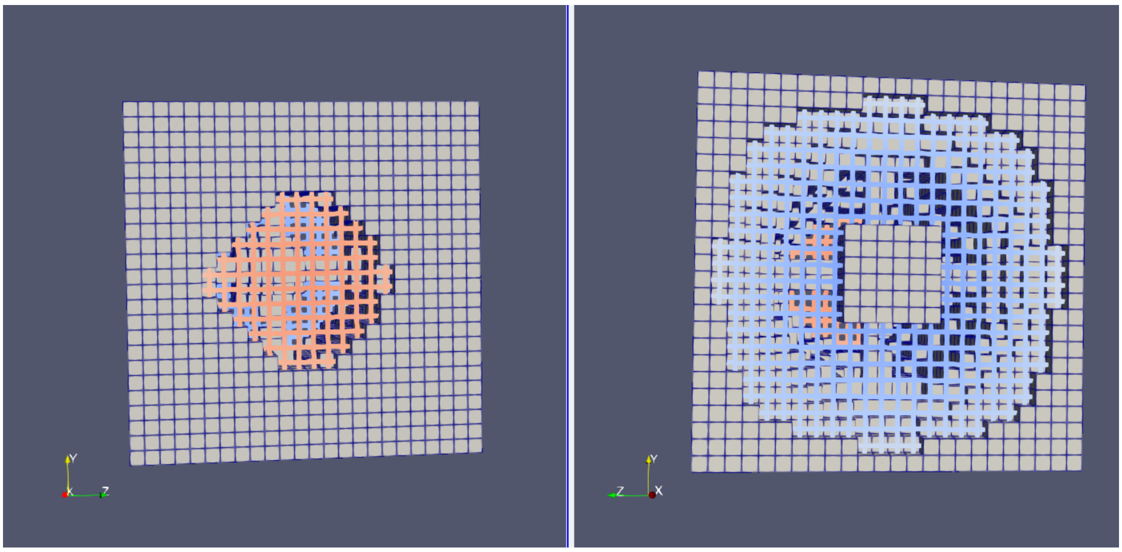

Figure 29: Shows the pressure field **for the combined case**

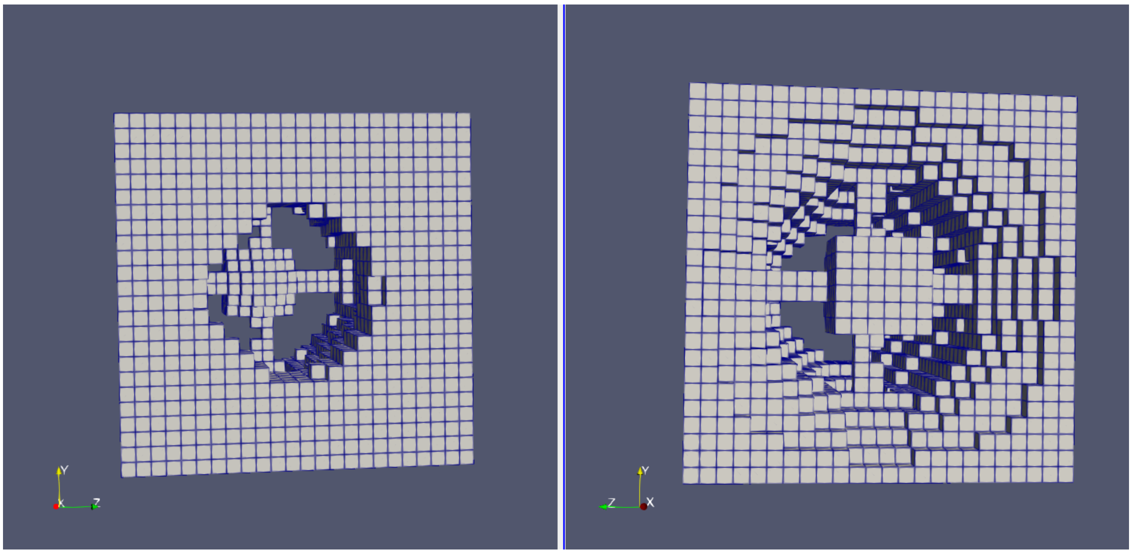

Figure 30: Shows the optimization **for the combined case**

You can see that the optimization has created a circular cross-section inside the rectangular valve, almost like a pipe. An "isosurface" was made in Paraview in order to make the structure look more realistic.

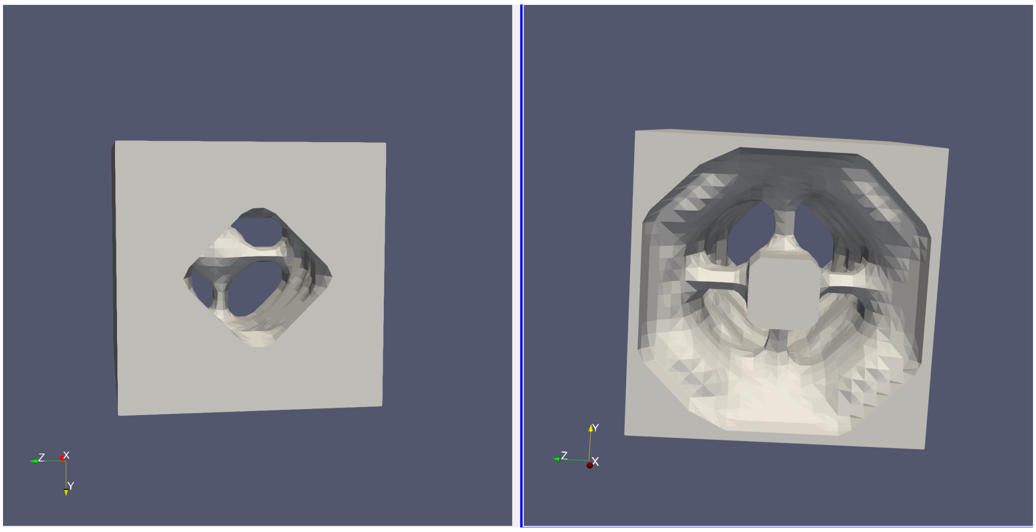

Figure 31: Shows the smoothed optimization for the combined case

4.2.4 FSI solution

The results for the FSI case can be seen in Figures 32 - 34. Where C_4 in Equation (24) was set to $C_4 = 2 \cdot 10^{-9}$

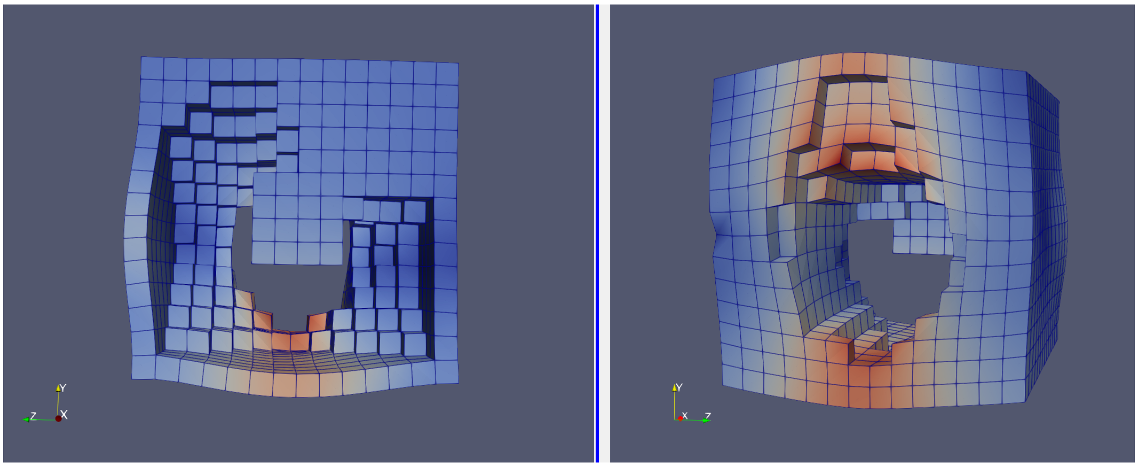

Figure 32: Shows the deformation **for the FSI case**

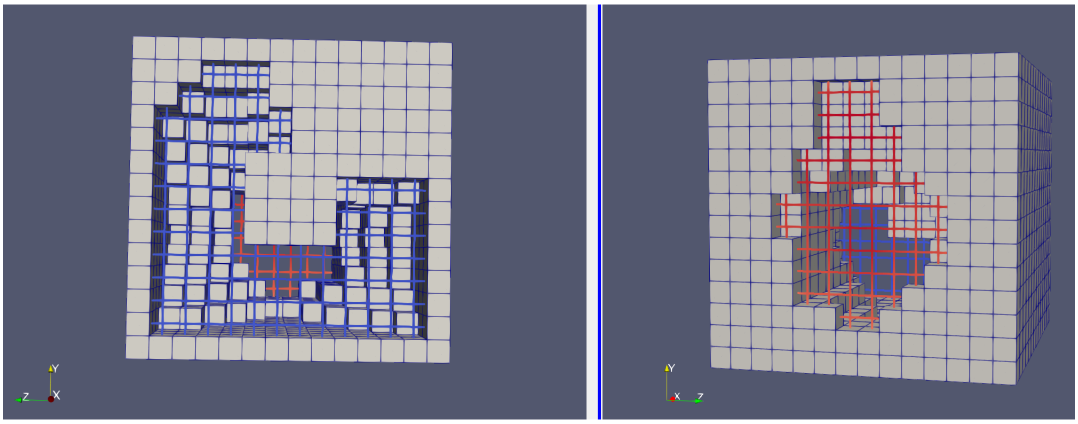

Figure 33: Shows the pressure field **for the FSI case**

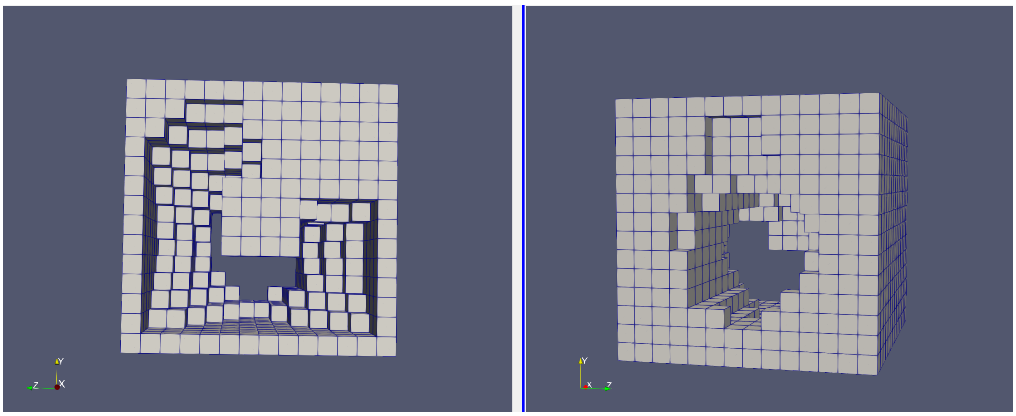

Figure 34: Shows the optimization **for the FSI case**

5 Discussion

The threshold for ρ was chosen to be $0.6 \leq \rho \leq 1$ for both the wall and simplified valve problem for *all* load cases in order to find the optimal design. If the threshold had been chosen, for example, to be $0.5 \leq \rho \leq 1$ or $0.7 \leq \rho \leq 1$, the optimal design would have not been the same as the one presented in this report.

5.1 Pure mechanical solution

In this section, the results of the pure mechanical solutions for the wall and simplified valve problems have been discussed.

5.1.1 Wall problem

The results for the mechanical wall optimization are shown in Figure 10. There, a structurally efficient design that minimizes the deformation under mechanical loads is apparent. The optimization has placed materials very tactically on both sides of the wall in a staircase-like structure. This pattern suggests that the optimization algorithm placed material sensibly in important areas of the wall, reinforcing it in a manner that effectively reduces the deformation. The optimized design has satisfied the volume constraint in terms of restriction on the material occupancy, expressed as $0 \leq \rho \leq 1$ and $V_{max} \leq C_0 V_{tot}$. Here, for the wall problem, the constant takes the value $C_0 = 10\%$.

These results contribute valuable insights to the overall project, showing that optimization methods are effective in achieving optimal material distribution for mechanical performance.

5.1.2 Simplified valve problem

The results for the simplified valve model under a pure mechanical load are presented in Figures 22 and 23. These are reasonable results if you consider the load case that was used in Equation (26). The reason for the non-symmetric load case was that if the $\vec{F_y}$ and $\vec{F_z}$ had the same magnitude, the walls would just get thicker and no material would build inside the valve.

By looking at Figure 22 you can see that more material has been built in the Y-direction, stopping the hat from moving in the X-direction as well as stopping the walls from deforming too much. Again, this is a reasonable response to the loading.

The optimized valve (Figure 23) is designed according to satisfy the constraints $0 \leq \rho \leq 1$ and $V_{max} \leq C_0 V_{tot}$. Where, for this valve, the constant took the value $C_0 = 70\%$ since the valve was already taking up a lot of volume from the start.

5.2 Pure fluid solution

In this section, the results of the pure fluid solutions for the wall and simplified valve problems have been discussed.

5.2.1 Wall problem

Prescribed velocity profile - The Wall problem with the pure fluid solution was first solved for the velocity profile. The objective function of the optimization was to minimize the pressure drop. Obviously, the best way to do this would be to remove all the material. To prevent this, the constant from Equation (21) was set to $C_1 = 0.1$ which means that the optimization had to fill at least 10% of the domain with solid material. By looking at Figure 11 and 12 it can be seen that the optimization had placed the material to the sides of the wall which is reasonable because the flow is higher in the middle, see Figure 37. This also explains why it places material in the upper corners. Furthermore, when running the optimization again with a much higher velocity the optimization solution nearly stayed the same. This is reasonable because the goal of the optimization is still the same and it doesn't depend on the velocity speed.

Prescribed traction - When doing the pure fluid with prescribed traction, the constraint of the optimization is the same as for the prescribed velocity case. Here, the optimization still wants to minimize the pressure drop and $C_1 = 0.1$.
Comparing Figures 12 and 14, one can see they are relatively similar visually with some slight differences. On the left side of the wall the one with prescribed traction isn't as long as the prescribed velocity but instead is thicker. This result places more material against the domain sides. On the right side of the wall, the two different figures look the same. This indicates that after passing the wall the prescribed traction and the prescribed velocity act the same which is a great indication that both have been implemented correctly.

5.2.2 Simplified valve problem

Note that the wall was solved for the prescribed pressure and velocity profile, while the valve was only solved for the prescribed pressure. The solutions for the valve can be seen in Figures 25 and 26. There the pressure field and the optimized valve are shown. You can see that there is higher pressure on the inlet and lower pressure on the outlet, as it should be. The reason for this was the optimization would not be able to place material on the regions where the velocity profile was prescribed, resulting in a less feasible solution.

The optimization interestingly placed material to make the outlet look like a fan. It also placed material under the hat creating a cone-like shape to lower the drag on the fluid in order to make it more fluid-dynamic.

The valve was solved with the condition according to Equation (21) with the constant $C_1 = 0.7$ which means that the optimization has to fill the domain with at least 70% of solid material.

5.3 Combined solution

For both the wall and the valve, the optimization differs quite a lot from the three different load cases. One reason for this lies in the optimization formulations. Let's compare the three (Equations (20), (21) and (22)).

- **Pure mechanical** - The optimization wants to add material to minimize the mechanical compliance (in order to minimize the displacement in the load direction), with the trivial solution of filling the domain with material. However, the volume constraint only allows the optimization to place material on a given fraction of the domain's volume.

- **Pure fluid** - The optimization wants to remove material to maximize the fluid compliance (in order to minimize the pressure drop), with the trivial solution of removing all the material from the domain. However, the volume constraint forces the optimization to place material in a given fraction of the domain's volume.

- **Combined** - The optimization here also wants to maximize the fluid compliance, with the same trivial solution of removing all material. But the constraint here is that a given value of the mechanical compliance has to be fulfilled. This means that the fluid compliance wants to remove material and the mechanical compliance wants to add material. This results in that the combined solution minimizes the pressure drop while maintaining a set displacement in the direction of the added mechanical load on the wall.

5.3.1 Wall problem

If you look at Figures 15 -17 you can see the result of the combined load case, with the deformation, pressure field and optimized model. Both the deformation and pressure field look reasonable since the wall deforms in the load direction and there is higher pressure on the inlet and lower pressure on the outlet.

If you compare Figure 17 to Figures 10 and 12, you can see that it is the sturdiest of the three load cases, though it is almost identical to the pure mechanical just with a thicker top. The reason for this is described above.

The wall was solved with the condition according to Equation (22) with the constant $C_2 = 10^6$ Nm.

5.3.2 Simplified valve problem

The optimization of the combined valve problem can be seen in Figure 31. Even though the original shape of the valve model was a rectangle, the optimization made the cross-section inside the valve into more of a circle. This is a reasonable solution when thinking about how equation 22 works. For the fluid to flow with as little pressure drop as possible and still make the mechanical compliance as small as possible, making the walls of the valve thicker but leaving as much space as possible in the middle where the flow velocity is the highest is very feasible. The optimization also made the cross-section nearly constant in the valve, which is reasonable since a change in area results in pressure losses. With the constant C_2 in Equation (22) is set to $C_2 = 7 \cdot 10^5$. This specific number had to be fine tuned, this was in order for the optimization to build material between the hat and the walls of the valve.

When looking at the hat, material has been placed under it which makes it more of a cone shape. This is mainly to decrease the pressure drop. Furthermore, arms were placed on the sides of the hat attaching it to the body of the valve. This makes sense due to the want to minimize the displacement of the hat itself.

5.4 FSI solution

In this section the results for the FSI solutions for the wall and simplified valve problems.

5.4.1 Wall problem

It becomes evident very quickly that the wall problem is not correct, see Figure 18. The reason for this can be seen in Figure 35 below.

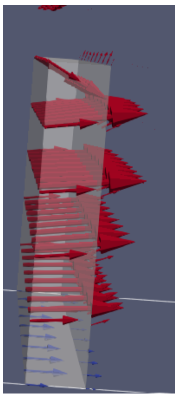

Figure 35: Shows the FSI forces on the wall

Here, you can see that the resultant of the FSI forces on the top elements are pushing the wall downwards while the other forces are pushing the wall in the flow direction. **These forces indicate that the results for the wall can not be considered correct**. The volume constraint in Equation (23) was set to $C_3 = 0.3$.

5.4.2 Simplified valve problem

The results for the valve can be quite hard to interpret, see Figures 32 - 34. To see if these results were reasonable, the FSI forces were visualized for iteration 0, just as in the wall problem. These forces can be seen in Figure 36 below.

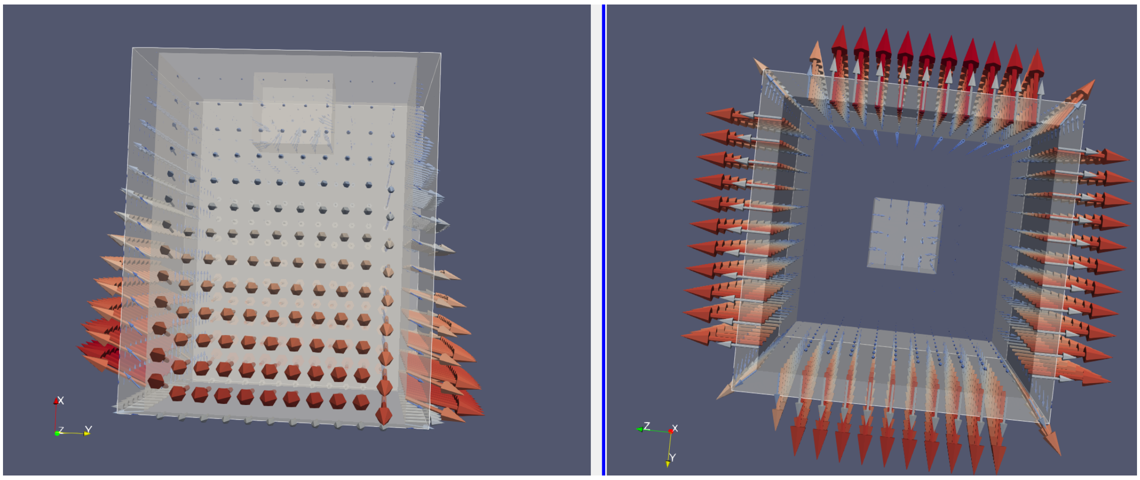

Figure 36: Shows the FSI forces on the simplified valve model

This shows that the forces are applied symmetrically across the valve, as they should be. **These forces indicate that the results for the valve can be considered to be correct**.

The value for the maximum compliance constraint was set to a very low value, the reason for this was that the FSI forces were also very low. The constant in Equation (24) was set to $C_4 = 2 \cdot 10^{-9}$.

By looking at the results, you can see that for the first time, the valve model has become very asymmetric. The reason for this is that the FSI load vector is heavily dependent on the design (ρ) since it adds force between elements with different ρ values. One thing that must be considered here as well is the intermediate ρ values, the SIMP penalization will not be able to successfully eliminate all of the values that are not 0 or 1. This means that if there is to be to elements next to each other, one with $\rho = 0$ and one with $\rho = 0.3$, an FSI load will still be added. When it in reality should not and this can alter the final design.

6 Conclusions

The conclusions will *only* include results and discussions for the simplified valve model. The wall was only solved for reference.

- **Pure mechanical** - Created a model that minimized the mechanical compliance (i.e. maximizing stiffness) with a volume constraint, looking at Figures 22 and 23 you can draw the conclusion that the optimization provided a feasible solution for the optimization formulation in Equation (20).

- **Pure fluid** - Created a model that maximized the fluid compliance (i.e. minimized the pressure drop) with a volume constraint, looking at Figures 25 and 26 you can draw the conclusion that the optimization provided a feasible solution for the optimization formulation in Equation (21).

- **Combined** - Created a model that maximized the fluid compliance (i.e. minimizing pressure drop) with a mechanical compliance constraint (i.e. stiffness constraint), looking at Figures 28, 29 and 31 you can draw the conclusion that the optimization provided a feasible solution for the optimization formulation in Equation (22).

- **FSI** - Created a model that maximized the fluid compliance with a mechanical compliance constraint. By looking at Figures 32, 33 and 34 you can see that a solution has been provided, but it can be quite difficult to interpret if it feasible or not. But by looking at Figure 36 you can see that the FSI load is implemented correctly and therefore you can draw the conclusion that the results provided are feasible.

7 Appendix

7.1 Velocity profile

The behavior of the fluids through and around the wall is analyzed. For the pure fluid problem, the prescribed velocity profile was needed (see the velocity profile in Figure 4). It was chosen to be a parabolic function, which means that it was assumed to be a fully developed flow. The function used to get the profile was.

$$v(y) = C_v((\frac{y_{max}}{2})^2 - (y - \frac{y_{max}}{2})^2) \tag{27}$$

Where $v(y)$ is the velocity at a given point y in the flow and y_{max} is the maximum Y-value of the inlet and C_v is just a constant to scale the value of the velocity. If we try to analyze the Equation (27), we can observe that if $y = 0$ and $y = y_{max}$, which gives the Equation (28).

$$\begin{cases} v(0) = C_v((\frac{y_{max}}{2})^2 - (-\frac{y_{max}}{2})^2 = C_v(\frac{y_{max}^2}{4} - \frac{y_{max}^2}{4}) = 0 \\ v(y_{max}) = C_v((\frac{y_{max}}{2})^2 - (y_{max} - \frac{y_{max}}{2})^2) = C_v((\frac{y_{max}}{2})^2 - (\frac{y_{max}}{2})^2) = 0 \end{cases} \tag{28}$$

To check that the velocity profile was implemented correctly, it was visualized by itself in Paraview. Since the flow was only prescribed to the inlet, it should only effect the first set of nodes. The velocity profile from Paraview can be seen in Figure 37 below.

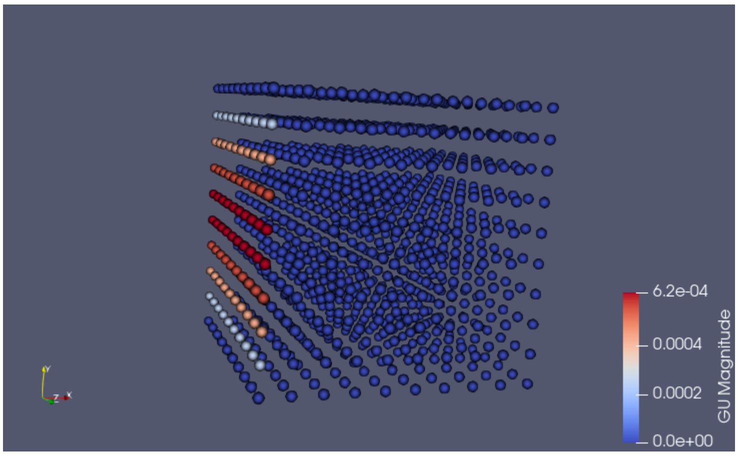

Figure 37: Shows the velocity profile in Paraview for $C_v = 1$

The top and bottom of Γ_{in} ($y = 0$ and $y = y_{max}$) are dark blue, which is zero velocity, as it should be. You can also see that the peak value is in the middle of the element, as it should be. The numerical values of the velocity is not of importance in this case, since the constant C_v is set to 1 the velocity is unscaled.

7.2 Basic strong Form and Weak Form BVP Derivation for 1D FEM

Moving forward the weak form of the BVP will be found. To start off, multiply the ordinary differential equation (ODE) in the last line of (35) by an arbitary function v=V(x) and integrate over the domain:

$$\int_0^L \frac{\partial}{\partial x}(AE\frac{\partial u}{\partial x})vdx = 0 \tag{29}$$

$$[AE\frac{\partial u}{\partial x}v]_0^L - \int_0^L AE\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}dx = 0 \tag{30}$$

$$AE\frac{\partial u}{\partial x}(L)v(L) - AE\frac{\partial u}{\partial x}(0)v(0) - \int_0^L AE\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}dx = 0 \tag{31}$$

Requiring that v(0)=0 we got an equation in u only:

$$AE\frac{\partial u}{\partial x}(L) = F \tag{32}$$

$$Fv(L) - \int_0^L AE\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}dx = 0 \tag{33}$$

Now introducing function space

$$V = v = v(x) : v(0) = o \tag{34}$$

$$Weak\ Form\ BVP = \begin{cases} u \in V : \\ \int_0^L AE\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}dx = Fv(L) \quad \forall v \in V \end{cases} \tag{35}$$

7.3 Sensitivities

Here the sensitivity for the constraint is derived. Let's start by implementing 2 new terms. **Note:** to simplify the notation for the reader, the "vector arrow" has been neglected but the following terms should still be viewed as vectors.

$$g_1 = F_{FSI}^T u_s + \lambda^T (K_f u_f - F_{traction}) + \kappa^T (K_s u_s - F_{FSI}) \tag{36}$$

Where $K_f u_f - F_{traction}$ and $K_s u_s - F_{FSI}$ are the equilibrium equations for the fluid and the solid, i.e. they are both equal to zero.

$$\frac{\partial g_1}{\partial \rho} = \frac{\partial F_{FSI}^T}{\partial \rho} u_s + \left(\frac{\partial F_{FSI}}{\partial u_f}\frac{\partial u_f}{\partial \rho}\right)^T + F_{FSI}^T \frac{\partial u_s}{\partial \rho} + \lambda^T(A) + \kappa^T(B) \tag{37}$$

Where

$$\begin{cases} A = \frac{\partial K_f}{\partial \rho} u_f + K_f \frac{\partial u_f}{\partial \rho} \\ B = \frac{\partial K_s}{\partial \rho} u_s + K_s \frac{\partial u_s}{\partial \rho} - \frac{\partial F_{FSI}}{\partial \rho} - \frac{\partial F_{FSI}}{\partial u_f}\frac{\partial u_f}{\partial \rho} \end{cases} \tag{38}$$

Now, Equation (37) can be expressed as one explicit and one implicit term. Such that $\frac{\partial g_1}{\partial \rho} = \frac{\partial g_1}{\partial \rho}_{explicit} + \frac{\partial g_1}{\partial \rho}_{implicit}$

$$\frac{\partial g_1}{\partial \rho}_{explicit} = \frac{\partial F_{FSI}^T}{\partial \rho} u_s + \lambda^T \frac{\partial K_f}{\partial \rho} u_f + \kappa^T \frac{\partial K_s}{\partial \rho} u_s - \kappa^T \frac{\partial F_{FSI}}{\partial \rho} \tag{39}$$

and

$$\frac{\partial g_1}{\partial \rho}_{implicit} = F_{FSI}^T \frac{\partial u_s}{\partial \rho} + \kappa^T K_s \frac{\partial u_s}{\partial \rho} + \left(\frac{\partial F_{FSI}}{\partial u_f}\frac{\partial u_f}{\partial \rho}\right)^T u_s + \lambda^T K_f \frac{\partial u_f}{\partial \rho} - \kappa^T \frac{\partial F_{FSI}}{\partial u_s}\frac{\partial u_s}{\partial \rho} \tag{40}$$

Once again rearranging the components, you will end up with

$$\frac{\partial g_1}{\partial \rho} = \frac{\partial g_1}{\partial \rho}_{explicit} + (F_{FSI}^T + \kappa^T K_s)\frac{\partial u_s}{\partial \rho} + (...) \tag{41}$$

Where you can see that

$$(F_{FSI}^T + \kappa^T K_s) = (K_s \kappa + F_{FSI})^T \tag{42}$$

You can now see that if you choose $\kappa = -u_s$, the term in Equation (42) becomes zero. Where κ is a so-called "adjoint variable". This will now give

$$\frac{\partial g_1}{\partial \rho} = 2\frac{\partial F_{FSI}^T}{\partial \rho}u_s - u_s^T\frac{\partial K_s}{\partial \rho}u_s + \lambda^T\frac{\partial K_f}{\partial \rho}u_f + (u_s^T\frac{\partial F_{FSI}}{\partial u_f} + u_s^T\frac{F_{FSI}}{\partial u_f} + \lambda^T K_f)\frac{\partial u_f}{\partial \rho} \quad (43)$$

Now, lets look at the last term in Equation (43). that term will be zero if you choose λ such that it solves the following system of equations.

$$K_f\lambda = -2\frac{\partial F_{FSI}}{\partial u_f}u_s \quad (44)$$

This will give the final result.

$$\frac{\partial g_1}{\partial \rho} = 2\frac{\partial F_{FSI}^T}{\partial \rho}u_s - u_s^T\frac{\partial K_s}{\partial \rho}u_s + \lambda^T\frac{\partial K_f}{\partial \rho}u_f \quad (45)$$

7.4 The General HEX8 Finite Element Equations

To understand the work surrounding the FSI it is important to have a basic understanding of the HEX8 element. The reference, [4], was used heavily within this section.

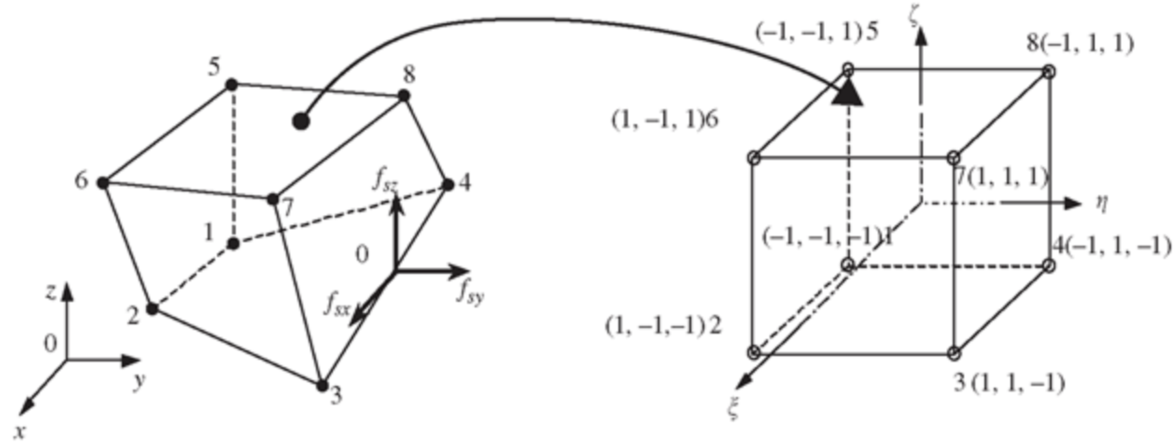

Figure 38

Looking at Figure (38) multiple HEXA8 elements and their global node numbers are shown. Furthermore, it should be noted that a global(x,y,z) and local(u,v,w) coordinate system can be seen. The displacement field involved in the body can then be computed within the following vector:

$$\vec{u} = \begin{bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{bmatrix} \tag{46}$$

The global node displacement vector and concentrated node loads are shown below:

$$\vec{d} = \begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2. \\ . \\ . \end{bmatrix} \tag{47}$$

$$\vec{f_c} = \begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{1z} \\ F_{2x} \\ F_{2y} \\ F_{2z}. \\ . \\ . \end{bmatrix} \tag{48}$$

Moving on to the element's nodal system, the element's nodal displacement vector is:

$$\vec{d_e} = \begin{bmatrix} u_{1e} \\ v_{1e} \\ w_{1e} \\ . \\ . \end{bmatrix} \tag{49}$$

The elements displacement displacement field vector is:

$$\vec{u_e} = \bar{N_e}\vec{d_e} \tag{50}$$

Where N_e is the elemental shape function matrix and be visualized as the following equation:

$$\begin{cases} N_1 = \frac{1}{8}(1-\xi)(1-\eta)(1-\zeta) \\ N_2 = \frac{1}{8}(1+\xi)(1-\eta)(1-\zeta) \\ N_3 = \frac{1}{8}(1+\xi)(1+\eta)(1-\zeta) \\ N_4 = \frac{1}{8}(1-\xi)(1+\eta)(1-\zeta) \\ N_5 = \frac{1}{8}(1+\xi)(1-\eta)(1+\zeta) \\ N_6 = \frac{1}{8}(1+\xi)(1-\eta)(1+\zeta) \\ N_7 = \frac{1}{8}(1+\xi)(1-\eta)(1-\zeta) \\ N_8 = \frac{1}{8}(1-\xi)(1+\eta)(1+\zeta) \end{cases} \tag{51}$$

$$\vec{f} = \bar{K}\vec{d} \tag{52}$$

Where $\vec{K}$ is equal to:

$$\bar{K} = \sum_{e=1}^{n} \quad (\bar{C_e})^T(\bar{K_e})(\bar{C_e}) \tag{53}$$

And $\bar{K_e}$ is equal to:

$$\bar{K_e} = \int_{V_e} \bar{B_e}^T (\bar{E_e})(\bar{B_e}) dV_e \tag{54}$$

$$\vec{f_t} = \sum_{e=1}^{n} \quad (\bar{C_e})^T \int_{S_t} \bar{N}^T \vec{t} dS \tag{55}$$

7.5 Solid Mechanics Equations for FSI

$$div(\vec{\epsilon}(\vec{u}_s)) = \frac{1}{2}[\nabla \vec{u}_s + \nabla (\vec{u}_s)^T] \tag{56}$$

7.6 Fluid Mechanics Equations for FSI

$$div(\vec{D}(\vec{u}_f)) = \frac{1}{2}[\nabla \vec{u}_f + \nabla (\vec{u}_f)^T] \tag{57}$$

7.7 FEA Discretization for FSI

$$\vec{u}_s \approx \vec{u}_s^h = \vec{N}_e(x, y, z)\vec{d}_{s_e} = \vec{N}_e \vec{C}_e \vec{d}_s \tag{58}$$

$$\vec{d}_s = \begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ . \\ . \\ . \end{bmatrix} \tag{59}$$

$$\vec{u_f} \approx \vec{u_f}^h = \vec{N}_e \vec{C}_e \vec{d}_s \tag{60}$$

$$\vec{p} \approx \vec{p^h} = \vec{p_e} = \vec{e_e} \vec{p} \tag{61}$$

$$\rho \approx \rho^h = \rho_e \tag{62}$$

$$\int_{\Omega} \rho \vec{\epsilon}(\vec{u}_s)^T \bar{E} \vec{\epsilon}(\vec{v}) \; dV \approx \sum_{e=1}^{m} \int_{\Omega_e} \rho^n \vec{\epsilon}(\vec{u_s}^h)^T \bar{E} \vec{\epsilon}(\vec{v}^h) \; dV \tag{63}$$

$$//\vec{\epsilon}(\vec{u_s}^h) = \vec{d} \bar{N}_e \vec{d}_s = \bar{B}_e \bar{C}_e \vec{d}_s// \tag{64}$$

$$\sum_{e=1}^{m} \int_{\Omega_e} \rho_e \vec{d_s}^T \vec{C_e}^T \vec{B_e}^T \vec{E} \vec{B_e} \vec{C_e} \vec{d_s} \; dV \tag{65}$$

$$\int_{\Omega} \alpha(\rho) \vec{u_f}^T \vec{v} \; dV \approx \vec{d_f}^T \vec{M}(\vec{\rho}) \tilde{d}_f \tag{66}$$

$$\int_{\Omega} \vec{\epsilon}(\vec{u_f})^T \vec{C} \vec{\epsilon}(\vec{v}) \; dV \approx \vec{d_f}^T \vec{K_f} \tilde{d}_f \tag{67}$$

$$\int_{\Omega} p div \vec{v} \; dV \approx \; \vec{p}^T \vec{G}^T \; \tilde{d}_f \tag{68}$$

$$\int_{\Gamma_{fs}(\vec{\rho})} -p \vec{n}_s^T \vec{v} \; dA \approx \sum_{e=1}^{m} \; \frac{1}{2} \; \int_{\Gamma_e \cap \Gamma_{fs}(\vec{\rho})} -p_e \vec{n}_s \vec{N}_e^T \; dA \; \vec{C}_e \; \tilde{d}_f \tag{69}$$

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