Chapter 3: Recursion

CS401

Michael Y. Choi, Ph.D.

Department of Computer Science
Illinois Institute of Technology

Revised Nell Dale Presentation

Chapter 3: Recursion

3.1 – Recursive Definitions, Algorithms and Programs

3.2 – The Three Questions

3.3 – Recursive Processing of Arrays

3.4 – Recursive Processing of Linked Lists

3.5 – Towers of Hanoi

3.6 – Fractals

3.7 – Removing Recursion

3.8 – Deciding Whether to Use a Recursive Solution

3.1 Recursive Definitions, Algorithms and Programs

Recursive Definitions

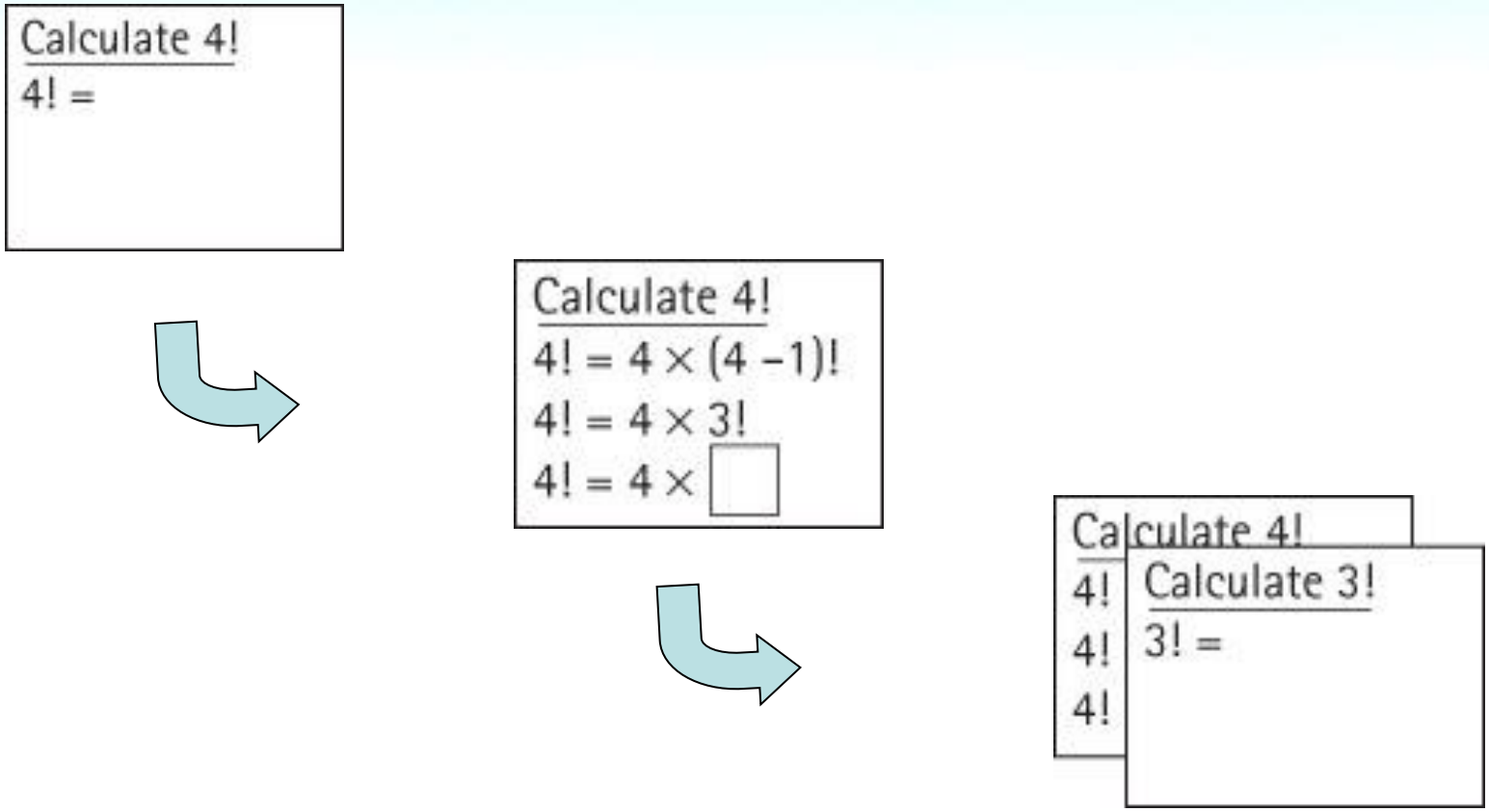
- **Recursive definition** A definition in which something is defined in terms of smaller versions of itself. For example:
  - A *directory* is an entity in a file system which contains a group of files and other *directories*.
  - A *compound sentence* is a *sentence* that consists of two *sentences* joined together by a coordinating conjunction.
  - $n! = 1 \quad \text{if } n = 0$

    $n \times (n - 1)! \quad \text{if } n > 0$

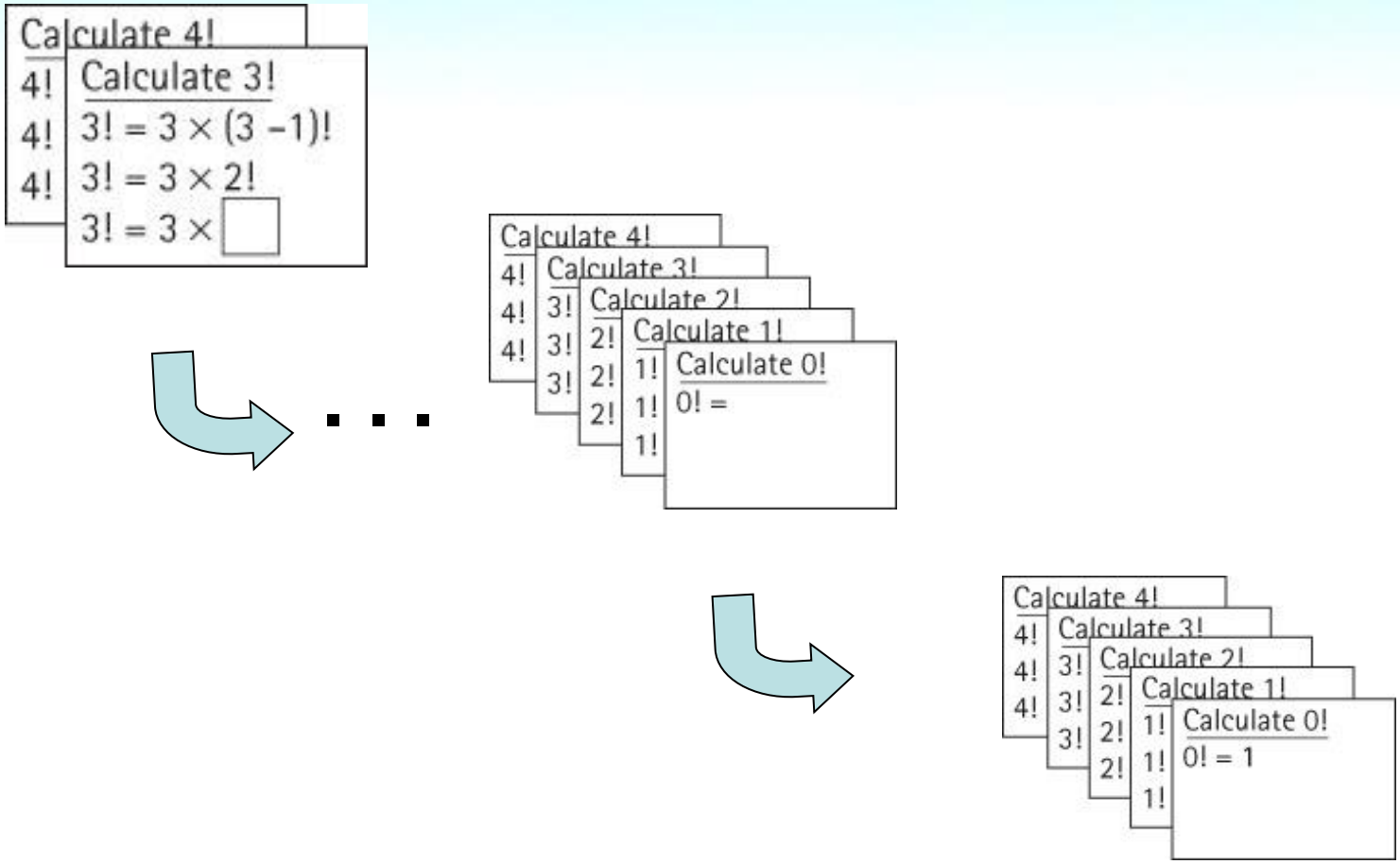
Recursion

- You must be careful when using recursion.
- Recursive solutions can be less efficient than iterative solutions.
- Still, many problems lend themselves to simple, elegant, recursive solutions.

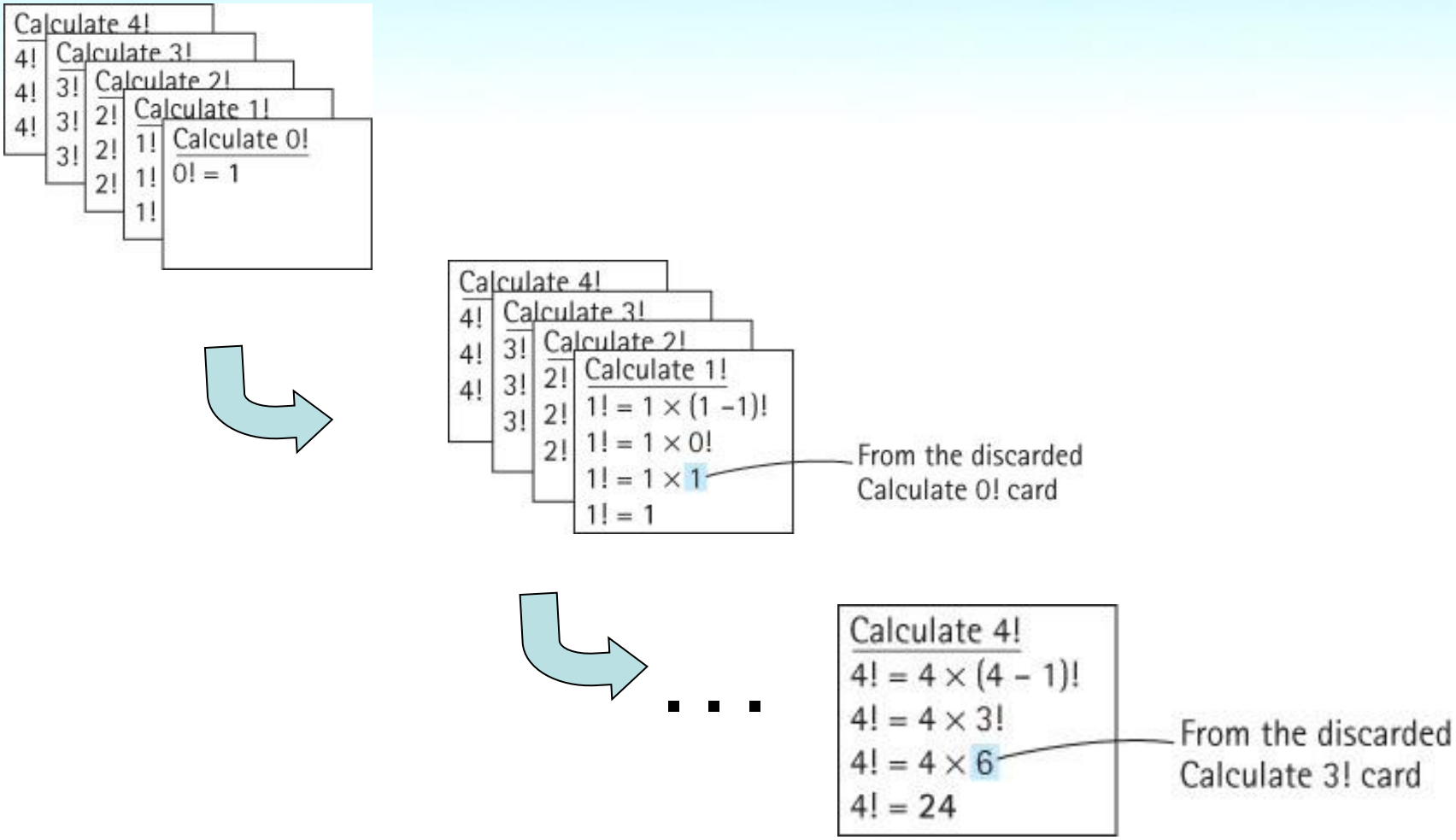
Example: Calculate 4!



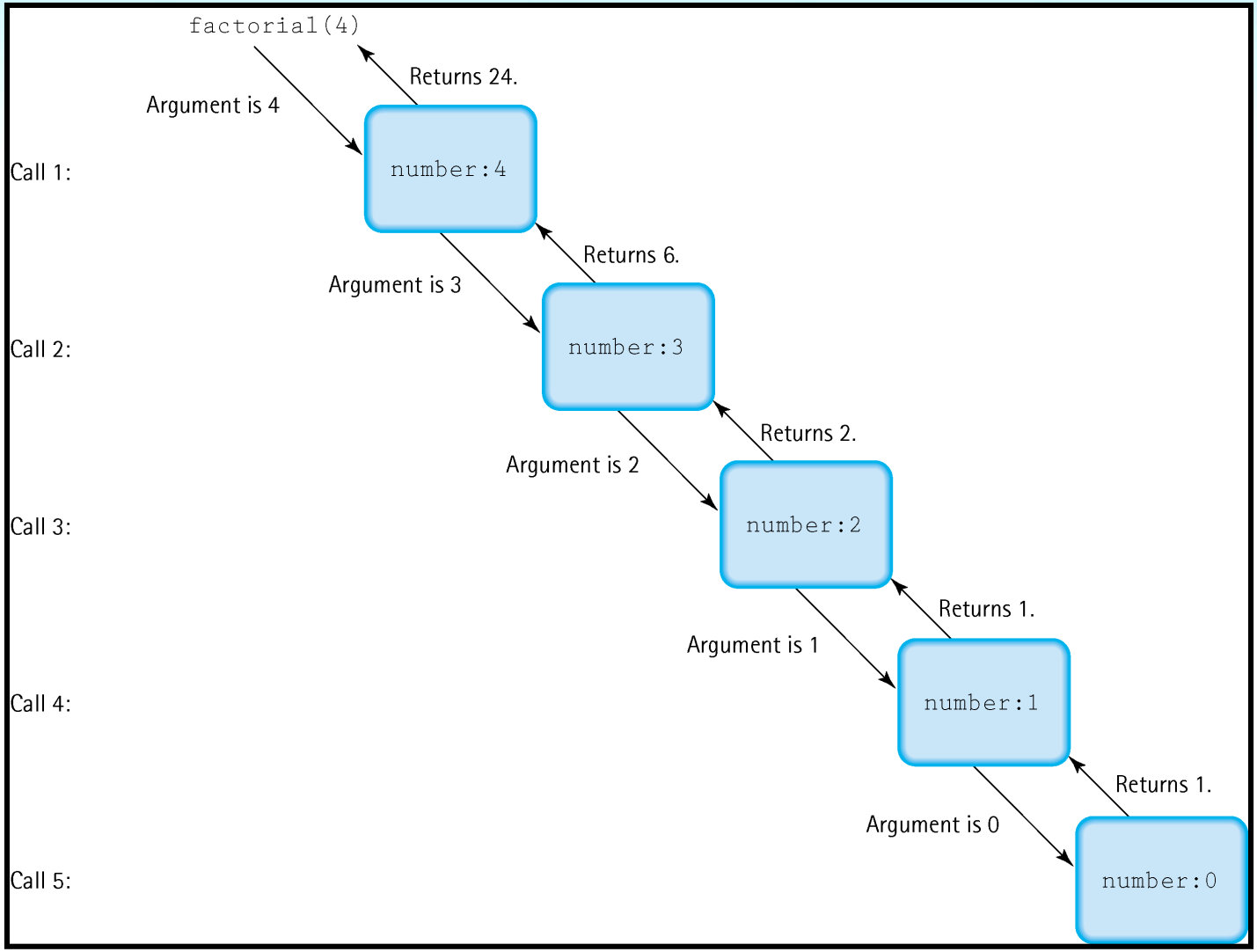
Example: Calculate 4!



Example: Calculate 4!



Example: Calculate 4!



Recursive Algorithms

- **Recursive algorithm** A solution that is expressed in terms of
  - smaller instances of itself and
  - a base case
- **Base case** The case for which the solution can be stated non-recursively
- **General (recursive) case** The case for which the solution is expressed in terms of a smaller version of itself

Examples of a Recursive Algorithm and a Recursive Program

Factorial (int n)
// Assume n >= 0
if (n == 0)
  return (1)
else
  return (n * Factorial (n – 1))

```
public static int factorial(int n)
// Precondition: n is non-negative
//
// Returns the value of "n!".
{
  if (n == 0)
    return 1;           // Base case
  else
    return (n * factorial(n - 1));     // General case
}
```

Augmented recursive factorial

```
private static int factorial(int n)
// Precondition: n is non-negative
//
// Returns the value of "n!".
{
  int retValue;   // return value
  System.out.println(indent + "Enter factorial " + n);
  indent = indent + "  ";
  if (n == 0)
    retValue = 1;
  else
    retValue = (n * factorial (n - 1));

  indent = indent.substring(2);
  System.out.println(indent + "Return " + retValue);

  return(retValue);
}
```

Output if argument is 9

```
Enter factorial 9
  Enter factorial 8
    Enter factorial 7
      Enter factorial 6
        Enter factorial 5
          Enter factorial 4
            Enter factorial 3
              Enter factorial 2
                Enter factorial 1
                  Enter factorial 0
                  Return 1
                Return 1
              Return 2
            Return 6
          Return 24
        Return 120
      Return 720
    Return 5040
  Return 40320
Return 362880
```

Recursion Terms

- **Recursive call** A method call in which the method being called is the same as the one making the call
- **Direct recursion** Recursion in which a method directly calls itself, like the factorial method.
- **Indirect recursion** Recursion in which a chain of two or more method calls returns to the method that originated the chain, for example method A calls method B which in turn calls method A

e.g. method A calls method B and method B calls back method A

Iterative Solution for Factorial

- We have used the factorial algorithm to demonstrate recursion because it is familiar and easy to visualize. In practice, one would never want to solve this problem using recursion, since a straightforward, more efficient iterative solution exists:

```
public static int factorial(int n)
{
  int retValue = 1;
  while (n != 0)
  {
    retValue = retValue * n;
    n = n - 1;
  }
  return(retValue);
}
```

3.2 The Three Questions

- In this section we present three questions to ask about any recursive algorithm or program.
- Using these questions helps us verify, design, and debug recursive solutions to problems.

Verifying Recursive Algorithms

To verify that a recursive solution works, we must be able to answer “Yes” to all three of these questions:

1. *The Base-Case Question:* Is there a nonrecursive way out of the algorithm, and does the algorithm work correctly for this base case?
2. *The Smaller-Caller Question:* Does each recursive call to the algorithm involve a smaller case of the original problem, leading inescapably to the base case?
3. *The General-Case Question:* Assuming the recursive call(s) to the smaller case(s) works correctly, does the algorithm work correctly for the general case?

We next apply these three questions to the factorial algorithm.

The Base-Case Question

```
Factorial (int n)
// Assume n >= 0
if (n == 0)
  return (1)
else
  return ( n * Factorial ( n – 1 ) )
```

Is there a nonrecursive way out of the algorithm, and does the algorithm work correctly for this base case?

The base case occurs when *n* is 0.

The Factorial algorithm then returns the value of 1, which is the correct value of 0!, and no further (recursive) calls to Factorial are made.

The answer is yes.

The Smaller-Caller Question

```
Factorial (int n)
// Assume n >= 0
if (n == 0)
  return (1)
else
  return ( n * Factorial ( n – 1 ) )
```

Does each recursive call to the algorithm involve a smaller case of the original problem, leading inescapably to the base case?

The parameter is *n* and the recursive call passes the argument *n - 1*. Therefore each subsequent recursive call sends a smaller value, until the value sent is finally 0.

At this point, as we verified with the base-case question, we have reached the smallest case, and no further recursive calls are made. The answer is yes.

The General-Case Question

```
Factorial (int n)
// Assume n >= 0
if (n == 0)
  return (1)
else
  return ( n * Factorial ( n – 1 ) )
```

Assuming the recursive call(s) to the smaller case(s) works correctly, does the algorithm work correctly for the general case?

Assuming that the recursive call Factorial(n – 1) gives us the correct value of *(n - 1)!*, the *return* statement computes *n * (n - 1)!*.

This is the definition of a factorial, so we know that the algorithm works in the general case.
The answer is yes.

Constraints on input arguments

- Constraints often exist on the valid input arguments for a recursive algorithm. For example, for Factorial, n must be >= 0.
- You can use the three question analysis to determine constraints:
  - Check if there are any starting argument values for which the smaller call does not produce a new argument that is closer to the base case.
  - Such starting values are invalid.
  - Constrain your legal input arguments so that these values are not permitted.

Steps for Designing Recursive Solutions

1. Get an exact definition of the problem to be solved.
2. Determine the size of the problem to be solved on this call to the method.
3. Identify and solve the base case(s) in which the problem can be expressed non-recursively. This ensures a yes answer to the base-case question.
4. Identify and solve the general case(s) correctly in terms of a smaller case of the same problem—a recursive call. This ensures yes answers to the smaller-caller and general-case questions.

Another example where recursion comes naturally

- **From mathematics, we know that**

  $2^0 = 1$ **and** $2^5 = 2 * 2^4$

- **In general,**

  $x^0 = 1$ **and** $x^n = x * x^{n-1}$

  for integer x, and integer n > 0.

- **Here we are defining x^n recursively, in terms of x^{n-1}**

```
// Recursive definition of power function

int  Power (  int   x,   int   n )

   //  Pre:     n >= 0.   x, n are not both zero
   //  Post:   Function value = x raised to the power n.

{
   if ( n == 0 )
        return  1;                  //  base case
   else                            // general case
        return (  x * Power ( x , n-1 ) ) ;

}
```

Of course, an alternative would have been to use looping instead of a recursive call in the function body.

3.3 Recursive Processing of Arrays

- Many problems related to arrays lend themselves to a recursive solution.
- A subsection of an array (a “subarray”) can also be viewed as an array.
- If we can solve an array-related problem by combining solutions to a related problem on subarrays, we may be able to use a recursive approach.

Binary Search

- Problem: find a target element in a sorted array
- Approach
  - examine the midpoint of the array and compare the element found there to our target element
  - eliminate half the array from further consideration
  - **recursively** repeat this approach on the remaining half of the array until we find the target or determine that it is not in the array

Example

- We have a sorted array of `int` named `values` of size 8
- Our target is 20
- Variables `first` and `last` indicate the sub-array currently under consideration
- Therefore, the starting configuration is:

target: 20

	first=0							last=7
	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
values:	4	6	7	15	20	22	25	27

Example

- The midpoint is the average of `first` and `last`

```
midpoint = (first + last) / 2
```

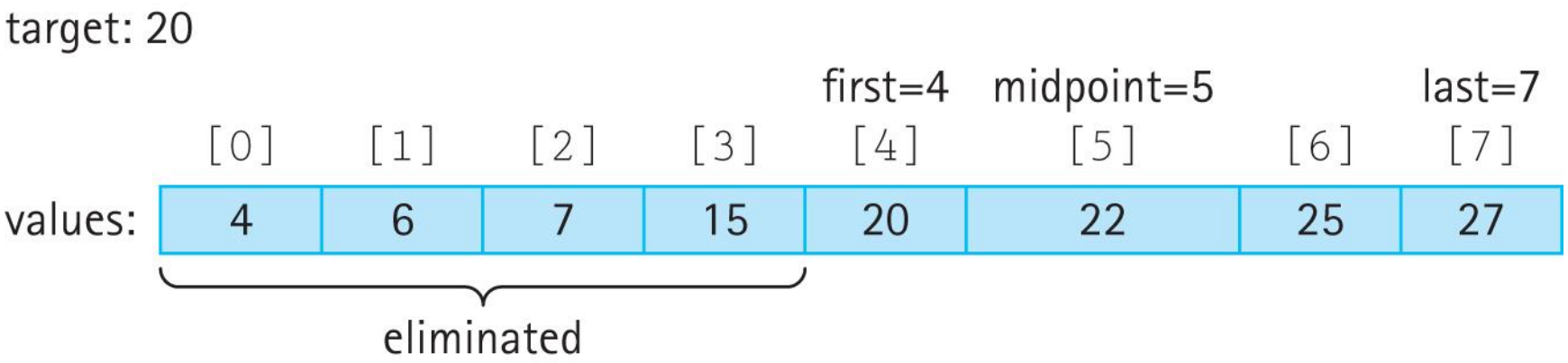
target: 20

	first=0			midpoint=3				last=7
	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
values:	4	6	7	15	20	22	25	27

- Since `values[midpoint]` is less than `target` we eliminate the lower half of the array from consideration

Example

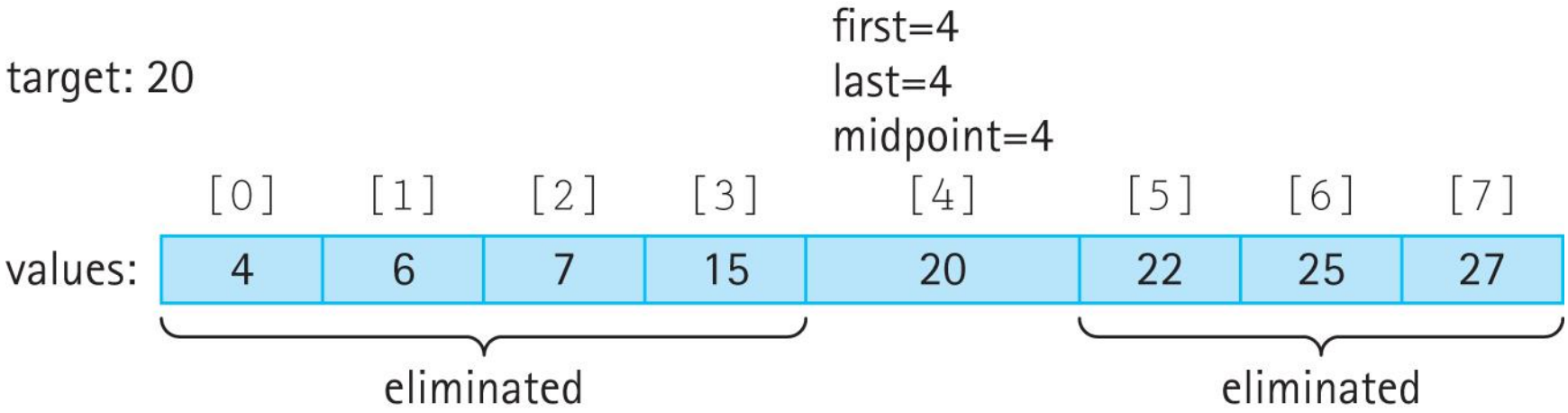
- Set `first` to `midpoint + 1` and calculate a new `midpoint` resulting in



- Since `values[midpoint]` is greater than `target` we eliminate the upper half of remaining portion of the array from consideration.

Example

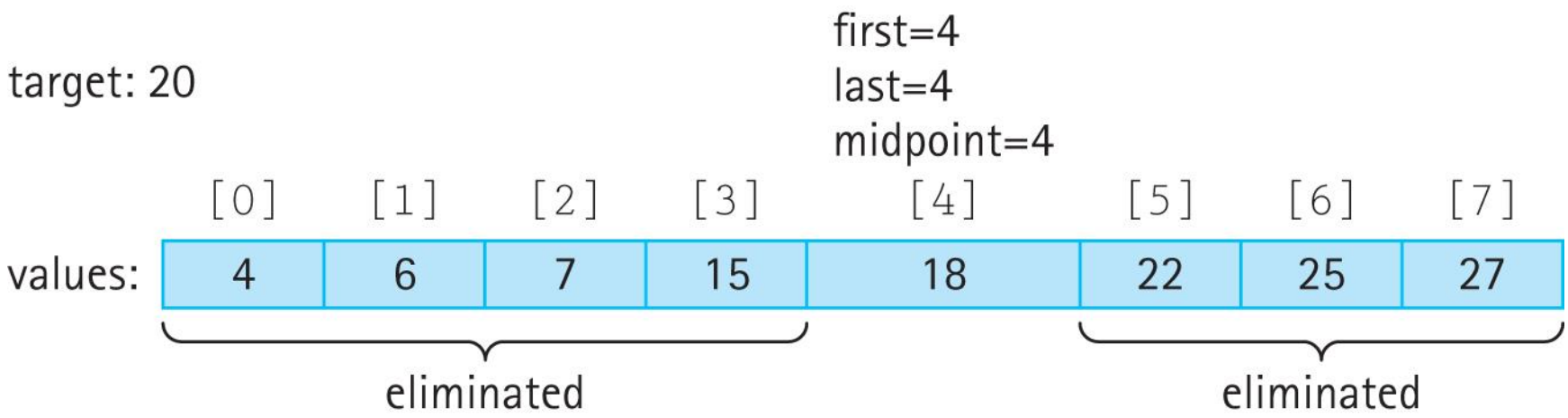
- Set `last` to `midpoint - 1` and calculate a new midpoint resulting in



- Since `values[midpoint]` equals `target` we are finished and return `true`

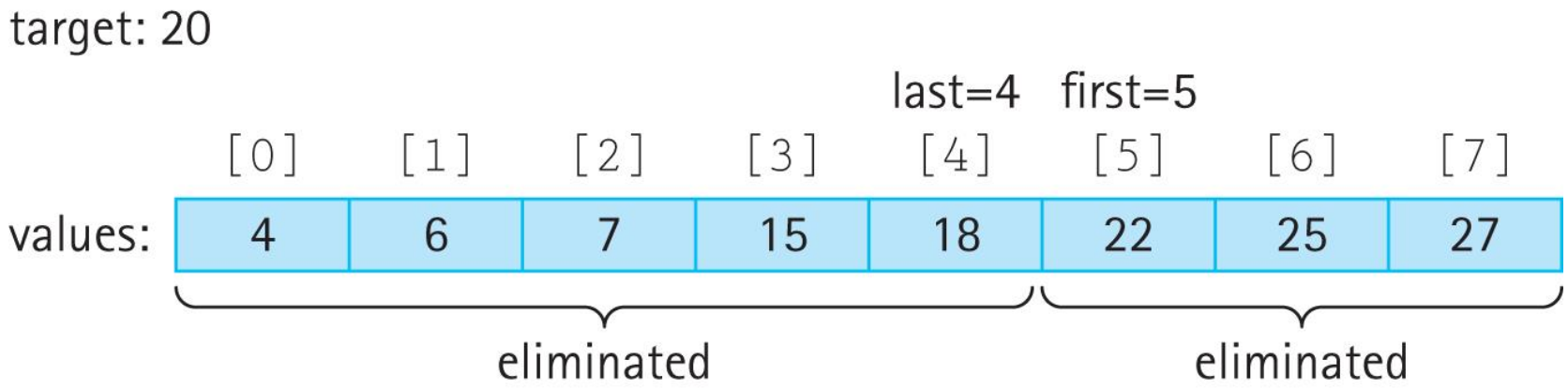
Example – `target` not in array

- Consider the above example again, but this time with 18 replacing 20 as the fifth element of `values`. The same sequence of steps would occur until at the very last step we have the following:



Example – `target` not in array

- Since `values[midpoint]` is less than `target` we set `first` to `midpoint + 1`:



- The entire array has been eliminated (since `first > last`) and we return `false`.

Code for the Binary Search

```
boolean binarySearch(int target, int first, int last)
// Precondition: first and last are legal indices of values
//
// If target is contained in values[first,last] return true
// otherwise return false.
{
   int midpoint = (first + last) / 2;
   if (first > last)
      return false;
   else
      if (target == values[midpoint])
         return true;
      else
      if (target > values[midpoint])
         return binarySearch(target, midpoint + 1, last);
      else
         return binarySearch(target, first, midpoint - 1);
}
```

recursive call

recursive call

3.4 Recursive Processing of Linked Lists

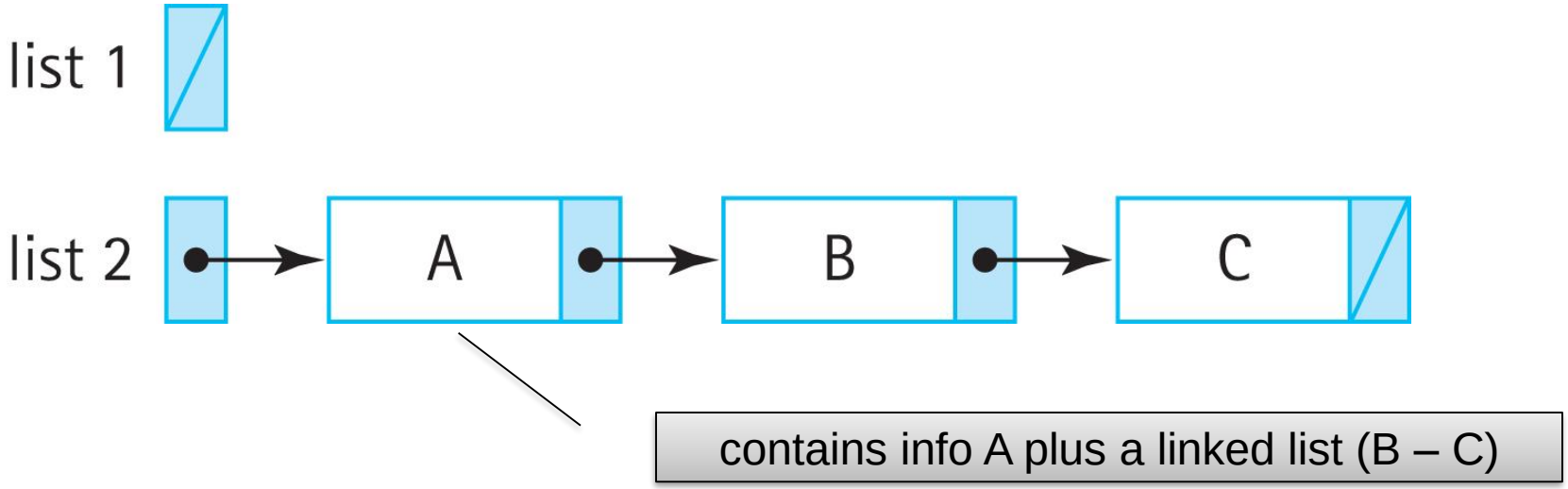
- A linked list is a recursive structure
- The `LLNode` class, our building block for linked lists, is a self-referential (recursive) class:

```
public class LLNode<T>
 {
    protected T info; // information stored in list
    protected LLNode<T> link; // reference to a node
  . . .
```

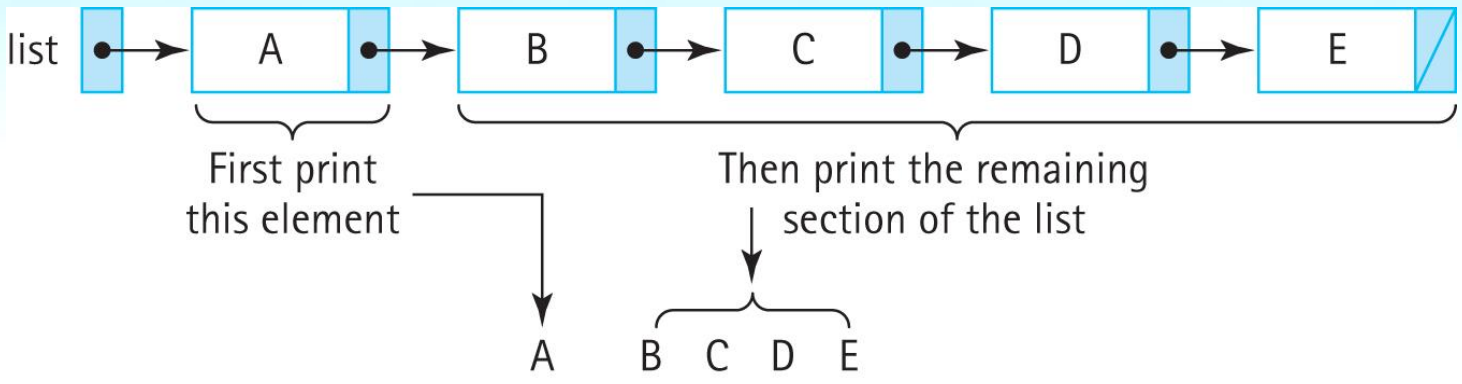
recursive reference

Recursive nature of Linked Lists

- A linked list is either empty or consists of a node containing two parts:
  - information
  - a linked list



Printing a Linked List Recursively




```
void recPrintList(LLNode<String> listRef)
{
   if (listRef != null)
   {
      System.out.println(listRef.getInfo());
      recPrintList(listRef.getLink());
   }
}
```

Comparison

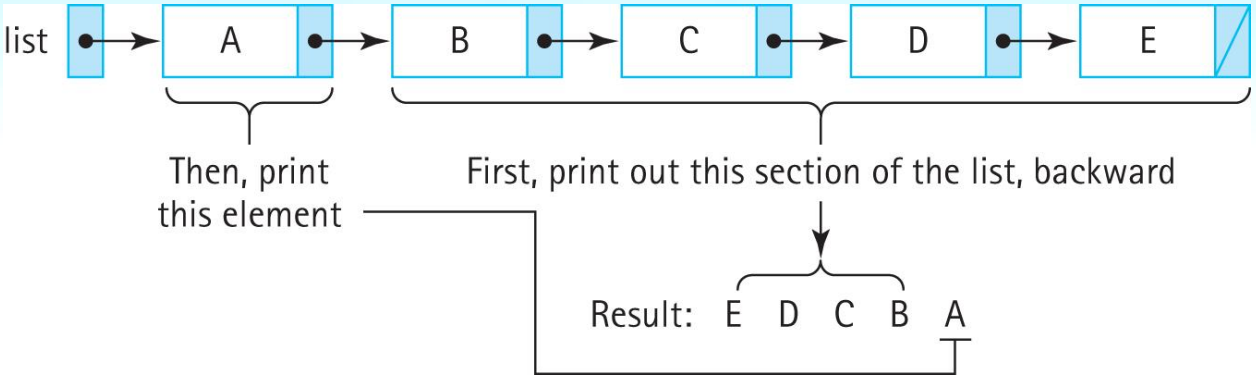
```
void recPrintList(LLNode<String> listRef)
{
   if (listRef != null)
   {
      System.out.println(listRef.getInfo());
      recPrintList(listRef.getLink());
   }
}
```

Iterative approach is better

```
void iterPrintList(LLNode<String> listRef)
{
   while (listRef != null)
   {
      System.out.println(listRef.getInfo());
      listRef = listRef.getLink();
   }
}
```

but what if you want to print the list in reverse???

Reverse Printing a Linked List




```
void recPrintList(LLNode<String> listRef)
{
   if (listRef != null)
   {
      recPrintList(listRef.getLink());
      System.out.println(listRef.getInfo());
   }
}
```

Solving this problem iteratively is not easy

Transforming a linked list recursively

- For example, insert an element at the end of the list

```
void recInsertEnd(String newInfo, LLNode<String> listRef)
// Adds newInfo to the end of the listRef linked list
{
  if (listRef.getLink() != null)
    recInsertEnd(newInfo, listRef.getLink());
  else
    listRef.setLink(new LLNode<String>(newInfo));
}
```

Although this works in the general case it does not work if the original list is empty

Problem with the approach

```
void recInsertEnd(String newInfo, LLNode<String> listRef)
// Adds newInfo to the end of the listRef linked list
{
  if (listRef.getLink() != null)
    recInsertEnd(newInfo, listRef.getLink());
  else
    listRef.setLink(new LLNode<String>(newInfo));
}
```

If we invoke, for example,
`recInsertEnd("Z",myList)`
the temporary variable `listRef` will hold a linked list consisting of "Z" but `myList` will still be `null`.

Solution

- Rather than return void where we invoke as

```
recInsertEnd("Z", myList)
```

- Need a solution that returns a linked list so we can invoke as

```
myList = recInsertEnd("Z", myList)
```

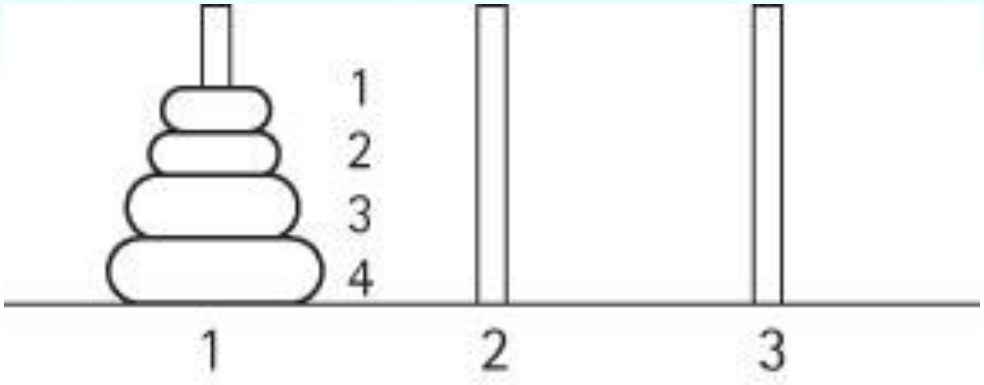
- This is the only way we can change the value of `myList` in the case that the list is empty

Solution

```
LLNode<String> recInsertEnd(String newInfo, LLNode<String> listRef)
// Adds newInfo to the end of the listRef linked list
{
   if (listRef != null)
      listRef.setLink(recInsertEnd(newInfo, listRef.getLink()));
   else
      listRef = new LLNode<String>(newInfo);
   return listRef;
}
```

This works in both the general case and the case where the original list is empty.

3.5 Towers of Hanoi



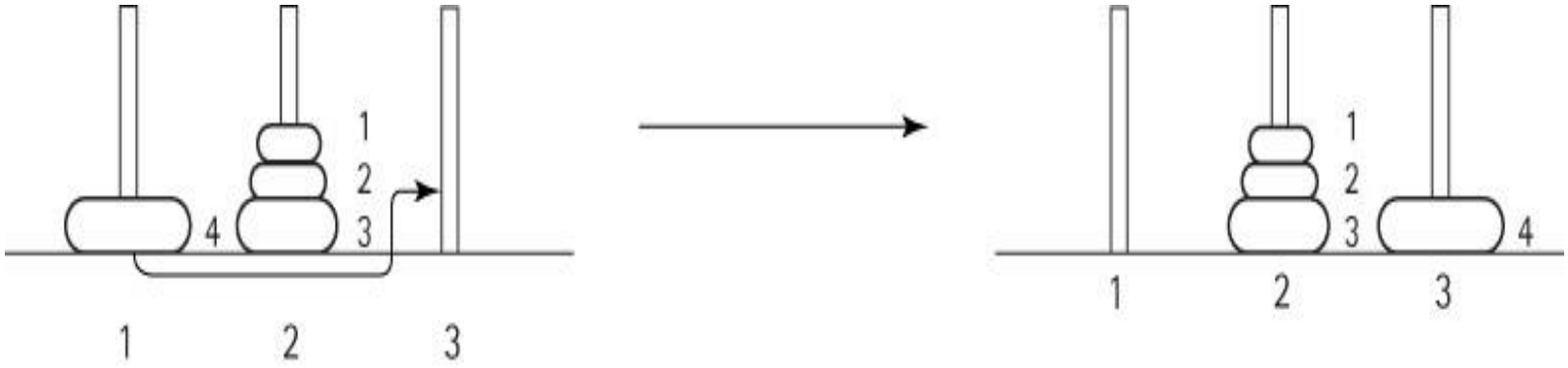
- Move the rings, one at a time, to the third peg.
- A ring cannot be placed on top of one that is smaller in diameter.
- The middle peg can be used as an auxiliary peg, but it must be empty at the beginning and at the end of the game.
- The rings can only be moved one at a time.

General Approach

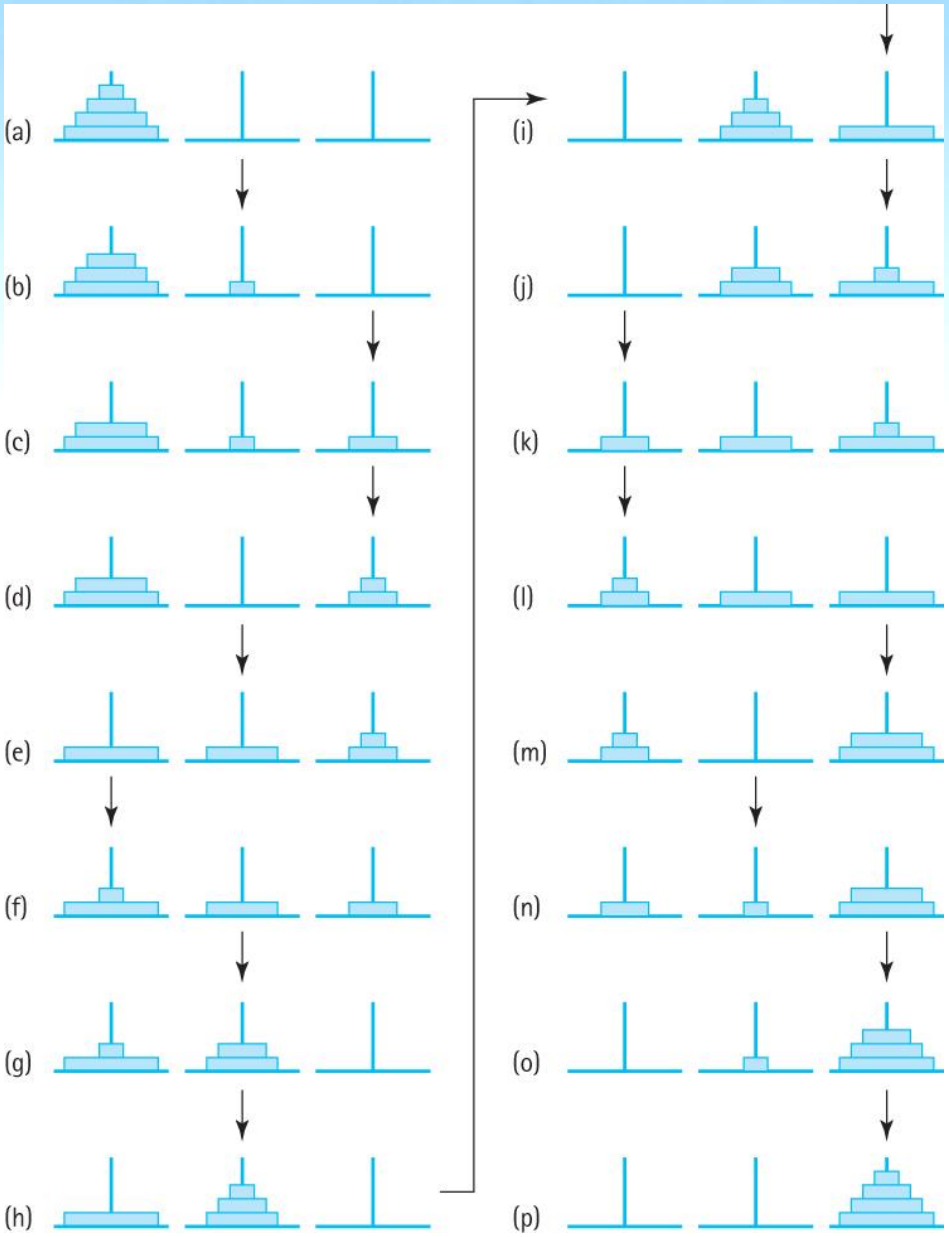
To move the largest ring (ring 4) to peg 3, we must move the three smaller rings to peg 2 (this cannot be done with 1 move).

Let's assume we can do this.

Then ring 4 can be moved into its final place:

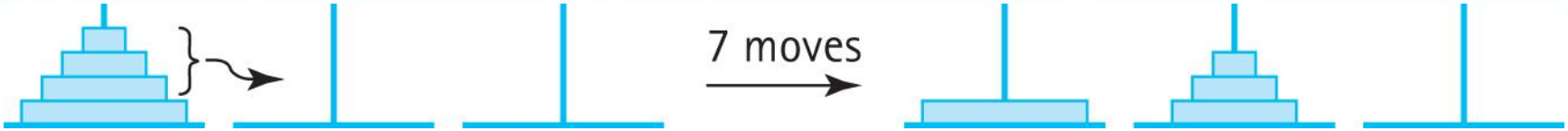


Sample Solution

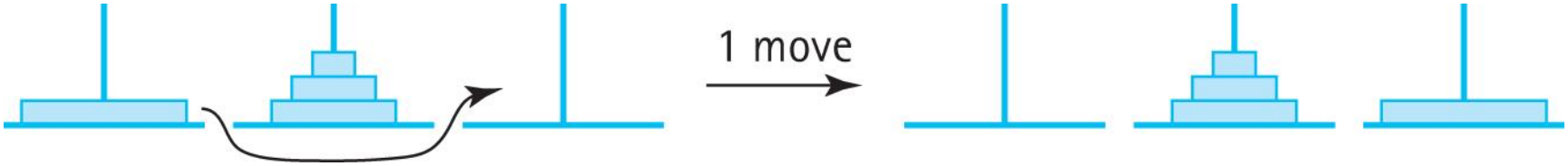


General Approach

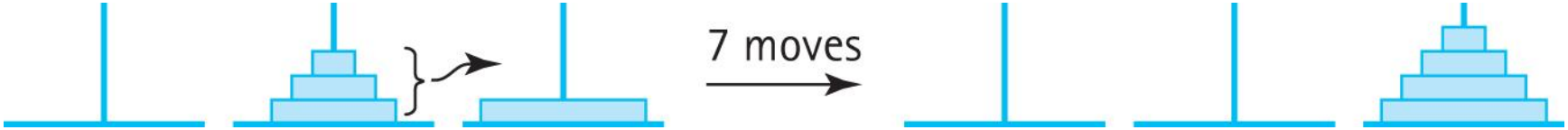
To move the largest ring to peg 3, we must move the three smaller rings to peg 2:



Then the largest ring can be moved to peg 3:



And finally the three smaller rings are moved from peg 2 to peg 3:



Recursion

- Can you see that our assumption (that we move the three smaller rings to peg 2) involved solving a smaller version of the problem? We have solved the problem using recursion.

- The general recursive algorithm for moving *n* rings from the starting peg to the destination peg 3 is:

Move n rings from Starting Peg to Destination Peg
Move n - 1 rings from starting peg to auxiliary peg
Move the nth ring from starting peg to destination peg
Move n - 1 rings from auxiliary peg to destination peg

Recursive Method

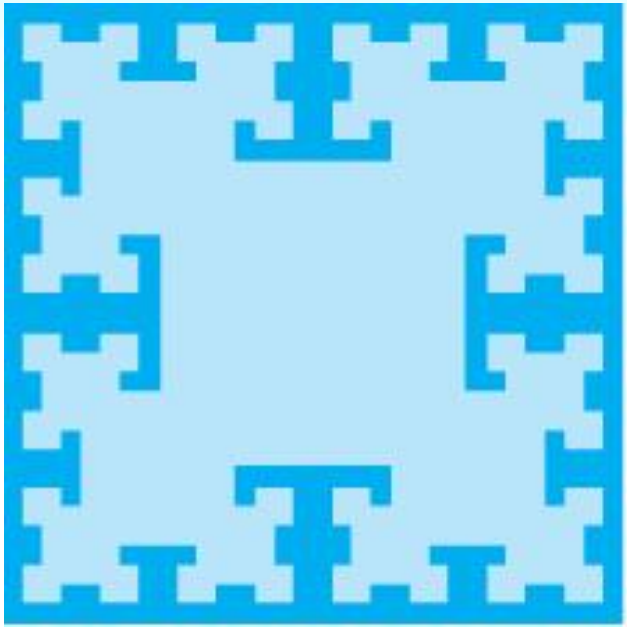
```
public static void doTowers(
   int n,              // Number of rings to move
   int startPeg,       // Peg containing rings to move
   int auxPeg,         // Peg holding rings temporarily
   int endPeg      )   // Peg receiving rings being moved
{
   if (n == 1) // Base case – Move one ring
      System.out.println("Move ring " + n + " from peg " + startPeg
                         + " to peg " + endPeg);
   else
   {
      // Move n - 1 rings from starting peg to auxiliary peg
      doTowers(n - 1, startPeg, endPeg, auxPeg);

      // Move nth ring from starting peg to ending peg
      System.out.println("Move ring " + n + " from peg " + startPeg
                          + " to peg " + endPeg);

      // Move n - 1 rings from auxiliary peg to ending peg
      doTowers(n - 1, auxPeg, startPeg, endPeg);
   }
}
```

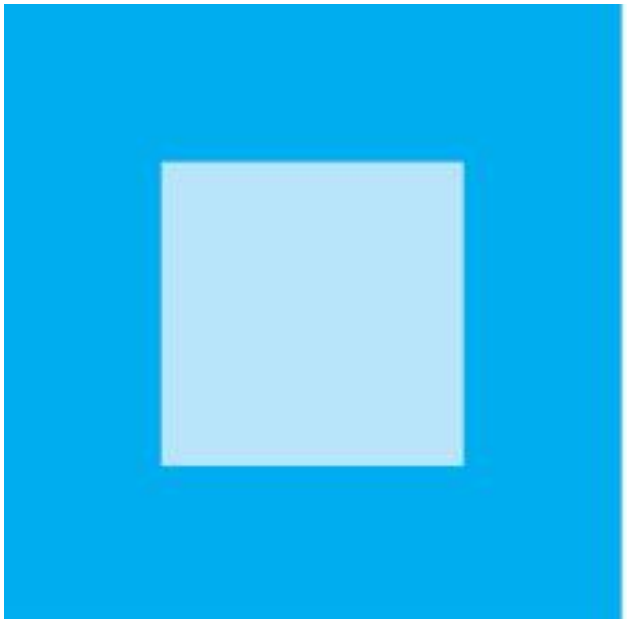
3.6 Fractals

- There are many different ways that people define the term “fractal.”
- For our purposes we define a fractal as an image that is composed of smaller versions of itself.



A T-Square Fractal

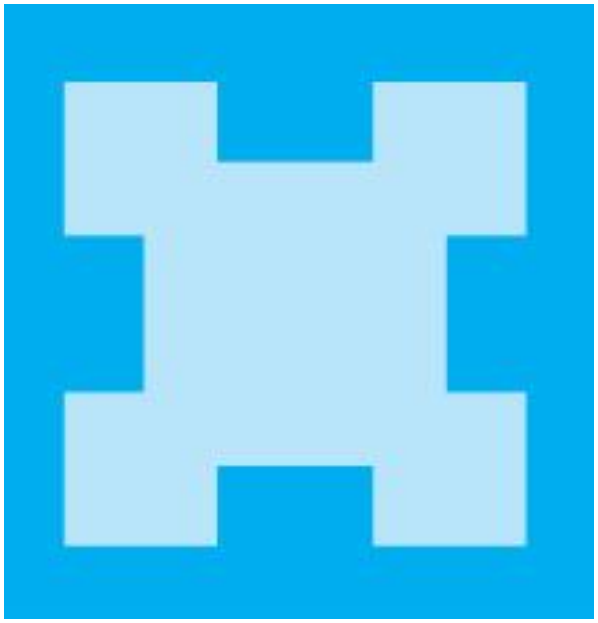
- In the center of a square black* canvas we draw a white* square, one-quarter the size of the canvas:



*Although our publisher supplied figures are tinted blue, our code generates black and white images.

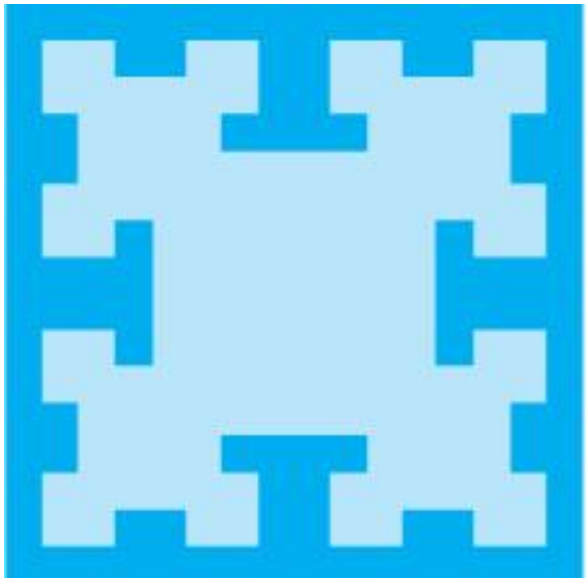
A T-Square Fractal

- We then draw four more squares, each centered at a corner of the original white square, each one-quarter the size of the original white square:



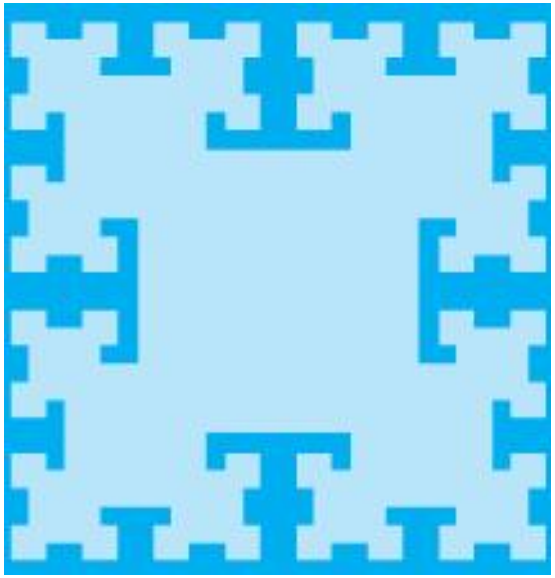
A T-Square Fractal

- For each of these new squares we do the same, (recursively) drawing four squares of smaller size at each of their corners :



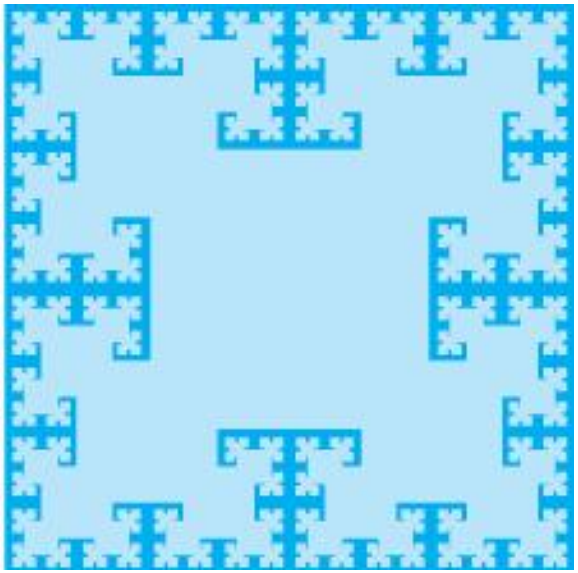
A T-Square Fractal

- And again:



A T-Square Fractal

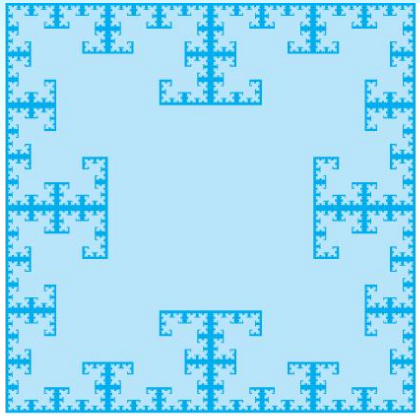
- And again:



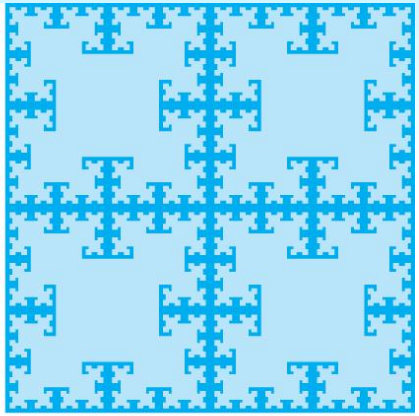
until we can no longer draw any more squares

Code and Demo

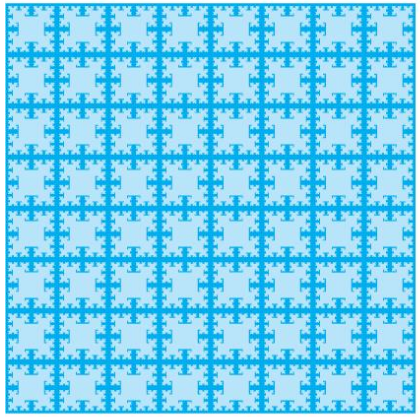
- Instructors can now walk through the code contained in `TSquare.java` in the `ch03.fractals` package and demonstrate the running program.
- Be sure to also check out
  - `TSquareThreshold.java` that allows the user to indicate when to start stop drawing squares
  - `TSquareGray.java` that uses different gray scale levels for each set of differently sized squares



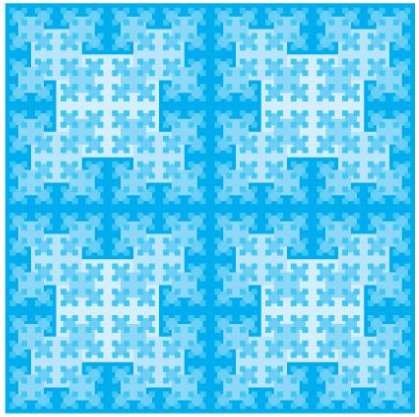
(a) Output from TSquare



(b) Ouput from TSquareThreshold 10 500



(c) Output from TSquareThreshold 1 80



(d) Output from TSquareGray 10 500

3.7 Removing Recursion

- We consider two general techniques that are often substituted for recursion
  - iteration
  - stacking.

Why need to consider non-Recursion solutions?

-Some languages do not support recursion
-Recursive solution is too costly: space and time

- First we take a look at how recursion is implemented.
  - Understanding how recursion works helps us see how to develop non-recursive solutions.

Static Storage Allocation

- A compiler that translates a high-level language program into machine code for execution on a computer must
  - Reserve space for the program variables.
  - Translate the high level executable statements into equivalent machine language statements.

Example of Static Allocation

Consider the following program:

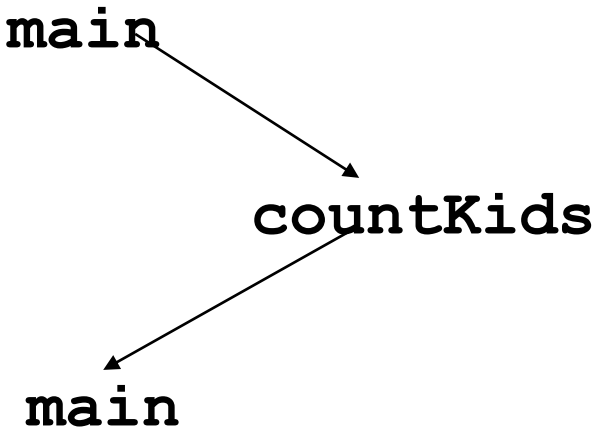
```
public class Kids
{
  private static int countKids(int girlCount, int boyCount)
  {
    int totalKids;
    . . .
  }

  public static void main(String[] args)
  {
    int numGirls; int numBoys; int numChildren;
    . . .
  }
}
```

A compiler could create two separate machine code units for this program, one for the `countKids` method and one for the main method. Each unit would include space for its variables plus the sequence of machine language statements that implement its high-level code.

Limitations of static allocation

- Static allocation like this is the simplest approach possible. But it does not support recursion.



Limitations of static allocation

- Static allocation like this is the simplest approach possible. But it does not support recursion.
- The space for the countKids method is assigned to it at compile time. This works fine when the method will be called **once** and then **always return before it is called again**. But a recursive method can be called again and again before it returns. Where do the second and subsequent calls find space for their parameters and local variables?
- Therefore dynamic storage allocation is needed.

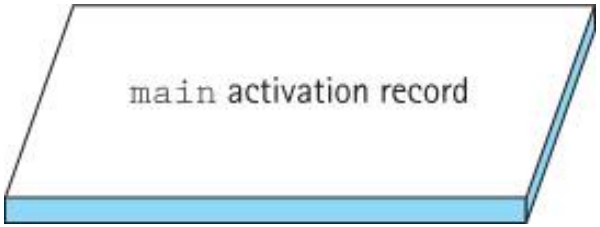
So, some HL only supports static memory allocation cannot use recursion techniques.

Dynamic Storage Allocation

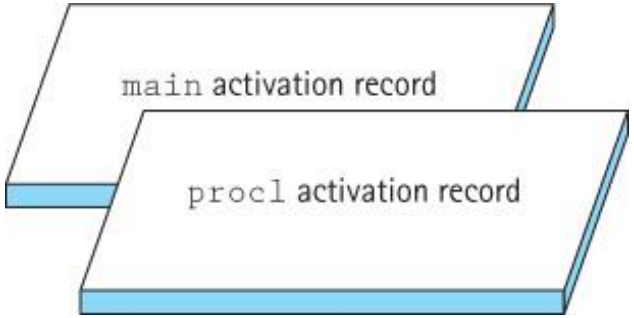
- Dynamic storage allocation provides memory space for a method when it is called.
- When a method is invoked, it needs space to keep its parameters, its local variables, and the return address (the address in the calling code to which the computer returns when the method completes its execution).
- This space is called an **activation record** or **stack frame**.

Dynamic Storage Allocation

Consider a program whose main method calls proc1, which then calls proc2. When the program begins executing, the “main” activation record is generated:

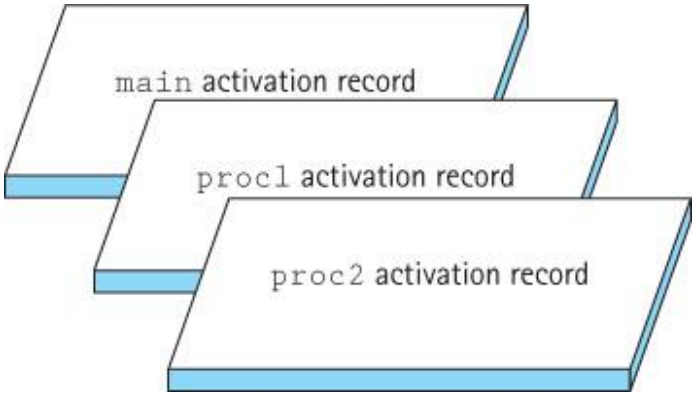


At the first method call, an activation record is generated for proc1:

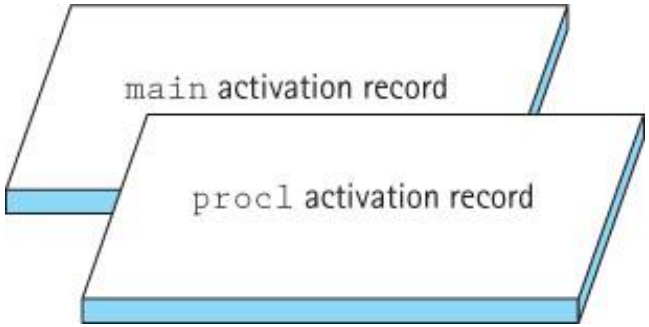


Dynamic Storage Allocation

When proc2 is called from within proc1, its activation record is generated. Because proc1 has not finished executing, its activation record is still around:



When proc2 finishes executing, its activation record is released:



Dynamic Storage Allocation

- The order of activation follows the Last-In-First-Out rule.
- **Run-time or system stack** A system data structure that keeps track of activation records during the execution of a program
- Each nested level of method invocation adds another activation record to the stack. As each method completes its execution, its activation record is popped from the stack. Recursive method calls, like calls to any other method, cause a new activation record to be generated.
- **Depth of recursion** The number of activation records on the system stack, associated with a given a recursive method

Removing Recursion - Iteration

- Suppose the recursive call is the last action executed in a recursive method (tail recursion)
  - The recursive call causes an activation record to be put on the run-time stack to contain the invoked method's arguments and local variables.
  - When this recursive call finishes executing, the run-time stack is popped and the previous values of the variables are restored.
  - Execution continues where it left off before the recursive call was made.
  - But, because the recursive call is the last statement in the method, there is nothing more to execute and the method terminates without using the restored local variable values.
- In such a case the recursion can easily be replaced with iteration.

Example of eliminating tail recursion

```
public static int factorial(int n)
{
  if (n == 0)
    return 1;           // Base case
  else
    return (n * factorial(n - 1));     // General case
}
```

Declare a variable to hold the intermediate values; initialize it to the value returned in the base case. Use a *while* loop so that each time through the loop corresponds to one recursive call. The loop should continue processing until the base case is met:

```
private static int factorial(int n)
{
  int retValue = 1;   // return value
  while (n != 0)
  {
    retValue = retValue * n;
    n = n - 1;
  }
  return(retValue);
}
```

Remove Recursion - Stacking

- When the recursive call is not the last action executed in a recursive method, we cannot simply substitute a loop for the recursion.
- In such cases we can “mimic” recursion by using our own stack to save the required information when needed, as shown in the Reverse Print example on the next slide.

Remove Recursion - Stacking

```
static void iterRevPrintList(LLNode<String> listRef)
// Prints the contents of the listRef linked list to standard output
// in reverse order
{
   StackInterface<String> stack = new LinkedStack<String>();
   while (listRef != null) // put info onto the stack
   {
      stack.push(listRef.getInfo());
      listRef = listRef.getLink();
   }

   // Retrieve references in reverse order and print elements
   while (!stack.isEmpty())
   {
      System.out.println(stack.top());
      stack.pop();
   }
}
```

3.8 Deciding Whether to Use a Recursive Solution

- In this section we consider factors used in deciding whether or not to use a recursive solution to a problem.
- The main issues are the efficiency and the clarity of the solution.

Recursive Solution

- You must be careful when using recursion.
- Recursive solutions can be less efficient than iterative solutions.
- Still, many problems lend themselves to simple, elegant, recursive solutions.

Efficiency Considerations

- Recursion Overhead
  - A recursive solution usually has more “overhead” than a non-recursive solution because of the number of method calls
    - time: each call involves processing to create and dispose of the activation record, and to manage the run-time stack
    - space: activation records must be stored
- Inefficient Algorithms
  - Another potential problem is that a particular recursive solution might just be inherently inefficient. This can occur if the recursive approach repeatedly solves the same sub-problem, over and over again

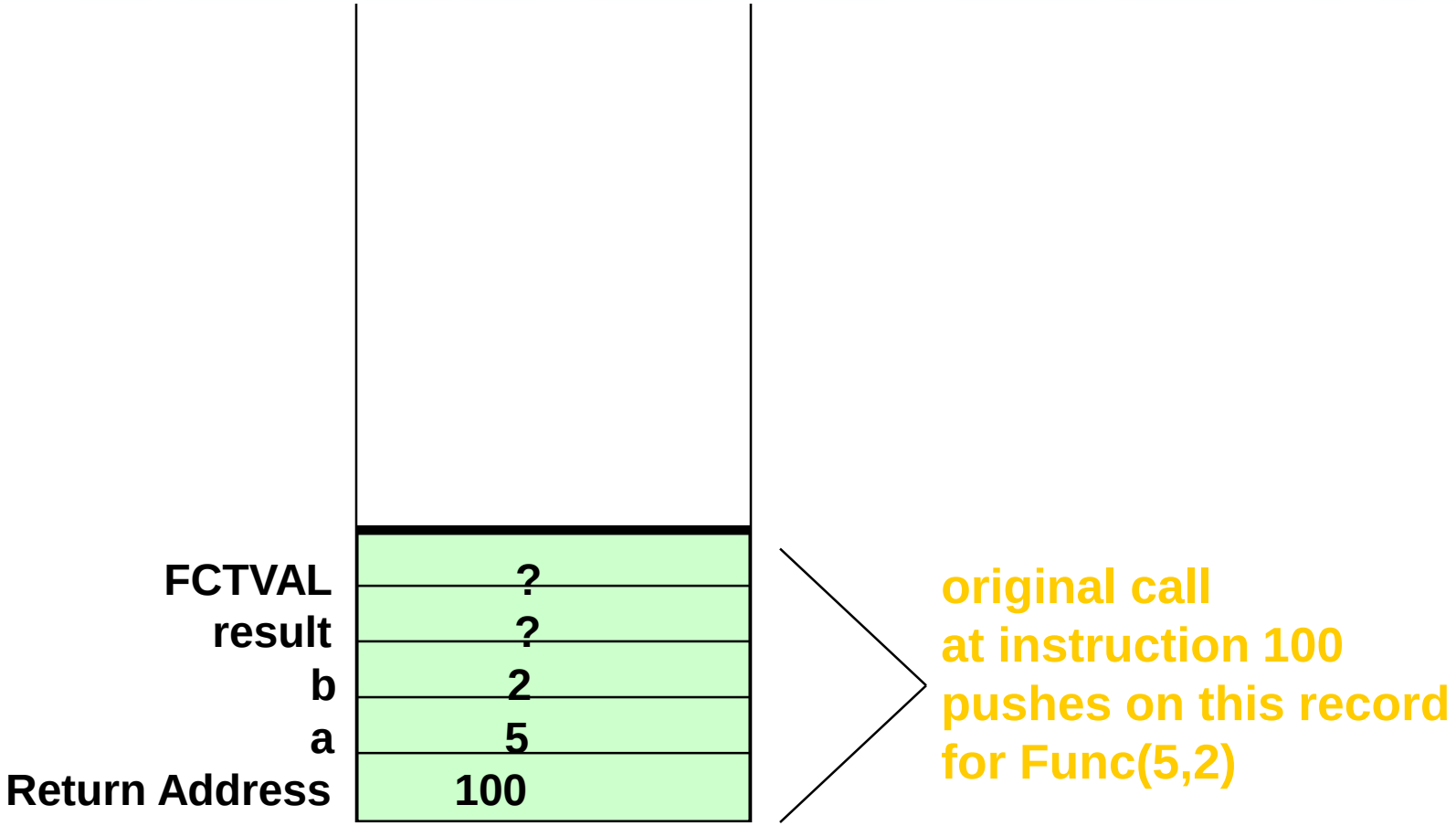
Clarity

- For many problems, a recursive solution is simpler and more natural for the programmer to write. The work provided by the system stack is “hidden” and therefore a solution may be easier to understand.
- Compare, for example, the recursive and nonrecursive approaches to printing a linked list in reverse order that were developed previously in this chapter. In the recursive version, we let the system take care of the stacking that we had to do explicitly in the nonrecursive method.

So….

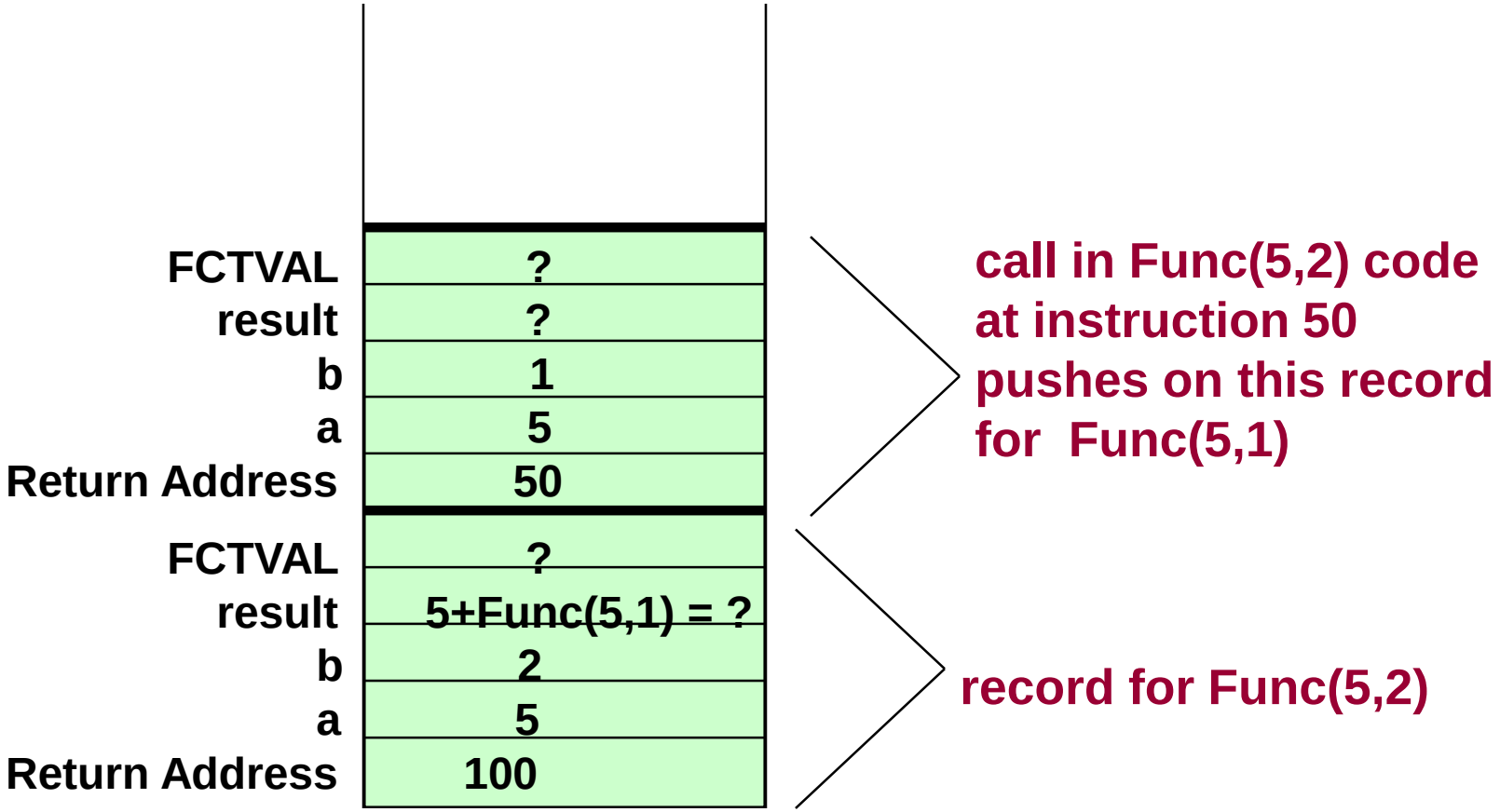
- Recursion is a very powerful problem-solving technique. Why? A complex algorithm with recursion approach make the flow simple, shorter and easy to understand the source codes.
- Trade offs: time and space vs. simplicity of algorithm
- Caution: must have a single base case to stop recursion calls; otherwise infinite loops

x = Func(5, 2); // original call is instruction 100

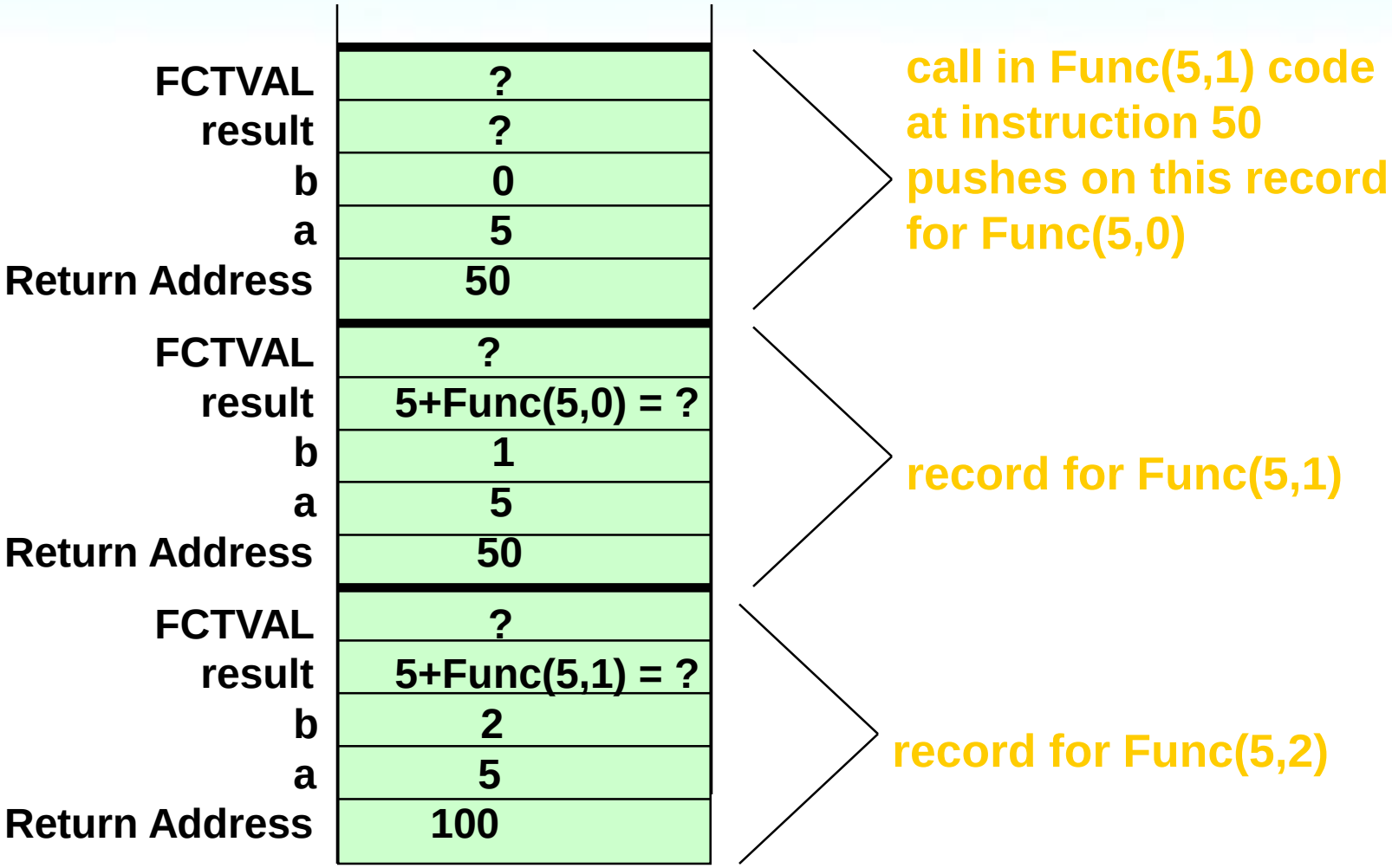


```
x = Func(5, 2);
```

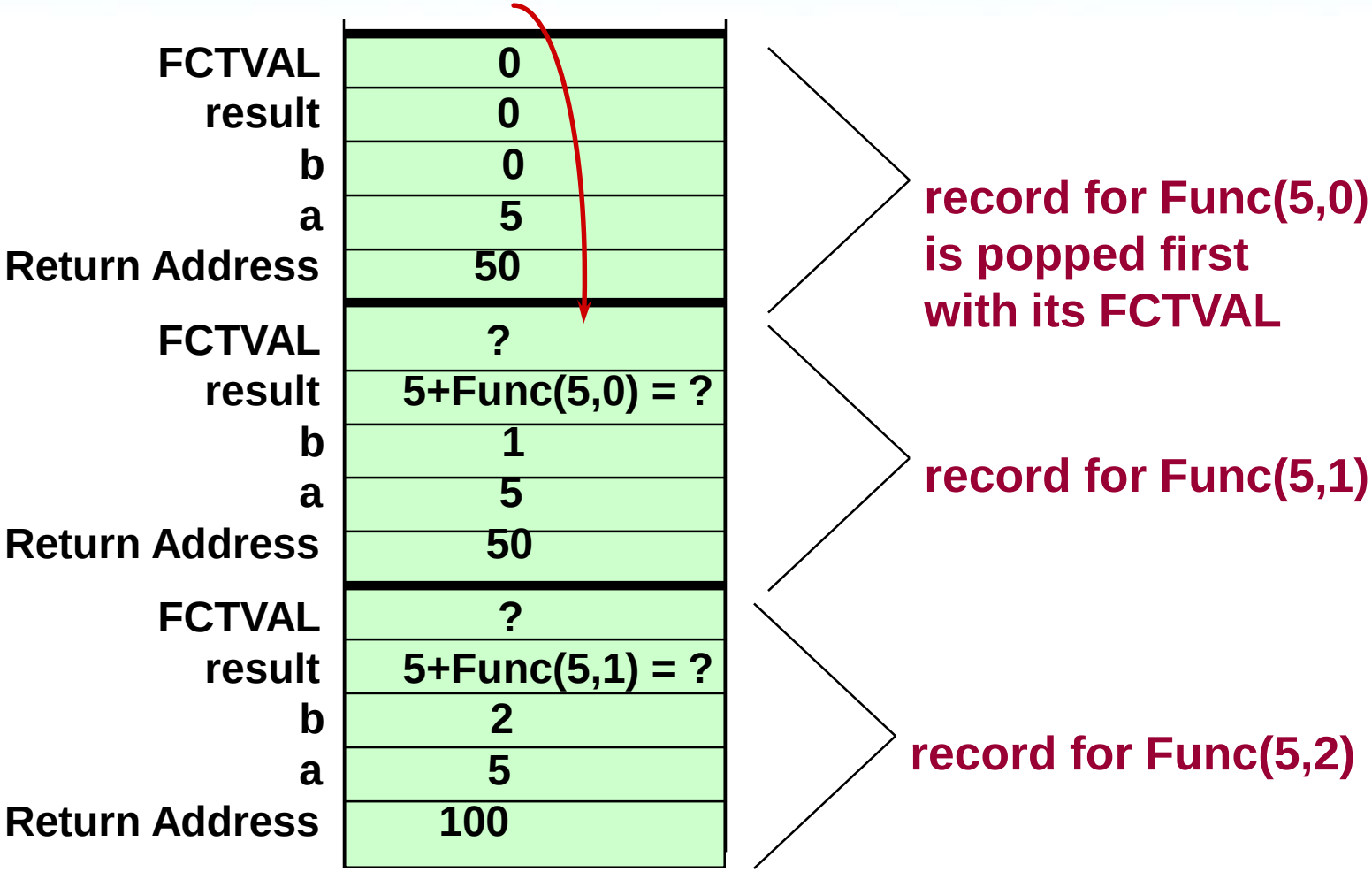
// original call at instruction 100



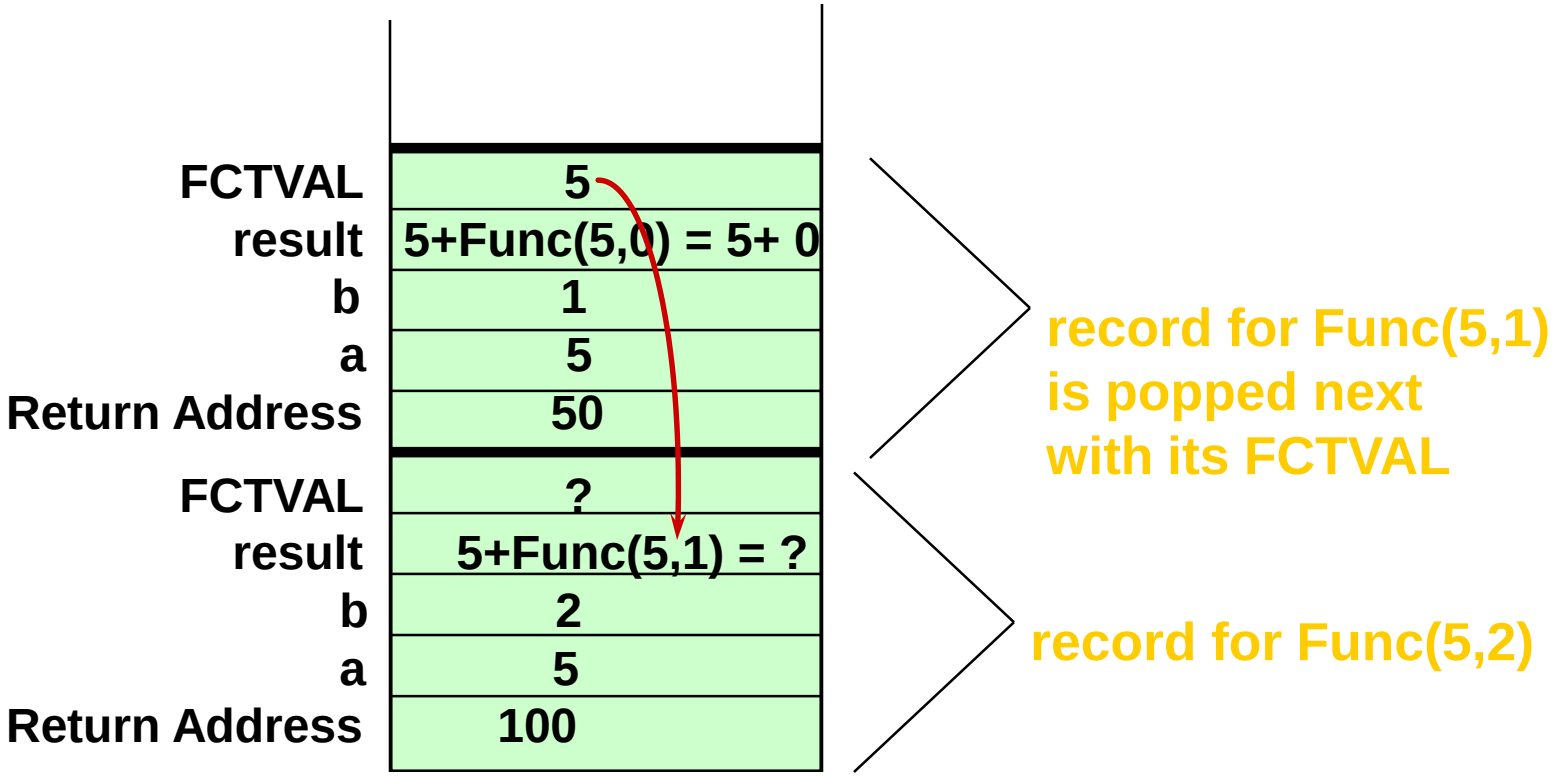
x = Func(5, 2); // original call at instruction 100



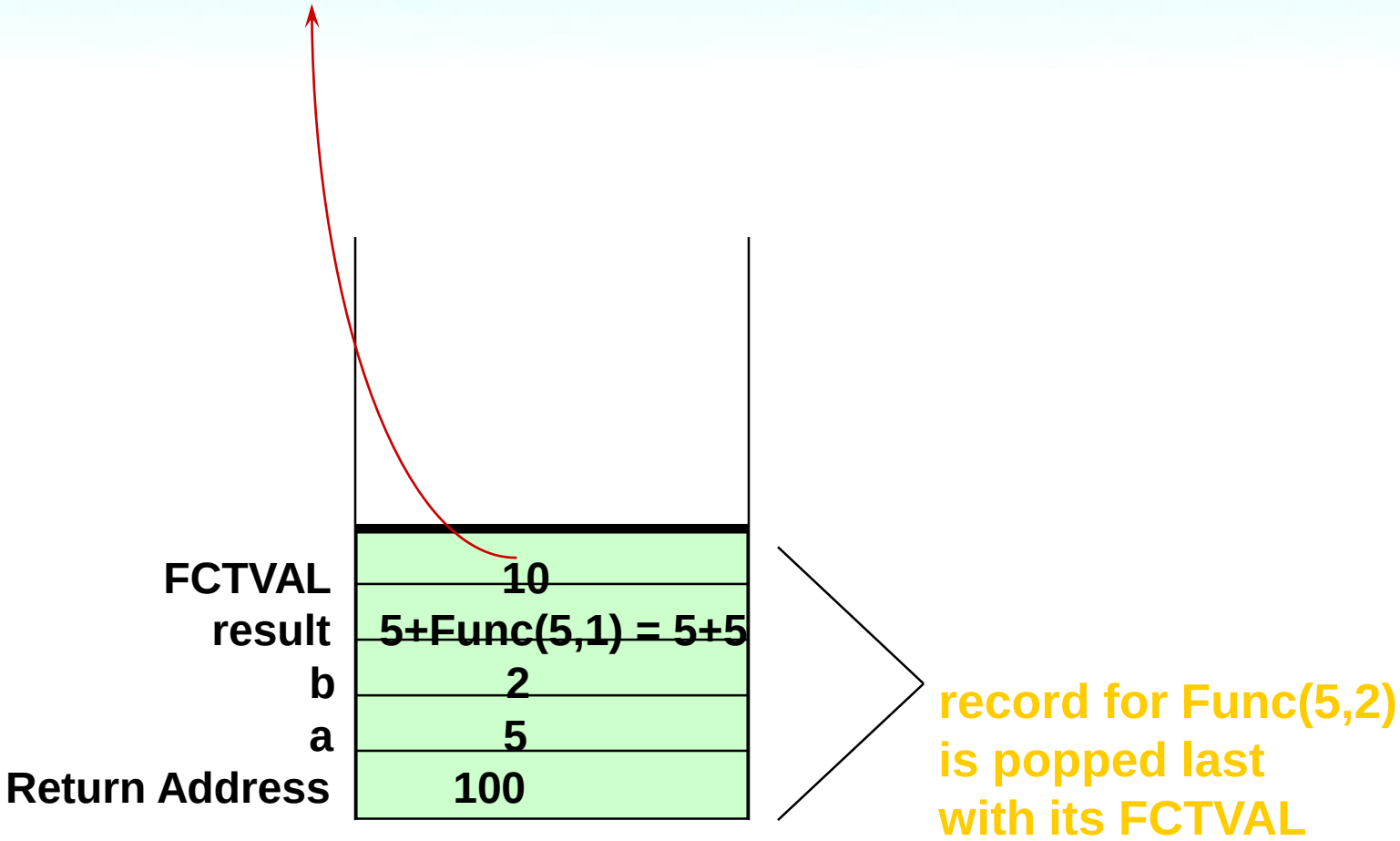
x = Func(5, 2); // original call at instruction 100



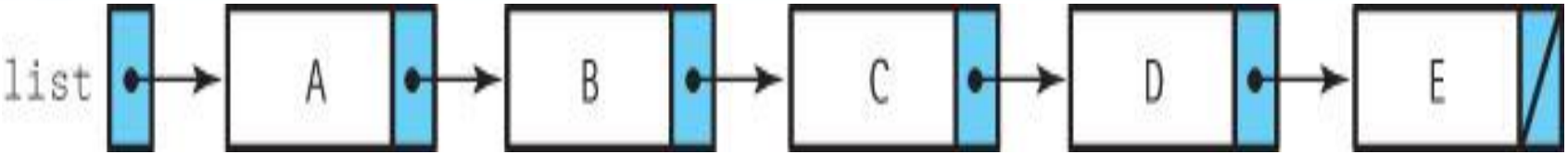
x = Func(5, 2); // original call at instruction 100



x = Func(5, 2); // original call at line 100



3.A Recursive Linked-List Processing



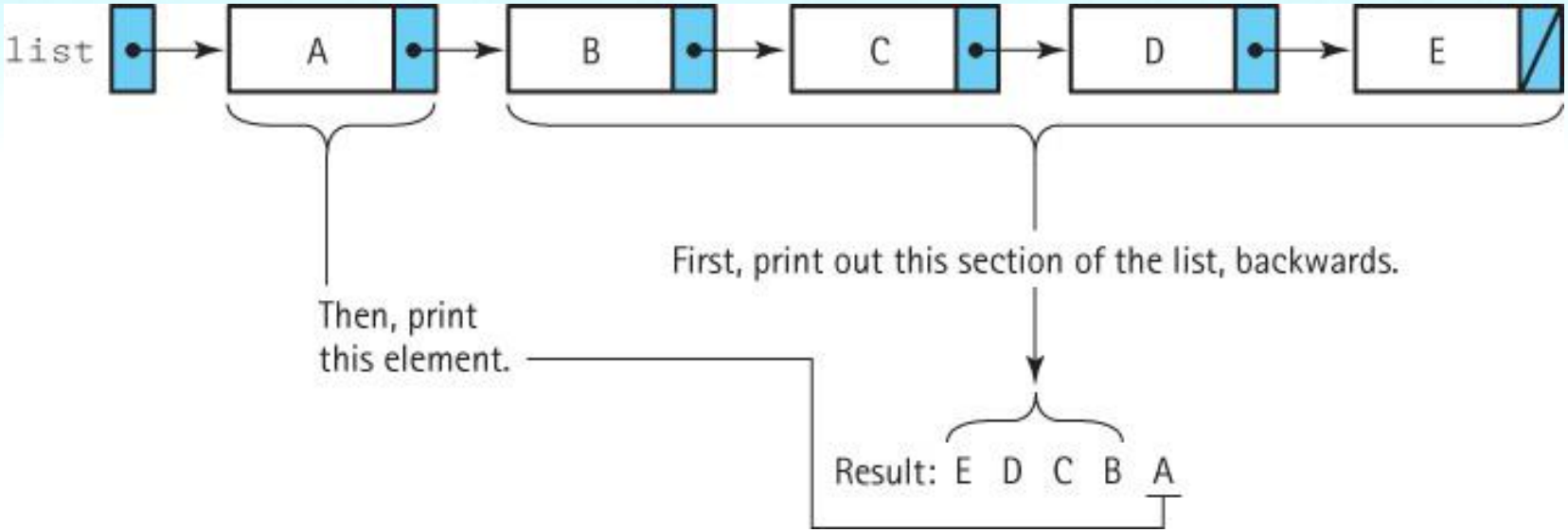
- Reverse Printing: Our goal is to print the elements of a linked list in *reverse* order.
- This problem is much more easily and elegantly solved recursively than it is iteratively.

3.A Recursive Linked-List Processing

Reverse Print (listRef)

Definition:	Prints out the list referenced by listRef in reverse order.
Size:	Number of elements in the list referenced by listRef.
Base Case:	If the list is empty, do nothing.
General Case:	Reverse Print the list referenced by listRef.next, then print listRef.info.

Recursive Reverse Print



Recursive revPrint (listRef)

Print out the second through last elements in the list referenced by listRef in reverse order.

Then print the first element in the list referenced by listRef

Extending LinkedStack with a Reverse Print

```
import ch03.stacks.*;
import support.LLObjectNode;

public class LinkedStack2 extends LinkedStack
{
  private void revPrint(LLObjectNode listRef)
  {
    if (listRef != null)
    {
      revPrint(listRef.getLink());
      System.out.println(" " + listRef.getInfo());
    }
  }

  public void printReversed()
  {
    revPrint(top);
  }
}
```