CS 480

Introduction to Artificial Intelligence

March 2, 2023

Announcements / Reminders

- **Please follow the Week 08 To Do List instructions**

Plan for Today

- **Predicate / First-Order Logic**

Proof by Resolution

The process of proving by resolution is as follows:

A. **Formalize the problem: "English to Predicate Logic"**

B. **Derive** $KB \wedge \neg Q$ [in general: NOT(Some Sentence)]

C. **Convert** $KB \wedge \neg Q$ **into CNF ("standardized") form**

D. **Apply resolution rule to resulting clauses. New clauses will be generated (add them to the set if not already present)**

E. **Repeat (D) until:**

   a. **no new clause can be added (KB does NOT entail Q)**

   b. **last two clauses resolve to yield the empty clause (KB entails Q)**

Proof by Resolution

The process of proving by resolution is as follows:

A. **Formalize the problem: "English to Predicate Logic"**
B. **Negate the input statement/claim C to obtain $\neg C$**
C. **Convert $\neg C$ into CNF ("standardized") form**
D. **Apply resolution rule to resulting clauses. New clauses will be generated (add them to the set if not already present)**
E. **Repeat (D) until:**
   a. **no new clause can be added (C is false)**
   b. **last two clauses resolve to yield the empty clause (C is true)**

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$
2. Reduce the scope of all $\neg$ to single term (De Morgan)
3. Make all variable names unique (standardize apart)
4. Move quantifiers left (convert to prenex normal form)
5. Eliminate Existential quantifiers (skolemization)
6. Eliminate Universal quantifiers
7. Convert to conjunction of disjuncts
8. Create separate clause for each conjunct

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall w\ (P_5(w))]$$

Predicate (First-Order) Logic to CNF

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$$\forall w\ ([P_1(w) \lor P_2(w) \Rightarrow P_3(w)] \lor [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall w\ (P_5(w))]$$

By Implication Law ($p \Rightarrow q \equiv \neg p \lor q$):

$$\forall w\ ([\underline{\mathbf{P_1(w) \lor P_2(w) \Rightarrow P_3(w)}}] \lor [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall w\ (P_5(w))]$$

becomes:

$$\forall w\ ([\underline{\mathbf{\neg(P_1(w) \lor P_2(w)) \lor P_3(w)}}] \lor [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall w\ (P_5(w))]$$

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becomes:

$$\forall w\ ([\neg(P_1(w) \lor P_2(w)) \lor P_3(w)] \lor [\exists x\ (\exists y\ (\underline{\mathbf{\neg P_6(x, y) \lor P_4(w, x)}})))]) \land [\forall w\ (P_5(w))]$$

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By De Morgan's Law ($\neg(p \lor q) \equiv \neg p \land \neg q$):

$$\forall w\ ([\underline{\neg(P_1(w) \lor P_2(w))} \lor P_3(w)] \lor [\exists x\ (\exists y\ (\neg P_6(x, y) \lor P_4(w, x))]]) \land [\forall w\ (P_5(w))]$$

becomes:

$$\forall w\ ([\underline{(\neg P_1(w) \land \neg P_2(w))} \lor P_3(w)] \lor [\exists x\ (\exists y\ (\neg P_6(x, y) \lor P_4(w, x))]]) \land [\forall w\ (P_5(w))]$$

Predicate (First-Order) Logic to CNF

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Original sentence S:

$$\forall w\ ([P_1(w) \lor P_2(w) \Rightarrow P_3(w)] \lor [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall w\ (P_5(w))]$$

Variable w (w and w) is bound to two different quantifiers:

$$\forall w\ ([\neg(P_1(w) \lor P_2(w)) \lor P_3(w)] \lor [\exists x\ (\exists y\ (\neg P_6(x, y) \lor P_4(w, x))]) \land [\forall w\ (P_5(w))]$$

Replace w with z and the sentence S becomes:

$$\forall w\ ([(\neg P_1(w) \land \neg P_2(w)) \lor P_3(w)] \lor [\exists x\ (\exists y\ (\neg P_6(x, y) \lor P_4(w, x))]) \land [\forall z\ (P_5(z))]$$

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$
2. Reduce the scope of all $\neg$ to single term (De Morgan)
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Original sentence S:

$$\forall w\ ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall z\ (P_5(z))]$$

Quantified variables unique, move quantifiers left (order!):

$$\forall w\ ([\neg(P_1(w) \vee P_2(w)) \vee P_3(w)] \vee [\underline{\exists \mathbf{x}\ (\exists \mathbf{y}}\ (\neg P_6(x, y) \vee P_4(w, x))]) \wedge [\underline{\forall \mathbf{z}}\ (P_5(z))]$$

becomes:

$$\forall w\ \underline{\exists \mathbf{x}\ \exists \mathbf{y}\ \forall \mathbf{z}}\ ([(\neg P_1(w) \wedge \neg P_2(w)) \vee P_3(w)] \vee [(\neg P_6(x, y) \vee P_4(w, x))]) \wedge [(P_5(z))]$$

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$
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Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall w\ (P_5(w))]$$

We have two existential quantifiers to remove here:

$$\forall w\ \underline{\exists x}\ \exists y\ \forall z\ ([(\neg P_1(w) \wedge \neg P_2(w)) \vee P_3(w)] \vee [(\underline{\neg P_6(x, y)} \vee \underline{P_4(w, x)})]) \wedge [(P_5(z))]$$

and:

$$\forall w\ \exists x\ \underline{\exists y}\ \forall z\ ([(\neg P_1(w) \wedge \neg P_2(w)) \vee P_3(w)] \vee [(\underline{\neg P_6(x, y)} \vee P_4(w, x))]) \wedge [(P_5(z))]$$

Both $\exists x$ and $\exists y$ are inside the scope of the universal quantifier $\forall w$. We need to use Skolem function substitution (Skolemization).

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \lor P_2(w) \Rightarrow P_3(w)] \lor [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall z\ (P_5(z))]$$

Let's start with $\exists x$ and replace x with a Skolem function:

$$\forall w\ \underline{\exists x}\ \exists y\ \forall z\ ([(\neg P_1(w) \land \neg P_2(w)) \lor P_3(w)] \lor [(\underline{\neg P_6(x, y)} \lor \underline{P_4(w, x)})]) \land [(P_5(z))]$$

becomes:

$$\forall w \exists y \forall z\ ([(\neg P_1(w) \land \neg P_2(w)) \lor P_3(w)] \lor [(\underline{\neg P_6(f(w), y)} \lor \underline{P_4(w, f(w))})]) \land [(P_5(z))]$$

Quantified variable x was replaced with Skolem function $f(w)$. Existential quantifier $\exists x$ was removed.

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall z\ (P_5(z))]$$

Now: remove $\exists y$ and replace y with a Skolem function:

$$\forall w \exists y \forall z\ ([(\neg P_1(w) \wedge \neg P_2(w)) \vee P_3(w)] \vee [(\underline{\neg P_6(f(w), y)} \vee P_4(w, f(w)))]) \wedge [(P_5(z))]$$

becomes:

$$\forall w \forall z\ ([(\neg P_1(w) \wedge \neg P_2(w)) \vee P_3(w)] \vee [(\underline{\neg P_6(f(w), g(w))} \vee P_4(w, f(w)))]) \wedge [(P_5(z))]$$

Quantified variable y was replaced with Skolem function $g(w)$. Existential quantifier $\exists y$ was removed.

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Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \lor P_2(w) \Rightarrow P_3(w)] \lor [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall z\ (P_5(z))]$$

Remaining quantified variables are universally quantified:

$$\forall w \forall z\ ([(\neg P_1(w) \land \neg P_2(w)) \lor P_3(w)] \lor [(\neg P_6(f(w), g(w)) \lor P_4(w, f(w)))]) \land [(P_5(z))]$$

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \lor P_2(w) \Rightarrow P_3(w)] \lor [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall z\ (P_5(z))]$$

We can simply "drop" universal quantifiers:

$$\forall w \forall z\ ([(\neg P_1(w) \land \neg P_2(w)) \lor P_3(w)] \lor [(\neg P_6(f(w), g(w)) \lor P_4(w, f(w)))]) \land [(P_5(z))]$$

becomes:

$$([(\neg P_1(w) \land \neg P_2(w)) \lor P_3(w)] \lor [(\neg P_6(f(w), g(w)) \lor P_4(w, f(w)))]) \land [(P_5(z))]$$

We are "dropping" universal quantifiers for inferential purposes only. Equivalence is lost, but we can still use the remaining sentence to infer.

Predicate (First-Order) Logic to CNF

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8. Create separate clause for each conjunct

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall z\ (P_5(z))]$$

By Associative Law (($p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$):

$$([(\neg P_1(w) \wedge \neg P_2(w)) \vee P_3(w)] \vee [(\neg P_6(f(w), g(w)) \vee P_4(w, f(w)))]) \wedge [(P_5(z))]$$

becomes:

$$([P_3(w) \vee (\neg P_1(w) \wedge \neg P_2(w))] \vee [(\neg P_6(f(w), g(w)) \vee P_4(w, f(w)))]) \wedge [(P_5(z))]$$

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \lor P_2(w) \Rightarrow P_3(w)] \lor [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall w\ (P_5(w))]$$

By Associative Law $((p \lor q) \lor r \Leftrightarrow p \lor (q \lor r))$**:**

$$([\underline{\mathbf{P_3(w) \lor (\neg P_1(w) \land \neg P_2(w))}}] \lor [(\neg P_6(f(w), g(w)) \lor P_4(w, f(w)))]) \land [(P_5(z))]$$

becomes:

$$([\underline{\mathbf{P_3(w) \lor (\neg P_1(w) \land \neg P_2(w))}}] \lor [(\neg P_6(f(w), g(w)) \lor P_4(w, f(w)))]) \land [(P_5(z))]$$

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall w\ (P_5(w))]$$

By Distributive Law ($p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$):

$$([\underline{\mathbf{P_3(w) \vee (\neg P_1(w) \wedge \neg P_2(w))}}] \vee [(\neg P_6(f(w), g(w)) \vee P_4(w, f(w)))]) \wedge [(P_5(z))]$$

becomes:

$$([\underline{\mathbf{(P_3(w) \vee \neg P_1(w)) \wedge (P_3(w) \vee \neg P_2(w))}}] \vee [(\neg P_6(f(w), g(w)) \vee P_4(w, f(w)))]) \wedge [(P_5(z))]$$

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \lor P_2(w) \Rightarrow P_3(w)] \lor [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall w\ (P_5(w))]$$

Let's make some substitutions:

$$([(P_3(w) \lor \neg P_1(w)) \land (P_3(w) \lor \neg P_2(w))] \lor [(\neg P_6(f(w), g(w)) \lor P_4(w, f(w)))]) \land [(P_5(z))]$$

$A \equiv (P_3(w) \lor \neg P_1(w))$

$B \equiv (P_3(w) \lor \neg P_2(w))$

$C \equiv (\neg P_6(f(w), g(w)) \lor P_4(w, f(w)))$

so the sentence becomes:

$$([A \land B] \lor [C]) \land [(P_5(z))]$$

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \lor P_2(w) \Rightarrow P_3(w)] \lor [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall w\ (P_5(w))]$$

By Distributive Law ($p \lor (q \land r) \Leftrightarrow (p \lor q) \land (p \lor r)$):

$$([A \land B] \lor [C]) \land [(P_5(z))]$$

becomes:

$$((A \lor C) \land (B \lor C)) \land [(P_5(z))]$$

where:

$A \equiv (P_3(w) \lor \neg P_1(w))$

$B \equiv (P_3(w) \lor \neg P_2(w))$

$C \equiv (\neg P_6(f(w), g(w)) \lor P_4(w, f(w)))$

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \vee P_2(w) \Rightarrow P_3(w)] \vee [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \wedge [\forall w\ (P_5(w))]$$

Remove substitutions:

$$((A \vee C) \wedge (B \vee C)) \wedge [(P_5(z))]$$

becomes:

$$(((P_3(w) \vee \neg P_1(w)) \vee (\neg P_6(f(w), g(w)) \vee P_4(w, f(w)))) \wedge ((P_3(w) \vee \neg P_2(w)) \vee (\neg P_6(f(w), g(w)) \vee P_4(w, f(w))))) \wedge [(P_5(z))]$$

where:

$A \equiv (P_3(w) \vee \neg P_1(w))$

$B \equiv (P_3(w) \vee \neg P_2(w))$

$C \equiv (\neg P_6(f(w), g(w)) \vee P_4(w, f(w)))$

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w\ ([P_1(w) \lor P_2(w) \Rightarrow P_3(w)] \lor [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall w\ (P_5(w))]$$

We can remove some parentheses:

$$(((P_3(w) \lor \neg P_1(w)) \lor (\neg P_6(f(w), g(w)) \lor P_4(w, f(w)))) \land ((P_3(w) \lor \neg P_2(w)) \lor (\neg P_6(f(w), g(w)) \lor P_4(w, f(w))))) \land [(P_5(z))]$$

becomes:

$$(P_3(w) \lor \neg P_1(w) \lor \neg P_6(f(w), g(w)) \lor P_4(w, f(w)))$$
$$\land\ (P_3(w) \lor \neg P_2(w) \lor \neg P_6(f(w), g(w)) \lor P_4(w, f(w)))$$
$$\land\ (P_5(z))$$

Converting FOL to CNF: Example 1

Original sentence S:

$$\forall w \; ([P_1(w) \lor P_2(w) \Rightarrow P_3(w)] \lor [\exists x \; (\exists y \; (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall w \; (P_5(w))]$$

We obtained sentence S in CNF form:

$$(P_3(w) \lor \neg P_1(w) \lor \neg P_6(f(w), g(w)) \lor P_4(w, f(w)))$$
$$\land \; (P_3(w) \lor \neg P_2(w) \lor \neg P_6(f(w), g(w)) \lor P_4(w, f(w)))$$
$$\land \; (P_5(z))$$

Predicate (First-Order) Logic to CNF

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1. **Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$**
2. **Reduce the scope of all $\neg$ to single term (De Morgan)**
3. **Make all variable names unique (standardize apart)**
4. **Move quantifiers left (convert to prenex normal form)**
5. **Eliminate Existential quantifiers (skolemization)**
6. **Eliminate Universal quantifiers**
7. **Convert to conjunction of disjuncts (CNF)**
8. **Create separate clause for each conjunct**

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Original sentence S:

$$\forall w\ ([P_1(w) \lor P_2(w) \Rightarrow P_3(w)] \lor [\exists x\ (\exists y\ (P_6(x, y) \Rightarrow P_4(w, x)))]) \land [\forall w\ (P_5(w))]$$

Let's number all clauses:

$$(P_3(w) \lor \neg P_1(w) \lor \neg P_6(f(w), g(w)) \lor P_4(w, f(w)))_1$$
$$\land\ (P_3(w) \lor \neg P_2(w) \lor \neg P_6(f(w), g(w)) \lor P_4(w, f(w)))_2$$
$$\land\ (P_5(z))_3$$

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6. Eliminate Universal quantifiers
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8. Create separate clause for each conjunct

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

“Everyone who loves all animals is loved by someone”

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

By Implication Law ($p \Rightarrow q \equiv \neg p \vee q$):

$$\forall x\ \underline{[\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]}$$

becomes:

$$\forall x\ \underline{\neg[\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \vee [\exists y\ Loves(y, x)]}$$

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

By Implication Law ($p \Rightarrow q \equiv \neg p \vee q$):

$$\forall x\ \neg[\forall y\ \underline{\mathbf{(Animal(y) \Rightarrow Loves(x, y))}}] \vee [\exists y\ Loves(y, x)]$$

becomes:

$$\forall x\ \neg[(\forall y\ (\neg \mathbf{Animal(y)}) \vee \mathbf{Loves(x, y)}))] \vee [\exists y\ Loves(y, x)]$$

Predicate (First-Order) Logic to CNF

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8. Create separate clause for each conjunct

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

By the equivalence ($\neg\forall x\ (p) \equiv \exists x\ (\neg p)$):

$$\forall x\ \underline{\neg[(\forall y\ (\neg \mathbf{Animal}(y) \vee \mathbf{Loves}(x, y))]} \vee [\exists y\ Loves(y, x)]$$

becomes:

$$\forall x\ \underline{[(\exists y\ \neg(\neg \mathbf{Animal}(y) \vee \mathbf{Loves}(x, y))]} \vee [\exists y\ Loves(y, x)]$$

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

By De Morgan's Law ($\neg(p \lor q) \equiv \neg p \land \neg q$):

$$\forall x\ [(\exists y\ \underline{\mathbf{\neg(\neg Animal(y) \lor Loves(x, y))}}] \lor [\exists y\ Loves(y, x)]$$

becomes:

$$\forall x\ [(\exists y\ \underline{\mathbf{(\neg\neg Animal(y) \land \neg Loves(x, y))}}] \lor [\exists y\ Loves(y, x)]$$

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

By Double Negation Law ($\neg(\neg p) \equiv p$):

$$\forall x\ [(\exists y\ \underline{\mathbf{(\neg\neg Animal(y)}} \wedge \neg Loves(x, y))] \vee [\exists y\ Loves(y, x)]$$

becomes:

$$\forall x\ [(\exists y\ \underline{\mathbf{(Animal(y)}} \wedge \neg Loves(x, y))] \vee [\exists y\ Loves(y, x)]$$

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$
2. Reduce the scope of all $\neg$ to single term (De Morgan)
3. Make all variable names unique (standardize apart)
4. Move quantifiers left (convert to prenex normal form)
5. Eliminate Existential quantifiers (skolemization)
6. Eliminate Universal quantifiers
7. Convert to conjunction of disjuncts
8. Create separate clause for each conjunct

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (\text{Animal}(y) \Rightarrow \text{Loves}(x, y))] \Rightarrow [\exists y\ \text{Loves}(y, x)]$$

Variable y (y and y) is bound to two different quantifiers:

$$\forall x\ [(\exists y\ \underline{\mathbf{(\neg\neg Animal(y)}}\mathbf{)} \wedge \neg\text{Loves}(x, y))] \vee [\exists y\ \text{Loves}(y, x)]$$

Replace y with z and the sentence S becomes:

$$\forall x\ [(\exists y\ \underline{\mathbf{(Animal(y)}}\mathbf{)} \wedge \neg\text{Loves}(x, y))] \vee [\exists z\ \text{Loves}(z, x)]$$

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. **Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$**
2. **Reduce the scope of all $\neg$ to single term (De Morgan)**
3. **Make all variable names unique (standardize apart)**
4. **Move quantifiers left (convert to prenex normal form)**
5. **Eliminate Existential quantifiers (skolemization)**
6. **Eliminate Universal quantifiers**
7. **Convert to conjunction of disjuncts**
8. **Create separate clause for each conjunct**

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

We COULD move $\exists z$ left here:

$$\forall x\ [(\exists y\ (Animal(y) \land \neg Loves(x, y))] \lor [\exists z\ Loves(z, x)]$$

But it will be removed with Skolemization in next step anyway (textbook).

Both approaches will work.

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

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3. **Make all variable names unique (standardize apart)**
4. **Move quantifiers left (convert to prenex normal form)**
5. **Eliminate Existential quantifiers (skolemization)**
6. **Eliminate Universal quantifiers**
7. **Convert to conjunction of disjuncts**
8. **Create separate clause for each conjunct**

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

We have two existential quantifiers to remove ($\exists y$, $\exists z$) :

$$\forall x\ [(\underline{\exists y}\ (Animal(y) \wedge \neg Loves(x, y))] \vee [\exists z\ Loves(z, x)]$$

and:

$$\forall x\ [(\exists y\ (Animal(y) \wedge \neg Loves(x, y))] \vee [\underline{\exists z}\ Loves(z, x)]$$

Both $\exists y$ and $\exists z$ are inside the scope of the universal quantifier $\forall x$. We need to use Skolem function substitution (Skolemization).

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

Let's start with $\exists y$ and replace y with a Skolem function:

$$\forall x\ [(\underline{\exists y}\ (Animal(y) \wedge \neg Loves(x, y))] \vee [\exists z\ Loves(z, x)]$$

becomes:

$$\forall x\ [(Animal(F(x)) \wedge \neg Loves(x, F(x)))] \vee [\exists z\ Loves(z, x)]$$

Quantified variable y was replaced with Skolem function $F(x)$. Existential quantifier $\exists y$ was removed.

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

Now, remove $\exists z$ and replace y with a Skolem function:

$$\forall x\ [(\exists y\ (Animal(y) \land \neg Loves(x, y))] \lor [\underline{\exists z}\ Loves(z, x)]$$

becomes:

$$\forall x\ [(Animal(F(x)) \land \neg Loves(x, F(x)))] \lor [Loves(G(x), x)]$$

Quantified variable z was replaced with Skolem function $G(x)$. Existential quantifier $\exists z$ was removed.

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$
2. Reduce the scope of all $\neg$ to single term (De Morgan)
3. Make all variable names unique (standardize apart)
4. Move quantifiers left (convert to prenex normal form)
5. Eliminate Existential quantifiers (skolemization)
6. Eliminate Universal quantifiers
7. Convert to conjunction of disjuncts
8. Create separate clause for each conjunct

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (\text{Animal}(y) \Rightarrow \text{Loves}(x, y))] \Rightarrow [\exists y\ \text{Loves}(y, x)]$$

Remaining quantified variables are universally quantified:

$$\forall x\ [(\text{Animal}(F(x)) \land \neg\text{Loves}(x, F(x)))] \lor [\text{Loves}(G(x), x)]$$

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (\text{Animal}(y) \Rightarrow \text{Loves}(x, y))] \Rightarrow [\exists y\ \text{Loves}(y, x)]$$

We can simply "drop" universal quantifiers:

$$\forall x\ [(\text{Animal}(F(x)) \wedge \neg \text{Loves}(x, F(x)))] \vee [\text{Loves}(G(x), x)]$$

becomes:

$$[(\text{Animal}(F(x)) \wedge \neg \text{Loves}(x, F(x)))] \vee [\text{Loves}(G(x), x)]$$

We are "dropping" universal quantifiers for inferential purposes only. Equivalence is lost, but we can still use the remaining sentence to infer.

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. **Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$**
2. **Reduce the scope of all $\neg$ to single term (De Morgan)**
3. **Make all variable names unique (standardize apart)**
4. **Move quantifiers left (convert to prenex normal form)**
5. **Eliminate Existential quantifiers (skolemization)**
6. **Eliminate Universal quantifiers**
7. **Convert to conjunction of disjuncts**
8. **Create separate clause for each conjunct**

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

By Commutative Law ($p \vee q \Leftrightarrow q \vee p$):

$$[(Animal(F(x)) \wedge \neg Loves(x, F(x)))] \vee [Loves(G(x), x)]$$

becomes:

$$[Loves(G(x), x)] \vee [(Animal(F(x)) \wedge \neg Loves(x, F(x)))]$$

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$
2. Reduce the scope of all $\neg$ to single term (De Morgan)
3. Make all variable names unique (standardize apart)
4. Move quantifiers left (convert to prenex normal form)
5. Eliminate Existential quantifiers (skolemization)
6. Eliminate Universal quantifiers
7. Convert to conjunction of disjuncts
8. Create separate clause for each conjunct

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

By Distributive Law ($p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$):

$$[Loves(G(x), x)] \vee [(Animal(F(x)) \wedge \neg Loves(x, F(x)))]$$

becomes:

$$(Loves(G(x), x) \vee Animal(F(x))) \wedge (Loves(G(x), x) \vee \neg Loves(x, F(x)))$$

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$
2. Reduce the scope of all $\neg$ to single term (De Morgan)
3. Make all variable names unique (standardize apart)
4. Move quantifiers left (convert to prenex normal form)
5. Eliminate Existential quantifiers (skolemization)
6. Eliminate Universal quantifiers
7. Convert to conjunction of disjuncts
8. Create separate clause for each conjunct

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

Sentence S is now in CNF form:

$$(Loves(G(x), x) \lor Animal(F(x))) \land (Loves(G(x), x) \lor \neg Loves(x, F(x)))$$

Predicate (First-Order) Logic to CNF

Variables and quantifiers are a challenge:

1. Eliminate all equivalences $\Leftrightarrow$ and implications $\Rightarrow$
2. Reduce the scope of all $\neg$ to single term (De Morgan)
3. Make all variable names unique (standardize apart)
4. Move quantifiers left (convert to prenex normal form)
5. Eliminate Existential quantifiers (skolemization)
6. Eliminate Universal quantifiers
7. Convert to conjunction of disjuncts
8. Create separate clause for each conjunct

Converting FOL to CNF: Example 2

Original sentence S:

$$\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$$

Let's number all clauses:

$$(Loves(G(x), x) \vee Animal(F(x))) \wedge (Loves(G(x), x) \vee \neg Loves(x, F(x)))$$

becomes:

$$(Loves(G(x), x) \vee Animal(F(x)))_1 \wedge (Loves(G(x), x) \vee \neg Loves(x, F(x)))_2$$

Predicate Logic Resolution: Example

Consider following sentences in English

A. Everyone who loves all animals is loved by someone
B. Anyone who kills an animal is loved by no one
C. Jack Loves all animals
D. Either Jack or Curiosity killed the cat, who is named Tuna

Q. Did Curiosity kill the cat?

FOL: The Resolution Inference Rule

Two clauses, which are assumed to be standardized apart, so that they share no variables, can be resolved if they contain complementary literals:

- **Propositional literals are complementary if one is the negation of the other**
- **Predicate logic literals are complimentary if one unifies with the negation of the other**

$$\frac{(l_1 \vee \ldots \vee l_k),\ (m_1 \vee \ldots \vee m_n)}{\mathrm{SUBST}(\theta,\ l_1 \vee \ldots \vee l_{i-1} \vee l_{i+1} \ \vee \ldots \vee l_k \vee m_1 \vee \ldots \vee m_{j-1} \vee m_{j+1} \ \vee \ldots \vee m_n)}$$

where $\theta = \mathrm{UNIFY}(l_{i-1}, m_j)$.

FOL: The Resolution Inference Rule

For example, the following two clauses:

$$[\text{Animal}(F(x)) \vee \text{Loves}(G(x), x)]$$ **and**

$$[\neg\text{Loves}(u, v) \vee \neg\text{Kills}(u, v)]$$

can be resolved by eliminating complementary literals

$\text{Loves}(G(x), x)$ **and** $\neg\text{Loves}(u, v)$

with the unifier

$$\theta = \{u/G(x), v/x\},$$

to produce the resolvent clause:

$$[\text{Animal}(F(x)) \vee \neg\text{Kills}(G(x), x)]$$

Predicate Logic Resolution: Example

Now, let's turn them into predicate logic sentences/KB:

A. $\forall x\ [\forall y\ (Animal(y) \Rightarrow Loves(x, y))] \Rightarrow [\exists y\ Loves(y, x)]$

B. $\forall x\ [\exists z\ (Animal(z) \wedge Kills(x, z))] \Rightarrow [\forall y\ \neg Loves(y, x)]$

C. $\forall x\ [Animal(x) \Rightarrow Loves(Jack, x)]$

D. $Kills(Jack, Tuna) \vee Kills(Curiosity, Tuna)$

E. $Cat(Tuna)$

F. $\forall x\ [Cat(x) \Rightarrow Animal(x)]$

Q. $Kills(Curiosity, Tuna)$, **so** $\neg Q \equiv \neg Kills(Curiosity, Tuna)$

Predicate Logic Resolution: Example

Let's turn them into predicate logic CNF sentences/KB:

A1. $(Animal(F(x)) \lor Loves(G(x), x))$ **(A1 and A2 related)**

A2. $(\neg Loves(x, F(x)) \lor Loves(G(x), x))$

B. $(\neg Loves(y, x) \lor \neg Animal(z) \lor \neg Kills(x, z))$

C. $(\neg Animal(x) \lor Loves(Jack, x))$

D. $(Kills(Jack, Tuna) \lor Kills(Curiosity, Tuna))$

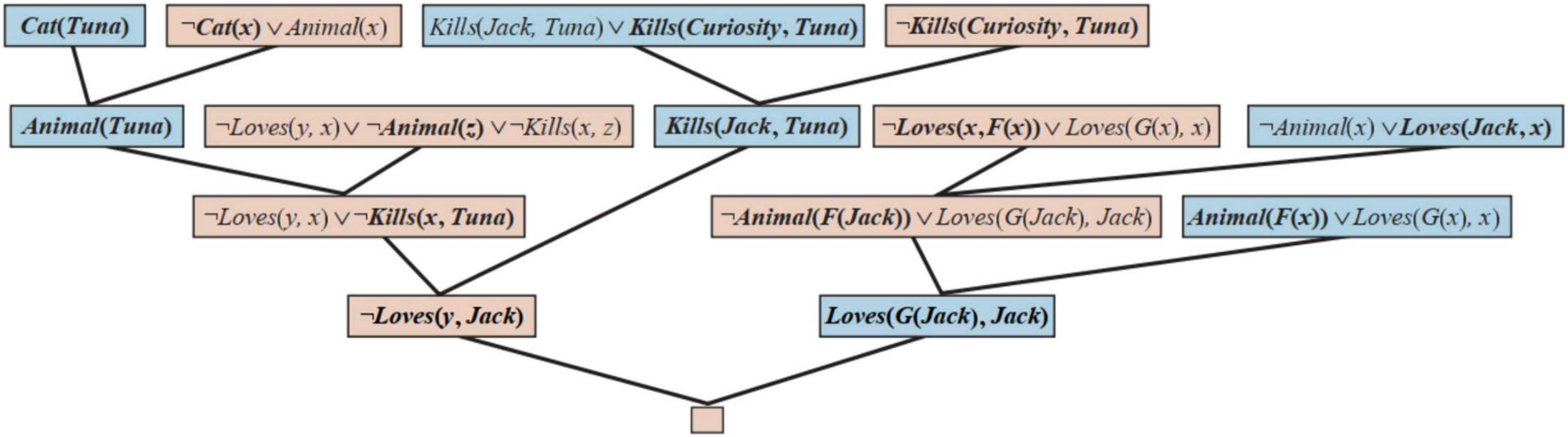
E. $(Cat(Tuna))$

F. $(\neg Cat(x) \lor Animal(x))$

Q. $Kills(Curiosity, Tuna)$**, so** $\neg Q \equiv (\neg Kills(Curiosity, Tuna))$

Predicate Logic Resolution: Example

Resolution process with substitutions:



Notice the use of factoring in derivation of the clause(Loves(G(Jack), Jack)**)**