Homework #8

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Question 1

$$L(\theta) = \prod_{i \in N} \theta e^{-\theta * x_i}$$

$$log(L(\theta)) = \prod_{i \in N} log(\theta e^{-\theta * x_i}) = \sum_{i \in N} log(\theta e^{-\theta * x_i}) = \sum_{i \in N} log(\theta) + log(e^{-\theta * x_i}) = \sum_{i \in N} log(\theta) - \theta * x_i$$

$$\frac{\partial}{\partial \theta} log(L(\theta)) = \frac{\partial}{\partial \theta} \sum_{i \in N} log(\theta) - \theta * x_i = \sum_{i \in N} \frac{1}{\theta} - x_i = 0$$

$$\sum_{i \in N} x_i = \frac{N}{\theta}$$

$$\boxed{\text{Therefore, } \theta = \frac{N}{\sum_{i \in N} x_i}}.$$

Question 2

(a) $\boxed{\text{The MLE for day 1 is 2.5, for day 2 is 1.67, for day 3 is 2.4.}}$.

The MLE for a Poisson distribution is represented by $\frac{\sum_i n_i}{N}$. Therefore the MLE for day 1 is $\frac{3+1+4+2}{4} = 2.5$, for day 2 is $\frac{2+1+2}{3} = 1.67$, for day 3 is $\frac{3+2+2+1+4}{5} = 2.4$.

(b) $\boxed{\text{The MLE for day 4 is 2.167.}}$.

The MLE for day 4 is $\frac{13}{6} = 2.167$.

(c) $\boxed{\text{The MLE for all days is 2.2}}$.

The MLE for all days is $\frac{10+5+12+13}{4+3+5+6} = \frac{40}{18} = 2.2$.

Question 3

(a) The MLE is $\boxed{\dfrac{36}{r} + 36}$.

The probability of landing on a zero slot is $P(zero) = \frac{x}{x+36}$.

The MLE of a geometric model is $\theta = \frac{1}{r+1}$, meaning $\frac{1}{r+1} = \frac{x}{x+36}$, which simplifies to $x = 36/r$.

Adding the total number of zero and non-zero slots, we get $\frac{36}{r} + 36$.

(b) This estimate is **not reliable** as we only saw this occur once.

(c) The MLE is $\boxed{\dfrac{36k}{\theta \sum_{i=1}^{k} r_i} + 36}$

$$L(\theta) = \prod_{i=1}^{k} \theta * (1-\theta)^{r_i}$$

$$log(L(\theta)) = \sum_{i=1}^{k} log(\theta) * log((1-\theta)^{r_i})$$

$$log(L(\theta)) = \sum_{i=1}^{k} log(\theta) * log((1-\theta)^{r_i}) = \sum_{i=1}^{k} log(\theta) + (r_i) * log((1-\theta))$$

$$\frac{\partial}{\partial\theta} log(L(\theta)) = \frac{\partial}{\partial\theta} \sum_{i=1}^{k} log(\theta) + (r_i) * log((1-\theta))$$

$$0 = \sum_{i=1}^{k} \frac{1}{\theta} - \frac{r_i}{1-\theta}$$

$$\frac{k}{\theta} = \sum_{i=1}^{k} \frac{r_i}{1-\theta}$$

$$\frac{k}{\theta} = \frac{1}{1-\theta} \sum_{i=1}^{k} r_i$$

$$k - k\theta = k\theta \sum_{i=1}^{k} r_i$$

$$\theta = \frac{k}{k+\theta \sum_{i=1}^{k} r_i} = \frac{x}{x+36}$$

$$x = \frac{36k}{\theta \sum_{i=1}^{k} r_i}$$

Since the total number of slots is $x + 36$, our MLE is therefore $\frac{36k}{\theta \sum_{i=1}^{k} r_i} + 36$.

Question 4

(a) $P(n|z=0) = \frac{0}{0+36} = \frac{0}{36} = 0$, $P(n|z=1) = \frac{1}{1+36} = \frac{1}{37}$, $P(n|z=2) = \frac{2}{2+36} = \frac{2}{38} = \frac{1}{19}$, $P(n|z=3) = \frac{3}{3+36} = \frac{3}{39} = \frac{1}{13}$

(b) $P(z=0|observations)$ is not 0 implies that the ball never lands on a zero slot. If were to land in a zero slot, this would mean $P(z=0|observations) = 0$.

(c) $P(2; 36; \frac{x}{x+36})$

$P(observations|z=x) = \binom{36}{2} * \frac{x}{x+36}^2 * (1 - \frac{x}{x+36})^3 * 6 - 2$

There are 4 possible situations and respective probabilities:

$$P(observations|z=0) = 0$$

$$P(observations|z=1) = 0.181$$

$$P(observations|z=2) = 0.278$$

$$P(observations|z=3) = 0.245$$

Using this, we can calculate the $P(observations) = 0 + 0.2 * 0.181 + 0.3 * 0.245 + 0.4 * 0.278 = 0.221$

Therefore the $P(z|observations)$:

$$P(0|observations) = 0$$

$$P(1|observations) = 0.164$$

$$P(0|observations) = 0.503$$

$$P(0|observations) = 0.333$$

Question 5

(a) $\boxed{L(\theta) = \binom{50}{35}\theta^3 * 5^2 * (1-\theta)}$

(b) $\boxed{L(\theta) = \binom{99}{54}\theta^5 * 4 * (1-\theta)^4 * 5}$

(c) $\boxed{L(\theta) = \binom{10}{5} * \theta^6 * (1-\theta)^7}$.

We can write the likelihood function for tossing a coin until it comes up heads on the third time based on the geometric distribution which would be: $L(\theta) = \theta * (1-\theta)^2$. Then we can write the likelihood function for coming up heads 5 times in 10 more tosses based on the binomial distribution which would be: $L(\theta) = \binom{10}{5}\theta^5 * (1-\theta)^5$. Combining these two likelihood functions would create $L(\theta) = \binom{10}{5} * \theta^6 * (1-\theta)^7$