Homework #8

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Question 1

How do we set up the likelihood function here?

$$L(\theta) = \prod_{i \in N} \theta e^{-\theta \cdot x_i}$$
$$\log(L(\theta)) = \sum_{i \in N} (\log(\theta) - \theta \cdot x_i)$$
$$\frac{\partial}{\partial \theta} \log(L(\theta)) = \sum_{i \in N} \frac{1}{\theta} - x_i = 0$$
$$\sum_{i \in N} x_i = \frac{N}{\theta}$$
$$\theta = \frac{N}{\sum_{i \in N} x_i}$$

Question 2

What's the MLE for each day separately?

(a) For Day 1, MLE is 2.51, Day 2 is 1.66, and Day 3 is 2.39.

Using the MLE for Poisson, which is sum of counts divided by observations, we get:

$$\text{Day 1: } \frac{3+1+4+2}{4} = 2.51$$
$$\text{Day 2: } \frac{2+1+2}{3} = 1.66$$
$$\text{Day 3: } \frac{3+2+2+1+4}{5} = 2.39$$

(b) For Day 4, the MLE is 2.16.

Using the same method, Day 4's calculation is $\frac{13}{6} = 2.16$.

(c) For all days combined, the MLE is 2.19.

Adding everything together, we get $\frac{10+5+12+13}{4+3+5+6} = \frac{40}{18} = 2.19$.

Question 3

How do we find the MLE for this setup with a geometric distribution?

(a) The MLE we get here is $\frac{36}{r} + 36$.
Here, the probability of a zero outcome is $P(\text{zero}) = \frac{x}{x+36}$.
Setting $\theta = \frac{1}{r+1}$ and matching it with $P(\text{zero})$, we simplify to $x = \frac{36}{r}$.
Then, summing up zero and non-zero cases, the MLE we get is $\frac{36}{r} + 36$.

(b) This isn't very reliable since it's only based on one instance we observed.

(c) The MLE for more trials comes out to $\frac{36k}{\theta \sum_k r_i + 36}$.

$$L(\theta) = \prod_k \theta \cdot (1-\theta)^{r_i}$$
$$\log(L(\theta)) = \sum_k (\log(\theta) + r_i \cdot \log(1-\theta))$$
$$\frac{\partial}{\partial \theta} \log(L(\theta)) = \sum_k \left(\frac{1}{\theta} - \frac{r_i}{1-\theta} \right) = 0$$

So the final MLE here is $\frac{36k}{\theta \sum_k r_i + 36}$.

Question 4

Can we calculate the probabilities based on different outcomes?

(a) The probabilities for each case are calculated as follows:

For $z = 0$: $P(n|z=0) = \frac{0}{0+36} = 0$

For $z = 1$: $P(n|z=1) = \frac{1}{1+36} = \frac{1}{37}$

For $z = 2$: $P(n|z=2) = \frac{2}{2+36} = \frac{2}{38} = \frac{1}{19}$

For $z = 3$: $P(n|z=3) = \frac{3}{3+36} = \frac{3}{39} = \frac{1}{13}$

So, we have that for each possible value of z, the probability $P(n|z)$ decreases as z increases.

(b) Here, $P(z=0|\text{observations})$ isn't zero, so the ball didn't land in a zero slot. If it had, we'd have $P(z=0|\text{observations}) = 0$.

(c) To find $P(2; 36; \frac{x}{x+36})$:

$$P(\text{observations}|z = x) = \binom{36}{2} \cdot \frac{x}{x+36}^2 \cdot \left(1 - \frac{x}{x+36}\right)^3 \cdot (6-2)$$
$$P(\text{observations}) = 0 + 0.2 \cdot 0.181 + 0.3 \cdot 0.245 + 0.4 \cdot 0.278 = 0.221$$

We get the probabilities $P(z|\text{observations})$ as:

$$P(0|\text{observations}) = 0$$
$$P(1|\text{observations}) = 0.165$$
$$P(2|\text{observations}) = 0.504$$
$$P(3|\text{observations}) = 0.334$$

Question 5

How do we write likelihood functions for these different cases?

(a) $L(\theta) = \binom{50}{35}\theta^3 \cdot 52 \cdot (1-\theta)$

(b) $L(\theta) = \binom{99}{54}\theta^5 \cdot 4 \cdot (1-\theta)^4 \cdot 5$

(c) $L(\theta) = \binom{10}{5} \cdot \theta^6 \cdot (1-\theta)^7$

For a coin toss stopping on heads the third time, we use a geometric distribution, so $L(\theta) = \theta \cdot (1-\theta)^2$. For getting heads 5 out of 10 tosses, it's binomial: $L(\theta) = \binom{10}{5}\theta^5 \cdot (1-\theta)^5$. Combining these:

$$L(\theta) = \binom{10}{5} \cdot \theta^6 \cdot (1-\theta)^7$$

Question 6

(a) **What is the MAP estimate of μ?**

$$\mu_{\text{MAP}} = \frac{\sum_{i=1}^{3} x_i + \frac{\mu_0}{\sigma_0^2}}{\frac{3}{\sigma^2} + \frac{1}{\sigma_0^2}}$$

where $\sigma = 1$ (known standard deviation of the likelihood) and $\sigma_0 = 10$ (standard deviation of the prior).

Substituting the values:

$$\mu_{\text{MAP}} = \frac{(-1 + 0 + 20) + \frac{0}{100}}{\frac{3}{1} + \frac{1}{100}} = \frac{19}{3.01} \approx 6.31$$

(b) **A new data point with value 1 arrives. What is the new MAP estimate of μ?**

Adding the new data point $x = 1$, we update the MAP estimate:

$$\mu_{\text{MAP,new}} = \frac{(-1 + 0 + 20 + 1) + \frac{0}{100}}{\frac{4}{1} + \frac{1}{100}} = \frac{20}{4.01} \approx 4.99$$