S. No.	Date	Title	Page No.	Teacher's Sign. / Remarks

Date 28 | 09 | 21
Page 02

SEMESTER :-01

SYLLABUS

UNIT :-01 Differential calculus

UNIT :-02 Integral calculus

UNIT :-03 Vector calculus

UNIT :-04 Analytical geometry of 2D

UNIT :-05 Real Analysis

UNIT :- 01

DIFFERENTIAL CALCULUS

Given, $y = f(x)$ —— ①

Let δx be the small change in x as a result of this δy be the corresponding change in y.

then eq^n ① reduces to

$$y + \delta y = f(x + \delta x) \quad \text{—— (ii)}$$

eq^n ② − eq^n ①

$$y + \delta y - y = f(x + \delta x) - f(x)$$

$$\delta y = f(x + \delta x) - f(x)$$

dividing both sides by δx and taking limit as $\delta x \to 0$

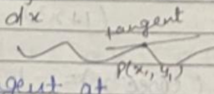
$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}}$$

where, $\frac{dy}{dx}$ = differentiation (derivative) of y w.r.t x

or, Rate of change of y w.r.t x.

* Geometrical meaning of $\frac{dy}{dx}$:-



$\left(\frac{dy}{dx}\right)_{P(x_1, y_1)}$ = slope of tangent at point P.

* List of formulas :-

1. $\frac{d}{dx}(x^n) = nx^{n-1}$
2. $\frac{d(x)}{dx} = 1$, $\frac{d}{dx}(\text{constant}) = 0$
3. $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$
4. $\frac{d}{dx}(\log_e x) = \frac{1}{x}$
5. $\frac{d}{dx}(a^x) = a^x \log_e a$
6. $\frac{d}{dx}(e^x) = e^x$
7. $\frac{d}{dx}(\sin x) = \cos x$
8. $\frac{d}{dx}(\cos x) = -\sin x$
9. $\frac{d}{dx}(\tan x) = \sec^2 x$
10. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$
11. $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$
12. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
13. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
14. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

15. $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{(1+x^2)}$

16. $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$

17. $\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}$

18. $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{(1+x^2)}$

* $\frac{d}{dx}(f(x)) = f'(x)$

* $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$

$\frac{d}{dx}(x^{-1}) = -1\,x^{-1-1}$
$= -x^{-2}$
$= \frac{-1}{x^2}$

* $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

$\frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{\frac{1}{2}-1}$
$= \frac{1}{2}x^{-1/2}$
$= \frac{1}{2x^{1/2}}$
$= \frac{1}{2\sqrt{x}}$

* $\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{-1}{2x\sqrt{x}}$

$\frac{d}{dx}\left(\frac{1}{x^{1/2}}\right)$
$= \frac{-1}{2}x^{-1/2-1}$
$= -\frac{1}{2}x^{-3/2}$
$= \frac{-1}{2x^{3/2}} = \frac{-1}{2x\sqrt{x}}$

* RULES OF DIFFERENTIATION :-

① $\frac{d}{dx}(KU) = K\frac{d}{dx}(U)$; where K is any constant

Ex:- $\frac{d}{dx}\left(\frac{-4}{x}\right) = -4\frac{d}{dx}\left(\frac{1}{x}\right) = -4\left(\frac{-1}{x^2}\right)$

$= \frac{4}{x^2}$

② $\frac{d}{dx}(u \pm v \pm w \pm \ldots) = \frac{d}{dx}u \pm \frac{d}{dx}v \pm \frac{d}{dx}w \ldots\ldots$

Ex:- $\frac{d}{dx}\left(7\sin x - \frac{3}{\cos x}\right) = 7\frac{d}{dx}(\sin x) - 3\frac{d}{dx}(\sec x)$
$= 7\cos x - 3\sec x \cdot \tan x$

③ Product rule :-

Form (I)

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{d(u)}{dx}$

Ex:- $\frac{d}{dx}(\sqrt{x}\tan x) = \sqrt{x}\frac{d}{dx}(\tan x) + \tan x\frac{d(\sqrt{x})}{dx}$
$= \sqrt{x}\sec^2 x + \frac{\tan x}{2\sqrt{x}}$ Ans.

Form (II)

$\frac{d}{dx}(uvwxy) = (uvwxy\ldots)\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \ldots\ldots\right)$

Ex:- $\frac{d}{dx}(\sqrt{x}\tan x)$

$(\sqrt{x}\tan x)\left(\frac{1}{\sqrt{x}}\frac{d}{dx}(\sqrt{x}) + \frac{1}{\tan x}\frac{d}{dx}\tan x\right)$

$\sqrt{x}\tan x\left(\frac{1}{2\sqrt{x}\sqrt{x}} + \frac{1}{\tan x}\sec^2 x\right)$

$= \frac{\tan x}{2\sqrt{x}} + \sqrt{x}\sec^2 x$ A

(4) Quotient Rule :-

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d(u)}{dx} - u\frac{d(v)}{dx}}{v^2}$$

Ex:- $\frac{d}{dx}\left(\frac{\cot x}{x^3}\right) = \frac{x^3\frac{d(\cot x)}{dx} - \cot x\frac{d(x^3)}{dx}}{(x^3)^2}$

$= \frac{x^3 \cdot -\operatorname{cosec}^2 x - \cot x \, 3x^2}{x^6}$

$= \frac{x^2(x - \operatorname{cosec}^2 x - 3\cot x)}{x^6}$

$= \frac{-x\operatorname{cosec}^2 x - 3\cot x}{x^4}$

$= -\frac{(x\operatorname{cosec}^2 x + 3\cot x)}{x^4}$

Q. $\frac{d}{dx}\left(\frac{\sin x - x\cos x}{x\sin x + \cos x}\right)$

Solⁿ:- $\frac{d}{d}$ $\frac{x\sin x + \cos x \frac{d(\sin x - x\cos x)}{dx} - \sin x - x\cos x \, d(x\sin x + \cos x)}{(x\sin x + \cos x)^2}$

$= \frac{x\sin x + \cos x[\cos x - (x - \sin x + \cos x)] - \sin x - x\cos x(x\cos x + \sin x - \sin x)}{(x\sin x + \cos x)^2}$

$= \frac{x^2\sin^2 x + \cos x \, x\sin x \cos x - x\sin x\cos x + x^2\cos^2 x}{(x\sin x + \cos x)^2}$

$= \frac{x^2}{(x\sin x + \cos x)^2}$

Q (H.W)

(1) $\frac{d}{dx}\left(\frac{ax^2+bx+c}{\sqrt{x}}\right)$ (2) $\frac{d}{dx}\left(\sqrt{\frac{x}{a}} + \sqrt{\frac{a}{x}}\right)^2$

(3) $\frac{d}{dx}\left(\frac{\sin x - \cos x}{\sin x + \cos x}\right)$ (4) $\frac{d}{dx}\left(\frac{\sec x - \tan x}{\sec x + \tan x}\right)$

(5) $\frac{d}{dx}\left(\frac{e^x \cot x}{x\sqrt{x}}\right)$

29.09.21

* SUCCESSIVE DIFFERENTIATION (n^{th} order derivative)

Given, $y = f(x)$

differentiating above successively w.r.to x, we get

$\frac{dy}{dx} = y_1 = y' = f'(x)$

$\frac{d^2y}{dx^2} = y_2 = y'' = f''(x)$

$\frac{d^3y}{dx^3} = y_3 = y''' = f'''(x)$

. . . . .

$\frac{d^ny}{dx^n} = y_n$

Q. Find y_n or $\frac{d^ny}{dx^n}$

1. $y = x^n$
2. $y = (ax+b)^n$
3. $y = e^x$
4. $y = e^{ax}$
5. $y = a^x$
6. $y = \frac{1}{(ax+b)}$
7. $y = \log(ax+b)$
8. $y = \sin x$
9. $y = \sin(ax+b)$
10. $y = \cos(ax+b)$
11. $y = e^{ax}\sin(bx+c)$
12. $y = e^{ax}\cos(bx+c)$
13. $y = \frac{1}{(x+1)(x-2)}$
14. $y = \frac{1}{(2x+3)(5x-4)}$
15. $y = \frac{1}{(x-1)(x-2)^2}$

Solutions :-

(1) $y = x^n$

differentiating above successively w.r.t x we get

$y_1 = \frac{dy}{dx} = nx^{n-1}$

$y_2 = \frac{d^2y}{dx^2} = n(n-1)x^{n-2}$

$y_3 = \frac{d^3y}{dx^3} = n(n-1)(n-2)x^{(n-3)}$

.....

.....

$y_{n-1} = n(n-1)(n-2)\ldots\ldots(n-(n-2))x^{n-(n-1)}$

$\therefore$

$y_n = n(n-1)(n-2)\ldots(n-(n-2))(n-(n-1))x^{n-n}$

$= n(n-1)(n-2)\ldots 2\times 1$

$= n!$

Hence, If $y = x^n$

then, $\boxed{y_n = n!}$

(2) $y = (ax+b)^n$

differentiating above successively w.r.t x, we get

$y_1 = \frac{dy}{dx} = n(ax+b)^{n-1}\cdot a$

$y_2 = \frac{d^2y}{dx^2} = n(ax+b\ a^2 n(n-1)(ax+b)^{n-2}$

$y_3 = \frac{d^3y}{dx^3} = a^3 n(n-1)(n-2)(ax+b)^{n-3}$

.....

.....

$y_{(n-1)} = a^{n-1}\, n(n-1)(n-2)\ldots\ldots(n(n-2))(n-(n-1))(ax+b)^{n-(n-1)}$

$y_n = a^n(n(n-1)(n-2)\ldots\ldots n(n-2)(n-(n-1))(ax+b)^{n-n}$

$y_n = a^n n!$

Hence, If $y = (ax+b)^n$

then, $\boxed{y_n = a^n n!}$

(3) $y = e^x$

differentiating above successively w.r.t x we get

$y_1 = \frac{dy}{dx} = e^x$

$y_2 = \frac{d^2y}{dx^2} = e^x$

$y_3 = \frac{d^3y}{dx^3} = e^x$

. . .

$y_{(n-1)} = e^x$

$y_n = e^x$

Hence, if $y = e^x$

then, $\boxed{y_n = e^x}$

(4) $y = e^{ax}$

differentiating above successively w.r.t. x we get:-

$y_1 = \frac{dy}{dx} = e^{ax} \cdot a$

$y_2 = \frac{d^2y}{dx^2} = a \cdot e^{ax} \cdot a = a^2 e^{ax}$

$y_3 = \frac{d^3y}{dx^3} = a^3 \cdot e^{ax}$

. . . . .

. . . . .

$y_{(n-1)} = a^{(n-1)} e^{ax}$

$\therefore y_n = a^n e^{ax}$

Hence, If $y = e^{ax}$

then, $\boxed{y_n = a^n e^{ax}}$

(5) $y = a^x$

differentiating above successively w.r.t. x we get:-

$y_1 = \frac{dy}{dx} = a^x \log a$

$y_2 = \frac{d^2y}{dx^2} = a^x (\log a)^2$

$y_3 = \frac{d^3y}{dx^3} = a^x (\log a)^3$

. . . . .

. . . .

$y_{(n-1)} = a^x (\log a)^{n-1}$

$\therefore y_n = a^x (\log a)^n$

Hence,

If $y = a^x$

then, $\boxed{y_n = a^x (\log a)^n}$

(6) $y = \frac{1}{(ax+b)}$

differentiating above successively w.r.t. x we get:

$y_1 = \frac{dy}{dx} = (ax+b)^{-1}$
$= (-1)\, a\, (ax+b)^{-2}$

$y_2 = \frac{d^2y}{dx^2} = (-1)(-2)\, a^2 (ax+b)^{-3}$

$y_3 = \frac{d^3y}{dx^3} = (-1)(-2)(-3)\, a^3 (ax+b)^{-4}$

$$\therefore y_n = (-1)(-2)(-3)\ldots(-n)a^n(ax+b)^{-(n+1)}$$

$$= \frac{(-1)^n a^n n!}{(ax+b)^{n+1}}$$

Hence, if $y = \frac{1}{ax+b}$

then, $\boxed{y_n = \frac{(-1)^n a^n n!}{(ax+b)^{n+1}}}$

(7) $y = \log(ax+b)$

differentiating above successively w.r.t. x we get

1st Method :-

$$y_1 = \frac{dy}{dx} = \frac{a}{(ax+b)} = a(ax+b)^{-1}$$

$$y_2 = \frac{d^2y}{dx^2} = (-1)a^2(ax+b)^{-2}$$

$$y_3 = \frac{d^3y}{dx^3} = (-1)(-2)a^3(ax+b)^{-3}$$

$$y_n = (-1)(-2)\ldots\ldots(-(n-1))a^n(ax+b)^{-n}$$

$$= \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$$

If $y = \log(ax+b)$

then,

$$\boxed{y_n = \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}}$$

2nd Method

$$y_1 = \frac{dy}{dx} = \frac{a \cdot 1}{(ax+b)} = a(ax+b)^{-1}$$

since,

$$y_n = \frac{(-1)^n a^n n!}{(ax+b)^{n+1}}$$

$$\therefore y_n = \frac{a(-1)^{n-1}a^{n-1}(n-1)!}{(ax+b)^{n-1+1}}$$

$$y_n = \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$$

~~Hence,~~

If $y =$

30.09.21

8. $y = \sin x$

differentiating above successively w.r.t. x we get:

$$y_1 = \cos x$$

or, $y_1 = \sin\left(\frac{\pi}{2}+x\right)$

$$\therefore y_2 = \cos\left(\frac{\pi}{2}+x\right)$$

or, $y_2 = \sin\left(\frac{\pi}{2}+\frac{\pi}{2}+x\right)$

or, $y_2 = \sin\left(\frac{2\pi}{2} + x\right)$

$y_3 = \cos\left(\frac{2\pi}{2} + x\right)$

or $y_3 = \sin\left(\frac{3\pi}{2} + x\right)$

....

....

$y_n = \sin\left(\frac{n\pi}{2} + x\right)$

Hence,

If $y = \sin x$

then, $\boxed{y_n = \sin\left(\frac{n\pi}{2} + x\right)}$

9. $y = \sin(ax+b)$

differentiating above successively w.r.t. x, we get:

$y_1 = a\cos(ax+b)$

or, $y_1 = a\sin\left(\frac{\pi}{2} + ax+b\right)$

$y_2 = a^2\cos\left(\frac{\pi}{2} + ax+b\right)$

or,

$y_2 = a^2\sin\left(\frac{\pi}{2} + \frac{\pi}{2} + ax+b\right)$

or $y_2 = a^2\sin\left(\frac{2\pi}{2} + ax+b\right)$

$\therefore y_3 = a^3\cos\left(\frac{2\pi}{2} + ax+b\right)$

or, $y_3 = a^3\sin\left(\frac{3\pi}{2} + ax+b\right)$

....

$\therefore y_n = a^n\sin\left(\frac{n\pi}{2} + ax+b\right)$

Hence,

$y = \sin(ax+b)$

then, $\boxed{y_n = a^n\sin\left(\frac{n\pi}{2} + ax+b\right)}$

10. $y = \cos(ax+b)$

differentiating successively w.r.t. x, we get:-

$y_1 = -a\sin(ax+b)$

or $y_1 = -a\cos\left(\frac{\pi}{2} + ax+b\right)$

$y_2 = -a^2\sin\left(\frac{\pi}{2} + ax+b\right)$

or

$y_2 = a^2\cos\left(\frac{\pi}{2} + \frac{\pi}{2} + ax+b\right)$

or, $y_2 = a^2\cos\left(\frac{2\pi}{2} + ax+b\right)$

$y_3 = -a^3\sin\left(\frac{2\pi}{2} + ax+b\right)$

or $y_3 = a^3\cos\left(\frac{3\pi}{2} + ax+b\right)$

$y_n = a^n\cos\left(\frac{n\pi}{2} + ax+b\right)$

Hence, $y = \cos(ax+b)$, then,

$\boxed{y_n = a^n\cos\left(\frac{n\pi}{2} + ax+b\right)}$

13. $y = \frac{1}{(x+1)(x+2)}$

Soln:- $y = \frac{1}{3}\left\{\frac{(x+1)-(x-2)}{(x+1)(x-2)}\right\}$

$y = \frac{1}{3(x-2)} - \frac{1}{3(x+1)}$

∵ If $y = \frac{1}{(ax+b)}$ then,

$y_n = \frac{(-1)^n a^n n!}{(ax+b)^{n+1}}$

$y_n = \frac{1}{3}\frac{(-1)^n(1)^n n!}{(x-2)^{n+1}}$

$= \frac{1}{3}\frac{(-1)^n(1)^n n!}{(x+1)^{n+1}}$

Hence,

$y_n = \frac{(-1)^n n!}{3}\cdot\left(\frac{1}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}}\right)$ Ans

14. $y = \frac{1}{(2x+3)(5x-4)}$

Soln:- $\therefore \frac{1}{23}\left\{\frac{5(2x+3)-2(5x-4)}{(2x+3)(5x-4)}\right\}$

$= \frac{5}{23}\times\frac{1}{(5x-4)} - \frac{2}{23}\times\frac{1}{(2x+3)}$

$\therefore y_n = \frac{5}{23}\frac{(-1)^n 5^n n!}{(5x-4)^{n+1}} - \frac{2}{23}\frac{(-1)^n 2^n n!}{(2x+3)^{n+1}}$

$= \frac{(-1)^n n!}{23}\left\{\frac{5^{n+1}}{(5x-4)^{n+1}} - \frac{2^{n+1}}{(2x+3)^{n+1}}\right\}$ Ans

01.10.21

14. $y = \frac{1}{(2x+3)(5x-4)}$

By partial fraction :-

$\frac{1}{(2x-3)(5x+4)} = \frac{A}{(2x+3)} + \frac{B}{(5x-4)}$ — (1)

$1 = A(5x-4) + B(2x+3)$

Put, $x = \frac{-3}{2}$, $1 = A\left(\frac{-15}{2} - 4\right)$

$1 = \frac{-23}{2}A \Rightarrow A = \left(\frac{-2}{23}\right)$

Put, $x = \frac{4}{5}$

$1 = B\left(\frac{8}{5} + 3\right) \Rightarrow 1 = \frac{23}{5}B \Rightarrow B = \frac{5}{23}$

from eqn (1)

$\frac{1}{(2x+3)(5x-4)} = \frac{(-2/23)}{(2x+3)} + \frac{(5/23)}{(5x-4)}$

15. $y = \frac{1}{(x+1)(x-2)^2}$ *

Soln:- $\frac{1}{L_1 L_2} = \frac{A}{L_1} + \frac{B}{L_2}$

$$\frac{1}{L_1 L_2^2} = \frac{A}{L_1} + \frac{B}{L_2} + \frac{C}{L_2^2}$$

$$\frac{1}{LQ} = \frac{A}{L} + \frac{Bx+C}{Q}$$

$$\frac{1}{L_1^2 L_2^2} = \frac{A}{L_1} + \frac{B}{L_1^2} + \frac{C}{L_2} + \frac{D}{L_2^2}$$

where, L, L_1, L_2 are linear and Q is quadratic

Sol^n:- $$\frac{1}{(x-1)(x-2)^2} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2} \quad \text{--- (1)}$$

$1 = A(x-2)^2 + B(x-1)(x-2) + C(x-1)$

Put, $x=1$ | put $x=0$
$1 = A$ | $1 = 4A + 2B - C$
$\Rightarrow A = 1$ | $\Rightarrow 1 = 4 + 2B - 1$
put $x=2$ | $2B = -2$
$1 = C$ | $B = -1$
$\Rightarrow C = 1$

$$\frac{1}{(x-1)(x-2)^2} = \frac{1}{(x-1)} + \frac{-1}{x-2} + \frac{1}{(x-2)^2} \quad \text{--- (ii)}$$

Let, $u = \frac{1}{(x-2)^2} = (x-2)^{-2}$

$u_1 = (-2)(x-2)^{-3}$

$u_2 = (-2)(-3)(x-2)^{-4}$

$u_3 = (-2)(-3)(-4)(x-2)^{-5}$

$\ldots\ldots$

$u_n = (-2)(-3)(-4)\cdots\cdots\{-(n+1)\}(x-2)^{-(n+2)}$

$$\therefore u_n = \frac{(-1)^n (n+1)!}{(x-2)^{n+2}}$$

from eq^n (2)

$$y_n = \frac{(-1)^n n!}{(x-1)^{n+1}} + \frac{(-1)(-1)^n n!}{(x-2)^{n+1}} + \frac{(-1)^n (n+1)!}{(x-2)^{n+2}}$$

11. Given, $y = e^{ax}\sin(bx+c)$

NOTE:- $u = a\sin x + b\cos x$ --- (1)
put $a = r\cos\theta$ --- (ii)
$b = r\sin\theta$ --- (iii)

squaring and adding eq^n (ii) & (iii)

$a^2 + b^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$a^2 + b^2 = r^2$

$r = \sqrt{a^2+b^2}$

$$\frac{\text{eq}^n \text{(iii)}}{\text{eq}^n \text{(ii)}} : \frac{b}{a} = \frac{r\sin\theta}{r\cos\theta}$$

$\frac{b}{a} = \tan\theta$

$\theta = \tan^{-1}\frac{b}{a}$

from eq^n (1)

$u = a\sin x + b\cos x$ --- (1)

$u = r\cos\theta\sin x + r\sin\theta\cos x$

$u = r\sin(x+\theta)$

$u = \sqrt{a^2+b^2}\,\sin\left(x + \tan^{-1}\frac{b}{a}\right)$

Sol^n:- differentiating above equation w.r.t. x we get:-

$y_1 = b e^{ax}\cos(bx+c) + a\sin(bx+c)e^{ax}$

$y_1 = e^{ax}\{a\sin(bx+c) + b\cos(bx+c)\}$

Date
Page 21

Put $a = r\cos\theta$ and $b = r\sin\theta$

$\therefore r = \sqrt{a^2+b^2}$

$\theta = \tan^{-1}\frac{b}{a}$

$\therefore y_1 = e^{ax}\{r\cos\theta\sin(bx+c) + r\sin\theta\cos(bx+c)\}$

$\Rightarrow y_1 = re^{ax}\cdot\sin(bx+c+\theta)$ — (ii)

$y_2 = \{rbe^{ax}\cos(bx+c+\theta) + ra\sin(bx+c+\theta)e^{ax}\}$

$y_2 = re^{ax}\{r\cos\theta\sin(bx+c+\theta) + r\sin\theta\cos(bx+c+\theta)\}$

$y_2 = r^2e^{ax}\sin(bx+c+\theta+\theta)$

$y_2 = r^2e^{ax}\sin(bx+c+2\theta)$ — (iii)

$\therefore y_3 = r^2\{e^{ax}\cos(bx+c+2\theta) + \sin(bx+c+2\theta)e^{ax}\}$

$y_3 = r^2e^{ax}\{r\cos\theta\sin(bx+c+2\theta) + r\sin\theta\cos(bx+c+2\theta)\}$

$y_3 = r^3e^{ax}\sin(bx+c+3\theta)$ — (iv)

...

...

$\therefore y_n = r^ne^{ax}\sin(bx+c+n\theta)$

$y_n = (a^2+b^2)^{n/2}e^{ax}\sin(bx+c+n\tan^{-1}\frac{b}{a})$

Date 04/10/21
Page 22

Note:- Let $P(n)$ be a statement and we have to show that $P(n)$ is true $\forall\ n \in N$, by principal of mathematical induction (PMI). we follow the following steps.

Step 1:- Prove that $P(1)$ is true.

Step 2:- Let $P(k)$ be true, where $k \in N$

Step 3:- Prove that $P(k+1)$ is true.

Eg:- $1+2+3+\dots+n = \frac{n(n+1)}{2}\ \forall\ n \in N$

proof that Let $P(n) = 1+2+3+\dots+n = \frac{n(n+1)}{2}$

Step (1): for $n=1$

L.H.S $= 1$

R.H.S $= \frac{1(1+1)}{2} = \frac{2}{2} = 1$

$P(1)$ is true.

Step 2: $1+2+3+\dots+k = \frac{k(k+1)}{2}$

Step 3: for $n = k+1$

$1+2+3+\dots+k+(k+1)$

$(1+2+3+\dots+k)+(k+1)$

$= \frac{k(k+1)}{2} + (k+1)$ using (i)

$\frac{(k+1)(k+2)}{2}$

$= P(k+1)$ is true

Since, $P(k)$ is true

$P(k+1)$ is true

hence,
by P.M.I $P(n)$ is true $\forall\ n \in N$ proved.

(2) Combination (selection) :-

① $^nC_r = \dfrac{n!}{r!\,(n-r)!}$

where $0 \le r \le n$

Remark :- $0! = 1$

② $n! = n(n-1)(n-2)$

(3) factorial of -ve number and fractional no. are defined

Note :-
① $^nC_0 = 1$
② $^nC_1 = n$
③ $^nC_n = 1$

* Properties of ^nC_r :-

① $^nC_r = {^nC_{n-r}}$ (ii) $^nC_r + {^nC_{r+1}} = {^{n+1}C_{r+1}}$

* Leibnitz Theorem :-
Let u and v are functions of x which are n times differential then,

$(uv)_n = u_n v + {^nC_1} u_{n-1} v_1 + {^nC_2} u_{n-2} v_2 + {^nC_3} u_{n-3} v_3 + \dots + {^nC_{n-1}} u_1 v_{n-1} + {^nC_n} u v_n$

Principle of mathematical induction

Proof :- we prove Leibnitz theorem by P.M.I

Step 1 :- for $n = 1$

$(uv)_1 = \dfrac{d}{dx}(uv) = u_1 v + u v_1$

for $n = 2$ (differentiating second time)

$(uv)_2 = \dfrac{d^2}{dx^2}(uv)$

$= \dfrac{d}{dx}(u_1 v + u v_1) = u_2 v + u_1 v_1 + u_1 v_1 + u v_2$

$= {^2C_0} u_2 v + {^2C_1} u_1 v_1 + {^2C_2} u v_2$

i.e., given statement is true for $n=1$ and $n=2$

Step 2 :- Let given statement is true for $n = 1$ and $n = 2$

$(uv)_k = u_k v + {^kC_1} u_{k-1} v_1 + {^kC_2} u_{k-2} v_2 + {^kC_3} u_{k-3} v_3 + \dots + {^kC_{k-1}} u_1 v_{k-1} + {^kC_k} u v_k$ — ①

Step 3 :- differentiating eqⁿ ① w.r.t x we get :-

$(uv)_{k+1} = (u_{k+1} v + u_k v_1) + {^kC_1}(u_k v_1 + u_{k-1} v_2) + {^kC_2}(u_{k-1} v_2 + u_{k-2} v_3) + \dots + {^kC_{k-1}}(u_2 v_{k-1} + u_1 v_k) + {^kC_k}(u_1 v_k + u v_{k+1})$

$(uv)_{k+1} = u_{k+1} v + {^kC_0}(1 + {^kC_1}) u_k v_1 + ({^kC_1} + {^kC_2}) u_{k-1} v_2 + \dots + ({^kC_{k-1}} + {^kC_k}) u_1 v_k + {^kC_k} u v_{k+1}$

$(uv)_{k+1} = u_{k+1} v + ({^kC_0} + {^kC_1}) u_k v_1 + ({^kC_1} + {^kC_2}) u_{k-1} v_2 + \dots + ({^kC_{k-1}} + {^kC_k}) u_1 v_k + {^kC_k} u v_{k+1}$

given statement is true for $n = (k+1)$

hence, by PMI, given statement is true for all $n \in N$.

Date / / Page 25 05.10.21

v.v.I. Q. If $y = e^{a\sin^{-1}x}$ prove that

(i) $(1-x^2)y_2 - xy_1 - a^2y = 0$

1st Proof:- Given, $y = e^{a\sin^{-1}x}$ — (i)

differentiating eqn (i) w.r.t x we get:-

$$y_1 = \frac{dy}{dx} = \frac{ae^{a\sin^{-1}x}}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\, y_1 = ay \quad \text{— (ii)}$$

Again differentiating w.r.t x

$$\sqrt{1-x^2}\, y_2 + y_1 \frac{1}{2\sqrt{1-x^2}}(-2x) = ay_1$$

$$(1-x^2)y_2 + xy_1 = a(y_1\sqrt{1-x^2})$$

$$(1-x^2)y_2 - xy_1 = a(y_1\sqrt{1-x^2})$$

$$(1-x^2)y_2 - xy_1 = a(ay) \quad \text{(using (ii))}$$

$$(1-x^2)y_2 - xy_1 - a^2y = 0$$

Proved

2nd Proof:- $(1-x^2)y_2 - xy_1 - a^2y = 0$ — (iii)

differentiate eqn (iii) by n times

$$(1-x^2)y_{n+2} + {}^nC_1 y_{n+1}(-2x) + {}^nC_2 y_n(-2) - \{xy_{n+1} + {}^nC_1 y_n\} - a^2y_n = 0$$

$$(1-x^2)y_{n+2} - 2nx\, y_{n+1} - \frac{n(n-1)}{2} y_n 2 - ny_{n+1} - ny_n - a^2y_n = 0$$

$${}^nC_2 = \frac{n!}{2!(n-2)!} \Rightarrow \frac{n(n-1)(n-2)!}{2(n-2)!}$$

Date / / Page 26

$$(1-x^2)y_{n+2} - 2nx\, y_{n+1} - n^2y_n + ny_n - xy_{n+1} - ny_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$$

Proved

06.10.21

(2) If $y = (\sin^{-1}x)^2$ prove that :-

(i) $(1-x^2)y_2 - xy_1 - 2 = 0$

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$

given, $y = (\sin^{-1}x)^2$ — (i)

$$y_1 = \frac{2(\sin^{-1}x)}{\sqrt{1-x^2}}$$

$$(\sqrt{1-x^2})\, y_1 = 2\sin^{-1}x \quad \text{— (ii)}$$

differentiating eqn (ii)

$$\sqrt{1-x^2}\, y_2 + y_1\left(\frac{1}{2\sqrt{1-x^2}}(-2x)\right) = \frac{2}{\sqrt{1-x^2}}$$

$$\frac{(1-x^2)y_2 - xy_1}{\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}$$

$$(1-x^2)y_2 - xy_1 - 2 = 0$$

differentiating above eqn n times by Leibnitz then,

$$(1-x^2)y_{n+2} + {}^nC_1 y_{n+1}(-2x) + {}^nC_2 y_n(-2) - \{xy_{n+1} + {}^nC_1 y_n\} = 0$$

$$(1-x^2)y_{n+2} - 2nx\, y_{n+1} - n(n-1)\frac{(2)}{2} y_n - xy_{n+1} - ny_n = 0$$

$$(1-x^2)y_{n+2} - 2nx\, y_{n+1} - n^2y_n + ny_n - xy_{n+1} - ny_n = 0$$

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$$

Proved.

(3) If $y = a\cos(\log x) + b\sin(\log x)$
prove that
$x^2 y_{n+2} + (2n+1)x y_{n+1} + (n^2+1) y_n = 0$

Proof:- Given, $y = a\cos(\log x) + b\sin(\log x)$ — (i)

$y_1 = -\dfrac{a\sin(\log x)}{x} + \dfrac{b\cos(\log x)}{x}$

$xy_1 = -a\sin(\log x) + b\cos(\log x)$ — (ii)

$xy_2 + y_1 = -\dfrac{a\cos(\log x)}{x} - \dfrac{b\sin(\log x)}{x}$

$x^2 y_2 + xy_1 = -(a\cos(\log x) + b\sin(\log x))$

$x^2 y_2 + xy_1 = -y$, using (1)

$x^2 y_2 + xy_1 + y = 0$ — (iii)

By leibnitz theorem,
$x^2 y_{n+2} + {}^nC_1 y_{n+1}(2x) + {}^nC_2 y_n(2) + x y_{n+1} + {}^nC_1 y_n \cdot 1 + y_n = 0$

$\Rightarrow x^2 y_{n+2} + 2nx y_{n+1} + n(n-1) y_n + x y_{n+1} + ny_n + y_n$

$\Rightarrow x^2 y_{n+2} + 2nx y_{n+1} + n^2 y_n + x y_{n+1} + y_n = 0$

$\Rightarrow x^2 y_{n+2} + (2n+1)x y_{n+1} + (n^2+1) y_n = 0$ proved.

08/10/21

(5) If $y = \log(x + \sqrt{1+x^2})^2$ prove that
$(1+x^2) y_{n+2} + (2n+1)x y_{n+1} + n^2 y_n = 0$

$y_1 = \log(x + \sqrt{1+x^2})^2 \Rightarrow \dfrac{2}{(x + \sqrt{1+x^2})}\left(1 + \dfrac{1}{2\sqrt{1+x^2}} 2x\right)$

$y_1 = \dfrac{2}{x + \sqrt{1+x^2}} \dfrac{(\sqrt{1+x^2} + x)}{(\sqrt{1+x^2})}$

$\sqrt{1+x^2}\, y_1 = 2$

$\therefore \sqrt{1+x^2}\, y_2 + \dfrac{y_1 (2x)}{2\sqrt{1+x^2}} = 0$

$(1+x^2) y_2 + xy_1 = 0$

By using leibnitz theorem,
$u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 v_{n-2} v_2$

$\Rightarrow y_{n+2}(1+x^2) + {}^nC_1 y_{n+1}(2x) + {}^nC_2 y_n(2)$

$\Rightarrow (1+x^2) y_{n+2} + (2n+1)x y_{n+1} + n^2 y_n = 0$

$\Rightarrow (1+x^2) y_{n+2} + 2nx y_{n+1} + n^2 y_n - n y_n + x y_{n+1} + n y_n = 0$

$\Rightarrow (1+x^2) y_{n+2} + 2x\, x y_{n+1} + n^2 y_n = 0$

* Note :-

$\log(mn) = \log m + \log n$
$\log(m/n) = \log m - \log n$
$\log m^n = n \log m$

Hyperbolic function

$\cosh x = \dfrac{e^x + e^{-x}}{2}$ $\qquad \dfrac{d}{dx}(\sinh x) = \cos x$

$\sinh x = \dfrac{e^x - e^{-x}}{2}$ $\qquad \dfrac{d}{dx}(\cosh x) = \sinh x$

Q. If $y^{1/m} + y^{-1/m} = 2x$,
prove that :-
(i) $(x^2-1) y_2 + xy_1 - m^2 y = 0$
(ii) $(x^2-1) y_{n+2} + (2n+1)x y_{n+1} + (n^2 - m^2) y_n = 0$

PARTIAL DIFFERENTIATION

Given, $u = f(x,y)$ — (1)

differentiating above partially w.r.t x

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

Here, we differentiate $u = f(x,y)$ w.r.t x treating y as constant

Similarly, differentiating partially eq^n (1) w.r.t y then we differentiate eq^n (1) partially w.r.t y treating x as a constant.

$$\frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

Let, $u = f(x,y,z)$

then,

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{f(x,y+k,z) - f(x,y,z)}{k}$$

and,

$$\frac{\partial u}{\partial z} = \lim_{l\to 0} \frac{f(x,y,z)+l) - f(x,y,z)}{l}$$

★ Partial differentiation of higher order :-

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

Q. If $u = \log(x^3+y^3+z^3+3xyz)$ find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$

Differentiating u partially w.r.t x we get:

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \log f(x^3+y^3+z^3+3xyz)$$

$$\frac{\partial u}{\partial x} = \frac{1}{(x^3+y^3+z^3+3xyz)}(3x^2+0+0+3yz)$$

$$= \frac{3(x^2+yz)}{(x^3+y^3+z^3+3xyz)}$$

$u = xy$, $\frac{du}{dx} = x\frac{dy}{dx} + y \cdot 1$

$\frac{\partial u}{\partial x} = y \cdot 1 = y$

$\frac{\partial u}{\partial x} = x \cdot 1 = x$

$$\frac{\partial u}{\partial y} = \frac{1}{x^3+y^3+z^3+3xyz}(0+3y^2+0+3xz) = \frac{3(y^2+xz)}{x^3+y^3+z^3+3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3+3yz}(0+0+3z^2+3xy) = \frac{3(z^2+xy)}{x^3+y^3+z^3+3yz}$$

Q. To calculate $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ and

$u = x^3 + y^3 - 3axy$

$$\frac{\partial u}{\partial x} = f(x^3+y^3-3axy) - f(x^3+y^3+-3axy)$$

$$= (x^3+y^3-3axy) - 3x^2+0-3ay$$

$$\frac{\partial u}{\partial x} = x^3+y^3-3axy - 3x^2-3ay$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}\right) = \cancel{3y}\,6ax$$

$$\frac{\partial^2 u}{\partial^2 u} = 3y^2 - 3ax$$

$$\frac{\partial^2 u}{\partial z} = 6y$$

Q. If $u = e^{xyz}$, find $\frac{\partial^3 u}{\partial x \partial y \partial z}$

Given, $u = e^{xyz}$

$$\frac{\partial u}{\partial z} = e^{xyz} \cdot xy$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial^2 u}{\partial y \partial z} \Rightarrow x\left\{e^{xyz} + ye^{xyz}\cdot xz\right\}$$

$$\frac{\partial^2 u}{\partial y \cdot \partial z} = xe^{xyz} + x^2yz\,e^{xyz}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial z}\right) = \frac{\partial^3 u}{\partial x \partial y \partial z}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial z}\right) = \frac{\partial^3 u}{\partial x \partial y \partial z} \Rightarrow \left\{xe^{xyz}yz + e^{xyz}\right\} + \left\{x^2 e^{xyz}\cdot yz + e^{xyz}\cdot 2x\right\}$$

$$\Rightarrow \frac{\partial^3 u}{\partial x \partial y \partial z} = xyz\,e^{xyz} + e^{xyz} + x^2y^2z^2 e^{xyz} + 2xyz\,e^{xyz}$$

$$\Rightarrow \frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz}\left(1 + 3xyz + x^2y^2z^2\right)$$

Q: If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Sol^n:- $$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \times \frac{1}{y} + \frac{1}{\left(1+\frac{y^2}{x^2}\right)}\left(\frac{-y}{x^2}\right)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{y}{\sqrt{y^2-x^2}} \times \frac{1}{y} + \frac{x^2}{x^2+y^2} \times \frac{-y}{x^2}$$

$$\Rightarrow x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} \quad \text{— (I)}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \times \frac{x}{-y^2} + \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x}$$

$$\frac{\partial u}{\partial y} = \frac{-y}{\sqrt{y^2-x^2}}\left(\frac{x}{y^2}\right) + \frac{x^2}{x^2+y^2} \times \frac{1}{x}$$

$$\Rightarrow \frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

$$\Rightarrow y\frac{\partial u}{\partial y} = \frac{-xy}{y\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \quad \text{— (II)}$$

adding corresponding sides of eq^n (I) & (II)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} - \frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$$

$= 0$ proved.

18|10|21

* HOMOGENEOUS FUNCTION :-

A function $f(x,y)$ is said to be homogeneous function if it can be expressed as the form of $x^n f\left(\frac{y}{x}\right)$ or $y^n f\left(\frac{x}{y}\right)$

where, n = degree of function $f(x,y)$

eg.:- $f(x,y) = 4x^3 - 7y^3 - x^2y + 2xy^2$

[If the total power of variable is same then it is homogenous function i.e., $2xy^2 = 1+2 = 3$]

[take maximum degree of x and y]

(I) $$f(x,y) = x^3\left(4 - \frac{7y^3}{x^3} - \frac{y}{x} + \frac{2y^2}{x^2}\right)$$

which is of the form of $x^n f(y/x)$

$$f(x,y) = y^3\left(\frac{4x^3}{y^3} - 7 - \frac{x^2}{y^2} + \frac{2x}{y}\right)$$ which is of the form of $y^n f\left(\frac{x}{y}\right)$

V.V.I

Euler's theorem on homogeneous function of two variables :-

Statement :- Let $u = f(x,y)$ be homogeneous function of two variables x and y; also $u = f(x,y)$ is of degree n then,

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu}$$

Proof :- It is given that $u = f(x,y)$ is homogeneous function of degree n then, it can be expressed as $u = x^n f\left(\frac{y}{x}\right)$ — (i)

differentiating eq^n (i) partially w.r.t x we get

$$\frac{\partial u}{\partial x} = x^n f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + f\left(\frac{y}{x}\right)\cdot n x^{n-1}$$

$$\frac{\partial u}{\partial x} = -x\frac{y}{x^2}\cdot x^n f'\left(\frac{y}{x}\right) + nx^{n-1}f\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x} = -x^{n-1}y f'\left(\frac{y}{x}\right) + nx^n f\left(\frac{y}{x}\right) \quad \text{— (ii)}$$

Differentiating eq^n (i) partially w.r.t y we get :

$$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right)\cdot\frac{1}{x}$$

$$y\frac{\partial u}{\partial y} = x^{n-1}y f'\left(\frac{y}{x}\right) \quad \text{— (iii)}$$

adding corresponding sides of eq^n (ii) & (iii) we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -x^{n-1}y f'\left(\frac{y}{x}\right) + nx^n f\left(\frac{y}{x}\right) + x^{n-1}y f'\left(\frac{y}{x}\right)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f\left(\frac{y}{x}\right)$$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu}$$ (from (i)) Proved.

Q. Verify Euler's theorem for the expression $x^n \sin\left(\frac{y}{x}\right)$

Proof :- Let, $u = x^n \sin\left(\frac{y}{x}\right)$ — (i)

$$\frac{\partial u}{\partial x} = x^n \cos\left(\frac{y}{x}\right)\cdot\left(\frac{-y}{x^2}\right) + \sin\left(\frac{y}{x}\right)\cdot nx^{n-1}$$

$$x\frac{\partial u}{\partial x} = -x^{n-1}\cos\left(\frac{y}{x}\right)y + nx^n \sin\left(\frac{y}{x}\right) \quad \text{— (ii)}$$

and $\frac{\partial u}{\partial y} = x^n \cos\left(\frac{y}{x}\right)\cdot\frac{1}{x}$

$$\Rightarrow y\frac{\partial u}{\partial y} = x^{n-1}y\cos\left(\frac{y}{x}\right) \quad \text{— (iii)}$$

Eq^n (ii) + Eq^n (iii)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -x^{n-1}\cancel{y\cos\left(\frac{y}{x}\right)} + nx^{n-1}\sin\left(\frac{y}{x}\right) + x^{n-1}\cancel{y\cos\left(\frac{y}{x}\right)}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^{n-1}\sin\left(\frac{y}{x}\right)$$

and using (i)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = nu$$ Proved

5/9 Q. If $u = \log\left(\frac{x^2+y^2}{x+y}\right)$ prove that :

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$$

Note:- $\log_b a = c$
$\Rightarrow a = b^c$

Given, $u = \log_e \left(\frac{x^2+y^2}{x+y}\right)$

$$e^u = \left(\frac{x^2+y^2}{x+y}\right)$$

$$e^u = \frac{x^2\left(1+\frac{y^2}{x^2}\right)}{\cancel{x}\left(1+\frac{y}{x}\right)}$$

$e^u = x^1 f\left(\frac{y}{x}\right)$ which is homogeneous function of degree 1.

let, $e^u = v$ —(i)

and since v is homogeneous function of degree 1.

$\therefore$ By Euler's theorem we can write:-

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = v \quad \text{—(ii)}$$

Now, from Eqn (i)

$$v = e^u$$

$$\frac{\partial u}{\partial x} = e^u\frac{\partial u}{\partial x} \text{ and } \frac{\partial u}{\partial y} = e^u\frac{\partial u}{\partial y}$$

from Eqn (ii)

$$x e^u \frac{\partial u}{\partial x} + y e^u \frac{\partial u}{\partial y} = e^u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1 \quad \text{Proved.}$$

Q. If $u = \sin^{-1}\frac{x^2+y^2}{x+y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$

Solⁿ:- from the given question,

$$\sin u = \frac{x^2+y^2}{x+y} = \frac{x^2\left(1+\frac{y^2}{x^2}\right)}{x\left(1+\frac{y}{x}\right)}$$

$\therefore$ $\sin u$ is a homogeneous function of first order in x and y. Let $v = \sin u$.
Therefore v is a homogeneous function of order 1 in x and y.
Hence, according to Euler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1 \cdot v$$

But, since $v = \sin u$ $\therefore \frac{\partial v}{\partial x} = \cos u\frac{\partial u}{\partial x}$,

$$\frac{\partial v}{\partial y} = \cos u\frac{\partial u}{\partial y}$$

Therefore, from (i) $x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \sin u$

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$, on dividing both sides by $\cos u$.

Hence the result.

Q. $y = \tan^{-1}\frac{x^3+y^3}{x-y}$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

Solⁿ $\tan u = \frac{x^3+y^3}{x-y} = \frac{x^3\left(1+\frac{y^3}{x^3}\right)}{x\left(1-\frac{y}{x}\right)} = x^2\phi\left(\frac{y}{x}\right)$

∴ tan u is a homogeneous function of order 2 in x and y.

Let $v = \tan u$

∴ according Euler's theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 2v \quad \text{—(1)}$$

But, from $v = \tan u$, we have :

$$\frac{\partial v}{\partial x} = \sec^2 u \frac{\partial u}{\partial x}$$

$$\frac{\partial v}{\partial y} = \sec^2 u \frac{\partial u}{\partial y}$$

from (1), $x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2\tan u$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u}$$

$$= 2\frac{\sin u \cos u}{\cos u}$$

$$= 2\sin u$$

proved

(3) $u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$.

$$\sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x\left(1+\frac{y}{x}\right)}{\sqrt{x}\left(1+\frac{\sqrt{y}}{\sqrt{x}}\right)} = x^{1/2}\phi\left(\frac{y}{x}\right)$$

Let, $v = \sin u$

according Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial y} = \frac{1}{2}v \quad \text{—(1)}$$

$v = \sin u$, we have $\frac{\partial v}{\partial x} = \cos u \frac{\partial u}{\partial x}$

$$\frac{\partial v}{\partial y} = \cos u \frac{\partial u}{\partial y}$$

from (1) $x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = \frac{1}{2}\sin u$

i.e., $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$ Proved.

20/10/21

Q. If $u = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$ show that $\frac{\partial u}{\partial x} = \frac{-y}{x}\frac{\partial u}{\partial y}$

Proof:- $\sin u = \left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right) \Rightarrow \frac{\sqrt{x}\left(1-\sqrt{x}/\sqrt{y}\right)}{\sqrt{x}\left(1+\sqrt{x}/\sqrt{y}\right)}$

Let, $v = \sin u$

$$v = \frac{\left(1-\frac{\sqrt{y}}{\sqrt{x}}\right)}{\left(1+\frac{\sqrt{y}}{\sqrt{x}}\right)}$$ = which is homogeneous of degree = 0

according to Euler's theorem;

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv$$

$$x\frac{\partial \sin u}{\partial x} + y\frac{\partial \sin u}{\partial y} = 0 \times \sin u$$

$$\frac{\partial v}{\partial x} = \cos\frac{\partial u}{\partial x}, \quad \frac{\partial v}{\partial y} = \cos u \frac{\partial u}{\partial y}$$

$$x \cos u \frac{\partial u}{\partial x} + y \cos\frac{\partial u}{\partial y} = 0$$

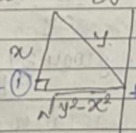
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} = \frac{-y}{x}\frac{\partial u}{\partial y}$$ Proved.

Q. If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$ prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Sol^n:- $\because \sin^{-1}\frac{x}{y}$

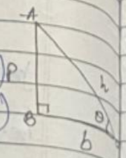


$\tan^{-1}\left(\frac{x}{\sqrt{y^2-x^2}}\right)$ — ①

Note :-

$\sin\theta = \frac{P}{H} \Rightarrow \theta = \sin^{-1}\left(\frac{P}{H}\right)$

$\tan\theta = \frac{P}{B} \Rightarrow \theta = \tan^{-1}\left(\frac{P}{B}\right)$



$\sin^{-1}\frac{P}{H} = \tan^{-1}\frac{P}{B}$

$u = \sin^{-1}\frac{x}{y} + \tan^{-1}\left(\frac{y}{x}\right)$

$u = \tan^{-1}\left(\frac{x}{\sqrt{y^2-x^2}}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

Note :-

$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

$$u = \tan^{-1}\left\{\frac{\dfrac{x^2 + y\sqrt{y^2-x^2}}{x\sqrt{y^2-x^2}}}{\dfrac{x\sqrt{y^2-x^2} - xy}{x\sqrt{y^2-x^2}}}\right\}$$

$$u = \tan^{-1}\left(\frac{x^2 + y\sqrt{y^2-x^2}}{x\sqrt{y^2-x^2} - xy}\right)$$

$$\tan u = \left\{\frac{x^2 + y\sqrt{x^2\left(\frac{y^2}{x^2}-1\right)}}{x\sqrt{x^2\left(\frac{y^2}{x^2}-1\right)} - xy}\right\}$$

$$\tan u = \left\{\frac{x^2 + xy\sqrt{\frac{y^2}{x^2}-1}}{x^2\sqrt{\frac{y^2}{x^2}-1} - xy}\right\}$$

Let, $v = \tan u = \dfrac{x^2\left(1 + \frac{y}{x}\sqrt{\frac{y^2}{x^2}-1}\right)}{x^2\left(\sqrt{\frac{y^2}{x^2}-1} - xy\right)}$ = homogeneous mixture, order = 0

By Euler's theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv$$

$$x\frac{\partial(\tan u)}{\partial x} + y\frac{\partial(\tan u)}{\partial y} = 0$$

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 0$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \quad \text{Proved}$$

Q. If $u = f(x, y)$ is a homogeneous function of degree, n, then

(i) $x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial x \partial y} = (n-1)\dfrac{\partial u}{\partial x}$

(ii) $x\dfrac{\partial^2 u}{\partial x \partial y} + y\dfrac{\partial^2 u}{\partial y^2} = (n-1)\dfrac{\partial u}{\partial y}$

V.V.I (iii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = n(n-1)u$

Sol^n: $u = f(x, y)$ is a homogeneous function of n^{th} order, so by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad \text{— ①}$$

differentiating ① w.r.t x

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\cdot 1 + y\frac{\partial^2 u}{\partial x \cdot \partial y} = n\frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \cdot \partial y} = n\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}$$

$$= (n-1)\frac{\partial u}{\partial x} \quad \text{— (ii)}$$

Proved.

differentiating eqn (i) w.r.t. y

$$x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = n\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial x \cdot \partial y} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}$$

Proved.

Now, multiply (2) by (x) and 3 by y then adding

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy + y^2\frac{\partial^2 u}{\partial y^2} = (n-1)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x}\right)$$

$$= (n-1)nu$$

$$= n(n-1)u \quad \text{Proved}$$

21/10/16

EXPANSION

(1) TAYLOR SERIES :-

Let $f(x)$ be a function of x. If the function $f(x+h)$ can be expanded in a convergent series of ascending positive integral powers of h then,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$

$$\cdots + \frac{h^n}{n!}f^{(n)}(x) + \cdots$$

$$\boxed{f(a+x-a) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots}$$

Proof :- It is given that $f(x)$ can be expanded in ascending powers of h then we can write,

$$f(x+h) = A_0 + A_1 h + A_2 h^2 + A_3 h^3 + A_4 h^4 + \cdots \quad \text{— (1)}$$

where,

A_0, A_1, A_2, A_3, A_4 are functions of x independent of h.

~~where, A_0, A_1, A_2, A_3, A_4 are functions of x independent of h.~~

here, we have to find $A_0, A_1, A_2, A_3, A_4 \cdots$ — (1)

differentiating eqn (1) successively w.r.t. h,

$\therefore$ we get :-

$$f'(x+h) = A_0 + A_1 + A_2(2h) + A_3(3h^2) + A_4(4h^3) + \cdots$$

$$f''(x+h) = 2A_2 + 6A_3 h + A_4(12h^2) + \cdots$$

$$f'''(x+h) = 24A_4 + \phi(h) + \cdots$$

$f^{iv}(x+h) = 24A_4$ Put, $h=0$ in above eqn one by one

$$f(x) = A_0 \Rightarrow A_0 = f(x)$$

$$f'(x) = A_1 \Rightarrow A_1 = f'(x)$$

$$f''(x) = 2A_2 \Rightarrow A_2 = f''(x)/2 \Rightarrow f''(x)/2!$$

$$f'''(x) = 3A_3 \Rightarrow A_3 = f'''(x)/3 \Rightarrow f'''(x)/3!$$

$$f^{iv}(x) = 4A_4 \Rightarrow A_4 = f^{iv}(x)/4 \Rightarrow f^{iv}(x)/4!$$

$\therefore$ from eqn (1)

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3}f'''(x) + \frac{h^4}{4}f^{iv}(x) + \cdots$$

Proved

h is function of x independent of h.

(2) MACLAURIN'S SERIES :-

If the function $f(x)$ possesses derivative of every order in $[0 \text{ to } x]$ and can be expanded in a convergent series of (ascending powers of) positive integral powers of x then,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \ldots$$

Proof :- By above statement $f(x)$ can be written as.

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots \quad -(1)$$

where,

$a_0, a_1, a_2, a_3 \ldots$ are constant which are independent which we have to be determined.

differentiating eq^n (1) successively w.r.t x

$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots$

$f''(x) = 2a_2 + 6a_3x + 12a_4x^2 + \ldots$

$f'''(x) = 6a_3 + 24a_4x + \ldots$

Put, $x=0$, in all above eq^n we get:-

$f(0) = a_0 = a_0 = f(0)$

$f'(0) = a_1 \Rightarrow a_1 = f'(0)$

$f''(0) = 2a_2 \Rightarrow a_2 = \frac{f''(0)}{2!}$

$f'''(0) = 6a_3 = a_3 = \frac{f'''(0)}{3!}$

Now, from eq^n (1), we get:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

Proved

Prove that :-

(1) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

Proof :- $f(x) = \sin x$

$f(0) = \sin 0 = 0$

$f'(x) = \cos x = f'(0) = 1$

$f''(x) = -\sin x = f''(0) = 0$

$f'''(x) = -\cos x = f'''(0) = -1$

$f^{iv}(x) = \sin x = f^{iv}(0) = 0$

$f^{v}(x) = \cos x = f^{v}(0) = 1$

$\ldots$

By maclaurin's series we know that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

$\Rightarrow \sin x = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + \ldots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$

Proved.

(2) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

Proof :- $f(x) = \cos x \Rightarrow f(0) = 1$

$f'(x) = -\sin x \Rightarrow f'(0) = 1$

$f''(x) = -\cos x = f''(0) = -1$

$f'''(x) = \sin x \Rightarrow f'''(0) = 0$

$f^{iv}(x) = \cos x \Rightarrow f^{iv}(0) = 1$

By maclaurin's series we know that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

$f(x) = f(0) + ?$

$\cos x = 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!}$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots$

Note:- Indeterminate form:-

$\frac{0}{0}, \frac{\infty}{\infty} = 0 \cdot \infty$

$\infty - \infty, 0^0, \infty^0, 1^\infty$

(3) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

Proof :- Given, $f(x) = e^x \Rightarrow f(0) = e^0 = 1$

$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1$

$f''(x) = e^x \Rightarrow f''(0) = e^0 = 1$

$f'''(x) = e^x \Rightarrow f'''(0) = e^0 = 1$

$f^{iv}(x) = e^x \Rightarrow e^{iv}(x) = e^0 = 1$

By maclaurin's series, we know that,

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$ Proved.

(4) Prove that:-

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

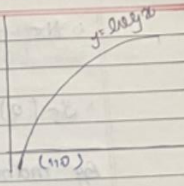
Proof:- Given, $f(x) = \log(1+x) = f(0) = 0$

$f'(x) = \left(\frac{1}{1+x}\right) = f'(0) = 0$

$f''(x) = \frac{-1}{(1+x)^2} = f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3} \Rightarrow f'''(0) = 2$

$f^{iv}(x) = \frac{-6}{(1+x)^4} \Rightarrow f^{iv}(0) = -6$



By maclaurin's series, we know that:

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$

$\log(1+x) = 0 + x - \frac{x^2}{2} + \frac{2x^3}{3} - \frac{x^4}{4} + \dots$

Note:-

This series is convergent only when $-1 < x < 1$

(5) Prove that:-

$\tan x = x + \frac{x^3}{3} + \frac{2}{15} x^5 + \dots$

Proof:- Given, $y = f(x) = \tan x \Rightarrow f(0) = 0$

$\therefore y_1 = \sec^2 x \Rightarrow y_1(0) = 1$

$y_2 = 2\sec^2 \tan x \Rightarrow y_2(0) = 0$

$y_3 = 2(\sec^4 x + 2\tan x \sec^2 x + \tan x)$

$\Rightarrow y_3(0) = 2$

$y_3 = 2(\sec^4 x + 2\sec^2 x + \tan^2 x)$

$y_3 = 2(\sec^4 x + 2\sec^2 x(\sec^2 x - 1))$

$y_3 = 2(\sec^4 x + 2\sec^4 x - 2\sec^2 x)$

$y_3 = (3\sec^4 x - 2\sec^2 x)$

$y_4 = 2(12 - \sec^3 x \sec x \tan x - 4\sec x \sec x \tan$

$y_4 = 0$

$y_4 = 2(12\sec^4 x \tan x - 4\sec^2 x \tan x)$

$\therefore y_5 = 2\{12\sec^4 x \sec^2 x + 12\tan x\,(4\sec^3 x)\sec x \cdot \tan x\} - 4(\sec^2 x \cdot \sec^2 x + 2\tan x \cdot \sec x \cdot \sec x \cdot \tan x)$

$y_5(0) = 24 - 8 = 16$

By maclaurin's series,

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$

$\tan x = 0 + x + 0 + \frac{2}{6}x^3 + 0 + \frac{16}{120}x^5$

$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots$ Proved

23/10/21

Q. Expand $e^{\sin x}$ by maclaurin's theorem as far as the term involving x^4

Soln. Given,

$y = e^{\sin x} \Rightarrow y(0) = e^0 = 1$

$y_1 = e^{\sin x}\cos x \Rightarrow y_1(0) = 1$

$\therefore y_1 = y\cos x$

$y_2 = -y\sin x + (\cos x) y_1$

$y_2(0) = 1$

$y_3 = -y\cos x - (\sin x) y_1 + \cos x\, y_2 - y_1 \sin x$

$y_3 = -1 + 1 = 0$

$y_4 = y\sin x - (\cos x) y_1 - (\sin x) y_2 - y_1\cos x + (\cos x) y_3 - y_2\sin x - y_1\cos x - (\sin x) y_2$

$y_4(0) = -1 - 1 - 1 = 3$

$\sin x \Rightarrow y_3(0) = -1 + 1 = 0$

By maclaurin's theorem, we know that

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} + f''(0) \dots$

$f(x) = y(0) + x y_1(0) + \frac{x^2 y_2(0)}{2!} + \frac{x^3}{3!} y_3(0) + \dots$

$\therefore e^{\sin x} = 1 + x + \frac{x^2}{2} + 0 - \frac{3}{4!}x^4 + \dots$

Q. Expand $\frac{e^x}{\cos x}$

Soln. let, $y = \frac{e^x}{\cos x} \Rightarrow e^x \sec x \Rightarrow y(0) = 1$

$\therefore y_1 = e^x \sec x \tan x + (\sec x) e^x$

$y_1 = y\tan x + y$

$\Rightarrow y_1(0) = 1$

$y_2 = y\sec^2 x + (\tan x) y_1 + y_1$

$\Rightarrow y_2(0) = 1 + 0 + 1 = 2$

$y_3 = y(2\sec^2 x + \tan x) + (\sec^2 x) y_1 + (\tan x) y_2 + y_1 \sec^2 x + y_2$

$y_3(0) = 1 + 1 + 2 = 4$

$y_4 = 2\{y\sec^2 x \tan x\left(\frac{1}{y}y_1 + \frac{1}{\sec^2 x}2\sec^2 x + \tan x + \frac{1}{\tan x}\sec^2 x\right)\} + (\sec^2 x) y_2 + y_1(2\sec^2 x \cdot \tan x) + (\tan x) y_3 + y_2 \sec^2 x + 2y_1 \sec^2 x \cdot \tan x + (\sec^2 x) y_2$

$y_4(0) = 2 + 2 + 2 + 4 + 2 = 12$

By maclaurin's theorem :-

$\frac{e^x}{\cos x} = 1 + \frac{x}{1!} + \frac{2x^2}{2!} + \frac{4x^3}{3!} + \frac{12x^4}{4!} + \dots$

$\frac{e^x}{\cos x} = 1 + x + x^2 + \frac{2x^3}{3!} + \frac{2y}{?} + \dots$

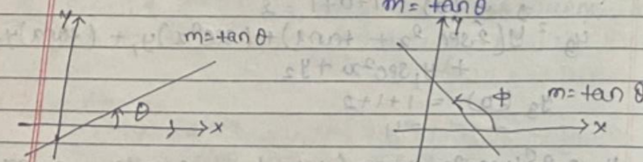
25/10/21

TANGENT AND NORMAL

Straight line

* BASIC CONCEPT :-

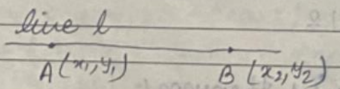
- Slope :- If a straight line makes a angle 'θ' with positive direction of x-axis in anti-clockwise direction then, '$\tan\theta$' is called slope of straight line. It is denoated by 'm' i.e., $m = \tan\theta$



Note :-

(i) Slope of x-axis $= m = \tan 0$
$m = 0$

(ii) Slope of $y = m = \tan 90^\circ$
$m = \infty$

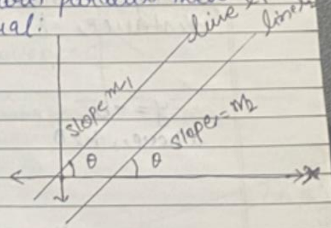
- 

$m =$ slope of line l

$= \frac{(y_2 - y_1)}{(x_2 - x_1)} \Rightarrow \frac{\text{delta } y}{\text{delta } x}$

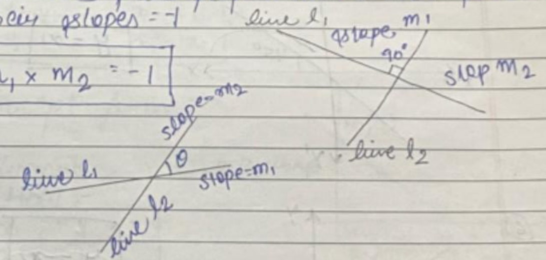
- If two straight lines are parallel then there slopes are equal:

$\boxed{m_1 = m_2}$



- If two lines are perpendicular then product of their slopes $= -1$

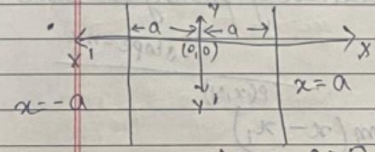
$\boxed{m_1 \times m_2 = -1}$



- Equation of straight lines :-

(i) Equation of x-axis $\Rightarrow y = 0$

(ii) Equation of y-axis $\Rightarrow x = 0$

- 

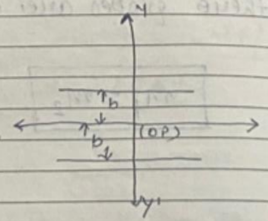
Eqn of straight line ∥ to y-axis at a distance a unit

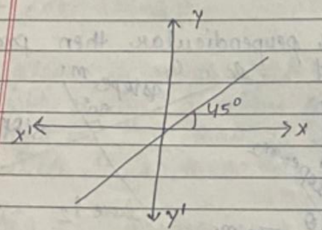
where $a > 0$

- Straight line parallel to x-axis at a distance b

$y = -b$

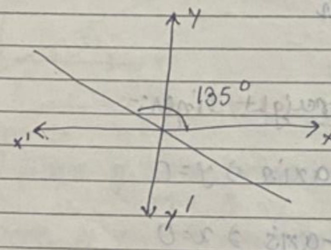
where, $b > 0$





$\boxed{y = x}$

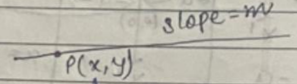
(6)



$\boxed{y = -x}$

- Various form for Equation of straight line

(1) point-slope form :-

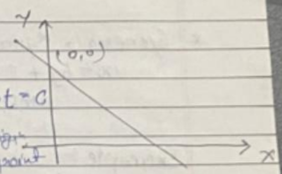


Equation $(y-y_1) = m(x-x_1)$

(2) Slope intercept form :-

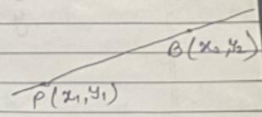
Equation $= mx + c$ → distance of origin and counting point

y-intercept = c



(3) Two point form :-

$y - y_1 = m(x - x_1)$



Equation :-

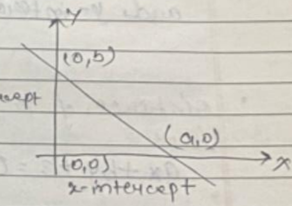
$$\boxed{\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}}$$

(4) Intercept form :

straight line formula

Equation :-

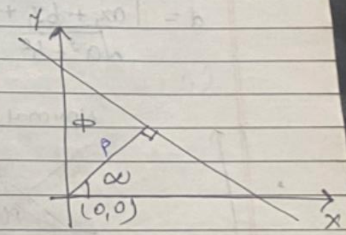
$$\boxed{\frac{x}{a} + \frac{y}{b} = 1}$$



(5) Normal form :

Equation :-

$\boxed{x\cos\alpha + y\sin\alpha = p}$



* General Equation of straight line :-

$ax + by + c = 0$

where, atleast one of a or b is non-zero.

Example :-

$by = -ax + c$

$y = -\left(\frac{a}{b}\right)x + \left(\frac{-c}{b}\right)$

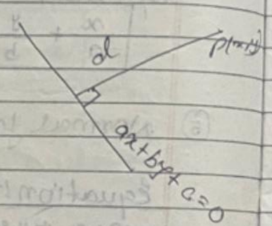
which is of the form of $y = mx + c$

$\therefore$ slope $= m = \frac{-a}{b}$

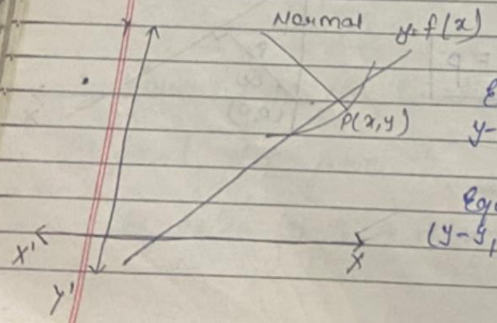
and y-intercept $= c = \frac{-c}{b}$

- Distance of a point P(x, y) from a straight line :-

$ax + by + c = 0$



$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$$



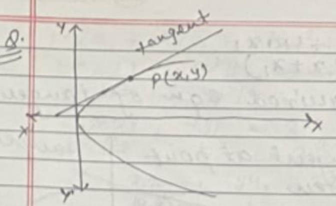
Equation of tangent :-

$$y - y_1 = \left(\frac{dy}{dx}\right)_{P(x_1,y_1)} (x - x_1)$$

Equation of normal :-

$$(y - y_1) = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}} (x - x_1)$$

Q.



$y^2 = 4ax$; $a > 0$

we have to find Eq^n of tangent at point $P(x_1, y_1)$ on the curve $y^2 = 4ax$

$\because y^2 = 4ax$

$\therefore$ differentiating w.r.t x

$2y \frac{dy}{dx} = 4a$

$\frac{dy}{dx} = \frac{2a}{y}$

$m \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \left(\frac{2a}{y}\right)$

since point $P(x_1, y_1)$ lies on the curve $y^2 = 4ax$

$\therefore$ it must satisfies Eq^n of curve i.e,

$y_1^2 = 4ax_1$ ——— (1)

$\therefore$ we know that,

Eq^n of tangent at point $P(x_1, y_1)$

$(y - y_1) = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} (x - x_1)$

$(y - y_1) = \left(\frac{2a}{y_1}\right)(x - x_1)$

$yy_1 - y_1^2 = 2a(x - x_1)$

$yy_1 - 4ax_1 = 2ax - 2ax_1$

$4y_1 = 2ax - 2ax_1 + 4ax_1$

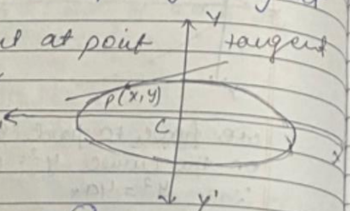
$= 2ax + 4ax_1$

$yy_1 = 2a(x+x_1)$

which is required eqn of tangent

(2) find eqn of tangent at point $P(x,y)$ on the given ellipse.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a>b$



soln:- $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ —— (1)

differentiating eqn (1) we get:-

$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\frac{y}{b^2}\frac{dy}{dx} = -\frac{x}{a^2}$

$\left(\frac{dy}{dx}\right) = -\frac{b^2}{a^2}\frac{dx}{dy}$

$m = \left(\frac{dy}{dx}\right) = -\frac{b^2}{a^2}\left(\frac{x_1}{y_1}\right)$ —— (ii)

Also point (x,y) satisfies eqn (1)

$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$ —— (iii)

$\therefore$ Equation of tangent

$(y - y_1) = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}(x - x_1)$

$(y - y_1) = \frac{b^2}{a^2}\left(\frac{x_1}{y_1}\right)(x - x_1)$

$a^2y_1(y - y_1) = -b^2x_1(x - x_1)$

$a^2yy_1 - a^2y_1^2 = -b^2xx_1 + b^2x_1^2$

$b^2xx_1 + a^2yy_1 = b^2x_1^2 + a^2y_1^2$

$\frac{b^2xx_1}{a^2b^2} + \frac{a^2yy_1}{a^2b^2} = \frac{b^2x_1^2}{a^2b^2} + \frac{a^2y_1^2}{a^2b^2}$

$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$

$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ using eqn (iii)

which is the required eqn of tangent

* shortcut way to find eqn of tangent replace,

$x^2 = xx_1$

$y^2 = yy_1$

$2x = x + x_1$

$2y = y + y_1$

$2xy = (xy_1 + x_1y)$

(i) $y^2 = 4ax$

Eqn of tangent $= y^2 = 2a(2x)$

$yy_1 = 2a(x + x_1)$

(ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Eqn of tangent $= \frac{xx_1}{a^2} + \frac{yy_1}{b^2}$

Q. find the eqn of tangent at (a,b) to the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$

soln:- Given,

Eqn of curve,

$\frac{x^n}{a^n} + \frac{y^n}{b^n} = 2$ —— (1)

differentiating w.r.t x, we get;

$\frac{nx^{n-1}}{a^n} + \frac{ny^{n-1}}{b^n}\left(\frac{dy}{dx}\right) = 0$

$\frac{\cancel{n}y^{n-1}}{b^n}\left(\frac{dy}{dx}\right) = -\frac{\cancel{n}x^{n-1}}{a^n}$

$$\left(\frac{dy}{dx}\right) = \frac{-x^{n-1}}{a^n} \times \frac{b^n}{y^{n-1}}$$

$$\therefore \left(\frac{dy}{dx}\right)_{(a,b)} = \frac{-a^{n-1}b^n}{a^n b^{n-1}}$$

$$\left(\frac{dy}{dx}\right)_{(a,b)} = \frac{-b}{a}$$

Eq^n of tangent at point (a,b)

$$(y-b) = \left(\frac{dy}{dx}\right)_{(a-b)} (x-a)$$

$$(y-b) = -\frac{b}{a}(x-a)$$

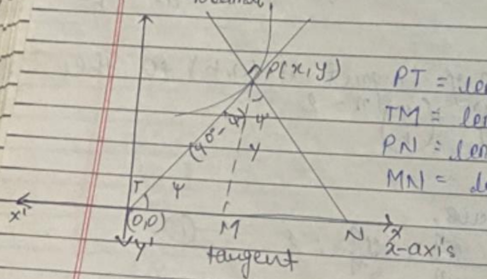
$$ay - ab = -bx + ab$$

$$bx + ay = 2ab$$

$$\frac{bx}{ab} + \frac{ay}{ab} = \frac{2ab}{ab}$$

$$\frac{x}{a} + \frac{y}{b} = 2$$

28/10/21



PT = length of tangent

TM = length of sub tangent

PN = length of Normal

MN = length of sub-normal

we know that,

$$\tan\psi = \frac{dy}{dx} = y_1 \quad \text{—(1)}$$

Now,

In Δ^{rt} PMT

$$\tan\psi = \frac{PM}{TM}$$

$$y_1 = \frac{y}{TM}$$

$$TM = \frac{y}{y_1}$$

length of sub tangent $= y/y_1$

In Δ^{rt} PMT

$$\sin\psi = \frac{PM}{PT}$$

$$\frac{1}{\operatorname{cosec}\psi} = \frac{y}{PT}$$

$$PT = y\operatorname{cosec}\psi$$

we know that,

$$\tan\psi = \frac{dy}{dx} = y_1 \quad \text{—(1)}$$

$$PT = y\sqrt{1+\cot^2\psi}$$

$$y = \sqrt{1 + \frac{1}{\tan^2\psi}}$$

$$y = \sqrt{1 + \frac{1}{y_1^2}}$$

length of tangent $= PT = \frac{y}{y_1}\sqrt{1+y_1^2}$

In Δ^{rt} PMN

$$\tan\psi = \frac{MN}{PM} \Rightarrow y_1 = \frac{MN}{y}$$

$$MN = yy_1$$

length of sub-normal $= yy_1$

Next

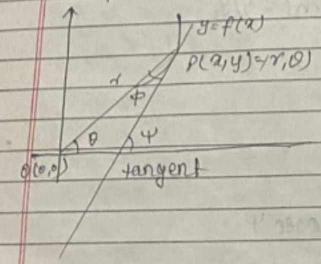
$$\cos\psi = \frac{PM}{PM} \Rightarrow PN = \frac{y}{\cos\psi}$$

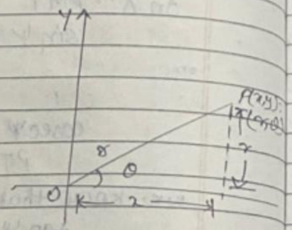
$PN = P\sec\psi$

$PN = y\sqrt{1+\tan^2\psi}$

length of normal $= y\sqrt{1+y_1^2}$

Q. Angle between radius vector and tangent at point $P(x,y)$ on the curve, $y = f(x)$





$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$

$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$

we know that,

$x = r\cos\theta$ and $y = r\sin\theta$

differentiating above eqn w.r.to θ

$\frac{dx}{d\theta} = -r\sin\theta + \cos\theta \frac{dr}{d\theta}$

$\frac{dx}{d\theta} = -r\sin\theta + r\cos\theta$

and $\frac{dy}{d\theta} = r\cos\theta + \sin\theta \frac{dr}{d\theta}$

$\frac{dy}{d\theta} = r\cos\theta + r\sin\theta$

$\tan\psi = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(r\cos\theta + r\sin\theta)}{(r\cos\theta - r\sin\theta)}$ — (1)

from figure,

$\psi = \theta + \phi$

$\phi = (\psi - \theta)$

$\tan\phi = \tan(\psi - \theta)$

$\tan\phi = \frac{\tan\psi - \tan\theta}{1 + \tan\psi \cdot \tan\theta}$

$\tan\phi = \frac{\frac{r\cos\theta + r\sin\theta}{r\cos\theta - r\sin\theta} - \frac{\sin\theta}{\cos\theta}}{1 + \left(\frac{r\cos\theta + r\sin\theta}{r\cos\theta - r\sin\theta}\right)\frac{\sin\theta}{\cos\theta}}$

$\tan\phi = \frac{r\cos^2\theta + r^{o}\sin\theta\cos\theta - r\sin\theta\cdot\cos\theta + r\sin^2\theta}{r^{o}\cos^2\theta - r\sin\theta\cos\theta + r\sin\theta\cos\theta + r^{o}\sin^2\theta}$

$\tan\phi = \frac{r}{r^{o}} \Rightarrow \frac{r}{\left(\frac{dr}{d\theta}\right)}$

$\boxed{\tan\phi = r\frac{d\theta}{dr}}$

29/10/21

Q. Length of perpendicular from pole (origin) to tangent at point $P(x,y)$ on the curve $y = f(x)$

$\tan\phi = r\frac{d\theta}{dr}$

Let $ON = P$

from figure, in Δ^{rt} PON

$$\sin\phi = \frac{ON}{OP} \Rightarrow \sin\phi = \frac{p}{r}$$

$$\boxed{p = r\sin\phi}$$

$$\frac{1}{p} = \frac{1}{r\sin\phi} \Rightarrow \frac{1}{p^2} = \frac{1}{r^2\sin^2\phi}$$

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi \Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left(1+\cot^2\phi\right)$$

$$= \frac{1}{p^2} = \frac{1}{r^2}\left(1+\frac{1}{\tan^2\phi}\right) \quad \frac{1}{p^2} = \frac{1}{r}\left(1+\frac{1}{r^2\left(\frac{d\phi}{dr}\right)^2}\right)$$

$$\boxed{\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2}$$

Put, $\frac{1}{r} = u$, $r = \frac{1}{u}$

$$\therefore \frac{dr}{d\theta} = \frac{-1}{u^2}\frac{du}{d\theta}$$

$$\frac{1}{p^2} = u^2 + u^4\left(\frac{-1}{u^2}\frac{du}{d\theta}\right)^2$$

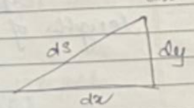
$$\frac{1}{p^2} = u^2 + u^4\frac{1}{u^4}\left(\frac{du}{d\theta}\right)^2$$

$$\boxed{\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2}$$

Result:-

$$\sin\phi = \frac{r\,d\theta}{ds} \qquad \cos\phi = \frac{dr}{ds} \qquad \tan\phi = \frac{r\,d\theta}{dr}$$

(i) $\sin\psi = \frac{dy}{ds}$ (ii) $\cos\psi = \frac{dx}{ds}$ (iii) $\tan\psi = \frac{dy}{dx}$



$$(ds)^2 = (dx)^2 + (dy)^2$$

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

similarly,

$$\left(\frac{ds}{dy}\right)^2 = \left(\frac{dx}{dy}\right)^2 + 1$$

$$\left(\frac{ds}{dy}\right) = \sqrt{1+\left(\frac{dx}{dy}\right)^2}$$

$$\boxed{\frac{ds}{d\theta} = \sqrt{r^2\left(\frac{dr}{d\theta}\right)^2}}$$

Q. Show that in the curve $by^2 = (a+x)^3$, the square of the sub-tangent varies as the sub-normal.

Soln:- Given,

$$by^2 = (a+x)^3 \quad \text{— (1)}$$

differentiating eqn (1) w.r.t. x, we get:-

$$2by\,y_1 = 3(a+x)^2$$

$$y_1 = \frac{3(a+x)^2}{2by}$$

length of sub-tangent $= y/y_1$

$$= y / \frac{3(a+x)^2}{2by}$$

$$(\text{subtangent})^2 = \left(\frac{2by^2}{3(a+x)^2}\right)^2$$

Page 59

$$= \frac{4b^2y^4}{9(a+x)^4} \quad \text{——— (1)}$$

length of sub normal :- yy_1

$$= \frac{y\,3(a+x)^2}{2by}$$

$$\text{Sub normal} = \frac{3(a+x)^2}{2b} \quad \text{——— (4)}$$

$$\frac{Eq^n(1)}{Eq^n(3)} = \frac{4b^2y^4}{9(a+x)^4} \times \frac{2b}{3(a+x)^2}$$

$$= \frac{8b^3y^4}{27b^2y^4} = \frac{8b}{27} = k$$

$$(\text{subtangent})^2 = \left(\frac{8b}{27}\right) \text{sub normal}$$

$$(\text{sub tangent})^2 \propto \text{sub normal}$$

$$\boxed{\begin{aligned} (1+\cos\theta) &= 2\cos^2\theta/2 \\ (1-\cos\theta) &= 2\sin^2\theta/2 \\ \sin 2\theta &= 2\sin\theta\cdot\cos\theta \end{aligned}}$$

20/10/21

* Pedal Equation :- The relation between p and r of a point on a curve is known as pedal equation or p-r equation.

V.Imp Q find the pedal equation of the cardiod $r = a(1-\cos\theta)$

Soln:- Given,

Page 60

$$r = a(1-\cos\theta) \quad \text{——— (1)}$$

taking log on both side, we get:

$$\log r = \log(a(1-\cos\theta))$$

$$\Rightarrow \log r = \log a + \log(1-\cos\theta)$$

differentiating above w.r.t. θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{(1)-(-\sin\theta)}{(1-\cos\theta)}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{(1-\cos\theta)}$$

$$r\frac{d\theta}{dr} = \frac{(1-\cos\theta)}{\sin\theta}$$

$$\tan\theta = \frac{2\sin^2\theta/2}{2\sin\theta/2\cdot\cos\theta/2}$$

$$\tan\phi = \tan\theta/2$$

$$\phi = \tan\theta/2$$

we know that,

$$p = r\sin\phi$$

$$p = r\sin\theta/2 \quad \text{——— (2)}$$

from eq (1)

$$r = a(1-\cos\theta)$$

$$r = a(2\sin^2\theta/2)$$

$$\frac{r}{2a} = \sin^2\theta/2 \quad \text{——— (3)}$$

from eq^n (2)

$$p = r\sin\theta/2$$

squaring both sides, we get

$$p^2 = r^2\sin^2\theta/2$$

$$p^2 = r^2\cdot\frac{r}{2a}$$

$$\boxed{p^2 = \frac{r^3}{2a}}$$

which is the required p-equation of a pa... on a curve $r = a(1-\cos\theta)$.

Q. find the pedal Equation of
$$r^m = a^m \cos m\theta$$

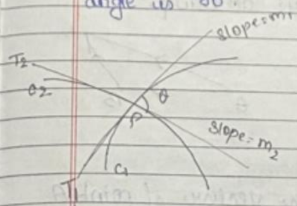
Solⁿ:- taking log on both sides we get :-
$$\log r^m = \log (a^m \cos\theta)$$
$$\log r^m = \log a^m + \log \cos m\theta$$
differentiating above eqⁿ w.r.t. θ we get,
$$m \log r = m \log a + \log \cos m$$
$$\frac{m\,dr}{r\,d\theta} = + \frac{1}{\cos m\theta}(-m \sin m\theta)$$
$$\frac{dr}{r\,d\theta} = -\tan m\theta$$
$$\frac{r\,d\theta}{dr} = -\cot m\theta$$

$$\tan\phi = -\tan(\pi/2 + m\theta)$$
$$\phi = (\pi/2 + m\theta) \quad -(1)$$
we know that,
$$p = r\sin\theta$$
$$p = r\sin(\pi/2 + m\theta)$$
$$p = r\cos m\theta$$
$$p = r\left(\frac{r^m}{a^m}\right) \quad ; \text{using (1)}$$
$$\boxed{p = \frac{r^{m+1}}{a^m}}$$

for Ex:- $r^2 = a^2\cos 2\theta$
pedal eqⁿ of above eqⁿ
$$\boxed{p = \frac{r^3}{a^2}}$$

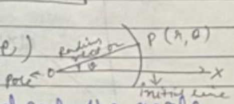
* Orthogonal curves:-

Two curves are said to be orthogonal curves if angle between them is 90°. Here, angle of the both curves means we draw tangents at the point of intersection of both the tangents and we find angle between both tangents in case of orthogonal curves angle is 90°.



If θ = 90°, then curves C_1 and C_2 are called orthogonal curves.
Also,
here $\boxed{m_1 m_2 = -1}$

(In case of Polar co-ordinate)



Note:-
In practical problems we find out the angle between two curves by first evaluating $\tan\phi_1$, $\tan\phi_2$ and then using $\boxed{\alpha = \phi_1 - \phi_2}$

Q. find the angle of intersection of the curves $r = a(1+\cos\theta)$ and $r = b(1-\cos\theta)$

parametric form:-
Given, $x = f(t)$ and
$y = g(t)$
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

UNIT :- III

VECTOR ANALYSIS / CALCULUS

Vector :- magnitude and direction

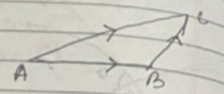
A • ——→ • B
West   East

$\vec{AB} = a$

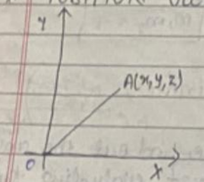
(AB = a unit) ⇒ magnitude
west to east ⇒ direction

* Triangle law of Vector addition :-

$\vec{AC} = \vec{AB} + \vec{BC}$

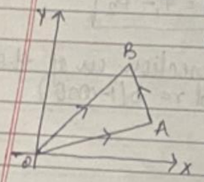


* POSITION VECTOR :-



position vector of point A

$\vec{OA} = x\hat{i} + y\hat{j} + z\hat{k}$



By triangle law of vector addition we can write:

$\vec{OA} + \vec{AB} = \vec{OB}$

$\vec{AB} = \vec{OB} - \vec{OA}$

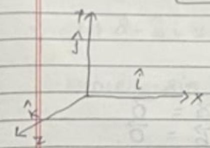
$\boxed{\vec{AB} = (\text{Position vector of point B}) - (\text{Position vector of point A})}$

* UNIT VECTOR :-

Let $\vec{a}$ be any vector then unit vector of $\vec{a}$ denoted by $\hat{a}$ and it is defined as

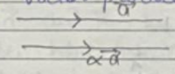
$$\boxed{\hat{a} = \frac{\vec{a}}{|\vec{a}|}}$$

Note :- $|\hat{a}| = 1$



$\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along x-axis, y axis and z-axis respectively.

* Any vector parallel to $\vec{a}$ is $\alpha\vec{a}$



where α is any scalar quantity.

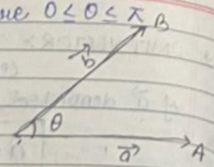
* Let, $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$, then, magnitude of $\vec{a}$ is denoted by $|\vec{a}|$ or simply 'a' and it is defined as

$$\boxed{|\vec{a}| = \sqrt{x^2 + y^2 + z^2}}$$

* Product of vectors :-

① Dot ~~p~~ product / ~~Scalar~~ scalar product :- Let $\vec{a}$ and $\vec{b}$ be two vectors and θ be the angle between them then,

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$ where $0 \le \theta \le \pi$



→ Properties of dot product :-

(i) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

(ii) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

Imp. (iii) $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

(iv) If $\vec{a} \cdot \vec{b} = 0$, then either $\vec{a} = \vec{0}$
or $\vec{b} = \vec{0}$
or $\vec{a} \perp \vec{b}$

(v) If $\vec{a} \perp \vec{b}$ then $\vec{a} \cdot \vec{b} = 0$

(vi) $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$

(vii) $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

(viii) Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$
$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$
then,
$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

Q. Let $\hat{a}$ and $\hat{b}$ be two unit vectors and θ be the angle between them then prove that $\sin\frac{\theta}{2} = \frac{1}{2}|\hat{a} - \hat{b}|$

Sol^n $|\hat{a} - \hat{b}|^2 = (\hat{a} - \hat{b}) \cdot (\hat{a} - \hat{b})$
$\Rightarrow |\hat{a} - \hat{b}|^2 = \hat{a} \cdot (\hat{a} - \hat{b}) - \hat{b} \cdot (\hat{a} - \hat{b})$
$\Rightarrow |\hat{a} - \hat{b}|^2 = \hat{a} \cdot \hat{a} - \hat{a} \cdot \hat{b} - \hat{b} \cdot \hat{a} + \hat{b} \cdot \hat{b}$
$\Rightarrow |\hat{a} - \hat{b}|^2 = |\hat{a}|^2 - 2\hat{a} \cdot \hat{b} + |\hat{b}|^2$
$|\hat{a} - \hat{b}|^2 = 1 - 2|\hat{a}||\hat{b}|\cos\theta + 1$

$\Rightarrow |\hat{a} - \hat{b}|^2 = 2 - 2|\hat{a}||\hat{b}|\cos\theta$
$\Rightarrow |\hat{a} - \hat{b}|^2 = 2 - 2\cos\theta$
$\Rightarrow |\hat{a} - \hat{b}|^2 = 2(1 - \cos\theta)$
$\Rightarrow |\hat{a} - \hat{b}|^2 = 2\left(2\sin^2\frac{\theta}{2}\right)$

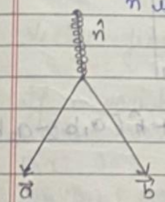
$|\hat{a} - \hat{b}|^2 = \cancel{2}$
$\Rightarrow |\hat{a} - \hat{b}| = 2\sin\frac{\theta}{2}$

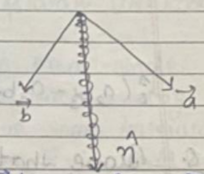
$\Rightarrow \frac{1}{2}|\hat{a} - \hat{b}| = \sin\frac{\theta}{2}$

Hence, $\sin\frac{\theta}{2} = \frac{1}{2}|\hat{a} - \hat{b}|$
Proved.

* Vector product (cross product) :-

Let $\vec{a}$ and $\vec{b}$ be two vectors and θ be the angle ~~btw~~ between them then, there cross product is denoted by $(\vec{a} \times \vec{b})$ and it is defined. $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, \hat{n}$ — (i)
where,
$\hat{n}$ is unit vector ⊥r to both $\vec{a}$ and $\vec{b}$.





from eq^n (i), $(\vec{a} \times \vec{b}) = |\vec{a}||\vec{b}| \sin\theta \, \hat{n}$ — (i)
$\therefore |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \cdot \sin\theta \, |\hat{n}|$
$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$ — (ii)

Now, $\dfrac{Eq^n (i)}{Eq^n (ii)} \Rightarrow \dfrac{(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$

$$\hat{n} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} \quad \text{—— (iii)}$$

* Properties of vector Product :-

(i) $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$

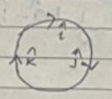
(ii) $\vec{a}\times(\vec{b}+\vec{c}) = (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c})$

(iii) $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$

(iv) $\hat{i}\times\hat{j} = \hat{k}$

$\hat{j}\times\hat{k} = \hat{i}$

$\hat{k}\times\hat{i} = \hat{j}$



(v) If $\vec{a} \parallel \vec{b}$ then, $(\vec{a}\times\vec{b}) = 0$

(vi) If $\vec{a}\times\vec{b} = \vec{0}$, then, $\vec{a} = \vec{0}$, $\vec{b} = \vec{0}$, $\vec{a}\parallel\vec{b}$

(vii) Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then,
$$(\vec{a}\times\vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\hat{i}(a_2b_3 - a_3b_2) - \hat{j}(a_1b_3 - a_3b_1) + \hat{k}(a_1b_2 - a_2b_1)$$

Q. Prove that
$$\vec{a}\times(\vec{b}+\vec{c}) + \vec{b}\times(\vec{c}+\vec{a}) + \vec{c}\times(\vec{a}+\vec{b}) = 0$$

Proof:- L.H.S $\Rightarrow \vec{a}\times(\vec{b}+\vec{c}) + \vec{b}\times(\vec{c}+\vec{a}) + \vec{c}\times(\vec{a}+\vec{b})$

$\Rightarrow (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c}) + (\vec{b}\times\vec{c}) + (\vec{b}\times\vec{a}) +$
$(\vec{c}\times\vec{a}) + (\vec{c}\times\vec{b})$

$= (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c}) + (\vec{b}\times\vec{c}) - (\vec{a}\times\vec{b}) - (\vec{a}\times\vec{c})$
$- (\vec{b}\times\vec{c})$

$= 0$

$=$ R.H.S

Proved.

Q. Prove that $|\vec{a}\times\vec{b}|^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix}$

Proof:- R.H.S $= \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix}$

$= (\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b}) - (\vec{a}\cdot\vec{b})(\vec{a}\cdot\vec{b})$

$= |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$

$= |\vec{a}|^2|\vec{b}|^2 - (|\vec{a}||\vec{b}|\cos\theta)^2$

$= |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta$

$= |\vec{a}|^2|\vec{b}|^2(1-\cos^2\theta)$

$= |\vec{a}|^2|\vec{b}|^2\sin^2\theta$

$= (|\vec{a}||\vec{b}|\sin\theta)^2$

$= |\vec{a}\times\vec{b}|^2$

$=$ L.H.S Proved.

* Vector function of a single scalar variable

If a variable vector $\vec{r}$ depends on a single scalar variable parameter t in such a way that as t varies continuously in some interval say (a, b) so, also does $\vec{r}$, then $\vec{r}$ is said to be a vector function of the scalar variable t defined in the interval $[a, b]$ and we write

$$\boxed{\vec{r} = \vec{r}(t)}$$

* Differentiation of a vector function:

Let $\vec{v}$ be a vector function of the scalar parameter t which we represent by the relation $\vec{u} = \vec{v}(t)$ then the derivative of the vector function with respect to scalar t is defined by the vector:

$$\frac{d\vec{v}}{dt} = \lim_{\delta t \to 0} \frac{\delta \vec{v}}{\delta t} = \lim_{\delta t \to 0} \frac{\vec{v}(t+\delta t) - \vec{v}(t)}{\delta t}$$

* Continuity of a vector function:

A vector function $\vec{v}(t) = v_1(t)\hat{i} + v_2(t)\hat{j} + v_3(t)\hat{k}$ is said to be continuous at $t=c$ if the three scalar functions $v_1(t)$, $v_2(t)$, $v_3(t)$ are continuous at $t=c$.

* Partial derivatives of vectors:

Let a vector function $\vec{v}(x,y,z)$ be a continuous vector function with scalar parameters x, y, z their partial derivative of $\vec{v}$ w.r.t. x

$$\frac{\partial \vec{v}}{\partial x} = \lim_{\delta x \to 0} \frac{\vec{v}(x+\delta x, y, z) - \vec{v}(x,y,z)}{\delta x}$$

Similarly,

$$\frac{\partial \vec{v}}{\partial y} = \lim_{\delta y \to 0} \frac{\vec{v}(x, y+\delta y, z) - \vec{v}(x,y,z)}{\delta y}$$

and,

$$\frac{\partial \vec{v}}{\partial z} = \lim_{\delta z \to 0} \frac{\vec{v}(x,y,z+\delta z) - \vec{v}(x,y,z)}{\delta z}$$

13/11/21

Higher derivative of a vector function w.r.t. a scalar parameter.

Let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

then,

$$\frac{d\vec{v}}{dt} = \frac{dv_1}{dt}\hat{i} + \frac{dv_2}{dt}\hat{j} + \frac{dv_3}{dt}\hat{k}$$

$$\frac{d^2\vec{v}}{dt^2} = \frac{d^2v_1}{dt^2}\hat{i} + \frac{d^2v_2}{dt^2}\hat{j} + \frac{d^2v_3}{dt^2}\hat{k} \ldots$$

$$\frac{d^n\vec{v}}{dt^n} = \frac{d^nv_1}{dt^n}\hat{i} + \frac{d^nv_2}{dt^n}\hat{j} + \frac{d^nv_3}{dt^n}\hat{k}$$

* Derivative of a constant vector:-

then, $\frac{d\vec{a}}{dt} = 0$

* The derivative of sum of vectors:-

let, $\vec{r} = \vec{r_1} + \vec{r_2} + \vec{r_3}$

then,

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r_1}}{dt} + \frac{d\vec{r_2}}{dt} + \frac{d\vec{r_3}}{dt}$$

* Differentiation of product:-

let, $\vec{r} = v\vec{a}$ where v is scalar then,

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}(v\vec{a})$$

$= v\frac{d\vec{a}}{dt} + \frac{dv}{dt}\vec{a}$

Note :- $\vec{a}$ is const. vector

$\frac{d\vec{r}}{dt} = v\frac{d\vec{a}}{dt} + \frac{dv}{dt}\vec{a}$

$\frac{d\vec{r}}{dt} = 0 + \frac{dv}{dt}\vec{a}$

case 2: $\frac{d}{dt}(\vec{r_1}\cdot\vec{r_2}) = \vec{r_1}\left(\frac{d\vec{r_2}}{dt} + r_2 \cdot \frac{d\vec{r_1}}{dt}\right)$

case 3: $\frac{d}{dt}(\vec{r_1}\times\vec{r_2}) = \vec{r_1}\times\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}\times\vec{r_2}$

Q. If $\vec{r} = \vec{a}\cos\omega t + \vec{b}\sin\omega t$, show that

(1) $\vec{r}\times\frac{d\vec{r}}{dt} = \omega(\vec{a}\times\vec{b})$

(2) $\frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r}$

Proof:- given, $\vec{r} = (\cos\omega t)\vec{a} + (\sin\omega t)\vec{b}$

then,

$\frac{d\vec{r}}{dt} = -\omega(\sin\omega t)\vec{a} + \omega(\cos\omega t)\vec{b}$ — (1)

$\vec{r}\times\frac{d\vec{r}}{dt} = \{(\cos\omega t)\vec{a} + (\sin\omega t)\vec{b}\}\times\{-\omega(\sin\omega t)\vec{a} + \omega(\cos\omega t)\vec{b}\}$

$\Rightarrow \omega(\cos\omega t)^2(\vec{a}\times\vec{b}) - \omega(\sin\omega t)^2(\vec{b}\times\vec{a})$

$\Rightarrow \omega(\cos\omega t)^2(\vec{a}\times\vec{b}) + \omega(\sin\omega t)^2(\vec{a}\times\vec{b})$

$\Rightarrow \omega(\vec{a}\times\vec{b})\{(\cos\omega t)^2 + (\sin\omega t)^2\}$

$\vec{r}\times\frac{d\vec{r}}{dt} = \omega(\vec{a}\times\vec{b})$

Proved

(ii) from eqn (1)

$\frac{d\vec{r}}{dt} = -\omega(\sin\omega t)\vec{a} + \omega(\cos\omega t)\vec{b}$

$\frac{d^2\vec{r}}{dt^2} = -\omega^2(\cos\omega t)\vec{a} + \omega^2(\sin\omega t)\vec{b}$

$\frac{d^2\vec{r}}{dt^2} = -\omega^2\{(\cos\omega t)\vec{a} + (\sin\omega t)\vec{b}\}$

$\frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r}$

Proved

Q. If $\vec{r_1} = t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}$

$\vec{r_2} = (2t-3)\hat{i} + \hat{j} - t\hat{k}$

find (i) $\frac{d}{dt}(\vec{r_1}\cdot\vec{r_2})$ (ii) $\frac{d}{dt}(\vec{r_1}\times\vec{r_2})$ where t=1

(i) $\vec{r_1}\cdot\vec{r_2} = 2t^3 - 3t^2 - t - 2t^2 - t$

$\vec{r_1}\cdot\vec{r_2} = 2t^3 - 5t^2 - 2t$

$\frac{d}{dt}(\vec{r_1}\cdot\vec{r_2}) = 6t^2 - 10t - 2$

$t = 1$

$\frac{d}{dt}(\vec{r_1}\cdot\vec{r_2}) = 6 - 10 - 2$

$= -6$

(ii) $(\vec{r_1}\times\vec{r_2}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t^2 & -t & (2t+1) \\ (2t-3) & 1 & -t \end{vmatrix}$

$\Rightarrow \{t^2 - (2t+1)\}\hat{i} - \hat{j}\{-t^3 - (4t^2+2t-6t-3)\} + k\{t^2 - 2t^2+3t\}$

$\Rightarrow (t^2-2t-1)\hat{i} + (t^3+4t^2-4t-3)\hat{j} + (3t^2-3t)\hat{k}$

$\frac{d}{dt}(\vec{r_1}\times\vec{r_2}) = (2t-2)\hat{i} + (3t^2+8t-4)\hat{j} + (6t-3)\hat{k}$

, if $t = 1$

$$\frac{d}{dt}(\vec{r_1}\times\vec{r_2}) = (0)\hat{i} + 7\hat{j} + 3\hat{k}$$

$$\frac{d}{dt}(\vec{r_1}\times\vec{r_2}) = 7\hat{j} + 3\hat{k} \quad \text{Ans}$$

* Scalar point function :-

If the physical quantity is non directed then to each point of space a scalar can be assigned and the point function in that case is called a scalar point function. and the region which it defined the physical quantity is known as scalar field.

Thus, if to each point $P(x,y,z)$ of a region 'R' of space, a scalar 'P' is assigned then the scalar function $F(P)$ is said to be scalar point function is the region 'R' and this region is known as scalar field.

* Vector point function:-

when to each point of space or to each point of a plane a vector is assigned, the point function is called a vector point function and the region in which it defined the physical quantity is known as vector field.

Thus if to each point $P(x,y,z)$ of a region 'R' of space, there is associated a vector $\vec{v}$ then the function $\vec{v}(P)$ is said to be a vector point function in the region 'R' and this region 'R' is called vector field.

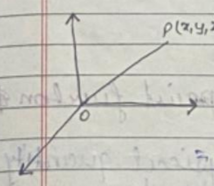
Del operator :-

It is a dual operator bcoz it satisfies the property of vector as well as differentiation. It is denoted by $(\vec{\nabla})$ and it is given by

$$\vec{\nabla} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

P.T $\vec{\nabla} r = \hat{r}$,

where $\vec{r}$ is position vector of any point let $P(x,y,z)$



$$\vec{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\therefore |\vec{OP}| = r = \sqrt{x^2+y^2+z^2}$$

now, $$\vec{\nabla} r = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{1/2}$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2} + \hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{1/2} + \hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{1/2}$$

$$= \hat{i}\frac{1(2x)}{2\sqrt{x^2+y^2+z^2}} + \frac{\hat{j}(2y)}{2\sqrt{x^2+y^2+z^2}} + \frac{\hat{k}\,2z}{2\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow \frac{x\hat{i}}{\sqrt{x^2+y^2+z^2}} + \frac{y\hat{j}}{\sqrt{x^2+y^2+z^2}} + \frac{z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{r}}{|\vec{r}|} = \hat{r}$$

then,

$$\vec{\nabla} r = \hat{r}$$

Prove that $\vec{\nabla} r^n = n r^{n-1} \hat{r}$

Proof:- R.H.S $= \vec{\nabla} r^n$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) (x^2+y^2+z^2)^{n/2}$$

$$\Rightarrow \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2} + \hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{n/2} + \hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{n/2}$$

$$\Rightarrow \hat{i}\frac{n}{2}(2x)(x^2+y^2+z^2)^{n/2-1} + \hat{j}\frac{n}{2}(2y)(x^2+y^2+z^2)^{n/2-1} + \hat{k}\frac{n}{2}(2z)(x^2+y^2+z^2)^{n/2-1}$$

$$\Rightarrow n(x^2+y^2+z^2)^{n/2-1}(x\hat{i}+y\hat{j}+z\hat{k})$$

$$\Rightarrow \frac{n(x^2+y^2+z^2)^{n/2}(x\hat{i}+y\hat{j}+z\hat{k})}{x^2+y^2+z^2}$$

main point $\Rightarrow$ $\frac{n(x^2+y^2+z^2)^{n/2}}{(x^2+y^2+z^2)^{1/2}} \times \frac{x\hat{i}+y\hat{j}+z\hat{k}}{(x^2+y^2+z^2)^{1/2}}$

$$\Rightarrow n(x^2+y^2+z^2)^{\frac{n-1}{2}}\hat{r}$$

$$\Rightarrow n\left(\sqrt{x^2+y^2+z^2}\right)^{n-1}\hat{r}$$

$$\Rightarrow \boxed{n\, r^{n-1}\hat{r}}$$

* Gradient of a scalar point function ϕ:-

Maximum change in any physical quantity w.r.t distance is called gradient

Gradient of a scalar point function is vector point function i.e.,

i.e., $\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$

$$\Rightarrow \vec{\nabla}\phi = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

* Divergence of a vector point function:-

Let, $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

then,

Divergence of $\vec{v}$ denoted by $\vec{\nabla}\cdot\vec{v}$

$$\vec{\nabla}\cdot\vec{v} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(v_1\hat{i} + v_2\hat{j} + v_3\hat{k})$$

$$\boxed{\vec{\nabla}\cdot\vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}}$$

Remark:- Divergence of vector function is a scalar

* Curl of a vector function:-

Let, $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ then curl of $\vec{v}$ is cross product of $\vec{\nabla}$ and $\vec{v}$

i.e.,

curl $\vec{v} = \vec{\nabla}\times\vec{v}$

$$\vec{\nabla}\times\vec{v} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times(v_1\hat{i} + v_2\hat{j} + v_3\hat{k})$$

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\vec{\nabla}\times\vec{v} = \hat{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \hat{j}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \hat{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

Note:-

Curl of a vector function is very curl mean rotation.

Q. Prove that curl (grad ϕ) = 0

Proof:- we have to show

$= \vec{\nabla} \times (\vec{\nabla}\phi) = 0$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$

$\Rightarrow \hat{i}\left(\frac{\partial}{\partial y}\frac{\partial\phi}{\partial z} - \frac{\partial}{\partial z}\frac{\partial\phi}{\partial y}\right) - \hat{j}\left(\frac{\partial}{\partial x}\frac{\partial\phi}{\partial z} - \frac{\partial}{\partial z}\frac{\partial\phi}{\partial x}\right) + \hat{k}\left(\frac{\partial}{\partial x}\frac{\partial\phi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial\phi}{\partial x}\right)$

$\Rightarrow 0\hat{i} - 0\hat{j} + 0\hat{k}$

$\Rightarrow 0$

Q. div (curl $\vec{v}$) = 0

Let $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$

we have to show,

$\vec{\nabla}\cdot(\vec{\nabla}\cdot v) = 0$

$\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[\hat{i}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_z}{\partial z}\right) - \hat{j}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right) + \hat{k}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\right]$

$\Rightarrow \frac{\partial^2 v_z}{\partial x\partial y} - \frac{\partial^2 v_y}{\partial x\partial z} - \frac{\partial^2 v_z}{\partial x\partial y} + \frac{\partial^2 v_x}{\partial y\partial z} + \frac{\partial^2 v_y}{\partial x\partial z} - \frac{\partial^2 v_x}{\partial y\partial z}$

$= 0$

Hence proved

$\vec{\nabla}$ $\vec{v}$

VVI

Q. If $\vec{v} = x^2yz\hat{i} + xy^2z\hat{j} + xyz^2\hat{k}$ then find (i) div $\vec{v}$ (ii) curl $\vec{v}$ (iii) curl curl $\vec{v}$

(i) $\vec{\nabla}\cdot v = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(x^2yz\hat{i} + xy^2z\hat{j} + xyz^2\hat{k})$

$\Rightarrow \frac{\partial}{\partial x}x^2yz + \frac{\partial}{\partial y}xy^2z + \frac{\partial}{\partial z}xyz^2$

$\Rightarrow 2xyz + 2xyz + 2xyz$

$= 6xyz$

(ii) $\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix}$

$\Rightarrow \hat{i}\left(\frac{\partial}{\partial y}xyz^2 - \frac{\partial}{\partial z}xy^2z\right) - \hat{j}\left(\frac{\partial xyz^2}{\partial x} - \frac{\partial x^2yz}{\partial z}\right) + \hat{k}\left(\frac{\partial xy^2z}{\partial x} - \frac{\partial x^2yz}{\partial y}\right)$

$\Rightarrow x(z^2-y^2)\hat{i} - y(z^2-x^2)\hat{j} + z(y^2-x^2)\hat{k}$

$\Rightarrow x(z^2-y^2)\hat{i} + y(x^2-z^2)\hat{j} + z(y^2-x^2)\hat{k}$ Ans

(iii) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v})$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(x(z^2-y^2)\hat{i} + y(x^2-z^2)\hat{j} + z(y^2-x^2)\hat{k}\right)$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x(z^2-y^2) & y(x^2-z^2) & z(y^2-x^2) \end{vmatrix}$

$\Rightarrow \hat{i}\left(\frac{\partial}{\partial y}(zy^2-zx^2) - \frac{\partial}{\partial z}(x^2y-z^2y)\right) - \hat{j}\left(\frac{\partial}{\partial x}(zy^2-x^2z) - \frac{\partial}{\partial z}(xz^2-xy^2)\right)$

$+ \hat{k}\left(\frac{\partial}{\partial x}(x^2y-z^2y) - \frac{\partial}{\partial y}(xz^2-xy^2)\right)$

$= (2yz + 2yz)\hat{i} - (-2xz - 2xz)\hat{j} + (2xy + 2xy)\hat{k}$

$= (4yz)\hat{i} + 4xz\hat{j} + 4xy\hat{k}$

$= 4(yz\hat{i} + xz\hat{j} + xy\hat{k})$

Q. find div (curl $\vec{F}$) where

$\vec{F} = x^2y\,\hat{i} + xz\,\hat{j} + 2yz\,\hat{k}$

$\text{curl}\,\vec{F} = \vec{\nabla} \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xz & 2yz \end{vmatrix}$$

$\vec{\nabla} \times \vec{F} = \hat{i}(2z - x) - \hat{j}(0-0) + \hat{k}(z - x^2)$

$\vec{\nabla} \times \vec{F} = (2z - x)\hat{i} + (z - x^2)\hat{k}$ — (i)

$\text{div}(\text{curl}\,\vec{F}) = \vec{\nabla} \cdot (\vec{\nabla} \cdot \vec{F})$

$\Rightarrow \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left((2z - x)\hat{i} + (z - x^2)\hat{k}\right)$

$\Rightarrow -1 + 1$

$= 0$ Ans

Q. Prove that $\vec{\nabla} \cdot (r^n \vec{r}) = (n+3) r^n$

Proof:- $\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\frac{\vec{r}}{r^n}\right)$

Result

$$\boxed{\text{Div}(\phi\vec{a}) = \phi(\text{Div}\,\vec{a}) + \vec{a} \cdot (\text{grad}\,\phi)}$$

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

L.H.S: $\vec{\nabla} \cdot [\phi(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})]$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (a_1\phi\hat{i} + a_2\phi\hat{j} + a_3\phi\hat{k})$

$\Rightarrow \frac{\partial}{\partial x}a_1\phi + \frac{\partial}{\partial y}a_2\phi + \frac{\partial}{\partial z}a_3\phi$

$\Rightarrow a_1\frac{\partial\phi}{\partial x}$

L.H.S = $\vec{\nabla} \cdot \phi\vec{a}$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (\phi\vec{a})$

$\Rightarrow \hat{i}\frac{\partial}{\partial x}(\phi\vec{a}) + \hat{j}\frac{\partial}{\partial y}(\phi\vec{a}) + \hat{k}\frac{\partial}{\partial z}(\phi\vec{a})$

$\Rightarrow \hat{i}\left(\phi\left(\frac{\partial\vec{a}}{\partial x}\right) + \frac{\partial\phi}{\partial x}\vec{a}\right) + \hat{j}\left(\phi\frac{\partial\vec{a}}{\partial y} + \frac{\partial\phi}{\partial y}\vec{a}\right) + \hat{k}\frac{\partial\vec{a}}{\partial z} + \hat{k}\frac{\partial\phi}{\partial z}\vec{a}$

$\Rightarrow \phi\,\hat{i}\frac{\partial\vec{a}}{\partial x} + \hat{i}\frac{\partial}{\partial x}$

$\Rightarrow \phi\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\vec{a} + \vec{a}\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$

$\Rightarrow \phi(\vec{\nabla} \cdot \vec{a}) + \vec{a}(\vec{\nabla}\phi)$ Ans

Q. $\vec{v} = \vec{\nabla}(x^3+y^3+z^3-3xyz)$

$\vec{\nabla}(x^3+y^3+z^3-3xyz)$

$\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(x^3+y^3+z^3-3xyz)$

$i(3x^2-3yz)+j(3y^2-3xz)+\hat{k}(3z^2-3xy)$

$\text{div}\,\vec{v} = \vec{\nabla}\cdot\vec{v}$

$=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\{(3x^2-3yz)i+(3y^2-3xz)j+(3z^2-3xy)\hat{k}\}$

$\Rightarrow 6x+6y+6z$

$= 6(x+y+z)$ Ans

$\text{curl}\,\vec{v} = \vec{\nabla}\times\vec{v}$

$=\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\left((3x^2-3yz)i+(3y^2-3xz)j+(3z^2-3xy)\hat{k}\right)$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$=(-3x+3x)i - j(-3y+3y)+\hat{k}(-3z+3z)$

$=0$ Ans

* Physical meaning of curl of $\vec{F}$:- The curl of the velocity of a rigid body is twice its angular velocity. This is reason that the name rotation is also used for the word "curl".

* Physical meaning of Divergence :- The divergence of a vector function represent the rate of outward flow through unit volume.

* Directional Derivative :- The directional derivative of a function in a particular direction is the projection of the gradient of the given function in that particular direction.

Eg. :- The directional derivative of the function "ϕ" in the direction of the vector $\vec{u}$ is $\frac{\vec{\nabla}\phi\cdot\vec{u}}{|\vec{u}|}$ at the given point.

85

UNIT :-

ANALYTICAL GEOMETRY OF 2D :-

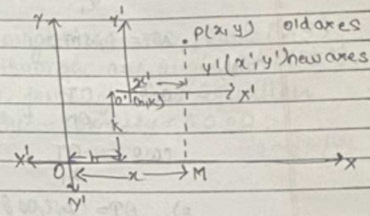
chapter 01 :-

CHANGE OF RECTANGULAR AXES

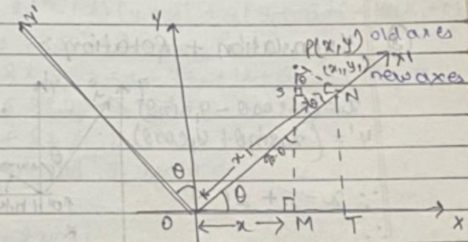
1. TRANSLATION
2. ROTATION
3. Translation + Rotation

(1) TRANSLATION :-

$$\boxed{\begin{aligned} x &= x' + h \\ y &= y' + k \end{aligned}}$$



(2) ROTATION :-



$OM = x$
$PM = y$
$ON = x_1$
$PN = y_1$

from figure;

In Δ^{rt} PSN,

$$\sin\theta = \frac{SN}{PN}$$

$$\sin\theta = \frac{SN}{y_1} \qquad SN = y_1 \sin\theta$$

Page 87

and, $\cos\theta = \frac{PS}{PN}$

$\Rightarrow \cos\theta = \frac{PS}{y_1}$

$PS = \cancel{\cos} y_1 \cos\theta$

In Δ^{rt} OTN

$\sin\theta = \frac{NT}{ON}$

$\sin\theta = \frac{NT}{x_1}$

$NT = x_1 \sin\theta$

and,

$\cos\theta = \frac{OT}{ON}$

$\cos\theta = \frac{OT}{x_1}$

$\Rightarrow OP = x_1 \cos\theta$

Now,

$OT = OM + MT$

$OT = OM + SN$

$x_1 \cos\theta = x + y_1 \sin\theta$

$$\boxed{x = x_1\cos\theta - y_1\sin\theta}$$

Next,

$PM = PS + SM$

$PM = PS + NT$

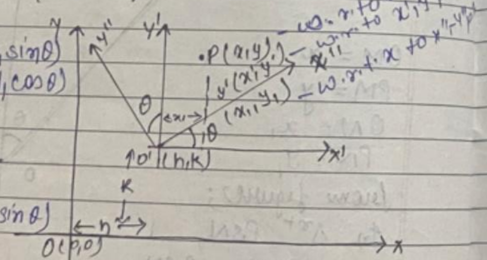
$\cancel{P} y = y_1\cos\theta + x_1\sin\theta$

$$\boxed{y = x_1\sin\theta + y_1\cos\theta}$$

(3) Translation + Rotation :-

$x' = (x_1\cos\theta - y_1\sin\theta)$

$y' = (x_1\sin\theta + y_1\cos\theta)$



$\because x = h + x'$

$\therefore$

$x = h + (x_1\cos\theta - y_1\sin\theta)$

$y = k + y'$

$y = k + (x_1\sin\theta + y_1\cos\theta)$

Page 87

Q. Transform the equation $4x^2 + 3y^2 - 2xy + 3x - 7y + 5 = 0$ to parallel axis through the point $(4, -1)$.

Put, $x = x_1 + 4$
and $y = y_1 - 1$ in given equation.

Q. Transform the equation
$17x^2 - 16xy + 17y^2 = 225$
to axes inclined at 45° to the original axis.

Q. By transforming the origin to the point $(2, 3)$ and turning axis through an angle $\pi/4\ (45^\circ)$, find the transform form of the equation
$$3x^2 + 2xy + 3y^2 - 18x + 22y + 50 = 0$$

19/11/21

S. Put $x = (x_1 + 2)$
and $y = (y_1 + 3)$ in Eqn (1)

$3(x_1+2)^2 + 2(x_1+2)(y_1+3) + 3(y_1+3)^2 - 18(x_1+2) - 22(y_1+3) + 50 = 0$

$\Rightarrow 3x_1^2 + 12x_1 + 2x_1y_1 + 6x_1 + 4y_1 + 12 + 3y_1^2 + 18y_1 + 27 - 18x_1 - 36 + 22y_1 - 66 + 50 = 0$

$\Rightarrow 3x_1^2 + 3y_1^2 + 2x_1y_1 = 1$

Replace x_1 by x and y_1 by y
then,

$3x^2 + 3y^2 + 2xy = 1$ — (ii)

for Rotation,

Given, $\theta = \pi/4 = 45^\circ$

Put $x = x_1\cos\pi/4 - y_1\sin\pi/4$

$\frac{x_1}{\sqrt{2}} - \frac{y_1}{\sqrt{2}}$

and $y = x_1 \sin\pi/4 + y_1 \cos\pi/4$
$= \left(\frac{x_1}{\sqrt2} + \frac{y_1}{\sqrt2}\right)$

$2\left(\frac{a_1}{\sqrt2} - \frac{y_1}{\sqrt2}\right)^2 + 3\left(\frac{x_1}{\sqrt2} + \frac{y_1}{\sqrt2}\right)^2 + 2\left(\frac{x_1}{\sqrt2} - \frac{x_1}{\sqrt2}\right)\left(\frac{x_1}{\sqrt2} - \ldots\right)$

$\Rightarrow 3\left(\frac{x_1^2}{2} + \frac{y_1^2}{2} - \frac{2x_1y_1}{2}\right) + 3\left(\frac{x_1^2}{2} + \frac{y_1^2}{2} + \frac{2xy}{2}\right) + 2\left(\frac{x_1}{\sqrt2} - \frac{y_1}{\sqrt2}\right)^2$

$\Rightarrow 3x_1^2 + 3y_1^2 - 3x_1y_1 + 3x_1y_1 + x_1^2 - y_1^2 = 1$

$\Rightarrow 4x_1^2 - 2y_1^2 = 1$

$\Rightarrow \boxed{4x^2 - 2y^2 = 1}$

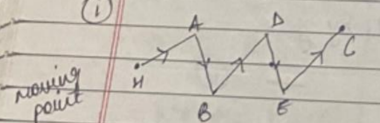
22/10/21

CONIC SECTION

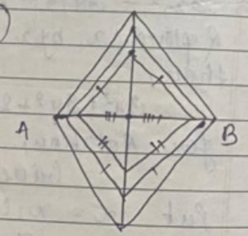
* Locus:-

Path traced by a moving ~~particle~~ point under certain geometric condition is called locus of that point.

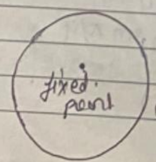
(1)



(2)



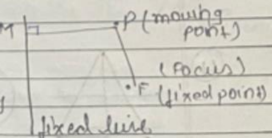
(3)



circle

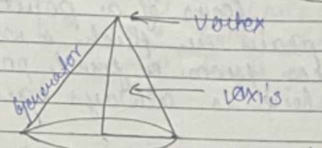
* Conic section :-

Locus of a point which moves on a plain in such a way that the ratio of its distances from a fixed point and a fixed line is always constant.

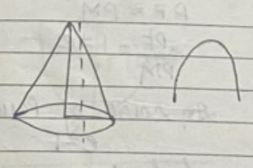


$\frac{PF}{PM}$ = constant
= Eccentricity

Here, fixed point is called "focus."
fixed line is called "directrix"
and fixed ratio is called "Eccentricity"

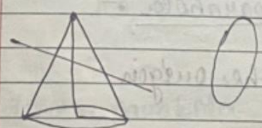


(I)

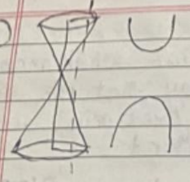


parabola $(e=1)$

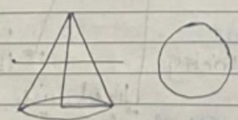
(II)



ellipse $(e<1)$

(iii) 

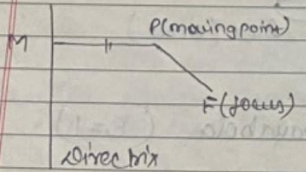
Hyperbola $(e>1)$

(iv) 

Circle $(e=0)$

① Parabola :-

locus of a point which moves on a plain in such a way that it's distances from a fixed point and a fixed line is always equal.



$PF = PM$

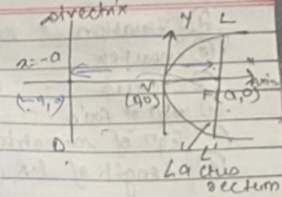
$\frac{PF}{PM} = 1 = e$

In case of Parabola $e=1$

* Standard form of parabola :-

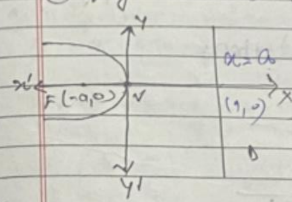
① when vertex is at the origin

① Right Handed parabola



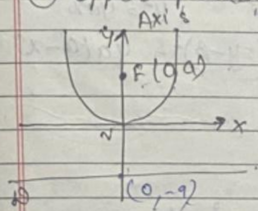
(i) Equation $y^2 = 4ax$; $a>0$
(ii) Vertex $V(0,0)$
(iii) focus $F(a,0)$
(iv) Eqn of axis ; $y=0$
(v) Eqn of Directrix ; $x = -a$
(vi) length of latus rectum $= 4a$

② Left handed Parabola :-



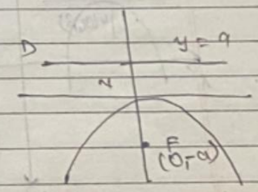
(i) Equation $y^2 = -4ax$; $a>0$
(ii) vertex $V(0,0)$
(iii) focus $F(-a,0)$
(iv) Eqn of Axis ; $y=0$
(v) Eqn of Directrix ; $x = a$
(vi) length of latus rectum $= 4a$

③ Upper parabola :-



(i) Equation , $x^2 = 4ay$, $a>0$
(ii) Vertex $V(0,0)$
(iii) focus $F(0,a)$
(iv) Equation of Axis ; $x=0$
(v) Equation of directrix ; $y=-a$
(vi) length of latus rectum $= 4a$

④ Lower parabola



Page 93

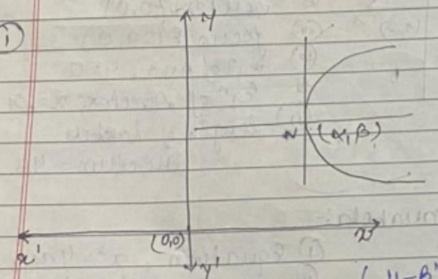
1. Equation = $x^2 = -4ay$, $a > 0$
2. vertex = $V(0,0)$
3. focus = $F(0,-a)$
4. Eqn of Axis = $x = 0$
5. Eqn of directrix = $y = a$
6. length of L.R = $4a$

23/11/21

(2) WHEN VERTEX IS NOT AT THE ORIGIN

Let vertex be $V(\alpha, \beta)$ then;

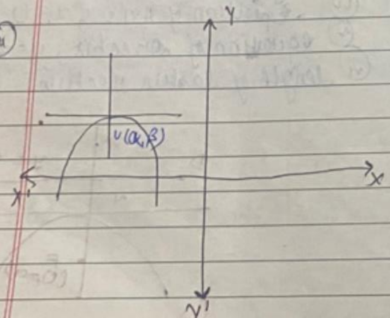
(1)



$y^2 = 4ax$

$(y-\beta)^2 = 4a(a-x)$

(4)



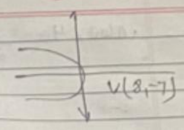
$x^2 = -4ay$

$(x-\alpha)^2 = 4a(y-\beta)$

Page 94

for ex:

$(y+7)^2 = 4a(x-8)$ $a > 0$



for finding vertex

Put, $x - 8 = 0$

$x = 8$

and $y + 7 = 0$

$y = -7$

vertex $(8,-7)$

Q: Equation :- $2x^2 + 5x - 7y + 3 = 0$

$\Rightarrow x^2 + \frac{5}{2}x - \frac{7}{2}y + \frac{3}{2} = 0$

$\Rightarrow \left(x^2 + 2 \cdot x \cdot \frac{5}{4}x + \frac{25}{16}\right) - \frac{25}{16} - \frac{7}{2}y + \frac{3}{2} = 0$

$\Rightarrow \left(x + \frac{5}{4}\right)^2 = \frac{7}{2}y + \frac{25}{16} - \frac{3}{2}$

$\Rightarrow \left(x + \frac{5}{4}\right)^2 = \frac{7}{2}y + \frac{25-24}{16}$

$\Rightarrow \left(x + \frac{5}{4}\right)^2 = \frac{7y}{2} + \frac{1}{16}$

$\Rightarrow \left(x + \frac{5}{4}\right)^2 = \frac{7}{2}\left(y + \frac{1}{56}\right)$

which is of the form of

$x^2 = 4ay$

for finding vertex:

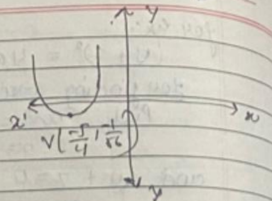
Put $x + \frac{5}{4} = 0 \Rightarrow x = \frac{-5}{4}$

and,

$y + \frac{1}{56} = 0 \Rightarrow y = \frac{-1}{56}$

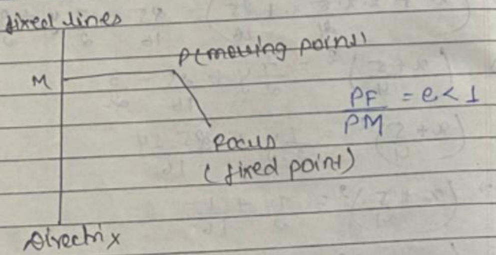
$\therefore$ vertex $V\left(\frac{-5}{4}, \frac{-1}{56}\right)$

also, $4a = \frac{7}{2} \Rightarrow a = \frac{7}{8}$



* Ellipse :-

Locus of a point which moves on a plain in such a way that the ratio of its distances from a fixed point and a fixed line is less than unity (1).



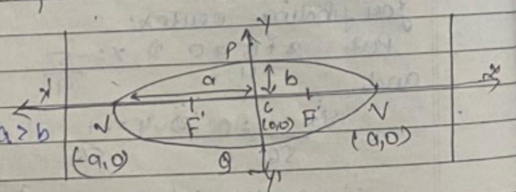
$\frac{PF}{PM} = e < 1$

Types of Ellipse :-

1. when center is at the origin

(1) Horizontal Ellipse :-

Equation of ellipse :- $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; $a > b$



center $C(0,0)$

eccentricity $= e = \sqrt{1 - \frac{b^2}{a^2}}$

vertices $V(\pm a, 0)$

foci $F(\pm ae, 0)$

length of major axis $= 2a$

length of minor axis $= 2b$

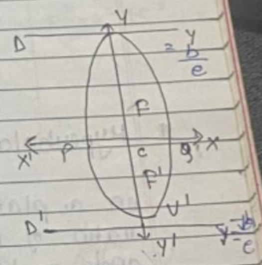
Equation of major axis; $y = 0$

Equation of minor axis; $x = 0$

Equation of directrix; $x = \pm \frac{a}{e}$

length of lactus rectum $= \frac{2b^2}{a}$

(ii) Vertical ellipse :-



Equation of ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a < b$

eccentricity $= e = \sqrt{1 - \frac{a^2}{b^2}}$

center $C(0,0)$

vertices $V(0, \pm b)$

foci $F(0, \pm be)$

length of major axis $2b$

length of minor axis $2a$

Eq^n of major axis, $x = 0$

Eq^n of minor axis, $y = 0$

Eq^n of directrices; $y = \pm \frac{b}{e}$

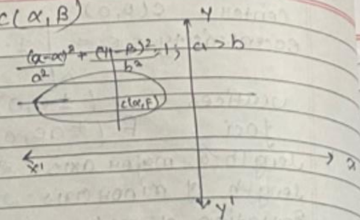
length of lactus rectum; $\frac{2a^2}{b}$

2) when center is at $C(\alpha, \beta)$

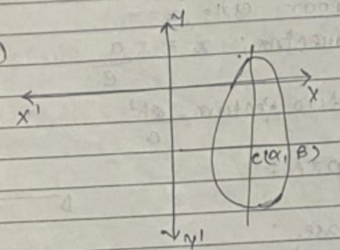
(i)

$$\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1$$

$a > b$



(ii)



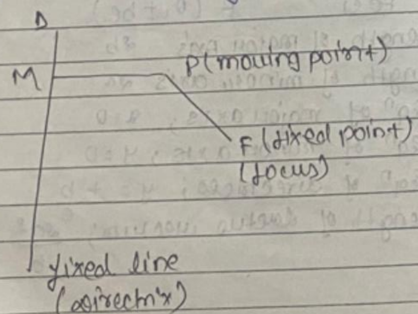
$$\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1$$

$(a < b)$

* Hyperbola :-

locus of a point which moves on a plain in such a way that the ratio of it's distances from a fixed point and a fixed line is greater than unity.

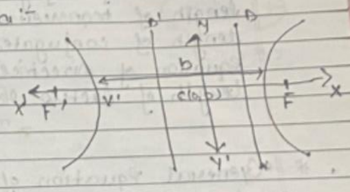
$\frac{PF}{PM} = e > 1$



23/11/21

Types of hyperbola :-

(1) Transverse hyperbola :-



(i) Equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(ii) Center $C(0,0)$

(iii) $e = \sqrt{1 + \frac{b^2}{a^2}}$

(iv) vertices $V(\pm a, 0)$

(v) foci $F(\pm ae, 0)$

(vi) Equation of transverse axis : $y = 0$

(vii) Equation of conjugate axis , $x = 0$

(viii) length of transverse Axis $= 2a$

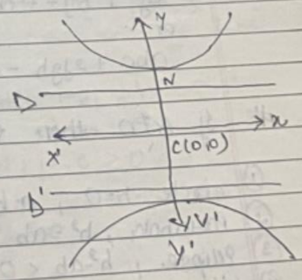
(ix) length of conjugate Axis $= 2b$

(x) Equation of directrices $x = \pm \frac{a}{e}$

(xi) length of lactus rectum $= \frac{2b^2}{a}$

eccentricity
$b^2 = a^2 (e^2 - 1)$

(2) Conjugate Hyperbola :-



(i) Equation $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(ii) Center $C(0,0)$

(iii) $e = \sqrt{1 + \frac{a^2}{b^2}}$

(iv) vertices $= V(0, \pm b)$

foci P $(0, \pm be)$

Equation of Transverse axis; $x=0$

Equation of conjugate Axis; $y=0$

length of transverse axis $= 2b$

length of conjugate axis $= 2a$

Equation of directrices $\Rightarrow \pm b/e$

length of latus rectum $= \frac{2a^2}{b}$

* General Equation of conic :-

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ —(1) is a 2nd degree equation in x and y

Equation (1) represents pair of straight if $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

i.e, $a(bc - f^2) + h(fg - ch) + g(fh - bg) = 0$

$\Rightarrow abc - af^2 + fgh - ch^2 + fgh - bg^2 = 0$

$\Rightarrow af^2 + bg^2 + ch^2 - 2fgh - abc = 0$

or

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

If $\Delta \neq 0$ then eqn (1) represents

1. circle $h=0$, $a=b$
2. Parabola, $h^2 = ab$
3. ellipse, $h^2 - ab < 0$
4. hyperbola, $h^2 - ab > 0$
5. rectangular hyperbola, $h^2 - ab > 0$, $a + b = 0$

$x^2 + y^2 + 2gx + 2fy + c = 0$

Centre $= (-g, -f)$, radius $= \sqrt{g^2 + f^2 - c}$

$\boxed{ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0}$

(h, g, f)

2hxy

Ex:-

$2x^2 + 5x - 7y + 3 = 0$

$a = 2$, $b = 0$, $2h = 0$

$h = 0$

$\therefore h^2 - ab = 0 - 2 \times 0$

$= 0$

Q. What conic section is represented by $2x^2 + 3y^2 - 4x - 12y + 13 = 0$

$a = 2$, $b = 3$, $h = 0$

$h^2 - ab$ $\quad 0 - 6 = 0$

$= -6 < 0$

given conic represents ellipse.

Q $5x^2 + 5y^2 + 2x + 3y + 4 = 0$

$h = 0$, $a = b = 5$

given conic represent ~~circle~~ ~~pair of straight line~~

Note :-

$x^2 + y^2 + 2gx + 2fy + c = 0$

represents circle with center $(-g, -f)$ and Radius $\sqrt{g^2 + f^2 - c}$ where

$g^2 + f^2 - c \geq 0$

soln $x^2 + y^2 + \frac{2}{5}x + \frac{3}{5}y + \frac{4}{5} = 0$

$\therefore 2g = \frac{2}{5} \Rightarrow g = \frac{1}{5}$; $2f = \frac{3}{5} \Rightarrow f = \frac{3}{10}$

$$g^2+f^2-c = \frac{1}{25}+\frac{9}{100}-\frac{4}{5}$$

$$\Rightarrow \frac{4+9-80}{100} = \frac{-67}{100} < 0$$

so, this doesn't represent circle

25 | 11 | 21

Q. To find equation of tangent on the curve
$S = ax^2+2hxy+by^2+2gx+2fy+c=0$ — (1)
at point (x_1, y_1) by calculus method
→ second degree general equation i.e.,

Sol^n

differentiating eq^n (1) partially w.r.to x we get,

$$\frac{\partial f}{\partial x} = 2ax+2hy+2g$$

$$\left(\frac{\partial f}{\partial x}\right)_{(x_1,y_1)} = 2ax_1+2hy_1+2g$$

Note:-
$$\frac{dy}{dx} = \left(\frac{-\frac{df}{dx}}{\frac{df}{dy}}\right) = \text{slope} = m$$

Next, differentiating eq^n (1) partially w.r.to y

$$\frac{\partial S}{\partial y} = 2hx+2by+2f$$

$$\left(\frac{\partial S}{\partial y}\right)_{(x_1,y_1)} = 2hx_1+2by_1+2f$$

we know eq^n of tangent (straight line) in point slope form

$$(y-y_1) = m(x-x_1)$$

$$(y-y_1) = \frac{-\left(\frac{\partial S}{\partial x}\right)_{(x_1,y_1)}}{\left(\frac{\partial S}{\partial y}\right)_{(x_1,y_1)}}(x-x_1)$$

$$\Rightarrow (y-y_1) = -\frac{(2ax_1+2hy_1+2g)}{(2hx_1+2by_1+2f)}(x-x_1)$$

$$\Rightarrow (y-y_1)(hx_1+by_1+f)+(x-x_1)(ax_1+hy_1+g) = 0$$

$$\Rightarrow hx_1y+byy_1+fy-hx_1y_1-by_1^2-fy_1+axx_1+hxy_1+gx-ax_1^2-hx_1y_1-gx_1=0$$

$$\Rightarrow axx_1+h(x_1y+xy_1)+byy_1+fy+gx = ax_1^2+2hx_1y_1+by_1^2+gx_1+fy_1$$

$$\Rightarrow axx_1+h(x_1y+xy_1)+byy_1+gx+gx_1+fy+fy_1+c = ax_1^2+2hx_1y_1+by_1^2+gx_1+fy_1+gx_1+fy_1+c$$

$$\Rightarrow axx_1+h(x_1y+xy_1)+byy_1+g(x+x_1)+f(y+y_1)+c = ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c \quad \text{— (ii)}$$

since point (x_1,y_1) lies on curve (1)
∴ it must satisfies eq^n (1)
then eq^n reduces to

$$ax_1^2+by_1^2+2hx_1y_1+2gx_1+2fy_1+c = 0 \quad \text{— (iii)}$$

from eq^n (ii) & (iii) eq^n of tangent will be

$$\boxed{axx_1+h(x_1y+xy_1)+byy_1+g(x+x_1)+f(y+y_1)+c=0}$$

* Equation of normal:-

Given,
$$S = ax^2+2hxy+by^2+2gx+2fy+c=0$$
we know that

(Slope of tangent)(Slope of normal) = −1

$$\Rightarrow \frac{\left(\frac{-\partial s}{\partial x}\right)}{\left(\frac{\partial s}{\partial y}\right)} = 1 \quad \text{(slope of normal)}$$

$$\text{slope of normal} = \frac{\left(\frac{\partial s}{\partial y}\right)}{\left(\frac{\partial s}{\partial x}\right)}$$

$$\frac{\partial s}{\partial x} = 2ax + 2hy + 2g$$

$$\Rightarrow \left(\frac{\partial s}{\partial x}\right)_{(x_1, y_1)} = (2ax_1 + 2hy_1 + 2g)$$

$$= 2hx + 2by + 2f$$

$$\left(\frac{\partial s}{\partial y}\right)_{(x_1, y_1)} = (2hx_1 + 2by_1 + 2f)$$

Eq^n of normal at point (x_1, y_1)

$$y - y_1 = (\text{slope of normal})(x - x_1)$$

$$(y - y_1) = \frac{\left(\frac{\partial s}{\partial y}\right)_{(x_1, y_1)}}{\left(\frac{\partial s}{\partial x}\right)_{(x_1, y_1)}} (x - x_1)$$

$$\Rightarrow (y - y_1) = \frac{(2hx_1 + 2by_1 + 2f)}{(2ax_1 + 2hy_1 + 2g)} (x - x_1)$$

$$\frac{(y - y_1)}{(hx_1 + by_1 + f)} = \frac{x - x_1}{(ax_1 + hy_1 + g)}$$

which is the required eq^n of normal.

UNIT :- IV

INTEGRAL CALCULUS

Syllabus :-

INDEFINITE INTEGRAL

Chapter 1 :- Basic concepts, Formulae, Rules, Questions.

Chapter 2 :- Methods of integration
(i) Transformation
(ii) Substitution

Chapter 3 :- Some special Integral

SET :- 01 Integral of the form $\int \frac{dx}{(x^2-a^2)}, \int \frac{dx}{(a^2-x^2)}$ and $\int \frac{dx}{(a^2+x^2)}$

Type (i) problems based upon direct Result and related to substitution

Type (ii) Integral of the form $\int \frac{dx}{(ax^2+bx+c)}$

Type (iii) Integral of the form $\int \frac{(px+q)}{ax^2+bx+c}\, dx$

Type (iv) Integral of the form $\int \left(\frac{x^2 \pm 1}{x^4+kx^2+1}\right) dx$

Type (v) Integral of the form
$\int \frac{dx}{a+b\sin^2 x}, \int \frac{dx}{a+b\cos^2 x}$ or $\int \frac{dx}{a\sin^2 x + b\cos^2 x + c}$

Type (vi) Integral of the form
$\int \frac{dx}{a+b\sin x}, \int \frac{dx}{a+b\cos x}$ or $\int \frac{dx}{a\sin x + b\cos x + c}$

Type (vii) Integral of the form $\int \frac{(a\sin x + b\cos x)\, dx}{(c\sin x + d\cos x)}$

Type (viii) Integral of the form $\int \left(\frac{a\sin x + b\cos x + c}{d\sin x + e\cos x + f}\right) dx$

SET :- 02 Integral of the form
$\int \frac{dx}{\sqrt{x^2-a^2}}, \int \frac{dx}{\sqrt{x^2+a^2}}$ and $\int \frac{dx}{\sqrt{a^2-x^2}}$ (with proof)

Type (i) Question based upon direct results and related to substitution.

Type (ii) Integral of the form $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

Type (iii) Integral of the form $\int \frac{(px+q)}{\sqrt{ax^2+bx+c}}\, dx$

SET 03 Integral of the form
$\int \sqrt{x^2-a^2}\, dx, \int \sqrt{x^2+a^2}\, dx$ and $\int \sqrt{a^2-x^2}\, dx$

(we proof result in next chapter by parts)

Type (i) problems based upon direct results and related to substitution

Type (ii) Integral of the form $\int \sqrt{ax^2+bx+c}\, dx$

Type (iii) Integral of the form
$\int (px+q)\sqrt{ax^2+bx+c}\, dx$

Chapter 04 :- Integration by parts / product Rule -

Type (i) Basic and standard problems

Type (ii) Integral of the form $\int f(x)\, dx$.

Type (iii) Integral of the form $\int e^x (f(x) + f'(x))\, dx$

Type (iv) Integral of the form $\int e^{ax}\sin(bx+c)$ and

$\int e^{ax} \cos(bx+c)\, dx$

CHAPTER:-05 Integration by partial fraction

Type (i) Basic and standard problems
Type (ii) Integration by substitution
Type (iii) Integration by replacement
Miscellaneous question

CHAPTER:-06

Integral of the form $\int \frac{dx}{X\sqrt{Y}}$ or $\int \frac{f(x)\,dx}{X\sqrt{Y}}$

(Integration of Irrational functions)

Type (1) X = linear, Y = linear

Type (2) X = quadratic, Y = linear

Type (3) X = linear, Y = Quadratic

Type (4) X = quadratic, Y = quadratic

29/11/21

Integration:-

It is the inverse process of differentiation. It is generally denoted by I and we can write $I = \int f(x)\,dx$ where, $f(x)$ is called integrand.

Ex:- (i) $\frac{d}{dx}(\tan x) = \sec^2 x$

$c + \tan x = \int \sec^2 x$

$\int \sec^2 x\, dx = \tan x + c$

(2) $\frac{d}{dx}(\sin x) = \cos x$

$\Rightarrow \int \cos x\, dx = \sin x + c$

$\int \sin x\, dx = -\cos x + c$

* Indefinite Integration

Formulae (Remember)

(i) $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$; $n \neq -1$

(ii) $\int dx = x + c$

(4) $\int \frac{1}{x}\, dx = \log|x| + c$

(iii) $\int e^x\, dx = e^x + c$

(5) $\int \sin x\, dx = -\cos x + c$

(iv) $\int a^x\, dx = \frac{a^x}{\log a} + c$

(7) $\int \sec^2 x\, dx = \tan x + c$

(6) $\cos x\, dx = \sin x + c$

(8) $\sec x \tan x\, dx = \sec x + c$

(9) $\int \operatorname{cosec}^2 x\, dx = -\cot x + c$

(10) $\int \operatorname{cosec} x \cot\, dx = -\operatorname{cosec} x + c$

(11) $\int \tan x\, dx = -\log|\cos x| + c$
or, $\log|\sec x| + c$

(12) $\int \cot x\, dx = \log|\sin x + c|$

(13) $\int \sec x\, dx = \log|\sec x + \tan x| + c$

or $\log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c$

(14) $\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + c$
$= \log\left|\tan \frac{x}{2}\right| + c$

(15) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + c$ (16) $\int \frac{dx}{(1+x^2)} = \tan^{-1}x + c$

(17) $\int \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1}x + c$

* $\int f'(x)\,dx = f(x) + c$

Imp * $\int f'\underbrace{(ax+b)}_{\text{linear}} \Rightarrow \frac{f(ax+b)}{a} + c$

Ex:- $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)}$

* $\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + c$ * $\int \frac{dx}{x^2} = \frac{-1}{x} + c$

Sol^n:- $I = \int \frac{1}{x^{1/2}}\,dx$
$\Rightarrow \int x^{-1/2}dx$
$\Rightarrow \frac{x^{-1/2+1}}{-1/2+1} \Rightarrow \frac{x^{1/2}}{1/2}$
$= 2\sqrt{x} + c$

Sol^n $I = \int \frac{1}{x^2}\,dx$
$= \int x^{-2}dx$
$= \frac{x^{-2+1}}{-2+1}$
$= \frac{x^{-1}}{-1} = \frac{-1}{x} + c$

* Rules of Integration :-

R1:- $\int k\,u\,dx = k\int u\,dx$
where
k is constant

Ex:- $I = \int \frac{-3}{\sqrt{x}}\,dx \Rightarrow -3\int \frac{dx}{\sqrt{x}}$
$= -3(2\sqrt{x}) = -6\sqrt{x} + c$

R2:- $\int (u \pm v \pm w \pm \dots)\,dx$
$= \int u\,dx \pm \int v\,dx \pm \int w\,dx \pm \dots$

Ex:- $\int \left(x^5 - \frac{7}{x} + \frac{3}{x^2}\right) dx$
$= \int x^5 dx - 7\int \frac{dx}{x} + 3\int \frac{dx}{x^2}$
$\Rightarrow \frac{x^6}{6} - 7\log|x| - \frac{3}{x} + c$

Question:-

(1) $\int \left(\frac{ax^2+bx+c}{\sqrt{x}}\right) dx$ (2) $\int \left(\sqrt{\frac{x}{a}} + \sqrt{\frac{a}{x}}\right)^2 dx$

(3) $\int \frac{dx}{\sin^2 x \cos^2 x}$ (4) $\int \sec x^\circ \tan x^\circ \, dx$

(5) $\int \cos(1-x)\,dx$ (6) $\int \left(\frac{e}{a}\right)^x dx$

(7) $\int \left(\frac{x^3-1}{x-1}\right) dx$

(1) $I = \int \left(\frac{ax^2+bx+c}{\sqrt{x}}\right) dx$
$\Rightarrow a\int \frac{x^2}{x^{1/2}}\,dx + b\int \frac{x}{x^{1/2}}\,dx + c\int \frac{dx}{x^{1/2}}$

$\Rightarrow a\int a^{3/2}\,dx + b\int x^{1/2}\,dx + c\int x^{-1/2}\,dx$

$\Rightarrow \dfrac{ax^{5/2}}{(5/2)} + \dfrac{bx^{3/2}}{(3/2)} + c\dfrac{x^{1/2}}{1/2}$

$\Rightarrow \dfrac{2a}{5}x^{5/2} + \dfrac{2b}{3}x^{3/2} + 2cx^{1/2} + C$ Ans.

04/12/21

(8) $I = \int\left(\dfrac{x^4+1}{x^2+1}\right)dx \Rightarrow$

* when degree of numerator is <= degree of denominator then we have to first divide.

soln: $x^2+1\,)\,x^4+1\,(\,x^2-1$

$\underline{x^4+x^2}$

$-x^2+1$

$\underline{-x^2-1}$

2

we know that,

$D = d\times q + R$

$\Rightarrow (x^4+1) = (x^2+1)(x^2-1)+2$

$\left(\dfrac{x^4+1}{x^2+1}\right) = \dfrac{(x^2+1)(x^2-1)}{(x^2+1)} + \dfrac{2}{(x^2+1)}$

$\Rightarrow \int\left(\dfrac{x^4+1}{x^2+1}\right)dx = \int(x^2-1)\,dx + 2\int\dfrac{dx}{(1+x^2)}$

$\therefore I = \dfrac{x^3}{3} - x + 2\tan^{-1}x + C$ Ans.

(9) $I = \int\left(\dfrac{x^2-1}{x^2+1}\right)dx$

soln: $I = \int\left(\dfrac{x^2+1-2}{x^2+1}\right)dx$

$I = \int\left(\dfrac{\cancel{x^2+1}}{\cancel{x^2+1}}\right)dx - 2\int\dfrac{1}{x^2+1}\,dx$

$\Rightarrow x - 2\tan^{-1}x + C$ Ans.

10. $I = \int\dfrac{dx}{(\sqrt{x+a}+\sqrt{x-b})}$

soln: $\int\dfrac{1}{(\sqrt{x+a}+\sqrt{x-b})}\times\dfrac{(\sqrt{x+a}-\sqrt{x-b})}{(\sqrt{x+a}-\sqrt{x-b})}\,dx$

$I = \int\dfrac{\sqrt{x+a}-\sqrt{x-b}}{\cancel{x}+a-\cancel{x}+b}\,dx$

$= \dfrac{1}{a+b}\int\left((x+a)^{1/2} - (x-b)^{1/2}\right)dx$

$\Rightarrow \dfrac{1}{a+b}\left(\dfrac{2}{3}(x+a)^{3/2} - \dfrac{2}{3}(x-b)^{3/2}\right)$

$\Rightarrow \dfrac{2}{3(a+b)}\cdot\left\{(x+a)^{3/2} - (x-b)^{3/2}\right\} + C$ Ans.

11. $I = \int\dfrac{dx}{\sqrt{2x+3}-\sqrt{2x-3}}$

soln: $I = \int\left(\dfrac{1}{\sqrt{2x+3}-\sqrt{2x-3}}\right)\times\left(\dfrac{\sqrt{2x+3}+\sqrt{2x-3}}{\sqrt{2x+3}+\sqrt{2x-3}}\right)dx$

$I = \int\left(\dfrac{\sqrt{2x+3}+\sqrt{2x-3}}{\cancel{2x}+3-\cancel{2x}+3}\right)dx$

$I = \dfrac{1}{6}\int\left((2x+3)^{1/2} + (2x-3)^{1/2}\right)dx$

$\Rightarrow \dfrac{1}{6}\left(\dfrac{2}{3}\dfrac{(2x+3)^{3/2}}{2} + \dfrac{2}{3}\dfrac{(2x-3)^{3/2}}{2}\right)$

$= \dfrac{1}{18}\left((2x+3)^{3/2} + (2x-3)^{3/2}\right) + C$ Ans.

12. $I = \int \frac{(x^3 + x^2 - x + 1)}{(x-1)} dx$

$I = \int (x^2 + 2x + 1) dx + \int \frac{2}{x-1} dx$

$x - 1 \overline{) x^3 + x^2 - x + 1} (x^2 + 2x + 1$
$\quad x^3 - x^2$
$\quad 2x^2 - x + 1$
$\quad 2x^2 - 2x$
$\quad x + 1$
$\quad x - 1$
$\quad 2$

$\Rightarrow \frac{x^3}{3} + \frac{2x^2}{2} + x + 2 \log|x-1|$

$\Rightarrow \frac{x^3}{3} + x^2 + x + 2 \log|x-1| + C$ Ans

* Methods of Integration :-

(A) Transformation

(1) $I = \int \tan^2 x \, dx$

$I = \int (\sec^2 x - 1) \, dx$

$= \tan x - x + c$ Ans

(2) $I = \int \cot^2 x \, dx$

$I = \int (\operatorname{cosec}^2 x - 1) \, dx$

$I = -\cot x - x + C$ Ans

(3) $I = \int \sin^2 x \, dx$

$= \int \cancel{(1 - \cos^2 x)} dx$

$= \cancel{\int x} -$

$I = \frac{1}{2} \int 2 \sin^2 x \, dx$

$= \frac{1}{2} \int (1 - \cos 2x) dx$

$= \frac{1}{2} \left(x - \frac{\sin 2x}{2} \right) + C$ Ans

Note :-

* $\cos 2x = \cos^2 x - \sin^2 x$
$= 1 - 2\sin^2 x$
$= 2\cos^2 x - 1$

* $1 - \cos 2x = 2 \sin^2 x$
$1 + \cos 2x = 2 \cos^2 x$

(4) $I = \int \cos^2 x \, dx$

$I = \frac{1}{2} \int 2 \cos^2 x \, dx$

$I = \frac{1}{2} \int (1 + \cos 2x) \, dx$

$I = \frac{1}{2} \left(x + \frac{\sin 2x}{2} \right) dx$

Note :-

$\sin^3 x = \frac{3 \sin x - \sin 3x}{4}$

$\sin 3x = 3 \sin x - 4 \sin^3 x$

$\cos^3 x = \frac{3 \cos x + \cos 3x}{4}$

$\cos 3x = 4 \cos^3 x - 3 \cos x$

$\tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$

(5) $I = \int \sin^3 x \, dx$

$I = \int \frac{3 \sin x - \sin 3x}{4} dx$

$= \frac{1}{4} \int 3 \sin x - \sin 3x \, dx$

$= \frac{1}{4} \int 3 \sin x \, dx - \int \sin 3x \, dx$

$= \frac{1}{4} 3 \left(-\cos x - \frac{\cos 3x}{3} + c \right)$

(6) $I = \int \cos^3 x \, dx$

$I = \int \frac{3 \cos x + \cos 3x}{4} dx$

$I = \frac{1}{4} 3 \left(\sin x + \frac{\sin 3x}{3} \right) + c$

(7) $I = \int \sin^4 x \, dx$

$I = \int (\sin^2 x)^2 dx$

$I = \int \left(\frac{1 - \cos 2x}{2} \right)^2 dx$

$I = \frac{1}{4} \int (1 - \cos 2x)^2 dx$

$I = \frac{1}{4} \int (1 + \cos^2 2x + 2 \cos 2x) dx$

$I = \frac{1}{4} \int 1 + \left(\frac{1 + \cos 4x}{2} \right) - 2 \cos 2x \, dx$

$I = \frac{1}{4} \int \frac{(3 + \cos 4x - 4 \cos 2x)}{2} dx$

$I = \frac{1}{4} \int \left(3x + \frac{\sin 4x}{4} - \frac{4 \sin 2x}{2} \right)$

(8) $I = \int \cos^4 x \, dx$

$I = \int (\cos^2 x)^2 dx$

$I = \int \left(\frac{1 + \cos 2x}{2} \right)^2 dx$

$= \frac{1}{4} \int (1 + \cos^2 2x + 2 \cos 2x) dx$

$= \frac{1}{4} \int 1 + \left(\frac{1 + \cos 4x}{2} \right) + 2 \cos 2x \} dx$

$= \frac{1}{4} \int \left(\frac{3 + \cos 4x + 4 \cos 2x}{2} \right) dx$

$= \frac{1}{8} \left(3x + \frac{\sin 4x}{4} + \frac{4 \sin 2x}{2} \right)$

$= \frac{3x}{8} + \frac{\sin 4x}{32} + \frac{\sin 2x}{4} + c$ Ans

$= \frac{3x}{8} + \frac{\sin 4x}{32} - \frac{\sin 2x}{4} + c$ Ans

02/12/21

Q9. $I = \int \sin 4x \cdot \cos 3x \, dx$

Sol^n:- $I = \frac{1}{2}\int 2\sin 4x \cdot \cos 3x \, dx$

$= \frac{1}{2}\int (\sin 7x + \sin x) \, dx$

$= \frac{1}{2}\left(-\frac{\cos 7x}{7} - \cos x\right)$

$= -\frac{1}{2}\left(\frac{\cos 7x}{7} + \cos x\right) + c$ Ans

SET:-01 Transformation formula

(i) $\sin(A+B) + \sin(A-B) = 2\sin A \cos B$

(ii) $\sin(A+B) - \sin(A-B) = 2\cos A \sin B$

(iii) $\cos(A+B) + \cos(A-B) = 2\cos A \cos B$

(iv) $\cos(A+B) - \cos(A-B) = -2\sin A \sin B$

(v) $\cos(A-B) - \cos(A+B) = 2\sin A \sin B$

Q10 $I = \int \cos 7x \cdot \cos 5x \, dx$

$I = \frac{1}{2}\int 2\cos 7x \cdot \cos 5x \, dx$

$= \frac{1}{2}\int \cos 12x + \cos 2x \, dx$

$= \frac{1}{2}\left(\frac{\sin 12x}{12} + \frac{\sin 2x}{2}\right) + c$ Ans

Q11 $I = \int \sin x \cdot \sin \frac{3x}{2} \, dx$

Sol^n: $I = \frac{1}{2}\int 2\sin x \cdot \sin \frac{3x}{2} \, dx$

$\frac{1}{2}\int 2 \cdot \cos x \cdot \cos \frac{3x}{2} \, dx$

$\frac{1}{2}\int \left(2 - \cos \frac{5x}{2} - \cos \frac{x}{2}\right)$

$= \frac{1}{2}\left(\frac{\sin \frac{x}{2}}{\frac{1}{2}} - \frac{\sin \frac{5x}{2}}{5/2}\right)$

$\Rightarrow \sin \frac{x}{2} - \frac{1}{5}\sin \frac{5x}{2} + c$ Ans

Note:-

(i) If sin sin or cos cos present in denominator then, in numerator write sin

(ii) If in denominator, sin cos or cos sin present then in numerator write cos.

Q12 $I = \int \frac{dx}{\sin(x-a)\cos(x-b)}$

$I = \frac{1}{\cos(b-a)}\int \frac{\cos((x-a)-(x-b))}{\sin(x-a)\cos(x-b)}$

$= \frac{1}{\cos(b-a)}\int \frac{\cos(x-a)\cos(x-b) + \sin(x-a)\sin(x-b)}{\sin(x-a)\cos(x-b)}$

$= \frac{1}{\cos(b-a)}\int (\cot(x-a) + \tan(x-b)) \, dx$

$= \frac{1}{\cos(b-a)} = \left\{\log|\sin(x-a)| - \log|\cos(x-b)|\right\}$

$= \frac{1}{\cos(b-a)} \log\left|\frac{\sin(x-a)}{\cos(x-b)}\right| + c$ Ans

Page 117

Q13. $I = \int \frac{dx}{\cos(x+a)\cos(x-b)}$

$I = \frac{1}{\sin(a+b)} \int \frac{\sin((x+a)-(x-b))}{\cos(x+a)\cos(x-b)} dx$

$I = \frac{1}{\sin(a+b)} \int \frac{\sin(x+a)\cos(x-b) - \cos(x+a)\sin(x-b)}{\cos(x+a)\cos(x-b)} dx$

$I = \frac{1}{\sin(a+b)} \int (\tan(x+a) - \tan(x-b))\, dx$

$= \frac{1}{\sin(a+b)} \left(-\log|\cos(x+a)| + \log|\cos(x-b)| \right)$

$= \frac{1}{\sin(a+b)} \log\left| \frac{\cos(x-b)}{\cos(x+a)} \right| + C$ Ans

14 Q. $I = \int \frac{dx}{\sin(x+a)\sin(x+b)}$

$I = \frac{1}{\sin(a-b)} \int \frac{\sin((x+a)-(x+b))}{\sin(x+a)\sin(x+b)} dx$

$I = \frac{1}{\sin(a-b)} \int \frac{\sin(x+a)\cos(x+b) - \cos(x+a)\sin(x+b)}{\sin(x+a)\sin(x+b)}$

$I = \frac{1}{\sin(a-b)} \int \cot(x+b) - \cot(x+a)$
$-\log|\cot(x+a)| + \log|\cos(x+b)|$

$I = \frac{1}{\sin(a-b)} \log\left| \frac{\sin(x+b)}{\sin(x+a)} \right| + C$ Ans

Page 118

15 Q. $I = \int \frac{\sin(x-a)}{\cos(x-b)} dx$

Soln:- $I = \int \frac{\sin((x-b)+(b-a))}{\cos(x-b)} dx$

$I = \int \frac{\sin(x-b)\cos(b-a) + \cos(x-b)\sin(b-a)}{\cos(x-b)} dx$

$I = \cos(b-a) \int \tan(x-b)\, dx + \sin(b-a)$

$= \cos(b-a) \log|\sec(x-b)| + x\sin(b-a) + C$ Ans

03-12-21

Substitution :-

Let, $I = \int \frac{f'(x)}{f(x)} dx$

let, $f(x) = t$
$\therefore f'(x)\, dx = dt$

$I = \int \frac{dt}{t} = \log|t| = \log|f(x)| + C$

Note :-
$\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + C$ Ans.

Question (Level 1)

(1) $I = \int \frac{\tan^{-1}x}{(1+x^2)} dx$

Soln- Let, $\tan^{-1}x = t$

$\therefore \frac{1}{1+x^2}\,dx = dt$

$I = \int t\,dt = \frac{t^2}{2} = \frac{(\tan^{-1}x)^2}{2} + c$ Ans

(2) $I = \int \frac{e^{m\sin^{-1}x}}{\sqrt{1-x^2}}\,dx \Rightarrow$ Let, $\sin^{-1}x = t$

$\therefore \frac{1}{\sqrt{1-x^2}}\,dx = dt$

$I = \int e^{mt}\,dt \Rightarrow \frac{e^{mt}}{m} \Rightarrow \frac{e^{m\sin^{-1}x}}{m} + c$ Ans

(3) $I = \int x^2\cos x^3 \sin x^3\,dx$

Sol^n:- Let, $\sin x^3 = t$

$\therefore (\cos x^3)(3x^2)\,dx = dt$

$\Rightarrow x^2\cos x^3\,dx = \frac{dt}{3}$

$I = \frac{1}{3}\int t\,dt$

$= \frac{t^2}{6} = \frac{(\sin x^3)^2}{6} + c$ Ans

(4) $I = \int \left(\frac{10^x\log 10 + 10x^9}{x^{10}+10^x}\right)dx$

Sol^n:- Let,

$x^{10}+10^x = t$

$\therefore (10x^9 + 10^x\log 10)\,dx = dt$

$I = \int \frac{dt}{t} \Rightarrow \log|t| = \log|x^{10}+10^x| + c$ Ans

Q. $I = \int \frac{(x+1)(\log x + x)^2}{x}\,dx$

Sol^n:- $I = \int \left(\frac{x+1}{x}\right)(\log x + x)^2\,dx$

$= \int \left(1+\frac{1}{x}\right)(\log x + x)^2\,dx$

Let, $\log x + x = t$

$\therefore \left(1+\frac{1}{x}\right)dx = dt$

$I = t^2\,dt$

$= \frac{t^3}{3} = \frac{(x+\log x)^3}{3} + c$ Ans

Q. $I = \int \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)dx$

Sol^n:- $(e^x + e^{-x}) = t$

$(e^x - e^{-x}) = dt$

$I = \int \frac{dt}{t}$

$I = \log|t|$

$I = \log|e^x + e^{-x}| + c$ Ans

Q. $I = \int e^{\sin^2 x}\sin 2x\,dx$

$\sin^2 x = t$

$\therefore 2\sin x\cos x = dt$

$\Rightarrow \sin 2x\,dx = dt$

$I = \int e^t\,dt$

$e^t = e^{\sin^2 x} + c$ Ans

Q. $I = \int (x+1) e^x \cos(xe^x)\, dx$

Sol^n:- $xe^x = t$

$\therefore (xe^x + e^x) dx = dt$

$\Rightarrow (x+1) e^x dx = dt$

$I = \int \cos t \, dt$

$= \sin t$

$= \sin(xe^x) + c$ Ans.

(iii) $I = \int \frac{dx}{x^{1/2} - x^{1/3}}$ $\Rightarrow$ Put, $x = t^6 \Rightarrow dx = 6t^5 dt$

$= \int \frac{6t^5 \, dt}{t^3 - t^2}$

$= 6\int \frac{t^3 dt}{t-1}$

Hint:- $x^{m/n} - x^{p/q}$ take lcm of n and q

$= 6\int \frac{(t^3 - 1 + 1)\, dt}{(t-1)}$

$= 6\int \frac{(t^3 - 1^3)}{(t-1)} + \frac{1}{(t-1)}\, dt$

$= 6\int\int \frac{(t-1)(t^2+1+t)}{(t-1)} dt + \int \frac{dt}{(t-1)}$

$= 6\left[\int (t^2+1+t)\, dt + \int \frac{dt}{(t+1)}\right]$

$= 6\left[\frac{t^3}{3} + t + \frac{t^2}{2} + \ln(t-1)\right] + c$

Put, $t = x^{1/6}$

$= 6\left[\frac{(x^{1/2})^3}{3} + x^{1/6} + \frac{(x^{1/3})^2}{2} + \ln(x^{1/6} - 1)\right] + c$

(iv) $I = \int \frac{dx}{x - \sqrt{x}}$ $\Rightarrow$ Put $x = t^2 \Rightarrow dx = 2t\, dt$

$\Rightarrow \int \frac{2t\, dt}{t^2 - t}$

$\Rightarrow \frac{2dt}{t-t}$

$= 2\ln|t-1| + c$

$= 2\ln|\sqrt{x} - 1| + c$ Ans.

(v) $I = \int 3^{3^{3^x}} \cdot 3^{3^x} \cdot 3^x \, dx$

Put, $3^{3^{3^x}} = t \Rightarrow 3^{3^{3^x}}(\ln 3) \times 3^{3^x} \cdot (\ln 3) \times 3^x \ln = dt$

$\Rightarrow 3^{3^{3^x}} \cdot 3^{3^x} \cdot 3^x \, dx = \frac{dt}{(\ln 3)^3}$

$= \int \frac{dt}{(\ln 3)^3}$

$= \frac{t}{(\ln 3)^3} = \frac{3^{3^{3^x}}}{(\ln 3)^3} + c$ Ans.

(6) $\int \frac{\sin x}{\sin(x-a)} dx \Rightarrow \int \frac{\sin[(x-a)+a]}{\sin(x-a)} dx$

$\sin(A+B) = \sin A \cos B + \cos B \sin B$, $A = x-a$, $B = a$

$\Rightarrow \int \frac{\sin(x-a)\cos a + \cos(x-a)\sin a}{\sin(x-a)}$

$= \cos a\, dx + \int \frac{\cos(x-a)\sin a\, dx}{}$

$\Rightarrow \cos a \int dx + \sin a \int \cot(x-a)\, dx$

$\Rightarrow (\cos a) \times x + (\sin a) \times \ln|\sin(x-a)| + c$.

7. $\rightarrow \sqrt{2} \times \sin^{-1}(t) + c \Rightarrow \sqrt{2}\sin^{-1}\frac{(\sin x - \cos x)}{2} + c$

Date 24/12/21

(2) $I = \int \frac{dx}{x^2(x^4+1)^{3/4}} \Rightarrow \int \frac{dx}{x^2\left[x^4\left(1+\frac{1}{x^4}\right)\right]^{3/4}} \Rightarrow \int \frac{dx}{x^2 x^3\left(1+\frac{1}{x^4}\right)^{3/4}}$

$I = \int \frac{dx}{x^5\left(1+\frac{1}{x^4}\right)^{3/4}}$ let, $1+\frac{1}{x^4} = t \Rightarrow 1+x^{-4} = t$

$\therefore -4x^{-5}\,dx = dt \Rightarrow \frac{1}{x^5}\,dx = \frac{-dt}{4}$

$I = \frac{-1}{4}\int \frac{dt}{(t)^{3/4}} \Rightarrow -\frac{1}{4}\int t^{-3/4}\,dt$

$\Rightarrow \frac{-1}{4}\frac{t^{1/4}}{1/4} \Rightarrow \left(1+\frac{1}{x^4}\right)^{1/4} + c$ Ans

H.W

chapter (3) some special integral
set I - Remember (Proof all)

(i) $\int \frac{dx}{(x^2-a^2)} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

(ii) $\int \frac{dx}{(a^2-x^2)} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

(iii) $\int \frac{dx}{(a^2+x^2)} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

(i) $I = \int \frac{dx}{x^2-a^2} \Rightarrow \int \frac{dx}{(x-a)(x+a)} = \int \frac{(x+a)-(x-a)}{2a(x-a)(x+a)}\,dx$

$\Rightarrow \frac{1}{2a}\left[\int \frac{(x+a)-(x-a)}{(x-a)(x+a)}\,dx\right] = \frac{1}{2a}\left[\int \frac{dx}{(x-a)} - \int \frac{dx}{x+a}\right]$

$\Rightarrow \frac{1}{2a}\left[\ln|x-a| - \ln|x+a|\right] = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$

(iii) $I = \int \frac{dx}{a^2+x^2} \Rightarrow$ Put $x = a\tan\theta$
$\theta = \tan^{-1}\frac{x}{a}$

$= \int \frac{a\sec^2\theta\,d\theta}{a^2+a^2\tan^2\theta} = \int \frac{a\sec^2\theta\,d\theta}{a^2(1+\tan^2\theta)}$

$= \frac{1}{a}\int \frac{\sec^2\theta\,d\theta}{\sec^2\theta} = \frac{1}{a}\int d\theta = \frac{1}{a}\theta = \frac{1}{a}\tan^{-1}\frac{x}{a}$

(ii) $I = \int \frac{dx}{(a^2-x^2)} = \int \frac{dx}{(a+x)(a-x)} = \frac{1}{2a}\int \frac{(a+x)+(a-x)}{(a+x)(a-x)}\,dx$

$\Rightarrow \frac{1}{2a}\int\left(\frac{1}{a-x}\right) + \left(\frac{1}{a+x}\right)dx$

$\Rightarrow \frac{1}{2a}\left(-\log|a-x| + \log|a+x|\right)$

$\Rightarrow \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

Page no:- 167, 168, 169, 170
Page no - 185, 186, 187,
nth order derivative

~~C156971285~~
~~C156972712~~

C156975120

C156971285

C156972712

(i) measurement of angles
(ii) trigonometric functions
(iii) compound angles
(iv) transformation formulae
set 1 :- four formulae
set 2 :- four formulae.
i.e., if $A+B=C$
and $A-B=D$
(v) multiple angles
(vi) sub-multiple angle
(vii) trigonometric ratios of some particular angles.
$\sin 18$, $\cos 18$, $\sin 36$, $\cos 36$
(viii) trigonometric equation (nine-formulae)

$\Rightarrow \frac{1}{2a}\left[\int \frac{(x+a)-(x-a)}{(x-a)(x+a)}\,dx\right] = \frac{1}{2a}\left[\int \frac{dx}{(x+a)} - \int \frac{dx}{x+a}\right]$

$\Rightarrow \frac{1}{2a}\left[\ln|x-a| - \ln|x+a|\right] = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$

If the human brain were a computer, it could perform 38 thousand-trillion operations per second. The world's most powerful supercomputer, BlueGene, can manage only

The total weight of all the ants on Earth is greater than total weight of all the humans on the planet.

Amazing Facts

Starfish can re-grow their arms. In fact, a single arm can regenerate a whole body.

The average hummingbird's heart rate is more than 1,200 beats per minute.