L-1 QM Physics

Quantum system: whose basis state form a vector space over complex number

↓

① finite dimension

② inner product / (scaler product) is defined

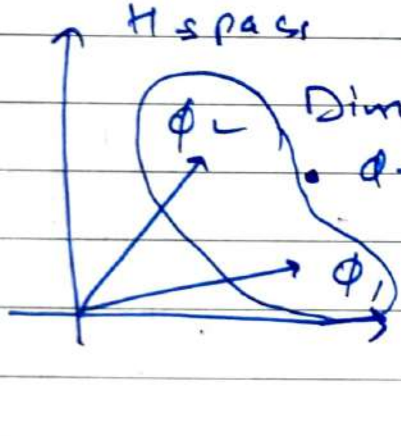
⇓

Hilbert state

Hilbert ~~state~~ space:- A Linear vector space over complex number.

Q - system:- An element in Hilbert space

↳ (state vector) $\in$ H.



Dim = 2

• Q-system = $\psi = c_1\phi + c_2\phi_2 = \sum_i c_i\phi_i$

$c_i \in \mathbb{C}$

$c_i \in \mathbb{C}$

linear transform $\hat{A}\phi_1 = \lambda\phi_1$

↳ linear operator

$$\hat{A}(a_1\psi_1 + a_2\psi_2) = a_1\hat{A}\psi_1 + a_2\hat{A}\psi_2$$

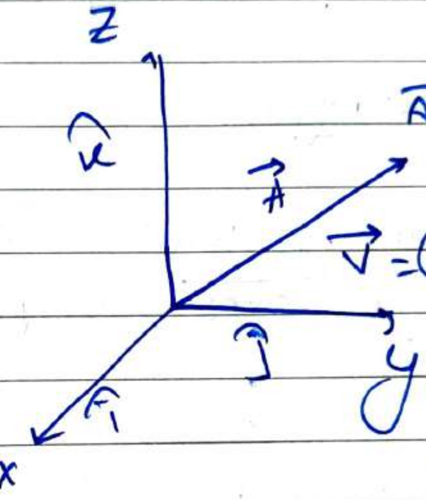
$$\boxed{\hat{A}\psi_1 = \lambda\psi_2}$$

Hilbert space → A linear vector space over complex number

* whose inner product is defined

Real space [3D]

↳ $c \in \mathbb{R}$



$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$\vec{V} = (2\hat{i} + 3\hat{j} + 5\hat{k})$

$\hat{i}\cdot\hat{i} = 1$

$\hat{j}\cdot\hat{j} = 1$

$\hat{k}\cdot\hat{k} = 1$ {normalized}

complete over R

$\hat{i}, \hat{j}, \hat{k}$ are called basis state

$\hat{i}\cdot\hat{j} = \hat{k}\cdot\hat{j} = \hat{j}\cdot\hat{k} = 0$ orthogonal

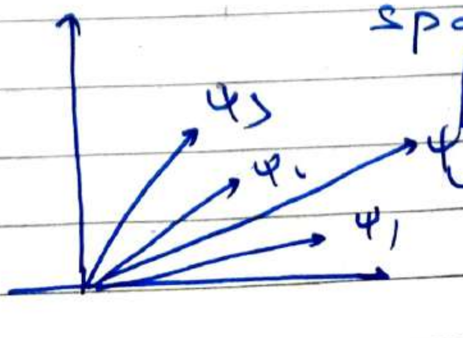
$[\hat{i}\ \hat{j}\ \hat{k}]$ ⇒ is basis which satisfy orthonormal condition (orthogonal + normalized)

⇓

"complete" no other state is required

Hilbert space

spanned by $[\psi_1, \psi_2, \psi_3]$



↳ φ system

$\psi = c_1\psi_1 + c_2\psi_2 + c_3\psi_3$

Basis : $[\psi_1, \psi_2, \psi_3]$

$\int \psi_i^* \psi_j \, d\tau = \delta_{ij} \Rightarrow \langle \psi_i | \psi_j \rangle = \delta_{ij}$ (Kronekar delta)

Integration volume

$i = j \;\; 1 \qquad i \neq j \;\; 0$

$\langle \psi_i | \psi_j \rangle = \delta_{ij} \Rightarrow$ orthogonal & normalized
↳ inner product

$\Downarrow$

Basis is complete

$\psi \in \mathcal{H} \Rightarrow \psi$ is normalised

$\psi = c_1\psi_1 + c_2\psi_2 + c_3\psi_3$

$\int \psi^* \psi \, dv = \int (c_1^*\psi_1^* + c_2^*\psi_2^* + c_3^*\psi_3^*)(c_1\psi_1 + c_2\psi_2 + c_3\psi_3)\, d\tau$

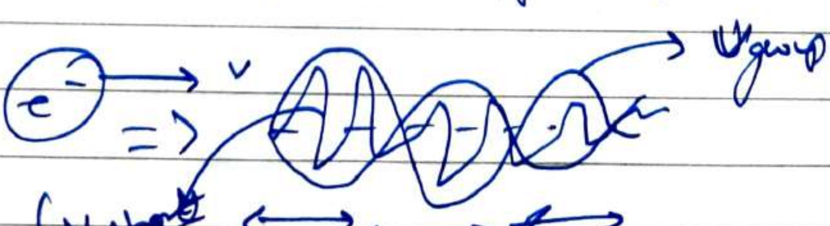
$= |c_1|^2 \underbrace{\int \psi_1^* \psi_1 \, d\tau}_{1} + |c_2|^2 \underbrace{\int \psi_2^* \psi_2 \, d\tau}_{1} + |c_3|^2 \underbrace{\int \psi_3^* \psi_3 \, d\tau}_{1}$

$\underbrace{\int \psi^* \psi \, dv}_{1} = |c_1|^2 + |c_2|^2 + |c_3|^2$

$\Rightarrow$ if $\int \psi^* \psi \, dv = 1 \Rightarrow \psi$ is normalized

$\Rightarrow |c_1|^2 + |c_2|^2 + |c_3|^2 = 1$

Sum of square of amplitudes = 1



$p_1 \quad p_2 \quad p_3 \rightarrow \neq 1$

$p_1 + p_2 + p_3 = 1$, ψ is normalized

$\lambda = \frac{h}{p}$, $h \to 10^{-34}$, $p \to 10^{-31}$

we are prominent in particle nature not in wave nature

① $\psi = |\psi_1\rangle + 2i|\psi_2\rangle$ find normalization constant?

$\hookrightarrow |1|^2 + |2i|^2 = 1 + 4 = 5$

$\psi = \frac{1}{\sqrt{5}} \left(|\psi_1\rangle + 2i|\psi_2\rangle\right)$

→ normalization constant

$\psi = a\psi_1 + b\psi_2$.

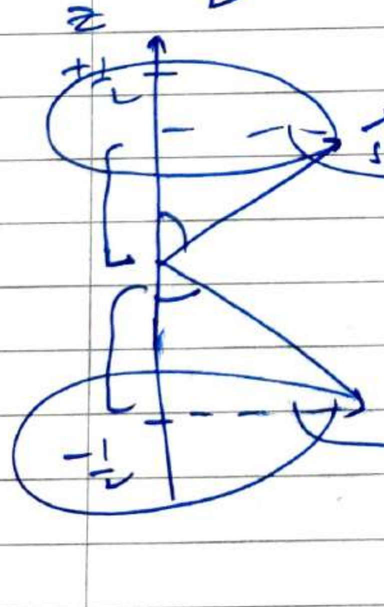
$N = \frac{1}{\sqrt{|a|^2 + |b|^2}}$

normalization required must because total probability is 1

eg Hilbert space

$e^- \to$ Basis $= |s, m_s\rangle$

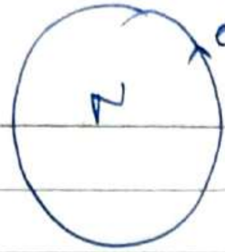
$\hookrightarrow s = \frac{1}{2}$ $\quad [-s \text{ to } +s]$



$\vec{s}$ → up

$|\uparrow\rangle = |\tfrac{1}{2}, \tfrac{1}{2}\rangle$

→ Down $|\downarrow\rangle = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$

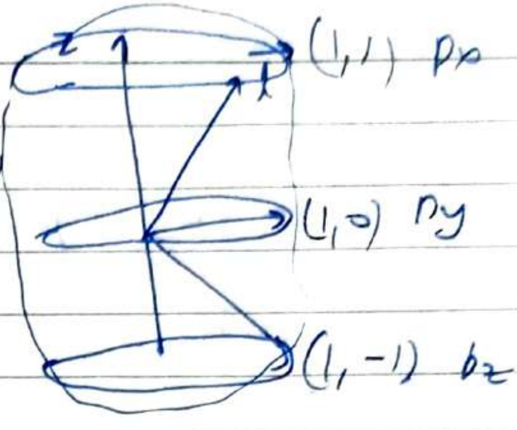


$e^- \to$ orbital $\vec{l}$

p-state

$l = 1$

$\Rightarrow m_l = +1, 0, -1$

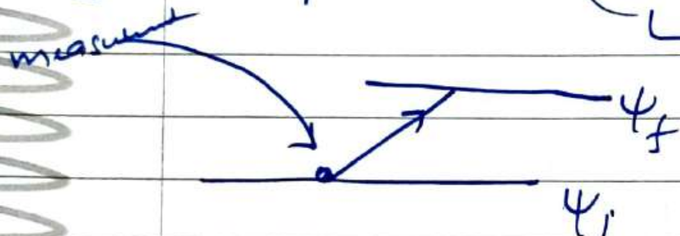


(II) Inner product & outer product

Let $|u\rangle, |v\rangle \in H$

(I) Inner product $\to$ (scalar product)

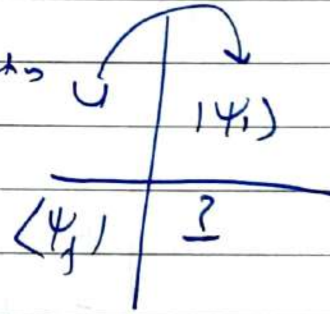
$\hookrightarrow \in \mathbb{C}$



Q what is the probability of finding the system in ψ_f after making a measurement on ψ_i?

Ans prob. amplitude $= \langle \psi_f | \psi_i \rangle$ $\to$ operator

final state



$= \text{prob} \, |\langle \psi_f | \psi_i \rangle|^2$

$\psi = \alpha|0\rangle + \beta|1\rangle$

$\langle 1|\psi\rangle$ probability amplitude

$= \langle 1|(\alpha|0\rangle + \beta|1\rangle)$

$= \alpha\cancel{\langle 1|0\rangle}_{\to 0} + \beta\underbrace{\langle 1|1\rangle}_{1}$

$= \beta$ probability $= |\beta|^2$.

$|\alpha|^2 + |\beta|^2 = 1$

$\langle \, | \, \rangle$

Target

Initial

$\psi_i = \alpha|0\rangle + \beta|1\rangle$

$P_{amp}^{(0)} =$

Outer product: let $|u\rangle$ & $|v\rangle \in H$

$|u\rangle\langle v|$ → outer product

ket, bra

$\psi_1 = \begin{bmatrix} 1 \\ 2 \\ 3i \end{bmatrix}$ $\psi_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

outer product $|\psi_1\rangle\langle\psi_2|$

$\begin{bmatrix} 1 \\ 2 \\ 3i \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$

$op = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3i & 3i & 6i \end{bmatrix}_{3\times 3}$

→ Complex matrix

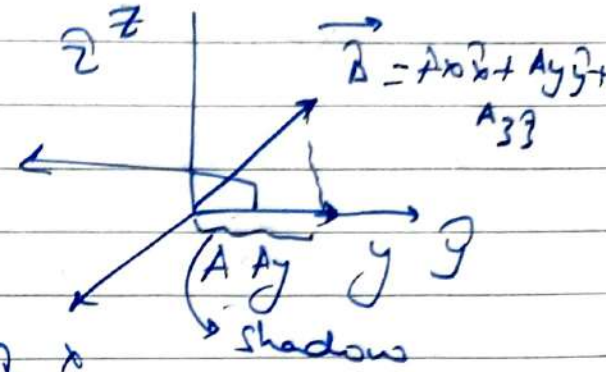
→ Operator

let op $|u\rangle\langle u|$ projection operator

same vector

Real space

projection of $\vec{A}$ along $\hat{y}$



$A_y\,\hat{y}$

$$(\vec{A}\cdot\hat{y}) = A_y$$

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

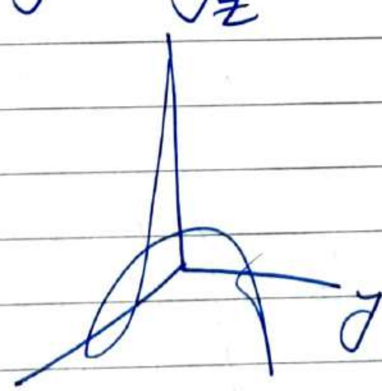
$$(\vec{A}\cdot\hat{x}) = A_x\,\hat{x}\cdot\hat{x} + A_y\,\cancel{\hat{y}\cdot\hat{x}} + A_z\,\cancel{\hat{z}\cdot\hat{x}}$$

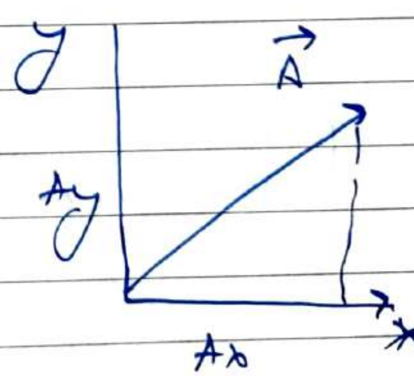
$$(\vec{A}\cdot\hat{x}) = A_x$$

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

$$\vec{A}\cdot\hat{y} = A_y$$





$$A_x\hat{x} + A_y\hat{y}$$

$$A_x\hat{x} = (\vec{A}\cdot\hat{x})\,\hat{x}$$

projection of $\vec{A}$ over $\hat{x}$

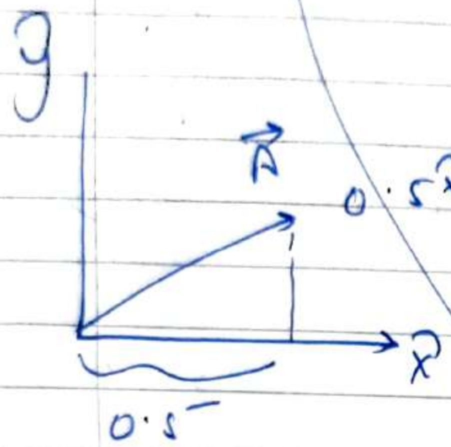
Quantum op

Projection is always inner product

$|u\rangle\langle u| \Rightarrow$ projection op $=$ $\vec{A}\cdot\vec{B} = AB\cos\theta$

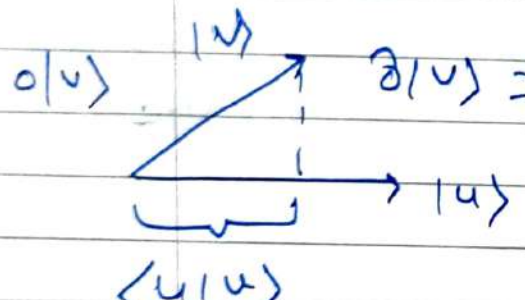
$\Downarrow$ $\quad (A\cos\theta)\,B$

$\quad\to$ projection of A over B



$0.5\hat{x} + 0.3\hat{y}$

$0.5\,\hat{x}$



$\langle u|v\rangle\,|u\rangle$

$\hat{O}|v\rangle = \langle u|v\rangle\,|u\rangle$

$\hat{O} = |u\rangle\langle v|$

$(|u\rangle\langle v|)\,|v\rangle)$

$$\boxed{\begin{aligned}&\langle u|v\rangle\,|u\rangle\\ &= c\,|u\rangle\end{aligned}}$$

project any vector on to the subspace of another vector

$$\psi_1 = \begin{bmatrix}1\\2\\1\end{bmatrix} \;\&\; \psi_2 = \begin{bmatrix}1\\0\\1\end{bmatrix} = \langle\psi_1|\psi_2\rangle$$

$\Downarrow$

$$\cancel{|\psi_1\rangle\langle\psi_1|} = \begin{bmatrix}1\\2\\1\end{bmatrix}$$

(1) projection opr $P_1 = |\psi_1\rangle\langle\psi_1|$

$$\begin{bmatrix}1\\2\\1\end{bmatrix}\begin{bmatrix}1 & 2 & 1\end{bmatrix} = \begin{bmatrix}1 & 2 & 1\\2 & 4 & 2\\1 & 2 & 1\end{bmatrix}_{3\times 3}$$

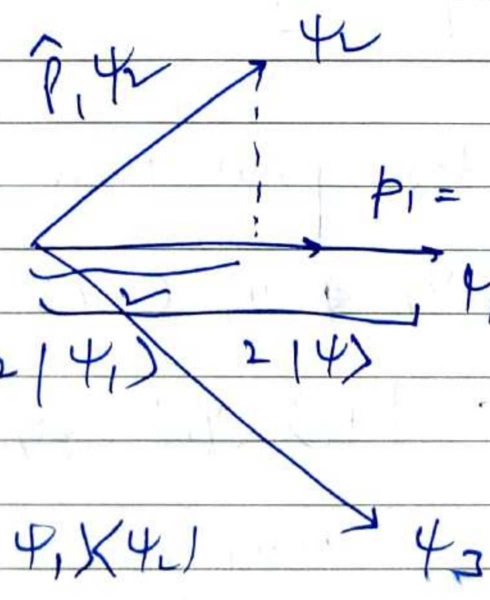
$$P_1|\psi\rangle = \left(|\psi_1\rangle\langle\psi_1|\right)|\psi_2\rangle$$

$$\langle\psi_1|\psi_2\rangle\,|\psi_1\rangle$$

$$\langle\psi_1|\psi_2\rangle \quad \begin{bmatrix}1, & 2 & 1\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix} = \underline{\underline{2}}$$

$$P_1|\psi_2\rangle = \underset{\substack{\downarrow\\ \text{projection}}}{2}\,|\psi_1\rangle$$

$\hat{P}_1\psi_2$



$P_1 = |\psi_1\rangle\langle\psi_1|$

$P_1|\psi_2\rangle = 2|\psi_1\rangle$ $\quad 2|\psi\rangle$

(1) Note : $|\psi_1\rangle\langle\psi_2|$

$$\begin{bmatrix}1\\2\\1\end{bmatrix}\begin{bmatrix}1 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 1\\2 & 0 & 2\\1 & 0 & 1\end{bmatrix}$$

Trace $= \sum_i \lambda_{ii}$ (sum of diagonal elements) Trace (sum of diagonal elements)

$1+0+1=2$

Trace must be equal to innerproduct.

$a_{11}=1, \; a_{22}=0, \; a_{33}=1$

$\boxed{\text{Tr of op} = \text{IP}}$

$\boxed{\text{Trace of } |\psi\rangle\langle\psi| = \langle\psi|\psi\rangle}$

H. $\bar{e} \Rightarrow Dm = 2$

$x = \begin{bmatrix} \; \end{bmatrix}$

$y = \begin{bmatrix} \; \end{bmatrix}$

$z = \begin{bmatrix} \; \end{bmatrix}$

L-2 Outer product and Postulates.

Let ψ_1 & $\psi_2 \in \mathcal{H}$

op $\rightarrow$ $|\psi_1\rangle\langle\psi_2|$ $\Rightarrow$ $\text{Tr}(|\psi_1\rangle\langle\psi_2|)$

$\hookrightarrow$ complex matrix $\quad = \langle\psi_1|\psi_2\rangle$

① Projection operator

$\hookrightarrow |u\rangle\langle u| = P_u$

$\hookrightarrow$ unit vector

Properties

(1) $P_u^{\dagger} = (|u\rangle\langle u|)^{\dagger} = |u\rangle\langle u| = \hat{P}_u$

(P_u is Hermitian)

$(|\psi\rangle\langle\psi_2|)^{\dagger} = |u\rangle\langle u|$

(2) $P_u^2 = \hat{P}_u \hat{P}_u = (|u\rangle\langle u|)(|u\rangle\langle u|) = |u\rangle\langle u|$

$\langle u|u\rangle \to 1$

$\hat{P}_u^2 = P_u$

(3) System $[|u\rangle, |v\rangle]$

$\sum \hat{P}_u = 1\,(I)$ $\qquad |u\rangle\langle u| + |v\rangle\langle v| = I$

$\hookrightarrow$ completeness rule $\qquad$ Identity

(3) Completeness Rule $\quad \sum_{\psi} P_{\psi} = 2$

Z-Basis (Computational basis)

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|0\rangle\langle 0| + |1\rangle\langle 1|$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

(2) X-Basis (Diagonal Basis)

$+ = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad |-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$

$\begin{pmatrix}1\\1\end{pmatrix}\begin{bmatrix}1 & 1\end{bmatrix}$

$|+\rangle\langle+| \quad + \quad |-\rangle\langle-|$

$\begin{matrix}1 & 1\\-1 & -1\end{matrix}$

$\frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} \quad \frac{1}{2}\begin{bmatrix}1 & 1\\-1 & -1\end{bmatrix} = \begin{matrix}1 & 0\\0 & 1\end{matrix} = I$

Expand an operator (square matrix) in outer product form

$\{\psi_1 \ \& \ \psi_2\}$ $\quad A = \begin{bmatrix}a & b\\c & d\end{bmatrix}$

ket → basis

A	$\|\psi_1\rangle$	$\|\psi_2\rangle$
Bra $\langle\psi_1\|$	a	b
$\langle\psi_2\|$	c	d

$\hat{A} = a|\psi_1\rangle\langle\psi_1| + b|\psi_1\rangle\langle\psi_2|$

$+ c|\psi_2\rangle\langle\psi_1| + d|\psi_2\rangle\langle\psi_2|$

X - gate $\quad$ —$\boxed{x}$— $\quad x = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ in Z basis

X	$\|0\rangle$	$\|1\rangle$
$\langle 0\|$	a 0	b 1
$\langle 1\|$	c 0	d 1

$\cancel{\underset{0}{a}|0\rangle\langle 0|} + \underset{1}{b}|0\rangle\langle 1| +$

$\underset{1}{c}|1\rangle\langle 0| + \underset{0}{d}|1\rangle\langle 1|$

$X = |0\rangle\langle 1| + |1\rangle\langle 0|$

② $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ in z axis

H	$\|0\rangle$	$\|1\rangle$
$\langle 0\|$	1	1
$\langle 1\|$	1	-1

$\frac{1}{\sqrt{2}}\left[1|0\rangle\langle 0| + 1|0\rangle\langle 1| + 1|1\rangle\langle 0| - 1|1\rangle\langle 1|\right]$

~~$\frac{1}{\sqrt{2}}[|0\rangle\langle 0| +$~~

② Matrix $\Rightarrow$ Outer product

Diagonal matrix

$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$

$\lambda \Rightarrow$ Diagonal elements.

a_{11} a_{22} a_{33}

$|0\rangle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $|1\rangle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $|2\rangle \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

A	$\|0\rangle$	$\|1\rangle$	$\|2\rangle$
$\langle 0\|$	a_{11}	0	0
$\langle 1\|$	0	a_{22}	0
$\langle 2\|$	0	0	a_{33}

$A = a_{11}|0\rangle\langle 0| +$
$a_{22}|1\rangle\langle 1| +$
$a_{33}|2\rangle\langle 2|$

$$a_{11}\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} + a_{22}\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix} + a_{33}\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

eigen values $\rightarrow$ $|0\rangle\langle 0|$ $\quad$ $\lambda \rightarrow |1\rangle\langle 1|$ $\quad$ $\lambda \rightarrow |2\rangle\langle 2|$

$$\lambda_1 \underline{|\psi_1\rangle\langle\psi_1|} + \lambda_2 \underline{|\psi_2\rangle\langle\psi_2|} + \lambda_3 \underline{|\psi_3\rangle\langle\psi_3|}$$

$\langle\psi_1|\psi_1\rangle$ $\quad$ $\langle\psi_2|\psi_2\rangle$ $\quad$ $\langle\psi_3|\psi_3\rangle$

Trace of numerator NR $\quad$ Tr of numerator NR $\quad$ Tr of NR

Postulates of Quantum Computation

$\downarrow$

set of assumptions to describe a system

Q. system:

How to study

(1) Representation of Q system $\Rightarrow$ (2) How the system change w.r.t. time

$\downarrow$

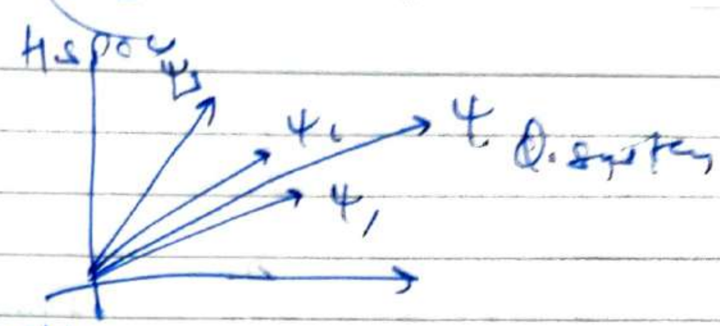
(3) measurement of Q system

$\downarrow$

(4) Composite system

(1) Postulate I (Representation)

Q-system



Spanned by bases vector of system $\{\psi_1, \psi_2, \psi_3\}$

ψ = Linear combination of basis

$= a_1\psi_1 + a_2\psi_2 + a_3\psi_3$

$$\boxed{\psi = \sum_i a_i \psi_i}$$

Quantum computation Basis

General

Z basis — X basis — Y basis

X basis, Y basis: Diagonal Basis

Z basis: $|0\rangle$ & $|1\rangle$

$\begin{bmatrix}1\\0\end{bmatrix}$ $\begin{bmatrix}0\\1\end{bmatrix}$

X basis: $|+\rangle$ $|-\rangle$

$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$

Y basis: $|\circlearrowright\rangle$ $|\circlearrowleft\rangle$

$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$ $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$

Postulate 2 :- $\psi = \sum_i a_i \psi_i$

Z Basis $\{|0\rangle, |1\rangle\}$ — X Basis $\{|+\rangle \,\&\, |-\rangle\}$

$\psi_{Qubit} = \alpha|0\rangle + \beta|1\rangle$ $\quad\psi_{Qubit} = \alpha|+\rangle + \beta|-\rangle$

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

$$|+\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle - |1\rangle\right]$$

$$\frac{1}{\sqrt{2}}\left[|0\rangle + \cancel{|1\rangle} + |0\rangle - \cancel{|1\rangle}\right]$$

$$\frac{2|0\rangle}{\sqrt{2}}$$

$$+\quad |+\rangle + |-\rangle = \frac{2}{\sqrt{2}}|0\rangle$$

$$|0\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$$

$$|1\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)$$

$$\psi_{Qubit} = \alpha|0\rangle + \beta|1\rangle$$

$$= \alpha\frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right] + \beta\frac{1}{\sqrt{2}}\left[|+\rangle - |-\rangle\right]$$

$$\psi_{Qubit} = \left(\frac{\alpha+\beta}{\sqrt{2}}\right)|+\rangle + \left(\frac{\alpha-\beta}{\sqrt{2}}\right)|-\rangle$$

Postulate 2 How ~~does~~ the system varying with respect to time

↳ schrodinger eqn. $i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$ ↳ Hamiltonian of closed system

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$$

continuous in time but our quantum computation in discrete time

Rewrite $i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$

$$\int_{\psi(0)}^{\psi(t)} \frac{\partial \psi}{\psi} = \int_0^t \frac{\hat{H}}{i\hbar}\, \partial t = \left[\ln \psi\right]_{\psi(0)}^{\psi(t)}$$

$$= \frac{-i\hat{H}}{\hbar} t$$

$$= \frac{1}{i}$$

$$\left[\ln \psi(t) - \ln \psi(0)\right] = \frac{-i\hat{H}t}{\hbar}$$

$$\ln\left[\frac{\psi(t)}{\psi(0)}\right] = \frac{-i\hat{H}t}{\hbar}$$

$$\psi(t) = \psi(0)e^{\frac{-i\hat{H}t}{\hbar}}$$

unitary opr

$$\boxed{\ln x = y \quad x = e^y}$$

$$\boxed{U^{+}U = I}$$

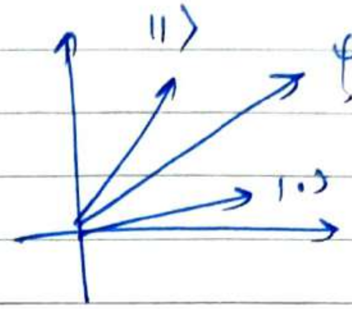
$$\psi(t) = U\psi|0\rangle$$

Discrete

$t_0 \to t \qquad \psi(t) = U\,\psi(t_0)$

$\hookrightarrow U(t_0, t_1)$

Postulate 3 (Measurement)



Quantum measurement

eg $\psi_{\text{Qubit}} = \alpha|0\rangle + \beta|1\rangle$

$|0\rangle, |1\rangle$

Measurement operator → projection operator

↳ project any vector on its subspace

↳ $M_y \Rightarrow \quad M_0 = |0\rangle\langle 0|$

↳ op $\quad M_1 = |1\rangle\langle 1|$

$\psi_1 = \alpha|0\rangle + \beta|1\rangle$

(1) $M_0\,\psi = |0\rangle \underbrace{\langle 0|(\alpha|0\rangle + \beta|1\rangle)}$

$= |0\rangle\,[\alpha\underbrace{\langle 0|0\rangle}_{1} + \beta\underbrace{\langle 0|1\rangle}_{0}]$

$M_0\psi_0 = \alpha|0\rangle$

(2) $M_1\psi_1 = |1\rangle\langle 1|(\alpha|0\rangle + \beta|1\rangle)$

$= |1\rangle(0 + \beta) = \beta|1\rangle$

$$M_1 \psi_i = \beta|1\rangle$$

② probability $\psi = \alpha|0\rangle + \beta|1\rangle \Rightarrow \langle\psi| = \alpha^*\langle 0| + \beta^*\langle 1|$

$$M_0\psi = \alpha|0\rangle$$

$$\langle\psi|M_0|\psi\rangle = \langle\psi|\alpha|0\rangle$$

$$= \alpha(\alpha^*\langle 0| + \beta^*\langle 1|)|0\rangle$$

$$= \alpha\alpha^* + 0$$

$$\langle\psi|M_0|\psi\rangle = |\alpha|^2$$

$$P_0 = \langle\psi|M_0|\psi\rangle$$

$$M_1|\psi\rangle = \beta|1\rangle$$

$$\langle\psi|M_1|\psi\rangle = \beta\langle\psi|1\rangle$$

$$= \beta\beta^*$$

$$\langle\psi|M_1|\psi\rangle = |\beta|^2$$

$$P_1 = |\beta|^2$$

Generally

$$P_m = \langle\psi|M_m|\psi\rangle$$

$$M^{\dagger}M = M_m$$

$$M_m^2 = M_m$$

$$M_m^2 = M_m M_m = M_m^{\dagger} M_m$$

$$M^2 = M$$

$$P_m = \langle\psi|M_m|\psi\rangle$$

$$= \langle\psi|M_m^2|\psi\rangle$$

$$P_m = \langle\psi|M_m^{\dagger}M_m|\psi\rangle$$

$M_1 \psi_1 = \beta|1\rangle$

② probability $\quad \psi = \alpha|0\rangle + \beta|1\rangle \Rightarrow \langle\psi| = \alpha^*\langle 0| + \beta^*\langle 1|$

$M_0\psi = \alpha|0\rangle$

$\langle\psi|M_0|\psi\rangle = \langle\psi|\alpha|0\rangle$

$= \alpha(\alpha^*\langle 0| + \beta^*\langle 1|)|0\rangle$

$= \alpha\alpha^* + 0$

$\langle\psi|M_0|\psi\rangle = |\alpha|^2$

$P_0 = \langle\psi|M_0|\psi\rangle$

$M_1|\psi\rangle = \beta|1\rangle$

$\langle\psi|M_1|\psi\rangle = \beta\langle\psi|1\rangle$

$= \beta\beta^*$

$\langle\psi|M_1|\psi\rangle = |\beta|^2$

$P_1 = |\beta|^2$

Generaly

$P_m = \langle\psi|M_m|\psi\rangle$

$M^\dagger M = M_m$

$M_m^2 = M_m$

$M_m^2 = M_m M_m = M_m^\dagger M_m$

$M^2 = M$

$P_m = \langle\psi|M_m|\psi\rangle$

$= \langle\psi|M_m^2|\psi\rangle$

$P_m = \langle\psi|M_m^\dagger M_m|\psi\rangle$

$P_m = \langle\psi| M_m^{\dagger} M_m |\psi\rangle$

Note completeness Rule

$$\sum_{\psi} |\psi\rangle\langle\psi| = I$$

$$\sum_{m} M_m = I$$

$$\boxed{\sum_{m} M_m^{\dagger} M_m = I}$$ — completeness Rule

(8) Post measurement state :-

$$\psi = \alpha|0\rangle + \beta|1\rangle$$

(1) $M_0 = |0\rangle\langle 0|$

$$M_0|\psi\rangle = \alpha|0\rangle$$

Post Measurement state generally is given by $\dfrac{M_0|\psi\rangle}{\sqrt{P_0}} = \dfrac{M_0|\psi\rangle}{\sqrt{\langle\psi|M_0|\psi\rangle}}$

$M_0 \rightarrow M_0^{\dagger} M_0$

$$\dfrac{M_0|\psi\rangle}{\sqrt{\langle\psi| M_0^{\dagger} M_0 |\psi\rangle}}$$

(2) $M_1 = |1\rangle\langle 1|$

$M_1|\psi\rangle = \beta|1\rangle$

Post measurement

$$\frac{M_1|\psi\rangle}{\sqrt{\langle\psi|M_1|\psi\rangle}} = \frac{M_1|\psi\rangle}{\sqrt{\langle\psi|M_1^{+}M_1|\psi\rangle}}$$

Generally

Post measurement of $|m\rangle$

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^{+}M_m|\psi\rangle}}$$

$\psi = \alpha|0\rangle + \beta|1\rangle$

(z - Basis) $|0\rangle$

Post measurement

$$\frac{M_0|\psi\rangle}{\sqrt{\langle\psi|M_0|\psi\rangle}} = \frac{\alpha|0\rangle}{\sqrt{|\alpha|^2}} = \frac{\alpha}{|\alpha|}|0\rangle = e^{i\theta}|0\rangle$$

$\hookrightarrow$ Global phase

$\downarrow$

$M_0^{+}M_0$

(2) post measurement of $|1\rangle$

$$\frac{M_1|\psi\rangle}{\sqrt{\langle\psi|M_1|\psi\rangle}} = \frac{\beta}{\sqrt{|\beta|^2}}|1\rangle = \frac{\beta}{|\beta|}|1\rangle = e^{i\theta}|1\rangle$$

$\to$ GP

① $M_m = |m\rangle\langle m|$

② $P_m = \langle\psi| M_m |\psi\rangle$

③ $\dfrac{M_m |\psi\rangle}{\sqrt{P_m}}$

x-Basis Diagonal Basis

$|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $|-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\psi = \alpha|0\rangle + \beta|1\rangle$

$\psi = \left(\frac{\alpha+\beta}{\sqrt{2}}\right)|+\rangle + \left(\frac{\alpha-\beta}{\sqrt{2}}\right)|-\rangle$

Measurement of $|+\rangle$

① $M_+ = |+\rangle\langle+|$

② $P_+ = \langle\psi| M_+ |\psi\rangle$

$$\begin{bmatrix} \frac{\alpha^*+\beta^*}{\sqrt{2}} & \frac{\alpha^*-\beta^*}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\alpha+\beta}{2} \\ \frac{\alpha-\beta}{2} \end{bmatrix}$$

	$\|+\rangle$	$\|-\rangle$
$\langle +\|$	1	0
$\langle -\|$	0	0

$$|+\rangle\langle +| \qquad |-\rangle\langle -|$$

③ Post measurement

$$\frac{M_+ |\psi\rangle}{\sqrt{P_+}} = \frac{\frac{\alpha+\beta}{\sqrt{2}} |+\rangle}{\sqrt{\left(\frac{\alpha+\beta}{\sqrt{2}}\right)^2}}$$

$$* \; M_+ |\psi\rangle = |+\rangle\langle +| \, |\psi\rangle$$

$$= \psi = \begin{bmatrix} \\ \end{bmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} \frac{\alpha+\beta}{\sqrt{2}} \\ \frac{\alpha-\beta}{\sqrt{2}} \end{bmatrix}$$

$$2\hat{i} + 3\hat{j}$$

$$\downarrow$$

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$M_+ |\psi\rangle = \begin{bmatrix} \frac{\alpha+\beta}{\sqrt{2}} \\ 0 \end{bmatrix}^{+}$$

$$= \frac{\alpha+\beta}{\sqrt{2}} |+\rangle$$

$$|\circlearrowleft\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} \quad = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} |\circlearrowright\rangle$$

$$\begin{array}{c|cc} & |+\rangle & |-\rangle \\ \hline \langle +| & 1 & 0 \\ \langle -| & 0 & 0 \end{array}$$

$|+\rangle\langle +| \quad |-\rangle\langle -|$

③ Post measurement

$$\frac{M_+ |\psi\rangle}{\sqrt{P_+}} = \frac{\frac{\alpha+\beta}{\sqrt{2}} |+\rangle}{\sqrt{\left(\frac{\alpha+\beta}{\sqrt{2}}\right)^2}}$$

$* \; M_+ |\psi\rangle = |+\rangle\langle +| \, [|\psi\rangle$

$= \psi = \begin{bmatrix} \\ \end{bmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} \frac{\alpha+\beta}{\sqrt{2}} \\ \frac{\alpha-\beta}{\sqrt{2}} \end{bmatrix}$$

$2\hat{i} + 3\hat{j}$

$\downarrow$

$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$M_+ |\psi\rangle = \begin{bmatrix} \frac{\alpha+\beta}{\sqrt{2}} \\ 0 \end{bmatrix}^T$$

$$= \frac{\alpha+\beta}{\sqrt{2}} |+\rangle$$

$$|\circlearrowleft\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \; |\circlearrowright\rangle$$

$y = |\circlearrowleft\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$, $|\circlearrowright\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$

$$\begin{array}{c|cc} & |\circlearrowleft\rangle & |\circlearrowright\rangle \\ \hline \langle\circlearrowleft| & 1 & 0 \\ \langle\circlearrowright| & 0 & 1 \end{array}$$

① $|\circlearrowleft\rangle \langle\circlearrowleft|$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix} \frac{1}{\sqrt{2}}[1 \cdot -1] \longrightarrow z\text{-basis}$$

$$= \frac{1}{2}\Bigg[\qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix}[1 \ -1]$$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Postulates 4

State vector of a composite system

$$0 = \psi = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}}[|0\rangle + |1\rangle] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$|A\rangle$ —

$|B\rangle$ — $\Big]$ $\psi_{AB} = |\psi_A \otimes \psi_B$

$\psi_A \otimes \psi_B$

$|0\rangle$ — $\psi_{comp.}$ → Tensor product of individual system

$|1\rangle$ —

$\psi = |0\rangle \otimes |0\rangle = |00\rangle$

composite systems

$\psi = \frac{1}{\sqrt{2}}\left[|00\rangle + |10\rangle\right.$

$= \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle \otimes |0\rangle\right]$

$\psi = \psi_1 \otimes \psi_2$

can split into two individual system

entangled state

$\psi = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right]$

no common factor

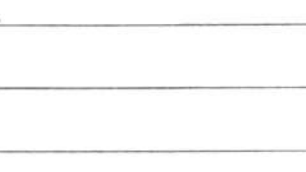
$\psi \neq \psi_A \otimes \psi_B$

$\beta_{00}, \beta_{01}, \beta_{10}, \beta_{11}.$

$\beta_{00} = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right.$

$\beta_{01} = \frac{1}{\sqrt{2}}\left[|01\rangle + |10\rangle\right)$

$(-1)^i$ sign

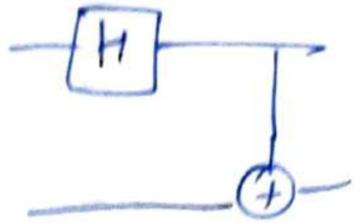


$\beta_{10} = \frac{1}{\sqrt{2}}\left[|00\rangle - |11\rangle\right.$

$\beta_{00} = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right]$

$\beta_{10} = \frac{1}{\sqrt{2}}\left[|00\rangle - |11\rangle\right]$

$\beta_{10} = \frac{1}{\sqrt{2}}\left[|00\rangle - |11\rangle\right]$



$$\boxed{\begin{aligned}\beta_{00} &= \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right] \\ \beta_{10} &= \frac{1}{\sqrt{2}}\left[|00\rangle - |11\rangle\right]\end{aligned}}$$

β_{01}

β_{11}

Density matrix (operator)

why?

Q. system

↓ Purestate

↓

superposition of normalized vector

↓

eg $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$ $|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$

$|+\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right] = \frac{1}{\sqrt{2}}\left\{\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right\}$

$|-\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle - |1\rangle\right] = \frac{1}{\sqrt{2}}\left\{\begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}\right\}$

$\psi = \frac{1}{\sqrt{2}}\left[|0\rangle + i|1\rangle\right]$

$M_0 = |0\rangle\langle 0|$ | $M_1 = |1\rangle\langle 1|$

Mixedstate

↓

	$\|0\rangle$	$\|1\rangle$
$\langle 0\|$	1	0
$\langle 1\|$	0	0

2 Bars

$|0\rangle\langle 0|$

Mixed state: Ensemble of two or more systems

eg: 50% $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ & 50% $|0\rangle$

75% $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ 25% $\frac{1}{\sqrt{2}}[|0\rangle - |1\rangle]$

$$\left[(|\psi_1\rangle, p_1), (|\psi_2\rangle, p_2), (|\psi_3\rangle, p_3) --- \right]$$

$\Downarrow$

can'nt Represent as a state vector

$\hat{O}$ ψ

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|01\rangle = |0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{array}{ll} \rightarrow |00\rangle & 0 \\ \rightarrow |01\rangle & 1 \\ \rightarrow |10\rangle & 2 \\ \rightarrow |11\rangle & 3 \end{array}$$

Density operator / Matrix is used for representation of a mixed state

Density Matrix/operator $\Rightarrow (\rho)$

$$\underbrace{M_0 = |0\rangle\langle 0| \qquad M_1 = |1\rangle\langle 1|}_{\text{projection operator}}$$

Q. system $\Rightarrow \psi \rightarrow$ pure state

$\Downarrow$

$\rho = |\psi\rangle\langle\psi|$ (pure state)

① Pure state ψ.

$\rho = |\psi\rangle\langle\psi|$

eg $\psi = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\rho = |\psi\rangle\langle\psi|$

$= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1, & 2 \end{bmatrix}$

$= \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Question find $\langle \sigma_x \rangle$? $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\rho =$

$\langle \sigma_x \rangle = \langle \psi | \sigma_x | \psi \rangle$

$\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle$

$\langle \sigma_x \rangle =$

operator $\hat{A}$

Debroglie wave

e^- Bohr

$\langle r \rangle = a_0$

Average for our system

$\frac{3}{2} a_0$

operator $\hat{A}$

$$\langle \hat{A} \rangle = \int \psi^* \hat{A} \psi \, dv$$

$$\langle \hat{A} \rangle = \langle \langle \psi | \hat{A} | \psi \rangle$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{5}(4) = \frac{4}{5}$$

why Density matrix

$$\rho = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\rho \sigma_x = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

$$Tr(\rho \sigma_x) = \frac{4}{5}$$

R for Row R for right
interchange rows

eg pure state $\psi = \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle] = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$

$\rho = |\psi\rangle\langle\psi|$

$$\rho = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

$$\psi = \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle]$$

$$\rho = |\psi\rangle\langle\psi|$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2[(|0\rangle + |1\rangle)(\langle 0| + \langle 1|)]$$

$$\rho = \frac{1}{2}[|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|]$$

$$\langle\sigma_z\rangle = \cup\langle\psi|\sigma_z|\psi\rangle$$

$$\hookrightarrow \rho\sigma_z = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}_{2\times 2}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}_{2\times 2}$$

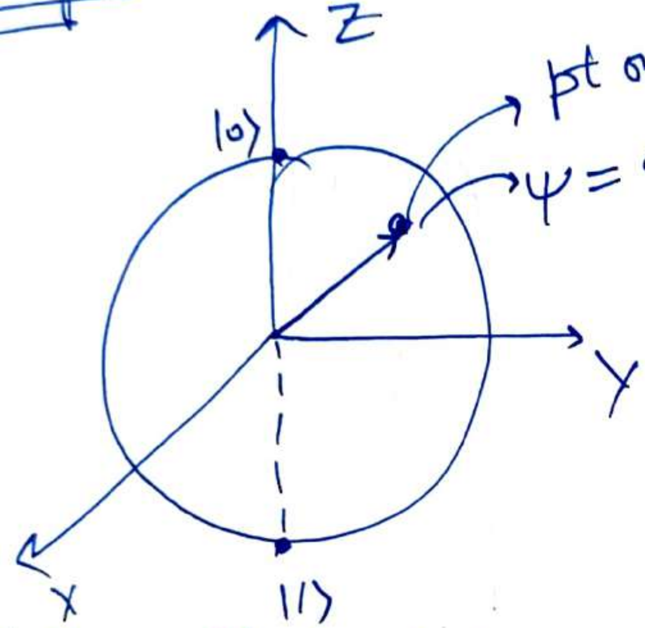
$$\rho\sigma_z = \frac{1}{2}\begin{bmatrix}1 & -1\\1 & -1\end{bmatrix}$$ $\underline{Tr = 1 - 1 = 0}$

$$\langle\sigma_z\rangle = Tr(\rho\sigma_z) = 0$$

$$\boxed{\langle \hat{A} \rangle = \mathrm{Tr}(\rho \hat{A})}$$

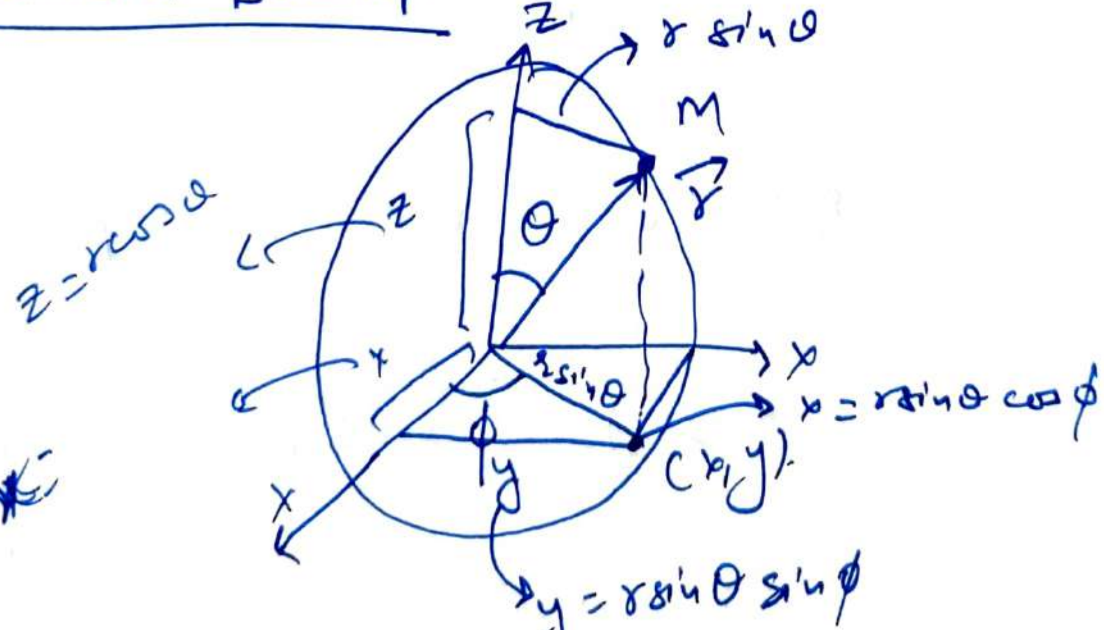
Find density mat. and predict the property of particle system

Bloch sphere



$$\psi = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle \quad (1)$$

(1) Points on Blochsphere



Position vector $\rightarrow \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\hat{r} = \frac{\vec{r}}{||r||}$

unit vector

$$\boxed{\vec{r} = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}}$$

$$\hat{n} = \hat{r} = \frac{\vec{r}}{|\vec{r}|} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\hat{n} = \hat{r} = \frac{\vec{r}}{|r|} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

come back to eq. (1)

$$\psi = \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\theta/2 \end{bmatrix}$$

Density matrix $\rho = |\psi\rangle\langle\psi|$

$$\begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\theta/2 \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} & e^{-i\phi}\sin\theta/2 \end{bmatrix}$$

$$\rho = \begin{bmatrix} \cos^2\frac{\theta}{2} & e^{-i\phi}\sin\theta/2\cos\theta \\ e^{i\phi}\sin\theta/2\cos\theta/2 & \sin^2\theta/2 \end{bmatrix}$$

$\frac{1}{2}(1+\cos\theta)$

$\frac{1}{2}(1-\cos\theta)$

$$\rho = \frac{1}{2}\begin{bmatrix} (1+\cos\theta) & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & 1-\cos\theta \end{bmatrix}$$

$1+\cos 2\theta$
$\hookrightarrow 2\cos^2\theta$

$1-\cos 2\theta$
$\hookrightarrow 2\sin^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

$= \frac{1}{2}\sin\theta$

$\sin\frac{2\theta}{2} = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$$\rho = \frac{1}{2}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \cos\theta & (\cos\phi - i\sin\phi)\sin\theta \\ (\cos\phi + i\sin\phi)\sin\theta & -\cos\theta \end{bmatrix} \right\}$$

$$\Downarrow$$

$$\cos\theta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & \cos\phi\sin\theta \\ & \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & \cos\phi\sin\theta - i\sin\phi\sin\theta \\ \cos\phi\sin\theta + i\sin\phi\sin\theta & 0 \end{bmatrix}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sigma_z \qquad \begin{bmatrix} 0 & \cos\phi\sin\theta \\ \cos\phi\sin\theta & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i\sin\phi\sin\theta \\ i\sin\phi\sin\theta & 0 \end{bmatrix}$$

$$\rho = \frac{1}{2}\left[I + \cos\theta\,\sigma_z + \sin\theta\cos\phi\,\sigma_x + \sin\theta\sin\phi\,\sigma_y \right]$$

$$\rho = \frac{1}{2}\left[I + \cos\theta\,\sigma_z + \sin\theta\cos\phi\,\sigma_x + \sin\theta\sin\phi\,\sigma_y \right]$$

$$\hat{n} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\boxed{(\hat{n}\cdot\vec{\sigma}) = \sin\theta\cos\phi\,\sigma_x + \sin\theta\sin\phi\,\sigma_y + \cos\theta\,\sigma_z}$$

$$\hookrightarrow \sigma_x\hat{i} + \sigma_y\hat{j} + \sigma_z\hat{k}$$

$\rho = \frac{1}{2}\left[I + \hat{n}\cdot\vec{\sigma}\right]$

↳ of point on Bloch sphere

ρ (point inside the Bloch sphere)

↳ mixed state

$$\boxed{\rho = \frac{1}{2}\left[I + \vec{a}\cdot\vec{\sigma}\right]}$$

① $\psi = \frac{1}{\sqrt{2}}\left[|0\rangle + i|1\rangle\right]$ find ρ ?

② $\psi = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right]$ find ρ?

↳ $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$

<u>Density matrix</u>   mixed $\Rightarrow$ Bell state

$\beta_{xy} = \frac{1}{\sqrt{2}}\left[|0y\rangle + (-1)^x|1\bar{y}\rangle\right]$

$\beta_{00} = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right]$

x y

pure state

$$\beta_{01} = \frac{1}{\sqrt{2}}\left[|01\rangle + |10\rangle\right]$$

$|0\,y\rangle$ $\quad |1\,\bar{y}\rangle$

$\downarrow$

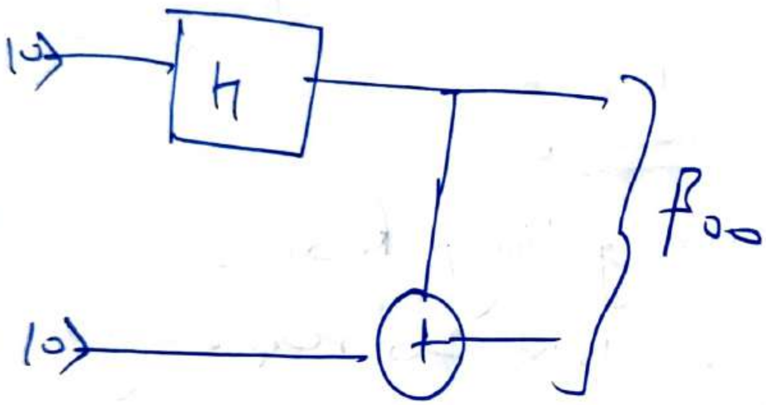
fixed state

$$\beta_{10} = \frac{1}{\sqrt{2}}\left[|00\rangle - |11\rangle\right]$$

$\downarrow$

$x=1$

(-1)



$$\beta_{00} = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right]$$

$|ab\rangle + |ac\rangle$

$a(|b\rangle + |c\rangle)$

$|ab\rangle + |ba\rangle$ not possible

$$\rho = |\psi\rangle\langle\psi| = |\beta_{00}\rangle\langle\beta_{00}|$$

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \Rightarrow \rho = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1&0&0&1\end{bmatrix}$$

$$= \frac{1}{2}\begin{array}{c|cccc} & |00\rangle & |01\rangle & |10\rangle & |11\rangle \\ \hline \langle 00| & 1 & 0 & 0 & 1 \\ \langle 01| & 0 & 0 & 0 & 0 \\ \langle 10| & 0 & 0 & 0 & 0 \\ \langle 11| & 1 & 0 & 0 & 1 \end{array}$$

$$\rho = \frac{1}{2}\Big[|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\Big]$$

$\rho = \cancel{\frac{1}{2}}[$

Partial trace!

we are getting Reduced density matrix corresponding to first system

$\rho_1 =$ Traceover 2^{nd} Qubit

$= Tr_2(\rho)$

$$Tr_2 \rho = \frac{1}{2}\Big[|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|\Big]$$

$$Tr_2 \rho = \frac{1}{2}\Big[|0\rangle\langle 0| \otimes Tr(|0\rangle\langle 0|) + |0\rangle\langle 1| \otimes \underbrace{Tr(|0\rangle\langle 1|)}_{0} + |1\rangle\langle 0| \otimes \underbrace{Tr(|1\rangle\langle 0|)}_{0} + |1\rangle\langle 1| \otimes \underbrace{Tr(|1\rangle\langle 1|)}_{1} \Big]$$

$$\rho_1 = \frac{1}{2}\Big[|0\rangle\langle 0| + |1\rangle\langle 1| \Big]$$

$$= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

Bloch sphere

$|\alpha|^2 + |\beta|^2 = 1$

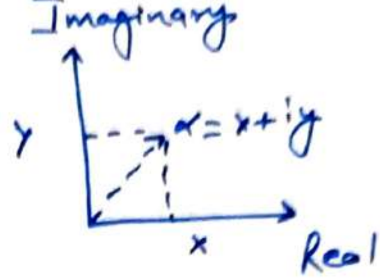
$\alpha, \beta \in \mathbb{C}$

$\alpha = I + Re$

$\beta = I + Re$

→ Blochsphere gives a geometric explanation to a single qubit operations

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$



$\alpha = x + iy$ (CCS)

$\alpha = r_0 e^{i\phi_0}$

$\beta = r_1 e^{i\phi_1}$

$$\psi = r_0 e^{i\phi_0}|0\rangle + r_1 e^{i\phi_1}|1\rangle$$

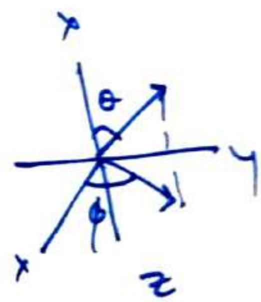
$$= e^{i\phi_0}\left(r_0|0\rangle + e^{i(\phi_1-\phi_0)}|1\rangle\right)$$

where $e^{i\phi_0}$ is a global phase and it has no significance

$e^{i(\phi_1-\phi_0)}$ relative phase. Let $\phi_1 - \phi_0 = \phi$

$$|\psi\rangle = r_0|0\rangle + r_1 e^{i\phi}|1\rangle$$

$$|r_0|^2 + |r_1|^2 = 1$$



$$\boxed{|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle}$$

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

together with the identity matrix they form the basis of real Hilbert space

$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$r_0 = \cos\frac{\theta}{2}$

$r_1 = \sin\frac{\theta}{2}$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

Density matrix

$$\rho = |\psi\rangle\langle\psi| \begin{pmatrix} \cos^2\theta & \frac{1}{2}e^{-i\phi}\sin\theta \\ \frac{1}{2}e^{i\phi}\sin\theta & \sin^2\theta \end{pmatrix}$$

$Tr(\rho) = 1.$ $\cos^2\theta + \sin^2\theta = 1$

$$\rho = \frac{1}{2}[I + \sigma s] \quad ; \; s = \text{Bloch vector}$$

$s = 1$ for pure state

we have one to one mapping to the surface of bloch sphere.

for mixed state

for : +ve z $\quad 0 < \theta < z$

$\theta = \phi = 0 \quad 0 < \phi < 2\pi$

$$|\psi\rangle = \cos 0|0\rangle + e^{i(0)}\sin 0|1\rangle$$

$$= |0\rangle + 0$$

$-z \quad \theta = \pi, \phi = 0$

$$\cos\frac{\pi}{2}|0\rangle + e^{i(0)}\sin\frac{\pi}{2}|1\rangle$$

$$|\psi = |1\rangle$$

for circular basis

$$0 < \theta < \pi$$

$$\theta = \frac{\pi}{2}, \ \phi = \frac{\pi}{2}, \quad j \in \mathbb{C} \qquad 0 < \phi < 2\pi$$

$$= \cos\frac{\pi}{4}|0\rangle + e^{j\frac{\pi}{2}} \sin\frac{\pi}{4}|1\rangle$$

$$\cos\frac{\pi}{2}|0\rangle + e^{i(0)} \sin\frac{\pi}{2}|1\rangle$$

$$|\psi\rangle = |1\rangle$$

$$\frac{1}{2}|0\rangle + \frac{j}{\sqrt{2}}|1\rangle$$

Diagonal basis :-

$x \longrightarrow$ axis

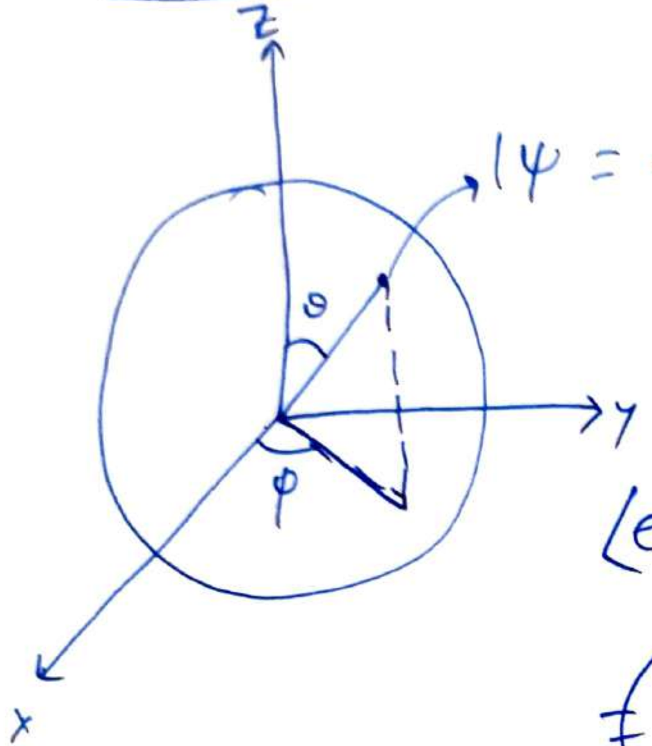
$$\phi = 0 \qquad |\psi\rangle = \cos\frac{\pi}{4}|0\rangle + e^{0} \sin\frac{\pi}{4}|1\rangle$$

$$\theta = \frac{\pi}{2} \qquad = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|\psi\rangle = |+\rangle$$

① Qubit Representation on bloch sphere



$|\psi\rangle = e^{i\gamma}\left[\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right]$

↳ Global phase

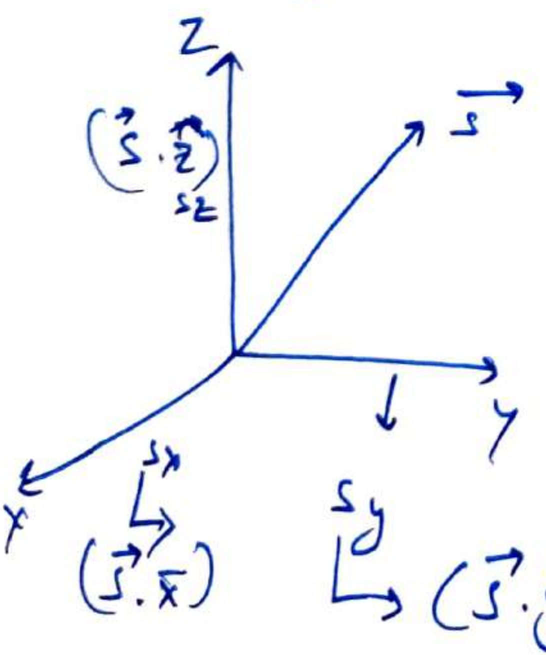
- unitary operation cannot effect of $e^{i\gamma}$

$\langle e^{-i\gamma}\psi|\, U\, |e^{i\gamma}\psi\rangle$

$= \underbrace{e^{i\gamma}e^{-i\gamma}}_{1}\langle\psi|U|\psi\rangle$

$$\boxed{\psi = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle = ??}$$ To be proved

Spin Angular momentum



① $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$

measurement operators corresponding spin measurement along $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$

$\frac{\hbar}{2}$ plancks constant

$S_x = \frac{\hbar}{2}\sigma_x$, $S_y = \frac{\hbar}{2}\sigma_y$, $S_z = \frac{\hbar}{2}\sigma_z$

$S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\downarrow$ $\downarrow$ $\downarrow$

eigenvalue $\frac{\hbar}{2}(\pm 1)$ $\frac{\hbar}{2}(\pm 1)$ $\frac{\hbar}{2}(\pm 1)$

$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \lambda = 1, -1$

$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \lambda^2 \cancel{+} tr\lambda + det = 0$

$|A - \lambda I| = 0$

$\lambda^2 - 0 - 1 = 0$

$\boxed{\lambda^2 = \pm 1}$

$det = ad - cb$

$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \lambda^2 \cancel{+} tr\lambda + det = 0$

$\lambda^2 - 0 - 1 = 0$

$\lambda^2 = 1$

$\boxed{\lambda = \pm 1}$

Eigen value of S_x, S_y, S_z $\Rightarrow$ $\boxed{\lambda = \pm \frac{\hbar}{2}}$

$(\vec{S}\cdot\vec{x})$ $(\vec{S},\vec{y})$ $(\vec{S},\vec{z})$

* eigen value are indendent of arbitrary direction

$\Rightarrow$ Eigen value of $(\vec{S}\cdot\vec{n}) = \pm \frac{\hbar}{2}$

arbitrary unit vector

Eigen vector of S_x $\rightarrow$ (Diagonal basis.)

$\hookrightarrow$ $\underbrace{\hat{A}}_{S_x}\psi = \lambda\psi$

$\boxed{\lambda = +\frac{\hbar}{2}}$ $\quad \cancel{\frac{\hbar}{2}}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \cancel{\frac{\hbar}{2}}\begin{bmatrix}x\\ y\end{bmatrix}$

$0 + y = x$

$y = x$

$\begin{bmatrix}1\\ 1\end{bmatrix}$ $\begin{bmatrix}2\\ 2\end{bmatrix}$ $\begin{bmatrix}-2\\ -2\end{bmatrix}$ -- ∞.

Least 1 is taken as idea

$\psi = \begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}x\\ x\end{bmatrix} \Rightarrow \psi = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$

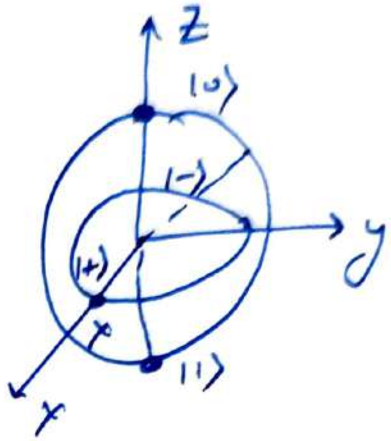
$+ = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$

$\lambda = -\frac{\hbar}{2} \qquad S_x \psi = \lambda \psi$

$$\cancel{\frac{\hbar}{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -\cancel{\frac{\hbar}{2}} \begin{bmatrix} x \\ y \end{bmatrix}$$

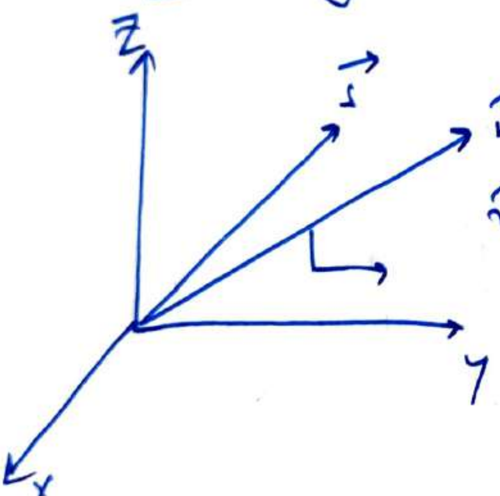
$$\boxed{y = -x}$$

$$\psi = \begin{bmatrix} x \\ -x \end{bmatrix} \Rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



$$-|-\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Arbitrary Direction!



$\hat{n} = \hat{r}$

$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

(Measure spin in arbitrary direction)

$$\vec{S} \cdot \hat{n}$$

$$\hat{A} = \vec{S} \cdot \hat{n} = S_x \cdot \sin\theta\cos\phi + S_y \sin\theta\sin\phi + S_z \cos\theta$$

$(S_x, S_y, S_z) \leftarrow$

$$= \frac{\hbar}{2} \begin{bmatrix} 0 & \sin\theta\cos\phi \\ \sin\theta\cos\phi & 0 \end{bmatrix} +$$

$$\frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sin\theta\cos\phi + \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \sin\theta\sin\phi + \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cos\theta$$

$$\frac{\hbar}{2}\left\{\begin{bmatrix} 0 & \sin\theta\cos\phi \\ \sin\theta\cos\phi & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i\sin\theta\sin\phi \\ i\sin\theta\sin\phi & 0 \end{bmatrix} + \begin{bmatrix} \cos\theta & 0 \\ 0 & -\cos\theta \end{bmatrix}\right\}$$

$$= \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta(\cos\phi - i\sin\phi) \\ \sin\theta(\cos\phi + i\sin\phi) & -\cos\theta \end{bmatrix}$$

$$= \frac{\hbar}{2}\begin{bmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & \cos\theta \end{bmatrix}$$

$$\hat{S}\cdot\hat{n} = \frac{\hbar}{2}\begin{bmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & \cos\theta \end{bmatrix} \Rightarrow \lambda \pm \frac{\hbar}{2}$$

(1) $\boxed{\lambda = +\frac{\hbar}{2}}$ $\quad A\psi = \lambda\psi$

$$\frac{\cancel{\hbar}}{\cancel{2}}\begin{bmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{\cancel{\hbar}}{\cancel{2}}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$x\cos\theta + y\,e^{-i\phi}\sin\theta = x$$

$$y\,e^{-i\phi}\sin\theta = x(1-\cos\theta)$$

$$\downarrow \qquad\qquad \downarrow$$

$$\cancel{2}\cancel{\sin\frac{\theta}{2}}\cos\frac{\theta}{2} \qquad \cancel{2}\sin^{\cancel{2}}\frac{\theta}{2}$$

$$y\,e^{-i\phi}\cos\frac{\theta}{2} = x\sin\frac{\theta}{2}$$

(1) $1-\cos\theta = 2\sin^2\frac{\theta}{2}$

(2) $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$\cancel{y = e}$ $y e^{-i\phi}\cos\frac{\theta}{2} = x\sin\frac{\theta}{2}$

$$y = \left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} e^{i\phi}\right) x$$

$$\psi = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ \left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} e^{i\phi}\right) x \end{bmatrix}$$

$$\psi = \begin{bmatrix} 1 \\ \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} e^{i\phi} \end{bmatrix}$$

$$\psi = \begin{bmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 \times 2 \\ \frac{1}{2} \times \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

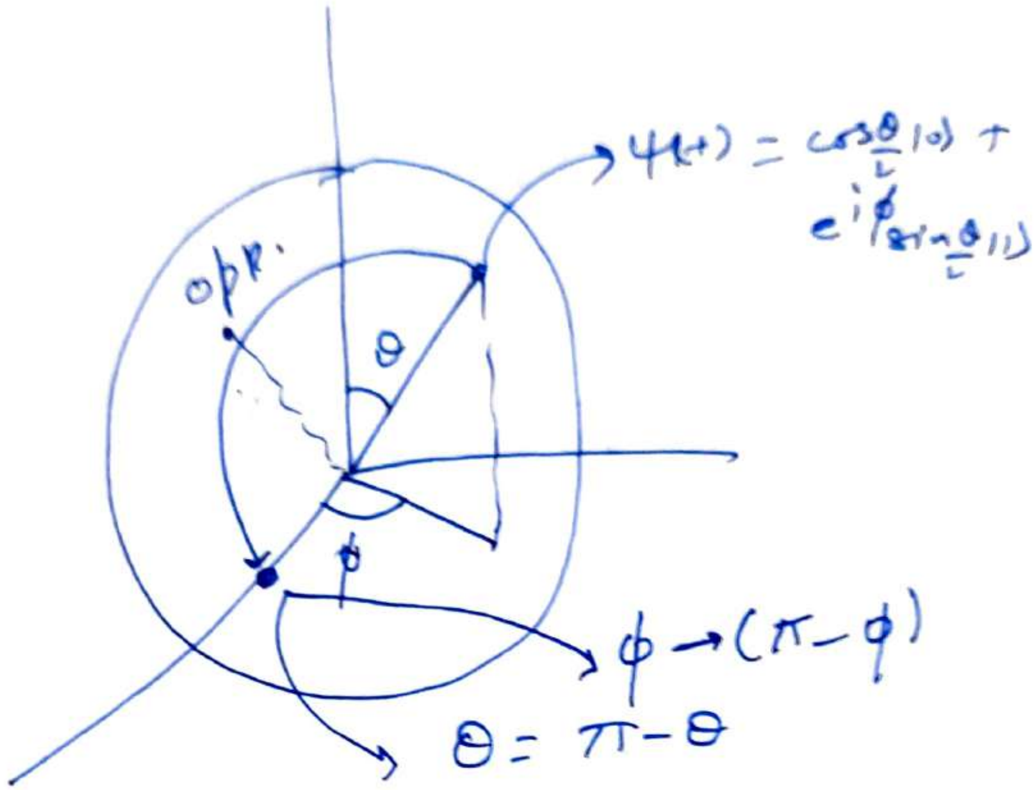
fixed

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow \underline{i\,i + j}$$

$$\begin{bmatrix} \frac{1}{3} \times 3 \\ 2 \times 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$\psi = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$

Qubit



$\cos\left(\frac{\pi-\theta}{2}\right)|0\rangle + e^{i(\pi+\phi)}\sin\left(\frac{\pi-\theta}{2}\right)|1\rangle$

$\cos(90+\theta/2)$ $\quad e^{i\pi}\cdot e^{i\phi}$ $\quad \sin(90-\theta/2)$

$\psi(-) = \sin\frac{\theta}{2}|0\rangle - e^{i\phi}\cos\frac{\theta}{2}|1\rangle$

Two opposite points on bloch sphere represent

Basis states.

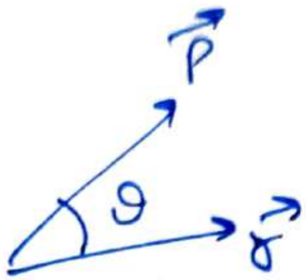
Anticlockwise +ve measurement

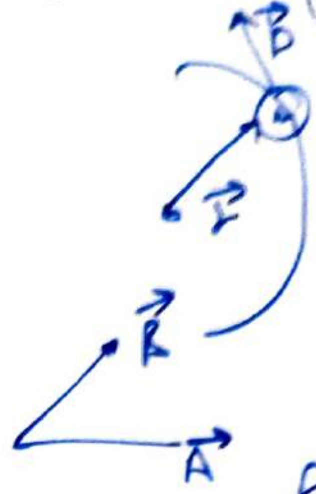
clockwise −ve measurement

7. Quantum gates from spin Rotation

Angular momentum :- movement of momentum

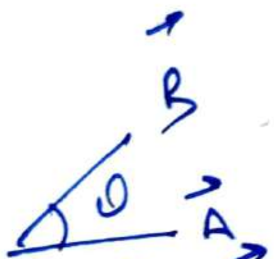
$$\vec{l} = \vec{r} \times \vec{p}$$





$\vec{A} \times \vec{B}$ Right hand rule

thumb shows the direction.



$\vec{B} \times \vec{A}$

orbital angular momentum

$$\vec{l} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$$\hat{i}\underbrace{(yp_z - zp_y)}_{L_x} + \hat{j}\underbrace{(zp_x - xp_z)}_{L_y} + \hat{k}\underbrace{(xp_y - yp_x)}_{L_z}$$

$$\vec{l} = l_x \hat{i} + l_y \hat{j} + l_y \hat{z}$$

$$\boxed{\begin{aligned} \hat{l}_x &= \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \\ \hat{l}_y &= \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \\ \hat{l}_z &= \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \end{aligned}}$$

First quantization rule

$$l_x = y p_z - z p_y$$

$$\vec{l} = \vec{r} \times \vec{p}$$

$x \to y$

$z \to x \to y \to z$

$$\hat{p}_x = -i\hbar \frac{\partial}{\partial x}$$

$$\hat{p}_y = -i\hbar \frac{\partial}{\partial y}$$

$$\hat{p}_z = -i\hbar \frac{\partial}{\partial z}$$

$$\boxed{\hat{p}_x = -i\hbar \frac{\partial}{\partial x}}$$

$n \longrightarrow$ +ve x-axis

$$\psi = A e^{i(kx - \omega t)}$$

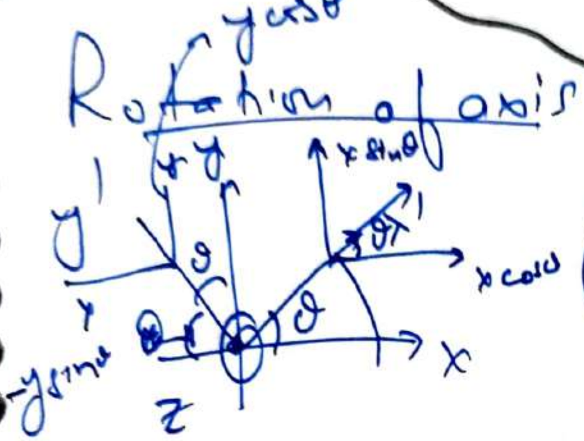
$\psi = A e^{i(kx-\omega t)}$ — in x-axis

$\hat{p}\psi = i\hbar \frac{\partial}{\partial x} A e^{i(kx-\omega t)} = A e^{i(kx-\omega t)} \times ik \times i\hbar$

$$\boxed{\begin{aligned}\frac{\partial}{\partial x}(e^x) &= e^x \\ \frac{\partial}{\partial x} e^{ax} &= a e^{ax}\end{aligned}}$$

$= -\hbar k \psi$

Rotation of axis



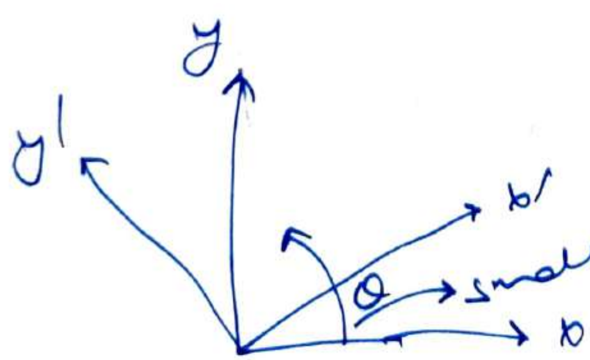
$$\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

↑ rotation matrix about z axis

$x' = x\cos\theta + y\sin\theta$

$y' = -x\sin\theta + y\cos\theta$

$$\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$



$x' = x\cos\theta + y\sin\theta$

$y' = -x\sin\theta + y\cos\theta$

when θ is small we approximate

$\cos\theta \simeq 1$

$\sin\theta \simeq \theta$

$x' = x\cos\theta + y\sin\theta \simeq x + y\theta$

$y' = -x\sin\theta + y\cos\theta = -x\theta + y = y - x\theta$

$x' = x + y\theta$

$y' = y - x\theta$

(2) Taylor series

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a)$$

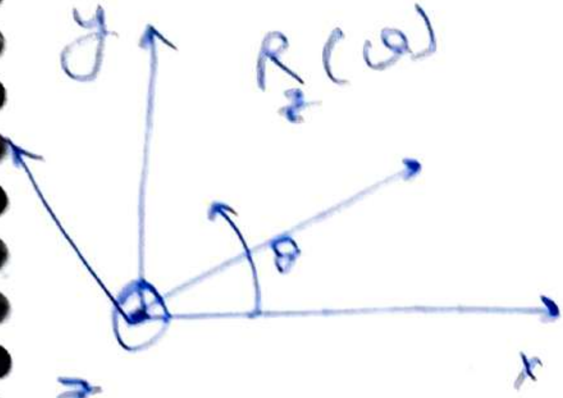
at $x = a$

$$f(x+a) = f(x) + a\frac{\partial f}{\partial x} + \cancel{\frac{a^2}{2!}\frac{\partial^2 f}{\partial x^2}} + \cancel{\frac{a^3}{3!}\frac{\partial^3 f}{\partial x^3}} + \cdots$$

a = very small

$$f(x+a) \simeq f(x) + a\frac{\partial f}{\partial x}$$

$$f(x+a, y+b) = f(x,y) + a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}$$



About Z-axis

$$\left.\begin{aligned} x' &= x + y\theta \\ y' &= y - x\theta \end{aligned}\right\} \Rightarrow f(x',y') = R_z(\theta) f(x,y)$$

2

$$f(\underset{(x+a)}{x'}, \underset{(y+b)}{y'}) = R_z(\theta) f(x,y)$$

$$f'(x',y') = \underbrace{f(x+y\theta,\ y-x\theta)}_{\text{use taylor series}}$$

$$= f(x,y) + y\theta\frac{\partial f}{\partial x}$$

$$= f(x,y) + a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}$$

$$= f(x,y) + y\theta\frac{\partial}{\partial x}f - x\theta\frac{\partial f}{\partial y}$$

$$f(x,y) = \left[1 + y\theta\frac{\partial}{\partial x} - x\theta\frac{\partial}{\partial y}\right] f(x,y)$$

$$f' = \left[1 + \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\theta\right] f$$

$-y - i\hbar\frac{\partial}{\partial x}$

$$= \left[1 + \frac{1}{i\hbar}\left(y\left(i\hbar\frac{\partial}{\partial x}\right) + x\left(i\hbar\frac{\partial}{\partial y}\right)\theta\right)\right] f$$

Assumption multiply by (-)

$$f' = \left[1 + \frac{1}{i\hbar}\left[x p_y - y p_x\right]\theta\right] f$$

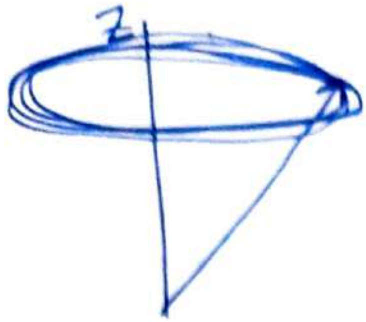
$$f' = \left[1 + \frac{1}{i\hbar}\left[\underbrace{\hat{x}\hat{p}_y - \hat{y}\hat{p}_x}_{\hat{l}_z}\right)\theta\right] f$$

$$f' = \left[1 + \frac{\hat{l}_z}{i\hbar}\theta\right] f$$

$$f' = \left[1 - \frac{i\hat{l}_z}{\hbar}\theta\right] f$$

$$f' = \left[1 - \frac{i l_z}{\hbar}\theta\right] f$$

↳ Rotation for one time



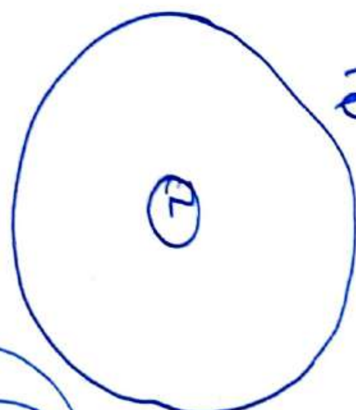
$$f' = \left(1 - \frac{i\hat{l}_z\theta}{\hbar}\right)^n f$$

$$f' = \left[\lim_{n\to\infty}\left(1 - \frac{i\hat{l}_z\theta}{\hbar}\right)^n\right] f$$

$$\boxed{\lim_{n\to\infty}(1+x)^n = e^x \;\Big|\; \lim_{n\to\infty}(1-x)^n = e^{-x}}$$

orbital motion $f' = \left(e^{-\frac{i\hat{l}_z\theta}{\hbar}}\right) f$

$\hookrightarrow$ Rotation about $R_z(\theta)$



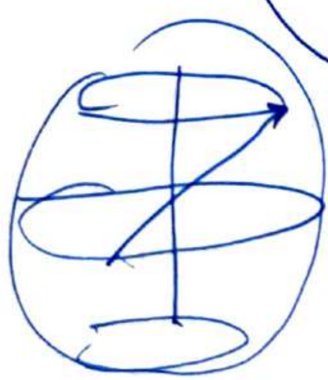
$\bar{e} \Rightarrow$ orbital spin

$\Downarrow$

Intrinsic

$$\vec{l} = \vec{r} \times \vec{p}$$

$$\vec{s} = \times\times$$



orbital $\vec{L}$

spin $\vec{S}$

total $\vec{J}$

Spin Rotation :-

orbital

$$f' = e^{-i\frac{S_z \theta}{\hbar}} f$$

$$= \quad S_x = \frac{\hbar}{2}\sigma_x \quad S_z = \frac{\hbar}{2}\sigma_z$$

$$f' = e^{-i\frac{\cancel{\hbar}\sigma_z\theta}{2\cancel{\hbar}}} f$$

$$\boxed{f' = e^{-i\sigma_z\frac{\theta}{2}} f}$$

Spin Rotation operators

$$R_z(\theta) = e^{-i\sigma_z\frac{\theta}{2}} \quad (\vec{\sigma}\cdot\hat{z})$$

$$\boxed{R_{\hat{n}}(\theta) = e^{-i(\vec{\sigma}\cdot\hat{n})\frac{\theta}{2}}}$$

single qubit quantum gate

$$R_{\hat{n}}(\theta) = e^{-i(\vec{\sigma}\cdot\hat{n})\frac{\theta}{2}}$$

** $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$\cancel{e^{+i\sigma_x\theta}} =$

$$e^{i\sigma\theta} = I + i\sigma\theta + \frac{(i\sigma\theta)^2}{2!} + \frac{(i\sigma\theta)^3}{3!} + \cdots$$

$-i$

$$\boxed{\begin{aligned} \sigma^2 &= I \\ \sigma^3 &= \sigma^2 \cdot \sigma = \sigma \\ \sigma^n &= \begin{cases} I & (n = \text{even}) \\ \sigma & (n = \text{odd}) \end{cases} \end{aligned}}$$

$$= I + i\sigma\theta + \frac{i^2\theta^2}{2!} I + \frac{i^3\theta^3}{3!}\sigma + \cdots$$

$$= I\underbrace{\left[1 - \frac{\theta^2}{2!} + \cdots\right]}_{\cos\theta} + i\sigma\underbrace{\left[\theta - \frac{\theta^3}{3!} + \cdots\right]}_{\sin\theta}$$

$$1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \qquad \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} - \cdots$$

$$\boxed{e^{\pm i\sigma\theta} = I\cos\theta \pm i\sigma\sin\theta}$$

$$e^{-i\underbrace{(\vec{\sigma}\cdot\hat{n})}\frac{\theta}{2}} = I\cos\frac{\theta}{2} - i(\vec{\sigma}\cdot\hat{n})\sin\frac{\theta}{2}$$

$$\boxed{R_{\hat{n}}(\theta) = e^{-i(\vec{\sigma}\cdot\hat{n})\frac{\theta}{2}} = I\cos\frac{\theta}{2} - i(\vec{\sigma}\cdot\hat{n})\sin\frac{\theta}{2}}$$

$\boxed{X}$ $\hat{n} = (1,0,0)$ $\quad$ $\boxed{Y}$ $\hat{n} = (0,1,0)$

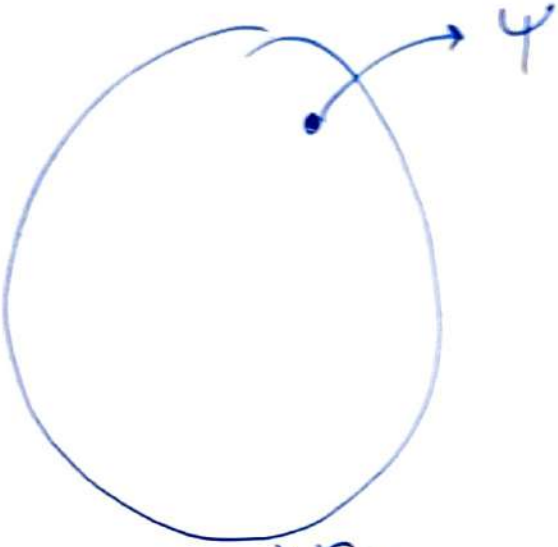
$\boxed{Z}$ $\hat{n} = (0,0,1)$

~~Icos~~ $\theta = \pi$ for $\boxed{x}$ − $\boxed{y}$; and $\boxed{z}$ gate

$-\boxed{x}-$

$\begin{bmatrix} a \\ b \end{bmatrix}$ rotate with an angle π on that axis;

$I \cos\frac{\pi}{2} - i\,\sigma_x \sin\frac{\theta}{2}$



$e^{i\gamma}\big(\qquad \big)$

$U(\theta) = e^{i\gamma}\left[I\cos\frac{\theta}{2} - i(\vec{\sigma}\cdot\hat{n})\sin\frac{\theta}{2}\right]$

(single qubit)

$\gamma = \pi/2$ $\quad e^{i\pi/2} \; (+i)$

$U(\theta) = (+i)\left[I\cos\frac{\theta}{2} - i(\vec{\sigma}\cdot\hat{n})\sin\frac{\theta}{2}\right]$

global phase

$-\boxed{x}-$ $\quad \theta = \pi \quad \cos\frac{\theta}{2} = 0, \; \sin\frac{\theta}{2} = 1$

$\hat{n} = (1, 0, 0)$

$2^0 \times 0$ $2^0 \times 1 = 1$

Quantum gates := Unitary operators

Basic states
1. Single Qubit
2. Two Qubit
3. Three Qubit

Single Qubit

Basis = $\left\{ \begin{matrix} |0\rangle & , & |1\rangle \\ \hookrightarrow 0 & & \hookrightarrow 1 \end{matrix} \right\}$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Basis $\left\{ \begin{matrix} |0\rangle & |1\rangle \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{matrix} \right\}$

Basis columns

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_2$

2. Two Qubit Basis states $\{$ ~~$|0\rangle \otimes |0\rangle$~~, $|0\rangle \otimes$

$\left\{ \begin{matrix} |0\rangle \otimes |0\rangle & , & |0\rangle \otimes |1\rangle & , & |1\rangle \otimes |0\rangle & , & |1\rangle \otimes |1\rangle \\ & & 2^1\,2^0 & & 2^1\,2^0 & & 2^1\,2^0 \\ |00\rangle & & |01\rangle & & |10\rangle & & |11\rangle \\ \hookrightarrow |0\rangle & & \hookrightarrow |1\rangle & & \hookrightarrow |2\rangle & & \hookrightarrow |3\rangle \end{matrix} \right\}$

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

A	B
0	0
0	1
1	0
1	1

$$|0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Basis states as columns

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_4 \quad D[1\,1\,1\,1] = I_4$$

(3) <u>3 Qubit Basis states</u>

$\{ ~~|000\rangle~~ ~~|001\rangle~~ |010\rangle\ |011\rangle\ |100\rangle$

$\{ |000\rangle\ |001\rangle\ |010\rangle\ |011\rangle\ |100\rangle\ |101\rangle\ |110\rangle\ |111\rangle$
$\quad \hookrightarrow 0 \quad \hookrightarrow 1 \quad \hookrightarrow 2 \quad \hookrightarrow 3 \quad \hookrightarrow 4 \quad \hookrightarrow 5 \quad \hookrightarrow 6 \quad \hookrightarrow 7$

$|0\rangle$ to $|2^3 - 1\rangle$

$\hookrightarrow [I_8]$

<u>Single Qubit Quantum gates</u>

$\hookrightarrow$ Unitary operators

* $UU^\dagger = I \Rightarrow U^{-1} = U^\dagger$ | $\det U = 1$ | columns satisfy orthonormal conditions

Any single Qubit Quantum gate is given by

$$U = e^{i\alpha} R_{\hat{n}}(\theta)$$

$\hookrightarrow e^{-i\frac{\theta}{2}(\vec{\sigma}\cdot\hat{n})}$ Rotational opr. abt arbitrary direction

$$= e^{i\alpha}\left[I \cos\frac{\theta}{2} - i(\vec{\sigma}\cdot\hat{n}) \sin\frac{\theta}{2} \right]$$

Let $\alpha = \frac{\pi}{2}$, $\theta = \pi$

$$U = \underbrace{e^{i\pi/2}}_{i}\left[\cancel{I \cos\frac{\pi}{2}}_{0} - i(\vec{\sigma}\cdot\hat{n}) \sin\frac{\pi}{2} \right] = (\vec{\sigma}\cdot\hat{n})$$

$U = \vec{\sigma}\cdot\hat{n} = \sigma_x n_x + \sigma_y n_y + \sigma_z n_z$

$\hat{n} = (1,0,0)$ $\quad U = \sigma_x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\hat{n} = (0,1,0)$ $\quad U = \sigma_y \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$\hat{n} = (0,0,1)$ $\quad U = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

—[X]— —[Y]— —[Z]—

Pauli gate (x, y, z)

$U = \vec{\sigma}\cdot\hat{n}$

$\hat{n} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = U = \frac{1}{\sqrt{2}}\sigma_x + \frac{1}{2}\sigma_z = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H$

(Hadamard Gate)

$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad H^2 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Pauli Gates:-

σ_x	$\|0\rangle$	$\|1\rangle$
$\langle 0\|$	0	1
$\langle 1\|$	1	0

$$\boxed{X = |0\rangle\langle 1| + |1\rangle\langle 0|}$$

$X|0\rangle = (|0\rangle\langle 1| + |1\rangle\langle 0|)|0\rangle$

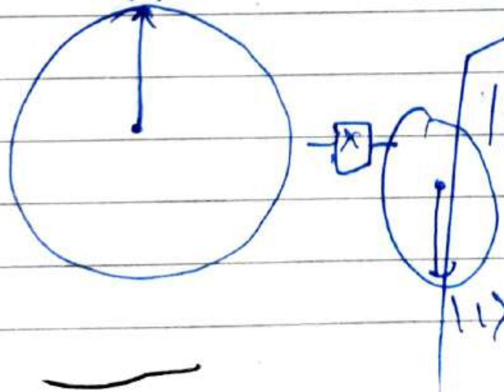
$X|0\rangle = |1\rangle$

$X|1\rangle = (|0\rangle\langle 1| + |1\rangle\langle 0|)|1\rangle$

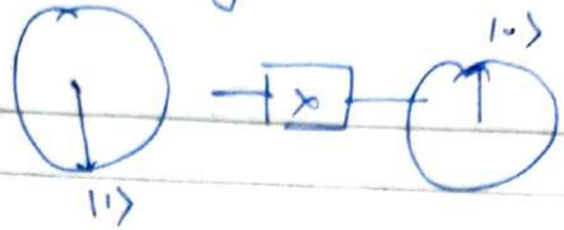
$X|1\rangle = |0\rangle$

$|0\rangle$ —[X]— $|1\rangle$

$|1\rangle$ —[X]— $|0\rangle$



NOT GATE Quantum   Fliping zero to one and one to zero



2) σ_y

σ_y	$\|0\rangle$	$\|1\rangle$
$\langle 0\|$	0	$-i$
$\langle 1\|$	i	0

$$Y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$$

$$\boxed{Y|0\rangle = i|0\rangle} \quad \boxed{Y|1\rangle = -i|0\rangle}$$

$|0\rangle$ — $\boxed{Y}$ — $i|1\rangle$ $|||^{ly}$ $|1\rangle$ — $\boxed{Y}$ — $-i|0\rangle$

3) σ_z

σ_z	$\|0\rangle$	$\|1\rangle$
$\langle 0\|$	1	0
$\langle 1\|$	0	-1

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$\boxed{Z|0\rangle = |0\rangle} \qquad Z|1\rangle = (|0\rangle\langle 0| - |1\rangle\langle 1|)|1\rangle$$

$$= -|1\rangle$$

$$\boxed{Z|1\rangle = -|1\rangle}$$

$|0\rangle$ — $\boxed{Z}$ — $|0\rangle$ $|1\rangle$ — $\boxed{Z}$ — $-|1\rangle$ → Phase change

4) Hadmord gate

H	$\|0\rangle$	$\|1\rangle$
$\langle 0\|$	$1/\sqrt{2}$	$1/\sqrt{2}$
$\langle 1\|$	$1/\sqrt{2}$	$-1/\sqrt{2}$

$$H = \frac{1}{\sqrt{2}}\left[|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right]$$

$$H|0\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle - |1\rangle\right]$$

$|0\rangle$ —— $\boxed{H}$ —— $\frac{1}{\sqrt{2}}[|0\rangle + |1\rangle]$

$|1\rangle$ —— $\boxed{H}$ —— $\frac{1}{\sqrt{2}}[|0\rangle - |1\rangle]$

Hadmard superposition state
gin

sq root of not gate

Note: $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

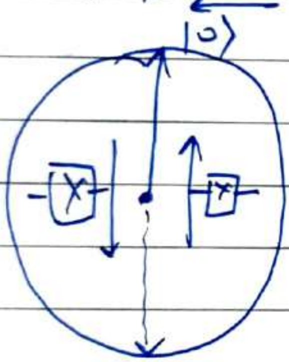
$H = \sqrt{NOT}$ | $\cancel{H^2 = NOT}$

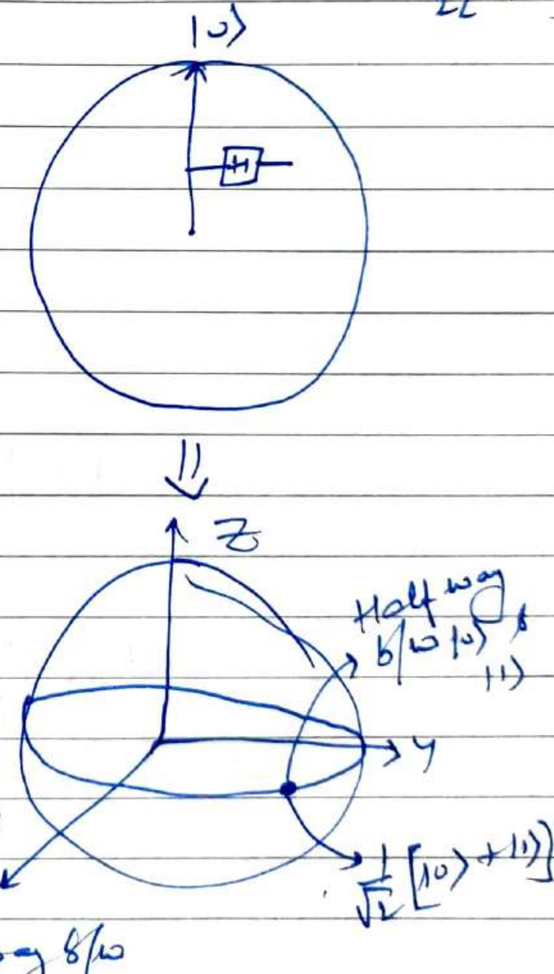
Q Hadamard Gate is called as $\sqrt{NOT}$? $H^2 = I$

$\boxed{NOT}$ $|0\rangle$ — $\boxed{X}$ — $|1\rangle$

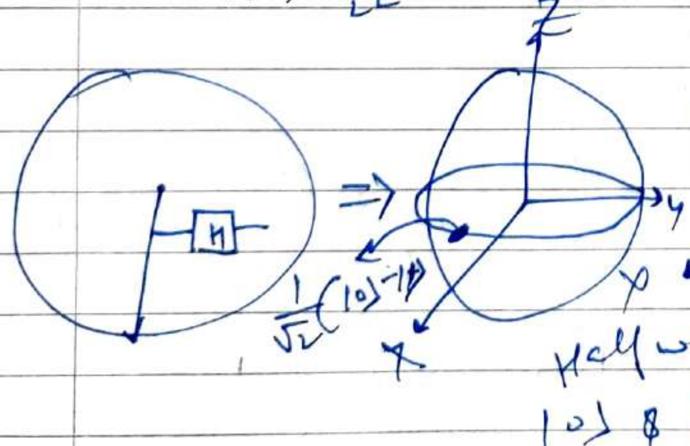
$|0\rangle \underset{\longleftarrow}{\xrightarrow{X}} |1\rangle$

$H|0\rangle = \frac{1}{2}[|0\rangle + |1\rangle]$





$H|1\rangle = \frac{1}{2}[|0\rangle - |1\rangle]$



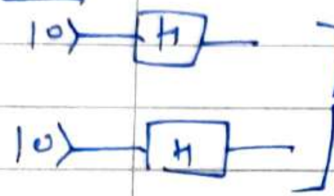
$\boxed{H = \sqrt{NOT}}$ due to operation mid way b/w

$$H = \frac{1}{\sqrt{2}}\left[X + Z\right] = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \quad H|x\rangle$$

$$H|0\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right] \;\&\; H|1\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle - |1\rangle\right]$$

$$H|x\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + (-1)^x|x\rangle\right]$$

Task



$\psi_{output} = ?$

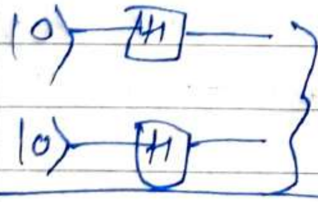
$$\psi_{Final} = H|0\rangle \otimes H|0\rangle$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{(\sqrt{2})^2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

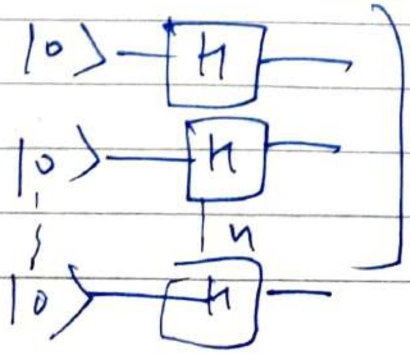
$$\cancel{\frac{1}{(\sqrt{2})^2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}$$

$$\psi_F^* = \frac{1}{(\sqrt{2})^2}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$$

$$\frac{1}{(\sqrt{2})^2}\left[\underset{\to |0\rangle}{|00\rangle} + \underset{\to 1\rangle}{|01\rangle} + \underset{\to 2}{|10\rangle} + \underset{\to 3}{|11\rangle}\right]$$



$$\frac{1}{(\sqrt{2})^2}\sum_{j=0}^{2^2-1}|j\rangle$$



$$\frac{1}{(\sqrt{2})^n}\sum_{j=0}^{2^n-1}|j\rangle$$

$\Rightarrow$ Random no. lie b/w 0 to $2^n - 1$

L-2 1st Gates $R_{\hat{n}}(\phi) = e^{-i\frac{\phi}{2}\hat{n}\cdot\vec{\sigma}}$

__/__/____

About X-Axis

$$R_x(\phi) = e^{-i\frac{\phi}{2}\sigma_x}$$

$$= I\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\sigma_x$$

$$= \begin{bmatrix} \cos\frac{\phi}{2} & 0 \\ 0 & \cos\frac{\phi}{2} \end{bmatrix} - \begin{bmatrix} 0 & i\sin\frac{\phi}{2} \\ i\sin\frac{\phi}{2} & 0 \end{bmatrix}$$

$$R_x(\phi) \begin{bmatrix} \cos\frac{\phi}{2} & -i\sin\frac{\phi}{2} \\ -i\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{bmatrix}$$

lll'y $R_y(\phi)$ and $R_z(\phi)$

About Y-axis

$$R_y(\phi) = e^{-i\frac{\phi}{2}\sigma_y}$$

$$I\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\sigma_y$$

$$\begin{bmatrix} \cos\frac{\phi}{2} & 0 \\ 0 & \cos\frac{\phi}{2} \end{bmatrix} - \begin{bmatrix} 0 & \sin\frac{\phi}{2} \\ \sin\frac{\phi}{2} & 0 \end{bmatrix}$$

$$R_y(\phi) = \begin{bmatrix} \cos\phi/_2 & -\sin\phi/_2 \\ +\sin\frac{\phi}{2} & \cos\phi/_2 \end{bmatrix}$$

About Z-axis

$$R_z(\phi) = e^{-i\frac{\phi}{2}\sigma_z}$$

$$I\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\sigma_z$$

$$\begin{bmatrix} \cos\frac{\phi}{2} & 0 \\ 0 & \cos\frac{\phi}{2} \end{bmatrix} - \begin{bmatrix} i\sin\frac{\phi}{2} & 0 \\ 0 & -i\sin\frac{\phi}{2} \end{bmatrix}$$

$$R_z(\phi) = \begin{bmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{+i\phi/2} \end{bmatrix}$$

$$R_{\hat{z}}(\phi) = \begin{bmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{bmatrix}$$

$$\cancel{\begin{bmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{bmatrix}} \quad \cancel{\begin{bmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{bmatrix}}$$

Adding a global phase multiply $e^{i\frac{\phi}{2}}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix} \Rightarrow$$ "operation Remain same but there is change of Global phase"

$$R_Z(\phi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}$$

$\boxed{T}$ Let $\phi = 2\pi$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

ψ —$\boxed{I}$— ψ

Identy

$\boxed{S}$ Let $\phi = \frac{\pi}{2}$

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

—$\boxed{S}$—

$\cancel{S}^*$

$$S^{\dagger} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\pi/2} \end{bmatrix}$$

face

$\boxed{T} = \sqrt{S}$

Let $\phi = \frac{\pi}{4}$

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \left(e^{i\frac{\pi}{2}}\right)^{\frac{1}{2}} \end{bmatrix}$$

—$\boxed{T}$—

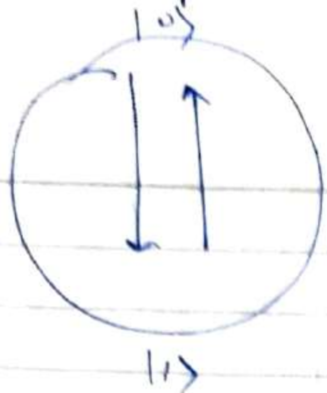
$$T^* = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\frac{\pi}{4}} \end{bmatrix}$$

squareroot of S

NOT GATE

Quantum NOT Gate $\equiv$ Pauli X-gate

$|0\rangle$ —$\boxed{X}$— $|1\rangle$ $\qquad$ $|1\rangle$ —$\boxed{X}$— $|0\rangle$



$\Rightarrow$ Pauli X gate is giving opposite point on bloch sphere

Let us consider an arbitrary state

$$\psi = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

$$X = (|0\rangle\langle 1| + |1\rangle\langle 0|)$$

$$X|\psi\rangle = (|0\rangle\langle 1| + |1\rangle\langle 0|)\left(\cos\frac{\theta}{2}|0\rangle + e^{i\frac{\theta}{2}}\sin\frac{\theta}{2}|1\rangle\right) \quad =1$$

$$X|\psi\rangle = \underline{\cos\frac{\theta}{2}|1\rangle + e^{i\phi}\sin\frac{\theta}{2}|0\rangle} \quad \text{—— (1)}$$

$$\psi_{opp} = \begin{cases} \theta \rightarrow \pi - \theta \\ \phi \rightarrow \pi + \phi \end{cases}$$

$$\psi_{opp} = \sin\frac{\theta}{2}|0\rangle - e^{i\phi}\cos\frac{\theta}{2}|1\rangle \quad \text{—— (2)}$$

$\boxed{(1) \neq (2)}$ so pauli X ~~not~~ gate is not universal ~~gate~~ quantum Not gate

$$|0\rangle \text{ —— } \boxed{X} \rightarrow |1\rangle$$

$$|1\rangle \text{ —— } \boxed{X} \rightarrow |0\rangle$$

$$\alpha|0\rangle + \beta|1\rangle \text{ —— } \boxed{X} \text{ —— } \alpha|1\rangle + \beta|0\rangle$$

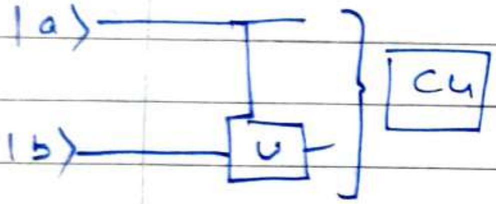
Not opposite point on Bloch sphere

2 Qubit quantum Gates

Basis: $\left\{ \begin{matrix} |00\rangle & |01\rangle & |10\rangle & |11\rangle \\ \hookrightarrow 0 & \hookrightarrow 1 & \hookrightarrow 2 & \hookrightarrow 3 \end{matrix} \right\} = \{0, 1, 2, 3\}$

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} = \underline{I}$$

2 Qubit quantum Gate $\Rightarrow$ Controlled unitary operation $\boxed{CU}$



$$CU = \begin{bmatrix} I_2 & O_2 \\ O_2 & U_2 \end{bmatrix}$$

$U = X = \begin{bmatrix}0&1\\1&0\end{bmatrix}$ [CNOT]

$U = Z = \begin{bmatrix}1&0\\0&-1\end{bmatrix}$ [CZ]

$U = S = \begin{bmatrix}1&0\\0&i\end{bmatrix}$ [Cphase]

~~CNOT~~

① $\boxed{CNOT}$

$$\hookrightarrow \begin{bmatrix} I_2 & O_2 \\ O_2 & X \end{bmatrix}_{4\times4} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} = \underline{I}$$

In CNOT
3rd and 4th column are interchanged

$2 \rightleftarrows 3$

$|10\rangle \rightleftarrows |11\rangle$

$|10\rangle \xrightarrow{CNOT} |11\rangle$

$|11\rangle \xrightarrow{CNOT} |10\rangle$

CNOT
* if first qubit is $|1\rangle$ then second qubit is flipped

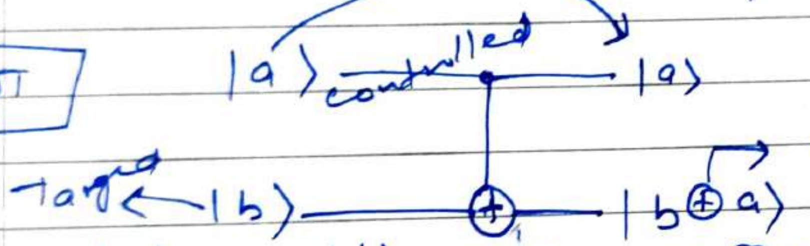
$|00\rangle \xrightarrow{CNOT} |00\rangle$

$|01\rangle \xrightarrow{CNOT} |01\rangle$

No change

$\boxed{CNOT}$

$\boxed{U = X}$



$0+0=0$
$0+1=1$
$1+0=1$
$1+1=2_n$
$=0$

$\|a\rangle$	$\|b\rangle$	$\|b\rangle$	$\|b \oplus a\rangle$
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

$|10\rangle \longrightarrow |11\rangle$

$|11\rangle \longrightarrow |10\rangle$

2) $U = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$CU = \begin{bmatrix} I_2 & 0_2 \\ 0_2 & Z \end{bmatrix} = \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$= I$$

$$3 \longrightarrow -3$$

$$|11\rangle \xrightarrow{CZ} |11\rangle$$

(3) $U = S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

c phase

$$\left\{ |000\rangle \; |001\rangle \; |010\rangle \; |011\rangle \; |100\rangle \; |101\rangle \; |110\rangle \; |111\rangle \right]$$

0 1 2 3 4 5 6 7

$\boxed{I_8}$

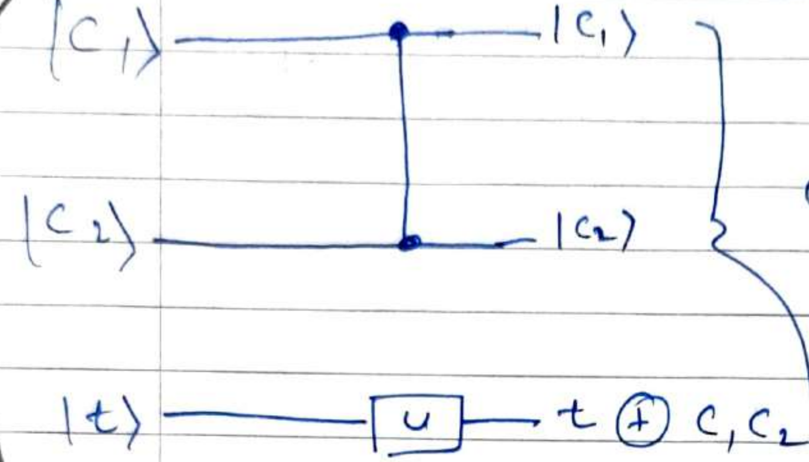
D $[11111111]$

(1) $U = e^{i\gamma} R_{\hat{n}}(\theta)$

(2) $CU = \begin{bmatrix} I_2 & 0 \\ 0 & U \end{bmatrix}$

(3) $CCU = C^2U = \begin{bmatrix} I_4 & 0_4 \\ 0_4 & CU \end{bmatrix}$

↳ controlled controlled unitary $\Rightarrow$ [C^2U] [C^2U]



$$C^2U = \begin{bmatrix} I_4 & O_4 \\ O_4 & U_4 \end{bmatrix}$$

[CNOT] (Toffoli) [SWAP] [FREDKIN Gate]

$U = $ CNOT

Toffoli =

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 7 & 6 \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

It is interchanging $6 \to 7$ and $7 \to 6$

$|110\rangle \rightleftarrows |111\rangle$

If first 2 qubits are $|1\rangle \Rightarrow 3^{rd}$ qubit is flipped.

I. Single Qubit Gates

① Pauli X gate

① —$\boxed{X}$— $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $X|0\rangle \rightarrow |1\rangle$

$X|1\rangle \rightarrow |0\rangle$

② —$\boxed{Y}$— $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $Y|0\rangle \rightarrow i|1\rangle$

$Y|1\rangle = i|0\rangle$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

③ —$\boxed{Z}$— $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $Z|0\rangle = |0\rangle$

$Z|1\rangle = -|1\rangle$

④ —$\boxed{H}$— $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $H|0\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]$

$H|1\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle - |1\rangle\right]$

⑤ —$\boxed{S}$— $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ $S|0\rangle = |0\rangle$

$S|1\rangle = i|1\rangle$

⑥ —$\boxed{T}$— $\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$ $T|0\rangle = |0\rangle$

$T|1\rangle = e^{i\frac{\pi}{4}}|1\rangle$

2 Qubit Quantum gates

① CNOT

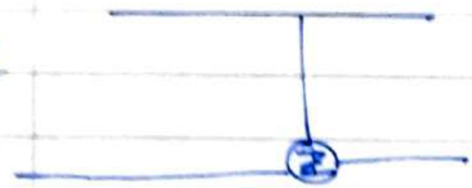
[CU]

$$\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$2 \rightleftarrows 3$

$|10\rangle \rightleftarrows |11\rangle$

② CZ

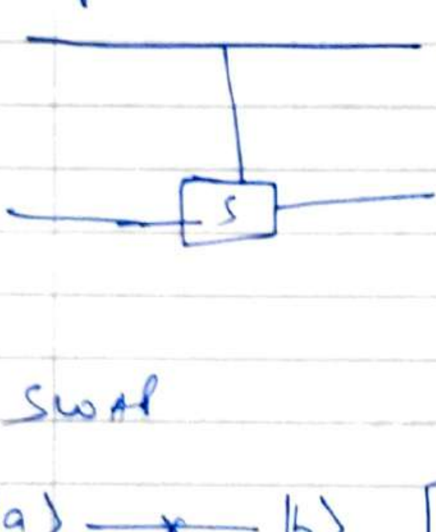


$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$3 \to -3$

$|11\rangle \to |11\rangle$

③ C-phase



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix}$$

$3 \to i3$

$|11\rangle \to i|11\rangle$

④ SWAP

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

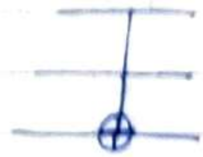
$|01\rangle \to |10\rangle$

$1 \to 2$

3 Qubit Quantum Gates $[C^2 U]$

① TOFFOLI $U = CNOT$

$$\begin{bmatrix} I_4 & 0 \\ 0 & CNOT \end{bmatrix}_{8\times 8} \Rightarrow |110\rangle \xrightarrow{6 \rightleftarrows 7} |111\rangle$$



② FREDKIN $U = SWAP$

$$\begin{bmatrix} I_4 & 0 \\ 0 & SWAP \end{bmatrix} \Rightarrow 5 \rightleftarrows 6$$

$|1\underline{01}\rangle \rightarrow |1\underline{10}\rangle$

——— o ———

except that by new scale which is equal to the scale matrix and the state of each form is 90° apart from each other

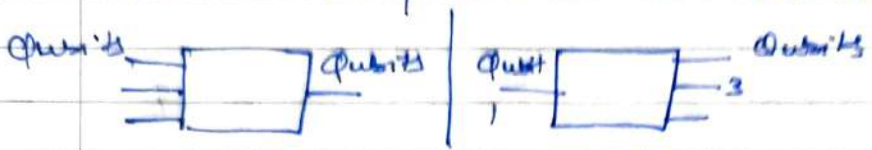
Quantum circuits

(1)

properties :- 1. Quantum circuits are acyclic (no loops)

* No feed back in Quantum circuits

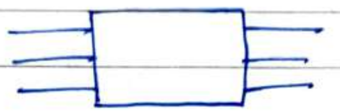
2. FAN-in & FAN-out are not allowed



A Qubit neither created nor destroyed

3. Quantum circuits are Reversible

input



output

$$\# \text{input} = \# \text{ of outputs}$$

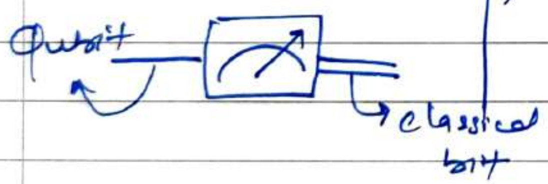
* Measurment of Quantum circuit (QC)

$$\psi_{\text{Qubit}} = \alpha|0\rangle + \beta|1\rangle$$

Measurement $\Rightarrow$ $|0\rangle = P = \dfrac{|\alpha|^2}{|\alpha|^2+|\beta|^2}$

$|1\rangle \Rightarrow P = \dfrac{|\beta|^2}{|\alpha|^2+|\beta|^2}$

classical Result



Measurments are irreversible

I. Quantum Gates in series

Let A & B are two single Qubit quantum gates

Right to left

(1) $\psi_1 - \boxed{A} - \boxed{B} - \;=\; \psi_1 - \boxed{BA} \Rightarrow BA|\psi_1\rangle$

$|0\rangle - \boxed{Y} - \boxed{X} - \;=\; |0\rangle - \boxed{XY} \Rightarrow XY|0\rangle = iZ|0\rangle$ $\quad Z = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$

$= i|0\rangle$

$|1\rangle - \boxed{Y} - \boxed{X} - \;=\; |1\rangle - \boxed{XY} \qquad XY|1\rangle = iZ|1\rangle$

$= -i|1\rangle$

Note $-\boxed{I}-$ NoP Gate ⇒ No operation gate

$\psi_1 - \boxed{I} - \;=\; \psi_1 -$ a bare line → $\boxed{I}$

(2) <u>Quantum Gates in parallel</u>

Let A & B are 2 single Qubit Quantum Gates

$|\psi_1\rangle - \boxed{A} -$

$|\psi_2\rangle - \boxed{B} -$ $\Big\}\ \psi_{o/p} = ??$ $\qquad \psi_1, \psi_2 - \boxed{(A\otimes B)} -$

$\Rightarrow (A\otimes B)|\psi_1\rangle \otimes |\psi_2\rangle$

$A\otimes B|\psi_1\psi_2\rangle$

$$\boxed{(A\otimes B)|\psi_1\psi_2\rangle}$$

$|0\rangle - \boxed{H} -$

$|1\rangle - I -$ $\Big\}$ $= (H\otimes I)(|0\rangle \otimes |1\rangle)$

$(H\otimes I)|01\rangle$

$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \otimes \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \left[|01\rangle + |11\rangle \right]$$

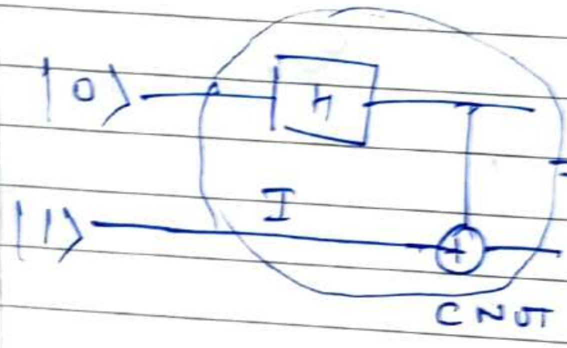
Basis

$\{ |0\rangle \otimes |1\rangle \}$

$\{ [\, |00\rangle \; |01\rangle \; |10\rangle \; |11\rangle \,] \}$

$\{ |000\rangle \; |001\rangle \; \ldots \; |111\rangle \}$

(2)



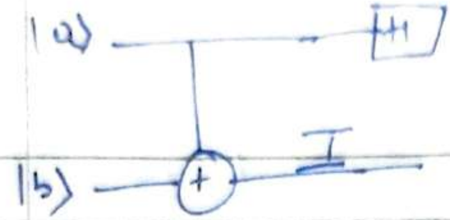
$\Rightarrow \Rightarrow (CNOT)(H \otimes I)$

$$CNOT \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$CNOT \; \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xleftarrow{} CNOT \quad \times \; \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

(3)



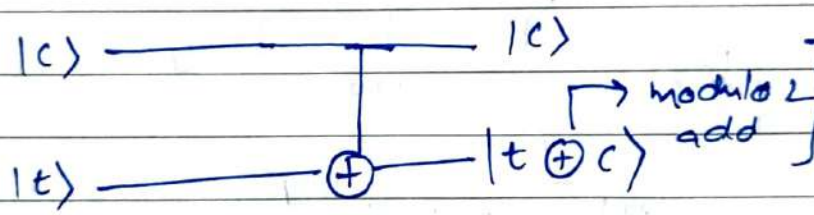
$\Rightarrow (H \otimes I)\, CNOT$

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

All quantum circuits are Unitary and Reversible"

Quantum circuits with CNOT

$$CNOT = \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



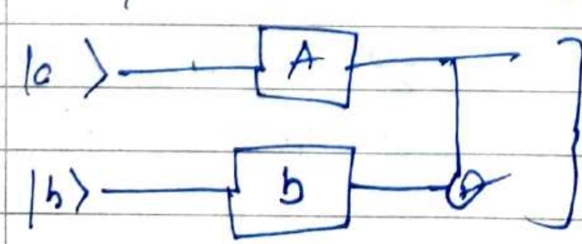
$2 \rightleftarrows 3$

$|10\rangle \rightleftarrows |11\rangle$

Combination with CNOT

Let A & B are two single Qubit states

(1)



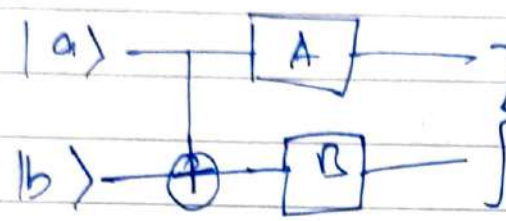
$\Psi_F = ??$

$\psi_i = |a_i b\rangle$ $\quad$ $\mathrm{CNOT}\,(A \otimes B)\,|\psi_i\rangle$

$$\mathrm{CNOT} \quad \begin{bmatrix} \\ \\ \\ \end{bmatrix}_{4\times 4} \qquad |01\rangle \longleftrightarrow |11\rangle$$

Interchange 3rd and 4th columns.

(2)



$\psi_f = ?$

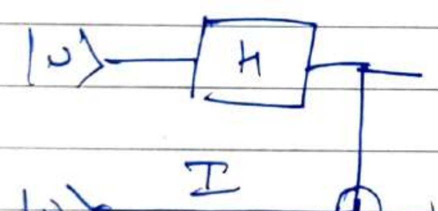
$$(A \otimes B)\ \mathrm{CNOT}\ |\psi_i\rangle$$

$$\left(\begin{bmatrix} \text{Right} \\ \Downarrow \end{bmatrix} \ \mathrm{CNOT} \right) \psi_i$$

3rd & 4th rows are interchang

3rd are 4th columns are interchange

(1)



$$\psi_f = \mathrm{CNOT}\,(H \otimes I)$$

$$\mathrm{CNOT}\ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} =$$

$$\psi_i = |0\rangle \otimes |0\rangle = |00\rangle$$

(3rd and 4th rows are interchanged)

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&1&0&-1\\1&0&-1&0\end{bmatrix}$$

$$\Psi_f = \frac{1}{\sqrt{2}}\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&1&0&-1\\1&0&-1&0\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} =$$

$$\boxed{\Psi_f = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right]}$$ Bell state or entangled state

$$\boxed{\text{Entangled state} \quad \Psi \neq \Psi_1 \otimes \Psi_2}$$

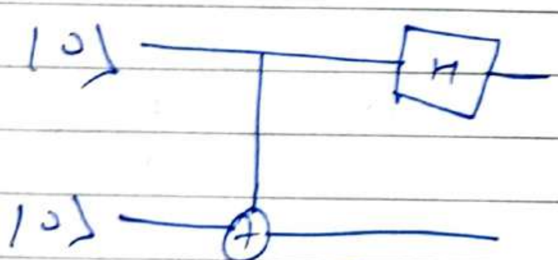
$$\Psi = \frac{1}{\sqrt{2}}\left[|01\rangle + |11\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right] \otimes |1\rangle$$

Creating a quantum state which can't be broken

$$= \frac{1}{\sqrt{2}}\left[|01\rangle + |11\rangle\right]$$

(2)



$$\Psi_f = \left((H \otimes I)\, CNOT\right)\Psi_1$$

$$\Downarrow$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{bmatrix} CNOT$$

3rd & 4th column interchange

$$\hookrightarrow \Psi_1 = |00\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1&0&0&1\\0&1&1&0\\1&0&0&-1\\0&1&-1&0\end{bmatrix}$$

$$\Psi_{final} = \frac{1}{\sqrt{2}}\begin{bmatrix}1&0&0&1\\0&1&1&0\\1&0&0&-1\\0&1&-1&0\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\left[|00\rangle + |10\rangle\right]$$

(Not a Bell state)

$$\frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right] \otimes |0\rangle$$

$$\frac{1}{\sqrt{2}}\left[|00\rangle + |10\rangle\right]$$

right

CNOT (A ⊗ B) → 3rd & 4th rows are interchange

(A ⊗ B) CNOT → 3rd and 4th columns are interchange

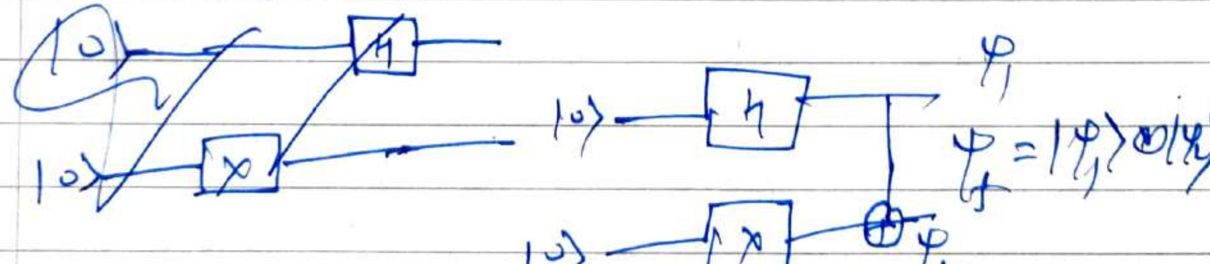
Circuits with CNOT

R

CNOT (A ⊗ B)   3rd & 4th rows are interchanged

(A ⊗ B) CNOT ⇒ 3rd & 4th columns are interchanged

(1)



$\Rightarrow CNOT(H \otimes X)$

$\Downarrow$

$\Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix} \otimes \begin{bmatrix}0&1\\1&0\end{bmatrix}$

~~CNOT~~ CNOT $\frac{1}{\sqrt{2}}\begin{bmatrix}0&1&0&1\\1&0&1&0\\0&1&0&-1\\1&0&-1&0\end{bmatrix}$

$\Psi_f = U\Psi_1 = U|0\rangle\otimes|0\rangle$

$\hookrightarrow |00\rangle$

$\Psi_f = \frac{1}{\sqrt{2}}\begin{bmatrix}0&1&0&1\\1&0&1&0\\1&0&-1&0\\0&1&0&-1\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} =$

$= \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\\0\end{bmatrix}$

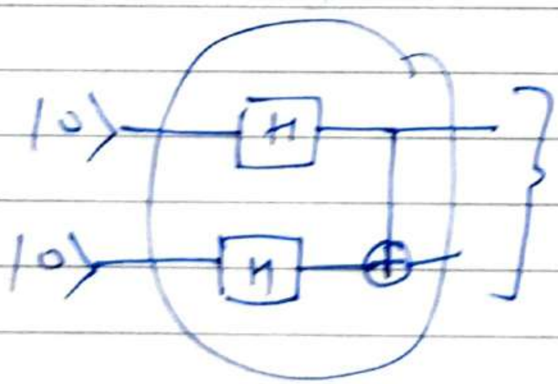
$Qc = \; = \frac{1}{\sqrt{2}}\begin{bmatrix}0&1&0&1\\1&0&1&0\\1&0&-1&0\\0&1&0&-1\end{bmatrix}$

$\Psi_f = \frac{1}{\sqrt{2}}\left[|01\rangle + |10\rangle\right]$

(Bell state)

$\neq \boxed{\cancel{|\Psi_1\rangle \otimes |\Psi_2\rangle}}$

(3)



$\Psi_f = U$

$U = CNOT\,(H \otimes H)$

$\Downarrow$

$\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}$

$$\text{CNOT} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

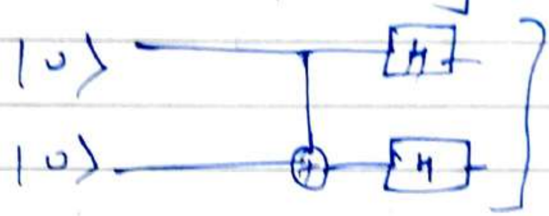
$$U = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

$$\psi_f = U\psi_i$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2}\left[|00\rangle + |01\rangle + |10\rangle + |11\rangle\right]$$

Homework

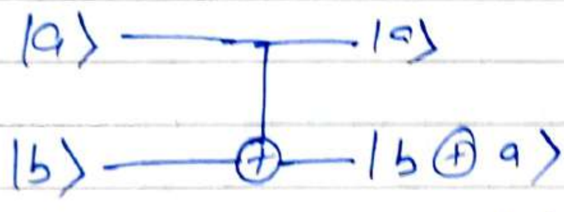


$\psi_f = ??$

Upside down CNOT

CNOT

$\boxed{\text{CNOT}}$



→ modulo 2

a	b	a	|b + a⟩
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

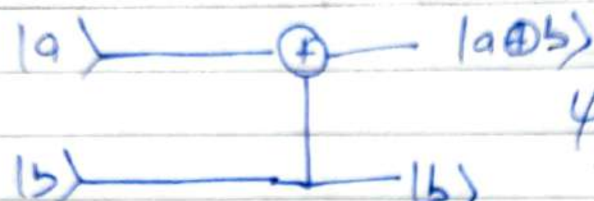
$|10\rangle \rightleftarrows |11\rangle$

$2 \rightleftarrows 3$

CNOT $(A \otimes B)$ 3rd and 4th rows are interchange

$(A \otimes B)$ CNOT 3rd & 4th columns interchange

<u>upside down CNOT</u> :- $\boxed{\text{CNOT}}$



$\psi_f = |a \oplus b\rangle |b\rangle$

a	b	$\|a \oplus b\rangle$	$\|b\rangle$	
0	0	0	0	
0	1	1	1	$\|01\rangle \rightarrow \|11\rangle$
1	0	1	0	
1	1	0	1	$\|11\rangle \rightarrow \|01\rangle$

if $|b\rangle = 1 \Rightarrow |a\rangle$ is flipped

~~Cnot~~ CNOT = $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

upside down <u>CNOT</u>

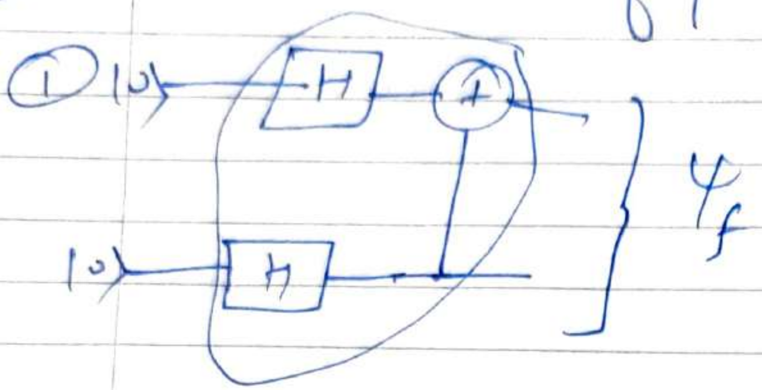
2nd and 4th columns interchange

If 2nd qubit is 1 then first qubit is fliped.

2nd and 4th rows are interchange

$(A \otimes B)$ CNOT $\Rightarrow$ 2nd & 4th columns or interchange

① Combination of Quantum circuit CNOT



$$U = \underline{CNOT}\,(H \otimes H)$$

R

$$\text{CNOT} \quad \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

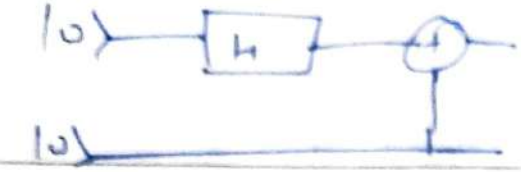
2nd and 4th row interchange

$$U = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

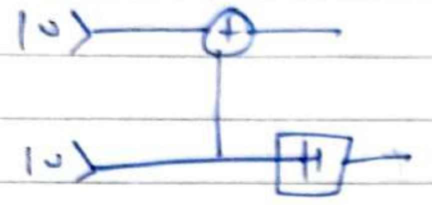
$$\psi_f = U\psi_i = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2}\left[|00\rangle + |01\rangle + |10\rangle + |11\rangle\right]$$

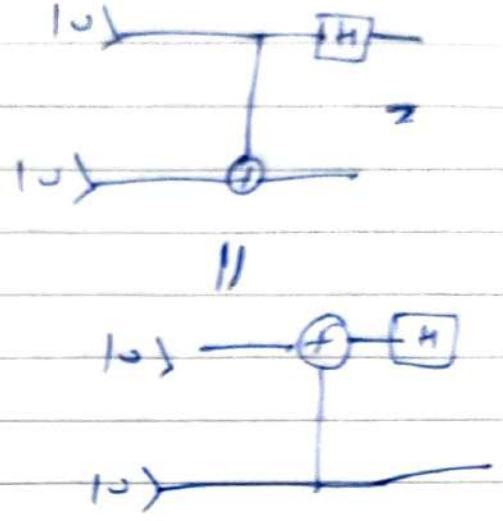
①



②

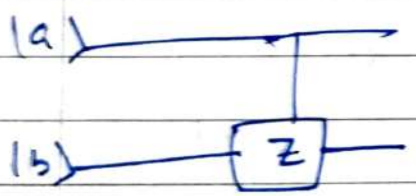


check



③

Controlled Z CZ Gate :-



$$U = \begin{bmatrix} I & 0 \\ 0 & Z \end{bmatrix}$$

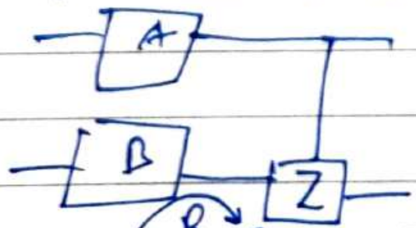
$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$|3\rangle = -|3\rangle$

$|11\rangle \rightarrow |11\rangle$

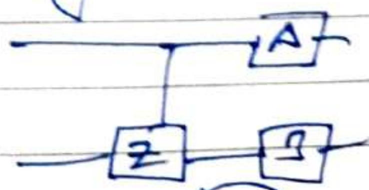
4th column is multiplied by (-1)

*



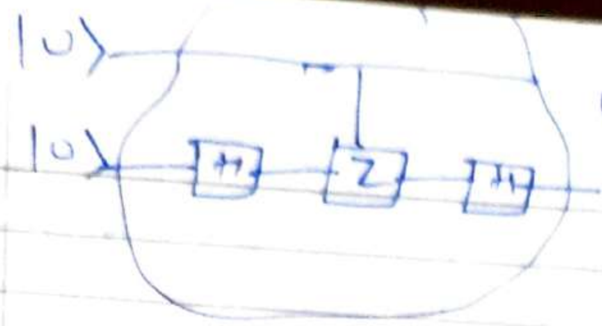
$CZ \xrightarrow{R} (A \otimes B)$

4th is multiplied by (-1)



$(A \otimes B)\ CZ$

4th column is multiplied by (-1)

(1) 

$\psi_f = ??$ Hermitian operator

Remember be careful (A) (B) $A \otimes B$

$$U = (I \otimes H)\, CZ\, (I \otimes H)$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} CZ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

4th row multiply by (-1)

$$\frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$=$ [

Quantum fourier Transformation

Integral transformations which can change one basis to another basis

— Changes computational basis to fourier basis

$\underline{CB}$ $|0\rangle$ & $|1\rangle$ $\longrightarrow$ FB → Diagonal basis
↳ basis state of polarisation

$|+\rangle$ & $|-\rangle$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \;\&\; \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\underline{QFT}$

$$|0\rangle \longrightarrow |+\rangle$$
$$|1\rangle \longrightarrow |-\rangle$$

$$|0\rangle \text{———} \boxed{H} \text{———} |+\rangle$$
$$|1\rangle \text{———} \boxed{H} \text{———} |-\rangle$$

Integral transformation

$$\{x\} \xrightarrow[IT]{} \{y\}$$

$$IT\,[x] = k(x, y)\,\{y\}$$

Kernel is function of Input and output

eg $$\mathcal{L}\left[f(t)\right] = k(t, s)\, f(s)$$

QFT

kernel

$QFT|x\rangle = K(x,y)|y\rangle$

$$\boxed{QFT = \frac{1}{\sqrt{2}}\sum_{y=0}^{N-1}\underbrace{\left(e^{2\pi i\frac{xy}{N}}\right)}_{\text{kernel.}}|y\rangle}$$

$(\omega_N)^{xy}$

$(1)^{\frac{1}{N}}$

$\sqrt[N]{1} = \omega_N$

$\left(e^{2\pi i}\right)^{\frac{1}{N}} = \omega_N = \left(e^{\frac{2\pi}{N}i}\right)$

$$\boxed{QFT|x\rangle = \frac{1}{\sqrt{N}}\sum_{y}^{N-1}\omega_N^{xy}|y\rangle}$$

other type of eqn.

— modification form

$$QFT|x\rangle = \frac{1}{\sqrt{N}}\sum_{y=0}^{N-1}e^{2\pi i\frac{xy}{N}}|y\rangle$$

What is x, y ??

x, y are Decimal numbers

$x \cdot y \Rightarrow$ multiplications.

1 Qubits $\quad N = 2^1 = 2$

$$QFT|x\rangle = \frac{1}{\sqrt{2}}\sum_{y=0}^{1}e^{2\pi i\frac{xy}{2}}|y\rangle$$

$|\tilde{x}\rangle$

$$|\tilde{x}\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + e^{2\pi i \frac{x}{2}}|1\rangle\right]$$

$$\boxed{e^{\pi i} = -1}$$

$x = 0$ $x = 1$

$\frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]$ $\frac{1}{\sqrt{2}}\left[|0\rangle - |1\rangle\right]$

$|+\rangle$ $|-\rangle$

$|\tilde{0}\rangle = |+\rangle$ $|\tilde{1}\rangle = |-\rangle$

$$\boxed{e^{\pi i} = -1}$$

eg $\psi = \alpha|0\rangle + \beta|1\rangle$

$$\downarrow QFT$$

$$QFT|\psi\rangle = \alpha|\tilde{0}\rangle + \beta|\tilde{1}\rangle$$

$$= \alpha\left[|+\rangle\right] + \beta\left[|-\rangle\right]$$

$$= \alpha\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right] + \beta\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$QFT|\psi\rangle = \frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle$$

$$QFT|x\rangle = \frac{1}{\sqrt{N}}\sum_{y=0}^{N-1} e^{2\pi i \frac{xy}{N}}|y\rangle$$

what is y ??

$$y = \underbrace{|y_1 \; y_2 \; y_3 \cdots y_n\rangle}_{\text{Binary values}}$$

$y \downarrow$ Decimal value

$$= y_1 \times 2^{n-1} + y_2 \times 2^{n-2} + \cdots + y_{n-1} \times 2^1 + y_n \times 2^0$$

$y = 5 = 101$

$n = 3$

0 0 0

0 0 1

1

1

1 1 1 $\longrightarrow 7$

1 2^2 2^1 2^0 $\longleftarrow$

$(2^2 \times 1) + (2^1 \times 1) + (2^0 \times 1) = 7$

$$= y_1 * 2^{n-1} + y_2 * 2^{n-2} + \cdots y_{n-1} * 2^1 + y_n 2^0$$

Eg. 3 Qubit $\{|0\rangle, |1\rangle, |2\rangle \cdots |7\rangle\}$

$\{|000\rangle, \cdots |111\rangle\}$

y is a decimal value represented in a Binary form

$$y = \sum_{k=1}^{n} y_k 2^{n-k}$$

QFT $|x\rangle$

$$|\tilde{x}\rangle = \frac{1}{\sqrt{N}} \sum_{y} e^{2\pi i x \frac{\sum_k y_k 2^{n-k}}{N}} |y_1 y_2 \cdots y_n\rangle$$

$N \to 2^n$

$$e^{x+y+z} = e^{x} e^{y} e^{z}$$

$$e^{\sum_i x_i} = \prod_i e^{x_i}$$

exponential summation is equal to product of exponential

$$= \frac{1}{\sqrt{N}} \sum_{y} \prod_{k=1}^{n} e^{2\pi i x \frac{y_k}{2^k}} |y_1 y_2 y_3 \cdots y_n\rangle$$

$$\left[\sum_{y_1=0}^{1} \sum_{y_2=0}^{1} \cdots \sum_{y_n=0}^{1}\right] \qquad |y_k\rangle$$

$$\frac{1}{\sqrt{N}} \bigwedge_{k=1}^{n} \left[\sum_{y_1=0}^{1} \sum_{y_2=0}^{1} \cdots \sum_{y_n=0}^{1} e^{\frac{2\pi i x y_k}{2^k}} |y_k\rangle \right]$$

$|y_1 y_2 \cdots y_n\rangle$ Binary

$$\Downarrow$$

$$\left[\left(|0\rangle + e^{2\pi i \frac{x}{2^k}} |1\rangle \right) \right.$$

$$\bigotimes_{k=1}^{n} \frac{1}{\sqrt{N}} \left[|0\rangle + e^{2\pi i \frac{x}{2^k}} |1\rangle \right]$$

expns

$$\boxed{\begin{gathered}(\alpha_1 |0\rangle + \beta_1 |1\rangle) \otimes (\alpha_2 |0\rangle + \beta_2 |1\rangle) \\ \alpha_1 \alpha_2 |00\rangle + \beta_1 \beta_2 |0 - - \\ 4\end{gathered}}$$

$$|\tilde{x}\rangle = \frac{1}{\sqrt{N}} \left[\left(|0\rangle + e^{2\pi i \frac{x}{2}} |1\rangle \right) \otimes \left(|0\rangle + e^{2\pi i \frac{x}{2^2}} |1\rangle \right) \otimes \left(|0\rangle + e^{2\pi i \frac{x}{2^n}} |1\rangle \right) \right.$$

$$QFT |x\rangle = \frac{1}{\sqrt{N}} \left[\left(|0\rangle + e^{2\pi i \frac{x}{2}} |1\rangle \right) \otimes \left(|0\rangle + e^{2\pi i \frac{x}{2^2}} |1\rangle \right) + \cdots \right.$$

$$\cdots \otimes \left(|0\rangle + e^{2\pi i \frac{x}{2^4}} \right) \otimes \left(|0\rangle + e^{2\pi i \frac{x}{2^n}} |1\rangle \right)$$

Decimal

$$\frac{x}{2^n} = ??$$

$$\frac{x}{2^n} \xrightarrow{\text{Decimal}} \frac{x_1 x_2 x_3 \cdots x_n}{2^n} \xrightarrow{\text{Binary}}$$

$$= \frac{1}{2^n}\left[x_1 * 2^{n-1} + x_2 * 2^{n-2} + \cdots + x_n * 2^0\right]$$

$$\frac{1}{2^n}\begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n \\ 2^{n-1} & 2^{n-2} & 2^{n-3} & \cdots & 2^1 & 2^0 \end{bmatrix}$$

$$= \frac{1}{2^n}\left[x_1 * 2^{n-1} + x_2 * 2^{n-2} + \cdots + x_{n-1} * 2^1 + x_n * 2^0\right]$$

$= 5 \leftarrow 101$

$1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$

eg ~~101~~

$$\begin{array}{ccc} 1 & 0 & 1 \\ 2^2 & 2^1 & 2^0 \end{array} = (5)$$

$\longleftarrow$

$1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$

$4 + 0 + 1 = 5$

$$\boxed{\frac{x}{2^n} = \frac{x_1}{2} + \frac{x_2}{2^2} + \frac{x_3}{2^3} \cdots + \frac{x_n}{2^n}}$$

$$\frac{5}{2^n} \rightarrow 101 = \frac{1}{2} + \frac{0}{2^2} + \frac{1}{2^3}$$

$$\frac{5}{2^3} \rightarrow 101 = \frac{1}{2^1} + \frac{0}{2^2} + \frac{1}{2^3}$$

$$= \frac{1}{2} + \frac{1}{2^3} = \frac{5}{8}$$

Decimal → Binary

$$\frac{123}{10^2} = 1.23 \Rightarrow$$

$$\Rightarrow \frac{x_1 x_2 x_3}{2^2} = x_1 . x_2 x_3$$

$$\frac{x_1 x_2 x_3}{2^1} = x_1 x_2 . x_3$$

$$\frac{x_1 x_2 x_3}{2^3} = 0 . x_1 x_2 x_3$$

$$\frac{x}{2^n} = \frac{x_1 x_2 x_3 \cdots x_n}{2^n}$$

$$= 0 . x_1 x_2 x_3 \cdots x_n$$

$$\frac{5}{2^3} \rightarrow 101 = 0.101$$

$$\frac{1}{2} + \cancel{\frac{0}{2^2}} + \frac{1}{2^3}$$

$$= \frac{4+1}{8} = 5/8$$

$$|\tilde{x}\rangle \frac{1}{\sqrt{N}}\Big[\left(|0\rangle + e^{2\pi i \frac{x}{2}}|1\rangle\right) \otimes$$

$$\left(|0\rangle + e^{2\pi i \frac{x}{2^2}}|1\rangle\right) \otimes \text{---}$$

$$\otimes \left(|0\rangle + e^{2\pi i \frac{x}{2^n}}|1\rangle\right)\Big]$$

$$\frac{x}{2^n} = \frac{x_1 x_2 \cdots x_n}{2^n} = 0.x_1 x_2 x_3 \cdots x_n$$

$$\frac{x}{2^{n-1}} = \frac{x}{2^n} \times 2 = 2\left(\frac{x}{2^n}\right) = 2\left(0.x_1 x_2 x_3 \cdots x_n\right)$$

$$= \underline{\underline{(x_1 . x_2 x_3 \cdots x_n)}}$$

Decimal

$10(0.123)$

$\Downarrow$

1.23

Binary

$2(0.x_1 x_2 x_3)$

$\Downarrow$

$x_1 . x_2 x_3$

$$e^{2\pi i \frac{x}{2^{n-1}}} =$$

$$= e^{2\pi i (x_1 . x_2 x_3 \cdots x_n)} \longrightarrow x_1 + 0.x_2 x_3 \cdots x_n$$

$$= \left(e^{2\pi i x_1}\right)\left(e^{2\pi i (0.x_2 x_3 \cdots x_n)}\right)$$

1.23

$\downarrow$

$1 + 0.23$

1

if x_1 is qubit if put it $|1\rangle$ then exponential is one $|0\rangle$ then also exponential $\underline{1}$

$$e^{2\pi i (0.x_2 x_3 \cdots x_n)}$$

$$e^{2\pi i \frac{x}{2^{n-1}}} = e^{2\pi i (0.x_2 x_3 \cdots x_n)}$$

<u>3 Qubits</u>

$|5\rangle \Rightarrow 101$

$x = 5$

$QFT|5\rangle$

$|5\rangle \Rightarrow 101$

$$\boxed{\frac{x}{2^n} = \frac{5}{2^3} = 0.101}$$

$$\frac{5}{2^2} = \frac{5}{2^3} \times 2 = 2(0.101) \quad \text{Binary}$$

$$= 1.01$$

$$e^{2\pi i (\cancel{1}.01)}$$

$$\boxed{e^{2\pi i \frac{5}{2^3}} = e^{2\pi i (0.101)}}$$

$$e^{2\pi i \frac{5}{2^2}} = e^{2\pi i (0.01)}$$

③ $$\frac{5}{2} = \frac{5}{2^3} \times 2^2 = 2^2(0.101)$$

$$= 10.1$$

$$e^{2\pi i \left(\frac{5}{2}\right)} = e^{2\pi i (0.1)}$$

$$\boxed{\begin{aligned} e^{2\pi i \frac{5}{2}} &= e^{2\pi i (0.1)} \\ e^{2\pi i \frac{5}{2^2}} &= e^{2\pi i (0.01)} \\ e^{2\pi i \frac{5}{2^3}} &= e^{2\pi i (0.101)} \end{aligned}}$$

$$\frac{7}{2^3} = 0.111 \Rightarrow e^{2\pi i \frac{7}{2^3}} = e^{2\pi i (0.111)}$$

$$\frac{7}{2^2} = \cancel{1}1.11 \Rightarrow e^{2\pi i \frac{7}{2^2}} = e^{2\pi i (0.11)}$$

$$\frac{7}{2^1} = \cancel{11}.1 \Rightarrow e^{2\pi \frac{7}{2^1}} = e^{2\pi i (0.1)}$$

$$|\tilde{x}\rangle = \frac{1}{\sqrt{N}}\left[\left(|0\rangle + e^{2\pi i \frac{x}{2}}|1\rangle\right) \otimes \left(|0\rangle + e^{2\pi i \frac{x}{2^2}}|1\rangle\right) \otimes \cdots \left(|0\rangle + e^{2\pi i \frac{x}{2^4}}|1\rangle\right)\right.$$

Decimal

$$\leftarrow \frac{x}{2} = \frac{x_1 x_2 \cdots x_4}{2} = 0.x_4$$

$$\frac{x}{2^2} = \frac{x_1 x_2 \cdots x_4}{2^2} = 0.x_{4-1}x_4$$

$$\frac{x}{2^3} = 0.x_{n-2}x_{n-1}x_n$$

$$\frac{x}{2^n} = 0.x_1x_2 \ldots x_n$$

$$= \frac{1}{\sqrt{N}} \Big[\big(|0\rangle + e^{2\pi i (0.x_n)} |1\rangle\big) \otimes$$

$$\big(|0\rangle + e^{2\pi i (0.x_{n-1}x_n)} |1\rangle\big) \otimes \ldots$$

$$\otimes \big(|0\rangle + e^{2\pi i (0.x_1x_2 \ldots x_n)} |1\rangle\big)$$

QFT $|x\rangle$

3 Qubits

$|x\rangle \to |x_1x_2x_3\rangle$

$$= \frac{1}{\sqrt{8}} \Big[\big(|0\rangle + e^{2\pi i \frac{x}{2}} |1\rangle\big) \otimes \big(|0\rangle + e^{2\pi i \frac{x}{2^2}} |1\rangle\big) \otimes \big(|0\rangle + e^{2\pi i \frac{x}{2^3}} |1\rangle\big) \Big]$$

$$= \frac{1}{\sqrt{8}} \Big[\big(|0\rangle + e^{2\pi i (0.x_3)} |1\rangle\big) \otimes \big[|0\rangle + e^{2\pi i\, 0.x_2x_3} |1\rangle\big)$$

$$\otimes \big(|0\rangle + e^{2\pi i (0.x_1x_2x_3)} |1\rangle\big)$$

$$|0\rangle + e^{2\pi i (0.0)} |1\rangle$$

Ckt of Quantum fourier transform

$$|\tilde{x}\rangle = \frac{1}{\sqrt{N}}\Big[\left(|0\rangle + e^{2\pi i \frac{x}{2}}|1\rangle\right) \otimes$$

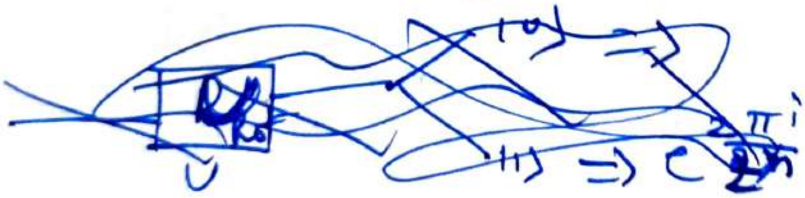
$$\left(|0\rangle + e^{2\pi i \frac{x}{2^2}}|1\rangle\right) \otimes$$

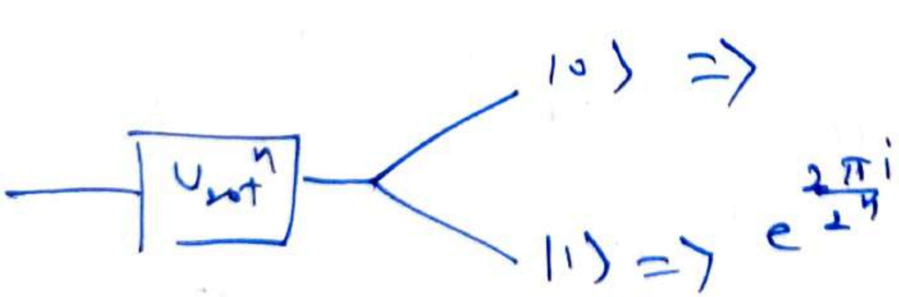
$$\left(|0\rangle + e^{2\pi i \frac{x}{2^3}}|1\rangle\right) \otimes \cdots$$

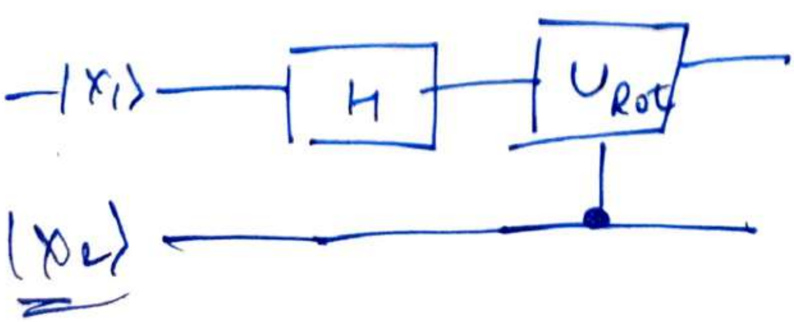
$$\cdots \left(|0\rangle + e^{2\pi i \frac{x}{2^n}}|1\rangle\right)\Big]$$

$|x\rangle$ ———— $\left(|0\rangle + e^{2\pi i \frac{x}{2}}|1\rangle\right)$

$|x\rangle$ —$\boxed{H}$— $= \left(|0\rangle + e^{2\pi i \frac{x}{2}}|1\rangle\right)$



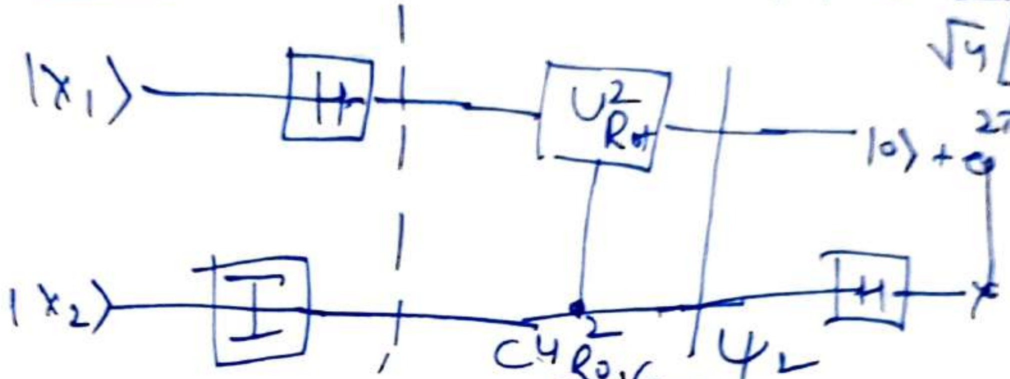




$$U^{n}_{Rot} = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{2^n}} \end{bmatrix}$$

<u>2 Qubits</u>

QFT $|x_1 x_2\rangle = \frac{1}{\sqrt{4}}\left[|0\rangle + e^{2\pi i \frac{x_2}{2}}|1\rangle \otimes |0\rangle + e^{2\pi i \frac{x}{2^2}}|1\rangle \right]$



$\psi_1 = |0\rangle + e^{2\pi i \frac{x}{2}}|1\rangle$

$\Downarrow$

$\left(|0\rangle + e^{2\pi i\, 0.x_2}|1\rangle\right)$

$(H \otimes I)\,|x_1 x_2\rangle$

$H|x_1\rangle \otimes |x_2\rangle$

$\left(|0\rangle + e^{2\pi i\, 0.x_1}|1\rangle\right)$

$\cancel{\left(|0\rangle + e^{2\pi i \frac{x}{2}}\right.}$

$\left(|0\rangle + e^{2\pi \frac{x_1}{2}}|1\rangle\right) \otimes |x_2\rangle$

$\left(|0\rangle + e^{2\pi \frac{x_1}{2}}|1\rangle\right) \otimes U^2_{Rot}(|x_2\rangle)$

$$U^{n}|x\rangle = |0\rangle + e^{2\pi i \frac{x}{2}}|1\rangle$$

$$U^{2}|x_2\rangle = \left(|0\rangle + e^{2\pi i \frac{x_2}{2^2}}|1\rangle\right)$$

$$\psi_2 = \left(|0\rangle + e^{2\pi i \frac{x_1}{2}}|1\rangle\right) \otimes \left(|0\rangle + e^{2\pi i \frac{x_2}{2^2}}|1\rangle\right)$$

$$= \left(|0\rangle + e^{2\pi i\left(\frac{x_1}{2} + \frac{x_2}{2^2}\right)}|1\rangle\right)$$

$$= |0\rangle + e^{2\pi i\left(0.x_1 + 0.0x_2\right)} \quad 0.x_1x_2$$

$$\psi_2 = \left(|0\rangle + e^{2\pi i(0.x_1x_2)}|1\rangle\right)$$

$$\psi_f = \left(|0\rangle + e^{2\pi i(0.x_1x_2)}|1\rangle\right)\left(|0\rangle + e^{2\pi i\, 0.x_2}|1\rangle\right)$$

$$\boxed{QFT|x_1x_2\rangle = \frac{1}{\sqrt{4}}\left[|0\rangle + e^{2\pi i(0.x_2)}|1\rangle \otimes |0\rangle + e^{2\pi i(0.x_1x_2)}|1\rangle\right]}$$

① $\boxed{\psi_1}$ $(H \otimes I)\,|x_1\, x_2\rangle$

$$\left(|0\rangle + e^{2\pi i \frac{x_1}{2}}|1\rangle\right) \otimes |x_2\rangle$$

$$CU_{Rot}^{2}|\psi_1\rangle = \left(|0\rangle + e^{2\pi i \frac{x_1}{2}}|1\rangle\right) \otimes U_{Rot}^{2}|x_2\rangle$$

$$\psi_2 = \left(|0\rangle + e^{2\pi i \frac{x_1}{2}}|1\rangle\right)\left(|0\rangle + e^{2\pi i \frac{x_2}{2^2}}|1\rangle\right)$$

$$= \left(|0\rangle + e^{2\pi i \left(\frac{x_1}{2} + \frac{x_2}{2^2}\right)}|1\rangle\right)$$

$$\psi_2 = \left(|0\rangle + e^{2\pi i (0.x_1 x_2)}|1\rangle\right)$$

$$\cancel{\psi_f = \psi_2}$$

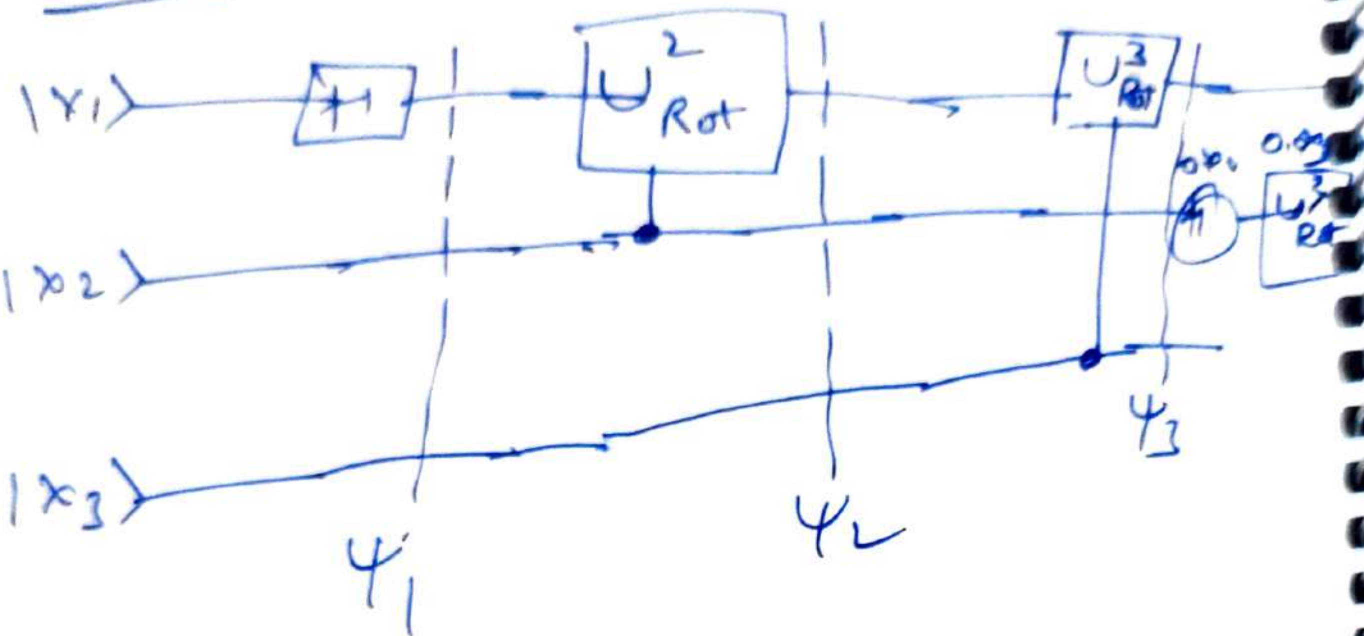
$$\psi_f = (I \otimes H)(\psi_2 \otimes |x_2$$

$$\psi_2 \hat{H}|x_2\rangle$$

$$\psi_f = \left(|0\rangle + e^{2\pi i (0.x_1 x_2)}|1\rangle\right)\left(|0\rangle + e^{2\pi i (0.x_2}|1\rangle\right)$$

<u>~~B Qubit~~</u>

3 Qubits $= |x_1 x_2 x_3\rangle = \frac{x_1}{2} + x_2$



$$\psi_1 = \left(|0\rangle + e^{2\pi i \frac{x_1}{2}}|1\rangle\right)|x_2 x_3\rangle$$

$$\psi_2 = \left(|0\rangle + e^{2\pi i \frac{x_1}{2}}|1\rangle\right) \otimes \left(|0\rangle + e^{2\pi i \frac{x_2}{2^2}}|1\rangle\right)|x_3\rangle$$

$$\psi_3 = \left(|0\rangle + e^{2\pi i \frac{x_1}{2}}|1\rangle\right) \otimes \left(|0\rangle + e^{2\pi i \frac{x_2}{2^2}}|1\rangle\right) \otimes \left(|0\rangle + e^{2\pi i \frac{x_3}{2^3}}|1\rangle\right)$$

$$= \left(|0\rangle + e^{2\pi i (0.x_1 x_2 x_3)}|1\rangle\right)$$