Noise and decoherence: Randomness change in property

Amplitude damping

$$= \alpha e^{i\phi_\alpha}|0\rangle + \beta e^{i\phi_\beta}|1\rangle$$

amplitude damping   phase damping

$$I = |0\rangle\langle 0| + |1\rangle\langle 1|$$

$$\text{Pure state} \xrightarrow{\text{Decoherence}} \text{mixed state}$$

$$\rho = \begin{pmatrix} - & \bar{0} \\ \underline{} & - \end{pmatrix}$$

superposition lost in Decoherence

$$|\psi\rangle = |\alpha| e^{i\phi_\alpha}|0\rangle + \beta e^{i\phi_\beta}|1\rangle$$

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \beta\alpha^* & |\beta|^2 \end{pmatrix}$$

$$= \begin{pmatrix} |\alpha|^2 & |\alpha||\beta| e^{i(\phi_\alpha - \phi_\beta)} \\ |\alpha||\beta| e^{-i(\phi_\alpha - \phi_\beta)} & |\beta|^2 \end{pmatrix}$$

$$\begin{pmatrix} f(t)|\alpha|^2 & e^{-\frac{t}{\tau}} \\ e^{-\frac{t}{\tau}} & 1 - f(t)|\alpha|^2 \end{pmatrix}$$

$\tau \rightarrow$ Decoherence time

T_1 time t_g = gate time

T_2 time τ = decoherence $N.P = \frac{\tau}{t_g}$

DFS Decoherence free subspace

Logical Qubits

$|0\rangle \rightarrow |0\rangle_L$

$|1\rangle \rightarrow |1\rangle_L$

$e^{i\phi_{global}}(\alpha|0\rangle_L + \beta|1\rangle_L$

$|0\rangle \longrightarrow e^{i\langle\phi_0\rangle}|0\rangle \longrightarrow |0\rangle_L = |\phi^-\rangle$

$|1\rangle \rightarrow e^{i\langle\phi_1\rangle}|1\rangle \longrightarrow |1\rangle_L = |\phi^+\rangle$

$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \longrightarrow \frac{e^{i\langle\phi_0\rangle}|0\rangle}{\sqrt{2}} + \frac{e^{i\langle\phi_1\rangle}|1\rangle}{\sqrt{2}}$

$|\phi^{\mp}\rangle = e^{i\langle\phi\rangle}|0\rangle \; e^{i\langle\phi_1\rangle}|1\rangle$

coherence main

span

gemetric explanation to a single qubit operation

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ $\alpha^2 + \beta^2 = 1$

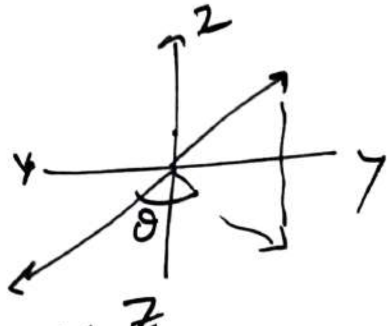
$\alpha = I + Rr$ $\alpha, \beta \in$ complex no $\mathbb{C}$

$\beta = I + Ro$ θ & ϕ this block

$\alpha = x + iy$ (CCS)

$\alpha = r_0 e^{i\phi_0}$ ~~Cartesian~~ Polar coordinate system

$\beta = r_1 e^{i\phi_1}$



$|\psi\rangle = r_0 e^{i\phi_0}|0\rangle + r_1 e^{i\phi_1}|1\rangle$

$\cancel{e^{i\phi_0}(r_0|0\rangle + r_1 e^{i(\phi_1 - \phi_0)}|1\rangle)}$

$e^{i\phi_0}(r_0|0\rangle + r e^{i(\phi_1 - \phi_0)}|1\rangle)$

$e^{i\phi_0}$ = global is no significance physically

$e^{i(\phi_1 - \phi_0)}$ relative phase. more physically significance

$\phi_1 - \phi_0 = \phi$

$|\psi\rangle = r_0|0\rangle + r_1 e^{i\phi}|1\rangle$

$|r_0|^2 + |r_1|^2 = 1$

$r_0 = \cos\frac{\theta}{2}$

$r_1 = \sin\frac{\theta}{2}$

$$\boxed{|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle}$$

only two parameter used i.e θ and ϕ

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ together with identity matrix

they form basis of real Hilbert space

$|\psi\rangle = \cos\frac{\theta}{2}|1\rangle + e^{i\phi}\sin\frac{\theta}{2}$

$\rho = |\psi\rangle\langle\psi|$

$$= \begin{pmatrix} \cos^2\theta & \frac{1}{2}e^{-i\phi}\sin\theta \\ \frac{1}{2}e^{i\phi}\sin\theta & \sin^2\theta \end{pmatrix}$$

$\rho = \frac{1}{2}[I + \sigma s]$

s = Bloch vector

$s = 1$ pure state

we have one to one mapping to surface of bloch sphere for mixed state

$0 < \theta < \pi$

$0 < \phi < 2\pi$

For +ve z

$\theta = \phi = 0$

$|\psi\rangle = \cos\theta|0\rangle + e^{i0}\sin\theta|1\rangle$

$$\boxed{|\psi\rangle = |0\rangle}$$

for -ve z $\quad \theta = \pi, \; \phi = 0$

$\cos\frac{\pi}{2}|0\rangle + e^{i(0)}\sin\frac{\pi}{2}|1\rangle$

$|\psi\rangle = |1\rangle$

circular basis

$y = \; \theta = \frac{\pi}{2}, \phi = \frac{\pi}{2} \leftrightarrow j \in \mathbb{C}$

$\cos\frac{\pi}{4}|0\rangle + e^{i\frac{\pi}{2}}\sin\frac{\pi}{4}|1\rangle$

$\frac{1}{\sqrt{2}}|0\rangle + \frac{j}{\sqrt{2}}$

$-y = \frac{1}{\sqrt{2}}|0\rangle - \frac{j}{\sqrt{2}}|1\rangle$

Diagonal basis

$x \rightarrow$ axis

$\phi = 0$

$\theta = \frac{\pi}{2}$

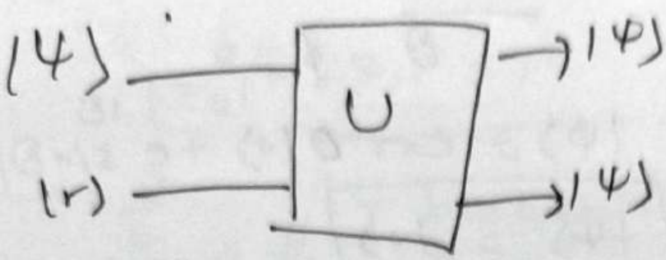
$|\psi\rangle = \cos\frac{\pi}{4}|0\rangle + e^{i}\sin\frac{\pi}{4}|1\rangle$

$= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

$|\psi\rangle = |+\rangle$

No cloning theorem



$$U^{T}U = I$$

$$U|\psi\rangle|r\rangle = |\psi\rangle|\psi\rangle$$

$$\psi_1 \Rightarrow U|\psi_1\rangle|r\rangle \longrightarrow |\psi_1\rangle|\psi_1\rangle$$

$$\psi_2 \Rightarrow U|\psi_2\rangle|r\rangle \longrightarrow |\psi_2\rangle|\psi_2\rangle$$

$$\langle\psi_1|\psi_2\rangle$$

$$r_1\langle\psi_1|\langle r|\underbrace{U^{\dagger}U}_{I}|\psi_2\rangle|r_2\rangle$$

$$\cancel{r\,|\psi\rangle}\ \underline{r_1}\langle\psi_1|\psi_2\rangle\underline{|r_2\rangle} \qquad |r|^2$$

$$r = r_1|0\rangle + r_2|1\rangle$$

$$\langle r|r\rangle$$

$$\underline{\underline{\langle\psi_1|\psi_2\rangle}}$$

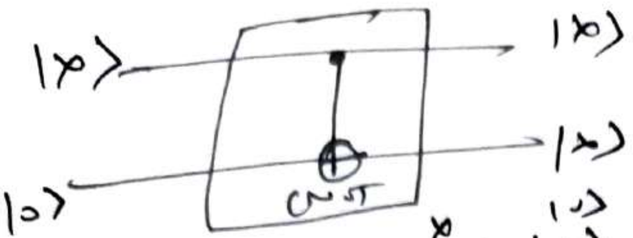
$$\begin{bmatrix} r_1 & r_2 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$\underline{\underline{(\langle\psi_1|\psi_2\rangle)^2}}$$

$a = a^2$ $\qquad$ $\rightarrow$ same $\Rightarrow r_1^2 + r_2^2 = 1$.

either $a = 0$

$a = 1 \rightarrow$ orthogonal.



$\text{CNOT}\ \overset{x}{|0\rangle}\overset{|0\rangle}{|0\rangle} \longrightarrow |0\rangle|0\rangle \text{ or } |00\rangle$

$\text{CNOT}\ |1\rangle|0\rangle \longrightarrow |1\rangle|1\rangle - |11\rangle$

$x = a|0\rangle + b|1\rangle$

$\text{CNOT}\left[(a|0\rangle + b|1\rangle)(|0\rangle) \right]$

$\text{CNOT}\left[a|00\rangle + b|10\rangle \right]$

$= a|00\rangle + b|11\rangle]$

$|xx\rangle$ expected outn

$(a|0\rangle + b|1\rangle)(a|0\rangle + b|1\rangle)$

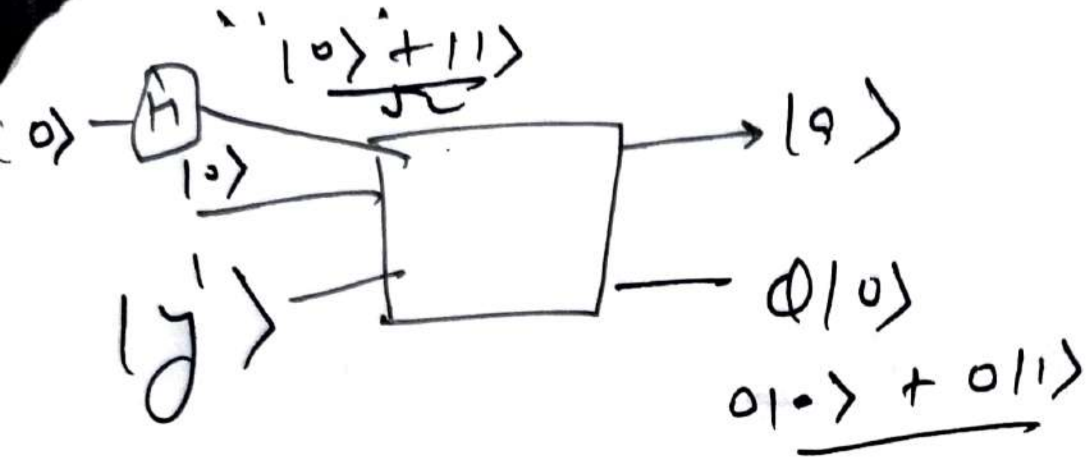
$= a^2|00\rangle + ab|01\rangle + ba|10\rangle + b^2|11\rangle$

$$|0\rangle \xrightarrow{\boxed{H}} \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \xrightarrow{\boxed{H}} |0\rangle$$

$$|1\rangle \xrightarrow{\boxed{H}} \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$\frac{|0\rangle - |1\rangle}{\sqrt{2}} \xrightarrow{\boxed{H}} |1\rangle$$



limitation of measurement

$\alpha|0\rangle + \beta|1\rangle$

$|0\rangle \quad |1\rangle$

Quantum Error Correction interference, emnoises

t sequential steps prob. p $0 < p < 1$

p^t if $p = 1$ error free

$p = 0$ max error.

- decrease exponentially as t grows.

classical error correction suppose that

b is a bit. Repeating state $b \rightarrow \underbrace{bbb}_{\text{codeword}}$

constraint in quantum computation

we cannot apply this in QC.

1. No clonning theorem
2. classical errors boil down to bit flip errors

In Q.C there are continuum many errors

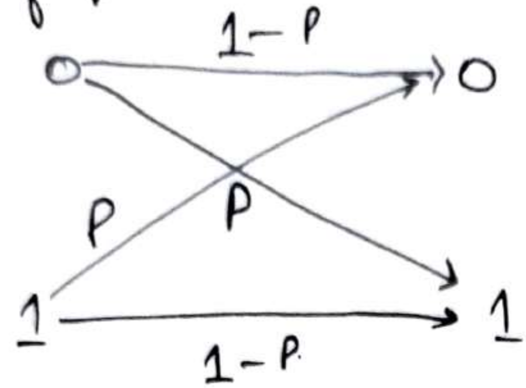
3. Measurement on quantum state lead to the collapse of the state

There are three stages in error correction

1. characterization of the error model (type of error)
2. Encoding (this introduce redundant information)
3. error recovery procedure

The Error Model

Example to a simple error model the bit flip error



classical bit flip channel

$$\mathcal{E}^c_i , = P_i$$

Encoding: we encode our collection of bits and obtain a codeword (encoded bits)

Suppose we have n-bits

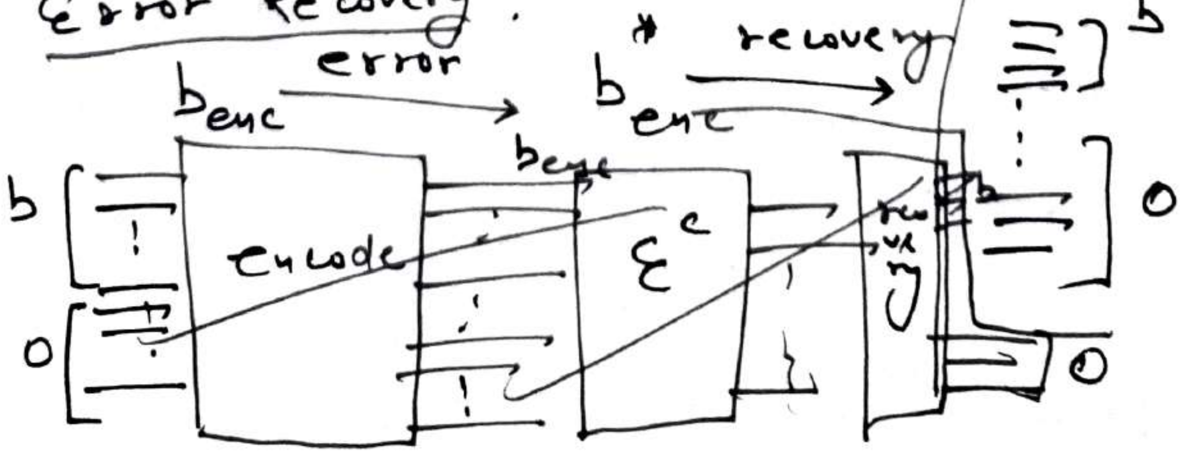
↓ encode

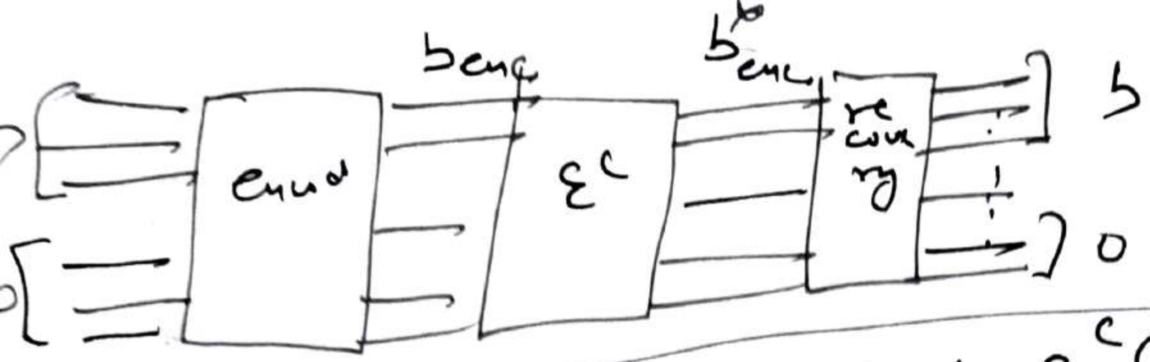
(n + m) - bit length

b : original bit we want to protect

$b_{enc.}$: codeword for b

Error Recovery:





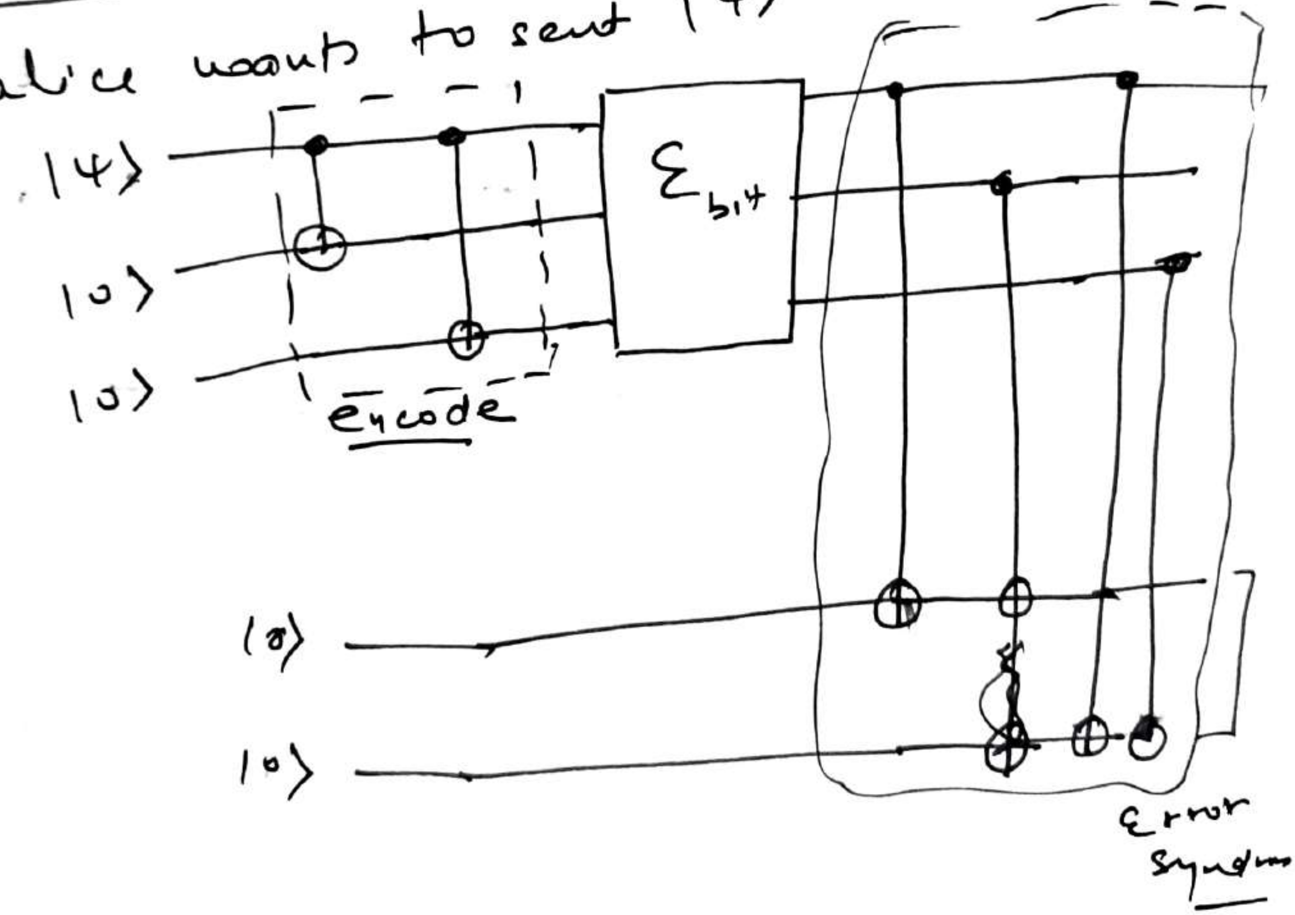
$$\mathcal{E}_i^c \qquad \mathcal{E}_i^c(k_{enc}) \neq \mathcal{E}_j^c(l_{enc})$$

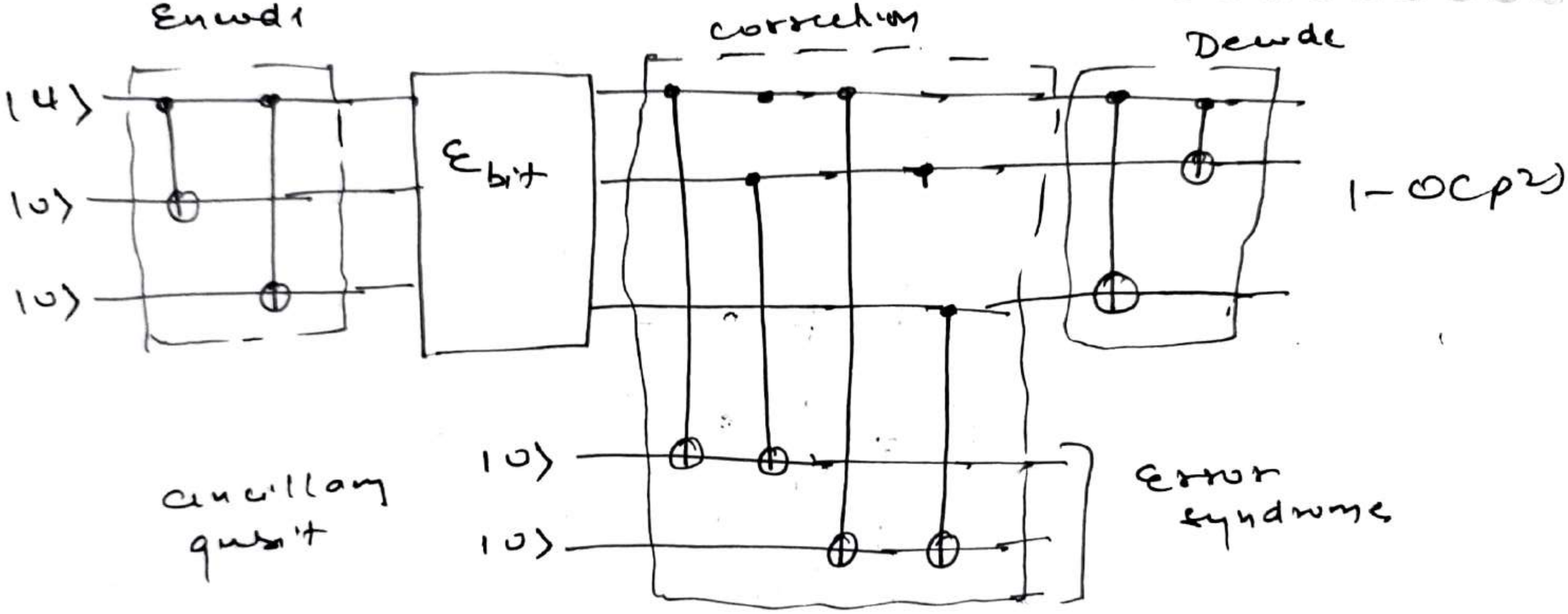
$$\forall k \neq l$$

essential conditions for error correct

Three qubit bit flip code

alice wants to sent $|\psi\rangle = a|0\rangle + b|1\rangle$




Encoder
correction
Decode
$|\psi\rangle$
$|0\rangle$
$|0\rangle$
$\mathcal{E}_{bit}$
$|0\rangle$
$|0\rangle$
ancillary qubit
error syndrome
$1 - O(p^2)$

alice wants to send $a|0\rangle + b|1\rangle$

$a|0\rangle + b|1\rangle (|00\rangle)$

$= a|000\rangle + b|100\rangle$

after the encoding stage we have.

$a|000\rangle + b|111\rangle$

$|0\rangle \longrightarrow |000\rangle$

$|1\rangle \longrightarrow |111\rangle$

state	probability	change
$a\|000\rangle + b\|111\rangle$ $\|00\rangle$	$(1-P)^3$	no change
$a\|100\rangle + b\|011\rangle$ $\|11\rangle$	$P(1-P)^2$	1^{st} bit
$a\|010\rangle + b\|101\rangle$ $\|10\rangle$	$P(1-P)^2$	2^{nd} bit
$a\|001\rangle + b\|110\rangle$ $\|01\rangle$	$P(1-P)^2$	3^{rd} bit
$a\|110\rangle + b\|001\rangle$ $\|01\rangle$	$P^2(1-P)$	1^{st} & 2^{nd} bits
$a\|101\rangle + b\|010\rangle$ $\|10\rangle$	$P^2(1-P)$	1^{st} & 3^{rd} bits.
$a\|011\rangle + b\|100\rangle$ $\|11\rangle$	$P^2(1-P)$	2^{nd} & 3^{rd} bits
$a\|111\rangle + b\|000\rangle$ $\|00\rangle$	P^3	all bits.

Measured Syndrom	action
0 0	do nothing
0 1	flip the third qubit
1 0	flip the 2^{nd} qubit
1 1	flip the first qubit

suppose Bob measure 10

$(a|0\rangle + b|1\rangle)|00\rangle$ or $(a|1\rangle + b|0\rangle)|00\rangle$

$1-P$

The prob. Bob fails to obtain alice state is $O(p^2)$ (with error correction)

$O(p)$ (w/o error correction)

Phase flip error

$a|0\rangle + b|1\rangle \longrightarrow a|0\rangle - b|1\rangle$

$a|0\rangle - b|1\rangle \longrightarrow a|0\rangle + b|1\rangle$

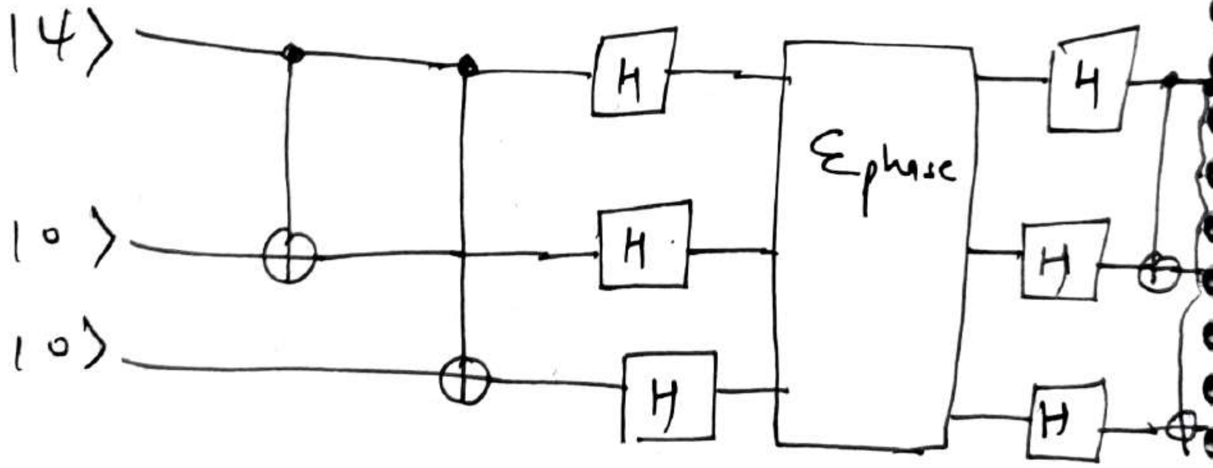
$e^{i\theta}$

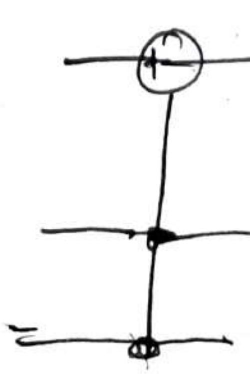
we ~~turn~~ the phase flip channel to a bit flip channel using change of basis. Suppose we ~~want~~ work in the

$|+\rangle, |-\rangle$

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$





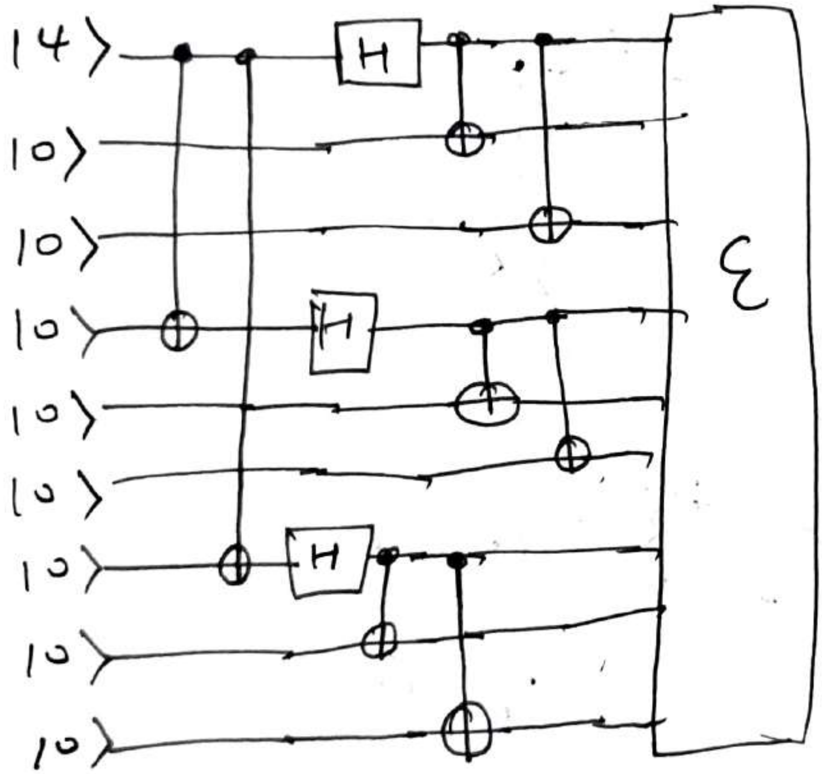
The shor code (9 qubit code)

$|0\rangle \longmapsto |+++\rangle$, $|1\rangle \longmapsto |---\rangle$

$|+\rangle \longmapsto \frac{(|000\rangle + |111\rangle)}{\sqrt{2}}$, $|-\rangle \longmapsto \frac{(|000\rangle - |111\rangle)}{\sqrt{2}}$

$$|0\rangle \longmapsto |0\rangle_L = \frac{\overset{|+\rangle}{(|000\rangle + |111\rangle)}\overset{|+\rangle}{(|000\rangle + |111\rangle)}\overset{|+\rangle}{(|000\rangle + |111\rangle)}}{2\sqrt{2}}$$

$$|1\rangle \longmapsto |0\rangle_L = \frac{(|000\rangle - |111\rangle)(|000\rangle + |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$



$Z \otimes Z \otimes I$ $\quad Z_1 Z_2$

$\lambda_1 = 1, \lambda_2 = -1$

compare the first and second qubit to tell whether they are different

next we compare the 2nd and the 3rd qubit

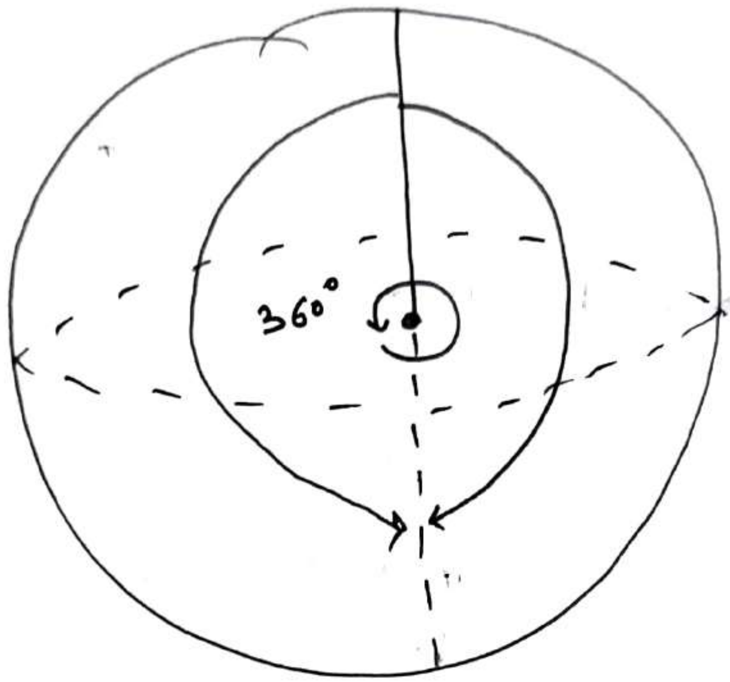
$Z_2 Z_3 : I \otimes Z \otimes Z$

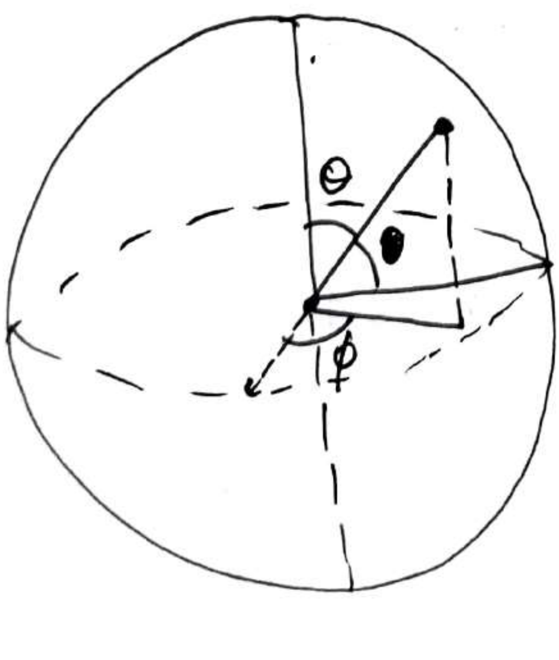
$(|000\rangle + |111\rangle)$ $(|000\rangle + |111\rangle)$

Blochsphere Representation of qubit

Prachodayan:- Enlightening Mind

Just like a sphere (Ball



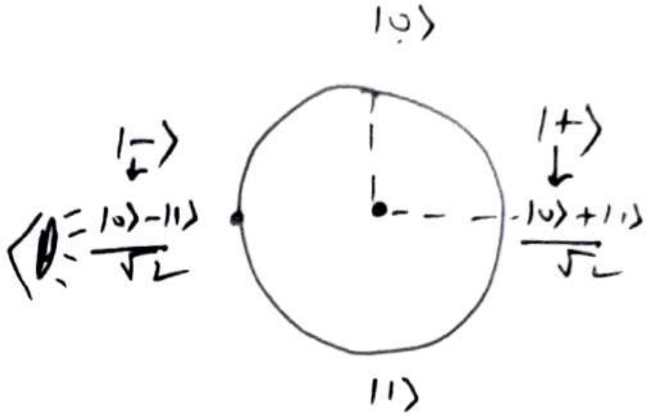


$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$

Based on this formula, you can predict any point on the sphere

Measurement in Basis other than computation
<u>Basis</u>

$|+\rangle \qquad |-\rangle$



$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ other way to write this eqn.

$|\psi\rangle = \left(\frac{\alpha + \beta}{2}\right)|+\rangle + \left(\frac{\alpha - \beta}{2}\right)|-\rangle$

$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

Gates output in states of other bases
$(|+\rangle, |-\rangle)$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \underbrace{1|0\rangle + 0|1\rangle}_{|0\rangle}$$

for $0|0\rangle + 1|1\rangle \qquad |1\rangle \rightarrow |0\rangle$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\boxed{\mathrm{Not}|+\rangle = |+\rangle}$

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = -\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$$

$\boxed{= -|-\rangle}$

CNOT $\begin{pmatrix} 0 & 0 \end{pmatrix} = 01$

$01 = 0$

$10 = 11$

$11 = 10$

CNOT$(|+\rangle, |0\rangle)$

$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)(|10\rangle + 0|1\rangle)$

$\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)(|10\rangle + 0|1\rangle)$

$\Rightarrow \frac{1}{\sqrt{2}}|00\rangle + \cancel{0|01\rangle} + \frac{1}{\sqrt{2}}|10\rangle + \cancel{0|11\rangle}$

$= \frac{1}{\sqrt{2}}|00\rangle + |10\rangle$

$\xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$

$= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

Quantum teleportation

from one place to another travering a particle nature with more than speed of light

Entanglement: They will not seprated at anycost

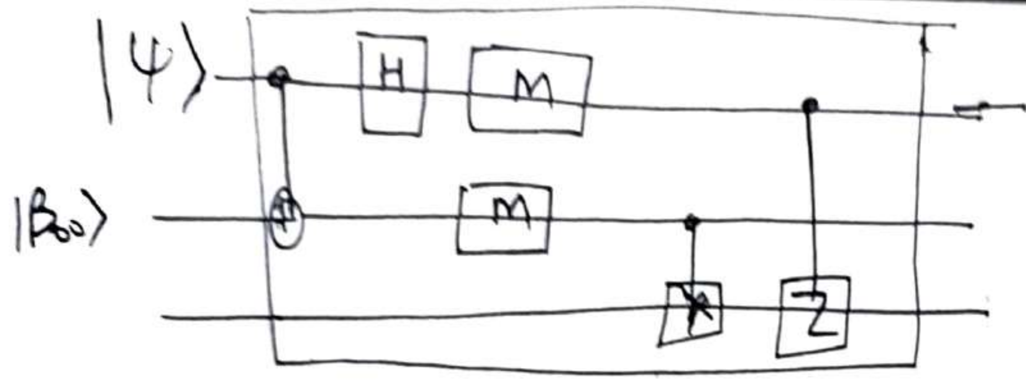
00, 11   0   0 / 1   1

01, 10   0   1 / 1   0

Bell states:

$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$, $\frac{|00\rangle - |11\rangle}{\sqrt{2}}$, $\frac{|01\rangle + |10\rangle}{\sqrt{2}}$, $\frac{|01\rangle - |10\rangle}{\sqrt{2}}$

entangled qubit value of one qubit depends on another qubit



$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ I/P $\Rightarrow (\alpha|0\rangle + \beta|1\rangle)\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)$

$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ $|a\rangle|b\rangle = |ab\rangle$

$\psi_0 = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$

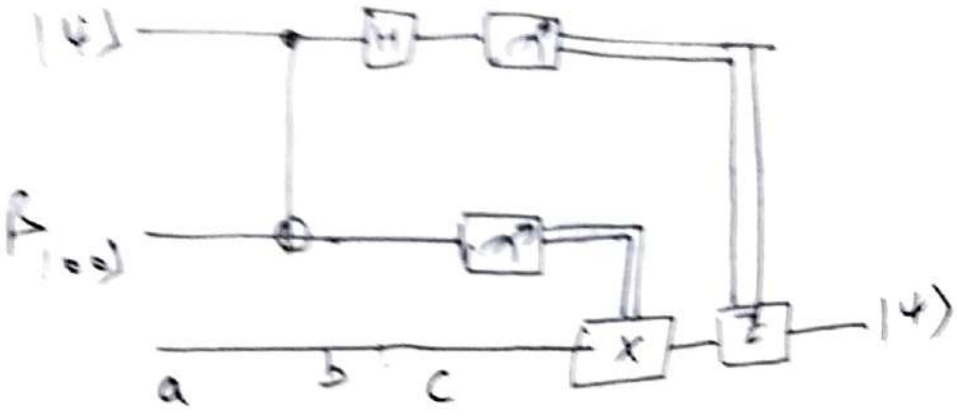
$|\psi_1\rangle \Rightarrow \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle)$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\alpha\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|00\rangle + \alpha\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|11\rangle + \beta\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]|10\rangle + \beta\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]|01\rangle$$

$$= \frac{1}{2}\alpha\left[|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle\right]$$

Take $|00\rangle$ as common, $|01\rangle$ as common

$$\Rightarrow \frac{1}{2}\Big[|00\rangle[\alpha|0\rangle + \beta|1\rangle] + |01\rangle[\alpha|1\rangle + \beta|0\rangle] + |10\rangle[\alpha|0\rangle - \beta|1\rangle] + |11\rangle[\alpha|1\rangle - \beta|0\rangle]\Big]$$



$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$|\beta_{00}\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$

$\alpha|0\rangle + \beta|1\rangle) \left(\dfrac{|00\rangle + |11\rangle}{\sqrt{2}}\right)$

$\Rightarrow \dfrac{\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle}{\sqrt{2}}$

$\frac{1}{\sqrt{2}}\left[\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle\right] \Rightarrow a$

step 1

$\Rightarrow \frac{1}{\sqrt{2}}\left[\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle\right] \Rightarrow b$

step 2:

$\frac{1}{\sqrt{2}}\left[\left[\alpha\frac{|0\rangle + \beta|1\rangle}{\sqrt{2}}\right]|00\rangle + \beta\left[\frac{\alpha|0\rangle + \beta|1\rangle}{\sqrt{2}}\right]|11\rangle\right.$

$\cancel{\alpha} \frac{1}{\sqrt{2}}\left[\alpha|0\rangle(|00\rangle + |11\rangle + \beta|1\rangle[|10\rangle + |01\rangle]\right]$

$= \frac{1}{\sqrt{2}}\left[\alpha\frac{|0\rangle + |1\rangle}{\sqrt{2}}(|00\rangle + |11\rangle + \beta\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right][|10\rangle + |01\rangle\right.$

$$c = \frac{1}{\sqrt{2}} = \Big[\alpha|000\rangle + \alpha|011\rangle + \alpha|100\rangle + \alpha|111\rangle + \beta|010\rangle + \beta|001\rangle + \beta|110\rangle - \beta|101\rangle\Big]$$

$\boxed{X}$

$$\frac{1}{2}\Big[|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)\Big]$$

$\boxed{Z}$

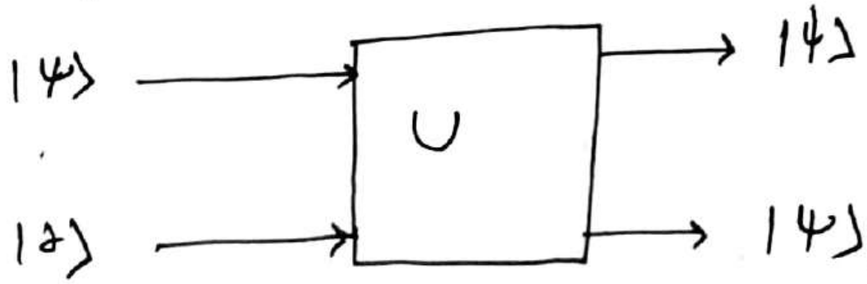
$\boxed{X}$ — $-\beta|0\rangle + \alpha|1\rangle$

$\boxed{Z}$ — $\alpha|0\rangle + \beta|1\rangle$

No cloning theorm — whether there is a chance of copying qubit

$|\psi\rangle \longrightarrow |0\rangle$ & $|1\rangle$ $\qquad |\psi\rangle \longrightarrow$ CNOT $\rightarrow |\psi\rangle|\psi\rangle$

$\hookrightarrow \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$



If U is valid quantum operation i.e

$$\boxed{U^{\dagger}U = I}$$

$$U = \begin{bmatrix} a+bi & c+di \\ e+fi & g+hi \end{bmatrix}$$

$$U^{+} = \begin{bmatrix} a-bi & c-di \\ e-fi & g-hi \end{bmatrix}$$

$c = \frac{1}{\sqrt{2}} = \Big[\alpha|000\rangle + \alpha|011\rangle + \alpha|100\rangle + \alpha|111\rangle +$

$\beta|010\rangle + \beta|001\rangle - \beta|110\rangle - \beta|101\rangle\Big]$

$\boxed{X}$

$\frac{1}{2}\Big[|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle)$

$+ |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)$

$\boxed{Z}$

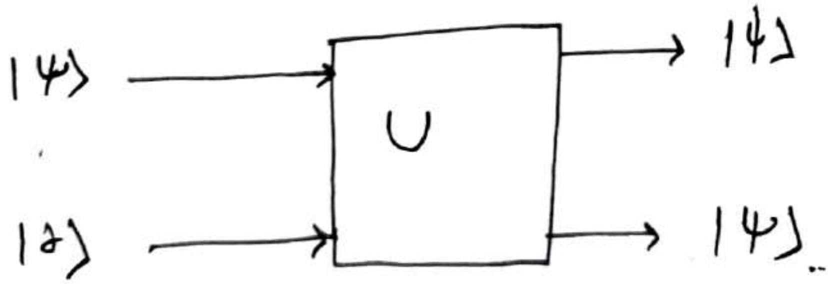
$\boxed{X}$ — $-\beta|0\rangle + \alpha|1\rangle$

$\boxed{Z}$ — $\alpha|0\rangle + \beta|1\rangle$

No cloning theorem whether there is a chance of copying qubit

$|\psi\rangle \longrightarrow |0\rangle$ & $|1\rangle$ $\qquad |\psi\rangle \longrightarrow CNOT \rightarrow |\psi\rangle|\psi\rangle$

$\hookrightarrow \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$



If U is valid quantum operation i.e.

$$\boxed{U^{\dagger}U = I}$$

$$U = \begin{bmatrix} a+bi & c+di \\ e+fi & g+hi \end{bmatrix}$$

$$U^{\dagger} = \begin{bmatrix} a-bi & c-di \\ e-fi & g-hi \end{bmatrix}$$

Inner product of qubits

$|\psi_1\rangle, |\psi_2\rangle$ $\qquad$ $\begin{bmatrix} a \\ b \end{bmatrix}$

$\langle\psi_1|\psi_2\rangle$ $\qquad$ $\begin{bmatrix} a & b \end{bmatrix}$

$\langle\psi^{\dagger}|\psi\rangle$

(i) $|\psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$ & $|\psi_2\rangle = \begin{bmatrix} c \\ d \end{bmatrix}$

$\langle\psi_1|\psi_2\rangle \Rightarrow \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = ac + bd.$

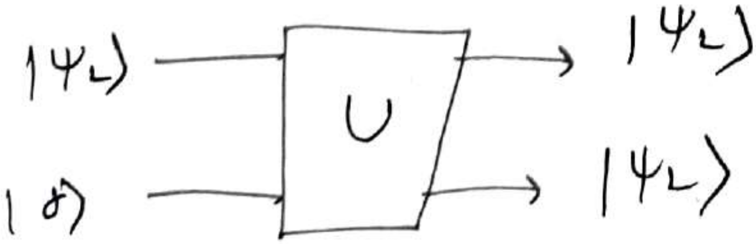
(ii) $|\psi\rangle = \begin{bmatrix} a + bi \\ c + di \end{bmatrix}$ $\quad$ $|\psi_2\rangle = \begin{bmatrix} e + fi \\ g + hi \end{bmatrix}$

$\langle\psi_1|\psi_2\rangle = \begin{bmatrix} a - bi & c - di \end{bmatrix} \begin{bmatrix} e + fi \\ g + hi \end{bmatrix}$

$\Downarrow$

scalar

No cloning theorm



$U|\psi_1\rangle|\gamma\rangle = |\psi_1\rangle|\psi_1\rangle$ — ①

$U|\psi_2\rangle|\gamma\rangle = |\psi_2\rangle|\psi_2\rangle$ — ②

Inner product of ① and ② we get

$|a\rangle|b\rangle = \langle a^\dagger|b\rangle$

$|a\rangle|b\rangle\ |c\rangle|d\rangle = \langle b|\langle a^\dagger|c\rangle|d\rangle$

$\langle\gamma|\langle\psi_1|\underbrace{U^\dagger U}_{I}|\psi_2\rangle|\gamma\rangle = \langle\psi_1|\langle\psi_1|\psi_2\rangle|\psi_2\rangle$

if U is valid quantum operation $U^\dagger U = I$

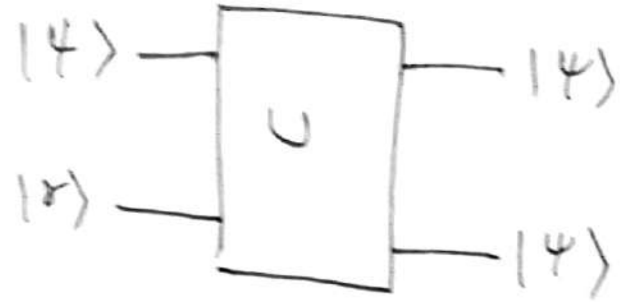
$\langle\gamma|\langle\psi_1|\psi_2\rangle|\gamma\rangle = (\langle\psi_1|\psi_2\rangle)^2$

Inner product of $\langle\gamma|\gamma\rangle$ $|\gamma\rangle$ is a valid qubit $[\alpha\ \beta]\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \alpha^2+\beta^2$

if $a|0\rangle + b|1\rangle$
then $a^2+b^2=1$

$\langle\psi_1|\psi_2\rangle = (\langle\psi_1|\psi_2\rangle)^2$

$a = a^2$ when $a=0$:- same qubits
$a=1$:- orthogonal to each other



$U^{\dagger}U = I$ if this condition satisfy then valid operation occur

<u>Proof</u> $U|\psi\rangle|r\rangle \longrightarrow |\psi\rangle|\psi\rangle$

$\psi_1 \Rightarrow U|\psi_1\rangle|r\rangle \longrightarrow |\psi_1\rangle|\psi_1\rangle$

$\psi_2 \Rightarrow U|\psi_2\rangle|r\rangle \longrightarrow |\psi_2\rangle|\psi_2\rangle$

$$\boxed{\begin{array}{ll} |r_1\rangle\,|\alpha_1\rangle & |r_2\rangle\,|\alpha_2\rangle \\ \langle\alpha_1|\cancel{\alpha_2}\rangle & \langle r_1|r_1|\,r_2|r_2\rangle \end{array}}$$

$\langle r|\langle\psi_1|\underline{U^{\dagger}U}|\psi_2\rangle|r\rangle = \langle\psi_1|\langle\psi_1|\psi_2\rangle|\psi_2\rangle$

$U^{\dagger}U = I \quad |r_1\rangle|r_2\rangle = \langle r_1|r_2\rangle$

$\langle r|\langle\psi_1|I|\psi_2\rangle|r\rangle = (\langle\psi_1|\psi_2\rangle)^2$

$\langle r|\langle\psi_1|\psi_2\rangle|r\rangle$ — ①

product of r
$|r\rangle$ is a product or blank paper

$r = r_1|0\rangle + r_2|1\rangle \qquad \begin{bmatrix} r_1 & r_2 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$

$= \langle r|r\rangle$

$= \gamma_1^2 + \gamma_2^2 = 1$

ignore value in eq (1)

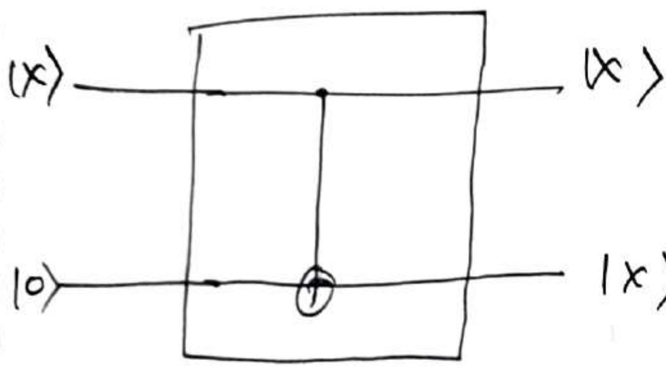
$\langle \psi_1 | \psi_2 \rangle = \langle \psi_1 | \psi_2 \rangle^2$ — (2)

if ψ_1 and ψ_2 can be cloned then according relation eq. (2) full fill. this is occur only when ~~ψ_1 = 0 or~~ $a = a^2$

$a = 1$ same so it means

0 orthogonal it

No cloning theorem part (2)



$$\text{CNOT } \overset{x}{|0\rangle} \overset{|0\rangle}{|0\rangle} \longrightarrow |0\rangle|0\rangle \text{ or } |00\rangle$$

$$\text{CNOT } \overset{x}{|1\rangle} \overset{|0\rangle}{|0\rangle} \longrightarrow |1\rangle|1\rangle \text{ or } |11\rangle$$

$x = a|0\rangle + b|1\rangle$

$$\text{CNOT } \overset{x}{[a|0\rangle + b|1\rangle]} \overset{|0\rangle}{|0\rangle} \longrightarrow$$

$\text{CNOT } a|00\rangle + b|10\rangle$

$= a|00\rangle + b|11\rangle$

$|x\,x\rangle$ expected

$[a|0\rangle + b|1\rangle]$

$[a|0\rangle + b|1\rangle]$

$a^2|00\rangle + ab|01\rangle +$
$ab|10\rangle + b^2|11\rangle$

Classical operations on a quantum computer

Classical computation on a quantum computer

Toffoli Gate or CCNOT gate

CNOT $|01\rangle = |01\rangle$

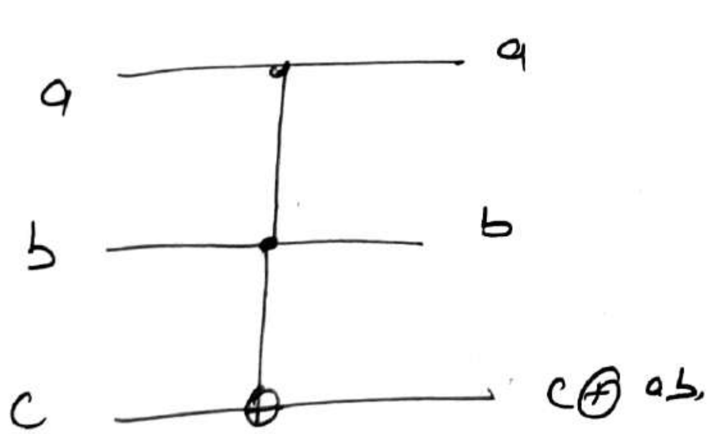
$|11\rangle = |10\rangle$

CCNOT $|xyz\rangle$ z = z → flip if x, y both are 1

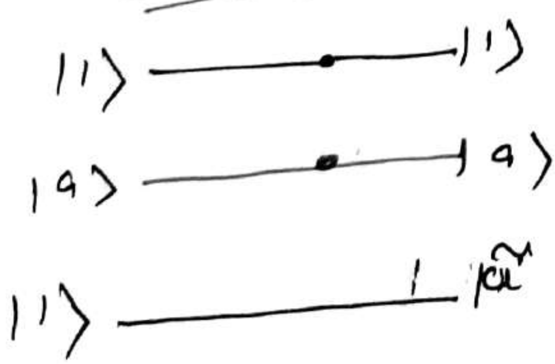
$|011\rangle = |011\rangle$

$|000\rangle = |000\rangle$

$|110\rangle = |111\rangle$



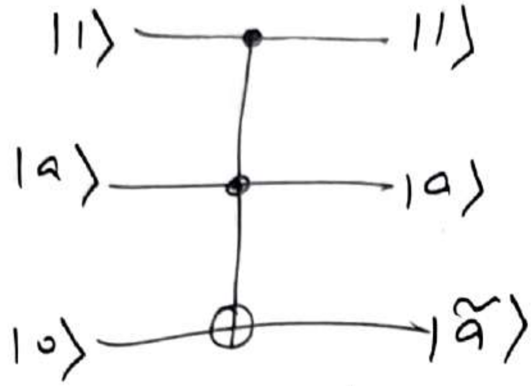
Not gate



if a = |0⟩
o/p = 1

if a = |1⟩
o/p = 0

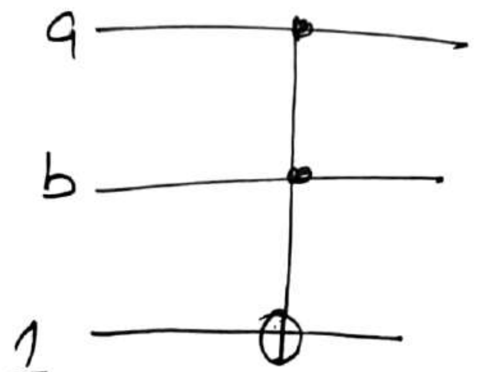
Fanout



a = |0⟩
o/p = |0⟩

a = |1⟩
o/p = |1⟩

Nand Gate



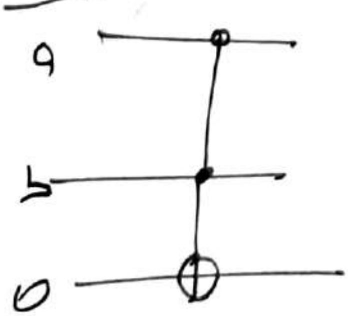
a = 0
b = 0
o/p = 1

If a = 0, b = 1
o/p = 1

If a = 1, b = 0
o/p = 1

If a = 1, b = 1
o/p = 0

AND gate



a b	o/p
0 0	0
0 1	0
1 0	0
1 1	1

" 1

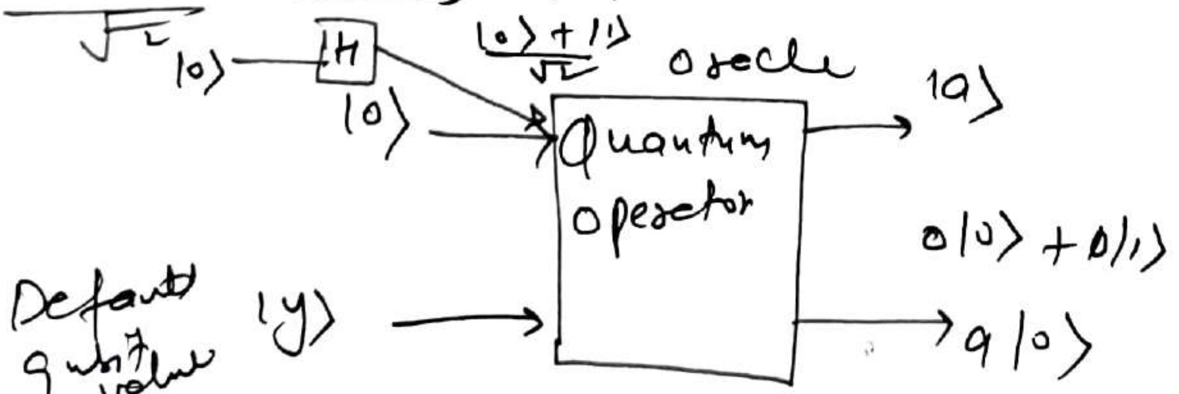
OR Gate $A+B = (\bar{A}\cdot\bar{B})'$

Quantum parallelism more efficient than classical computation due to parallelism $|0\rangle \xrightarrow{H} \frac{|0\rangle+|1\rangle}{\sqrt{2}} \; (|+\rangle)$

$$\frac{|0\rangle+|1\rangle}{\sqrt{2}} \xrightarrow{H} |0\rangle$$

$$|1\rangle \xrightarrow{H} \frac{|0\rangle-|1\rangle}{\sqrt{2}} \; (|-\rangle)$$

$$\frac{|0\rangle-|1\rangle}{\sqrt{2}} \xrightarrow{H} |1\rangle$$



$|00\rangle \rightarrow \boxed{H} \rightarrow$

00
01
10
11

Limitation Measurement

$\alpha|0\rangle + \beta|1\rangle$

Super dense coding

$|\psi_1\rangle = |00\rangle$

$$|\psi_2\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Alice 00 $|\psi_3\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ (1)

01: $|\psi_3\rangle = X|\psi_2\rangle$ $\qquad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$= \frac{|10\rangle + |01\rangle}{\sqrt{2}}$$

$X|1\rangle = |0\rangle$

$X|0\rangle = |1\rangle$

10: $|\psi_3\rangle = Z|\psi_2\rangle$ $\qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$|\psi_3\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$Z|0\rangle = |0\rangle$

11: $|\psi_3\rangle = X(Z|\psi_2\rangle)$ $\qquad Z|1\rangle = -|1\rangle$

$$= \frac{X|00\rangle - X|11\rangle}{\sqrt{2}} = \frac{|10\rangle - |01\rangle}{\sqrt{2}}$$

s b

$|\psi_4\rangle$ on eq (1)

$$\text{CNOT}\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)$$

$$= \frac{|00\rangle + |10\rangle}{\sqrt{2}}$$

$$= \frac{|0\rangle + |1\rangle}{\sqrt{2}} |0\rangle + \frac{|0\rangle - |1\rangle}{2} |0\rangle$$

$$= \frac{|00\rangle + \cancel{|10\rangle} + |00\rangle - \cancel{|10\rangle}}{2}$$

$$= \frac{\cancel{2}|00\rangle}{\cancel{2}} = |00\rangle$$

$$|\psi_4\rangle = H\left(\frac{|11\rangle + |01\rangle}{\sqrt{2}}\right)$$

$$|\psi_4\rangle = \frac{|0\rangle - |1\rangle}{2} |1\rangle + \frac{|0\rangle + |1\rangle}{\sqrt{2}} |1\rangle$$

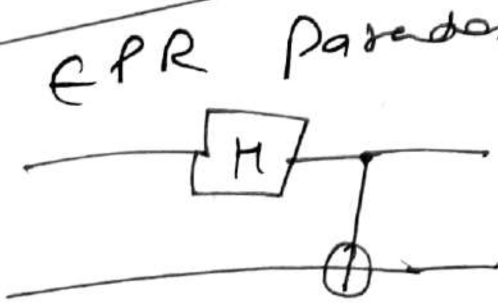
$$= \frac{|01\rangle - \cancel{|11\rangle}}{2} + \frac{|01\rangle + \cancel{|11\rangle}}{2}$$

$$= \frac{\cancel{2}|01\rangle}{\cancel{2}} = |01\rangle$$

lly $|\psi_4\rangle = |10\rangle$, $\underline{\underline{|11\rangle}}$

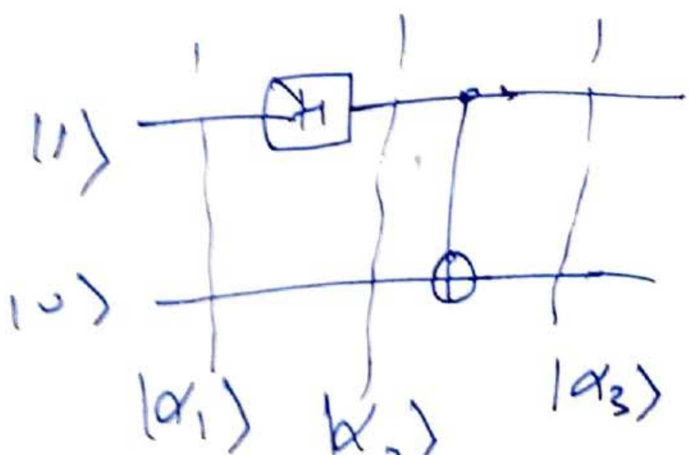
How to create Bell state

EPR Paradox



$$|xy\rangle \longrightarrow \frac{|0y\rangle + (-1)^x|1\bar{y}\rangle}{\sqrt{2}}$$

Input	Bellstate	
$\|00\rangle$	$\frac{\|00\rangle + \|11\rangle}{\sqrt{2}}$	$\|\phi^+\rangle$
$\|01\rangle$	$\frac{\|01\rangle + \|10\rangle}{\sqrt{2}}$	$\|\psi^+\rangle$
$\|10\rangle$	$\frac{\|00\rangle - \|11\rangle}{\sqrt{2}}$	$\|\phi^-\rangle$
$\|11\rangle$	$\frac{\|01\rangle - \|10\rangle}{\sqrt{2}}$	$\|\psi^-\rangle$



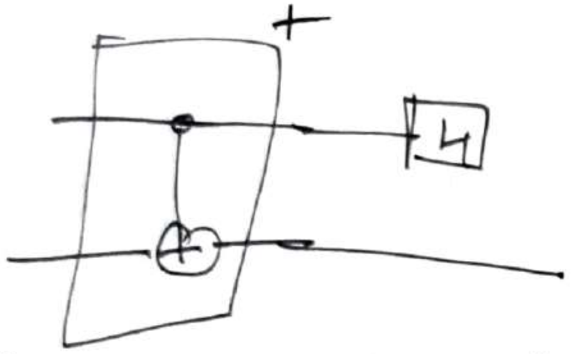
$$|\alpha_1\rangle = |10\rangle$$

$$|\alpha_2\rangle = H|1\rangle\,|0\rangle$$

$$= \frac{|0\rangle - |1\rangle}{\sqrt{2}}\,|0\rangle$$

$$= \frac{|00\rangle - |10\rangle}{\sqrt{2}}$$

$$|\alpha_3\rangle = \frac{|01\rangle - |11\rangle}{\sqrt{2}}$$



Quantum algorithms.

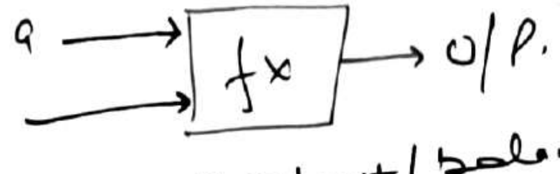
Deutsch algorithm

Problem: Given a logical function $f(b)$ constant or balanced

a	b	O/P1	O/P2	O/P3	O/P4	
0	0	0	1	0	1	0
0	1	0	1	0	1	1
1	0	0	1	1	0	1
1	1	0	1	1	0	0

constant functions (O/P1, O/P2)

balanced functions (O/P3, O/P4)

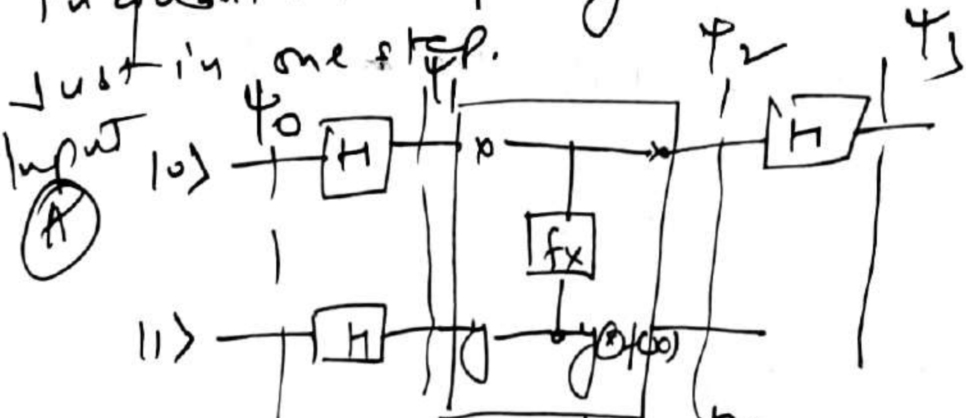


constant / balanced ?

for classical computation $\frac{4}{2}+1$

i.e $\frac{2^n}{2}+1$

in quantum computing this can be predicted just in one step.



Deutsch algorithm

A	O/P$_1$	O/P$_2$	O/P$_3$	O/P$_4$
0	0	1	0	1
1	0	1	1	0

constant | balanced function

$|\psi_0\rangle = |0\rangle \otimes |1\rangle = |01\rangle$

$|\psi_1\rangle = \left[\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]$

$$= \left[\frac{|00\rangle + |01\rangle + |10\rangle - |11\rangle}{2}\right]$$

$$\boxed{= \frac{1}{2}\left[|00\rangle - |01\rangle + |10\rangle - |11\rangle\right]}$$

If $f(0)$ is constant

if $f(0) = 0$

$$\frac{1}{2}\left[|00\rangle - |01\rangle + |10\rangle + |11\rangle\right]$$

$$= \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$|+\rangle \; |-\rangle$

If $f(0)$ is constant and equal to one

$f(0) = 1$ (constant)

$$\frac{1}{2}\left[|01\rangle - |00\rangle + |11\rangle - |10\rangle\right]$$

$$= \frac{1}{2}\left[-|0\rangle(|0\rangle - |1\rangle) - |1\rangle(|0\rangle - |1\rangle)\right]$$

$$= \frac{1}{2}\left[-(|0\rangle + |1\rangle)(|0\rangle - |1\rangle)\right.$$

$$\Rightarrow -\frac{(|0\rangle + |1\rangle)}{\sqrt{2}} \frac{(|0\rangle - |1\rangle)}{\sqrt{2}}$$

$$= -|+\rangle|-\rangle$$

if $f(x)$ is constant

$|\psi\rangle = \pm |+\rangle |-\rangle$ state

if $f(x)$ is balanced

$f(0) = 0 , f(1) = 1$

$I/P = \frac{1}{2}\left[|00\rangle - |01\rangle + |10\rangle - |11\rangle\right]$

CNOT

$= \frac{1}{2}\left[|00\rangle - |01\rangle + |11\rangle - |10\rangle\right]$

$= \frac{1}{2}\left[|0\rangle(|0\rangle - |1\rangle) - |1\rangle(|1\rangle - |0\rangle)\right]$

$= \frac{1}{2}\left[|0\rangle - |1\rangle \quad |0\rangle - |1\rangle)\right]$

$= \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] = |-\rangle|-\rangle$

if $f(0) = 1 , f(1) = 0$

$= \frac{1}{2}\left[|01\rangle - |00\rangle + |10\rangle - |11\rangle\right]$

$= \frac{1}{2}\left[-|0\rangle(|0\rangle - |1\rangle) + |1\rangle[|0\rangle - |1\rangle\right]$

$= \frac{1}{2} \; -(|0\rangle - |1\rangle) \; (|0\rangle - |1\rangle)$

$= \frac{1}{2} \; -\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \; \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

$- |-\rangle |-\rangle$

If $f(x)$ is constant

$|\psi_2\rangle = \pm |+\rangle |-\rangle$

If $f(x)$ is balanced

$|\psi_2\rangle = \pm |-\rangle |-\rangle$

o/p $\Rightarrow \pm |0\rangle |-\rangle$ if $f(x)$ is constant

$\Rightarrow \pm |1\rangle |-\rangle$ if $f(x)$ is balanced

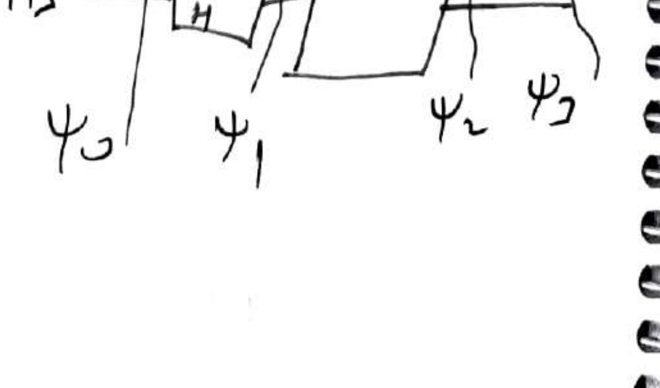
$\pm f(0) \oplus f(1) \, [|-\rangle]$

Deutsch's Jozsa algorithm

Generalized version of Deutsch's algorithm

Deutch Algo

DJ Algo



DJ → Constant → 0_s or 1_s

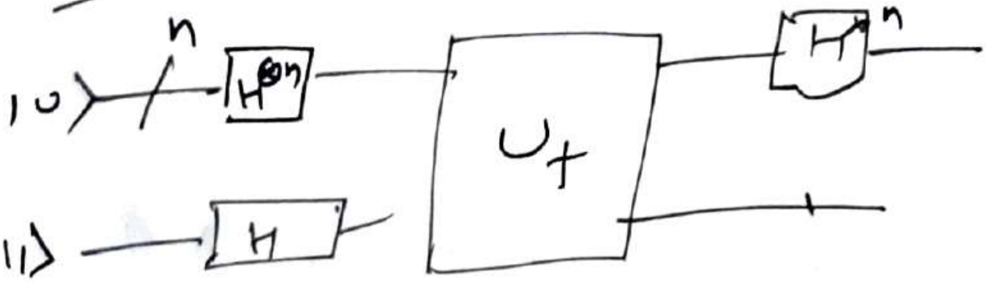
DJ → balanced → same equal no. of 0_s and 1_s

A	O/P	O/P	O/P	O/P
0	0	1	0	1
1	0	1	1	0

A	B						
0	0	0	1	0	1	1	0
0	1	0	1	0	1	0	1
1	0	0	1	1	0	1	0
1	1	0	1	1	0	0	1

~~constant~~ Balanced

DJA Circuit



let $n = 2$

let $f(x)$ be constant, $O/P = 0$,

$$|\psi_0\rangle = |001\rangle$$

$$|\psi_1\rangle = \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$\Rightarrow \left[\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$|\psi_1\rangle = \frac{|000\rangle - |001\rangle + |010\rangle - |011\rangle + |100\rangle - |101\rangle + |110\rangle - |111\rangle}{2\sqrt{2}}$$

$$|\psi\rangle = \frac{|000\rangle - |001\rangle + |010\rangle - |011\rangle + |100\rangle - |101\rangle + |110\rangle - |111\rangle}{2\sqrt{2}}$$

$f(00) = 0 \qquad f(10) = 0$

$f(01) = 0 \qquad f(11) = 0$

$$|\psi_2\rangle = \frac{|000\rangle - |001\rangle + |010\rangle - |011\rangle + |100\rangle - |101\rangle + |110\rangle - |111\rangle}{2\sqrt{2}}$$

$$|\psi_3\rangle = \frac{|00\rangle(|0\rangle - |1\rangle) + |01\rangle(|0\rangle - |1\rangle) + |11\rangle(|0\rangle - |1\rangle) + |10\rangle(|0\rangle - |1\rangle)}{2\sqrt{2}}$$

$= |+\rangle |+\rangle |-\rangle$

$= |0\rangle |0\rangle |\blacksquare\rangle$

If $f(x)$ is constant $f(x) = 1$

$$|\psi_2\rangle = \frac{(|001\rangle - |000\rangle + |011\rangle - |010\rangle + |101\rangle - |100\rangle + |111\rangle - |110\rangle}{2\sqrt{2}}$$

$$|\psi_3\rangle = \frac{|00\rangle(|1\rangle - |0\rangle + |01\rangle(|1\rangle - |0\rangle) + |10\rangle(|1\rangle - |0\rangle) + |11\rangle(|1\rangle - |0\rangle}{2\sqrt{2}}$$

$= |++\rangle(|1\rangle - |0\rangle)$
$- |00\rangle |-\rangle$

Balanced function $f(00) = 0$ $f(10) = 1$
$f(01) = 0$ $f(11) = 1$

$$|\psi\rangle = \frac{|000\rangle - |001\rangle + |010\rangle - |011\rangle + |101\rangle - |100\rangle + |111\rangle - |110\rangle}{2\sqrt{2}}$$

$= |00\rangle |-\rangle + |01\rangle (|-\rangle - |10\rangle |-\rangle - |11\rangle |-\rangle$

$$= \frac{|00\rangle + |01\rangle - |10\rangle - |11\rangle}{2} (|-\rangle)$$

$$= \left(\frac{|0\rangle}{\sqrt{2}} |+\rangle - \frac{|1\rangle}{\sqrt{2}} |+\rangle\right) |-\rangle$$

$|-\rangle |+\rangle |-\rangle$ $= 10 |-\rangle$

① $|0\rangle^{\otimes n}$ $|1\rangle$

② $\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n - 1} |x\rangle \left[|-\rangle\right]$

③ $\sum_{x=0}^{2^{n}-1} (-1)^{f(x)} |x\rangle |-\rangle$

④ $\sum_{z=0}^{2^n - 1} \sum_{x=0}^{2^n-1} \frac{(-1)^{x \cdot z + f(x)}}{\sqrt{2^n}} |z\rangle |-\rangle$

$|0\rangle^{\otimes n}$
$\Rightarrow \sum_{i=0}^{2^n - 1} \frac{1}{\sqrt{2^n}} |i\rangle$

$|0\rangle^{\otimes 2}$
$$\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{\sqrt{2^2}}$$
$$\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$$

(5) Measure & constant or balanced

Grovers algorithm search element

in unorderd list

O(N)

-2 0 5 -10 14 8 15 20

Binary ordered the element

(1) N :- No. of elements

$O(\sqrt{N})$

No. of elements

(2) n :- no. of qubit Required to store the element

N	n
2	1
4	2
8	3
16	4

$N = 2^n$

8, 10, 12, 15

(3)

Address		element				op	a'b'c'd'
x	y	a	b	c	d		
0	0	1	0	0	0	0	
0	1	1	0	1	0	0	
1	0	1	1	0	0	1	
1	1	1	1	1	1	0	

search 12 and adding value
a.b c'd!.

$$f(12) = by'$$

(3) $I_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $I_4 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$I_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(4) $|s\rangle = \begin{bmatrix} +\frac{1}{\sqrt{2}} \\ +\frac{1}{\sqrt{2}} \end{bmatrix}$ $|s\rangle\langle s| = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$2|s\rangle\langle s| = 2\begin{bmatrix} \frac{1}{\cancel{2}} & \frac{1}{\cancel{2}} \\ \frac{1}{\cancel{2}} & \frac{1}{\cancel{2}} \end{bmatrix}$$

(5) $|00\rangle \longrightarrow H \longrightarrow \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$

$$|000\rangle \xrightarrow{H} \frac{|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle}{\sqrt{8}}$$

search 12 and add values

a.b c'd'.

$f(12) = by'$

(3) $I_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $I_4 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$I_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(4) $|s\rangle = \begin{bmatrix} +\frac{1}{\sqrt{2}} \\ +\frac{1}{\sqrt{2}} \end{bmatrix}$ $|s\rangle\langle s| \Rightarrow \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$2|s\rangle\langle s| = 2\begin{bmatrix} \cancel{\frac{1}{2}} & \cancel{\frac{1}{2}} \\ \cancel{\frac{1}{2}} & \cancel{\frac{1}{2}} \end{bmatrix}$

(5) $|00\rangle \longrightarrow H \longrightarrow \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

$|000\rangle \xrightarrow{H} \frac{|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle}{\sqrt{8}}$

$$|0^{\otimes 4}\rangle \xrightarrow{H^{\otimes 4}} \sum_{i=0}^{2^4-1} \frac{1}{\sqrt{4}} \cdot |i\rangle$$

$$U_w = \frac{\overset{a}{|00\rangle} + \overset{b}{|01\rangle} + \overset{c}{|10\rangle} + \overset{d}{|11\rangle}}{2}$$

$f(x) \rightarrow$ element to be search

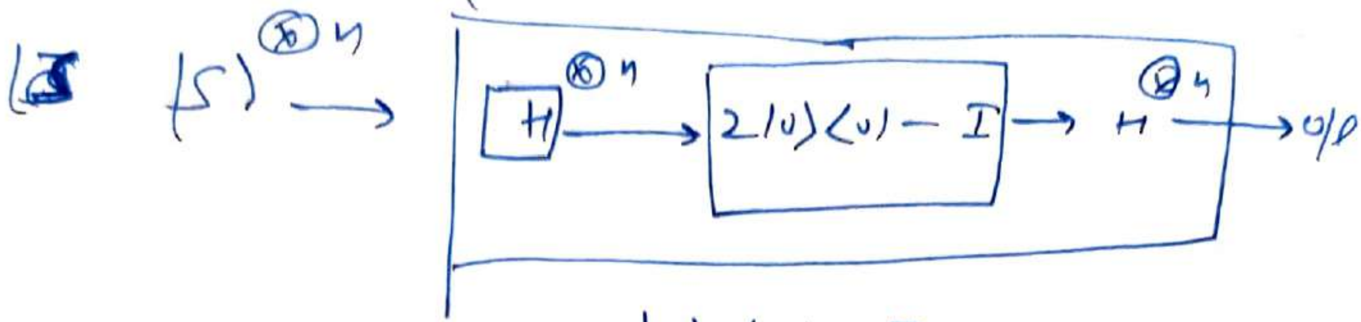
$f(x) = 1$ if $x = s$ — search element
$ 0$ if $x \neq s$

$$\begin{bmatrix} (-1)^{f(x)} & 0 & 0 & 0 \\ 0 & (-1)^{f(01)} & 0 & 0 \\ 0 & 0 & (-1)^{f(10)} & 0 \\ 0 & 0 & 0 & (-1)^{f(11)} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

Grovers Diffusion operator



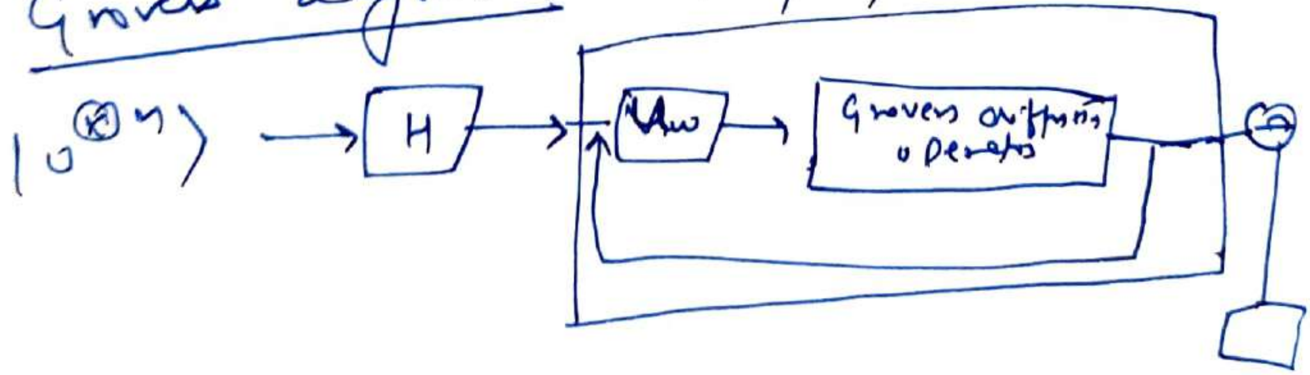
$$2|s\rangle\langle s| - I$$

$$|s\rangle = 2\begin{pmatrix}\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\end{pmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix} - I_2$$

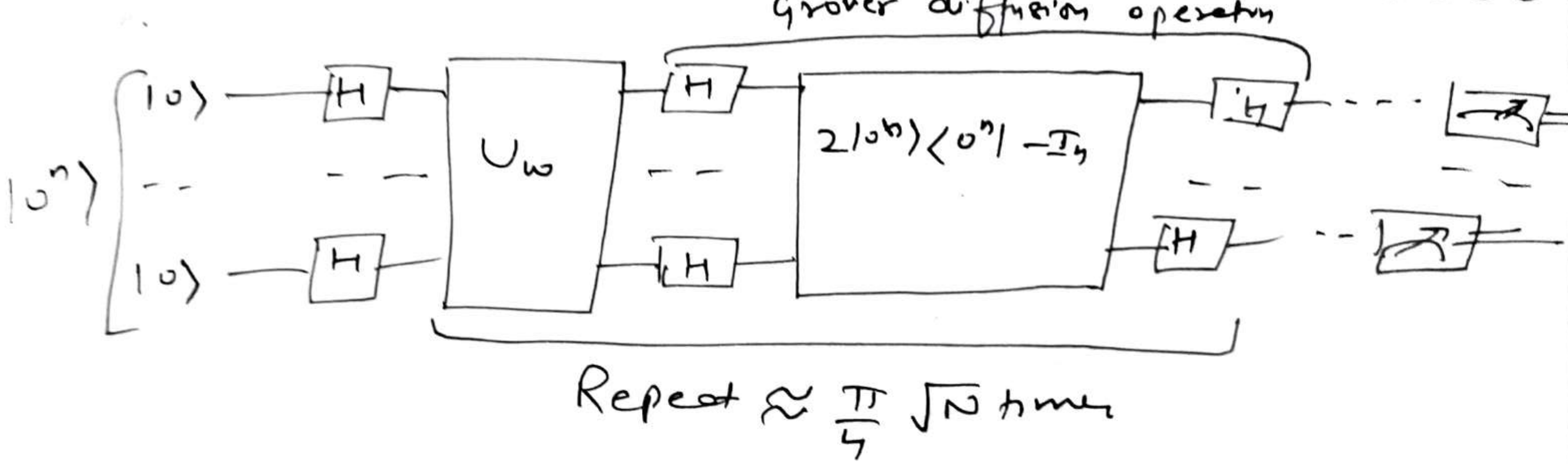
$$= 2\begin{bmatrix}\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\end{bmatrix} - I = \begin{bmatrix}1 & 1 \\ 1 & 1\end{bmatrix} - \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}$$

Grovers algorithm N, M



$t\sqrt{N}$ times $\Longrightarrow$ Grovers iteration


Grover diffusion operator
$|0^n\rangle$
$|0\rangle$
$|0\rangle$
H
U_ω
$2|0^n\rangle\langle 0^n| - I_n$
Repeat $\approx \frac{\pi}{4}\sqrt{N}$ times

Numerical example of grovers algo

$$\begin{array}{cccccccc} 8 & 4 & 12 & 3 & 0 & 10 & 11 & 15 \end{array}$$

$$\frac{1}{\sqrt{8}}\left[|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle\right]$$

$|000\rangle - \boxed{H} \rightarrow (\quad)$ $\sqrt{N}$ times repetition

$\underline{\underline{0}}$ $\quad + \underline{\underline{1}} = 0.99.$

0.001

Diffusion operator $\left[2|s\rangle\langle s| - I\right]$

soln → 1

non-soln → 0

$$\frac{1}{2}\left[\overset{a}{\widetilde{|00\rangle}} + \overset{b}{\widetilde{|01\rangle}} + \overset{c}{\widetilde{|10\rangle}} + \overset{d}{\widetilde{|11\rangle}}\right]$$

$\cancel{0|00\rangle} + 1|01\rangle$ ~~$+ 0|10\rangle + 0|11\rangle$~~

$\boxed{01}$ $\quad U_w \rightarrow$ sign ~~inverted~~ inverted at soln

$$\begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 \end{array}$$

avg $\Rightarrow \dfrac{\frac{1}{2} - \frac{1}{2} + \frac{1}{2} + \frac{1}{2}}{4} \Rightarrow \dfrac{1}{4}$

$\delta_{00} \Rightarrow$ avg − prev. coeff $\Rightarrow \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$

$\delta_{01} = \frac{1}{4} - \left(-\frac{1}{2}\right) \Rightarrow \frac{3}{4}$

$\delta_{10} \Rightarrow -\frac{1}{4}$

$\delta_{11} \Rightarrow -\frac{1}{4}$

coeff $\Rightarrow$ avg + δ $\qquad \cancel{\frac{1}{4}} + \cancel{-\frac{1}{4}}$

$$\frac{1}{4} + \frac{3}{4} = 1$$

avg coef

δ = avg − prev. coeff

updated coeff :- avg + δ

$\sqrt{N} \Rightarrow \approx I \Rightarrow$ coeff of sol

$\approx 0 \Rightarrow$ coeff of non sol

eg

$$|0\rangle^{\otimes n} \longrightarrow H^{\otimes n} \longrightarrow U_\omega \longrightarrow 2|s\rangle\langle s| - I$$

$\sqrt{N}$ times

$$|00\rangle \longrightarrow \frac{1}{2}\left[|00\rangle + |01\rangle + |10\rangle + |11\rangle\right]$$

$$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \xrightarrow{U_\omega} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$2\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\boxed{\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ & |11\rangle & & \end{matrix}}$$

eg

$$\underset{0000}{a} \quad \underset{0001}{b} \quad \underset{0010}{c} \quad \underset{0011}{d} \quad \underset{0100}{e} \quad \underset{0101}{f} \quad \underset{0110}{g} \quad \underset{0111}{h} \quad \underset{1000}{i} \quad \underset{1001}{j} \quad \underset{1010}{k} \quad \underset{1011}{l} \quad \underset{1100}{m} \quad \underset{1101}{n} \quad \underset{1110}{o} \quad \underset{1111}{p}$$

$$|0000\rangle \xrightarrow{h}$$

$$2|s\rangle\langle s| - I$$

$$2\begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \vdots \\ \frac{1}{4} \end{bmatrix}\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots & \frac{1}{4} \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \cdots & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & \cdots & \frac{1}{8} \\ -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & \cdots & \frac{1}{8} \\ \vdots & & & & & \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -\frac{7}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \cdots \\ \frac{1}{8} & -\frac{7}{8} & \cdots & & \\ -\frac{1}{8} & -\frac{1}{8} & \cdots & \frac{7}{8} & \cdots \end{bmatrix}$$

$$\begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{7}{4} \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} -\frac{7}{32} & \frac{1}{32} & \frac{1}{32} & -\frac{1}{32} & \cdots & \frac{1}{32} \\ \frac{1}{32} & \frac{1}{32} & \cdots & & \frac{1}{32} \\ \frac{1}{32} & \frac{1}{32} & -\frac{7}{32} & - & \frac{1}{32} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{6}{32} \\ \frac{6}{32} \\ \frac{6}{32} \\ 22/32 \\ \frac{6}{32} \end{bmatrix}$$

$$\text{avg} = \frac{1}{4} + \frac{1}{4}(15) \Big/ 16$$

$$\Rightarrow \frac{\cancel{1}\,\overset{7}{\cancel{4}}}{4} \times \frac{1}{\cancel{16}\,8}$$

$$= \frac{7}{32}$$

$$\delta_{S_1} = \frac{7}{32} - \left(-\frac{1}{4}\right) \Rightarrow 15/32$$

$$\delta_{\text{nonsm}}\ \frac{7}{32} - \frac{1}{4} \Rightarrow -\frac{1}{32}$$

$$\boxed{\begin{aligned} S_{S_1} &= \frac{7}{32} + \frac{15}{32} \\ &\Rightarrow \frac{22}{32} \\ \text{nonsm}\ &\frac{7}{32} - \frac{1}{32} \\ &= \frac{6}{32} \end{aligned}}$$

Shors algorithm

$\hookrightarrow$ factorization

$p, q = N$.

RSA $p \times q = N$ $\quad N \Rightarrow p, q$.

(1) Pick 'a' $\forall$ coprime N

$\gcd(a, N) = 1$

$1 < a < N-1$

(2) cal. order 'r' of the func

$a^x \bmod N$.

$r \rightarrow$ smallest no ($\neq 0$)

$a^r \bmod N = 1$

(3) if r is even ✓

$x = a^{r/2} \bmod N$

if $x \bmod N \neq 1$ ✓

if $x \neq -1 \pmod N$

$x \neq N-1$ ✓

(4) else

go to step 1

select 'a' value

5)

if all three conditions satisfy

$\left.\begin{array}{l} \gcd(x+1, N) \\ \gcd(x-1, N) \end{array}\right\}$ at least one factor

eg

$N = 15$ ($p = 5$, $q = 3$)

1. Pick 'a'

$a = 13$

$\gcd(13, 15) = 1$

$1 < a < (15-1)$

2. order 'r'

$x =$	0	1	2	3	4	5	6	7	8	---
$a^x \bmod N = 1$	1	13	4	7	1	13	4	7	1	--

$13^x \bmod 15$

$r = 4$

if r = even

$x = a^{r/2} \bmod N$

$= (13)^{\cancel{4}/\cancel{2}\,2} \bmod 15$

$= 169 \bmod 15$

$x = 4$

$x \bmod N \not\equiv 1$

$4 \bmod 15 = 4 \not\equiv 1$

$x \neq N-1$

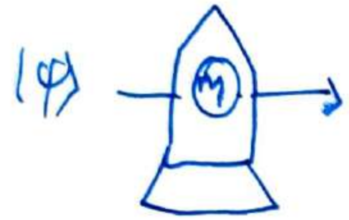
$4 = 15-1$

$\gcd(b+1, N) \Rightarrow \gcd(5, 15) \Rightarrow 5$

$\gcd((b-1, N) \Rightarrow \gcd(3, 15) \Rightarrow 3$

$b, 9.$

Elitzur - Vaidman Bomb



Dud out = input

Bomb

M = 1
then device explodes
otherwise
output = 0

Goal

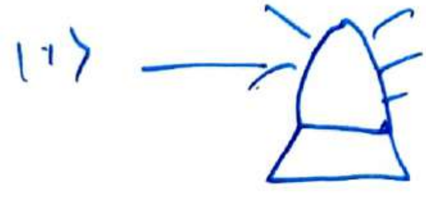
Bomb or Dud
(no explosion)

there are three approaches:

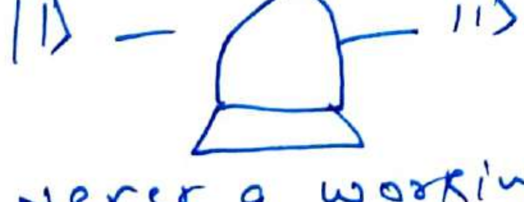
① Naive approach

Dud
output = ~~output~~ input
|0⟩

Bomb
output = |0⟩
useless input



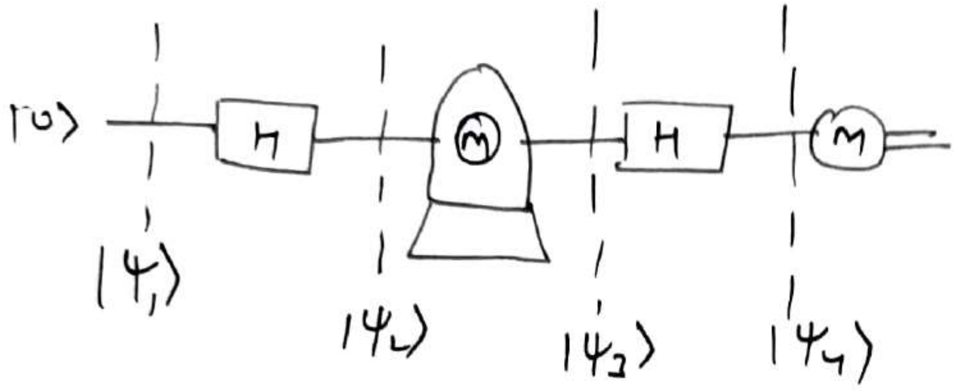
explodes



Dud

Never a working Bomb

② Approach #2



$|\psi_1\rangle = |0\rangle$, $|\psi_2\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

Incase <u>Dud</u>, $|\psi_3\rangle = |\psi_2\rangle$

$H|\psi\rangle = |0\rangle = |\psi_4\rangle$

<u>Dud</u> $= |0\rangle$

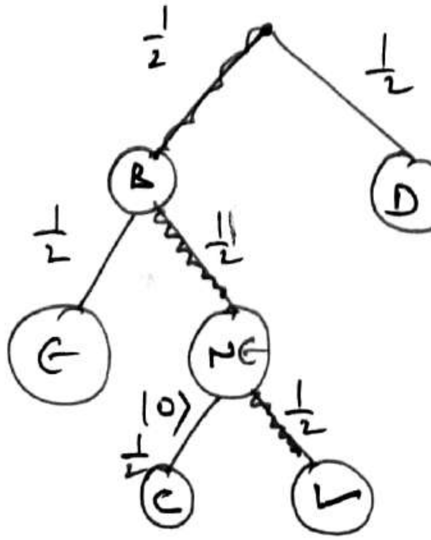
<u>Bomb</u>:- $|\psi_2\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

$|\psi_3\rangle = |0\rangle$

$|\psi_4\rangle = H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

$|0\rangle$ — 50%

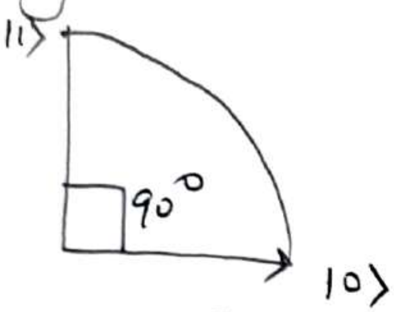
$|1\rangle$ — 50%



Prob of detecting working bomb, without explosion

$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \boxed{= \frac{1}{8}}$

Background

① 

$\langle 0|1\rangle = 0$ orthogonal

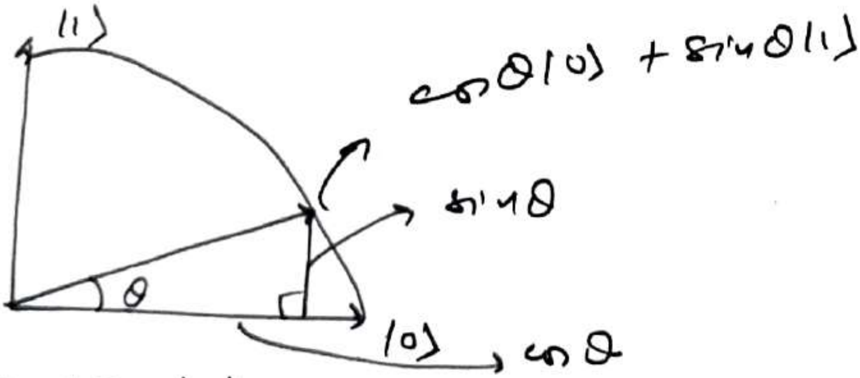
$\| |0\rangle \| = \| |1\rangle \| = 1$
on the unit circle

② $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

θ +ve = anticlockwise rotation

-ve = clockwise rotation.

$R_\theta^\dagger R_\theta = I$ unitary

③ 

$|0\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$

$= |0\rangle$

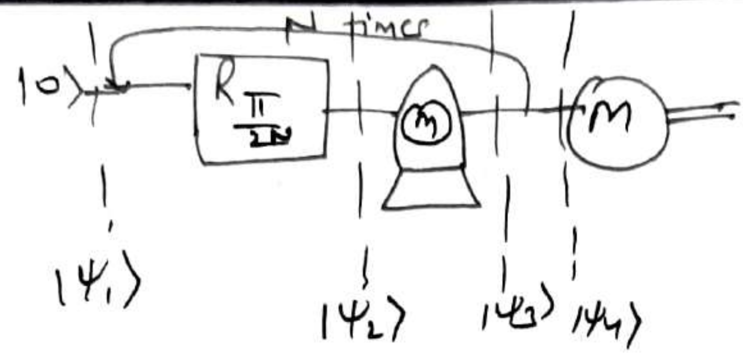
Prob of $|0\rangle = |\cos\theta|^2$

Prob of $|1\rangle = |\sin\theta|^2$

for small θ $|\cos^2\theta| > |\sin^2\theta|$

$\sin\theta \approx \theta$ for small θ

$\cos\theta \approx 1 - \frac{\theta^2}{2}$ for $\cos\theta|0\rangle$



Dud

1st iteration

$|\psi_1\rangle = |0\rangle$

$|\psi_2\rangle = \cos\frac{\pi}{2N}|0\rangle + \sin\frac{\pi}{2N}|1\rangle$

$|\psi_3\rangle = |\psi_2\rangle$

2nd iteration

$|\psi_2\rangle = \cos\frac{2\pi}{2N}|0\rangle + \sin\frac{2\pi}{2N}|1\rangle$

$|\psi_3\rangle = |\psi_2\rangle$

k^{th} iteration

$|\psi_2\rangle = \cos k\frac{\pi}{2N}|0\rangle + \sin k\frac{\pi}{2N}|1\rangle$

— — —

n^{th}

$|\psi_2\rangle = \cos\frac{\pi}{2}|0\rangle + \sin\frac{\pi}{2}|1\rangle$

$= |1\rangle$

$\cos 90^\circ = 0$

$\sin 90^\circ = 1$

$|\psi_3\rangle = |\psi_2\rangle$

$|\psi_4\rangle = |1\rangle$

outcome $|1\rangle$ for Dud

For Dud output is $|1\rangle$ Always

For Bomb

$|\psi_1\rangle = |0\rangle$

1st iteration

$|\psi_2\rangle = \cos\frac{\pi}{2N}|0\rangle + \sin\frac{\pi}{2N}|1\rangle$

M

Prob. $\left|\sin\frac{\pi}{2N}\right|^2$
we measure $|1\rangle$
explodes

Probability $\left|\cos\frac{\pi}{2N}\right|^2$
$|0\rangle$

$|\psi_3\rangle = |0\rangle$

2nd iteration

$|\psi_3\rangle = |0\rangle$

After n iterations

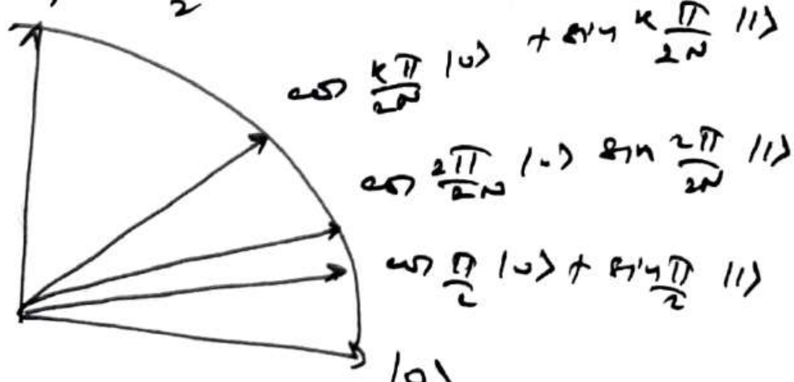
$|\psi_3\rangle = |0\rangle$

$|\psi_4\rangle = |0\rangle$

Bomb $|0\rangle$

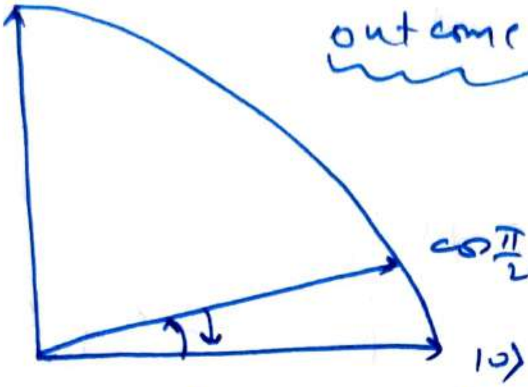
Graphical summary

Dud



outcome — $|1\rangle$

Bomb



$\cos\frac{\pi}{2N}|0\rangle + \sin\frac{\pi}{2N}|1\rangle$

outcome $= |1\rangle$

$$P(\text{no expl}) = P\left(\underset{\text{1st}}{\text{no expl}}\right) P\left(\underset{\text{2nd}}{\text{no expl}}\right) \ldots P\left(\underset{\text{nth}}{\text{no expl}}\right)$$

$$P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B)$$
$$= P(A)\, P(B)\, P(C)$$

$$\left|\cos\frac{\pi}{2N}\right|^2 \cdot \left|\cos\frac{\pi}{2N}\right|^2 \cdots \left|\cos\frac{\pi}{2N}\right|^2$$

$$\boxed{= \cos^{2N}\frac{\pi}{2N}}$$

N	No explosion (detecting Bomb)
2	~~[illegible]~~ 85%
10	94%
20	97%
50	99.99%

~~cos 20~~

$\cos\frac{\pi}{20}$

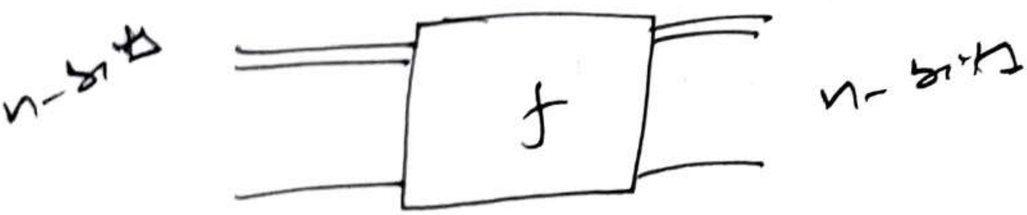
500 ≈ 1

Simon's algorithm

Problem Given a 2-1 function $f:\{0,1\}^n \longrightarrow \{0,1\}^n$, such that $f(x)$ $f(x \oplus s)$ for $s \in \{0,1\}^n$

Our goal is to find s

2-1 function



$$f(x) = f(x \oplus s)$$

eg

$n = 3$, $s = 101$

domain x	range $f(x)$
0 0 0	1 1 1
0 0 1	0 0 0
0 1 0	1 1 0
0 1 1	0 1 0
1 0 0	0 0 0
1 0 1	1 1 1
1 1 0	0 1 0
1 1 1	1 1 0

$$\begin{array}{r} 001 \\ 101 \\ \hline 100 \end{array}$$

$$\begin{array}{r} 110 \\ 101 \\ \hline 011 \end{array}$$

$x, y \to s$

$$\boxed{x \oplus y = s}$$

domain $= 2^n$

half input i.e. 2^{n-1} produce unique outputs

$2^{n-1} + 1 \longrightarrow O(2^n)$

$2^{n/2} \longrightarrow O(2^n)$

Quantum Algo

$O(n)$

Schmidt Decomposition

Pure

Theorem:- If you have a two-qubit quantum state $|\psi\rangle$ such that $|\psi\rangle \in H_A \otimes H_B$, then $\exists$ [$\{|a_i\rangle\} \in H_A$ & $[|b_i\rangle] \in H_B$ and such that

$$\boxed{|\psi\rangle_{AB} = \sum_i \lambda_i |a_i\rangle \otimes |b_i\rangle} \quad \text{—} \quad (1)$$

Schmidt bases

λ_i = it's a +ve number
↓
Schmidt's coefficient

$$\sum \lambda_i^2 = 1$$

just looking schmidt coefficient we predict that the state is entangled or not

$$\rho = |\psi\rangle_{AB}\langle\psi|_{AB} = \left(\sum_i \lambda_i |a_i\rangle \otimes |b_i\rangle\right)\left(\sum_j \lambda_j \langle a_j| \otimes \langle b_j|\right)$$

$$= \cancel{\sum_{i,j} \lambda_i \lambda_j |a_i\rangle\langle a_j| \otimes |b_i\rangle\langle b_j|}$$

$$\sum_{i,j} \lambda_i \lambda_j \left(|a_i\rangle \otimes |b_i\rangle\right)\left(\langle a_j| \otimes \langle b_j|\right)$$

From eq (1) λ_i square root of eigenvalue of ρ^A and ρ^B

$$\rho^A = Tr_B(\rho) \; ; \; \rho^B = Tr_A(\rho)$$

Schmidt Decomposition

Purple

Theorem:- If you have a two-qubit quantum state $|\psi\rangle$ such that $|\psi\rangle \in H_A \otimes H_B$, then $\exists \left[\{|a_i\rangle\} \in H_A \;\&\; [|b_i\rangle] \in H_B\right]$ and such that

$$\boxed{|\psi\rangle_{AB} = \sum_i \lambda_i |a_i\rangle \otimes |b_i\rangle} \quad \text{--- (1)}$$

Schmidt bases

λ_i = it's a +ve number

$\downarrow$

Schmidt's coefficient

$$\sum \lambda_i^2 = 1$$

just looking schmidt coefficient we predict that the state is entangled or not

$$\rho = |\psi\rangle_{AB}\langle\psi|_{AB} = \left(\sum_i \lambda_i |a_i\rangle \otimes |b_i\rangle\right)\left(\sum_j \lambda_j \langle a_j| \otimes \langle b_j|\right)$$

$$= \cancel{\sum_{i,j} \lambda_i \lambda_j |a_i\rangle\langle a_j| \otimes |b_i\rangle\langle b_j|}$$

$$\sum_{i,j} \lambda_i \lambda_j \left(|a_i\rangle \otimes |b_i\rangle\right)\left(\langle a_j| \otimes \langle b_j|\right)$$

From eq (1) λ_i square root of eigenvalue of ρ^A and ρ^B

$$\rho^A = Tr_B(\rho) \;;\; \rho^B = Tr_A(\rho)$$

$$\rho = |\psi\rangle_{AB}\,{}_{AB}\langle\psi|_{AB} = \sum_{ij}\lambda_{ij}\left(|a_i\rangle\otimes|b_i\rangle\right)\left(\langle a_j|\otimes\langle b_j|\right)$$

$$\rho^A = Tr_B(\rho) = \sum_{ij}\lambda_{ij}\,|a_i\rangle\langle a_j|\,Tr\left(|b_i\rangle\langle b_j|\right)$$

$$= \sum_{ij}\lambda_{ij}\,|a_i\rangle\langle a_j|\,\underbrace{\langle b_i|b_j\rangle}_{\delta_{ij}}$$

$$= \sum_i \lambda_i^2\,|a_i\rangle\langle a_i|\,\cancel{\langle b_i|b_j\rangle}^{\,1}$$

$$= \boxed{\rho^2 = \sum \lambda_i^2\,|a_i\rangle\langle a_i|}$$

$$\boxed{\begin{aligned}\rho^B &= \sum_i \lambda_i^2\,|b_i\rangle\langle b_i| \\ \rho^A &= \sum_i \lambda_i^2\,|a_i\rangle\langle a_i|\end{aligned}}$$

$$\rho = \rho^B\,(\quad)_{2\times 2} \longrightarrow 2$$

$$\rho^A\,(\;)_{2\times 2} \longrightarrow 2$$

If only one of the eigenvalue is non-zero then it is an entangled state.

~~If~~ number of non-zero schmidt coefficient are called as schmidt number.

no. of nonzero eigen value of ρ^A or ρ^B

$\# S = 1 \Rightarrow$ it is separable state or non entangled state

$\# r > 1$ It is entangled state

eg $1|01\rangle + 0|00\rangle + 0|10\rangle + 0|11\rangle$

eg $\frac{|01\rangle + |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$

schmidt coeffi

eg $\frac{|00\rangle + |01\rangle}{\sqrt{2}} \quad |0\rangle \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$

$= 1(0+)$

example $|\psi\rangle = \frac{1}{2}\left(|00\rangle - |01\rangle - |10\rangle + |11\rangle\right)$

$\rho^{AB} = |\psi\rangle\langle\psi| = \_\_ \_\_ = 16$ terms

$\rho^A = Tr_B(\rho^{AB}) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$\cancel{\rho^B} = \quad \lambda_1 = 1, 0$

$\# S = 1 \Rightarrow$ separable state

$\frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

$$|s\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

$$\rho = \left(\frac{|01\rangle - |10\rangle}{\sqrt{2}}\right)\left(\frac{\langle 01| - \langle 10|}{\sqrt{2}}\right) = \frac{1}{2}\begin{bmatrix} |01\rangle\langle 01| - |01\rangle\langle 10| \\ - |10\rangle\langle 01| + |10\rangle\langle 10| \end{bmatrix}$$

$$\rho^{A} = Tr_B \, \rho^{AB} = \frac{1}{2}\Big[|0\rangle\langle 0| \langle 0| Tr |1\rangle\langle 1| - |0\rangle\langle 1| \underbrace{Tr |1\rangle\langle 0|}_{\langle 0|1\rangle}$$

$$- |1\rangle\langle 0| \underbrace{Tr |0\rangle\langle 1|}_{\langle 1|0\rangle} + |1\rangle\langle 1| Tr |0\rangle\langle 0| \Big]$$

$$\frac{1}{2}\left[|0\rangle\langle 0| + |1\rangle\langle 1| \right] = \frac{1}{2} I_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$\lambda_i = \frac{1}{2}, \frac{1}{2}$$

eg $|\psi_3\rangle \in H_A \otimes H_B \otimes H_C$

$$|\psi\rangle_3 \neq \sum_i \lambda |a_i\rangle |b_i\rangle |c_i\rangle$$

Schmid Decomposition + Purification
mixed $\rightarrow$ Pure state

$|\psi\rangle$

$$\rho = \sum_i \lambda_i |a_i\rangle |b_i\rangle$$

* Singular value Decomposition

Purification of mixed states

ρ^A = mixed state $\qquad \{|a_i\rangle\} \in \mathcal{H}_A$

$$\rho = \sum_i p_i |a_i\rangle\langle a_i|$$

$$|AR\rangle = \sum_i \sqrt{p_i}\, |a_i\rangle |r_i\rangle \qquad \{|r_i\rangle\} \in \mathcal{H}_R$$

R = Reference

$$Tr_R\left(|AR\rangle\langle AR|\right) = Tr_R\left(\sum_i \sqrt{p_i}\, |a_i\rangle|r_i\rangle \sum_j \sqrt{p_j} \langle a_j|\langle r_j|\right)$$

$$= \sum_{ij} \sqrt{p_i}\sqrt{p_j}\, |a_i\rangle\langle a_j|\, \underbrace{Tr\, |r_i\rangle\langle r_j|}_{\underbrace{\langle r_j|r_i\rangle}_{\delta_{ij}}}$$

$$= \sum_{i,j} \sqrt{p_i}\sqrt{p_j}\, |a_i\rangle\langle a_j|\, \delta_{ij} = \sum_i \sqrt{p_i}\sqrt{p_i}\, |a_i\rangle\langle a_i|$$

$$= \sum_i p_i |a_i\rangle\langle a_i| = \rho^{A}$$

Purification of mixed state :- Example

$$\rho^A = 0.64|0\rangle\langle 0| + 0.36|1\rangle\langle 1|$$

$$\sum_i p_i |i\rangle\langle i|$$

$\underline{\underline{\text{Sol}^n}}$ $|AR\rangle = \sqrt{0.64}\,|00\rangle + \sqrt{0.36}\,|11\rangle$

$Tr(\rho^2) \neq 1$ $\quad \rho^{AR}$, $|AR\rangle\langle AR|$

$|AR\rangle\langle AR|$

$= 0.64|00\rangle\langle 00| + 0.36|11\rangle\langle 11|$
$+ 0.48|00\rangle\langle 11| + 0.48|11\rangle\langle 00|$

$$(\rho^{AR})^2 = \begin{pmatrix} 0.64 & 0 & 0 & 0.48 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.64 & 0 & 0 & 0.36 \end{pmatrix} \times$$

$$\begin{pmatrix} 0.64 & 0 & 0 & 0.48 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.64 & 0 & 0 & 0.36 \end{pmatrix}$$

$$= \begin{pmatrix} 0.64^2 + 0.48^2 & 0 & 0 & 0.64(0.48) + 0.48(0.36) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.64(0.45) + 0.48(0.64) & 0 & 0 & 0.48^2 + 0.36^2 \end{pmatrix}$$

$$\mathrm{Tr}\left(\rho^{AR}\right)^2 = 0.64^2 + 0.48^2 + 0.48^2 + 0.36^2$$

$$= 1$$

$$\rho^{AR} = \sqrt{0.64}\,|00\rangle + \sqrt{0.36}\,|11\rangle$$

Po Quantum language!!

State ⇒ Possible state combination → state space

$\Downarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Downarrow$

$\psi(x)$ $\qquad\qquad\qquad\qquad\qquad\qquad$ wave function space

$\psi_1(x), \psi_2(x), \psi_3(x)$ ---

Linear vector space

$|\psi_1\rangle$ represent state (Ket) ~~represent state~~ Symbol

P.A.M. Dirac

$\langle\psi_1|$ (Bra) Symbol $\qquad \langle\ \rangle$ Bra - ket

To represent wave function

Unit vector $\hat{e}$:- whose magnitude is one

Normalized wave function

$$\int_{-\infty} \psi^*(x)\,\psi(x)\,dx = 1$$ normalized wave function

$\vec{A}\cdot\vec{B}$

scalar product :- give number

$\psi_1(x), \psi_2(x)$
inner product

$\langle\vec{A}|\vec{B}\rangle$

$\vec{A}\cdot\vec{A} = |A|^2$

$$\int_{-\infty}^{\infty} \psi_1(x)^*\,\psi_2(x)\,dx$$ inner product

$\langle\psi_1|\psi_2\rangle$

$\langle\psi(x)|\psi(x)\rangle$

$$= \int_{-\infty} \psi^*(x)\psi(x)\,dx$$

$\infty\ |\psi(x)|^2$

=

$\vec{A} \cdot \vec{B} = 0$
$\vec{A}$ and $\vec{B}$
are orthogonal to each other

If $\langle \psi_1 | \psi_2 \rangle = 0$ then $\psi_1(x)$, $\psi_2(x)$ are orthogonal to each other

$\hat{i}, \hat{j}, \hat{k}$ vectors
all are $\perp$ to each other and having magnitude one

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

Three dimensional space

$$\langle \hat{i} | \hat{i} \rangle = 1 \quad \langle \hat{j} | \hat{j} \rangle = 1, \quad \langle \hat{k} | \hat{k} \rangle = 1$$

$$\langle \hat{i} | \hat{j} \rangle = \langle \hat{j} | k \rangle = \langle \hat{k} | \hat{i} \rangle = \langle \hat{i} | \hat{j} \rangle = 0$$

orthonormal basis.

$$|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle \text{ ——— .}$$

$$\langle \phi_1 | \phi_1 \rangle = 1 \quad \langle \phi_2 | \phi_2 \rangle ; \langle \phi_3 | \phi_3 \rangle = 1$$

$$\langle \phi_1 | \phi_2 \rangle = \langle \phi_2 | \phi_3 \rangle = \langle \phi_3 | \phi_1 \rangle = 0$$

$$|\psi\rangle = \sum_i c_i |\phi_i\rangle$$

Infinite dimensional space

$$A_x = \langle \hat{i} | \vec{A} \rangle, \quad A_y = \langle \hat{j} | \vec{A} \rangle = A_z \ \langle \hat{z} | \vec{A} \rangle$$

$$c_i = \langle \phi_i | \psi \rangle$$

$$|p_0\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} p_0 x} \qquad |p_0\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} p_0 x}$$

$p_0 \rightarrow -\infty \text{ to } \infty$

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int a(p_0) e^{\frac{i}{\hbar} p_0 x} \, dp_0$$

$$|\psi\rangle = \sum_i c_i |\phi_i\rangle$$

$$c_i = \langle \phi_i | \psi \rangle$$

$$\int_{-\infty}^{\infty} \phi_i^*(x) \, \psi(x) \, dx$$

$$a(p_0) = \langle p_0 | \psi \rangle$$

$$= \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar} p_0 x} \psi(x)\, dx$$

$$\langle p_0 | p_0 \rangle = \int_{-\infty}^{\infty} \frac{1}{2\pi\hbar} e^{-\frac{i}{\hbar} p_0 x} \frac{1}{2\pi\hbar} e^{\frac{i}{\hbar} p_0 x} dx$$

$$= \text{infinity} \quad (\text{not normalized})$$

$$\langle p_1 | p_2 \rangle = \int_{-\infty}^{\infty} \left(\frac{1}{2\pi\hbar}\right) e^{-\frac{i}{\hbar} p_1 x} e^{\frac{i}{\hbar} p_2 x} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{(2\pi\hbar)} e^{\frac{i}{\hbar}(p_2 - p_1) x} dx$$

$$= \delta(p_2 - p_1)$$

Position pure state

$$|x_0\rangle = \delta(x - x_0) \quad , \quad x_0 \text{ from } -\infty \text{ to } +\infty$$

$$\psi(x) = \int_{-\infty}^{\infty} \psi(x_0)\, \delta(x - x_0)\, dx_0$$

$$|\psi\rangle = \int_{-\infty}^{\infty} \psi(x_0) |x_0\rangle\, dx_0$$

$$\psi(x_0) = \langle x_0 | \psi \rangle$$

$$= \int_{-\infty}^{\infty} \delta(x - x_0) \psi(x)\, dx = \psi(x_0)$$

orthonormal continuous basis

$$\langle \phi_i | \phi_j \rangle = \delta_{ij} = \quad \delta_{ij} = 1$$

Kronecker delta $\quad = 0 \text{ if } i \neq j$

<u>Operators</u>

$$\psi = |p_0\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} p_0 x}$$

$$\frac{\partial \psi}{\partial x} = \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{i}{\hbar} p_0\right) e^{\frac{i}{\hbar} p_0 x}$$

$$-i\hbar \frac{\partial \psi}{\partial x} = p_0 \psi$$

$$\left(-i\hbar \frac{\partial}{\partial x}\right) \psi = (p_0) \psi.$$

$-i\hbar \frac{\partial}{\partial x} \Leftarrow p_0$ momentum

linear momentum operator

$$P = -i\hbar \frac{\partial}{\partial x}$$

<u>Position</u> operator X

X = multiply by x

pure state $\delta(x - x_0)$

$$X \, \delta(x - x_0) = x \, \delta(x - x_0)$$

$$= x_0 \delta(x - x_0)$$

$$K = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} = K = \frac{p^2}{2m}$$

$P \cdot E = V(x)$
operation $\Rightarrow$ multiply by $V(x)$

$T \cdot E = K \cdot E + P \cdot E$

Eigen value and Eigen function.

Position operator $\Rightarrow$ Multiplication by $x \Rightarrow X$
momentum $-i\hbar \frac{\partial}{\partial x} \Rightarrow P$

$$X\,\delta(x-x_0) = x\,\delta(x-x_0) = x_0\,\delta(x-x_0)$$

$|x_0\rangle$

$X|x_0\rangle = x_0|x_0\rangle$ Eigenvalue equation

$X\psi = x_0\psi$ Eigenstate of X

eigenvalue — Eigenfunction of X

$$\frac{d}{dx} e^x = 1 \cdot e^x$$

Linear momentum

$$P\psi = p_0\psi$$

$$\frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} p_0 x} = p_0(\_)$$

$$\psi(x) = \int \psi(x_0)\,\underline{\delta(x-x_0)}\,dx_0$$

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int a(p_0)\, e^{\frac{i}{\hbar} p_0 x}\, dp_0$$

$$|\psi\rangle = \int_{-\infty}^{\infty} \psi(x_0)|x_0\rangle\, dx$$

$$|\psi\rangle = \int a(p_0)\,|p_0\rangle\, dp_0$$

Observable $a \Rightarrow$ operator A

$A\psi = a\psi$

$(A+B)\psi = A\psi + B\psi$

$(AB)\psi = A(B\psi)$

x, p

$(ax+bp)\psi \quad (ax)\psi + (bp)\psi$

$= ax\psi + bp\psi$

$(xp)\psi = x\left(-i\hbar \frac{\partial}{\partial x}\psi\right)$

$= -i\hbar x \frac{\partial \psi}{\partial x}$

$(px)\psi = -i\hbar \frac{\partial}{\partial x}(x\psi)$

$= -i\hbar\left[x\frac{\partial \psi}{\partial x} + \psi \frac{\partial x}{\partial x}\right]$

$\therefore (xp)\psi \neq (px)\psi$

$xp \neq px$

x, p do not commute with each other

$(xp - px)\psi = +i\hbar\psi$

$xp - px = i\hbar$

$[x, p] = i\hbar$

commutator of x with p

a, b

$[A, B] = i\hbar$

$\Delta a, \Delta b, \Delta c$

momentum $P = -i\hbar \frac{\partial}{\partial x}$

K.E $K = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$

$(PK)\psi = P(K\psi)$

$(KP)\psi = K(P\psi)$

$[P, K] = 0$

commute $\frac{1}{2\pi\hbar} e^{\frac{i}{\hbar} p_0 x}$

$K = \frac{p_0^2}{2m}$

$p \to p_0$

Hermitian operator

A observable a

$A\,\phi(x) = a\,\phi(x)$

eigen functions of A

corresponding eigen value

momentum p

$P = -i\hbar \frac{\partial}{\partial x}$

$P(\phi) = p\,\phi(x)$

$\phi(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} p_0 x}$

K.E $\quad K = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$

$= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}(\sin kx)$

$\frac{\hbar^2 k^2}{2m}(\sin kx)$

$$\frac{\partial \sin kx}{\partial x}$$

$$k\cos kx$$

$$k\frac{\partial}{\partial x}\cos kx$$

$$-k\cdot k\sin kx$$

$\sin kx$ is an eigenfunction K

$\Rightarrow$ eigenvalue $= \frac{\hbar^2 k^2}{2m}$

$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}(\cos kx) = \frac{\hbar^2 k^2}{2m}(\cos kx)$

eigenvalues $= \frac{\hbar^2 k^2}{2m}$

[$c\sin kx$]

$\cos kx + i\sin kx = e^{ikx}$

$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}e^{ikx} = \frac{\hbar^2 k^2}{2m}(e^{ikx})$

eigenvalue $\quad$ eigenfunct

$\cos kx - i\sin kx = e^{-ikx}$

$\rightarrow$ only two independent vector linear combinations

Non degenerate eigenvalue
doublydegenerate / or two fold degenerate
$\to A|\phi\rangle = a|\phi\rangle$

Doubly degenerate

$A|\phi_1\rangle = a|\phi_1\rangle$ independent

$A|\phi_2\rangle = a|\phi_2\rangle$

doubly degenerate / two fold degenerate

---

n - fold degenerate

Hermitian operator :- corresponding to observable quantity

$A, |\phi_1\rangle, |\phi_2\rangle$

scaler product $|\phi_1\rangle$ with $A|\phi_2\rangle$

$|\phi_1\rangle, |\phi_2\rangle$

$$\langle \phi_1|\phi_2\rangle = \int_{-\infty}^{\infty} \phi_1^*(x)\,\phi_2(x)\,dx$$

$\langle \phi_1|A\phi_2\rangle$

scaler product of $A|\phi_1\rangle$ with $|\phi_2\rangle$

$= \langle A\phi_1|\phi_2\rangle$

if $\langle \phi_1|A\phi_2\rangle = \langle A\phi_1|\phi_2\rangle$ for all $|\phi_1\rangle, |\phi_2\rangle$

then A is called Hermitian operator

$\langle \phi_1 | A | \phi_2 \rangle$

Hermitial Adjoint   If A is not Hermitian

~~If then~~ $\langle \phi_1 | A \phi_2 \rangle \neq \langle \phi_1 A | \phi_2 \rangle$

$= \langle B\phi_1 | \phi_2 \rangle$

the B is a hermitian adjoint of A

and vice versa

$+ \rightarrow$ Dager

$B = A^{+}$

$A = B^{+}$

$(A + B)^{+} = A^{+} + B^{+}$

$(AB)^{+} = B^{+} A^{+}$ [change the position of A and B with each other in output]

- Real.
- orthogonal

How close are two quantum states?

Trace distance

Two quantum

$$\boxed{D = \frac{1}{2} Tr \left| \rho - \sigma \right|}$$

compute the trace distance b/w

(i) ρ and σ (ii) ρ and π

$\rho = \frac{3}{4} |0\rangle\langle 0| + \frac{1}{4} |1\rangle\langle 1|$

$\sigma = \frac{2}{3} |0\rangle\langle 0| + \frac{1}{3} |1\rangle\langle 1|$

$\pi = \frac{1}{8} |0\rangle\langle 0| + \frac{7}{8} |1\rangle\langle 1|$

$D(\rho, \sigma) = \frac{1}{2} Tr |\rho - \sigma|$

$\rho - \sigma = \frac{3}{4} |0\rangle\langle 0| + \frac{1}{4} |1\rangle\langle 1| - \left(\frac{2}{3} |0\rangle\langle 0| + \frac{1}{3} |1\rangle\langle 1|\right)$

$= \frac{3}{4} |0\rangle\langle 0| + \frac{1}{4} |1\rangle\langle 1| - \frac{2}{3} |0\rangle\langle 0| - \frac{1}{3} |1\rangle\langle 1|$

$= \left(\frac{3}{4} - \frac{2}{3}\right) |0\rangle\langle 0| + \left(\frac{1}{4} - \frac{1}{3}\right) |1\rangle\langle 1|$

$= \frac{9-8}{12} |0\rangle\langle 0| + \frac{-1}{12} |1\rangle\langle 1|$

$$\boxed{\rho - \sigma = \frac{1}{12} |0\rangle\langle 0| - \frac{1}{12} |1\rangle\langle 1|}$$

$$= \frac{1}{2}\,\mathrm{Tr}\left|\frac{1}{\sqrt{2}}\underbrace{(|0\rangle\langle 0| - |1\rangle\langle 1|)}\right|$$

$$= \frac{1}{2\sqrt{2}}\,\mathrm{Tr}\,\big|\,|0\rangle\langle 0| - |1\rangle\langle 1|\,\big|$$ Trace is a linear vecto

$$\frac{1}{\sqrt{2}}\,\mathrm{Tr}\,|0\rangle\langle 0| - \frac{1}{\sqrt{2}}\,|1\rangle\langle 1|$$

$$\frac{1}{2\sqrt{2}}\Big(\big|\mathrm{Tr}\,|0\rangle\langle 0|\big| + \big|-\mathrm{Tr}\,|1\rangle\langle 1|\big|\Big)$$

$$\frac{1}{2\sqrt{2}}\big(\langle 0|0\rangle + \langle 1|1\rangle\big)$$

$$= \frac{1}{2\sqrt{2}}\cdot 2 = \frac{1}{\sqrt{2}}$$

||ly $D(\rho,\pi) = \frac{1}{2}\,\mathrm{Tr}\,|\rho - \pi| = \frac{5}{8}$

$$D(\rho,\sigma) < D(\rho,\pi)$$

ρ & σ are closer to each other than ρ & π

Properties of Trace Distance :-

$$D = \frac{1}{2}\,\mathrm{Tr}\,|\rho - \sigma|$$

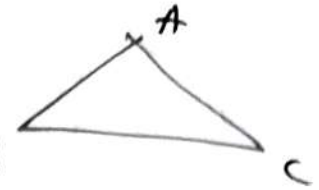
(I) $1 \geq D \geq 0$

$D = 1$ orthogonal

$D = 0$ Same state

② Symmetry, $D(\rho,\sigma) = D(\sigma,\rho)$

③ Triangle inequality



$$AB \le BC + CA$$

$$D(\rho,\sigma) \le D(\rho,\pi) + D(\pi,\sigma)$$

④ If ρ is a pure state, $\rho = |\psi\rangle\langle\psi|$

$$D(\rho,\sigma) = \sqrt{1-\langle\psi|\sigma|\psi\rangle}$$

⑤ If $\rho-\sigma$ has eigenvalues λ_i

$$D(\rho,\sigma) = \frac{1}{2}\sum_i |\lambda_i|$$

⑥ If $\vec{r}$ and $\vec{s}$ are the Bloch vector of ρ and σ respectively,

$$D(\rho,\sigma) = \frac{1}{2}|\vec{r}-\vec{s}|$$

⑦ If ρ and σ commute and $\rho = \sum_x p_x |x\rangle\langle x|$

$$\sigma = \sum_x q_x |x\rangle\langle x|$$

$$D(\rho,\sigma) = \sum_x |p_x - q_x|$$

Proof Trace distance Distance formula

Prove that if ρ and σ commute with each other such that $\rho = \sum_x p_x |x\rangle\langle x|$ & $\sigma = \sum_x q_x |x\rangle\langle x|$

then $D(\rho, \sigma) = \frac{1}{2} \sum_x |p_x - q_x|$

Proof :- $D(\rho, \sigma) = \frac{1}{2} Tr|\rho - \sigma|$

$= \frac{1}{2} Tr \left| \sum_x p_x |x\rangle\langle x| - \sum_x q_x |x\rangle\langle x| \right|$

$= \frac{1}{2} Tr \left| \sum_x (p_x - q_x) |x\rangle\langle x| \right|$

$= \frac{1}{2} \left| \sum_x (p_x - q_x) \underbrace{Tr |x\rangle\langle x|}_{\langle x|x\rangle = 1} \right|$

$= \cancel{\frac{1}{2} \left| \sum_x |p_x - q_x| \right|} \; \frac{1}{2} \left| \sum_x p_x - q_x \right|$

$= \frac{1}{2} \sum_x |p_x - q_x|$

Finding trace distance

$\rho = 0.64 |0\rangle\langle 0| + 0.36 |1\rangle\langle 1| \quad \begin{pmatrix} 0.64 & 0 \\ 0 & 0.36 \end{pmatrix}$

$\sigma = 0.36 |0\rangle\langle 0| + 0.64 |1\rangle\langle 1|$

$\begin{pmatrix} 0.36 & 0 \\ 0 & 0.36 \end{pmatrix}$

$(\rho - \sigma)^T = \quad \rho - \sigma = \begin{pmatrix} 0.28 & 0 \\ 0 & -0.28 \end{pmatrix}$

$$D(\rho,\sigma) = \frac{1}{2}\sum_x |p_x - q_x|$$

x_0, x_1

$$= \frac{1}{2}\left(|p_0 - q_0| + |p_1 - q_1|\right)$$

$$= \frac{1}{2}\left(|0.64 - 0.36| + |0.36 - 0.64|\right)$$

$$= \frac{1}{2}\left(0.28 + 0.28\right)$$

$$= \underline{\underline{0.28}}$$

$$\boxed{\begin{aligned} D &= \frac{1}{2}\sum_x |p_x - q_x| \\ D &= \frac{1}{2}\mathrm{Tr}|\rho - \sigma| \end{aligned}}$$

$\underline{\text{Same eigenbasis}}$

$$D = \frac{1}{2}\underbrace{\mathrm{Tr}|\rho - \sigma|}$$

$$= \frac{1}{2}\mathrm{Tr}\sqrt{(\rho-\sigma)^{\dagger}\cdot(\rho-\sigma)}$$

$$\frac{1}{2}\mathrm{Tr}\sqrt{((\rho-\sigma)(\rho-\sigma)}|$$

$$= \frac{1}{2}\mathrm{Tr}\sqrt{(\rho-\sigma)^2}|$$

$$= \frac{1}{2}\mathrm{Tr}|(\rho-\sigma)|$$

$$= \frac{1}{2}\mathrm{Tr}|(\rho-\sigma)|$$

$$= \frac{1}{2}\mathrm{Tr}\begin{vmatrix} 0.28 & 0 \\ 0 & -0.28 \end{vmatrix}$$

$\underbrace{\mathrm{Tr}|A|}_{\text{Norm.}}$: Trace norm

$\mathrm{Tr}|A| = \|A\|$

$|A| = \sqrt{A^{\dagger}A}$

$\mathrm{Tr}\sqrt{A^{\dagger}A}$

$$= \begin{matrix} f(a) & 0 \\ 0 & f(b) \end{matrix}$$

$$\frac{1}{2}\,Tr\begin{pmatrix} 0.28 & 0 \\ 0 & 0.28 \end{pmatrix} = \frac{1}{2}(0.28 + 0.28)$$
$$= 0.28$$

Quantum fidelity

- Closeness relation b/w Two quantum states

- ρ and σ are Two quantum states

$$F = Tr\sqrt{\sqrt{\rho}\,\sigma\,\sqrt{\rho}}$$

Statistical overlap

$$|\psi\rangle = a|\phi_1\rangle + b|\phi_2\rangle$$

$$|\langle\phi_2|\psi\rangle|^2$$

If ρ is a pure state: $\rho = |\psi\rangle\langle\psi|$

$$\rho^2 = \rho$$
$$\rho = \sqrt{\rho}$$

$$F = Tr\sqrt{\rho\,\sigma\,\rho}$$

$$= Tr\sqrt{|\psi\rangle\underbrace{\langle\psi|\sigma|\psi\rangle}\langle\psi|}$$

$$= Tr\sqrt{\langle\psi|\sigma|\psi\rangle\,|\psi\rangle\langle\psi|}$$

$$= \sqrt{\langle\psi|\sigma|\psi\rangle\,\underbrace{Tr\,|\psi\rangle\langle\psi|}_{\langle\psi|\psi\rangle = 1}}$$

$= \sqrt{\langle \psi | \sigma | \psi \rangle}$

$$\boxed{F = \sqrt{\langle \psi | \sigma | \psi \rangle}}$$

when both ρ and σ are pure states.

$\rho = |\psi\rangle\langle\psi|$ and

$\sigma = |\phi\rangle\langle\phi|$

$F = \mathrm{Tr}\sqrt{\rho\sigma\rho}$

$$= \mathrm{Tr}\sqrt{|\psi\rangle\underbrace{\langle\psi|\phi\rangle}\underbrace{\langle\phi|\psi\rangle}\langle\psi|}$$

- inner product
- conjugate of each other

$|\langle\psi|\phi\rangle|^2$

$$= \mathrm{Tr}\sqrt{|\langle\psi|\phi\rangle|^2 |\psi\rangle\langle\psi|}$$

$$= \sqrt{|\langle\psi|\phi\rangle|^2}\,\underbrace{\mathrm{Tr}\,|\psi\rangle\langle\psi|}_{1} = |\langle\psi|\phi\rangle|$$

$F = |\langle\psi|\phi\rangle|$

Probability of finding $|\psi\rangle$ at $|\phi\rangle$

$\mathrm{Prob}(|\psi\rangle, |\phi\rangle) = |\langle\psi|\phi\rangle|^2 = F^2$

$= [f(|\psi\rangle, |\phi\rangle)]^2$

$= 0 \leq f \leq 1$

orthogonal equali

* $f(\rho, \sigma) = f(U\rho U^\dagger, U\sigma U^\dagger)$

* Bures Distance, $D_B^{(\rho,\sigma)} = \sqrt{2(1 - f(\rho,\sigma))}$

If ρ and σ are commuting: $\rho = \sum_x p_x |x\rangle\langle x|$

$\sigma = \sum_x q_x |x\rangle\langle x|$

$= \sum_x \sqrt{p_x q_x}$

Proof of fidelity theorm

Quantum fidelity If ρ and σ commute with each other such that

$\rho = \sum_x p_x |x\rangle\langle x|$ and $\sigma = \sum_x q_x |x\rangle\langle x|$

$f(\rho, \sigma) = \sum_x \sqrt{p_x q_x}$

Proof: $f(\rho, \sigma) = Tr\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}$

$= Tr\sqrt{\sqrt{\sum_x p_x |x\rangle\langle x|}\ \sum_x q_x |x\rangle\langle x| \sqrt{\sum_x p_x |x\rangle\langle x|}}$

$$\mathrm{Tr}\sqrt{\sum_x \sqrt{p_x}\, q_x \sqrt{p_x}\, \sqrt{|x\rangle\langle x|}\sqrt{|x\rangle\langle x|}\; |x\rangle\langle x|}$$

$$\mathrm{Tr}\sqrt{\sum_x p_x q_x\, |x\rangle\underbrace{\langle x|x\rangle}_{1}\langle x|} = \mathrm{Tr}\sqrt{\sum_x p_x q_x |x\rangle\langle x|}$$

$$= \sqrt{\sum_x p_x q_x} = \sum_x \sqrt{p_x q_x}$$

computing fidelity example

compute the fidelity b/w

$\rho_1 = \frac{4}{5}|0\rangle\langle 0| + \frac{1}{5}|1\rangle\langle 1|$ & $\rho_2 = \frac{3}{5}|0\rangle\langle 0| + \frac{2}{5}|1\rangle\langle 1|$

$$\rho_1 = \begin{pmatrix} \frac{4}{5} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} \frac{3}{5} & 0 \\ 0 & 2/5 \end{pmatrix}$$

$$\Rightarrow \sqrt{\rho_1} \begin{pmatrix} \sqrt{4/5} & 0 \\ 0 & \sqrt{1/5} \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} & 0 \\ 0 & \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$F = \mathrm{Tr}\sqrt{\sqrt{\rho_1}\,\rho_2\sqrt{\rho_1}} = \mathrm{Tr}\sqrt{\begin{pmatrix} 2/\sqrt{5} & 0 \\ 0 & \frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} 3/5 & 0 \\ 0 & 2/5 \end{pmatrix}\begin{pmatrix} 2/\sqrt{5} & 0 \\ 0 & \frac{1}{\sqrt{5}} \end{pmatrix}}$$

$$\mathrm{Tr}\sqrt{\begin{pmatrix} \frac{12}{25} & 0 \\ 0 & \frac{2}{25} \end{pmatrix}} = \mathrm{Tr}\begin{pmatrix} \frac{2\sqrt{3}}{5} & 0 \\ 0 & \frac{\sqrt{2}}{5} \end{pmatrix}$$

$$= \frac{2\sqrt{3}+\sqrt{2}}{5} = \underline{\underline{0.976}}$$

$\underline{\underline{\text{Soln}}}$

$$F = \sum_{x} \sqrt{p_x q_x} = \sqrt{p_0 q_0} + \sqrt{p_1 q_1}$$

$x = 0, 1$

$$= \sqrt{\frac{4}{5}\cdot\frac{3}{5}} + \sqrt{\frac{1}{5}\cdot\frac{2}{5}}$$

$$= \sqrt{\frac{12}{25}} + \sqrt{\frac{2}{25}} = \frac{2\sqrt{3}+\sqrt{2}}{5}$$

Relation b/w Distance Measure

More similar: High fidelity
low trace distance

Less similar: Low fidelity
high trace distance

(1)

$$1-F(\rho,\sigma) \leq D(\rho,\sigma) \leq \sqrt{1-(F(\rho,\sigma))^2}$$

(2)

$$\boxed{D_B(\rho,\sigma) = \sqrt{2(1-F(\rho,\sigma))}}$$

③ $1 - F^2(|\psi\rangle, \sigma) \leq D(|\psi\rangle, \sigma)$

Concurrency as a measure of entanglement

- How much of entanglement a state have? → Concurrency
- Cost of creating a given entangled state? → Entanglement of formation.

2-qubit state
↓
$|\psi\rangle \longrightarrow$ How much of ent--?

Definition

$$\boxed{\begin{array}{l} C(|\psi\rangle) = |\langle\psi|\tilde{\psi}\rangle| \\ |\tilde{\psi}\rangle = Y \otimes Y |\psi^*\rangle \end{array}}$$

$|\psi^*\rangle$ → complex conjugate of $|\psi\rangle$

Using density matrix

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\rho = |\psi\rangle\langle\psi|$$

$$\tilde{\rho} = \rho (Y \otimes Y) \rho^{+} (Y \otimes Y)$$

eigenvalue of $\tilde{\rho} = \lambda_1, \lambda_2, \lambda_3, \lambda_4$

$$C = \max[0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4]$$
$$; \lambda_1 > \lambda_2 > \lambda_3, \lambda_4$$

Problems on Concurrency

$0 < c < 1$

min. entanglement $\leftarrow$ 0 ; 1 $\rightarrow$ max entanglement

Example - 1 $\quad |\psi\rangle = |0\rangle \otimes |1\rangle$

$$|\tilde{\psi}\rangle = Y \otimes Y |\psi^*\rangle = [Y \otimes Y (|0\rangle \otimes |1\rangle)]$$

$$= \cancel{y \otimes} \; Y|0\rangle \otimes Y|1\rangle$$

$$i|1\rangle \otimes (-i)|0\rangle$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Y|0\rangle = i|1\rangle$$

$$Y|1\rangle = -i|0\rangle$$

$$|\tilde{\psi}\rangle = |1\rangle \otimes |0\rangle$$

$$\langle \psi | \tilde{\psi} \rangle = (\langle 0| \otimes \langle 1|)(|1\rangle \otimes |0\rangle)$$

$$= \underset{0}{\langle 0|1\rangle} \; \underset{0}{\langle 1|0\rangle} = 0$$

$$\boxed{c = |\langle \psi | \tilde{\psi} \rangle| = 0}$$

example 2 $\quad |s\rangle = \dfrac{|01\rangle - |10\rangle}{\sqrt{2}}$

$$\rho = |s\rangle\langle s| = \left(\frac{|01\rangle - |10\rangle}{\sqrt{2}}\right)\left(\frac{\langle 01| - \langle 10|}{\sqrt{2}}\right)$$

$$= \frac{1}{2}\left[|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|\right]$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{\rho} = \rho(Y \otimes Y)\rho^{+}(Y \otimes Y)$$

$$Y \otimes Y \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \cancel{\rho^{+}}$$

$$\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\tilde{\rho} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\lambda_1 = 1, \lambda_2 = \lambda_3, \lambda_4 = 0$$

$$C(\rho) = \max[0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4] = \max[0, 1]$$

$$C(\rho) = \underline{1}$$

Entanglement of formation.

$$E(\rho) = h\Big(\underbrace{\frac{1 + \sqrt{1 - c^2(\rho)}}{2}}_{x}\Big)$$

$$h(x) = -x\log_2 x - (1-x)\log_2(1-x)$$

Given: $\rho = \frac{5}{6}|\phi^{+}\rangle\langle\phi^{+}| + \frac{1}{24} I_4$

$$|\phi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$\begin{pmatrix} \frac{11}{24} & 0 & 0 & \frac{5}{12} \\ 0 & \frac{1}{24} & 0 & 0 \\ 0 & 0 & \frac{1}{24} & 0 \\ \frac{5}{12} & 0 & 0 & \frac{11}{24} \end{pmatrix}$$

$$\tilde{\rho} = \rho(Y \otimes Y)\rho^{*}(Y \otimes Y)$$

$$= \begin{bmatrix} \frac{221}{576} & 0 & 0 & \frac{55}{144} \\ 0 & \frac{1}{576} & 0 & 0 \\ 0 & 0 & \frac{1}{576} & 0 \\ \frac{55}{144} & 0 & 0 & \frac{221}{576} \end{bmatrix}$$

$$\lambda_1 = \frac{49}{69}$$

$$\underbrace{\frac{1}{576}}_{\lambda_2},\ \underbrace{\frac{1}{576}}_{\lambda_3},\ \underbrace{\frac{1}{576}}_{\lambda_4}$$

$$C(\rho) = \max\left[0, \lambda_1 - \lambda_2\right] = \max\left[0, \underbrace{\frac{49}{64} - \frac{1}{576}}_{0.76}\right]$$

$$= 0.76$$

$$x = \frac{1 + \sqrt{1-(0.76)^2}}{2} = \underline{\underline{0.82}}$$

$$E(P) = -0.82\log_2 0.82 - (1-0.82)\log_2(1-0.82)$$

$$= \underline{\underline{0.67}}$$

$$E(\rho) = h\underbrace{\left(\frac{1+\sqrt{1-c^2(\rho)}}{2}\right)}_{x}$$

$$h(x) = -x\log_2 x - (1-x)\log_2(1-x)$$

$C = 0$

No entanglement

$$x = \frac{1+\sqrt{1+0^2}}{2} = \frac{2}{2} = 1$$

$$E(P) = -1\log_2 1 - \cancel{(1-1)\log(1-1)}$$
$\qquad\qquad 0 \qquad 0$

max entanglement $= \underline{\underline{0}}$

$C = 1$

$$x = \frac{1+\sqrt{1-1^2}}{2} = \frac{1}{2}$$

$$E(P) = -\frac{1}{2}\log_2\frac{1}{2} - \left(\frac{1}{2}\right)\log_2\left(\frac{1}{2}\right)$$

$$= -\cancel{2} \cdot \frac{1}{\cancel{2}} \log_2\left(\frac{1}{2}\right)$$

$$= -\log_2\left(\frac{1}{2}\right) = -\log_2 2^{-1}$$

$$= \underline{\underline{1}}$$

Normalization of wave function

- What, why, How, time evolution

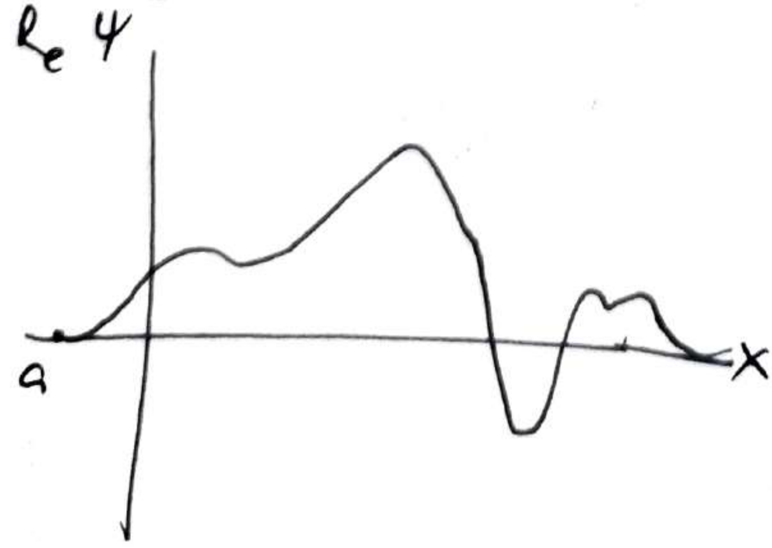
$\underline{\underline{F = ma}}$ in classical computation

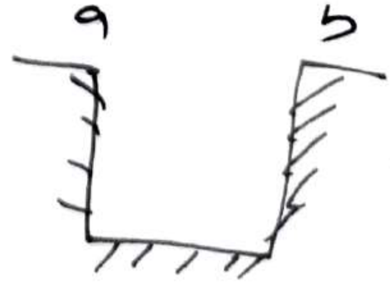
$\Psi(r, t)$

$$\boxed{i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi}$$

Ψ —: Position of time and position

mathematical describs





$\psi^* \psi$ $\qquad z = a + ib$

$\parallel$

Probability distribution $\qquad z^* = a - ib$

$$z^* z = (a - ib)(a + ib)$$

$$= a^2 - (ib)^2 \qquad i = \sqrt{-1}$$

$$= a^2 + b^2 \qquad (i)^2 = -1$$

$$a^2 = -b^2$$

$$a^2 = (ib)^2$$

$$\psi^* \psi \longrightarrow |\psi_{()}|^2$$

$$\int_{-\infty}^{+\infty} \psi^* \psi \, dx > 1$$

$$\psi_1 = A\psi$$

$\downarrow$

Normalization constant

$$\int_{-\infty}^{+\infty} \psi_1^* \psi \, d_x = 1$$

$$i\hbar \frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi$$

$$i\hbar \frac{\partial}{\partial t}(A\psi) = -\frac{\hbar^2}{2m}\frac{\partial^2 (A\psi)}{\partial x^2} + VA\psi$$

$$= i\hbar \cancel{A}\left(\frac{\partial\psi}{\partial t}\right) = -\frac{\hbar^2}{2m} A \frac{\partial^2\psi}{\partial x^2} + AV\psi$$

$$= \cancel{A}\left(-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi\right)$$

$$\psi = \begin{cases} e^{ik_n x} & 0 \le x \le 2\pi \\ 0 & \text{everywhere else} \end{cases}$$

$$\int^{2\pi} \psi^* \psi \, dx = 1$$

identify $e^{ik_n x}$ is normalized

$$\psi^* = e^{-ik_n x}$$

$$\int_0^{2\pi} |\psi|^2 dx$$

$$\psi^*\psi = |\psi|^2$$

$$= \int_0^{2\pi} e^{-ik_nx} e^{ik_nx} \, dx$$

$$\int_0^{2\pi} dx = [x]_0^{2\pi} = 2\pi - 0 = \underline{\underline{2\pi}}$$

$$= 2\pi \times \frac{1}{2\pi}$$

$$A^2 = \frac{1}{2\pi} \;\; A = \sqrt{\frac{1}{2\pi}}$$

$$\psi = Ae^{ik_nx}$$

$$\psi^* = Ae^{-ik_nx}$$

$$\int_0^{2\pi} \psi^* \psi \, dx = 1$$

$$= \int_0^{2\pi} Ae^{-ik_nx} \cdot Ae^{ik_nx}$$

$$= A^2 \int_0^{2\pi} dx = 1$$

$$= A^2 [x]_0^{2\pi} = 1$$

$$A^2 [2\pi - 0] = 1$$

$$A^2 = \frac{1}{2\pi} \quad ; \quad A = \sqrt{\frac{1}{2\pi}}$$

$$\psi = \begin{cases} \sqrt{\frac{1}{2\pi}} e^{iknx} & 0 \le x \le 2\pi \\ 0 & \text{otherwis} \end{cases}$$

$$i\hbar \frac{\partial \Psi_{(x,t)}}{\partial t} = -\frac{\hbar^2}{2\mu} \frac{\partial^2 \Psi_{(x,t)}}{\partial x^2} + V_{(x)} \Psi_{(x,t)}$$

tim = t

$$\Psi_{(x,t)}$$

$$\frac{d}{\partial t} \left(\int_{-\infty}^{\infty} \Psi^* \Psi . dx \right) = 0$$

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial t} (\psi^* \psi) dx \quad - (1)$$

$$\frac{\partial}{\partial t} (\psi^* \psi) = \psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t}$$

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2\mu} \frac{\partial^2 \psi}{\partial t} + V\psi$$

$-\frac{i}{\hbar}$ $\quad i\cancel{\hbar}x = \frac{i}{\hbar}$

$$\boxed{\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar}V}$$

$$\frac{\partial \psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + \frac{i}{\hbar}V\psi^*$$

$$\frac{\partial \psi^*\psi}{\partial t} = \frac{i\hbar}{2m}\psi^*\frac{\partial^2 \psi}{\partial x^2} \cancel{- \frac{i}{\hbar}V\psi\psi^*}$$

$$- \frac{i\hbar}{2m}\psi\frac{\partial^2 \psi^*}{\partial x^2} \cancel{+ \frac{i}{\hbar}V\psi\psi^*}$$

$$= \frac{i\hbar}{2m}\left(\psi^*\frac{\partial^2 \psi}{\partial x^2} - \psi\frac{\partial^2 \psi^*}{\partial x^2}\right)$$

$$= \frac{i\hbar}{2m}\frac{\partial}{\partial x}\left[\psi^*\frac{\partial \psi}{\partial x} - \psi\frac{\partial \psi^*}{\partial x}\right]$$

$$\int_{-\infty}^{+\infty}\frac{\partial}{\partial t}(\psi^*\psi)\,dx$$

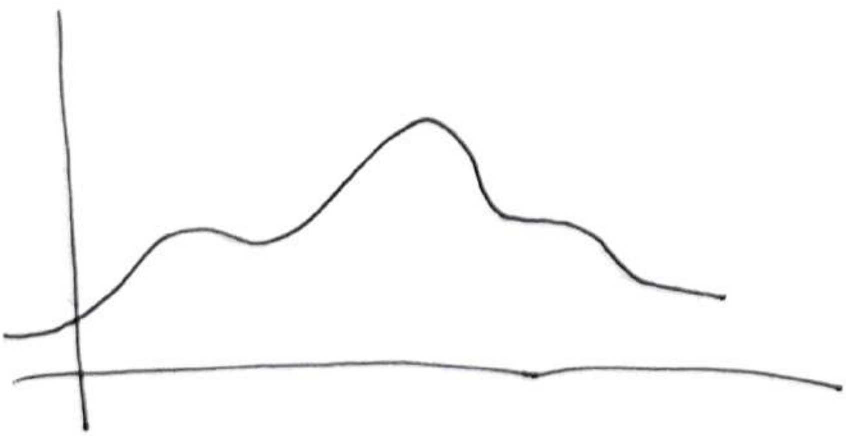
$$\int_{-\infty}^{\infty}\frac{i\hbar}{2m}\frac{\partial}{\partial x}\left(\psi^*\frac{\partial \psi}{\partial x} - \psi\frac{\partial \psi^*}{\partial x}\right)dx$$

$$\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\left(\psi^*\frac{\partial\psi}{\partial x}-\psi\frac{\partial\psi^*}{\partial x}\right)dx$$

$$\int\frac{d}{\cancel{dx}}(x)\,\cancel{dx}$$

$$\int 1\cdot dx = x$$

$$=\frac{i\hbar}{2m}\left[\psi^*\frac{\partial\psi}{\partial x}-\psi\frac{\partial\psi^*}{\partial x}\right]_{+\infty}^{+\infty}$$



$\psi \to 0$ at $\pm\infty$

$= 0$

$$\frac{d}{dt}\left(\int_{-\infty}^{+\infty}\psi^*\psi\,dx\right)$$

Quantum algorithm L-15 NPTEL

Quantum algorithm: history

What does it mean for a Quantum computer to be faster than a classical computer?

Complexity of algorithms
- Different notions of measuring performance
- Computational complexity
- Space complexity
- Communication
- Information
- Query (Query complexity)

Deutch and Jozsa alogrithm

Goal: To create a separation b/w the power of classical and quantum computer

Result: A problem which
- For all inputs, quantum computers can solve with certainity in polynomial time but
- Classical computer take exponential time to solve ~~certain~~ with certainity

Motivation for us
- Two perspective
1) Show why a quantum computer is powerful then any classical computer such setting

2. Learn fundamental quantum modules that enable quantum computers to be powerful

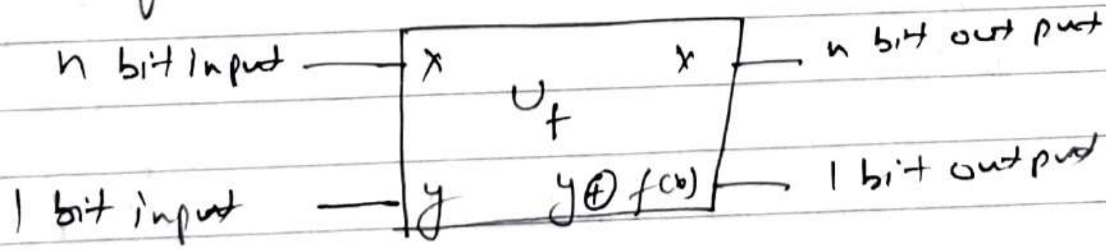
Problem:-

Problem statement:

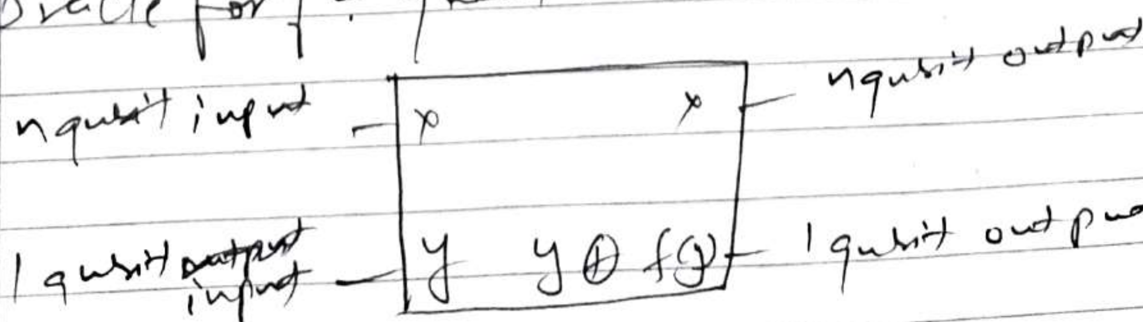
- Oracle or black box access to a function f
  - Input : n bits
  - Output: a single bit, either 0 or 1
- Guarantee that f(x) is either
  - constant - same output for all input or
  - balanced - half input output 0, and the other half output 1

- Problem solution
  - figure out whether f is constant or balanced

Oracle for f: classical



Oracle for f: quantum



Classical algorithm for DJ problem

- Need to find the correct answer always
- Must query more than half of input
- One space - no optimization ~~problem~~ possible to get the answer correctly always
- Query $2^{n-1}+1$ points to f
- Query complexity is exponential in input length n!!

!! solve this problem with certainty in polynomial time!!

step 1 Hadmard transform

$|0^{\otimes n}\rangle$ n

$|1\rangle$

step 0 input

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\left[|0\rangle - |1\rangle\right]$$

step 1 $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$H|a\rangle = \frac{1}{\sqrt{2}}((-1)^0|0\rangle + (-1)^a|1\rangle)$$

$$= \frac{1}{\sqrt{2}}\sum_{x\in\{0,1\}} (-1)^{a\cdot b}|x\rangle$$

$$|a\rangle \xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{a\cdot x}|x\rangle$$

$$|00\ldots0\rangle \xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle$$

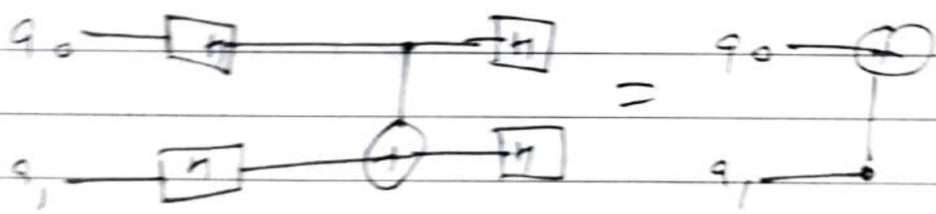
step 2 Phase kickback $\qquad CNOT|x\rangle|y\rangle = |x\rangle|y\oplus x\rangle$

$CNOT|00\rangle = |00\rangle$

$CNOT|01\rangle = |01\rangle$

$CNOT|10\rangle = |11\rangle$

$CNOT|11\rangle = |10\rangle$



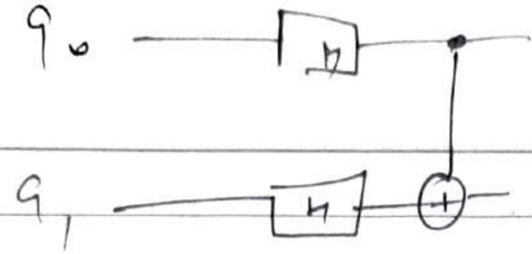
$H\cdot CNOT\cdot H|00\rangle = |00\rangle$

$H\cdot CNOT\cdot H|01\rangle = |11\rangle$

$CNOT|x\rangle|y\rangle = |x\rangle|y\oplus x\rangle$

$H\cdot CNOT\cdot H|10\rangle = |10\rangle$

$H\cdot CNOT\cdot H|11\rangle = |01\rangle$



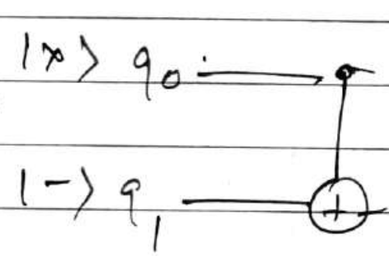
$\text{CNOT}\,|x-\rangle = (-1)^x\,|x-\rangle$

$\text{CNOT}\,|++\rangle = |++\rangle$

$\text{CNOT}\,|1-\rangle = |--\rangle$

$\text{CNOT}\,|-+\rangle = |-+\rangle$

$\text{CNOT}\,|--\rangle = |+-\rangle$



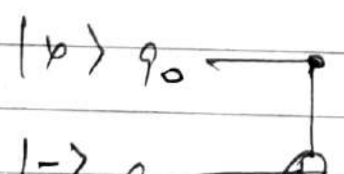
$\text{CNOT}\,|x\rangle|y\rangle = |x\rangle|y+x\rangle$

$$\text{CNOT}\,|x-\rangle = \frac{1}{\sqrt{2}}\,\text{CNOT}\big(|x\rangle(|0\rangle - |1\rangle)\big)$$

$$= \frac{1}{\sqrt{2}}\big(|x\rangle(|0+x\rangle - |1+x\rangle)\big)$$

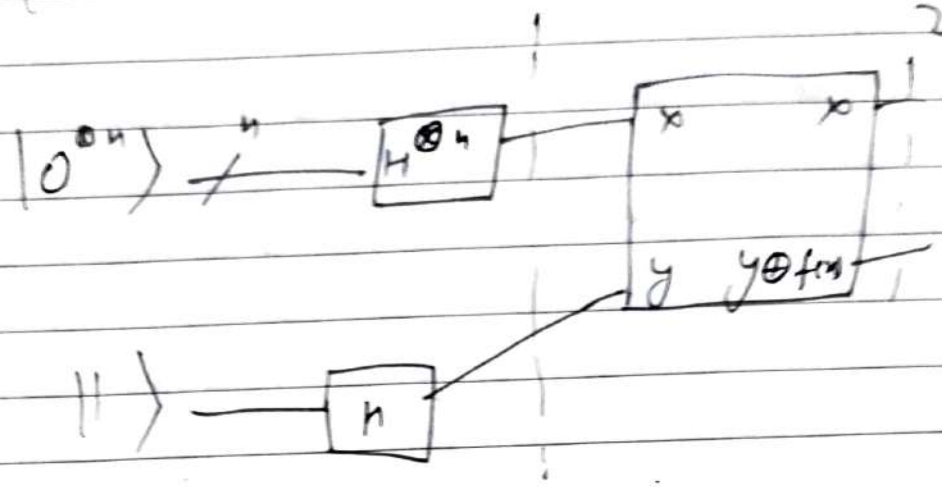
$$= \frac{1}{\sqrt{2}}\big(|x\rangle\,(-1)^x(|0\rangle - |1\rangle)\big)$$

$$= (-1)^x|x\rangle|-\rangle = (-1)^x|x-\rangle$$



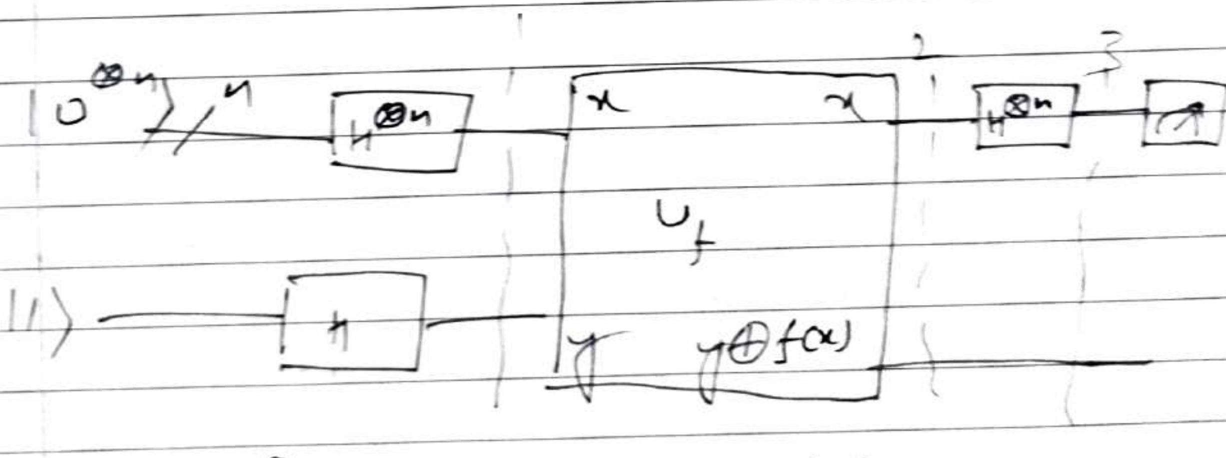
$$|x-\rangle \xrightarrow{CNOT} (-1)^{x} |x-\rangle$$

$$|x-\rangle \xrightarrow{U_f} (-1)^{f(x)} |x-\rangle$$



$$|00\ldots0\rangle \xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle \xrightarrow{f} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle$$

Step 3:- Hadamard transform



$$|a\rangle \xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{a \cdot x} |x\rangle$$

$$\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle \xrightarrow{H^{\otimes n}} \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} \left(\sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle \right)$$

Rewritten as $\frac{1}{2^n} \sum_{y \in \{0,1\}^n} \left(\sum_{x \in \{0,1\}^n} (-1)^{f(x)} (-1)^{x \cdot y} \right) |y\rangle$

If measurement output is all zeros: f is constant
for any other output: f is balanced

, iff f is constant for $y = |00\ldots0\rangle$

$$\sum_{x \in \{0,1\}^n} (-1)^{f(x)} (-1)^{x \cdot y} = \sum_{x \in \{0,1\}^n} (-1)^{f(x)}$$

$\Rightarrow (-1)^{f(x)} \cdot 2^n = \pm 2^n$ if $f(x)$ is constant

$\Rightarrow (-1)^0 \cdot 2^{n-1} + (-1)^1 \cdot 2^{n-1} = 0$ if $f(x)$ is balanced

Bernstein - Vazirani algorithm

Motivation of BV:

: DJ: quantum computers can solve a problem with certainity in polynomial time but classical computers take exponential time on some inputs with certainity

- Probablistic algorithm (a.k.a BPP) allows for wrong answer with small probability. Therefore considered a practical class for classical computers
- classical BPP algorithm take polynomial time to solve the DJ problem

P version : Query $2^{n-1}+1$ point

BPP version : Query random set of point

Intuition:

- for a balanced function,
  - Two buckets to choose from
  - Probability of input to keep coming from a single bucket reduces exponentially

- BV A problem which quantum computers can solve with very high probability but any classical computer cannot solve with probability $> \frac{1}{2}$ using the same time as a quantum computer.

Problem:

Given

- Oracle for function f,
  - Input: n bits
  - Output 1 bit, either 0 or 1
  - There exists an n-bit value a such that $f(x) = a \cdot x \pmod 2$

output

- n bit string a

Instead of the function being balanced or constant as in DJ, the function is BV is guaranteed to return the dot product of the input with some fixed string a classical solution

classical oracle: $f(x) = a \cdot x \pmod 2$

Algorithm

Query the oracle with sequence of inputs

- $100\ldots0$ output a_1
- $010\ldots0$: output a_2
- $001\ldots0$: output a_3
- $000\ldots1$: output a_n

$a = 1011$

$1000 \cdot 1011 = 1$

$0100 \cdot 1011 = 0$

$0010 \cdot 1011 = 1$

$0001 \cdot 1011 = 1$

classical solution: Lower bound.

- we can't do in less than n calls
- consider any function which has a associated with it
- After $n-1$ call to the oracle with input x_1 to x_{n-1}, we will have $n-1$ equations of the form

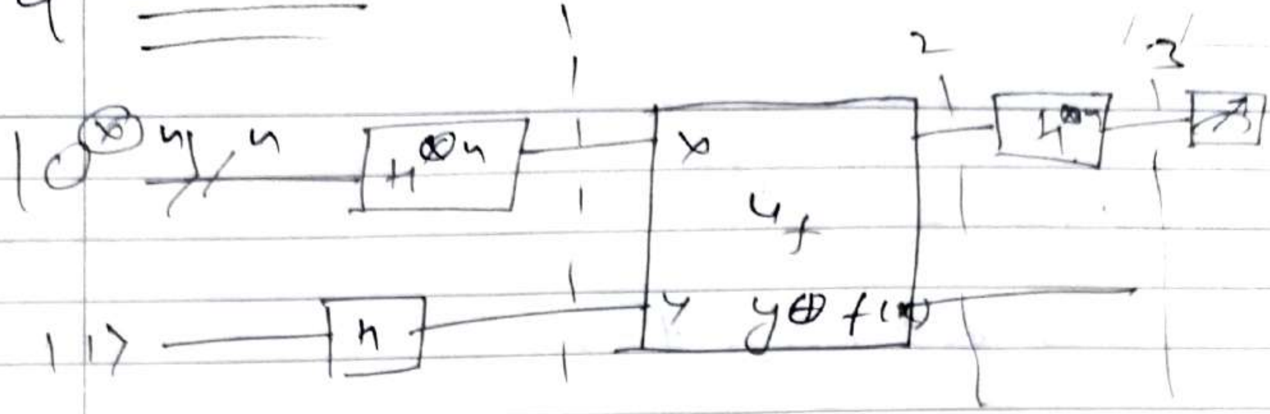
$$x_{1,1} a_1 + \ldots + x_{1,n} a_n = f(x_1)$$

$$\ldots$$

$$x_{n-1,1} a_1 + \ldots + x_{n-1,n} a_n = f(x_{n-1})$$

we have $n-1$ eqn. with n variables and we cannot find the unique soln a for this undetermined system of eqns

Quantum s/w:



$$|a\rangle \xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{a\cdot x} |x\rangle$$

$$|00\ldots0\rangle \xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle$$

$$|x\rangle \xrightarrow{f_a} (-1)^{a\cdot x} |x\rangle$$

Step 2

Phase kick

$$|00\ldots0\rangle \xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle \xrightarrow{f_a} \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{a\cdot x} |x\rangle$$

Step 3 ! Inverse Hadamard transform

$$|a\rangle \xrightarrow{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{a\cdot x} |x\rangle$$

Bra — $\langle|$

ket — $|\rangle$

$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{matrix} 0^{th} \\ 1^{st} \end{matrix}$

$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{matrix} 0^{th} \\ 1^{st} \end{matrix}$

computation basis of 2D

$B_2 = [|0\rangle, |1\rangle]$

Eg $\begin{bmatrix} 3+5i \\ 7 \end{bmatrix}$

write using 2D comp basis

sol: $\begin{bmatrix} 3+5i \\ 7 \end{bmatrix} = 3+5i|0\rangle + 7|1\rangle$

Bra notation

- $|\psi\rangle = (\langle\psi|)^{\dagger}$
- $\langle\psi| = (|\psi\rangle)^{\dagger}$

$\dagger$: conjugate transpose

$\dagger$: $*$ T

Eg. $|\psi\rangle = 3+5i|0\rangle + 7|1\rangle$

write $\langle\psi| = ?$

sol: $|\psi\rangle = \begin{bmatrix} 3+5i \\ 7 \end{bmatrix}$

conjugate $= \begin{bmatrix} 3-5i \\ 7 \end{bmatrix}$ | imaginary term change its sign. *

transpose operation

$|\psi^*\rangle^T = [|\psi\rangle]^{\dagger}$

$= [3-5i \quad 7]$

$\langle\psi| = [3-5i \quad 7]$

---

1. $\langle\psi|\ |\phi\rangle$
2. $|\psi\rangle\langle\phi|$
3. $\langle\psi|\ \langle\phi|$
4. $|\psi\rangle\ |\phi\rangle$

1. $\langle\psi|\ |\phi\rangle$ or $\langle\psi|\phi\rangle$

inner product

Eg find inner product ~~$|\psi\rangle = 3+5i$~~

$|\psi\rangle = 3+5i|0\rangle + 6|1\rangle$ 7

$|\phi\rangle = 5|0\rangle + 7|1\rangle$

$\langle\psi| = [3-5i \quad 6]$

$|\phi\rangle = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

$\langle\psi|\phi\rangle = [3-5i \quad 6] \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

$= 15-25i + 42$

$= -25i + 57$

Ans

2. $|\psi\rangle\langle\phi| =$ matrix

~~Eg~~ $|\psi\rangle = 3+5i|0\rangle + 6|1\rangle$

$|\phi\rangle = 5|0\rangle + 7|1\rangle$

Find $|\psi\rangle\langle\phi|$

soln $\langle\phi| = [5, 7]$

$$|\psi\rangle\langle\phi| = \begin{bmatrix} 3+5i \\ 6 \end{bmatrix} [5, 7]$$

$$\begin{bmatrix} 15+25i & 21+35i \\ 30 & 42 \end{bmatrix}$$

Ans

(5) $\langle\psi|\langle\phi|$ X not possible
because not fulfil the condition of same row and same column

(4) $|\psi\rangle|\phi\rangle$

$|\psi\rangle \otimes |\phi\rangle$

$|\psi\rangle|\phi\rangle = |\psi\rangle \otimes |\phi\rangle$

$= |\psi\phi\rangle$

$|\psi\rangle^{\otimes 5} = |\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle$

eg $v = \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}$. $w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$v \otimes w = ?$

soln $v \otimes w$ $\begin{bmatrix} 1w \\ 0w \\ 7w \end{bmatrix}$

$$= \begin{bmatrix} 1\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ 0\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ 7\begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 7 \\ 14 \end{bmatrix}$$

Ans

$|\psi\rangle|\phi\rangle = |\psi\rangle \otimes |\phi\rangle$

Find $|\psi\rangle\langle\phi|$

Eg $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$H^{\otimes 2} = H \otimes H = ?$

Soln. $H \otimes H = \begin{bmatrix} \frac{1}{\sqrt{2}}H & \frac{1}{\sqrt{2}}H \\ \frac{1}{\sqrt{2}}H & -\frac{1}{\sqrt{2}}H \end{bmatrix}$

$H^{\otimes 2} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

Ans

Properties: ① $A \otimes (B \otimes C) = (A \otimes B) \otimes C$

$= A \otimes B \otimes C$

② $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

③ $A \otimes (B+C) = (A \otimes B) + (A \otimes C)$

$(A+B) \otimes C = (A \otimes C) + (B \otimes C)$

④ α : scaler

$\alpha A \otimes B = A \otimes \alpha B$

$= \alpha (A \otimes B)$

freely floating scaler.

computation Basis

2D

$B_2 = [|0\rangle, |1\rangle]$

$4D = 2^2$

$B_2 \otimes B_2 = [|00\rangle, |01\rangle, |10\rangle, |11\rangle]$

$8D = 2^3$

$B_2 \times B_4 = [|000\rangle, |001\rangle, |010\rangle, \ldots |111\rangle]$

$|01\rangle$

length $= 2$

$2^2 = \underline{4}$.

$(0\,1)_2 = 1$

$$= \begin{array}{c} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array}$$

$|101\rangle = ?$

$2^3 =$

$(101)_2 = 5$

$$|101\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}$$

Ex

$|\psi\rangle = 7\,|101\rangle + 3i\,|111\rangle$

write in vector form

Sol $|101\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$(111)_2 = 7$

$|111\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} =$

$2^2\ 2^1\ 2^0$

$(1\cdot 2^2)(0\cdot 2^1)(1\cdot 2^0)$

$4 \quad 0 \quad 1 = 5$

$1\cdot 2^2 + 1\cdot 2^1 + 1\cdot 2^0$

$4 + 2 + 1$

7

classical unit of data

0, 1

QC

Qubit

states of qubit

① Pure state

0 or 1

② Superposition

0 and 1

Probability

Math Representation

$\psi = \alpha|0\rangle + \beta|1\rangle$

α, β are amplitudes

$\alpha, \beta \in \mathbb{C}$

prob of 0 $= |\alpha|^2$

prob of 1 $= |\beta|^2$

$\alpha^2 = \alpha^* \alpha$

$\beta^2 = \beta^* \alpha$

Normalization constant

$|\alpha|^2 + |\beta|^2 = 1$

$|\psi\rangle = \alpha\begin{bmatrix}1\\0\end{bmatrix} + \beta\begin{bmatrix}0\\1\end{bmatrix}$

$= \begin{bmatrix}\alpha\\\beta\end{bmatrix} \leftarrow$ unit vector

$\||\psi\rangle\| = 1$

or

$\langle\psi|\psi\rangle = 1$

eg $|\psi\rangle = \frac{-4}{5} i|0\rangle + \frac{3}{5}|1\rangle$

(a) If $|\psi\rangle$ a valid qubit

(b) what is the prob of 0

Part (a) $|\alpha|^2 + |\beta|^2 = 1$

$\||\psi\rangle\| = 1$

$\langle\psi|\psi\rangle = 1$

$|-\frac{4}{5} i|^2 + |\frac{3}{5}|^2 = 1$

$(\frac{4}{5} i \times -\frac{4}{5} i) + (\frac{3}{5} \times \frac{3}{5})$

$= \frac{16}{25} + \frac{9}{25} = 1$

Part (b) Prob of 0 $= |-\frac{4}{5} i|^2$

$= \frac{16}{25} = 0.64$

Classical comp

Prob of $|1\rangle = \left|\frac{3}{5}\right|^2$

$= \frac{9}{25} = 0.36$

$|\psi\rangle = -\frac{4}{5}i|0\rangle + \frac{3}{5}|1\rangle$

$|\phi\rangle = \frac{|0\rangle}{\sqrt{2}} + \frac{|1\rangle}{\sqrt{2}}$

$\cancel{|\psi\rangle \otimes \langle\phi}$ $|\psi\rangle \otimes |\phi\rangle$

$= |\psi \cdot \phi\rangle$

$= \left(-\frac{4}{5}i|0\rangle + \frac{3}{5}|1\rangle\right)$

$\otimes \left(\frac{|0\rangle}{\sqrt{2}} + \frac{|1\rangle}{\sqrt{2}}\right)$

$= \alpha\frac{4}{5\sqrt{2}}|00\rangle - \beta\frac{4}{5\sqrt{2}}|01\rangle$

$+ \gamma\frac{3}{5\sqrt{2}}|10\rangle + \delta\frac{3}{5\sqrt{2}}|11\rangle$

$|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$

Prob of 00 =

$= |\alpha|^2 = \left|\frac{-4i}{5\sqrt{2}}\right|^2$

$= \frac{16}{50} = 0.32$

Operation of qubits

- Measure
  - Full
  - Partial
- Transform using Quantum gate

L-3 Rules

(1) Superposition

$\xrightarrow{M}$ Purestate

(2) Normalization contant must always uphold.

$P(A \cup B) = P(A) + P(B)$

eg (1) $|\psi\rangle = \frac{1}{2}|00\rangle - \frac{i}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$

Measurement	Prob.	Resultant state
00	$\left\|\frac{1}{2}\right\|^2 = \frac{1}{4} = 0.25$	$\|\psi\rangle = \|00\rangle$
10	$\left\|\frac{-i}{2}\right\|^2 = \frac{i}{2} \times \frac{-i}{2} = \frac{1}{4} = 0.25$	$\|\psi\rangle = \|10\rangle$
11	$\left\|\frac{1}{\sqrt{2}}\right\|^2 = \frac{1}{2} = 0.5$	$\|\psi\rangle = \|11\rangle$

Square of amplitude

Measurement	Probability	Resultant state
1st qubit = 1 ;	$\left\|\frac{i}{2}\right\|^2 + \left\|\frac{1}{\sqrt{2}}\right\|^2 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$	$\|\psi\rangle = \dfrac{\frac{-i}{2}\|10\rangle + \frac{1}{\sqrt{2}}\|11\rangle}{\|\frac{-i}{2}\|10\rangle + \frac{1}{\sqrt{2}}\|11\rangle\|} = \dfrac{\frac{-i}{2}\|10\rangle + \frac{1}{2}\|11\rangle}{\sqrt{\frac{3}{4}}} = \frac{-i}{\sqrt{3}}\|10\rangle + \frac{\sqrt{2}}{\sqrt{3}}\|11\rangle$
2nd qubit = 0	$\left\|\frac{1}{2}\right\|^2 + \left\|\frac{-i}{2}\right\|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$	$\|\psi\rangle = \dfrac{\frac{1}{2}\|00\rangle - \frac{i}{2}\|10\rangle}{\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}}\|00\rangle - \frac{i}{\sqrt{2}}\|10\rangle$

Measurement	Probability	Resultant state
1st qubit = 1.	$\left\|\frac{-i}{2}\right\|^2+\left\|\frac{1}{\sqrt{2}}\right\|^2$ $=\frac{1}{4}+\frac{1}{2}=\frac{3}{4}$	$\|\psi\rangle=\dfrac{\frac{-i}{2}\|10\rangle+\frac{1}{\sqrt{2}}\|11\rangle}{\left\|\frac{-i}{2}\|10\rangle+\frac{1}{\sqrt{2}}\|11\rangle\right\|}$ $=\dfrac{\frac{-i}{2}\|10\rangle+\frac{1}{2}\|11\rangle}{\sqrt{\frac{3}{4}}}$ $=-\frac{i}{\sqrt{3}}\|10\rangle+\frac{\sqrt{2}}{\sqrt{3}}\|11\rangle$
2nd qubit = 0	$\left\|\frac{1}{2}\right\|^2+\left\|\frac{i}{2}\right\|^2$ $=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$	$\|\psi\rangle=\dfrac{\frac{1}{2}\|00\rangle-\frac{i}{2}\|10\rangle}{\frac{1}{\sqrt{2}}}$ $=\frac{1}{\sqrt{2}}\|00\rangle-\frac{i}{\sqrt{2}}\|10\rangle$

$P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$\left|\frac{-i}{2}\right|^2+\left|\frac{1}{\sqrt{2}}\right|^2\neq 1.$

Example $\frac{1}{\sqrt{5}}|0000\rangle-\sqrt{\frac{2}{5}}|0100\rangle+\sqrt{\frac{1}{5}}|1111\rangle+\sqrt{\frac{1}{5}}|0110\rangle$

What is the prob and resultant state if 1st and 4th qubit are 0

$P(A\cup B\cup C)=P(A)+P(B)+P(C)$

Soln Prob = $\left|\frac{1}{\sqrt{3}}\right|^2 + \left|-\sqrt{\frac{2}{3}}\right|^2 + \left|\frac{1}{\sqrt{3}}\right|^2 = \frac{4}{3}$

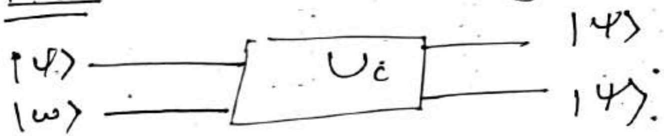
Resultant state $|\psi\rangle = \frac{1}{\sqrt{3}}|0000\rangle - \sqrt{\frac{2}{3}}|0100\rangle + \frac{1}{\sqrt{3}}|0110\rangle$

$$\frac{2}{\sqrt{3}}$$

$$= \frac{1}{2}|0000\rangle - \frac{1}{\sqrt{2}}|0100\rangle + \frac{1}{2}|0110\rangle$$

No cloning theorem L-4

Theorem There is no unitary operator that can clone arbitrary qubits



$$U_c|\psi\rangle|w\rangle = |\psi\rangle|\psi\rangle$$

- All unitary operator must be linear

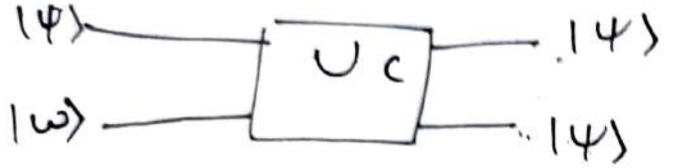
$U_c(a|\psi\rangle + b|\varphi\rangle)$

$a U_c|\psi\rangle + b U_c|\varphi\rangle$

no cloning theorem

there is no unitary operator that can clone arbitrary qubits

sign of 1/P = ...



$|\omega\rangle = |0\rangle$

$$\boxed{U_c|\psi\rangle|\omega\rangle = |\psi\rangle|\psi\rangle}$$

- All unitary operators is linear.

$U_c(a|\psi\rangle + b|\varphi\rangle)$

$= a\, U_c|\psi\rangle + b\, U_c|\varphi\rangle$

Proof 1

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$\therefore |\psi\rangle|0\rangle = \alpha|00\rangle + \beta|10\rangle$

$U_c(\alpha|00\rangle + \beta|10\rangle) = \alpha U_c|00\rangle + \beta U_c|10\rangle$

To prof LHS = RHS

LHS $U_c(\alpha|00\rangle + \beta|10\rangle)$

$= (\alpha|00\rangle + \beta|10\rangle)(\alpha|00\rangle + \beta|10\rangle)$

$$= \alpha^2|0000\rangle + \alpha\beta|0010\rangle + \beta\alpha|1000\rangle + \beta^2|0010\rangle \quad — ①$$

<u>RHS</u> $\alpha U_c|00\rangle + \beta U_c|10\rangle$

$$= \alpha|00\rangle|00\rangle + \beta|10\rangle|10\rangle$$

$$= \alpha|0000\rangle + \beta|1010\rangle \quad — ②$$

$$LHS \neq RHS$$

$$① \neq ②$$

No linear hence not exist

<u>Other proof</u>

Unitary operators preserve inner product.

$$A|x\rangle = |m\rangle$$

$$A|y\rangle = |n\rangle$$

$$\underbrace{\langle x|y\rangle}_{\text{Before operation}} = \underbrace{\langle m|n\rangle}_{\text{after operation.}}$$

<u>Proof</u> $\underline{U_c}$

$$U_c|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle$$

$$U_c|\varphi\rangle|0\rangle = |\varphi\rangle|\varphi\rangle$$

$$(\langle\psi|\langle 0|)(|\varphi\rangle|0\rangle) = (\langle\psi|\langle\psi|)(|\varphi\rangle|\varphi\rangle)$$

$$\langle\psi|\varphi\rangle\cdot\langle 0|0\rangle = \langle\psi|\varphi\rangle\cdot\langle\psi|\varphi\rangle$$

$$\langle 0|0\rangle = 1$$

$$\langle\psi|\varphi\rangle = \langle\psi|\varphi\rangle^2$$

$$1 = 1^2 \text{ orthogonal}$$

$$0 = 0^2$$

inner product = 0 means orthogonal vector

inner product = 1 means same vectors

Entanglement L-6

① Given a state of multiply entangled qubits and cannot express individual qubit qubits seperately.

Ex
$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} \neq (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$$

seperately

$$\frac{|01\rangle + |00\rangle}{\sqrt{2}} = |0\rangle \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

② Given a state of multiply entangled qubit, measuring any qubit individually reveals all other qubit.

Ex

Entangled

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

1st (M) 2nd

$1/0 \longrightarrow 0$

$\quad \longrightarrow 1$

2nd (M)

$1/0 \xrightarrow{1st} 0$

$\quad \longrightarrow 1$

seperably

$$\frac{|01\rangle + |00\rangle}{\sqrt{2}}$$

1st $|0\rangle$

2nd

$|0\rangle \sim \frac{1}{2}$

$|1\rangle \sim \frac{1}{2}$

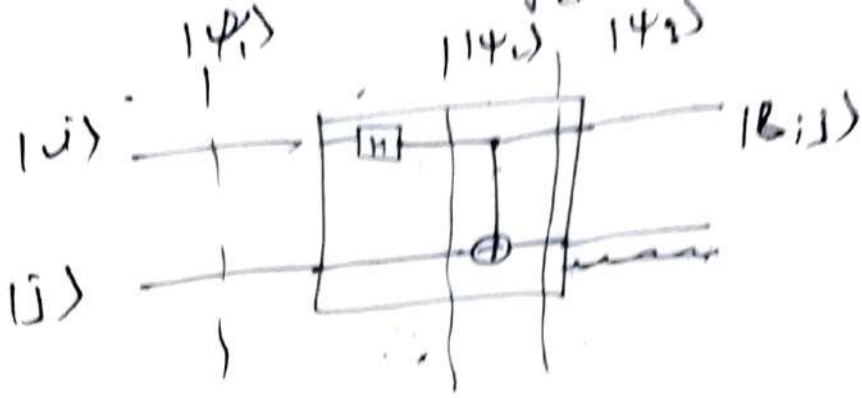
Bell state (EPR state)

$$|\phi^+\rangle = |B_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$|\phi^-\rangle = |B_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$|\psi^+\rangle = |B_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$|\psi\rangle = |B_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$



$|0\rangle$

$|1\rangle$

$|\psi_1\rangle = |0\rangle|1\rangle = |01\rangle$

$|\psi_2\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|1\rangle$

$= \frac{|01\rangle + |11\rangle}{\sqrt{2}}$

$|\psi_3\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} = |B_{01}\rangle$

Given $|\omega\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$.

find if $|\omega\rangle$ is entangled or seperate?

Proof Proof by contraditction (PBC)

$$\frac{|01\rangle - |10\rangle}{\sqrt{2}} = (\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$$

$$\frac{|01\rangle}{\sqrt{2}} - \frac{|10\rangle}{\sqrt{2}} = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$$

$\alpha\gamma = 0$

$\alpha\delta = \frac{1}{\sqrt{2}}$

$\beta\gamma = -\frac{1}{\sqrt{2}}$

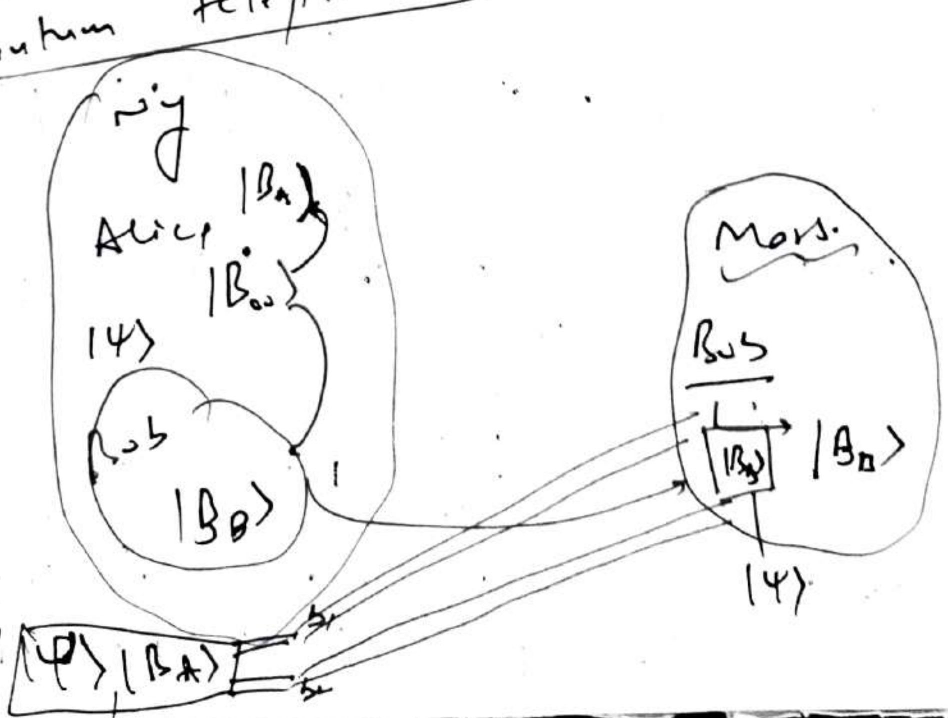
$\beta\delta = 0$ either $\beta = 0$
or $\delta = 0$

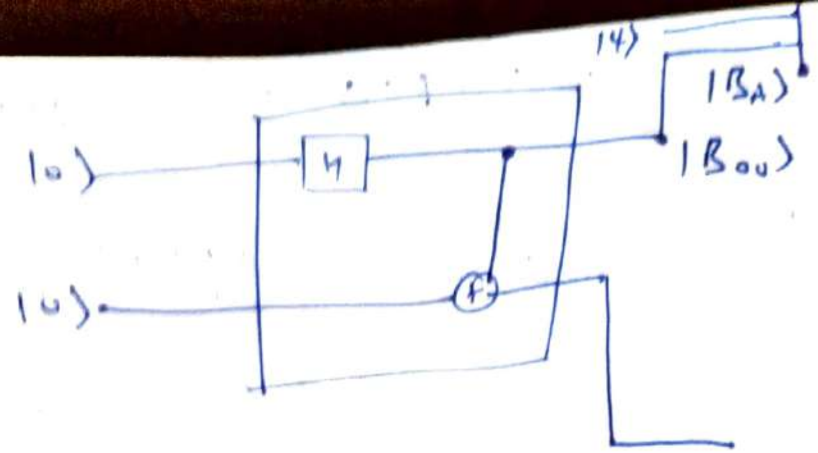
if $\beta = 0$ then $\beta\gamma = 0$ this is not possible

$\therefore \beta \neq 0$

$\delta \neq 0$

Quantum teleportation





$|\omega_1\rangle = |0\rangle|0\rangle$

$|\omega_2\rangle = H|0\rangle|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}|0\rangle = \frac{|00\rangle + |10\rangle}{\sqrt{2}}$

$|\omega_3\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = |B_{00}\rangle$

$|\omega_4\rangle = |\psi\rangle|B_{00}\rangle$ $\qquad |\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

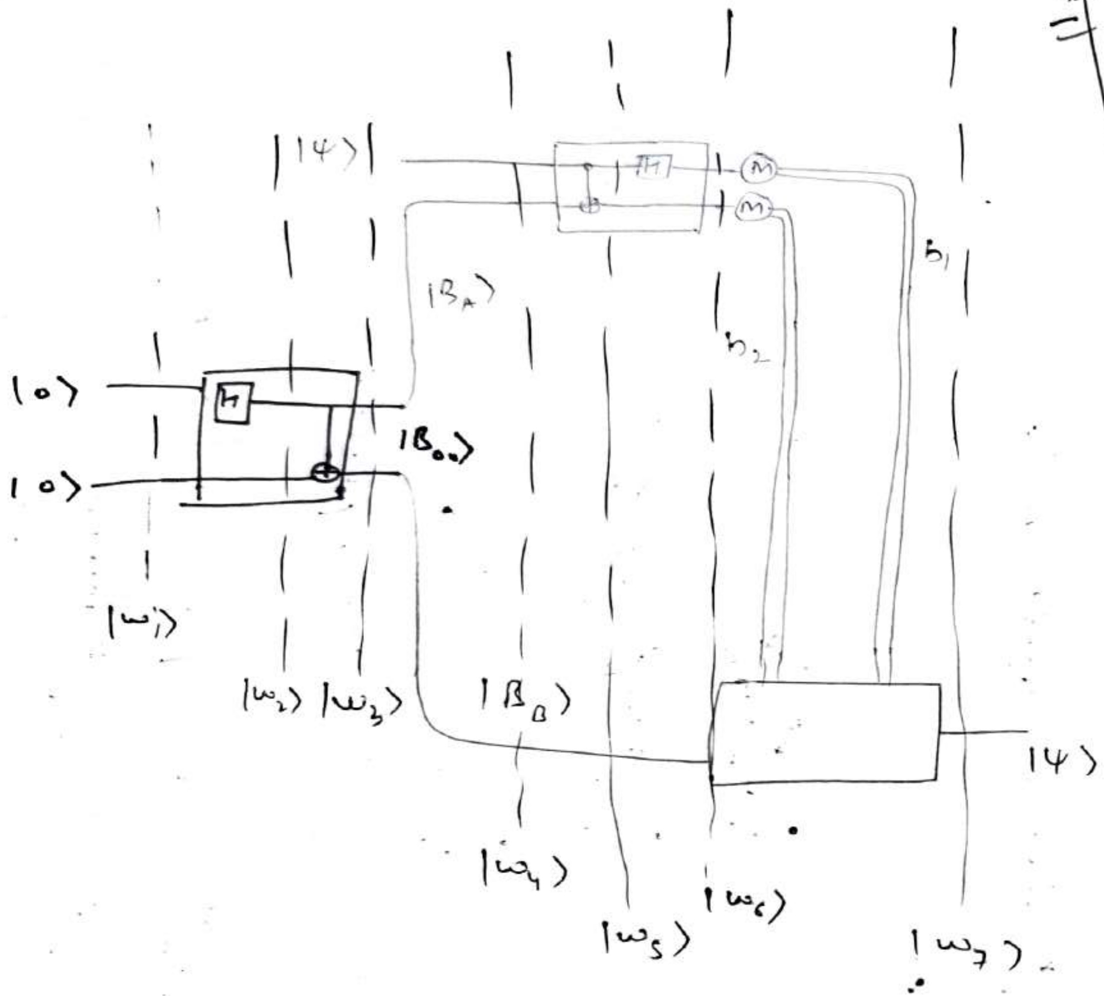
$= (\alpha|0\rangle + \beta|1\rangle)\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)$

$= \frac{\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle}{\sqrt{2}}$

$|\omega_5\rangle = \frac{\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle}{\sqrt{2}}$

$|\omega_6\rangle = \frac{1}{\sqrt{2}}\left[\alpha\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}|00\rangle\right) + \alpha\frac{|0\rangle + |1\rangle}{\sqrt{2}}|11\rangle + \beta \frac{|0\rangle - |1\rangle}{\sqrt{2}}|10\rangle + \beta\frac{|0\rangle - |1\rangle}{\sqrt{2}}|01\rangle\right]$

$|\omega_6\rangle = \frac{1}{2}\Big[\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle\Big]$



$$w_6 = \frac{1}{2}\Big[|00\rangle(\alpha|0\rangle+\beta|1\rangle) + |10\rangle(\alpha|0\rangle-\beta|1\rangle) + |01\rangle[\alpha|1\rangle+\beta|0\rangle] + |11\rangle(\alpha|1\rangle-\beta|0\rangle)\Big]$$

M	Prob	Resultant state	operations
00	$\frac{1}{4}$	$\alpha\|0\rangle+\beta\|1\rangle$	No operation
10	$\frac{1}{4}$	$\alpha\|0\rangle-\beta\|1\rangle$	Z gate $Z\|B_B\rangle = \alpha\|0\rangle+$
01	$\frac{1}{4}$	$\alpha\|1\rangle+\beta\|0\rangle$	$X\|B_B\rangle = \alpha\|0\rangle+\beta\|1\rangle$
11	$\frac{1}{4}$	$\alpha\|1\rangle-\beta\|0\rangle$	$X\|B_B\rangle = \alpha\|0\rangle-\beta\|1\rangle$, $Z(X B_B) = \alpha\|0\rangle+\beta\|1\rangle$

$$|\psi\rangle = \alpha|0\rangle+\beta|1\rangle$$

Quantum fourier transform

Basic of Discrete FT :

Primitive of roots of unitary

$z^n = 1$, z is $\mathbb{C}$

- n roots (sol^{n}s)

$$[\omega_n^0, \omega_n^1, \omega_n^2 -- \omega_n^{n-1}]$$

$$\omega_n^k = e^{2\pi i \frac{k}{n}}$$

1. ω_n^k lies on unit circle

$$|\omega_n^k|^n = 1$$

$$= \sqrt{\omega_n^{k^*} \cdot \omega_n^k}$$

$$= \sqrt{e^{-2\pi i \frac{k}{n}} e^{2\pi i \frac{k}{n}}}$$

$$= \sqrt{e^0}$$

$$= \underline{\underline{1}}$$ Hence proved.

2. ω_n^k is a periodic function

$$\omega_n^k = \omega_n^{k \bmod n}$$

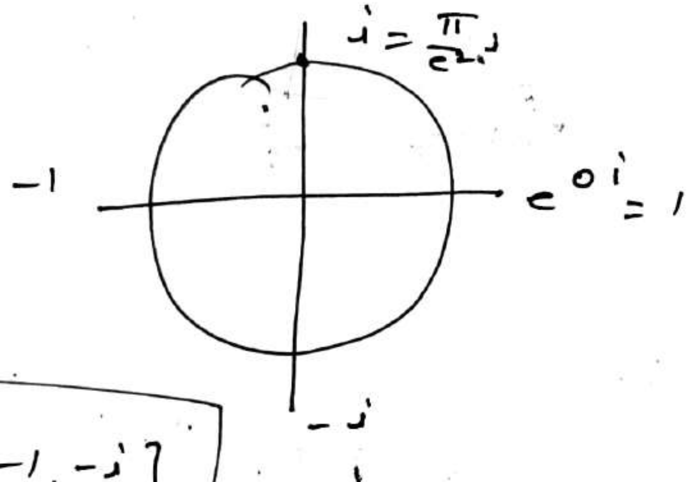
eg write all roots $z^4 = 1 \longrightarrow [1, -1, i, -i]$

soln $[\,w_4^0 \; w_4^1 \; w_4^2 \; w_4^3\,]$

$= [e^{2\pi i \frac{0}{4}}, e^{2\pi i \frac{1}{4}}, e^{2\pi i \frac{2}{4}}, e^{2\pi i \frac{3}{4}}]$

$= [e^{0i}, e^{\frac{\pi}{2} i}, e^{\pi i}, e^{\frac{3}{2}\pi i}]$

$= \{ e^{i\theta}$



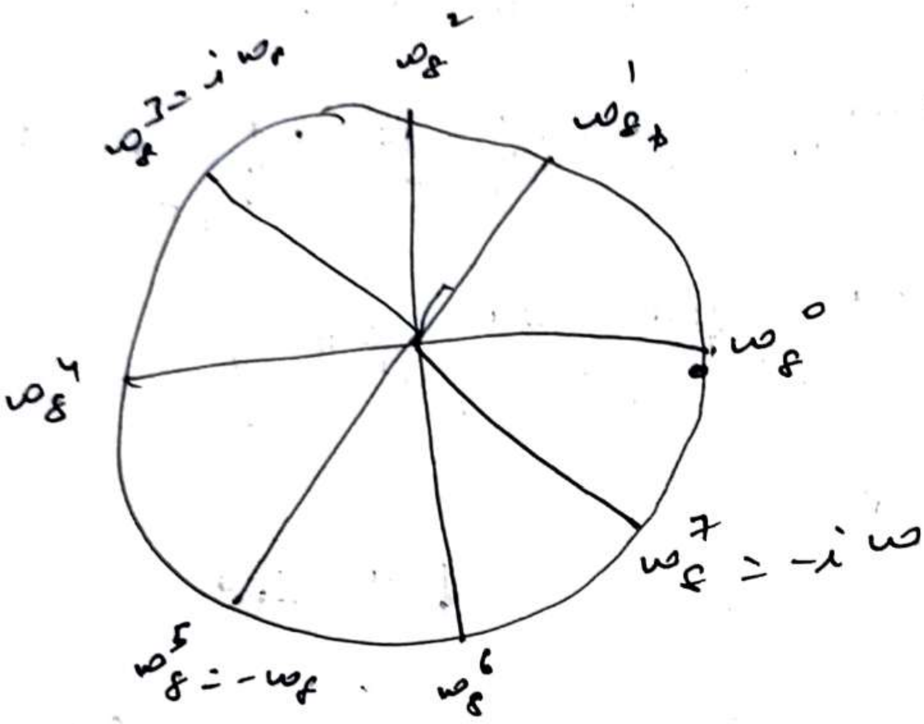
$$\boxed{[1, i, -1, -i]}$$

eg $w_4^4 = w_4^{4 \bmod 4} = w_4^0$

$w_4^7 = w_4^{7 \bmod 4} = w_4^3 = -i$

---

eg:2 write all possible roots $z^8 = 1$



$\frac{360}{8} = 45$

$\{1, \omega_8, i\omega_8, -1, -\omega_8, -i, -i\omega]$

$$F_n = \frac{1}{\sqrt{n}} \begin{pmatrix} \omega_n^{0\times 0} & \omega_n^{0\times 1} & \omega_n^{0\times 2} & \cdots & \omega_n^{0\times(n-1)} \\ & & & & \omega_n^{k\times(n-1)} \\ \omega_n^{k\times 0} & \omega_n^{k\times 1} & & & \\ & & & & \omega_n^{(n-1)(n-1)} \\ \cdots & & & & \end{pmatrix}$$

$$F_n = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \omega_n^3 & \cdots & \omega_n^{(n-1)} \\ 1 & \omega_n^2 & \omega_n^4 & \omega_n^6 & \omega_n^8 \cdots & \omega_n^{2(n-1)} \\ 1 & \omega_n^3 & \omega_n^6 & \omega_n^9 & \cdots & \omega_n^{3(n-1)} \\ 1 & & & & & \\ \vdots & & & & & \omega_n^{(n-1)(n-1)} \\ 1 & & & & & \end{pmatrix} \begin{array}{l} \to \text{row \#1} \\ \to \text{row \#2} \\ \\ \\ \\ \\ \\ \end{array}$$

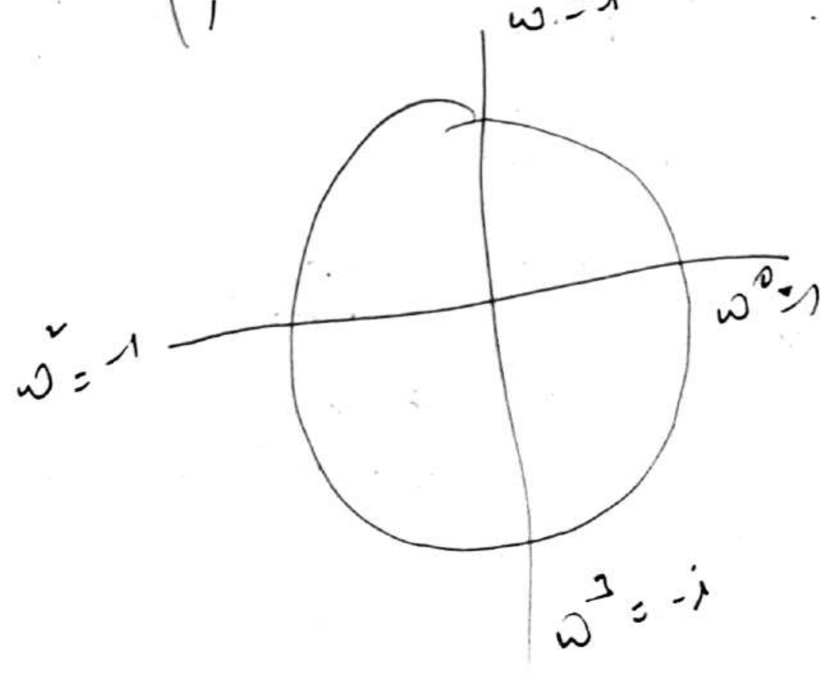
$\omega_n^{k \bmod n}$

<u>Example</u> Transform $|\psi\rangle = \dfrac{|0\rangle + |3\rangle}{\sqrt{2}}$ using F_4

$F_4 |\psi\rangle = |\varphi\rangle$

$$F_4 = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4 & \omega_4^2 & \omega_4^3 \\ 1 & \omega_4^2 & \omega_4^4 & \omega_4^6 \\ 1 & \omega_4^3 & \omega_4^6 & \omega_4^9 \end{pmatrix} \begin{array}{l} \to \text{1st} \\ \to \text{2nd} \\ \to \text{3rd} \\ \\ \end{array} \quad \frac{1}{\sqrt{2}} \begin{array}{c} \\ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array}$$

$$\begin{pmatrix} 1 \\ 0 \\ \cdot \\ 1 \end{pmatrix}$$



$$= \frac{1}{2\sqrt{2}}\begin{pmatrix}2\\1-i\\0\\1+i\end{pmatrix} = \varphi$$

$$\cancel{|\varphi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{\ldots}{2\sqrt{2}}|1\rangle + \frac{\ldots}{4\sqrt{2}}|3\rangle}$$

$$\boxed{|\varphi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1-i}{2\sqrt{2}}|1\rangle + \frac{1+i}{2\sqrt{2}}|3\rangle}$$

$$\begin{array}{ccccc} |0\rangle & \text{or} & |1\rangle & \text{or} & |3\rangle \\ \downarrow & & \downarrow & & \downarrow \\ \frac{1}{2} & & \left|\frac{1-i}{2\sqrt{2}}\right|^2 & & \left|\frac{1+i}{2\sqrt{2}}\right|^2 \end{array}$$

sampling

Properties of DFT

(1) DFT is unitary

$$\langle \vec{c}_j | \vec{c}_k \rangle = \begin{cases} 0 & j \neq k \text{ orthogonal} \\ 1 & j = k \text{ (normalized)} \end{cases}$$

$$\vec{c}_j = \frac{1}{\sqrt{n}} \begin{pmatrix} \omega^{0\times j} \\ \omega^{1\times j} \\ \omega^{2\times j} \\ \vdots \\ \omega^{(n-1)j} \end{pmatrix} = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ \omega^{j} \\ \omega^{2j} \\ \vdots \\ \omega^{(n-1)j} \end{pmatrix}$$

$$\vec{C}_k = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ \omega^{k} \\ \omega^{2k} \\ \omega^{3k} \\ \vdots \\ \omega^{(n-1)k} \end{pmatrix}$$

$$\langle c_j | c_k \rangle = \frac{1}{n} \sum_{m=0}^{n-1} \overline{\omega^{m\times j}}\, \omega^{m\times k}$$

$$\boxed{= \frac{1}{n} \sum_{m=0}^{n-1} \omega^{m(k-j)}}$$

Case 1 :- $j = k$

$$\langle c_j | c_k \rangle = \frac{1}{n} \sum_{m=0}^{n-1} \omega^{0}$$

$$= \frac{1}{n} [1 + 1 + \dots + 1]$$

each column is normalized

$$= \frac{n}{n} = \boxed{1}$$

case 2 : Geometric series

$$a + ra + r^2 a + \dots r^{n-1} a$$

$$S = a(r^n - 1)/r - 1$$

$$S = \frac{a(r^n-1)}{r-1} \qquad a=1, \quad r=\omega^{k-j}$$

case 2 $j \neq k$

$$\langle c_j | c_k \rangle = \frac{1}{n} \sum_{m=0}^{n-1} \omega^{m(k-j)}$$

$$\langle c_j | c_k \rangle = \frac{\omega^{(k-j)n} - 1}{\omega^{k-j} - 1} \quad \text{--- (1)}$$

$$\omega_n^{f_n} = 1 \qquad \omega_n^{f_n} = \omega_n^{f_n \bmod y}$$

$$= \omega_n^0 = 1$$

eq. (1) becomes

$$\frac{1-1}{\omega^{k-j}-1} = 0$$

(2) convolution ⟷ multiplication

$$\frac{1}{\sqrt{n}} \begin{pmatrix} \\ \\ \\ \\ \\ \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \end{pmatrix}$$

Measure output

$|k\rangle \longrightarrow |\beta_k|^2$

$$\begin{pmatrix} \alpha_{n-1} \\ \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-2} \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \omega \beta_1 \\ \omega^2 \beta_2 \\ \vdots \\ \omega^{n-1} \beta_{n-1} \end{pmatrix}$$

$|k\rangle \longrightarrow |\beta_k|^2$

$|\omega^2| = |\omega^3| = 1$

$$\beta_k = \alpha_0 + \omega^k \alpha_1 + \omega^{2k} \alpha_2 + \cdots + \omega^{(n-1)k} \alpha_{n-1}$$

$\uparrow$ k^{th} output before L.S

Linear shift by 1

$$\alpha_m = \alpha_{m+1}$$

$$\alpha_{n-1} + \omega^k \alpha_0 + \omega^{2k} \alpha_1 + \omega^{3k} \alpha_2 + \cdots + \omega^{(n-1)k} \alpha_{n-2}$$

k^{th} output after linear shift

$$\omega^k \beta_k = \omega^k \alpha_0 + \omega^{2k} \alpha_1 + \cdots \omega^{nk} \alpha_{n-1}$$

nk mode $n = 0$

⑧ Period wave function length Relationship

$$f:[0,1]^n \longrightarrow [0,1]^n$$

$$\text{QFT } f(x) = \hat{f}(x)$$

Period $= r$

$$\frac{N}{r} = \frac{2^n}{r}$$

$$\left(\text{QFT}\right)\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{r-1} \\ \alpha_0 \\ \alpha_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{\frac{N}{r}} \\ \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \end{pmatrix}$$

Period finding algorithm

Given a periodic function $f:[0,1]^n \to [0,1]^n$, find period r, such that $f(x) = f(x+kr)$, $\forall k \in \mathbb{Z} - \{0\}$

$f: \{0,1\}^4 \rightarrow \{0,1\}^4$

$f(x) = x \mod 3$

x	$f(x)$
0	0
1	1
2	2
3	0
4	1
5	2
6	1
7	:
8	:
9	:
10	
11	
12	
13	
14	
15	

$f(0) = 0$

$f(1) = 1$

$f(2) = 2$

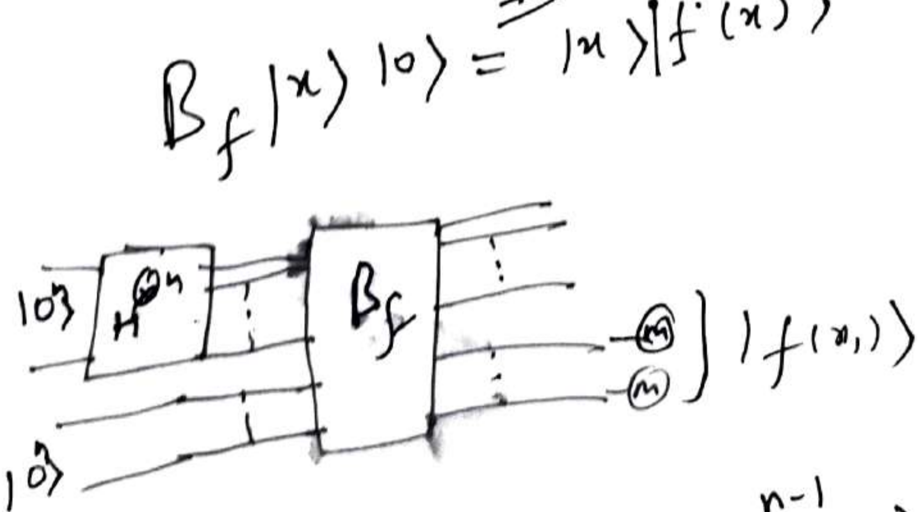
$f(3) = 0$

$r = |3 - 0|$

$= \boxed{3}$

$O(r) = \underbrace{O(2^n)}_{\text{Expo}}$ is not good.

$$B_f |x\rangle |0\rangle = |x\rangle |f(x)\rangle$$



$$H^{\otimes n} |0^n\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{n-1} |x\rangle$$

$$\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle$$

$$\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |f(x)\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle |f(x)\rangle$$

$$\frac{|x_1\rangle + |x_1+r\rangle + |x_1+2r\rangle + \left(\frac{2^n}{r}-1\right) r}{\sqrt{\frac{2^n}{r}}}$$

$\Rightarrow$ All operators of qubits must be unitary

Lect! 14 <u>Unitary matrices</u>

A matrix $U \in \mathbb{C}^{n \times n}$ is unitary if

$$U^{\dagger} \cdot U = U \cdot U^{\dagger} = 1$$

$$\Rightarrow U^{\dagger} = U^{-1}$$

<u>Eg</u> $$U = \frac{1}{\sqrt{5}} \begin{pmatrix} i & -2i \\ -2i & -i \end{pmatrix}$$

Verify if U is unitary or not?

<u>Soln</u>

$$U^{\dagger} = U^{*} \; U^{T}$$

U^{*} ↓ Conjugate operation ~~change~~ change the (sign of imaginary terms)

U^{T} ↓ rows to columns & vice versa

$$U^{\dagger} = \frac{1}{\sqrt{5}} \begin{pmatrix} -i & 2i \\ 2i & i \end{pmatrix}$$

$$U^{\dagger} U = \frac{1}{5} \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Properties of

(1) U^{+} is unitary

Proof U^{+} is unitary

$$(U^{+})^{+} \cdot U^{+} = U^{+} \cdot (U^{+})^{+} = I$$

$$U \cdot U^{+} = U^{+} \cdot U = I$$

(2) Cols of U make orthonormal basis

P_1 ~~Bt~~. orthonormal

P_2 linearly independent

Part 1

$$\langle c_i | c_j \rangle = \begin{cases} 0 & i \neq j \rightarrow \text{orthogonal} \\ 1 & i = j \rightarrow \text{normalized} \end{cases}$$

$$U^{+} U = I$$

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_n \end{pmatrix}^{+} \begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_n \\ & & \vdots & & \end{pmatrix}$$

$= I$

$$\begin{pmatrix} \vec{c}_1^{\,+} \\ \vec{c}_2^{\,+} \\ \vec{c}_3^{\,+} \\ \vdots \\ \vec{c}_n^{\,+} \end{pmatrix} (c_1\ c_2 \cdots c_n) = I$$

$$\begin{pmatrix} \langle c_1|c_1\rangle & \langle c_1|c_2\rangle & \langle c_1|c_3\rangle \cdots & \langle c_1|c_n\rangle \\ \langle c_2|c_1\rangle & \langle c_2|c_2\rangle & \langle c_2|c_3\rangle \cdots & \langle c_2|c_n\rangle \\ \vdots \\ \langle c_n|c_1\rangle & \langle c_n|c_2\rangle & \langle c_n|c_3\rangle \cdots & \langle c_n|c_n\rangle \end{pmatrix}$$

$c_1^+ c_1 = \langle c_1|c_1\rangle$

$= \langle c_1|c_1\rangle = \langle c_2|c_2\rangle = \cdots \langle c_n|c_n\rangle = 1$

$\because$ they all are diagonal entries and diagonal entries are 1

$\langle c_1|c_2\rangle = 0$

Linearly dependent

Def $\exists$ constant $a_1, a_2 \cdots a_n$ at least one $a_k \neq 0$ s.t

when $a_1\vec{c}_1 + a_2\vec{c}_2 + \cdots + a_n\vec{c}_n = 0$

$$\begin{array}{ccc} \vec{c}_1 & \vec{c}_2 & \vec{c}_3 \end{array}$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 1 & 0 & 2 \\ 1 & 1 & 3 \end{pmatrix}$$

cols are linearly dep}

$$a_1 \vec{c}_1 + a_2 \vec{c}_2 + a_3 \vec{c}_3 = \vec{0}$$

$$2\vec{c}_1 + \vec{c}_2 - \vec{c}_3 = \vec{0}$$

$$2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} 2+2-4 \\ 2+0-2 \\ 2+1-3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Proof PbC

Assume cols of U are dep. Hence

$$a_1 \vec{c}_1 + a_2 c_2 + \dots + a_n \vec{c}_n = 0 \quad \text{—— (1)}$$

mul eq (1) with c_1^{+}

$$a \langle c_1 | c_1 \rangle + a_2 \langle c_1 | c_2 \rangle + \dots + a_n \langle \vec{c}_1 | \vec{c}_n \rangle = 0$$

$$a_1 \cdot 1 = 0$$

$$\boxed{a_1 = 0}$$

Mult c_1 with $c_2^{\dagger}$

$$\boxed{a_2 = 0}$$

$$\vec{a}_1 = \vec{a}_2 = \dots = \vec{a}_n = 0$$

Hence my assumption was wrong

Hence colm of U are IND

(3) Rows of U make an orthogonal basis

(4) Unitary matrices preserve inner product

$$U|\psi_1\rangle = |\varphi_1\rangle$$

$$U|\psi_2\rangle = |\varphi_2\rangle$$

$$\langle\psi_1|\psi_2\rangle = \langle\varphi_1|\varphi_2\rangle$$

Proof $\langle\varphi_1|\varphi_2\rangle = \left(U|\varphi_1\rangle\right)^{\dagger}\left(U|\psi_2\rangle\right)$

$$= \langle\psi_1|U^{\dagger}U|\psi_2\rangle$$

$$= \langle\psi_1|\psi_2\rangle$$

(5) unitary matrices preserve lengths (norm) of vectors.

$$\| U |\psi_1\rangle \| = \| |\psi_1\rangle \|$$

<u>Proof</u> $\| U |\psi_1\rangle \|$

$$= \sqrt{U |\psi_1\rangle^{\dagger} \cdot (U |\psi_1\rangle)}$$

$$= \sqrt{\langle \psi_1 | U^{\dagger} U | \psi_1 \rangle}$$

$$= \sqrt{\langle \psi_1 | \psi_1 \rangle}$$

$$= \| |\psi_1\rangle \|$$

(6) unitary matrices preserve ~~bf~~ angles b/w vectors

$$\cos\theta = \frac{\langle \psi | \varphi \rangle}{\| |\psi\rangle \| \, \| |\varphi\rangle \|}$$

(7) Unitary matrices make a multipli-cative group

$G, \times$

(i) Closed under $\times$, if $a, b \in G$ then

$a \times b = c \in G$ (must)

(ii) $\exists$ an $i \in G$, such for each $a \in G$

$a \times i = a$

(iii) There must exist an inverse for each

$a \in G$ $\quad a^{-1} \in G$

$\cancel{a \times a^{-1} = G}$ $\quad a \times a^{-1} = i$

(iv) Associativity if $a, b, c \in G$

$(a \times b) \times c = a \times (b \times c)$

Closed

A, B are unitary. $A \times B = C$ must be unitary

$U^{\dagger} \cdot U = I$

$C^{\dagger} \times C \stackrel{?}{=} I$

$\therefore (AB)^{\dagger} \cdot (A \cdot B) \stackrel{?}{=} I$

$B^{\dagger} \underline{A^{\dagger} A} B \stackrel{?}{=}$

$B^{\dagger} B \stackrel{?}{=} I$

$I = I$

(2) Identity

$U \times \boxed{I} = U$

$I \in$ unitary

$I^{\dagger} I = I$

(3) inverse

U is unitary

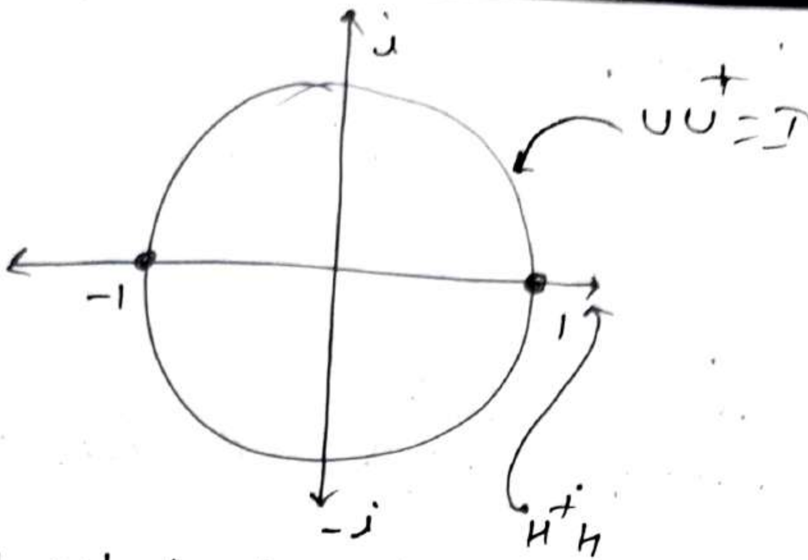
U^{-1} must also unitary.

$U \cdot U^{-1} = I$

$U^{-1} = U^{\dagger}$

(4) $(A \times B) \times C = A(B \times C)$ order of parentheses will not change the result

(8) eigenvalues of unitary matrices lie on complex unit circle



A^{+} A which both unitary and Hermitian

Range (U) ∩ Range (H)

$= \{1, -1\}$

$x + iy = r\cos\theta + ir\sin\theta$

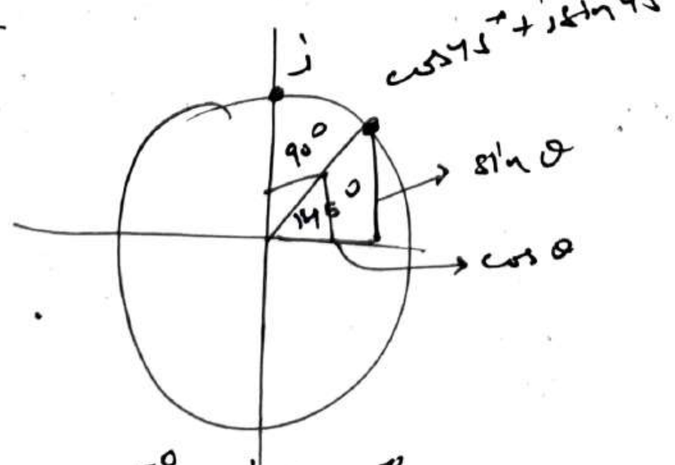
$= re^{i\theta}$

$r = |x + iy| = \sqrt{(x+iy)^{*}(x+iy)}$

$r = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1}\frac{y}{x}$

eg :-



$= \cos 45° + i\sin 45°$

$= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

$e^{i\pi/4}$ | $\cos 90^\circ + i \sin 90^\circ$

$= i$

$= e^{i\pi/2}$

eigenvalue λ

eigenvector $|\psi\rangle$

$U|\psi\rangle = \lambda|\psi\rangle$ (By definition)

Unitary matrix preserve norm

$$\| |\psi\rangle \| = \| U|\psi\rangle \|$$

$$= \| \lambda |\psi\rangle \|$$

$$\sqrt{\langle\psi|\psi\rangle} = \sqrt{(\lambda|\psi\rangle)^{+} (\lambda|\psi\rangle)}$$

$$\sqrt{\langle\psi|\psi\rangle} = \sqrt{\langle\psi|\underbrace{\lambda^* \lambda}|\psi\rangle}$$

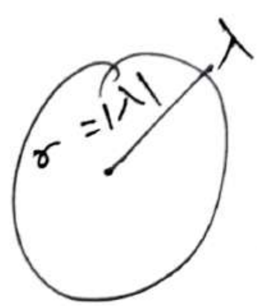
$$= \sqrt{\lambda^* \lambda \langle\psi|\psi\rangle}$$

$$\sqrt{\langle\psi|\psi\rangle} = \sqrt{\langle\psi|\psi\rangle} = \sqrt{\lambda^* \lambda}\, \sqrt{\langle\psi|\psi\rangle}$$

$$\sqrt{\langle\psi|\psi\rangle} = |\lambda| \sqrt{\langle\psi|\psi\rangle}$$

$$|\lambda| = 1$$

$$\sqrt{\lambda^* \lambda} = 1$$



(9) Normal matrix is always diagonalizable by a unitary similarity

Q1: What is normal matrix?
A matrix $N \in \mathbb{C}^{n \times n}$ is normal if

$$N^{\dagger} N = N N^{\dagger}$$

* Unitary $\quad U^{\dagger} U = U U^{\dagger} = I$

* Hermitian $\quad H^{\dagger} = H$

* Skew-symmetric

$$H^{\dagger} = -H.$$

Q2: What is diagonalization?
— Decomposing a matrix into three different matrix

$$A = P D P^{-1}$$

eigenvector of A are linearly independent

$$= \{\vec{e}_1, \vec{e}_2, \ldots \vec{e}_n\}$$

corresponding eigenvalues are $\lambda_1, \lambda_2, \ldots \lambda_n$

$$P = \begin{bmatrix} | & | & & | \\ \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \\ | & | & & | \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

Q What is diagonalization by unitary similarity?

$$N = U D U^{\dagger}$$

N's ~~eigenvalue~~ eigenvectors
— linearly indep.
— orthonormal

$$U = [\vec{e}_1 \ \vec{e}_2 \ \dots \ \vec{e}_n]$$

$$D = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \ \lambda_n \end{bmatrix}$$

Q: What are advantages of diagonalization?

N^{100}

e^{iN}

N^{-1}

$\cos(N) \cdot \sin(N)$

$N!$

$N = UDU^{+}$

$N^2 = (UDU^{+})(UDU^{+})$

$= UDU^{+}UDU^{+}$

$= UD^2U^{+}$

$N^{100} = UD^{100}U^{+}$

$$D^{100} = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & \ddots \ d_n \end{bmatrix}$$