Introduction to Algorithms

Dynamic Programming

Dynamic programming

- It is used, when the solution can be recursively described in terms of solutions to subproblems (*optimal substructure*)
- Algorithm finds solutions to subproblems and stores them in memory for later use
- More efficient than "*brute-force methods*", which solve the same subproblems over and over again

Optimal Substructure Property

- Definition:
  - If S is an optimal solution to a problem, then the components of S are optimal solutions to subproblems
- Examples:
  - True for knapsack
  - True for coin-changing
  - True for single-source shortest path
  - Not true for longest-simple-path

Dynamic Programming

- Works "bottom-up"
  - Finds solutions to small sub-problems first
  - Stores them
  - Combines them somehow to find a solution to a slightly larger subproblem
- Compare to greedy approach
  - Also requires optimal substructure
  - But greedy makes choice first, then solves

Problems Solved with Dyn. Prog.

- Coin changing
- Multiplying a sequence of matrices
  - Can do in various orders: (AB)C vs. A(BC)
  - Pick order that does fewest number of scalar multiplications
- Longest common subsequence
- All-pairs shortest paths (Floyd's algorithm)
- Constructing optimal binary search trees
- Knapsack problems (we'll do 0/1)

Remember Fibonacci numbers?

- Recursive code:

```
long fib(int n) {
    assert(n >= 0);
    if ( n == 0 ) return 0;
    if ( n == 1 ) return 1;
    return fib(n-1) + fib(n-2);
}
```

- What's the problem?
  - Repeatedly solves the same subproblems
  - Exponential time complexity

Memorization

- Before talking about dynamic programming, another general technique: **Memoization**
  - AKA using a *memory function*
- Simple idea:
  - Calculate and store solutions to subproblems
  - Before solving it (again), look to see if you've remembered it

Memorization

- Use a Table abstract data type
  - Lookup key: whatever identifies a subproblem
  - Value stored: the solution
- Could be an array/vector
  - E.g. for Fibonacci, store **fib(n)** using index **n**
  - Need to initialize the array
- Could use a map / hash-table

Memorization and Fibonacci

- Before recursive code below called, must initialize results[] so all values are -1

```
long fib_mem(int n, long results[]) {
    if ( results[n] != -1 )
        return results[n];  // return stored value
    long val;
    if ( n == 0 || n ==1 )
        val = n; // odd but right
    else
        val = fib_mem(n-1, results)
                + fib_mem(n-2, results);
    results[n] = val; // store calculated value
    return val;
}
```

Observations on fib_mem()

- Same elegant top-down, recursive approach based on definition
  - Without repeated subproblems
- Memory function: a function that remembers
  - Save time by using extra space
- Can show this runs in $\Theta(n)$

General Strategy of Dyn. Prog.

1. Structure: What's the structure of an optimal solution in terms of solutions to its subproblems?
2. Give a recursive definition of an optimal solution in terms of optimal solutions to smaller problems
   - Usually using min or max
3. Use a data structure (often a table) to store smaller solutions in a <u>bottom-up</u> fashion
   - Optimal value found in the table
4. (If needed) Reconstruct the optimal solution
   - I.e. what produced the optimal value

Dyn. Prog. vs. Divide and Conquer

- Remember D & C?
  - Divide into subproblems. Solve each. Combine.
- Good when subproblems do not overlap, when they're independent
  - No need to repeat them
- Divide and conquer: top-down
- Dynamic programming: bottom-up

Longest Common Subsequence (LCS)

Application: comparison of two DNA strings

Ex: X= {A B C B D A B }, Y= {B D C A B A}

Longest Common Subsequence:

X = A **B** **C** **B** D **A** B

Y = **B** D **C** A **B** **A**

Brute force algorithm would compare each subsequence of X with the symbols in Y

LCS Algorithm

- if $|X| = m$, $|Y| = n$, then there are 2^m subsequences of X; we must compare each with Y (n comparisons)
- So the running time of the brute-force algorithm is $O(n\ 2^m)$
- Notice that the LCS problem has *optimal substructure*: solutions of subproblems are parts of the final solution.
- Subproblems: "find LCS of pairs of *prefixes* of X and Y"

LCS Algorithm

- First we'll find the length of LCS. Later we'll modify the algorithm to find LCS itself.
- Define X_i, Y_j to be the prefixes of X and Y of length *i* and *j* respectively
- Define *c[i,j]* to be the length of LCS of X_i and Y_j
- Then the length of LCS of X and Y will be *c[m,n]*

$$c[i,j] = \begin{cases} c[i-1, j-1]+1 & \text{if } x[i] = y[j], \\ \max(c[i, j-1], c[i-1, j]) & \text{otherwise} \end{cases}$$

LCS recursive solution

$$c[i,j] = \begin{cases} c[i-1,j-1]+1 & \text{if } x[i] = y[j], \\ \max(c[i,j-1], c[i-1,j]) & \text{otherwise} \end{cases}$$

- We start with $i = j = 0$ (empty substrings of x and y)
- Since X_0 and Y_0 are empty strings, their LCS is always empty (i.e. $c[0,0] = 0$)
- LCS of empty string and any other string is empty, so for every i and j: $c[0, j] = c[i,0] = 0$

LCS recursive solution

$$c[i,j]=\begin{cases} c[i-1,j-1]+1 & \text{if } x[i]=y[j], \\ \max(c[i,j-1],c[i-1,j]) & \text{otherwise} \end{cases}$$

- When we calculate *c[i,j]*, we consider two cases:
- **First case:** *x[i]=y[j]*: one more symbol in strings X and Y matches, so the length of LCS X_i and Y_j equals to the length of LCS of smaller strings X_{i-1} and Y_{i-1} , plus 1

LCS recursive solution

$$c[i,j]=\begin{cases} c[i-1,j-1]+1 & \text{if } x[i]=y[j], \\ \max(c[i,j-1],c[i-1,j]) & \text{otherwise} \end{cases}$$

- **Second case:** *x[i] != y[j]*
- As symbols don't match, our solution is not improved, and the length of LCS(X_i , Y_j) is the same as before (i.e. maximum of LCS(X_i, Y_{j-1}) and LCS(X_{i-1},Y_j)

Why not just take the length of LCS(X_{i-1}, Y_{j-1}) ?

LCS Length Algorithm

```
LCS-Length(X, Y)
m = length(X)  // get the # of symbols in X
n  = length(Y) // get the # of symbols in Y
for i = 1 to m    c[i,0] = 0      // special case: Y_0
for j = 1 to n    c[0,j] = 0      // special case: X_0
for i = 1 to m                    // for all X_i
      for j = 1 to n                    // for all Y_j
            if ( X_i == Y_j )
                  c[i,j] = c[i-1,j-1] + 1
            else c[i,j] = max( c[i-1,j], c[i,j-1] )
return c[m,n]   // return LCS length for X and Y
```

LCS Example

We'll see how LCS algorithm works on the following example:

- X = ABCB
- Y = BDCAB

What is the Longest Common Subsequence of X and Y?

LCS(X, Y) = BCB

X = A B C B

Y = B D C A B

LCS Example (0)

ABCB
BDCAB

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi						
1	A						
2	B						
3	C						
4	B						

X = ABCB;   m = |X| = 4
Y = BDCAB; n = |Y| = 5
Allocate array c[5,6]

LCS Example (1)

ABCB
BDCAB

i		j	0	1	2	3	4	5
			Yj	B	D	C	A	B
0	Xi		0	0	0	0	0	0
1	A		0					
2	B		0					
3	C		0					
4	B		0					

for i = 1 to m $\quad$ *c[i,0] = 0*
for j = 1 to n $\quad$ *c[0,j] = 0*

LCS Example (2)

ABCB

BDCAB

i	j	0	1	2	3	4	5
		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0				
2	B	0					
3	C	0					
4	B	0					

if $(X_i == Y_j)$

$c[i,j] = c[i-1,j-1] + 1$

else $c[i,j] = max(c[i-1,j], c[i,j-1])$

LCS Example (3)

ABCB

BDCAB

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0	0	0		
2	B	0					
3	C	0					
4	B	0					

$$\begin{aligned}&\text{if } (X_i == Y_j)\\ &\qquad c[i,j] = c[i-1,j-1] + 1\\ &\text{else } c[i,j] = \max(c[i-1,j], c[i,j-1])\end{aligned}$$

LCS Example (4)

ABCB

BDCAB

i		j	0	1	2	3	4	5
			Yj	B	D	C	A	B
0	Xi		0	0	0	0	0	0
1	A		0	0	0	0	1	
2	B		0					
3	C		0					
4	B		0					

if $(X_i == Y_j)$

$c[i,j] = c[i-1,j-1] + 1$

else $c[i,j] = max(c[i-1,j], c[i,j-1])$

LCS Example (5)

ABCB
BDCAB

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0	0	0	1	→ 1
2	B	0					
3	C	0					
4	B	0					

$if\ (X_i == Y_j)$
$c[i,j] = c[i-1,j-1] + 1$
$else\ c[i,j] = max(c[i-1,j], c[i,j-1])$

LCS Example (6)

ABCB

BDCAB

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0	0	0	1	1
2	B	0	1				
3	C	0					
4	B	0					

if $(X_i == Y_j)$

$c[i,j] = c[i-1,j-1] + 1$

else $c[i,j] = max(c[i-1,j], c[i,j-1])$

LCS Example (7)

ABCB

BDCAB

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0	0	0	1	1
2	B	0	1	1	1	1	
3	C	0					
4	B	0					

if ($X_i == Y_j$)

$c[i,j] = c[i-1,j-1] + 1$

else $c[i,j] = \max(c[i-1,j], c[i,j-1])$

LCS Example (8)

ABCB

BDCAB

i		j	0	1	2	3	4	5
			Yj	B	D	C	A	B
0	Xi		0	0	0	0	0	0
1	A		0	0	0	0	1	1
2	B		0	1	1	1	1	2
3	C		0					
4	B		0					

if (X_i == Y_j)

$c[i,j] = c[i-1,j-1] + 1$

else $c[i,j] = max(c[i-1,j], c[i,j-1])$

LCS Example (10)

ABCB

BDCAB

	j	*0*	*1*	*2*	*3*	*4*	*5*
i		*Yj*	*B*	*D*	*C*	*A*	*B*
0	*Xi*	*0*	*0*	*0*	*0*	*0*	*0*
1	*A*	*0*	*0*	*0*	*0*	*1*	*1*
2	*B*	*0*	*1*	*1*	*1*	*1*	*2*
3	*C*	*0*	$\downarrow$ *1* $\rightarrow$	$\downarrow$ *1*			
4	*B*	*0*					

$$\textit{if}\ (X_i == Y_j)$$
$$c[i,j] = c[i-1,j-1] + 1$$
$$\textit{else}\ c[i,j] = \max(c[i-1,j], c[i,j-1])$$

LCS Example (11)

ABCB

BDCAB

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0	0	0	1	1
2	B	0	1	1	1	1	2
3	C	0	1	1	2		
4	B	0					

if $(X_i == Y_j)$

$c[i,j] = c[i-1,j-1] + 1$

else $c[i,j] = max(c[i-1,j], c[i,j-1])$

LCS Example (12)

ABCB
BDCAB

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0	0	0	1	1
2	B	0	1	1	1	1	2
3	C	0	1	1	2	2	2
4	B	0					

if ($X_i == Y_j$)

$c[i,j] = c[i-1,j-1] + 1$

else $c[i,j] = \max(c[i-1,j], c[i,j-1])$

LCS Example (13)

ABCB

BDCAB

	j	*0*	*1*	*2*	*3*	*4*	*5*
i		*Yj*	*B*	*D*	*C*	*A*	*B*
0	*Xi*	*0*	*0*	*0*	*0*	*0*	*0*
1	*A*	*0*	*0*	*0*	*0*	*1*	*1*
2	*B*	*0*	*1*	*1*	*1*	*1*	*2*
3	*C*	*0*	*1*	*1*	*2*	*2*	*2*
4	*B*	*0*	*1*				

$$\text{if } (X_i == Y_j)$$
$$c[i,j] = c[i-1,j-1] + 1$$
$$\text{else } c[i,j] = \max(c[i-1,j], c[i,j-1])$$

LCS Example (14)

ABCB

BDCAB

i		j	0	1	2	3	4	5
			Yj	B	D	C	A	B
0	Xi		0	0	0	0	0	0
1	A		0	0	0	0	1	1
2	B		0	1	1	1	1	2
3	C		0	1	1	2	2	2
4	B		0	1	1	2	2	

if $(X_i == Y_j)$

$c[i,j] = c[i-1,j-1] + 1$

else $c[i,j] = \max(c[i-1,j], c[i,j-1])$

LCS Example (15)

ABCB
BDCAB

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0	0	0	1	1
2	B	0	1	1	1	1	2
3	C	0	1	1	2	2	2
4	B	0	1	1	2	2	3

if $(X_i == Y_j)$

$c[i,j] = c[i-1,j-1] + 1$

else $c[i,j] = max(c[i-1,j], c[i,j-1])$

LCS Algorithm Running Time

- LCS algorithm calculates the values of each entry of the array c[m,n]
- So what is the running time?

  O(m*n)

  since each c[i,j] is calculated in constant time, and there are m*n elements in the array

How to find actual LCS

- So far, we have just found the *length* of LCS, but not LCS itself.
- We want to modify this algorithm to make it output Longest Common Subsequence of X and Y

Each *c[i,j]* depends on *c[i-1,j]* and *c[i,j-1]*

or *c[i-1, j-1]*

For each c[i,j] we can say how it was acquired:

2	*2*
2	*3*

For example, here

c[i,j] = c[i-1,j-1] +1 = 2+1=3

How to find actual LCS - continued

- Remember that

$$c[i,j]=\begin{cases} c[i-1,j-1]+1 & \text{if } x[i]=y[j], \\ \max(c[i,j-1],c[i-1,j]) & \text{otherwise} \end{cases}$$

  - So we can start from c[m,n] and go backwards
  - Look first to see if 2nd case above was true
  - If not, then c[i,j] = c[i-1, j-1]+1, so remember x[i] (because x[i] is a part of LCS)
  - When i=0 or j=0 (i.e. we reached the beginning), output remembered letters in reverse order

Algorithm to find actual LCS

- Here's a recursive algorithm to do this:

```
LCS_print(x, m, n, c) {
  if (c[m][n] == c[m-1][n]) // go up?
    LCS_print(x, m-1, n, c);
  else if (c[m][n] == c[m][n-1] // go left?
    LCS_print(x, m, n-1, c);
  else { // it was a match!
    LCS_print(x, m-1, n-1, c);
    print(x[m]); // print after recursive call
   }
}
```

Finding LCS

	j	0	1	2	3	4	5
i		Yj	B	D	C	A	B
0	Xi	0	0	0	0	0	0
1	A	0	0	0	0	1	1
2	B	0	1	1	1	1	2
3	C	0	1	1	2	2	2
4	B	0	1	1	2	2	3

Finding LCS (2)

i	j	0	1	2	3	4	5
		Yj	*B*	*D*	*C*	*A*	*B*
0	*Xi*	0	0	0	0	0	0
1	*A*	0	0	0	0	1	1
2	*B*	0	1	1	1	1	2
3	*C*	0	1	1	2	2	2
4	*B*	0	1	1	2	2	3

LCS (reversed order): B C B

LCS (straight order): B C B

(this string turned out to be a palindrome)

Review: Dynamic programming

- DP is a method for solving certain kind of problems
- DP can be applied when the solution of a problem includes solutions to subproblems
- We need to find a recursive formula for the solution
- We can recursively solve subproblems, starting from the trivial case, and save their solutions in memory
- In the end we'll get the solution of the whole problem

Properties of a problem that can be solved with dynamic programming

- Simple Subproblems
  - We should be able to break the original problem to smaller subproblems that have the same structure
- Optimal Substructure of the problems
  - The solution to the problem must be a composition of subproblem solutions
- Subproblem Overlap
  - Optimal subproblems to unrelated problems can contain subproblems in common

Review: Longest Common Subsequence (LCS)

- Problem: how to find the longest pattern of characters that is common to two text strings X and Y
- Dynamic programming algorithm: solve subproblems until we get the final solution
- Subproblem: first find the LCS of *prefixes* of X and Y.
- this problem has *optimal substructure*: LCS of two prefixes is always a part of LCS of bigger strings

Conclusion

- Dynamic programming is a useful technique of solving certain kind of problems
- When the solution can be recursively described in terms of partial solutions, we can store these partial solutions and re-use them as necessary
- Running time (Dynamic Programming algorithm vs. naïve algorithm):
  - LCS: **$O(m*n)$** vs. **$O(n * 2^m)$**
  - 0-1 Knapsack problem: **$O(W*n)$** vs. **$O(2^n)$**