中国科学技术大学
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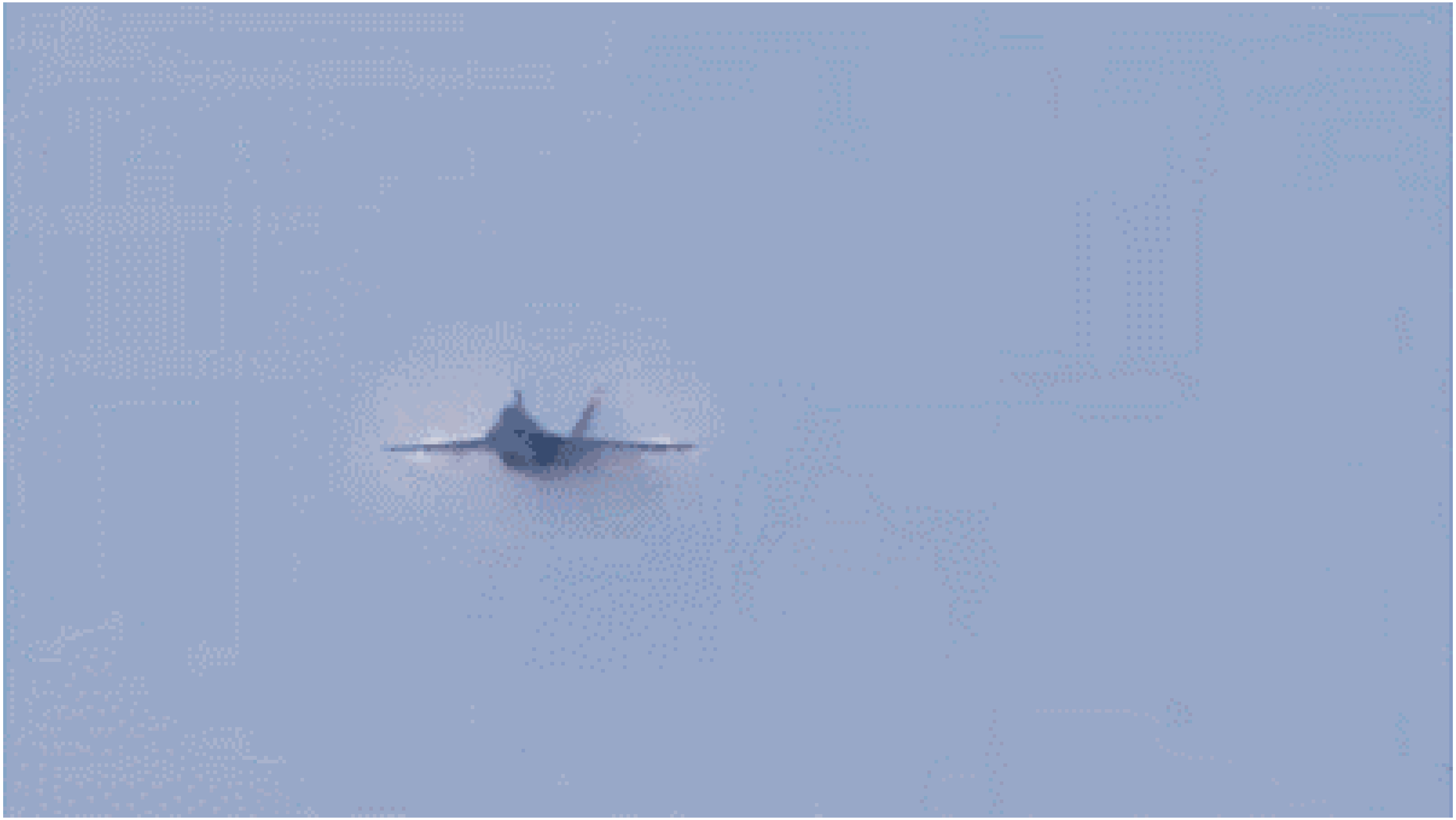
微分方程

激波简介

音爆



疏散波和压缩波

- 1860 年， Riemann 在他关于激波的数学理论中研究非线性守恒律方程标度不变初值问题，并用以求解空气动力学中一维等熵流动的 Euler 方程组，揭示了等熵流动的基本波：激波和疏散波。

- 追赶问题：

人头曲线$u(x,t)$满足一阶拟线性方程$\dfrac{\partial u}{\partial t}+a(u)\dfrac{\partial u}{\partial x}=0, x\in\mathbb{R}, t>0$

及初始条件$u(x,0)=\varphi(x)$.

特征方程为$\dfrac{dx}{\partial t}=a(u)$, 每一根特征线都是$(x,t)$平面上的直线，

但特征线的斜率$1/a(u)$是不相同的；沿着特征线, $u(x,\ t)=$常数.

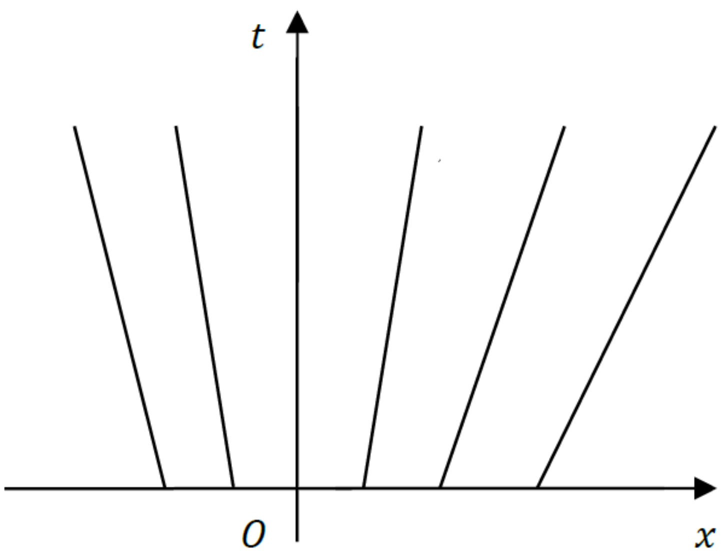
$\because \dfrac{du(x(t),t)}{dt}=\dfrac{\partial u}{\partial t}+\dfrac{dx}{dt}\dfrac{\partial u}{\partial x}=\dfrac{\partial u}{\partial t}+a(u)\dfrac{\partial u}{\partial x}=0$, 过任一点$(x_0,0)$的特征线

满足$\dfrac{x-x_0}{t-0}=\dfrac{dx}{dt}=a(u(x,t))=a(u(x_0,0))=a(\varphi(x_0))$.

疏散波和压缩波

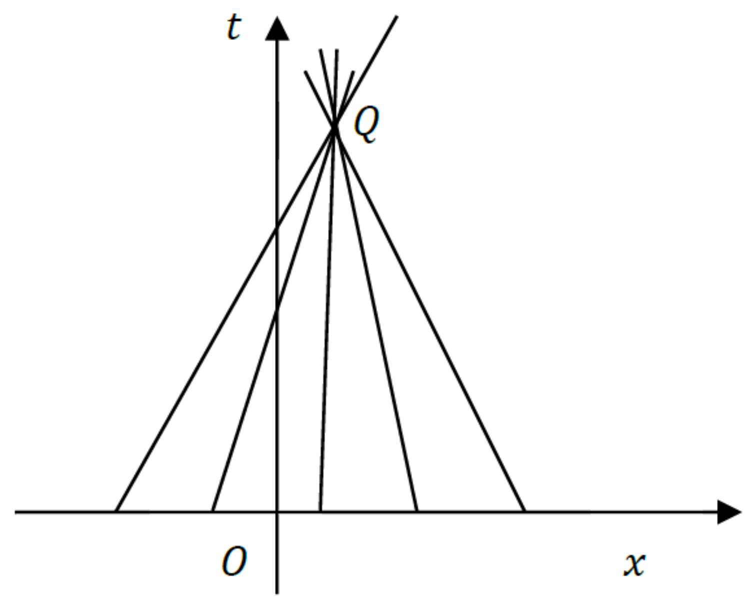
■ 追赶问题：

故$x = x_0 + a(\varphi(x_0))t$和$u(x,t) = \varphi(x_0)$.从第一式解出$x_0 = x - a(u(x,t))t$,
由第二式可得到隐式解$u(x,t) = \varphi(x - a(u(x,t))t)$.但仅从第一式并不能
唯一地解出$x_0 = x_0(x,t)$来得到显式解$u(x,t) = \varphi(x_0(x,t))$, 见如下的图2.



$a(\varphi(x_1)) < a(\varphi(x_2)), x_1 < x_2$

疏散波



$a(\varphi(x_1)) > a(\varphi(x_2)), x_1 < x_2$

压缩波

疏散波和压缩波

■ 追赶问题：

令$F(x,t;x_0)=a(\varphi(x_0))t-(x-x_0)=0$ (*)，则

$\dfrac{\partial F}{\partial x_0}=t\dfrac{da}{d\varphi}\dfrac{d\varphi}{dx_0}+1=ta'\varphi'+1$.(1)若$a'\varphi'\geq 0$，有$\dfrac{\partial F}{\partial x_0}>0$，由隐函数定理知

x_0可唯一地表示成x,t的函数.(2)若$a'\varphi'<0$，存在$t^*>0$使$\left.\dfrac{\partial F}{\partial x_0}\right|_{t=t^*}=0$.

(*)对t,x求偏微分，易有$\dfrac{\partial x_0}{\partial t}=-\dfrac{a}{1+ta'\varphi'}$，$\dfrac{\partial x_0}{\partial x}=\dfrac{1}{1+ta'\varphi'}$，

$$\therefore \frac{\partial u}{\partial t}=\frac{\partial\varphi(x_0)}{\partial t}=\varphi'\frac{\partial x_0}{\partial t}=-\frac{a\varphi'}{1+ta'\varphi'},\ \frac{\partial u}{\partial x}=\frac{\partial\varphi(x_0)}{\partial x}=\frac{\varphi'}{1+ta'\varphi'}.$$

$a'\varphi'<0$时这些偏导数在t^*为无穷大，初值问题将不存在唯一连续解，即出现爆破(blow-up)现象.

间断解：激波

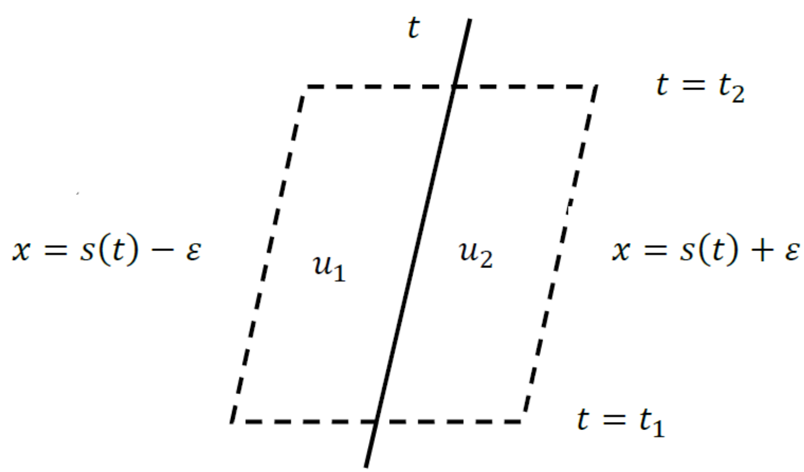
■ 考虑一般的一阶拟线性偏微分方程初值问题

$$\begin{cases} \dfrac{\partial u}{\partial t}+\dfrac{\partial q(u)}{\partial x}=0, x\in\mathbb{R}, t>0 \\ u(x,0)=\varphi(x), x\in\mathbb{R} \end{cases}$$

令L为O_{xt}平面逐段光滑的正向闭曲线, D为L围成的区域, 由Green定理有

$\iint_D\left[\dfrac{\partial u}{\partial t}+\dfrac{\partial q(u)}{\partial x}\right]dxdt=\oint_L udx-q(u)dt=0$ (守恒律).考察u的第一类间断,

则在间断线$x=x(t)$的两侧u取不同的值, 见下图.



间断解：激波

任意固定$t_1, t_2 > 0, \forall \varepsilon > 0$,由守恒律,

$$0 = \oint_L u dx - q(u) dt$$

$$= \int_{t_2}^{t_1} \left[u(s(t)-\varepsilon, t)\frac{ds}{dt} - q(u(s(t)-\varepsilon, t)) \right] dt + \int_{s(t_1)-\varepsilon}^{s(t_1)+\varepsilon} u(s(t), t_1) dx$$

$$+ \int_{t_1}^{t_2} \left[u(s(t)+\varepsilon, t)\frac{ds}{dt} - q(u(s(t)+\varepsilon, t)) \right] dt + \int_{s(t_2)-\varepsilon}^{s(t_2)+\varepsilon} u(s(t), t_2) dx$$

$$\Rightarrow \varepsilon \to 0\text{时} \int_{t_1}^{t_2} \left\{ \left[u_+ \frac{ds}{dt} - q(u_+) \right] - \left[u_- \frac{ds}{dt} - q(u_-) \right] \right\} dt = 0.$$

得到间断连接条件−Rankin-Hugoniot条件：

$$\frac{dx}{dt} = \frac{[q(u)]}{[u]} = \frac{q(u_+) - q(u_-)}{u_+ - u_-}.$$

还需要满足间断稳定性条件或熵条件：$q'(u_+) < \dfrac{dx}{dt} < q'(u_-)$.

同时满足Rankin-Hugoniot条件和熵条件的间断线称为激波.