3.3.1 "⇒" f 几乎处处可微 且 $f(x)=f(0)+\int_0^x f'(t)dt$

而 $|f'(x)|\le M$，故 ∃ $\{g_n\}$ 连续且 $|g_n(x)|\le M$

且 $\|g_n-f'\|_{L_1}\to 0$

取 $f_n(x)=f(0)+\int_0^x g_n(t)dt$ 即可

"⇐" $|f(x)-f(y)|=\lim_{n\to\infty}|f_n(x)-f_n(y)|=\lim_{n\to\infty}|f_n'(\xi_n)||x-y|$

$\le M|x-y|$

2. $g(x)=\sum_{n=1}^{\infty}f_n'(x)$ $\quad f(x)=\sum_{n=1}^{\infty}(f_n(x)-f_n(x))+\sum_{n=1}^{\infty}f_n(0)$

$=f(0)+\sum_{n=1}^{\infty}\int_0^x f_n'(t)dt$

而由条件(1)，定义 $g_n(x)=\sum_{i=1}^{n}f_i'(t)$，有 $|g_n(x)|\le\sum_{n=1}^{\infty}|f_n'(x)|dx$

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3. $\int_0^x f'(t)\,dt=f(x)-f(0)$，由 f' 可积 $\Rightarrow$ 令 $x\to\infty$ $f(+\infty)$ 存在，而 f 可积

$\Rightarrow \lim\limits_{x\to\infty} f(x)=0$，同理 $\lim\limits_{x\to-\infty} f(x)=0$

故 $\int_{\mathbb{R}} f'=\lim\limits_{x\to\infty}\int_0^x f'+\int_{-x}^0 f' \overset{\lim\limits_{x\to\infty}}{=} f(x)-f(0)+f(0)-f(-x)$

$=0$

4. $f_n(x)=\int_0^x f_n'(t)\,dt$ $|f_n(x)-f_m(x)|\le\int_0^1|f_n'(t)-f_m'(t)|\,dt\to0$

故 $\exists f\in C([0,1])$ 使 $f_n\rightrightarrows f$，由于 f_n' Cauchy 列 $\Rightarrow\exists g\in L^1$

使 $f_n'\xrightarrow{L^1} g$ 故 $\left|\int_0^x f_n'(t)-g(t)\,dt\right|\le\int_0^1|f_n'(t)-g(t)|\,dt\to0$

则 $f(x)=\lim\limits_{n\to\infty}\int_0^x f_n'(t)\,dt=\int_0^x g(t)\,dt$

1. 取 $a_n \to 0$ $b_n \to 0$ 且 $\lim\limits_{k\to\infty} f_k(a_n) = \lim\limits_{k\to\infty} f_k(b_n) = 1$

故 $\lim\limits_{k\to\infty}\int_{a_n}^{b_n} f_k'(x) \geq \int_{a_n}^{b_n} \lim\limits_{k\to\infty} f_k'(x)$

而

即 $\int_{a_n}^{b_n} \lim\limits_{k\to\infty} f_k'(x) \leq 0 \Rightarrow \lim\limits_{k\to\infty} f_k'(x) = 0$ a.e. on $[a_n, b_n]$

令 $n\to\infty$ 即可

补充：已知周期为 $2\pi, 1$ 的连续函数为常值，证明周期为 $1, 2\pi$ 的~~连续函~~数也为常值

局部可积

pf: $f \in L^1_{loc}(\mathbb{R})$ 则 $g_a(t) = \int_{x_0+t}^{x_0+a+t} f(x)\,dx$

则 $g_a(t)$ 连续，周期为 $2\pi, 1 \Rightarrow g_a(t) = C_a$

故 $\lim\limits_{a\to 0} \frac{g_a(t)}{a} = \lim\limits_{a\to 0} \frac{\int_{x_0+t}^{x_0+a+t} f(x)\,dx}{a} = \lim\limits_{a\to 0} \frac{C_a}{a} = f(x_0+t)$

⑥ $\forall \varepsilon>0$, $f(x)$在$[\varepsilon,1]$上绝对连续 $\int_0^1 x|f'(x)|^p < \infty$ ($p>2$). 证明 $\lim\limits_{x\to 0} f(x)$存在

记$\frac{1}{p}+\frac{1}{q}=1$ $\left(\int_a^b (x^{\frac{1}{p}} f'(x))^p \, dx\right)^{\frac{1}{p}} \cdot \left(\int_a^b (x^{-\frac{1}{p}})^q\right)^{\frac{1}{q}} \geq \int_a^b |f'(x)| \, dx \geq \left|\int_a^b f'(x) \, dx\right| = |f(b)-f(a)|$

而$\int_a^b x|f'(x)|^p dx < \int_0^1 x|f'(x)|^p dx < M$ $\left(\int_0^b x^{-\frac{q}{p}}\right)^{\frac{1}{q}} < C\left(b^{1-\frac{q}{p}} - a^{1-\frac{q}{p}}\right)$

即$|f(b)-f(0)| < MC\left(b^{1-\frac{q}{p}} - a^{1-\frac{q}{p}}\right)$, 当$p>2$时 $1-\frac{q}{p}>0$ $\Rightarrow$ 当$a,b\to 0$时

$b^{1-\frac{q}{p}} - a^{1-\frac{q}{p}} \to 0 \Rightarrow |f(b)-f(0)| \to 0 \Rightarrow \lim\limits_{x\to 0} f(x)$存在