(2) 证 Kolmogrov $\overset{\text{SLLN}}{\text{强大数律}}$.

对 $\forall \varepsilon>0$. 令 $Y_n=\frac{\tilde{S}_n}{n}$.

$$P(\max_{2^m\le n<2^{m+1}}|Y_n|>\varepsilon)\le P(\max_{2^m\le n<2^{m+1}}|\tilde{S}_n|>2^m\cdot\varepsilon)\le P(\max_{1\le n<2^{m+1}}|\tilde{S}_n|>2^m\cdot\varepsilon)$$

$$\le \frac{1}{(2^m\varepsilon)^2}\sum_{i=1}^{2^{m+1}-1}Var(X_i)$$

$$\sum_{m=1}^{+\infty}P(\max_{2^m\le n<2^{m+1}}|Y_n|>\varepsilon)\le\sum_{m=1}^{+\infty}\frac{1}{(2^m\varepsilon)^2}\sum_{i=1}^{2^{m+1}-1}Var(X_i)=\sum_{i=1}^{\infty}\frac{1}{\varepsilon^2}Var(X_i)\sum_{m=m(i)}^{\infty}\frac{1}{2^{2m}},\ m(i)=\min\{m:i<2^{m+1}\}$$

$$=\frac{1}{\varepsilon^2}\sum_{i=1}^{\infty}Var(X_i)\cdot\frac{\frac{1}{2^{2m(i)}}}{1-\frac14}=\frac{4}{3\varepsilon^2}\sum_{i=1}^{\infty}\frac{Var(X_i)}{2^{2m(i)}}=\frac{16}{3\varepsilon^2}\sum_{i=1}^{\infty}\frac{Var(X_i)}{2^{2m(i)+2}}<\infty$$

hw: 7.4.1, 7.11.17, 7.11.20

又 $\sum_{i=1}^{+\infty}\frac{Var(X_i)}{i^2}<+\infty$. $2^{m(i)}\le i<2^{m(i)+1}$ $\Rightarrow \sum_{i=1}^{+\infty}\frac{Var(X_i)}{2^{2m(i)+2}}<\sum_{i=1}^{+\infty}\frac{Var(X_i)}{i^2}<+\infty$

$P(\max_{2^m\le n<2^{m+1}}|Y_n|\ge\varepsilon\ \text{i.o.})=0$ $\quad P(|Y_n|\ge\varepsilon,\ \text{i.o.})=0$ $\quad Y_n\xrightarrow{a.s.}0$

定理 $\{X_n\}$ i.i.d. $S_n=\sum_{k=1}^{n}X_k$ $\quad \frac{S_n}{n}-\mu\xrightarrow{a.s.}0\iff E[X_n]=\mu$

引理 $X\ge0$. 则 $E(X^k)=\int_0^{+\infty}kx^{k-1}P(X>x)\,dx$

证: $\int_0^{+\infty}kx^{k-1}P(X>x)\,dx=\int_0^{+\infty}kx^{k-1}\int_\Omega I_{\{X>x\}}\,dP\,dx=\int_\Omega\int_0^{+\infty}kx^{k-1}I_{\{X>x\}}\,dx\,dP$

$$=\int_\Omega\int_0^X kx^{k-1}\,dx\,dP=\int_\Omega X^k\,dP.$$

$$E|X_n|=\int_0^{+\infty}P(|X_n|>x)\,dx=\sum_{n=1}^{+\infty}\int_{n-1}^{n}P(|X_n|>x)\,dx\ge\sum_{n=1}^{\infty}P(|X_n|>n)$$

$$E|X_n|\le\sum_{n=1}^{\infty}P(|X_n|>n-1)=\sum_{n=1}^{\infty}P(|X_n|>n)+P(|X_n|\in(n-1,n])\le\sum_{n=1}^{\infty}P(|X_n|>n)+1$$

证明 i.i.d. 情形下 SLLN

"$\Rightarrow$" $\frac{X_n}{n}=\frac{S_n-S_{n-1}}{n}\xrightarrow{a.s.}0$ $\quad$ 对 $\forall\varepsilon>0$, $\sum_{n=1}^{\infty}P(|\frac{X_n}{n}|>\varepsilon)<\infty$

取 $\varepsilon=1$, $\sum_{n=1}^{\infty}P(|X_n|>n)<\infty$ $\quad E[X_n]$ 存在 $\quad E[\frac{S_n}{n}]=E[X_1]=\mu$.

"$\Leftarrow$" 令 $Y_n=\begin{cases}X_n, & |X_n|\le n\\ 0, & |X_n|>n\end{cases}$

$P(X_n\ne Y_n)=P(|X_n|>n)$, $\sum_{n=1}^{\infty}P(X_n\ne Y_n)=\sum_{n=1}^{\infty}P(|X_n|>n)<\infty$

$\Rightarrow P(X_n\ne Y_n\ \text{i.o.})=0$ $\quad$ 验证 $\frac{\sum_{k=1}^{n}Y_k}{n}\xrightarrow{a.s.}\mu$ $\quad\{Y_n\}$ 独立.

$$\sum_{n=1}^{\infty}\frac{Var(Y_n)}{n^2}\le\sum_{n=1}^{\infty}\frac{E(Y_n^2)}{n^2}\le\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=1}^{n}k^2P(k-1\le|X_n|<k)=\sum_{k=1}^{+\infty}k^2P(k-1\le|X_n|<k)\sum_{n=k}^{\infty}\frac{1}{n^2}$$

$$\le\sum_{k=1}^{+\infty}k^2\left(\frac{1}{k^2}+\frac{1}{k}\right)P(k-1\le|X_n|<k)\le 2+\sum_{k=1}^{+\infty}(k-1)P(k-1\le|X_n|<k)$$

$$\le 2+E|X_n|<\infty$$

$$\Rightarrow \frac{Y_1+\cdots+Y_n}{n}\xrightarrow{a.s.}\mu \qquad \frac{X_1+\cdots+X_n}{n}\xrightarrow{a.s.}\mu$$

§7.4 中心极限定理

$\{X_n\}$ i.i.d. $S_n=X_1+\cdots+X_n$ $\quad \frac{S_n-E(S_n)}{\sqrt{Var(S_n)}}\xrightarrow{D}N(0,1)$

Feller CLT + $\lim\limits_{n\to\infty}\max\limits_{1\le k\le n}\frac{a_k^2}{a_1^2+\cdots+a_n^2}=0$

Lindeberg条件：$\{X_n\}$独立，$a_k=E[X_k]$，$b_k^2=Var(X_k)$ $\quad B_n^2=\sum_{k=1}^{n}b_k^2$ $\quad F_k(x)$为X_k的分布函数.

对$\forall\varepsilon>0$，$\lim\limits_{n\to\infty}\frac{1}{B_n^2}\sum_{k=1}^{n}\int_{|x-a_k|>\varepsilon B_n}(x-a_k)^2dF_k(x)=0$

$$\Leftrightarrow \lim_{n\to\infty}\frac{1}{B_n^2}\sum_{k=1}^{n}E\left((X_k-a_k)^2I_{\{|X_k-a_k|>\varepsilon B_n\}}\right)=0$$

注记：(1) $\{X_n\}$满足Lindeberg条件，则 $\max\limits_{1\le k\le n}\left|\frac{X_k-a_k}{B_n}\right|\xrightarrow{P}0$

(2) L条件成立，则Feller条件成立. $\lim\limits_{n\to\infty}\max\limits_{1\le k\le n}\frac{b_k^2}{B_n^2}=0$

证：(1) $\forall\varepsilon>0$. $P\left(\max\limits_{1\le k\le n}\left|\frac{X_k-a_k}{B_n}\right|>\varepsilon\right)=P\left(\max\limits_{1\le k\le n}|X_k-a_k|>\varepsilon B_n\right)$

$$=P\left(\bigcup_{1\le k\le n}|X_k-a_k|>\varepsilon B_n\right)\le\sum_{k=1}^{n}P(|X_k-a_k|>\varepsilon B_n)$$

$$\le\sum_{k=1}^{n}E\left[\frac{(X_k-a_k)^2}{\varepsilon^2B_n^2}1_{\{|X_k-a_k|>\varepsilon B_n\}}\right]\longrightarrow 0$$

(2) $\max\limits_{1\le k\le n}\frac{b_k^2}{B_n^2}=\frac{1}{B_n^2}\max\limits_{1\le k\le n}E((X_k-a_k)^2)$

$$=\frac{1}{B_n^2}\max_{1\le k\le n}\left(E((X_k-a_k)^2\cdot I_{\{|X_k-a_k|>\varepsilon B_n\}})\right.$$

$$\left.+E((X_k-a_k)^2I_{\{|X_k-a_k|\le\varepsilon B_n\}})\right)$$

$$\le \frac{1}{B_n^2}\sum_{k=1}^n E((X_k-a_k)^2 I_{\{|X_k-a_k|>\varepsilon B_n\}}) + \varepsilon^2 \to 0 \quad (令\ \varepsilon\to 0)$$

定理 Lindeberg-Feller CLT

$\{X_k\}$ 独立随机变量列 $E[X_k]=a_k$, $Var(X_k)=b_k^2$. $B_n^2=\sum_{k=1}^n b_k^2$, $S_n=X_1+\cdots+X_n$.

满足 Lindeberg 条件. 则 $\frac{S_n-E[S_n]}{B_n} \xrightarrow{D} N(0,1)$

证: 令 $X_{nk}=\frac{X_k-a_k}{B_n}$, $k=1,2,\cdots,n$

$E(X_{nk})=0$. $E(X_{nk}^2)=\frac{b_k^2}{B_n^2}$, X_{nk} 的特征函数为 $\varphi_{nk}(t)$.

$\frac{S_n-E(S_n)}{B_n}$ 特征函数记为 $\varphi_n(t)=\prod_{k=1}^n \varphi_{nk}(t)$. 下面证明 $\lim_{n\to\infty}\prod_{k=1}^n \varphi_{nk}(t)=e^{-\frac{t^2}{2}}$

$$\varphi_{nk}(t)=E[e^{itX_{nk}}]=1+\frac{i^2t^2}{2}E[X_{nk}^2]+r_{nk}(t)$$

tips: X 的特征函数 $\varphi(t)=E[e^{itX}]$

(1) $\left|e^{it}-1-\frac{it}{1}-\frac{(it)^2}{2!}\right|\le\frac{|t|^3}{3!}$ (2) $\left|e^{it}-1-it-\frac{(it)^2}{2!}\right|\le|e^{it}-1-it|+\frac{t^2}{2}\le\frac{t^2}{2}+\frac{t^2}{2}=t^2$

$$|r_{nk}(t)|=|E(e^{itX_{nk}}-(1+itX_{nk}+\frac{i^2t^2}{2}X_{nk}^2))$$

$$\le E((tX_{nk})^2\wedge|t^3X_{nk}^3|)$$

$$\le E(|t^3X_{nk}^3|\cdot I_{\{|X_{nk}|\le\varepsilon\}}+|t^2X_{nk}^2|\cdot I_{\{|X_{nk}|>\varepsilon\}})$$

$$\le |t^3|\cdot\varepsilon E(X_{nk}^2)+t^2E[X_{nk}^2\cdot I_{\{|X_{nk}|>\varepsilon\}}]$$

tips: $|a_k|\le1$, $|b_k|\le1$, $|a_1\cdots a_n-b_1\cdots b_n|\le\sum_{k=1}^n|a_k-b_k|$

$$\left|\prod_{k=1}^n\varphi_{nk}(t)-\prod_{k=1}^n(1-\frac{t^2b_k^2}{2B_n^2})\right|\le\sum_{k=1}^n\left|\varphi_{nk}(t)-(1+\frac{t^2b_k^2}{2B_n^2}\right|=\sum_{k=1}^n|r_{nk}|\cdot\varepsilon\sum_{k=1}^n\frac{b_k^2}{B_n^2}+t^2\sum_{k=1}^nE[X_{nk}^2I_{\{X_{nk}>\varepsilon\}}]$$

$$=\varepsilon\cdot|t|^3+t^2\cdot\frac{1}{B_n^2}\cdot E[(X_k-a_k)^2I_{\{|X_k-a_k|>\varepsilon B_n\}}]\to0 \quad 令\ \varepsilon\to0$$

$$\lim_{n\to\infty}\prod_{k=1}^n\varphi_{nk}(t)=\lim_{n\to\infty}\prod_{k=1}^n(1-\frac{t^2b_k^2}{2B_n^2})$$

tips: $e^{x(1-x)}\le1+x\le e^x$

$$\prod_{k=1}^n(1-\frac{t^2b_k^2}{2B_n^2})\le\prod_{k=1}^n e^{-\frac{t^2b_k^2}{2B_n}}=e^{-\frac{t^2}{2}},\quad\prod_{k=1}^n(1-\frac{t^2b_k^2}{2B_n^2})\ge\prod_{k=1}^n(e^{-\frac{tb_k^2}{2B_n^2}+\frac{tb_k^2}{4B_n^2}})$$

$$=e^{-\frac{t^2}{2}}\cdot\exp(-\frac{t^4}{4}\sum_{k=1}^n(\frac{b_k^2}{B_n^2})^2)$$

$$=e^{-\frac{t^2}{2}}\exp(-\underbrace{\max_{1\le k\le n}\frac{b_k^2}{B_n^2}}_{\to0}\cdot\frac{t^4}{4}\cdot\underbrace{\sum_{k=1}^n\frac{b_k^2}{B_n^2}}_{=1})$$

$$\to e^{-\frac{t^2}{2}}$$

定理 Lyapunov CLT $\{X_n\}$ 独立. 若 $\exists\,\delta>0$. s.t. $\frac{1}{B_n^{2+\delta}}\sum_{k=1}^nE(|X_k-a_k|^{2+\delta})\to0$

则 $\frac{S_n-ES_n}{B_n}\xrightarrow{D}N(0,1)$