微分方程

高维偏微分方程初边值问题的唯一性和分离变量法

一、能量法与高维波动方程初边值问题的唯一性

$$
(\mathrm{P})\begin{cases}
u_{tt}-c^2\Delta u = f(x,t), x\in D\subset\mathbb{R}^n, t>0\\
u\big|_{t=0}=\varphi(x), u_t\big|_{t=0}=\psi(x), x\in\bar{D}\\
u\,\big|_{\partial D}=g(x,t), x\in\partial D, t\geq 0 \qquad (\mathrm{D})\\
\left.\dfrac{\partial u}{\partial \nu}\right|_{\partial D}=g(x,t), x\in\partial D, t\geq 0 \quad (\mathrm{N})\\
\left.\left(\sigma(x)u+\dfrac{\partial u}{\partial \nu}\right)\right|_{\partial D}=g(x,t), x\in\partial D, t\geq 0 \quad (\mathrm{R})
\end{cases}
$$

其中已知函数满足相应的相容条件, 函数$\sigma(x)\geq 0$.

唯一性定理：问题(P)在任一边界下至多有一解.

证明：设(P)有两解u_1, u_2，令$w = u_1 - u_2$，则w满足

$$(\mathrm{w})\begin{cases} w_{tt} - c^2\Delta w = 0, x \in D, t > 0 \\ \text{齐次的初始和边界条件} \end{cases}, \text{须证} w \equiv 0.$$

对边界条件(D)和(N)，定义(w)的“能量”为

$$E_{\mathrm{DN}}(t) = \frac{1}{2}\int_D (w_t^2 + c^2 |\nabla w|^2)dx, \ t \geq 0.$$

对边界条件(R)，定义(w)的“能量”为

$$E_{\mathrm{R}}(t) = \frac{1}{2}\int_D (w_t^2 + c^2 |\nabla w|^2)dx + \frac{c^2}{2}\int_{\partial D} \sigma(x) w^2 dS, \ t \geq 0.$$

下证(w)能量守恒. 首先由Gauss公式或散度定理有

$$\int_D (w_t \Delta w + \nabla w_t \cdot \nabla w) dx = \int_D \nabla \cdot (w_t \nabla w) dx$$

$$= \int_{\partial D} w_t \nabla w \cdot \nu dS = \int_{\partial D} w_t \frac{\partial w}{\partial \nu} dS$$

$$\Rightarrow \int_D \nabla w \cdot \nabla w_t dx = \int_D \nabla w_t \cdot \nabla w dx = \int_{\partial D} w_t \frac{\partial w}{\partial \nu} dS - \int_D w_t \Delta w dx.$$

则由上式和w满足齐次波动方程及齐次边界条件有

$$\frac{dE_{\mathrm{DN}}(t)}{dt} = \int_D (w_t w_{tt} + c^2 \nabla w \cdot \nabla w_t) dx$$

$$= \int_D w_t (w_{tt} - c^2 \Delta w) dx + c^2 \int_{\partial D} w_t \frac{\partial w}{\partial \nu} dS = 0, \ t \geq 0.$$

$$\frac{dE_{\mathrm{R}}(t)}{dt}=\int_D (w_t w_{tt}+c^2\nabla w\cdot\nabla w_t)dx+c^2\int_{\partial D}\sigma(x)ww_t dS$$

$$=\int_D w_t(w_{tt}-c^2\Delta w)dx+c^2\int_{\partial D}w_t(\sigma(x)w+\frac{\partial w}{\partial \nu})dS=0,\ t\geq 0.$$

故由齐次初始条件有$E_{\mathrm{DN}}(t)=E_{\mathrm{DN}}(0)=E_{\mathrm{R}}(t)=E_{\mathrm{R}}(0)=0,$

即$w_t=0,\nabla w=\vec{0}\Rightarrow w\equiv$常数$=w|_{t=0}=0, u_1\equiv u_2$.证毕.

注：对高维热方程初边值问题的唯一性可类似采用能量法证明.例如对Dirichlet边界定义能量为

$E_{\mathrm{D}}(t)=\frac{1}{2}\int_D w^2(x,t)dx, t\geq 0,$证明能量衰减即可.

二、高维偏微分方程初边值问题的分离变量法

- 球域内三维波动方程的初边值问题

$$
\begin{cases}
\dfrac{\partial^2 u}{\partial t^2} - c^2\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right) = 0, & x^2 + y^2 + z^2 < a^2, t > 0 \\
u\big|_{t=0} = \varphi(x, y, z), & x^2 + y^2 + z^2 \le a^2 \\
u_t\big|_{t=0} = \psi(x, y, z), & x^2 + y^2 + z^2 \le a^2 \\
u\big|_{x^2+y^2+z^2=a^2} = 0, & t \ge 0
\end{cases}
$$

求解步骤如下：（完整求解过程自行补充）

第一步：首先将时间变量与空间变量分离开来，即求分离解

$$u(x,y,z,t)=v(x,y,z)T(t)$$

$$v(x,y,z):\ \begin{cases}\Delta v+\lambda v:=\Delta v+k^2 v=0\\ v\big|_{x^2+y^2+z^2=a^2}=0\end{cases}$$

（上述偏微分方程称为“亥姆霍兹(*Helmholtz*)方程”）

$$T(t):\quad T''(t)+k^2c^2T(t)=0$$

第二步：求解$T(t)$

$$\begin{cases} T(t) = C\cos(kct) + D\sin(kct), & k \neq 0 \\ T(t) = C + Dt, & k = 0 \end{cases}$$

第三步：求解$v(x,y,z)$

$$\begin{cases} \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial v}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial v}{\partial\theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 v}{\partial\varphi^2} + k^2 v = 0 \\ v\big|_{r=a} = 0 \\ v\big|_{r=0} = \text{有限值} \\ v(r,\theta,\varphi) = v(r,\theta,\varphi+2\pi) \\ v\big|_{\theta=0,\pi} = \text{有限值} \end{cases}$$

求如下形式的解 $v(r,\theta,\varphi)=R(r)Y(\theta,\varphi)$

$$R(r):\begin{cases}\dfrac{d}{dr}\left(r^2\dfrac{dR}{dr}\right)+\left[k^2r^2-l(l+1)\right]R=0\\ R(a)=0\\ R(0)=\text{有限值}\end{cases}$$

球Bessel方程

$$\sqrt{\frac{\pi}{2k_n r}}AJ_{l+\frac{1}{2}}\left(k_n r\right)$$

球Bessel函数

$$Y(\theta,\varphi):\begin{cases}\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial Y}{\partial\theta}\right)+\dfrac{1}{\sin^2\theta}\dfrac{\partial^2 Y}{\partial\varphi^2}+l(l+1)Y=0\\ Y(\theta,\varphi)=Y(\theta,\varphi+2\pi)\\ Y\big|_{\theta=0,\pi}=\text{有限值}\end{cases}$$

$$Y_l^m\left(\theta,\varphi\right)$$

球函数

第四步：利用初始条件求形式解

$$u(r,\theta,\varphi,t)=\sum_{n=1}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}P_l^m(\cos\theta)(A_{lm}\cos m\varphi+B_{lm}\sin m\varphi)$$

$$\times\frac{1}{\sqrt{k_n r}}J_{l+\frac{1}{2}}\left(k_n r\right)\left(C_n\cos(k_n ct)+D_n\sin(k_n ct)\right)$$

- 球域内三维热方程的初边值问题

$$\begin{cases} \dfrac{\partial u}{\partial t} - k\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right) = 0, & x^2 + y^2 + z^2 < a^2, t > 0 \\ u\big|_{t=0} = \varphi(x, y, z), & x^2 + y^2 + z^2 \le a^2 \\ u\big|_{x^2+y^2+z^2=a^2} = 0, & t \ge 0 \end{cases}$$

$$u(r,\theta,\varphi,t) = \sum_{n=1}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty} P_l^m(\cos\theta)(A_{lmn}\cos m\varphi + B_{lmn}\sin m\varphi) \times \frac{1}{\sqrt{k_n r}} J_{l+\frac{1}{2}}\left(k_n r\right)\exp\left(-k k_n^2 t\right)$$