5. Let X_r, $1 \le r \le n$, be independent random variables which are symmetric about 0; that is, X_r and $-X_r$ have the same distributions. Show that, for all x, $\mathbb{P}(S_n \ge x) = \mathbb{P}(S_n \le -x)$ where $S_n = \sum_{r=1}^{n} X_r$.

Is the conclusion necessarily true without the assumption of independence?

<u>3.2.5</u>: $(-X_r : 1 \le r \le n)$ 与 $(X_r : 1 \le r \le n)$ 联合分布相同

故 S_r 与 $-S_r$ 联合分布相同 (或对离散型直接展开.)

若连续型

$n=2$ 时 $\{X_1 + X_2 > x\}$

$= \bigcup_{\substack{q_1, q_2 \in \mathbb{Q} \\ q_1 + q_2 > x}} \{X_1 > q_1, X_2 > q_2\}$

$= \bigcup \{-X_1 > q_1, -X_2 > q_2\}$

先说明有限并概率相等

再利用集合升链来说明

可数并概率相等.

最后归纳

不独立反例: X.Y 联合分布

$$\begin{array}{c|c} Y = -1 & \\ 0 & \\ 1 & \end{array}\begin{pmatrix} \frac{1}{6} & \frac{1}{12} & \frac{1}{12} \\ 0 & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{12} & \frac{1}{12} \end{pmatrix}$$

$$X = -1 \quad 0 \quad 1$$

$\therefore$ X, Y 对称.

但 $P(X+Y=-2) = \frac{1}{6} \ne \frac{1}{12} = P(X+Y=2)$

2. Coupons. Every package of some intrinsically dull commodity includes a small and exciting plastic object. There are c different types of object, and each package is equally likely to contain any given type. You buy one package each day.

(a) Find the mean number of days which elapse between the acquisitions of the jth new type of object and the $(j+1)$th new type.

(b) Find the mean number of days which elapse before you have a full set of objects.

3. Each member of a group of n players rolls a die.

(a) For any pair of players who throw the same number, the group scores 1 point. Find the mean and variance of the total score of the group.

(b) Find the mean and variance of the total score if any pair of players who throw the same number scores that number.

<u>3.3.2</u>: (a) 每天买到新品种概率为 $\frac{c-j}{c}$

几何分布 期望为 $\frac{c}{c-j}$

(b) $\sum_{j=0}^{c-1} \frac{c}{c-j} = c \sum_{k=1}^{c} \frac{1}{k}$

<u>3.3.3</u>: (a) 令 I_{ij} 为事件 i, j 点数相同的示性函数.

则 $\mathbb{E}(I_{ij}) = P(I_{ij}=1) = \sum_{i=1}^{6} \left(\frac{1}{6}\right)^2 = \frac{1}{6}$

总分 $S = \sum_{i<j} I_{ij}$ $\therefore$ $\mathbb{E}(S) = \sum_{i<j} \mathbb{E}(I_{ij}) = \frac{1}{6}\binom{n}{2}$

claim: I_{ij} 与 I_{jk} 独立. $(i<j<k)$ (Ex 1.5.2)

$\therefore$ $\mathbb{E}(I_{ij} I_{jk}) = \mathbb{E}(I_{ij})\mathbb{E}(I_{jk})$

$\therefore$ $\mathrm{Var}(S) = \sum_{i<j} \mathrm{Var}(I_{ij}) = \binom{n}{2} \mathrm{var}(I_{12}) = \binom{n}{2} \frac{1}{6}\left(1 - \frac{1}{6}\right)$

(b) 设 X_{ij} 为 i,j 的得分, $X_i=0$ if i,j 得分不同

$$S=\sum_{i<j}X_{ij}.$$

$$\mathbb{E}(S)=\binom{n}{2}\mathbb{E}(X_{12})=\binom{n}{2}\frac{1}{6}\cdot\frac{7}{2}=\frac{7}{12}\binom{n}{2}$$

X_{ij} 不两两独立

$$\mathrm{Var}(S)=\mathbb{E}\left\{\left(\sum_{i<j}X_{ij}\right)^2\right\}-\left(\mathbb{E}(S)\right)^2$$

$$=\binom{n}{2}\mathbb{E}(X_{12}^2)+6\binom{n}{3}\mathbb{E}(X_{12}X_{23})+\left\{\binom{n}{2}^2-\binom{n}{2}-6\binom{n}{3}\right\}\mathbb{E}(X_{12})^2-\left(\frac{7}{12}\right)^2\binom{n}{2}^2$$

$$=\frac{35}{16}\binom{n}{2}+\frac{35}{72}\binom{n}{3}$$

2. An urn contains n balls numbered $1, 2, \dots, n$. We remove k balls at random (without replacement) and add up their numbers. Find the mean and variance of the total.

3.4.2: $T=\sum_{i=1}^{k}X_i$, X_i 为第 i 个球上的数.

$$\mathbb{E}(T)=k\,\mathbb{E}(X_i)=\frac{1}{2}k(n+1)$$

$$\mathbb{E}\left\{\left(\sum_{i=1}^{k}X_i\right)^2\right\}=k\,\mathbb{E}(X_1^2)+k(k-1)\,\mathbb{E}(X_1X_2)$$

$$=\frac{k}{n}\sum_{i=1}^{n}i^2+\frac{k(k-1)}{n(n-1)}2\sum_{i>j}ij$$

$$=\frac{k}{n}\left\{\frac{1}{3}n(n+1)(n+2)-\frac{1}{2}n(n+1)\right\}\frac{k(k-1)}{n(n-1)}\sum_{j=1}^{n}j\{n(n+1)-j(j+1)\}$$

$$=\frac{1}{6}k(n+1)(2n+1)+\frac{1}{12}k(k-1)(3n+2)(n+1)$$

$$\Rightarrow \mathrm{Var}(T)=\frac{1}{12}(n+1)k(n-k)$$

4. Urn R contains n red balls and urn B contains n blue balls. At each stage, a ball is selected at random from each urn, and they are swapped. Show that the mean number of red balls in urn R after stage k is $\frac{1}{2}n\{1+(1-2/n)^k\}$. This 'diffusion model' was described by Daniel Bernoulli in 1769.

3.4.4: 每个红球在 k 次操作后仍在 R 中当且仅当它被选了偶数次.

$$\text{概率为 } P=\sum_{\substack{m \text{ is even}\\ m\le k}}\binom{k}{m}\left(\frac{1}{n}\right)^m\left(1-\frac{1}{n}\right)^{k-m}$$

$$=\frac{1}{2}\left\{\left[\left(1-\frac{1}{n}\right)+\frac{1}{n}\right]^k+\left[\left(1-\frac{1}{n}\right)-\frac{1}{n}\right]^k\right\}$$

$$=\frac{1}{2}\left\{1+\left(1-\frac{2}{n}\right)^k\right\}.$$

期望为 $nP=\frac{n}{2}\left\{1+\left(1-\frac{2}{n}\right)^k\right\}$

3. Let X and Y be discrete random variables with joint mass function

$$f(x, y) = \frac{C}{(x+y-1)(x+y)(x+y+1)}, \qquad x, y = 1, 2, 3, \ldots.$$

Find the marginal mass functions of X and Y, calculate C, and also the covariance of X and Y.

4. Let X and Y be discrete random variables with mean 0, variance 1, and covariance ρ. Show that $\mathbb{E}(\max\{X^2, Y^2\}) \le 1 + \sqrt{1-\rho^2}$.

5. Mutual information. Let X and Y be discrete random variables with joint mass function f.

(a) Show that $\mathbb{E}(\log f_X(X)) \ge \mathbb{E}(\log f_Y(X))$.

(b) Show that the *mutual information*

$$I = \mathbb{E}\left(\log\left\{\frac{f(X,Y)}{f_X(X)f_Y(Y)}\right\}\right)$$

satisfies $I \ge 0$, with equality if and only if X and Y are independent.

3.6.3:
$$\mathbb{P}(X=x) = \sum_{y=1}^{\infty} \mathbb{P}(X=x, Y=y)$$
$$= \sum_{y=1}^{\infty} \frac{C}{2}\left\{\frac{1}{(x+y-1)(x+y)} - \frac{1}{(x+y)(x+y+1)}\right\}$$
$$= \frac{C}{2x(x+1)} = \frac{C}{2}\left(\frac{1}{x} - \frac{1}{x+1}\right)$$

$\therefore C = 2$, Y与X同分布

$$\mathbb{E}(X) = \sum_{x=1}^{\infty} \frac{1}{x+1} = \infty \qquad \therefore \mathrm{Var}(X)\text{不存在}$$

3.6.4:
$$\max\{x, y\} = \frac{1}{2}(x+y) + \frac{1}{2}|x-y|$$
$$\mathbb{E}(\max\{X^2, Y^2\}) = \frac{1}{2}\mathbb{E}(X^2+Y^2) + \frac{1}{2}\mathbb{E}|(X-Y)(X+Y)|$$
$$\overset{\text{Cauchy-Schwarz}}{\le} 1 + \frac{1}{2}\sqrt{\mathbb{E}((X-Y)^2)\,\mathbb{E}((X+Y)^2)}$$
$$= 1 + \frac{1}{2}\sqrt{(2-2\rho)(2+2\rho)}$$
$$= 1 + \sqrt{1-\rho^2}$$

3.6.5: (a) 由 $\log y \le y-1$，等号 $\Leftrightarrow y=1$

$$\therefore \mathbb{E}\left(\log \frac{f_Y(X)}{f_X(X)}\right) \le \mathbb{E}\left[\frac{f_Y(X)}{f_X(X)} - 1\right] = 0$$

等号 $\Leftrightarrow f_Y = f_X$

可不假设：$\mathrm{supp}(X) \supseteq \mathrm{supp}(Y)$

Remark: $f_X(x) = 0$ 时，$f_X(x)\log\frac{f_Y(x)}{f_X(x)} = 0$ 仍有定义.

(b) 与 (a) 同理

$$-I = \mathbb{E}\left(\log\left(\frac{f_X(X)f_Y(Y)}{f(X,Y)}\right)\right)$$

$$\leq \mathbb{E}\left(\frac{f_X(X)f_Y(Y)}{f(X,Y)} - 1\right)$$

$$= 0$$

$\therefore I \geqslant 0$, $I = 0 \Leftrightarrow f_X(x)f_Y(y) = f(x,y)$

$\Leftrightarrow$ X, Y 独立