微分方程

Riccati方程

1.Riccati方程

Riccati方程 $x' = p(t)x^2 + q(t)x + r(t)$

其中 $p(t) \neq 0, q(t), r(t) \in C(I)$

例： $x' = t^2 + x^2$

- 最简单的一类非线性方程
- 深受关注：Bernoulli家族，Euler，d’Alembert，Liouville……
- 仍是世界著名难题

4·75. *On the impossibility of integrating Riccati's equation in finite terms.*

By means of the result just obtained, we can discuss Riccati's equation

$$\frac{dy}{dz} = az^n + by^2$$

with a view to proving that it is, in general, not integrable in finite terms.

It has been seen (§ 4·21) that the equation is reducible to

$$\frac{d^2u}{d\zeta^2} - c^2 \zeta^{2q-2} u = 0,$$

where $n = 2q - 2$; and, by § 4·3, the last equation is reducible to Bessel's equation for functions of order $1/(2q)$ unless $q = 0$.

Hence the only possible cases in which Riccati's equation is integrable in finite terms are those in which q is zero or $1/q$ is an odd integer; and these are precisely the cases in which n is equal to -2 or to

$$-\frac{4m}{2m \pm 1}. \qquad (m = 0,\ 1,\ 2,\ \ldots)$$

Consequently the only cases in which Riccati's equation is integrable in finite terms are the classical cases discovered by Daniel Bernoulli (cf. § 4·11) and the limiting case discussed after the manner of Euler in § 4·12.

This theorem was proved by Liouville, *Journal de Math.* VI. (1841), pp. 1—13. It seems impossible to establish it by any method which is appreciably more brief than the analysis used in the preceding sections.

G. N. Waston, A Treatise on the Theory of Bessel Functions, 1944, P123

2. 两个有趣的结果

定理1(1725,Daniel Bernoulli:概率、数学物理先驱，Euler老师)

$Riccati$方程 $x' = at^n + bx^2$ $(a, b \neq 0, n:$常数$)$在

$n = -2, -\dfrac{4m}{2m \pm 1}(m = 0, 1, 2, \cdots)$时可化为分离型方程。

证: (1)$n = -2$时, 令$u = tx$得$u' = (a + u + bu^2)/t$, 分离型方程

(2)$n = 0$时显然; $n = -\dfrac{4m}{2m+1}$, $m \in \mathbb{N}$时,

令$t = \xi^{\frac{1}{n+1}}$, $x = \dfrac{a}{n+1}\eta^{-1}$, 有

$$\frac{d\eta}{d\xi} + \eta^2 = -\frac{ab}{(n+1)^2}\xi^{-\frac{4m}{2m-1}} \quad (*)$$

再令$\xi = z^{-1}, \eta = z - uz^2$，有

$$\frac{du}{dz} + u^2 = -\frac{ab}{(n+1)^2} z^{-\frac{4(m-1)}{2(m-1)+1}}.$$

重复上述变换m次方程可化为$n=0$情形

(3) $n = -\dfrac{4m}{2m-1}, m \in \mathbb{N}$时，做变量变换方程可化为

(*)的形式，故可化为$n=0$情形

注：对Riccati方程， Riccati本人还得到如下结论：若知道方程的一个，两个或三个特解，则通解可由这些特解表示

定理2：(2014, Lazhar Bougoffa)

若Riccati方程$x' = p(t)x^2 + q(t)x + r(t)$的系数满足

$$\frac{r'(t)}{r(t)} - \frac{p'(t)}{p(t)} - 2q(t) = k(t)\sqrt{p(t)r(t)},$$

其中$k(t)$为某一函数，则方程可化为分离型方程.

例：(1) $x' = t^{m-1}x^2 + (n-m)t^{-1}x + t^{2n-1}, k = 0:$

$$x = t^{n-m}\tan\left(\frac{t^{m+n}}{m+n} + C\right)$$

(2) $x' = ax^2 + be^{2t}, a, b > 0:$常数，$k = \dfrac{2e^{-t}}{\sqrt{ab}}:$

$$x = \sqrt{\frac{b}{a}}\left(\frac{2e^{-t}}{\sqrt{ab}} - \frac{1}{\sqrt{ab}e^t + C}\right)$$