5.8

5. Let $X_1, X_2, \ldots$ be independent $N(0,1)$ variables. Use characteristic functions to find the distribution of: (a) X_1^2, (b) $\sum_{i=1}^{n} X_i^2$, (c) X_1/X_2, (d) X_1X_2, (e) $X_1X_2 + X_3X_4$.

(e)

解. $\varphi_{X_1X_2+X_3X_4}(t) = E[e^{it(X_1X_2+X_3X_4)}] \overset{独立}{=} E[e^{itX_1X_2}]E[e^{itX_3X_4}] = (\varphi_{X_1X_2}(t))^2$

$\varphi_{X_1X_2}(t) = E[e^{itX_1X_2}] = E[E[e^{itX_1X_2}|X_2]] = E[\varphi_{X_1}(X_2t)] = E[e^{-\frac{1}{2}X_2^2t^2}]$

$= \int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2t^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{1+t^2}}$

故 $\varphi_{X_1X_2+X_3X_4}(t) = \frac{1}{1+t^2}$

由 5.8.9(a) 知. $f(x) = \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$

5.8

9. Find the characteristic functions of the following density functions:

(a) $f(x) = \frac{1}{2}e^{-|x|}$ for $x \in \mathbb{R}$,

(b) $f(x) = \frac{1}{2}|x|e^{-|x|}$ for $x \in \mathbb{R}$.

$f(x)=e^{-x}$ 的特征函数

解: (a) $\varphi(t) = E[e^{itX}] = \frac{1}{2}\left(\underset{\varphi_1(t)}{\frac{1}{1-it}} + \underset{\varphi_1(-t)}{\frac{1}{1+it}}\right) = \frac{1}{1+t^2}$

(b) 先求 $f(x) = xe^{-x}$, $x \geq 0$ 的特征函数

$\varphi_2(t) = E[e^{itX}] = \int_0^{+\infty} e^{itx} \cdot xe^{-x} dx = \frac{1}{(it-1)^2}$

$\varphi(t) = \frac{1}{2}\left(\frac{1}{(it-1)^2} + \frac{1}{(it+1)^2}\right) = \frac{1-t^2}{(1+t^2)^2}$

5.9

2. Let X_n have distribution function

$$F_n(x) = x - \frac{\sin(2n\pi x)}{2n\pi}, \qquad 0 \leq x \leq 1.$$

(a) Show that F_n is indeed a distribution function, and that X_n has a density function.

(b) Show that, as $n \to \infty$, F_n converges to the uniform distribution function, but that the density function of F_n does not converge to the uniform density function.

证：(a) $f_n(x)=1-\cos(2n\pi x)\geq 0$, $\int_0^1 f_n(x)\,dx=\int_0^1 1-\cos(2n\pi x)\,dx=1-\frac{\sin(2n\pi)}{2n\pi}=1$

$\Rightarrow f_n(x)$ 为密度函数，$F_n(x)$ 为分布函数.

(b) $n\to\infty$ 时, $F_n(x)\longrightarrow x$，但 $f_n(x)$ 不收敛.

5.9

5. Use the inversion theorem to show that

$$\int_{-\infty}^{\infty}\frac{\sin(at)\sin(bt)}{t^2}\,dt=\pi\min\{a,b\}.$$

证：设 r.v. X 在 $[-a,a]$ 上均匀分布，r.v. Y 在 $[-b,b]$ 上均匀分布. 且 X 与 Y 独立.

$\varphi_X(t)=E[e^{itX}]=\int_{-a}^{a}e^{itx}\cdot\frac{1}{2a}\,dx=\frac{1}{2a}\cdot\frac{1}{it}e^{itx}\Big|_{-a}^{a}=\frac{\sin(at)}{at}$

同理，$\varphi_Y(t)=\frac{\sin(bt)}{bt}$ $\qquad \varphi_{X+Y}(t)\overset{X,Y\text{独立}}{=\!=\!=}\varphi_X(t)\,\varphi_Y(t)=\frac{\sin(at)\sin(bt)}{abt^2}$

由反转公式知，$f(a)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-ita}\varphi(t)\,dt$. 代入 $a=0$.

$f_{X+Y}(0)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\varphi_{X+Y}(t)\,dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\sin(at)\sin(bt)}{abt^2}\,dt$

而 $f_{X+Y}(0)=\int_{-\min\{a,b\}}^{\min\{a,b\}}\frac{1}{2a}\cdot\frac{1}{2b}=\frac{\min\{a,b\}}{2ab}$

$\therefore \int_{-\infty}^{+\infty}\frac{\sin(at)\sin(bt)}{t^2}\,dt=\pi\cdot\min\{a,b\}$

5.9

8. Let X_1, X_2 have a bivariate normal distribution with zero means, unit variances, and correlation ρ. Use the inversion theorem to show that

$$\frac{\partial}{\partial\rho}\mathbb{P}(X_1>0,\ X_2>0)=\frac{1}{2\pi\sqrt{1-\rho^2}}.$$

Hence find $\mathbb{P}(X_1>0,\ X_2>0)$.

$(t_1\ \ t_2)\begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}\begin{pmatrix}t_1\\ t_2\end{pmatrix}$

证：$\varphi(t)\triangleq\varphi(t_1,t_2)=\exp(i\mu^Tt-\frac{1}{2}t^T\Sigma t)\overset{\mu=0}{=}\exp(-\frac{1}{2}t^T\Sigma t)$

$\frac{\partial}{\partial\rho}P(X_1>0,X_2>0)=\frac{\partial}{\partial\rho}\int_0^{+\infty}\int_0^{+\infty}f(x_1,x_2)\,dx_1dx_2$

$\overset{\text{反转公式}}{=\!=\!=}\frac{\partial}{\partial\rho}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{4\pi^2}\iint_{R^2}e^{-it_1x_1}e^{-it_2x_2}\exp(-\frac{1}{2}t^T\Sigma t)\,dt_1dt_2\,dx_1dx_2$

$\overset{\text{对x积分}}{=\!=\!=}\frac{\partial}{\partial\rho}\cdot\frac{1}{4\pi^2}\iint_{R^2}\frac{\exp(-\frac{1}{2}t^T\Sigma t)}{(it_1)(it_2)}\,dt_1dt_2$

$=\frac{1}{4\pi^2}\iint_{R^2}\exp(-\frac{1}{2}t^T\Sigma t)\,dt_1dt_2=\frac{2\pi|\Sigma^{-1}|^{\frac{1}{2}}}{4\pi^2}=\frac{1}{2\pi\sqrt{1-\rho^2}}$

由 ex.4.7.5 知 $P(X_1>0,X_2>0)=\frac{1}{4}+\frac{1}{2\pi}\sin^{-1}\rho$

5.10

1. Prove that, for $x \geq 0$, as $n \to \infty$,

(a) $$\sum_{\substack{k: \\ |k-\frac{1}{2}n| \leq \frac{1}{2}x\sqrt{n}}} \binom{n}{k} \sim 2^n \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du,$$

(b) $$\sum_{\substack{k: \\ |k-n| \leq x\sqrt{n}}} \frac{n^k}{k!} \sim e^n \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du.$$

证：(a) $X_i \overset{i.i.d.}{\sim} B(1, \frac{1}{2})$ $\quad S_n = X_1 + \cdots + X_n \quad Var(X_i) = \frac{1}{4}$, $E[X_i] = \frac{1}{2}$

$$P\left(\frac{|S_n - \frac{1}{2}n|}{\frac{1}{2}\sqrt{n}} \leq x\right) = \sum_{\frac{|k-\frac{1}{2}n|}{\frac{1}{2}\sqrt{n}} \leq x} P(S_n = k) = \sum_{\frac{|k-\frac{1}{2}n|}{\frac{1}{2}\sqrt{n}} \leq x} C_n^k \cdot 2^{-n}$$

又 $$\frac{S_n - \frac{1}{2}n}{\frac{1}{2}\sqrt{n}} \xrightarrow{D} N(0,1)$$

故 $$P\left(\frac{|S_n - \frac{1}{2}n|}{\frac{1}{2}\sqrt{n}} \leq x\right) \xrightarrow{n\to\infty} \Phi(x) - \Phi(-x) = \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

$$\sum_{\substack{k: \\ |k-\frac{1}{2}n| \leq \frac{1}{2}x\sqrt{n}}} \binom{n}{k} \sim 2^n \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

(b) $X_i \overset{i.i.d.}{\sim} Poi(1)$ $\quad S_n = X_1 + \cdots + X_n \quad Var(X_i) = 1$, $E[X_i] = 1$

$$P\left(\frac{|S_n - n|}{\sqrt{n}} \leq x\right) = \sum_{\frac{|k-n|}{\sqrt{n}} \leq x} P(S_n = k) = \sum_{\frac{|k-n|}{\sqrt{n}} \leq x} e^{-n} \cdot \frac{n^k}{k!}$$

又 $$\frac{S_n - n}{\sqrt{n}} \xrightarrow{D} N(0,1)$$

故 $$P\left(\frac{|S_n - n|}{\sqrt{n}} \leq x\right) \xrightarrow{n\to\infty} \Phi(x) - \Phi(-x) = \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

$$\sum_{\substack{k: \\ |k-n| \leq x\cdot\sqrt{n}}} \frac{n^k}{k!} \sim e^n \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

3. Let X have the $\Gamma(1, s)$ distribution; given that $X = x$, let Y have the Poisson distribution with parameter x. Find the characteristic function of Y, and show that

$$\frac{Y - \mathbb{E}(Y)}{\sqrt{\mathrm{var}(Y)}} \xrightarrow{\mathrm{D}} N(0, 1) \qquad \text{as } s \to \infty.$$

Explain the connection with the central limit theorem.

证: $\varphi_Y(t) = E[e^{itY}] = E[E[e^{itY}|X]] \overset{Y\sim Poi(X)}{=\!=\!=} E[\exp(X(e^{it}-1))]$

$\overset{X\sim\Gamma(1,s)}{=\!=\!=} \left(\frac{1}{1-(e^{it}-1)}\right)^s = \left(\frac{1}{2-e^{it}}\right)^s$

$E[Y] = \frac{1}{i}\phi_Y'(0) = s$, $E[Y^2] = -\phi_Y''(0) = s^2+2s$, $\mathrm{Var}(Y) = 2s$

令 $Z = \frac{Y-E[Y]}{\sqrt{\mathrm{Var}(Y)}} = \frac{Y-s}{\sqrt{2s}}$

$\varphi_Z(t) = E[\exp(itZ)] = E[\exp(it\cdot\frac{Y-s}{\sqrt{2s}})] = e^{-it\sqrt{\frac{s}{2}}}\,\phi_Y(\frac{t}{\sqrt{2s}})$

$\log(\phi_Y(\frac{t}{\sqrt{2s}})) = \log((2-e^{i\cdot\frac{t}{\sqrt{2s}}})^{-s}) = -s\log(2-\exp(i\cdot\frac{t}{\sqrt{2s}}))$

$\overset{Taylor}{=\!=} s(\exp(i\cdot\frac{t}{\sqrt{2s}})-1) + \frac{1}{2}s\cdot(\exp(i\cdot\frac{t}{\sqrt{2s}})-1)^2 + o(1)$

$= it\sqrt{\frac{1}{2}s} - \frac{1}{2}t^2$

$\log(\phi_Z(t)) \to e^{-\frac{1}{2}t^2}$, 故 $Z = \frac{Y-E[Y]}{\sqrt{\mathrm{Var}(Y)}} \xrightarrow{D} N(0,1)$

联系: 同样也可以用中心极限定理证明:

令 $Y = P_1 + P_2 + \cdots + P_X$, $P_i \sim Poi(1)$ 即可.

4. Let $X_1, X_2, \ldots$ be independent random variables taking values in the positive integers, whose common distribution is non-arithmetic, in that $\gcd\{n : \mathbb{P}(X_1 = n) > 0\} = 1$. Prove that, for all integers x, there exist non-negative integers $r = r(x)$, $s = s(x)$, such that

$$\mathbb{P}\big(X_1 + \cdots + X_r - X_{r+1} - \cdots - X_{r+s} = x\big) > 0.$$

证: 设 X_1 在 $\{n_1, \cdots, n_k\}$ 中每点取值的概率 >0.

且 $\gcd(n_1, \cdots, n_k) = 1$.

故 $\exists N$, 对 $\forall n \geq N$, $n \in \mathbb{Z}$, 都存在 $\alpha_1, \cdots, \alpha_k$ 非负.

s.t. $n = n_1\alpha_1 + \cdots + n_k\alpha_k$.

令 $N = n_1\beta_1 + \cdots + n_k\beta_k$, $N+x = n_1\gamma_1 + \cdots + n_k\gamma_k$, β_i, γ_i 非负

$$P(X_1+\cdots+X_r-X_{r+1}-\cdots-X_{r+s}=x) \geq P(X_1+\cdots+X_{r+s}=N+x)\,P(X_{r+1}+\cdots+X_{r+s}=N)$$

$$\begin{cases} P(X_{r+1}+\cdots+X_{r+s}=N) \geq \prod_{i=1}^{k} P(X_1=n_i)^{\beta_i} > 0. \\ P(X_1+\cdots+X_{r+s}=N+x) \geq \prod_{i=1}^{k} P(X_1=n_i)^{\gamma_i} > 0 \end{cases}$$

故 $P(X_1+\cdots+X_r-X_{r+1}-\cdots-X_{r+s}=x)>0$.

7.2

1. (a) Suppose $X_n \xrightarrow{r} X$ where $r \geq 1$. Show that $\mathbb{E}|X_n^r| \to \mathbb{E}|X^r|$.

(b) Suppose $X_n \xrightarrow{1} X$. Show that $\mathbb{E}(X_n) \to \mathbb{E}(X)$. Is the converse true?

(c) Suppose $X_n \xrightarrow{2} X$. Show that $\mathrm{var}(X_n) \to \mathrm{var}(X)$.

2. Dominated convergence. Suppose $|X_n| \leq Z$ for all n, where $\mathbb{E}(Z) < \infty$. Prove that if $X_n \xrightarrow{\mathrm{P}} X$ then $X_n \xrightarrow{1} X$.

1. 证：(a) 由 Minkowski 不等式，$(E|X^r|)^{\frac{1}{r}} \leq (E|(X-X_n)^r|)^{\frac{1}{r}} + (E|X_n^r|)^{\frac{1}{r}}$,

$(E|X_n^r|)^{\frac{1}{r}} \leq (E|(X-X_n)^r|)^{\frac{1}{r}} + (E|X^r|)^{\frac{1}{r}}$

$\because X_n \xrightarrow{r} X \quad \therefore E|(X-X_n)^r| \to 0$ as $n \to +\infty$. 结合上两式，有 $E|X_n^r| \to E|X^r|$

(b) $|E(X_n)-E(X)| = |E(X_n-X)| \leq E|X_n-X| \to 0 \quad \therefore E(X_n) \longrightarrow E(X)$

反之不对.

(c) 由(a)知，当 $X_n \xrightarrow{2} X$，则 $EX_n^2 \to EX^2$

$X_n \xrightarrow{2} X \Rightarrow X_n \xrightarrow{1} X$，则 $E|X_n| \to E|X| \Rightarrow EX_n \to EX$

$\therefore Var(X_n) = E[X_n^2] - E[X_n]^2 \longrightarrow E[X^2] - E[X]^2 = Var(X)$

2. 证：令 $Z_n = |X_n - X| \leq 2Z$ 　只需证 $EZ_n \to 0$

$\because X_n \xrightarrow{P} X \quad \therefore P(\lim_{n\to\infty} Z_n \geq \varepsilon) = P(\lim_{n\to\infty} |X_n - X| \geq \varepsilon) = 0$.

$\forall \varepsilon > 0$, $E[Z_n] = E[Z_n 1_{\{Z_n \leq \varepsilon\}}] + E[Z_n 1_{\{Z_n > \varepsilon\}}] \leq \varepsilon + 2E[Z 1_{\{Z_n > \varepsilon\}}]$

$= \varepsilon + 2E[Z] \cdot P(Z_n > \varepsilon)$

令 $\varepsilon \to 0$. $n \to +\infty$. 有 $E[Z_n] \longrightarrow 0$. $\Leftrightarrow X_n \xrightarrow{1} X$