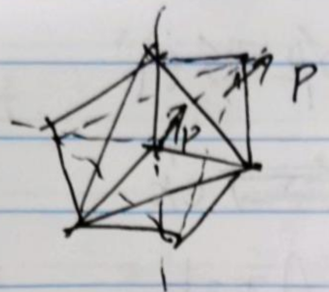
T =

作业 19.1

ℓ_1 范数球



$n=3$.

$P_C(x)$ 为在 ℓ_1 范数球上，$P_C(x)-x$ 垂直于

ℓ_1 范数球 C 表面，

若 $\|x\|_1 \le 1$. 则 x 在 C 内部. 则 $P_C(x)$ 即为 x 本身

$\Rightarrow \lambda=0$. 否则. 此时 $P_C(x)$ 应满足

$P_C(x)=(x-\lambda p)_+$, $p_k=\mathrm{sign}(x_k)$.

λ 应满足 $1^T(x-\lambda p)_+=\sum_{k=1}^{n}\max\{|x_k|-\lambda,0\}=1$

$$\Rightarrow \quad P_C(x)_k=\begin{cases} x_k-\lambda & x_k>\lambda \\ 0 & -\lambda\le x_k\le\lambda \\ x_k+\lambda & x<-\lambda \end{cases}$$

作业 20.1

1. 线性规划标准形式

$$\min \ c^T x$$
$$\text{s.t.} \ Ax=b.$$

引入 $z=x$，变为 $\min \ c^T z \quad \text{s.t.} \ Ax=b,\ x=z$

增广 L 函数：

$$L_\rho(x,z,\lambda)= c^T z+\lambda_1^T(Ax-b)+\lambda_2^T(x-z)$$
$$+\frac{\rho}{2}\|Ax-b\|_2^2+\frac{\rho}{2}\|x-z\|_2^2$$

迭代形式为

① $x^{k+1}=\arg\min\limits_{x}\{\lambda_{1,k}^T(Ax-b)+\lambda_{2,k}^T(z^k-x)+\frac{\rho}{2}\|Ax-b\|_2^2+\frac{\rho}{2}\|x-z^k\|_2^2\}$

② $z^{k+1}=\arg\min\limits_{z}\{c^T z+\lambda_{1,k}^T(z-x^{k+1})+\frac{\rho}{2}\|x^{k+1}-z\|_2^2$

$\lambda_{1,k+1}=\lambda_{1,k}+\rho(Ax^{k+1}-b)$

$\lambda_{2,k+1}=\lambda_{2,k}+\rho(z^{k+1}-x^{k+1})$

对偶问题形式为 $\min \ b^T y$

$$\text{s.t} \quad A^T y \le c$$

引入 $z=y$，变为 $\min \ b^T z$，s.t $A^T y \le c$，$z=y$

迭代形式为

③ $y^{k+1} = \arg\min_y \{ \lambda_{1,k}^T (A^T y - c) + \lambda_{2,k}^T (z^k - y) + \frac{\rho}{2} \|A^T y - c\|_2^2 + \frac{\rho}{2} \|z^k - y\|_2^2 \}$

④ $z^{k+1} = \arg\min_z \{ b^T z + \lambda_{2,k}^T (z - y^{k+1}) + \frac{\rho}{2} \|z - y^{k+1}\|_2^2 \}$.

$\lambda_{1,k+1} = \lambda_{1,k} + \rho (A^T y^{k+1} - b)$

$\lambda_{2,k+1} = \lambda_{2,k} + \rho (z^{k+1} - y^{k+1})$，考虑①②③④子问题求解 $(\nabla = 0)$

解 ① $x^{k+1} = (A^T A + I)^{-1} [b + z^k - \frac{1}{\rho} (A^T \lambda_{1,k} + \lambda_{2,k})]$

② $z^{k+1} = x^{k+1} - \frac{1}{\rho} (c + \lambda_{2,k})$

③ $y^{k+1} = (A A^T + I)^{-1} [c + z^k - \frac{1}{\rho} (A \lambda_{1,k} - \lambda_{2,k})]$

④ $z^{k+1} = y^{k+1} - \frac{1}{\rho} (b + \lambda_{2,k})$

2. 引入变量 z，s.t $x_i - z = 0$，$\forall 1 \le i \le n$.

从而原问题为 $\min \sum_{i=1}^{n} f_i(x_i)$，s.t $x_i - z = 0$，$1 \le i \le n$

$$L_\rho(x_i, z, \lambda_i) = \sum_{i=1}^{n} \left(f_i(x_i) + \lambda_i^T (x_i - z) + \frac{\rho}{2} \|x_i - z\|_2^2 \right)$$

$\Rightarrow$ 则迭代形式为

$$x_1^{k+1} = \arg\min_{x_1} \left\{ f_1(x_1) + \lambda_{1,k}^T (x_1 - z^k) + \frac{\rho}{2} \|x_1 - z^k\|_2^2 \right\}$$

$$= \text{solve} \left\{ \nabla f(x_1) + \lambda_{1,k} + \rho (x_1 - z^k) = 0 \right\}$$

如果用 $f_1(x_1^k) + \nabla f(x_1^k)^T (x_1 - x_1^k)$ 做近似，那么

$$x_1^{k+1} = \text{solve} \left\{ \nabla f(x_1^k) + \lambda_{1,k} + \rho (x_1 - z^k) = 0 \right\}$$

$$= z^k - \frac{1}{\rho} \left(\nabla f(x_1^k) + \lambda_{1,k} \right).$$

实际上由对称性，迭代格式为

$\forall\, 1 \le i \le n.$

$$x_i^{k+1} = z^k - \frac{1}{\rho}\left(\nabla f(x_i^k) + \lambda_{i,k}\right),$$

$$z^{k+1} = \arg\min_z \left\{ \sum_{i=1}^n \left[\lambda_{i,k}^T (x_i^k - z) + \frac{\rho}{2}\|x_i^k - z\|_2^2\right] \right\}$$

$$= \text{solve}\left\{ -\sum_{i=1}^n \lambda_{i,k} + \rho \sum_{i=1}^n (z - x_i^k) = 0 \right\}$$

$$= \frac{1}{n\rho}\left(\sum_{i=1}^n \rho x_i^k + \lambda_{i,k}\right)$$

$$\lambda_{i,k+1} = \lambda_{i,k} + \rho(x_i^{k+1} - z^{k+1}),$$

若 f_i 非光滑，则用 ∂f 代替 ∇f.

3. 引入变量 Z, s.t 问题变为

$$\min \operatorname{Tr}(CX) - \log\det X + \lambda\|Z\|_1 \quad \text{s.t } X = Z$$

增广 L-函数为，设 $U=(u_{ij}),\ X=(x_{ij}),\ Z=(z_{ij})$

$$L_\rho(X,Z,U) = \operatorname{Tr}(CX) - \log\det X + \lambda\|Z\|_1 + \rho\sum_{ij} u_{ij}(x_{ij} - z_{ij}) + \frac{\rho}{2}\|X - Z\|_F^2$$

迭代格式为

⇒ ① $X^{k+1} = \underset{X}{\arg\min} \left(\mathrm{Tr}(CX) - \log\det X + \frac{\rho}{2}\|X - Z^k + U^k\|_F^2 \right)$

② $Z^{k+1} = \arg\min \left(\lambda\|Z\|_1 + \frac{\rho}{2}\|X^{k+1} - Z + U^k\|_F^2 \right)$

$= \mathrm{prox}_{\frac{\lambda}{\rho}}(X^{k+1} + U^k)$

$U^{k+1} = U^k + (X^{k+1} - Z^{k+1})$

对于问题1. 求梯度得 (转成矩阵)

$$C - X^{-1} + \rho(X - Z^k + U^k) = 0$$

即 $\rho(Z^k - U^k) - C = \rho X - X^{-1}$，设 λ_i 为X特征值

对 $\rho(Z^k - U^k) - C = Q \Lambda Q^T$ 作分解.

计算得到 $\Lambda_{ii} = \rho\lambda_i - \frac{1}{\lambda_i} \Rightarrow \lambda_i = \frac{\Lambda_{ii} + \sqrt{\Lambda_{ii}^2 + 4\rho}}{2\rho}$

令 $D = \mathrm{diag}(\lambda_1 \ldots \lambda_n)$. 则

令 $X^{k+1} = QDQ^T$ 即可.

② 这是 LASSO 问题子问题, 有显式解.

$$Z_{ij}^{k+1} = \mathrm{sign}(X_{ij}^{k+1} + U_{ij}^{k}) \max\{0, |X_{ij}^{k+1} + U_{ij}^{k} - \frac{1}{\rho}|\}.$$