微分方程

特殊区域的Green函数求法以及一般有界区域定解问题

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一、镜像法：

物理原理：

Green函数=自由点电荷产生的电场+边界感应电荷产生的电场

而边界感应电荷产生的电场=虚设电荷产生的电场，虚设电荷满足：1.在域外 2.在边界上与自由点电荷产生的电场相同

数学原理：

$$G(x,y)=V(y-x)+H(y,x), x,y\in D,$$

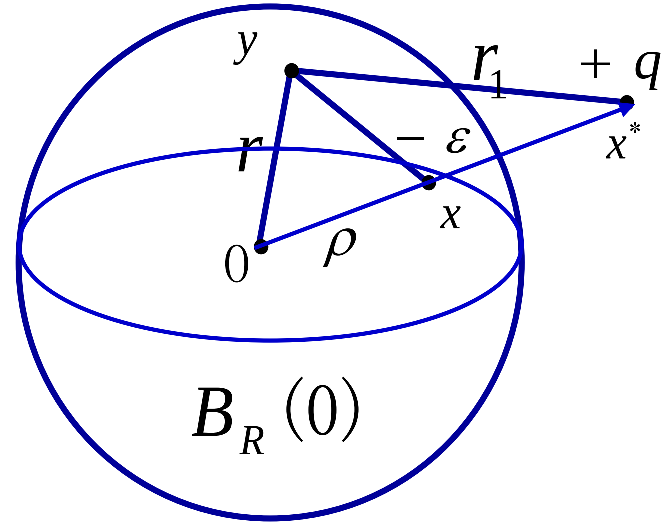
$$V=V(y-x)=V(x-y)=\begin{cases}\dfrac{1}{2\pi}\ln|x-y|, & N=2\\ -\dfrac{1}{4\pi|x-y|}, & N=3\end{cases}$$

$$\begin{cases}\Delta_y H(y,x)=0, x,y\in D\\ H|_{\partial D}=-V|_{\partial D}\end{cases}$$ (域外虚设电荷产生的电场)

想法示例：球内Green函数

若能在$B_R(0)$外的某点x^*放一适当的$+q$，则

$$\Delta_y\left(\frac{q}{4\pi\varepsilon r_1}\right)=0,\quad y\in B_R(0)$$

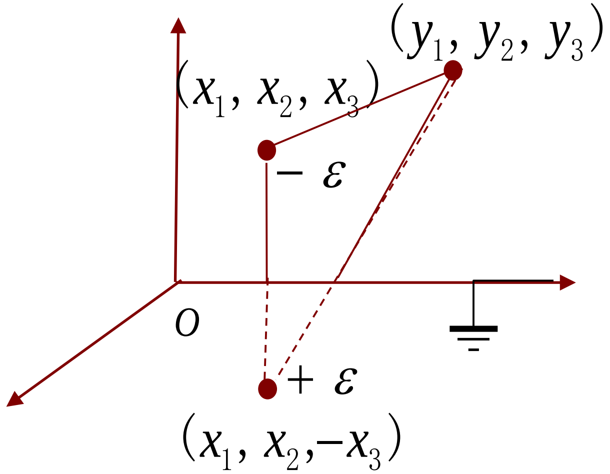


使：$\dfrac{q}{4\pi\varepsilon r_1}\Big|_{r=R}=\dfrac{1}{4\pi|y-x|}\Big|_{r=R}$

则 $H=\dfrac{q}{4\pi\varepsilon r_1}$

$\therefore$ 求$H(y,x)\Leftrightarrow$ 确定x^*的位置$+$确定q大小

例1. 上半空间第I边值问题的Green函数



$$\begin{cases} \Delta_3 G = \delta(x-y),\ x_3, y_3 > 0 \\ G\big|_{y_3=0} = 0 \end{cases}$$

物理方法：在一接地无限大导体板上方的点x处放置一电量为$-\varepsilon$的点电荷。为求导体板上方任一点y的电势，

令虚设电荷所带电量为$+\varepsilon$，位置如图所示，故金属薄板上方任一点的电势为

$$\boxed{\begin{aligned} G(x,y) &= -\frac{1}{4\pi|y-x|} + \frac{\varepsilon}{4\pi\varepsilon|y-x^*|} \\ &= -\frac{1}{4\pi}\frac{1}{\sqrt{(y_1-x_1)^2+(y_2-x_2)^2+(y_3-x_3)^2}} + \frac{1}{4\pi}\frac{1}{\sqrt{(y_1-x_1)^2+(y_2-x_2)^2+(y_3+x_3)^2}} \end{aligned}}$$

数学方法：

令$x^* = (x_1, x_2, -x_3) \notin D$为$x = (x_1, x_2, x_3)$关于平面$y_3=0$的对称点，

则$H(y,x) := -V(y-x^*), x, y \in D$满足：

(1) $\Delta H(y,x) = -\Delta V(y-x^*) = -\delta(y-x^*) = 0$

(2) $$H|_{y_3=0} = -V(y-x^*)|_{y_3=0} = \frac{1}{4\pi|y-x^*|}\Big|_{y_3=0}$$

$$= \frac{1}{4\pi|y-x|}\Big|_{y_3=0} = -V(y-x)|_{y_3=0}$$

$$\therefore \boxed{G(x,y) = V(y-x) + H(y,x) = V(y-x) - V(y-x^*)}$$

$$\Rightarrow \frac{\partial G}{\partial \nu}\Big|_{y_3=0} = -\frac{\partial G}{\partial y_3}\Big|_{y_3=0} = \frac{x_3}{2\pi[(y_1-x_1)^2+(y_2-x_2)^2+x_3^2]^{3/2}},$$ 由Poisson公式

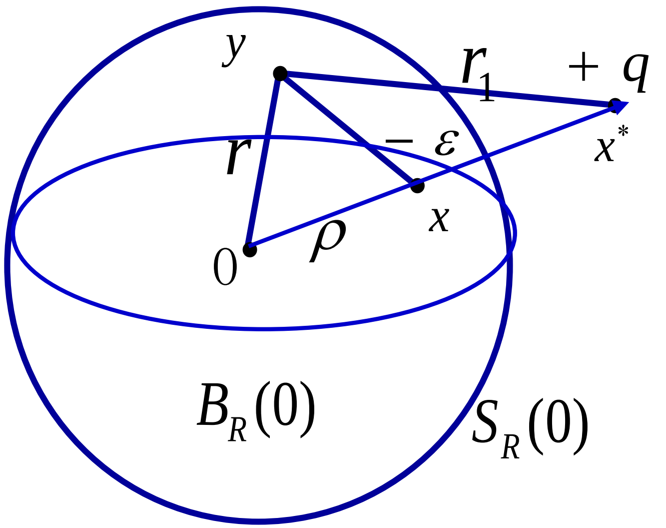
$$\Rightarrow \begin{cases} \Delta_3 u = 0,\ x_3 > 0 \\ u\big|_{x_3=0} = \varphi(x_1, x_2) \end{cases}$$ 解为 $$\boxed{u(x_1,x_2,x_3) = \frac{x_3}{2\pi}\int_{\mathbb{R}^2} \frac{\varphi(y_1,y_2)}{[(y_1-x_1)^2+(y_2-x_2)^2+x_3^2]^{3/2}} dy_1 dy_2}$$

$n=2$ 时 $$\begin{cases} \Delta_2 u = 0,\ x_2 > 0 \\ u\big|_{x_2=0} = \varphi(x_1) \end{cases}$$ 的解为 $$\boxed{u(x_1,x_2) = \frac{x_2}{\pi}\int_{-\infty}^{+\infty} \frac{\varphi(s)}{(x_1-s)^2+x_2^2} ds}$$

$$\boxed{\frac{q}{4\pi\varepsilon r_1}\Big|_{r=R}=\frac{1}{4\pi|y-x|}\Big|_{r=R}}$$

例2. 三维球第I边值问题的Green函数

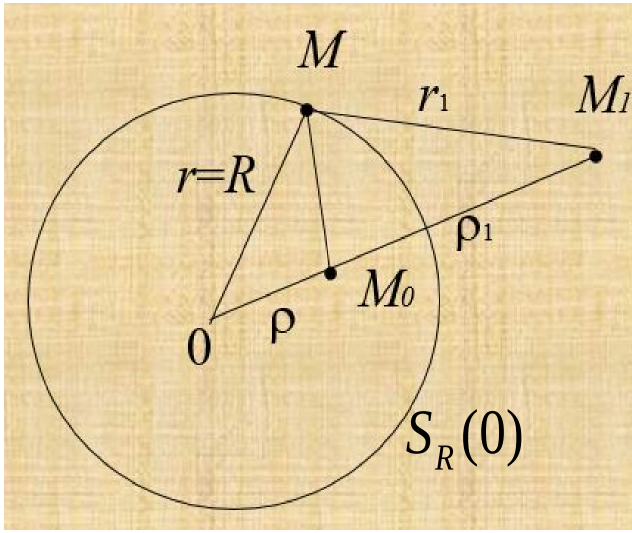
$$\begin{cases}\Delta_3 G=\delta(x-y),\ x,y\in B_R(0)\\ G\big|_{r=R}=0\ (r:=|y|)\end{cases}$$



其中 $B_R(0), S_R(0)$ 分别为半径 R 球心在原点的三维球和球面。记 $x=OM_0, y=OM,$

令 $\rho=|x|,\ \rho_1=|OM_1|,\ \rho\cdot\rho_1=R^2\Rightarrow\dfrac{R}{\rho}=\dfrac{\rho_1}{R},$

称 $x^*=M_1$ 为 M_0 关于球面$S_R(0)$的对称点。



$\because M$ 在球面上时 $\Delta OM_1M \backsim \Delta OM_0M,$

$$\therefore\frac{R}{\rho}=\frac{\rho_1}{R}=\frac{r_1}{|y-x|}\Rightarrow\frac{R}{4\pi\rho r_1}=\frac{1}{4\pi|y-x|}$$

$$\Rightarrow q=\frac{\varepsilon R}{\rho}\Rightarrow H=\frac{q}{4\pi\varepsilon r_1}=\frac{R}{4\pi\rho r_1}=\frac{R}{4\pi|x||y-x^*|}$$

$$\Rightarrow\ \boxed{G(x,y)=-\frac{1}{4\pi|y-x|}+\frac{R}{4\pi|x||y-x^*|}}$$

数学方法：令 $M_1=(\frac{R}{\rho})^2M_0$ 为 M_0 关于球面 $S_R(0)$ 的对称点，其中

$\rho=|OM_0|=|x|$. 这里记 $x=OM_0, y=OM, x^*=OM_1$, $r_0=|M-M_0|$, $r_1=|M-M_1|$.

令 $H(y,x)=-\frac{R}{\rho}V(y-x^*), x,y\in B_R(0), x^*\notin\overline{B_R(0)}$, 则

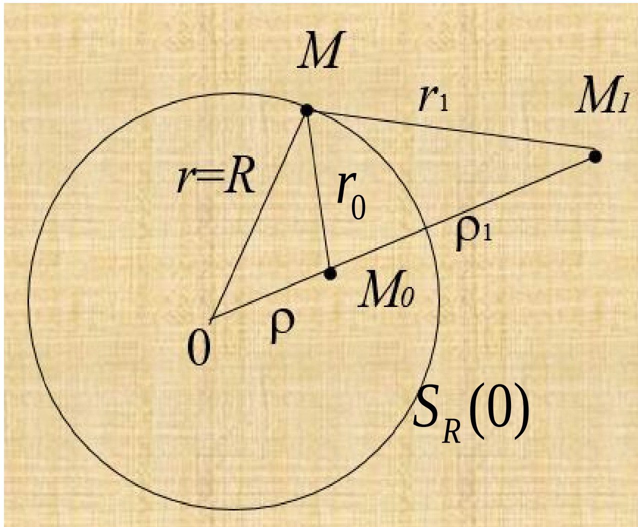
(1) $y\in B_R(0)$: $\Delta H=-\frac{R}{\rho}\Delta V(y-x^*)=-\frac{R}{\rho}\delta(y-x^*)=0$

(2) $y\in S_R(0)$: $\because r_1=|y-x^*|=|y-(\frac{R}{\rho})^2x|=[R^2-2(\frac{R}{\rho})^2y\cdot x+\frac{R^4}{\rho^2}]^{1/2}$

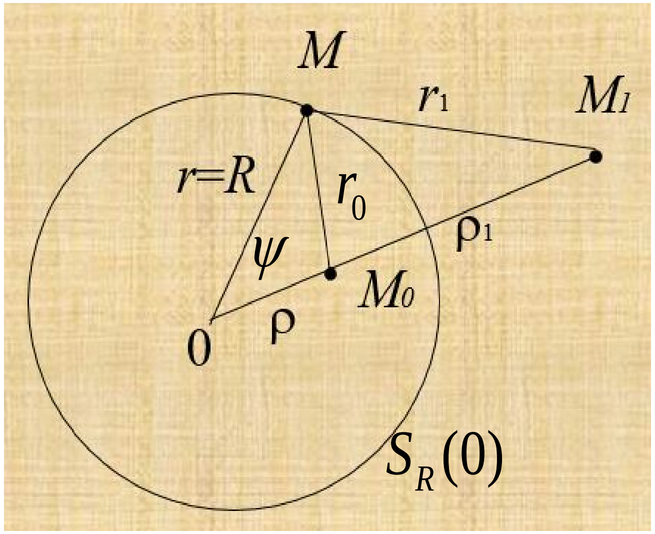
$$=\frac{R}{\rho}[\rho^2-2y\cdot x+R^2]^{1/2}=\frac{R}{\rho}|y-x|=\frac{R}{\rho}r_0$$

$$\therefore H|_{S_R(0)}=\frac{R}{\rho}\frac{1}{4\pi r_1}|_{S_R(0)}=\frac{1}{4\pi r_0}|_{S_R(0)}=-V(y-x)|_{S_R(0)}$$

$$\boxed{\begin{aligned}G(x,y)&=V(y-x)+H(y,x)\\&=V(y-x)-\frac{R}{\rho}V(y-x^*)\end{aligned}}$$



$$\frac{\partial G}{\partial \nu}=\frac{\partial G}{\partial r}=-\frac{1}{4\pi}\left[\frac{\partial}{\partial r}\left(\frac{1}{r_0}\right)-\frac{R}{\rho}\frac{\partial}{\partial r}\left(\frac{1}{r_1}\right)\right],$$



其中 $r_0=|M-M_0|=(r^2+\rho^2-2r\rho\cos\psi)^{1/2}$,

$$r_1=|M-M_1|=(r^2+\rho_1^2-2r\rho_1\cos\psi)^{1/2}.$$

$$\frac{\partial}{\partial r}\frac{1}{r_0}\Big|_{S_R(0)}=\frac{\rho^2-r^2-r_0^2}{2rr_0^3}\Big|_{S_R(0)}=\frac{\rho^2-R^2-r_0^2}{2Rr_0^3},$$

$$\frac{R}{\rho}\frac{\partial}{\partial r}\frac{1}{r_1}\Big|_{S_R(0)}=\frac{R}{\rho}\frac{\rho_1^2-R^2-r_1^2}{2Rr_1^3}=\frac{R^2-r_0^2-\rho^2}{2Rr_0^3}\text{(相似三角形)}$$

$$\Rightarrow \frac{\partial G}{\partial \nu}\Big|_{S_R(0)}=\frac{R^2-\rho^2}{4\pi Rr_0^3}$$

$$\boxed{u(x)=\int_D G(x,y)f(y)dy+\int_{\partial D}\varphi(y)\frac{\partial G(x,y)}{\partial \nu}dS(y)}$$

Poisson公式 $\Rightarrow$ $\begin{cases}\Delta_3 u=0,\ x\in B_R(0)\\ u\big|_{r=R}=\Phi(x)\end{cases}$ 的解为

$$\boxed{\begin{aligned}u(x)&=\iint_{S_R(0)}\Phi(y)\frac{\partial G}{\partial \nu}dS(y)\\&=\frac{R^2-\rho^2}{4\pi R}\iint_{S_R(0)}\frac{\Phi(y)}{\left(R^2+\rho^2-2R\rho\cos\psi\right)^{3/2}}dS(y)\\&=\frac{R^2-|x|^2}{4\pi R}\int_0^{2\pi}\int_0^{\pi}\frac{\Phi(\theta,\varphi)}{\left(R^2+|x|^2-2R|x|\cos\psi\right)^{3/2}}R^2\sin\theta d\theta d\varphi\\&=\frac{R(R^2-|x|^2)}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{\Phi(\theta,\varphi)\sin\theta}{\left(R^2+|x|^2-2R|x|\cos\psi\right)^{3/2}}d\theta d\varphi\end{aligned}}$$

(Poisson公式)

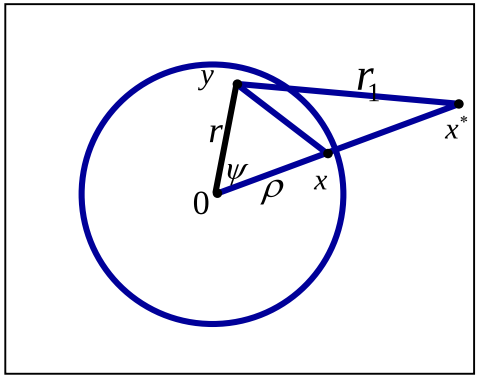
其中 $\cos\psi=\dfrac{OM\cdot OM_0}{|OM||OM_0|}=\sin\theta\sin\theta_0\cos(\varphi-\varphi_0)+\cos\theta\cos\theta_0$

$$\Rightarrow\quad \boxed{x=0,\rho=0:u(0)=\frac{1}{4\pi R^2}\iint_{S_R(0)}u(y)dS(y)}$$

(调和函数的球面平均值公式)

注1：二维圆域第I问题边值问题的Green函数为

$$\boxed{G(x,y)=\frac{1}{2\pi}\ln|y-x|+\frac{1}{2\pi}\ln\frac{R}{|x|\left|y-x^*\right|}}$$



x^*是x关于圆环$S_R(0)$的对称点.

注2：三维球外第I问题边值问题：

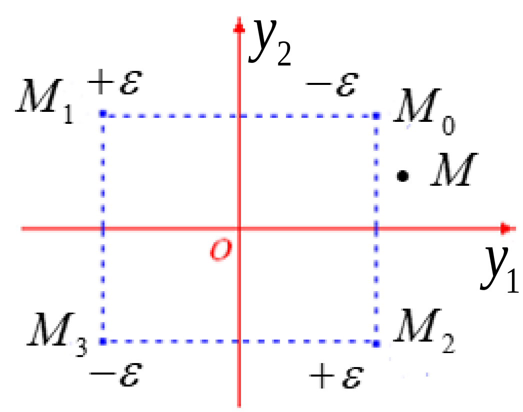
$$\begin{cases}\Delta u_1=0\,,|x|=r>R\\ u_1|_{r=R}=\Phi(x)\end{cases}$$

$$\boxed{u_1(x)=\frac{R(|x|^2-R^2)}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{\Phi(\theta,\varphi)\sin\theta}{\left(R^2+|x|^2-2R|x|\cos\psi\right)^{3/2}}d\theta d\varphi=-u(x)}$$

Green函数为 $\boxed{G_1(x,y)=V(y-x)-\frac{R}{\rho}V(y-x^*)=G(x,y)}$

例3. 求四分之一平面第I问题边值问题的Green函数。

$$\begin{cases} \Delta_y G = \delta(x-y),\ y_1 > 0, y_2 > 0 \\ G\big|_{y_1=0} = G\big|_{y_2=0} = 0 \end{cases}$$



设$x = M_0$点有电荷$-\varepsilon$，M_0关于y_2轴, y_1轴与原点的对称点分别为M_1, M_2与M_3, 相应的电荷见右图

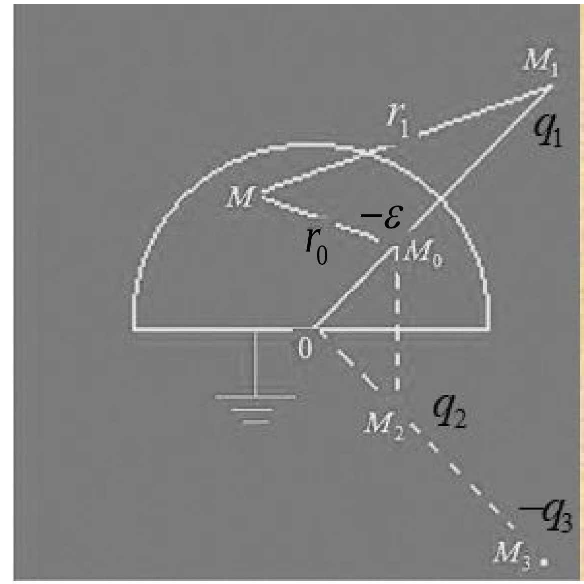
则等效电场 $H_k: -\dfrac{1}{2\pi}\ln r_1,\ -\dfrac{1}{2\pi}\ln r_1, \dfrac{1}{2\pi}\ln r_3$; 其中$r_k = |M - M_k|, 0 \le k \le 3.$

$$\Rightarrow \quad G(x,y) = V(y-x) + H_1(y,x) + H_2(y,x) + H_3(y,x)$$

$$\boxed{G(x,y) = \frac{1}{2\pi}\ln r_0 - \frac{1}{2\pi}\ln r_1 - \frac{1}{2\pi}\ln r_2 + \frac{1}{2\pi}\ln r_3 = \frac{1}{2\pi}\ln\frac{r_0 r_3}{r_1 r_2}}$$

例4. 求上半圆域第I问题边值问题的Green函数。

$$\begin{cases} \Delta_y G = \delta(x-y), x, y \in B_R^+(0) \subset \mathbb{R}^2 \\ G\big|_{S_R^+} = 0 \end{cases}$$



令$x = M_0 = (\rho, \theta_0)$，取关于圆的对称点$M_1(\frac{R^2}{\rho}, \theta_0)$

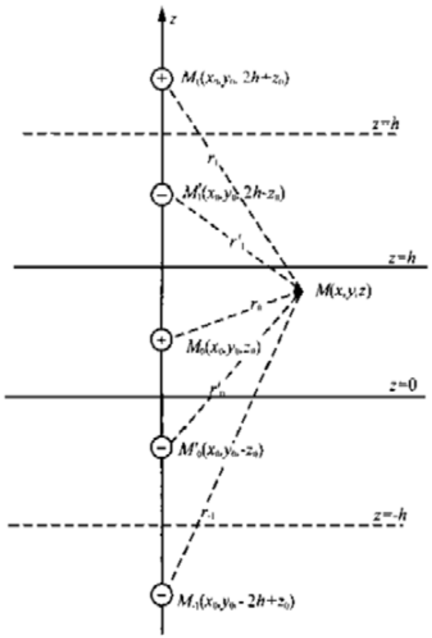
以及关于水平轴的对称点$M_2(\rho, -\theta_0), M_3(\frac{R^2}{\rho}, -\theta_0)$

则 $q_1 = \varepsilon \frac{R}{\rho}, q_2 = \varepsilon, q_3 = -\varepsilon \frac{R}{\rho}$

$$\boxed{G(x,y) = \frac{1}{2\pi}[(\ln r_0 + \ln \frac{R}{\rho r_1}) - (\ln r_2 + \ln \frac{R}{\rho r_3})] = \frac{1}{2\pi} \ln \frac{r_0 r_3}{r_1 r_2}}$$

讨论：层状空间与上半空间第III边值问题

1. 层状空间第I问题边值问题的Green函数。



$$\begin{cases} \Delta_M G = \delta(M - M_0), \quad 0 < z < h \\ G\big|_{z=0} = G\big|_{z=h} = 0 \end{cases}$$

$$M(x, y, z), M_0(x_0, y_0, z_0)$$

$$G(M, M_0) = \frac{1}{4\pi}\sum_{n=-\infty}^{+\infty}\left(\frac{1}{r_n^-} - \frac{1}{r_n^+}\right),$$

$$r_n^{\pm} = \sqrt{(x-x_0)^2 + (y-y_0)^2 + [(z-(2nh \pm z_0)]^2}$$

2015，Sadybekov

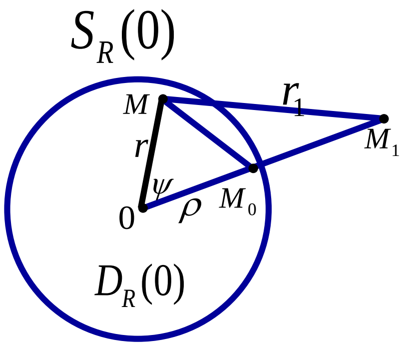
2. 上半空间(单位圆盘)第III边值问题的Green函数？单位圆盘

$$\alpha = \beta = 1, \rho = |OM_0|, r = |OM| : G(M, M_0) = \frac{1}{2\pi}\ln\frac{|M - M_0|}{|\rho M - \frac{M_0}{\rho}|} - \frac{1}{2\pi}\int_0^1 \frac{1-(r\rho s)^2}{1 - 2r\rho s\cos(\theta-\theta_0) + (r\rho s)^2}ds$$

$$= \frac{1}{2\pi}\ln\frac{|M - M_0|}{|\rho M - \frac{M_0}{\rho}|} - \frac{1}{2\pi}\left[1 - \frac{2\cos(\theta-\theta_0)}{r\rho}\ln\left|\rho M - \frac{M_0}{\rho}\right| + \frac{2|\sin(\theta-\theta_0)|}{r\rho}\arctan\frac{r\rho|\sin(\theta-\theta_0)|}{1 - r\rho\cos(\theta-\theta_0)}\right]$$

补充：圆域一般第I边值问题

$$\begin{cases} \Delta u = f(x_1, x_2), \quad (x_1, x_2) \in D_R(0) \subset \mathbb{R}^2 \\ u\big|_{r=R} = \varphi(x_1, x_2) \end{cases}$$



Green函数为

$$\boxed{G(x,y) = \frac{1}{2\pi}\ln|y-x| + \frac{1}{2\pi}\ln\frac{R}{|x|\left|y-x^*\right|}}$$

即

$$\boxed{G(x_1,x_2,y_1,y_2) = \frac{1}{2\pi}\ln\frac{R\sqrt{\rho^2+r^2-2\rho r\cos\psi}}{\sqrt{\rho^2 r^2+R^4-2\rho r R^2\cos\psi}}}$$

其中$x^* = (x_1^*, x_2^*) = M_1 = (\frac{R}{\rho})^2 M_0$是$x = (x_1, x_2) = M_0$关于圆环$S_R(0)$的对称点.

则 $$\frac{\partial G}{\partial \nu}\bigg|_{S_R(0)} = \frac{\partial G}{\partial r}\bigg|_{r=R} = \frac{1}{2\pi R}\frac{R^2-(x_1^2+x_2^2)}{(y_1-x_1)^2+(y_2-x_2)^2}\bigg|_{(y_1,y_2)\in S_R(0)}$$

由Poisson公式知

$$\boxed{u(x)=\int_D G(x,y)f(y)dy+\int_{\partial D}\varphi(y)\frac{\partial G(x,y)}{\partial \nu}dS(y)}$$

直角坐标下的解的形式:

$$u(x_1,x_2)=\frac{1}{2\pi}\iint_{D_R(0)}\ln\frac{R\sqrt{(y_1-x_1)^2+(y_2-x_2)^2}}{\sqrt{x_1^2+x_2^2}\sqrt{(y_1-x_1^*)^2+(y_2-x_2^*)^2}}f(y_1,y_2)dy_1dy_2$$

$$+\frac{R^2-(x_1^2+x_2^2)}{2\pi R}\int_{S_R(0)}\frac{\varphi(y_1,y_2)}{(y_1-x_1)^2+(y_2-x_2)^2}dS(y_1,y_2)$$

极坐标下的解的形式:

$$u(\rho,\theta)=\frac{1}{4\pi}\int_0^R dr\int_0^{2\pi}\ln\frac{\rho^2r^2+R^4-2\rho rR^2\cos(\theta-\alpha)}{R^2\left[\rho^2+r^2-2\rho r\cos(\theta-\alpha)\right]}f(r,\alpha)rd\alpha$$

$$+\frac{R^2-\rho^2}{2\pi}\int_0^{2\pi}\frac{\varphi(\alpha)}{\rho^2+R^2-2R\rho\cos(\theta-\alpha)}d\alpha$$

二、保形变换法：（仅适用二维区域）

定理(单连通区域的Green函数)：

Riemann映射定理

令 $D\subset\mathbb{R}^2$ 单连通，若保形变换(共形映射：解析，一对一)

$w:D\to D_1$ (单位圆盘)满足

$$w(z_0)=0\ (z_0=\xi+i\eta\in D),\ \ |w(z)|=1\ (z=x+iy\in\partial D),$$

则 D 上第I边值问题的Green函数为

$$\boxed{G(x,y,\xi,\eta)=G(z,z_0)=\frac{1}{2\pi}\ln|w(z)|}$$

即满足

$$\begin{cases}\Delta G=\delta(x-\xi,y-\eta),\ (x,y),(\xi,\eta)\in D\\ G\big|_{\partial D}=0\end{cases}$$

证： $\because G\big|_{\partial D}=\frac{1}{2\pi}\ln|w(z)|\Big|_{|w(z)|=1}=0,\ \Delta(\frac{1}{2\pi}\ln|z-z_0|)=\delta(x-\xi,y-\eta),$

$$G=\frac{1}{2\pi}\ln|z-z_0|+\frac{1}{2\pi}\ln\frac{|w(z)|}{|z-z_0|}$$

只须证 $\ln\frac{|w(z)|}{|z-z_0|}$ 在 D 内调和。

令

$$F(z)=\begin{cases}\dfrac{w(z)}{z-z_0}, & z\neq z_0\\[2ex] \lim\limits_{z\to z_0}\dfrac{w(z)-w(z_0)}{z-z_0}=w'(z_0)\neq 0, & z=z_0\end{cases}$$

则 $F(z)\neq 0,\ln F(z)$ 在 D 内单值解析(调和)。

$\therefore \operatorname{Re}\ln F(z)=\ln\frac{|w(z)|}{|z-z_0|}$ 在 D 内调和。

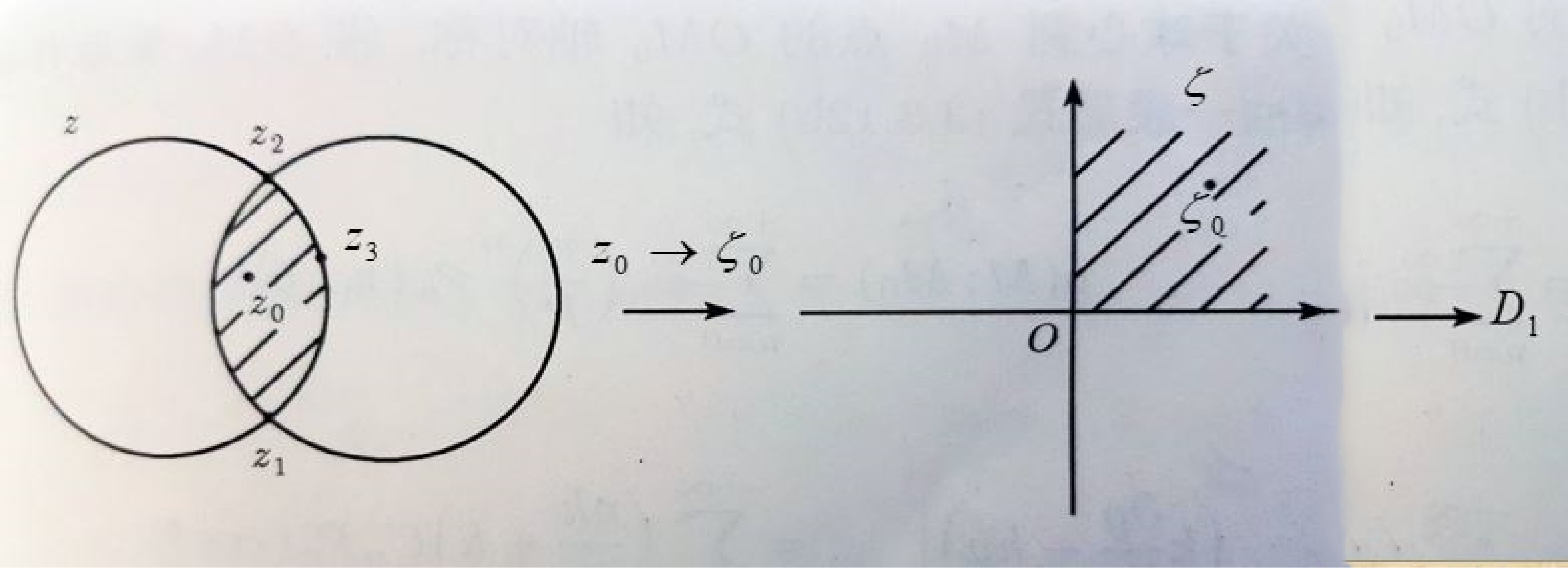
(任何解析函数的实部和虚部均调和, 复变：Cauchy-Riemann方程)

例.

1. $w(z)=\dfrac{z-z_0}{z-\overline{z}_0}:\mathbb{R}_+^2\to D_1\Rightarrow G(z,z_0)=\dfrac{1}{2\pi}\ln\left|\dfrac{z-z_0}{z-\overline{z}_0}\right|$

2. $w(z)=\dfrac{R(z-z_0)}{R^2-z\overline{z}_0}:D_R\to D_1\Rightarrow G(z,z_0)=\dfrac{1}{2\pi}\ln\left|\dfrac{R(z-z_0)}{R^2-z\overline{z}_0}\right|$

3. $w(z)=\dfrac{e^z-e^{z_0}}{e^z-e^{\overline{z}_0}}:\{0<\mathrm{Im}\,z<\pi\}\to D_1\Rightarrow G(z,z_0)=\dfrac{1}{2\pi}\ln\left|\dfrac{e^z-e^{z_0}}{e^z-e^{\overline{z}_0}}\right|$



4. $\zeta = g(z) = \dfrac{z_3 - z_2}{z_3 - z_1} \cdot \dfrac{z - z_1}{z - z_2}, f(\zeta) = \dfrac{\zeta^2 - \zeta_0^2}{\zeta^2 - \overline{\zeta}_0^2}, w(z) = f(g(z))$

$$\Rightarrow G(z, z_0) = \frac{1}{2\pi} \ln |w(z)| = \frac{1}{2\pi} \ln \left| \frac{g^2(z) - g^2(z_0)}{g^2(z) - \overline{g}^2(z_0)} \right|$$

三、Fourier展开法：

思想：将Green函数按某正交基(保持同样的齐次边界)作广义Fourier展开，再利用方程和边界条件确定相关系数。

例1(二维矩形区域).

$$\begin{cases} \Delta_2 G = \delta(x-\xi, y-\eta), \quad 0 < x, \xi < a, 0 < y, \eta < b \\ G\big|_{x=0,a} = G\big|_{y=0,b} = 0 \end{cases}$$

解：考虑特征值问题

$$\begin{cases} \Delta_2 v + \lambda v = 0, \quad 0 < x < a, 0 < y < b \\ v\big|_{x=0,a} = v\big|_{y=0,b} = 0 \end{cases}$$

将分离解 $X(x)Y(y)$ 代入方程有

$$\begin{cases} X''+\mu X=0, & 0<x<a \\ X(0)=X(a)=0 \end{cases} \Rightarrow \mu_n=(\frac{n\pi}{a})^2, X_n(x)=\sin\frac{n\pi x}{a}, n\geq 1$$

$$\begin{cases} Y''+\nu Y=0, & 0<y<b \\ Y(0)=Y(b)=0 \end{cases} \Rightarrow \nu_m=(\frac{m\pi}{b})^2, Y_m(x)=\sin\frac{m\pi y}{b}, m\geq 1$$

$$\Rightarrow \lambda=\lambda_{nm}=\mu_n+\nu_m=(\frac{n\pi}{a})^2+(\frac{m\pi}{b})^2, v_{nm}(x,y)=\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$$

令 $G=\sum_{n,m\geq 1} C_{nm} v_{nm}(x,y)=\sum_{n,m\geq 1} C_{nm}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$

$$\Rightarrow \Delta_2 G=-\sum_{n,m\geq 1} C_{nm}[(\frac{n\pi}{a})^2+(\frac{m\pi}{b})^2]\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$$

另外，$\Delta_2 G=\delta(x-\xi,y-\eta)=\sum_{n,m\geq 1}\frac{<\delta(x-\xi,y-\eta),v_{nm}(x,y)>}{\|v_{nm}(x,y)\|^2}v_{nm}(x,y)$

$$=\sum_{n,m\geq 1}\frac{\sin\frac{n\pi\xi}{a}\sin\frac{m\pi\eta}{b}}{\frac{a}{2}\frac{b}{2}}v_{nm}(x,y)\Rightarrow C_{nm}=-\frac{4\sin\frac{n\pi\xi}{a}\sin\frac{m\pi\eta}{b}}{ab[(\frac{n\pi}{a})^2+(\frac{m\pi}{b})^2]}$$

例2(三维球域第III边值问题).

$$\begin{cases} \Delta_y G = \delta(x-y), \quad x, y \in B_R(0) \subset \mathbb{R}^3 \\ \left.(\alpha G + \beta \dfrac{\partial G}{\partial r})\right|_{r=R} = 0 \end{cases}$$

解: $G(x,y) = V(y-x) + H(y,x),\ V(y-x) = -\dfrac{1}{4\pi|y-x|}$

$$\begin{cases} \Delta_y H = 0, \quad x, y \in B_R(0) \\ \left.(\alpha H + \beta \dfrac{\partial H}{\partial r})\right|_{r=R} = -\left.(\alpha V + \beta \dfrac{\partial V}{\partial r})\right|_{r=R} \end{cases}$$

(球轴对称问题)

利用球坐标和分离变量法知 $H(y,x) = \sum_{n\geq 0} C_n (\dfrac{r}{R})^n P_n(\cos\theta)$,

其中 $P_n(\cdot)$: Legendre多项式(参考季孝达《数学物理方程》),

再由边界条件有

$$\left.(\alpha H+\beta\frac{\partial H}{\partial r})\right|_{r=R}=\sum_{n\geq 0}(\alpha+\frac{n\beta}{R})C_nP_n(\cos\theta).$$

另外利用Legendre多项式的母函数知：$\rho=|x|<r=|y|$ 时

$$-4\pi V=\frac{1}{|y-x|}=(r^2+\rho^2-2r\rho\cos\theta)^{-1/2}=\frac{1}{r}\sum_{n\geq 0}(\frac{\rho}{r})^nP_n(\cos\theta)$$

$$\Rightarrow -\left.(\alpha V+\beta\frac{\partial V}{\partial r})\right|_{r=R}=\sum_{n\geq 0}\frac{\alpha R-(n+1)\beta}{4\pi R^{n+2}}\rho^nP_n(\cos\theta)$$

$$C_n=\frac{\alpha R-(n+1)\beta}{\alpha R+n\beta}\frac{\rho^n}{4\pi R^{n+1}}$$

$$\boxed{\begin{aligned}&G(x,y)=V(y-x)+H(y,x)\\&=\frac{1}{4\pi}[-\frac{1}{|y-x|}+\sum_{n\geq 0}\frac{\alpha R-(n+1)\beta}{(\alpha R+n\beta)R^{2n+1}}|x|^n|y|^n\,P_n(\frac{|x|^2+|y|^2-|y-x|^2}{2})]\end{aligned}}$$

例3(混合区域).

$$\begin{cases} \Delta_3 G = \delta(x-\xi, y-\eta, z-\zeta),\ (x,y,z),(\xi,\eta,\zeta) \in [0,1]\times\mathbb{R}^+\times\mathbb{R} \\ G|_{x=0,1} = G|_{y=0} = 0, G|_{y=+\infty} \text{ 和 } G|_{z=\pm\infty} \text{ 有界} \end{cases}$$

解：$\Delta_3 = \partial_x^2 + \partial_y^2 + \partial_z^2$， 针对解的区间和条件，可分别采取正弦级数展开，正弦变换和Fourier变换。

令 $\int_0^\infty \sin(\lambda y)dy \int_{-\infty}^{+\infty} G(x,y,z)e^{-i\gamma z}dz = \sum_{n\geq 1} \hat{G}(n,\lambda,\gamma)\sin(n\pi x).$

$\Delta_3 G$的变换是$-\sum_{n\geq 1}(n^2\pi^2 + \lambda^2 + \gamma^2)\hat{G}(n,\lambda,\gamma)\sin(n\pi x),$

$\delta(x-\xi, y-\eta, z-\zeta)$的变换是$2\sum_{n\geq 1}\sin(n\pi\xi)\sin(\lambda\eta)e^{-i\gamma\zeta}\sin(n\pi x).$

$$\Rightarrow \hat{G}(n,\lambda,\gamma) = -2\frac{\sin(n\pi\xi)\sin(\lambda\eta)e^{-i\gamma\zeta}}{n^2\pi^2 + \lambda^2 + \gamma^2}.$$

$$G(x,y,z)=\sum_{n\geq 1}\sin(n\pi x)\frac{2}{\pi}\int_0^\infty \sin(\lambda y)d\lambda\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{G}(n,\lambda,\gamma)e^{i\gamma z}d\gamma$$
$$=-\frac{2}{\pi^2}\sum_{n\geq 1}\sin(n\pi x)\sin(n\pi\xi)\int_0^\infty \sin(\lambda y)\sin(\lambda\eta)d\lambda\int_{-\infty}^{+\infty}\frac{e^{i\gamma(z-\zeta)}}{n^2\pi^2+\lambda^2+\gamma^2}e^{i\gamma z}d\gamma$$

例4. 二维环形区域$\{x\in\mathbb{R}^2 \mid a<|x|<b\}$ 上第I边值问题的Green函数为

$$\boxed{\begin{aligned}G(x,y)=\frac{1}{2\pi}[&\ln|x-y|-A_0(y)-B_0(y)\ln|x|\\&+\sum_{k\geq 1}\frac{1}{k}(A_k(y)|x|^k+B_k(y)|x|^{-k})\cos k(\theta-\theta_y)]\end{aligned}}$$

1928，Ann. Math. Hickey

$$A_0(y)=\ln b\ln(a/|y|)/\ln(a/b),B_0(y)=\ln(|y|/b)/\ln(a/b),$$

$$A_k(y)=\frac{|y|^k-(a^2/|y|)^k}{b^{2k}-a^{2k}},B_k(y)=\frac{a^{2k}[(b^2/|y|)^k-|y|^k]}{b^{2k}-a^{2k}}$$

四、对一般有界区域D上的定解问题

$$\begin{cases} u_t = k\Delta u,\ x\in D\subset R^N, t>0, N\ge 1 \\ u|_{t=0} = \varphi(x) \\ u|_{\partial D} = 0 \end{cases}$$
或
$$\begin{cases} u_{tt} = c^2\Delta u,\ x\in D\subset R^N, t>0, N\ge 1 \\ u|_{t=0} = \varphi(x), u_t|_{t=0} = \psi(x) \\ u|_{\partial D} = 0 \end{cases}$$

$$v(x)T(t) \Rightarrow \frac{T'(x)}{kT(t)} = \frac{\Delta v(x)}{v(x)} = -\lambda \quad \text{或} \quad \frac{T''(x)}{c^2T(t)} = \frac{\Delta v(x)}{v(x)} = -\lambda$$

代入边界条件有

$$\boxed{\begin{cases} \Delta v(x) + \lambda v(x) = 0 \quad in \quad D \\ v|_{\partial D} = 0 \end{cases}} \Rightarrow \boxed{0<\lambda_n \uparrow +\infty, \{v_n(x)\}_{n\ge 1}\text{为}L^2(D)\text{的正交基}}$$

PDE特征值问题，Helmholtz方程，隐形，

Science, 2006(312),1777-1780

令形式解为 $u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n kt} v_n(x)$

或 $u(x,t) = \sum_{n=1}^{\infty} (C_n \cos c\sqrt{\lambda_n} t + D_n \sin c\sqrt{\lambda_n} t) v_n(x)$

代入初始条件确定待定系数即可。

Pólya猜想(Pólya's Conjecture)

1954年George Pólya(1887-1985，美籍匈牙利数学家，斯坦福大学教授)提出如下Pólya猜想：

N=1时显然成立！

特征值问题 $\begin{cases} -\Delta v(x) = \lambda v(x) \quad in \quad D \subset \mathbb{R}^N, N \geq 2 \\ v|_{\partial D} = 0 \end{cases}$

的特征值λ_n满足$\lambda_n \geq (\frac{nC_N}{|D|})^{\frac{2}{N}}, n \geq 1,$ 其中$C_N = \frac{(2\pi)^N}{\omega_N}$.

目前仍是世界Open Problem！

- 目前最好的结果：1983，丘成桐 $\lambda_n \geq \frac{N}{N+2}(\frac{nC_N}{|D|})^{\frac{2}{N}}$
- 2019, Laugesen：对分数阶Laplace算子 $(-\Delta)^{\alpha/2}$

Pólya猜想不成立：(1)$N=1, D=(0,L), 0<\alpha<2$： $\lambda_n(\alpha) < (n\pi/L)^\alpha$

(2)$N=2, D=\{x \in \mathbb{R}^2 \,\|\, |x|<1\}, 0<\alpha<0.984$： $\lambda_1(\alpha) < 2^\alpha = (\frac{1 \cdot C_2}{\pi})^{\alpha/2}$.

五、讨论：

1.单位圆盘的第I特征值问题

$$\boxed{\begin{cases} \Delta v(x) + \lambda v(x) = 0 \quad in \quad D_1 \subset \mathbb{R}^2 \\ v|_{S_1} = 0 \end{cases}}$$

2.例3改变条件后如何?

比如： $G|_{x=0} = \partial_x G|_{x=1} = 0$或$\partial_y G|_{y=0} = 0$或$z \in [1,2]$或. . .