概率论

Homework 10

7.3.3 令 S_n 为每步向右移动概率为 p 的简单随机游走，$S_0 = 0$，记 $X_n = S_n - S_{n-1}$

(a) 证明 $\{S_n = 0 \ i.o.\}$ 不是序列 $\{X_n\}$ 的尾事件。

(b) 证明当 p 不等于 0.5 时,$\mathbb{P}(S_n = 0 \ i.o.) = 0$

解： (a) 设 $X_{2n} = 1, X_{2n+1} = -1 \ for \ n \geq 1$, 则 $\{S_n = 0 \ i.o.\}$ 当 $X_1 = -1$ 时发生，否则不发生. 与 X_1 的取值有关，因此不是尾事件。

(b)$\mathbb{P}(S_{2n} = 0) = \binom{2n}{n}[p(1-p)]^n$, 所以

$$\sum\nolimits_n \mathbb{P}(S_{2n} = 0) < \infty \ if \ p \neq \frac{1}{2}$$

由 B-C 引理得证。

7.3.7

7. Complete convergence. A sequence $X_1, X_2, \ldots$ of random variables is said to be *completely convergent* to X if

$$\sum_n \mathbb{P}(|X_n - X| > \epsilon) < \infty \qquad \text{for all } \epsilon > 0.$$

Show that, for sequences of independent variables, complete convergence is equivalent to a.s. convergence. Find a sequence of (dependent) random variables which converges a.s. but not completely.

解： 必要性：定理 (7.2.4)

充分性：$\{X_n\}$ 独立且 a.s. 收敛到 X，由习题 (7.2.8),X a.s. 等于一个常数，从而 X_n converges to $c \in \mathbb{R}$ almost surely.

所以 $\{|X_n - c| > \epsilon\}$ 只发生有限次的概率为 1。由第二 B-C 引理得证。

反例：$X_n = \mathbf{1}_{[0,\frac{1}{n}]}$

7.3.8

8. Let $X_1, X_2, \ldots$ be independent identically distributed random variables with common mean μ and finite variance. Show that

$$\binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} X_i X_j \xrightarrow{\mathrm{P}} \mu^2 \qquad \text{as } n \to \infty.$$

解：

$$\binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} X_i X_j = \frac{n}{n-1}(\frac{1}{n}\sum_{i=1}^{n} X_i)^2 - \frac{1}{n(n-1)}\sum_{i=1}^{n} X_i^2.$$

$$\frac{1}{n(n-1)}\sum X_i^2 \xrightarrow{P} 0$$

由 WLLN, $\frac{1}{n}\sum_{i=1}^n X_i \xrightarrow{D} \mu$; 因此由定理 (7.2.4a)$\frac{1}{n}\sum_{i=1}^n X_i \xrightarrow{P} \mu$, 所以 $(\frac{1}{n}\sum_{i=1}^n X_i)^2 \xrightarrow{D} \mu^2$ 再由上面的恒等式易得证。

7.3.10

10. Let $\{X_n : n \geq 1\}$ be independent $N(0, 1)$ random variables. Show that:

(a) $\mathbb{P}\left(\limsup_{n\to\infty} \frac{|X_n|}{\sqrt{\log n}} = \sqrt{2}\right) = 1,$

(b) $\mathbb{P}(X_n > a_n \text{ i.o.}) = \begin{cases} 0 & \text{if } \sum_n \mathbb{P}(X_1 > a_n) < \infty, \\ 1 & \text{if } \sum_n \mathbb{P}(X_1 > a_n) = \infty. \end{cases}$

解:（a）由习题（4.4.8）

$$\mathbb{P}\big(|X_n| \geq \sqrt{2\log n}(1+\epsilon)\big) \sim \frac{1}{\sqrt{2\pi \log n}(1+\epsilon)n^{(1+\epsilon)^2}}.$$

求和收敛当且仅当 $\epsilon > 0$, 由 B-C 引理得证。

(b) 直接用 B-C 引理

7.11.2

(ii) Show that if $X_n \xrightarrow{\text{a.s.}} X$ and $Y_n \xrightarrow{\text{a.s.}} Y$ then $X_nY_n \xrightarrow{\text{a.s.}} XY$. Does the corresponding result hold for the other modes of convergence?

解:

（a.s.）注意到 $\{X_nY_n \nrightarrow XY\} \subset \{X_n \nrightarrow X\} \cup \{Y_n \nrightarrow Y\}$

(rth) 对 r 阶矩收敛结论不成立，反例: $X_n = Y_n \sim density\ f$, 这里 $f = x^{-4} * \mathbf{1}_{x>1}$。其四阶矩不存在。

(P) 设 $X_n \xrightarrow{P} X,\ Y_n \xrightarrow{P} Y$

$$\begin{aligned}\mathbb{P}\big(|X_nY_n - XY| > \epsilon\big) &= \mathbb{P}\Big(\big|(X_n - X)(Y_n - Y) + (X_n - X)Y + X(Y_n - Y)\big| > \epsilon\Big) \\ &\leq \mathbb{P}\big(|X_n - X|\cdot|Y_n - Y| > \tfrac{1}{3}\epsilon\big) + \mathbb{P}\big(|X_n - X|\cdot|Y| > \tfrac{1}{3}\epsilon\big) \\ &\quad + \mathbb{P}\big(|X|\cdot|Y_n - Y| > \tfrac{1}{3}\epsilon\big).\end{aligned}$$

Now, for $\delta > 0$,

$$\mathbb{P}\big(|X_n - X|\cdot|Y| > \tfrac{1}{3}\epsilon\big) \leq \mathbb{P}\big(|X_n - X| > \epsilon/(3\delta)\big) + \mathbb{P}\big(|Y| > \delta\big),$$

$n \to \infty, \delta \to \infty$ 时，上式趋近于 0. 类似可得另外两项也趋近于 0，得证。

(D) 对依分布收敛不成立，设 X_n 有对称的分布，令 $Y_n = -X_n$ 即为反例

7.11.4

4. Let $Y_1, Y_2, \ldots$ be independent identically distributed variables, each of which can take any value in $\{0, 1, \ldots, 9\}$ with equal probability $\frac{1}{10}$. Let $X_n = \sum_{i=1}^{n} Y_i\, 10^{-i}$. Show by the use of characteristic functions that X_n converges in distribution to the uniform distribution on $[0, 1]$. Deduce that $X_n \xrightarrow{\text{a.s.}} Y$ for some Y which is uniformly distributed on $[0, 1]$.

解：用特征函数证明：

$$\mathbb{E}(e^{itX_n}) = \Pi_{j=1}^{n}\mathbb{E}(e^{itY_j/10^j}) = \Pi_{j=1}^{n}\left\{\frac{1}{10} \times \frac{1-e^{it/10^{j-1}}}{1-e^{it/10^j}}\right\} = \frac{1-e^{it}}{10^n(1-e^{it/10^n})} \to \frac{1-e^{it}}{it}$$

极限即是 [0,1] 上均匀分布的特征函数

因为 $X_n \le 1$ 递增，所以存在极限 Y。$X_n \overset{a.s.}{\to} Y$, 所以 $X_n \overset{D}{\to} Y$

7. Show that $X_n \xrightarrow{\text{a.s.}} X$ whenever $\sum_n \mathbb{E}(|X_n - X|^r) < \infty$ for some $r > 0$.

8. Show that if $X_n \xrightarrow{\text{D}} X$ then $aX_n + b \xrightarrow{\text{D}} aX + b$ for any real a and b.

7.11.7

解：由 Markov 不等式

$$\sum_n \mathbb{P}(|X_n - X| > \epsilon) \le \sum_n \frac{\mathbb{E}[|X_n - X|^r]}{\epsilon^r} < \infty$$

由第一 B-C 引理，$X_n \overset{a.s.}{\to} X$

7.11.8

解：由 Skorokhod representation, 存在 Y_n, Y 与 X_n, X 同分布，且 $Y_n \overset{a.s.}{\to} Y$, 从而 $aY_n + b \overset{a.s.}{\to} aY + b$, 又由 X,Y 同分布得证

或用特征函数证明