**1.** For what values of the parameters are the following functions probability density functions?

(a) $f(x) = C\{x(1-x)\}^{-\frac{1}{2}}$, $0 < x < 1$, the density function of the 'arc sine law'.

(b) $f(x) = C\exp(-x - e^{-x})$, $x \in \mathbb{R}$, the density function of the 'extreme-value distribution'.

(c) $f(x) = C(1+x^2)^{-m}$, $x \in \mathbb{R}$.

4.1.1(c): 换元 $v = (1+x^2)^{-1}$

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^m} = \int_0^1 v^{m-\frac{3}{2}}(1-v)^{-\frac{1}{2}}dv = B(\tfrac{1}{2}, m-\tfrac{1}{2})$$

$$C = \frac{1}{B(\frac{1}{2}, m-\frac{1}{2})}$$

**4.** **Survival.** Let $X$ be a positive random variable with density function $f$ and distribution function $F$. Define the *hazard function* $H(x) = -\log[1-F(x)]$ and the *hazard rate*

$$r(x) = \lim_{h \downarrow 0} \frac{1}{h}\mathbb{P}(X \le x+h \mid X > x), \qquad x \ge 0.$$

Show that:

(a) $r(x) = H'(x) = f(x)/\{1-F(x)\}$,

(b) If $r(x)$ increases with $x$ then $H(x)/x$ increases with $x$,

(c) $H(x)/x$ increases with $x$ if and only if $[1-F(x)]^\alpha \le 1 - F(\alpha x)$ for all $0 \le \alpha \le 1$,

(d) If $H(x)/x$ increases with $x$, then $H(x+y) \ge H(x) + H(y)$ for all $x, y \ge 0$.

(a) $r(x) = \lim\limits_{h\downarrow 0} \frac{1}{h} \dfrac{F(x+h)-F(x)}{1-F(x)} = \dfrac{f(x)}{1-F(x)} = H'(x)$

(b) $$\frac{d}{dx}\left(\frac{H(x)}{x}\right) = \frac{d}{dx}\left\{\frac{1}{x}\int_0^x r(y)dy\right\}$$

$$= \frac{r(x)}{x} - \frac{1}{x^2}\int_0^x r(y)dy$$

$$= \frac{1}{x^2}\int_0^x [r(x)-r(y)]dy \geq 0.$$

(c) $\dfrac{H(x)}{x}$ 增 $\iff \dfrac{1}{\alpha x}H(\alpha x) \leq \dfrac{1}{x}H(x)$ for all $x \geq 0$.

$\iff -\alpha^{-1}\log[1-F(\alpha x)] \leq -\log[1-F(x)]$.

(d) $\dfrac{H(x)}{x}$ 增 $\iff \dfrac{H(\alpha x)}{\alpha} \leq H(x)$

$\Rightarrow H(\alpha x) + H((1-\alpha)x) \leq H(x)$

**2.** Let $X$ and $Y$ be independent random variables with common distribution function $F$ and density function $f$. Show that $V = \max\{X, Y\}$ has distribution function $\mathbb{P}(V \le x) = F(x)^2$ and density function $f_V(x) = 2f(x)F(x)$, $x \in \mathbb{R}$. Find the density function of $U = \min\{X, Y\}$.

**3.** The annual rainfall figures in Bandrika are independent identically distributed continuous random variables $\{X_r : r \ge 1\}$. Find the probability that:

(a) $X_1 < X_2 < X_3 < X_4$,

(b) $X_1 > X_2 < X_3 < X_4$.

<u>4.2.2</u>:

$$P(V \le x) = P(X \le x, Y \le x)$$
$$= P(X \le x)\, P(Y \le x)$$
$$= F(x)^2$$
$$f_V(x) = \frac{d}{dx}[F(x)^2] = 2f(x)F(x)$$

$$P(U \le u) = 1 - P(U > u) = 1 - P(X > u)\,P(Y > u)$$
$$= 1 - [1 - F(u)]^2$$

$$f_U(u) = 2f(u)[1 - F(u)]$$

<u>4.2.3</u>: $(X_1, X_2, X_3, X_4)$有24种排序.

由对称性. 每种排序概率都等为 $\frac{1}{24}$.

$$\therefore P(X_1 < X_2 < X_3 < X_4) = \frac{1}{24}$$
$$P(X_1 > X_2 < X_3 < X_4) = \frac{3}{24}$$

**3.** Let $X$ be a non-negative random variable with density function $f$. Show that

$$\mathbb{E}(X^r) = \int_0^\infty r x^{r-1} \mathbb{P}(X > x)\, dx$$

for any $r \ge 1$ for which the expectation is finite.

$$r\int_0^\infty x^{r-1} P(X > x)\, dx = r\int_0^\infty x^{r-1} \left\{ \int_{y=x}^\infty f(y)\, dy \right\} dx$$

$$\underset{\text{Fubini}}{=} \int_{y=0}^\infty f(y) \left\{ \int_{x=0}^{y} r x^{r-1}\, dx \right\} dy$$

$$= \int_0^\infty y^r f(y)\, dy = E[X^r]$$

**5.** Let $X$ be a random variable with mean $\mu$ and continuous distribution function $F$. Show that

$$\int_{-\infty}^{a} F(x)\,dx = \int_{a}^{\infty} [1 - F(x)]\,dx,$$

if and only if $a = \mu$.

$$\mu = E(X^+) - E(X^-)$$
$$= \int_0^\infty P(X > x)\,dx - \int_0^\infty P(X < -x)\,dx$$
$$= \int_0^\infty [1 - F(x)]\,dx - \int_0^\infty F(-x)\,dx$$
$$= \int_0^\infty [1 - F(x)]\,dx - \int_{-\infty}^{0} F(x)\,dx.$$

注意到 $\mu = \int_0^\mu 1\,dx = \int_0^\mu F(x)\,dx + \int_0^\mu [1 - F(x)]\,dx$

$$\therefore \int_{-\infty}^{\mu} F(x)\,dx = \int_{\mu}^{\infty} [1 - F(x)]\,dx$$

只有 $a=\mu$ 时等式成立. 因为 等式 左边关于 $a$ ↑, 右边关于 $a$ ↓

**3.** Let $X$ have the uniform distribution on $[0, 1]$. For what function $g$ does $Y = g(X)$ have the exponential distribution with parameter 1?

若 $g$ 单调减 $P(g(X) \leq y) = P(X \geq g^{-1}(y))$
$$= 1 - g^{-1}(y)$$

$$\therefore P(g(X) \leq y) = 1 - e^{-y}$$
$$\Leftrightarrow g^{-1}(y) = e^{-y} \qquad \therefore g(x) = -\log x.$$

Remark: $g$ 不唯一.

**5.** **Log-normal distribution.** Let $Y = e^X$ where $X$ has the $N(0, 1)$ distribution. Find the density function of $Y$.

$$P(Y \leq y) = P(X \leq \log y) = \Phi(\log y)$$
$$\therefore f_Y(y) = \frac{1}{y} f_X(\log y) = \frac{1}{y\sqrt{2\pi}} e^{-\frac{1}{2}(\log y)^2}$$

**4.** Let $X$ and $Y$ be independent random variables each having the uniform distribution on $[0, 1]$. Let $U = \min\{X, Y\}$ and $V = \max\{X, Y\}$. Find $\mathbb{E}(U)$, and hence calculate $\operatorname{cov}(U, V)$.

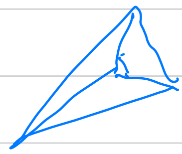
(i) $F_U(u) = 1-(1-u)^2$ $\therefore$ $E(U) = \int_0^1 2u(1-u)\,du = \frac{1}{3}$. , $E(V) = 1 - E(U) = \frac{2}{3}$

(ii) $UV = XY$, $\therefore$ $E(UV) = E(X)E(Y) = \frac{1}{4}$

$\therefore \operatorname{cov}(U,V) = E(UV) - E(U)E(V) = \frac{1}{4} - \frac{1}{3}\left(1-\frac{1}{3}\right)$

$= \frac{1}{36}$

**6.** Three points A, B, C are chosen independently at random on the circumference of a circle. Let $b(x)$ be the probability that at least one of the angles of the triangle ABC exceeds $x\pi$. Show that



$$b(x) = \begin{cases} 1-(3x-1)^2 & \text{if } \frac{1}{3} \le x \le \frac{1}{2}, \\ 3(1-x)^2 & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

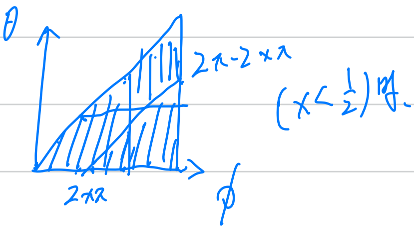
Hence find the density and expectation of the largest angle in the triangle.

**7.** Let $\{X_r : 1 \le r \le n\}$ be independent and identically distributed with finite variance, and define $\overline{X} = n^{-1}\sum_{r=1}^{n} X_r$. Show that $\operatorname{cov}(\overline{X}, X_r - \overline{X}) = 0$.

4.5.6: 固定 $A=(1,0)$ , 设 $B=(1,\Theta)$, $C=(1,\Phi)$, 且 $0<\Theta<\Phi$.

则 $\Theta, \Phi$ 有 联合密度 $f(\theta,\phi) = (2\pi^2)^{-1}$, $0<\theta<\phi<2\pi$

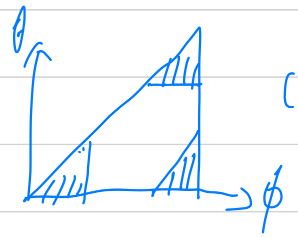
$\angle A = \frac{1}{2}(\Phi-\Theta)$ , $\angle B = \pi - \frac{1}{2}\Phi$, $\angle C = \frac{1}{2}\Theta$.




$\angle B > x\pi \iff (2-2x)\pi > \Phi$ $2\pi.$

$\angle C > x\pi \iff \Theta > 2x\pi$ $2$

$\angle A > x\pi \iff \Phi - \Theta > 2x\pi$.




区域形状以 $x=\frac{1}{2}$ 为分界,

计算出 $g(x) = \begin{cases} 6(3x-1) & \text{if } \frac{1}{3} \le x \le \frac{1}{2} \\ 6(1-x) & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$

期望为 $\frac{11}{18}\pi$

4.5.7: $E(\overline{X}) = \mu$. $E(X_r - \overline{X}) = 0$.

$\Rightarrow E(\overline{X}(X_r - \overline{X})) = \frac{1}{n} E\left(\sum_s X_r X_s\right) - E(\overline{X}^2)$

$= \frac{1}{n}\{\sigma^2 + n\mu^2\} - (\operatorname{var}(\overline{X}) + E(\overline{X})^2)$

$= \frac{1}{n}\{\sigma^2 + n\mu^2\} - (\frac{\sigma^2}{n} + \mu^2) = 0.$

$\therefore \operatorname{cov}(\bar{X}, X_r - \bar{X}) = 0.$

**9.** Let $X$ and $Y$ be independent continuous random variables, and let $U$ be independent of $X$ and $Y$ taking the values $\pm 1$ with probability $\frac{1}{2}$. Define $S = UX$ and $T = UY$. Show that $S$ and $T$ are in general dependent, but $S^2$ and $T^2$ are independent.

$S^2 = X,\ T^2 = Y.$ 独立

若 $X, Y$ 恒正，则 $\{X>0\} = \{Y>0\} = \{U>0\}$

$\therefore$ $X, Y$ dependent.