微分方程

算子法求特解

算子法(O. Heaviside发明)

定义运算子 $\dfrac{d}{dt}=D,\ \dfrac{d^2}{dt^2}=D^2,\cdots,\dfrac{d^n}{dt^n}=D^n$. 记 $D^0=1$.

对于函数 $x=x(t)$: $\dfrac{dx}{dt}=Dx$, $\dfrac{d^2x}{dt^2}=D^2x,\cdots,\dfrac{d^nx}{dt^n}=D^nx.$

$P(D):=a_nD^n+a_{n-1}D^{n-1}+\cdots+a_1D+a_0$ 称为 **n** 阶算子多项式.

其中 a_0 , a_1 , $\cdots,a_n$ 是常数.

定义 $P(D)x=a_nD^nx+a_{n-1}D^{n-1}x+\cdots+a_1Dx+a_0x$

$$=a_nx^{(n)}+a_{n-1}x^{(n-1)}+\cdots+a_1x'+a_0x.$$

基本实例: $1^\circ\ P(D)e^{at}=P(a)e^{at}$;

$2^\circ\ P(D)[e^{at}x(t)]=e^{at}P(D+a)x(t)$;

$3^\circ\ P(D^2)\sin\omega t=P(-\omega^2)\sin\omega t, P(D^2)\cos\omega t=P(-\omega^2)\cos\omega t.$

例如，二阶非齐次方程 $x'' + px' + qx = f(t)$

可记为 $D^2 x + pDx + qx = f(t)$.

常系数线性常微分方程 $P(D)x = f(t)$的特解可表示为

$x^* = \dfrac{1}{P(D)} f(t)$, 其中$\dfrac{1}{P(D)}$称为$P(D)$的逆算子.

例如, $\dfrac{1}{D^k}[f(t)] = \int \cdots \int f(t)(dt)^k$ (k重积分).

特别, 当$Q(t)$为n次多项式时, 得到$D^2 x + pDx + qx = Q(t)$

特解$x^* = \dfrac{1}{D^2 + pD + q} Q(t)$的方法是将 $\dfrac{1}{D^2 + pD + q}$

展开为泰勒级数到 n 次项为止并作用到$Q(t)$.

例1. 求$2x''+2x'+x=t^2+2t-1$的特解.

解. 方法1. $(2D^2+2D+1)x=t^2+2t-1$的特解

$$x^*=\frac{1}{2D^2+2D+1}(t^2+2t-1),$$ 先求逆算子$\dfrac{1}{2D^2+2D+1}$.

$$\begin{array}{r|l}
 & \mathbf{1-2D+2D^2} \\ \hline
\mathbf{1+2D+2D^2} & \mathbf{1+0D+0D^2+\cdots} \\
 & \mathbf{1+2D+2D^2} \\ \hline
 & \mathbf{-2D-2D^2} \\
 & \mathbf{-2D-4D^2-4D^3} \\ \hline
 & \mathbf{2D^2+4D^3}
\end{array}$$

$$\Rightarrow x^*=(1-2D+2D^2+\cdots)(t^2+2t-1)=t^2-2t-1.$$

例1. 求$2x''+2x'+x=t^2+2t-1$的特解.

解. 方法2. $(2D^2+2D+1)x=t^2+2t-1$的特解

$x^*=\dfrac{1}{2D^2+2D+1}(t^2+2t-1)$, 先求逆算子$\dfrac{1}{2D^2+2D+1}$.

利用$\dfrac{1}{1+x}=1-x+x^2-x^3+\cdots$, 有

$$\frac{1}{2D^2+2D+1}=\frac{1}{1+(2D^2+2D)}$$

$$=1-(2D^2+2D)+(2D^2+2D)^2-(2D^2+2D)^3+\cdots$$

$$=1-2D+2D^2+\cdots$$

$$\Rightarrow x^*=(1-2D+2D^2+\cdots)(t^2+2t-1)=t^2-2t-1.$$

例2. 求$x'' - x = t^4 + 1$的特解.

解：方程的算子形式为$(D^2 - 1)x = t^4 + 1$,

特解$x^* = \dfrac{1}{D^2 - 1}(t^4 + 1)$.

$$\because \frac{1}{D^2 - 1} = -1 - D^2 - D^4 - D^6 - \cdots$$

$$\therefore \text{特解} x^* = \frac{1}{D^2 - 1}(t^4 + 1) = (-1 - D^2 - D^4 + \cdots)(t^4 + 1)$$

$$= -t^4 - 12t^2 - 25.$$

注：$$\frac{1}{D^2 + D}(t^2 + 1) = \frac{1}{D(1 + D)}(t^2 + 1) = \frac{1}{D}[(1 - D + D^2 + \cdots)](t^2 + 1)$$

$$= \frac{1}{D}(t^2 - 2t + 3) = \frac{t^3}{3} - t^2 + 3t.$$

常见情形 $f(t)=e^{at}g(t)$, 其中a为复数，$g(t)$为实函数.

例3. 求$x''-2x'+x=te^t$的特解.

解：方程的算子形式为$(D-1)^2x=te^t$,

由2° $P(D)[e^{at}x(t)]=e^{at}P(D+a)x(t)$知

$$P(D)[e^{at}\frac{1}{P(D+a)}x(t)]=e^{at}P(D+a)\frac{1}{P(D+a)}x(t)$$

$=e^{at}x(t)$, 从而得 $\frac{1}{P(D)}[e^{at}x(t)]=e^{at}\frac{1}{P(D+a)}x(t)$.

$$\therefore 特解 x^*=\frac{1}{(D-1)^2}[te^t]=e^t\frac{1}{D^2}t=\frac{t^3}{6}e^t.$$

例4. 求$x''-6x'+13x=e^{3t}\sin 2t$的特解.

解：方程为$(D^2-6D+13)x=e^{3t}\sin 2t$, 特解为

$$x^*=\frac{1}{D^2-6D+13}e^{3t}\sin 2t=\frac{1}{D^2-6D+13}\operatorname{Im} e^{(3+2i)t}$$

$$=\operatorname{Im}[\frac{1}{D^2-6D+13}e^{(3+2i)t}]$$

$$=\operatorname{Im}[e^{(3+2i)t}\frac{1}{(D+3+2i)^2-6(D+3+2i)+13}1]$$

$$=\operatorname{Im}[e^{(3+2i)t}\frac{1}{D(D+4i)}1]=\operatorname{Im}[e^{(3+2i)t}\frac{1}{4iD}(1-\frac{D}{4i}+\cdots)1]$$

$$=\operatorname{Im}[e^{(3+2i)t}\frac{t}{4i}]=\operatorname{Im}[\frac{t}{4}e^{3t}(\sin 2t-i\cos 2t)]$$

$$=-\frac{t}{4}e^{3t}\cos 2t.$$

例5. 求微分方程组 $\begin{cases} x' + y' + x + y = 2t \\ x' + 2y' - y = 3t \end{cases}$ 的特解.

解：算子形式为 $\begin{cases} (D+1)x + (D+1)y = 2t \\ Dx + (2D-1)y = 3t \end{cases}$.

系数行列式$\Delta = \begin{vmatrix} D+1 & D+1 \\ D & 2D-1 \end{vmatrix} = D^2 - 1 \neq 0,$

令$\Delta_1 = \begin{vmatrix} 2t & D+1 \\ 3t & 2D-1 \end{vmatrix} = (2D-1)2t - (D+1)3t = 1 - 5t,$

$\Delta_2 = \begin{vmatrix} D+1 & 2t \\ D & 3t \end{vmatrix} = (D+1)3t - D(2t) = 3t + 1,$由克莱姆法则得

$$x^* = \frac{\Delta_1}{\Delta} = \frac{1-5t}{D^2-1} = (-1 - D^2 - D^4 - \cdots)(1-5t) = 5t - 1,$$

$$y^* = \frac{\Delta_2}{\Delta} = \frac{3t+1}{D^2-1} = (-1 - D^2 - D^4 - \cdots)(3t+1) = -3t - 1.$$