习题课讲义

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- 当$c=0$时，$\tau=c\kappa=0$，由曲线存在唯一性知$\mathbf{r}$ 为平面曲线。

由$|\mathbf{t}|=1$，不妨令$\dot{\mathbf{r}}=\mathbf{t}(s)=(cos(\theta(s)),sin(\theta(s)),0)$，

由$\kappa=|\dot{\mathbf{t}}|$有$\theta(u)=\pm\int_0^u\kappa(t)dt$，解得曲线$\mathbf{r}(s)=\int_0^s\mathbf{t}(u)du=(\int_0^s cos(\int_0^u\kappa(t)dt)du,\pm\int_0^s sin(\int_0^u\kappa(t)dt)du,0)$

由存在唯一性，其它符合条件的曲线均由$\mathbf{r}$刚体运动得到

- 当 $c\neq 0$时，考虑$Frent$标架

$$\begin{cases}\dot{\mathbf{t}}= & & \kappa\mathbf{n} & \\ \dot{\mathbf{n}}= & -\kappa\mathbf{t} & + & c\kappa\mathbf{b}\\ \dot{\mathbf{b}}= & & -c\kappa\mathbf{n} & \end{cases}$$

类似的我们令$\theta(s)=\int_0^s\kappa(t)dt$，做参数变换有

$$\begin{cases}\frac{d\mathbf{t}(\theta)}{d\theta}= & & \mathbf{n} & \\ \frac{d\mathbf{n}(\theta)}{d\theta}= & -\mathbf{t} & + & c\mathbf{b}\\ \frac{d\mathbf{b}(\theta)}{d\theta}= & & -c\mathbf{n} & \end{cases}$$

得到$\frac{d^2\mathbf{n}(\theta)}{d\theta^2}=-(1+c^2)\mathbf{n}$

这里我们只需要找到一组解即可，取一组解

$$\begin{cases}t=\frac{1}{\sqrt{1+c^2}}(sin(\sqrt{1+c^2}\theta),-cos(\sqrt{1+c^2}\theta,c)\\ n=(cos(\sqrt{1+c^2}\theta),sin(\sqrt{1+c^2}\theta,0)\\ b=-\frac{c}{\sqrt{1+c^2}}(sin(\sqrt{1+c^2}\theta),-cos(\sqrt{1+c^2}\theta,-\frac{1}{c})\end{cases}$$

最后解得

$\mathbf{r}(s)=\int_0^s t(u)du$

$= \frac{1}{\sqrt{1+c^2}}(\int_0^s sin(\sqrt{1+c^2}(\int_0^u \kappa(t)dt))du, -\int_0^s cos(\sqrt{1+c^2}(\int_0^u \kappa(t)dt))du, cs)$

其它符合条件的曲线均由$\mathbf{r}$刚体运动得到

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本题是存在唯一性的一个简单应用，主要思路是计算出两者的曲率和挠率并验证两者相等，需要用到前面第5题的结论

证明:

$\mathbf{r}'(t) = (1+\sqrt{3}cos(t), -2sin(t), \sqrt{3}-cos(t))$

$\mathbf{r}''(t) = (-\sqrt{3}sin(t), -2cos(t), sin(t))$

$\mathbf{r}'''(t) = (-\sqrt{3}cos(t), 2sin(t), cos(t))$

$\kappa(t) = \frac{|\mathbf{r}' \wedge \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{1}{4}$

$\tau(t) = \frac{(\mathbf{r}', \mathbf{r}'', \mathbf{r}''')}{|\mathbf{r}' \wedge \mathbf{r}''|^2} = -\frac{1}{4}$

$\bar{\mathbf{r}}'(t) = (-2sin(t), 2cos(t), -2)$

$\bar{\mathbf{r}}''(t) = (-2cos(t), -2sin(t), 0)$

$\bar{\mathbf{r}}'''(t) = (2sin(t), -2cos(t), 0)$

$\bar{\kappa}(t) = \frac{|\mathbf{r}' \wedge \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{1}{4}$

$\bar{\tau}(t) = \frac{(\mathbf{r}', \mathbf{r}'', \mathbf{r}''')}{|\mathbf{r}' \wedge \mathbf{r}''|^2} = -\frac{1}{4}$

得到$\kappa(t) = \bar{\kappa}(t)$，$\tau(t) = \bar{\tau}(t)$，由曲线存在唯一性得证。

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- 存在性同17中$c=0$的情况
- 唯一性考虑$\mathbf{r}$和$\bar{\mathbf{r}}$的$Frent$标架在$s=0$处相同，即$\mathbf{t}(0) = \bar{\mathbf{t}}(0)$，$\mathbf{r}(0) = \bar{\mathbf{r}}(0)$且曲率相同，则有

  $\frac{d}{ds}[|\mathbf{t} - \bar{\mathbf{t}}|^2 + |\mathbf{n} - \bar{\mathbf{n}}|^2]$

  $= 2[< \mathbf{t} - \bar{\mathbf{t}}, \kappa(\mathbf{n} - \bar{\mathbf{n}}) > + < -\kappa(\mathbf{t} - \bar{\mathbf{t}}), (\mathbf{n} - \bar{\mathbf{n}}) >] = 0$

得$\mathbf{r} = \bar{\mathbf{r}}$

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（1）

$$\begin{cases} x = & a\cos(u)\cos(v) \\ y = & b\cos(u)\sin(v) \\ z = & c\sin(u) \end{cases}$$

$u \in (-\frac{\pi}{2}, \frac{\pi}{2}]$，$v \in (0, 2\pi]$

(5)

$$\begin{cases} x = & a(u+v) \\ y = & b(u-v) \\ z = & 4uv \end{cases}$$

$u, v \in \mathbb{R}$

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（1）$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z$，且$\phi : (u, v) \to (x, y)$为双射，故为双曲抛物面

（2）$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z$，注意到$z = u^2 \geq 0$，或者说$\phi : (u, v) \to (x, y)$不为满射，为双曲抛物面的上半部分

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曲线嵌入$\mathbb{R}^3$为$\mathbf{r} = (x(t), y(t), 0)$，故曲面上的点均可表示成$\mathbf{r} + s\alpha = (x(t) + s\alpha_1, y(t) + s\alpha_2, s\alpha_3)$

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证明:

$\nabla F = (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}) = (-\frac{y}{x^2}F_1 - \frac{z}{x^2}F_2, \frac{1}{x}F_1, \frac{1}{x}F_2)$

曲面$\mathbf{r}$上一点$\mathbf{x}_0$处的切平面为

$\nabla F(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = 0$

将坐标原点代入得

$$\nabla F(\mathbf{x}_0)\cdot(-\mathbf{x}_0) = -x_0\cdot(-\frac{y_0}{x_0^2}F_1 - \frac{z_0}{x_0^2}F_2) + (-y_0)\frac{1}{x_0}F_1 + (-z_0)\frac{1}{x_0}F_2 = 0$$

因此原点在$\mathbf{r}$的任意一点的切平面上。

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证明:

令平面法向为$\mathbf{n}'$，取$h(u,v) =< \mathbf{r}(u,v) - \mathbf{r}(u_0,v_0), \mathbf{n}' >$

则$h \geq 0$或$h \leq 0$，$h(u_0,v_0) = 0$为极值点，因此

$$\nabla h(u_0,v_0) = 0 \Rightarrow < r_u(u_0,v_0), \mathbf{n}' >= 0, < r_v(u_0,v_0), \mathbf{n}' >= 0$$

$\mathbf{n}'$为该点法向，该平面为切平面

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考虑参数方程

$$\mathbf{r}(u,v) = (a\cos(u)\cos(v), b\cos(u)\sin(v), c\sin(u))$$

$$\mathbf{r}_u = (-a\sin(u)\cos(v), -b\sin(u)\sin(v), c\cos(u))$$

$$\mathbf{r}_v = (-a\cos(u)\sin(v), b\cos(u)\cos(v), 0)$$

$$\Rightarrow \begin{cases} E =< \mathbf{r}_u, \mathbf{r}_u >= a^2\sin^2(u)\cos^2(v) + b^2\sin^2(u)\sin^2(v) + c^2\cos^2(u) \\ F =< \mathbf{r}_u, \mathbf{r}_v >= (a^2 - b^2)\sin(u)\sin(v)\cos(u)\cos(v) \\ G =< \mathbf{r}_v, \mathbf{r}_v >= a^2\cos^2(u)\sin^2(v) + b^2\cos^2(u)\cos^2(v) \end{cases}$$

$$I = Edudu + 2Fdudv + Gdvdv = \ldots\ldots$$

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(2)

$$\mathbf{r}_u = (\cos(v), \sin(v), 0)$$

$$\mathbf{r}_v = (-u\sin(v), u\cos(v), b)$$

$$\Rightarrow \begin{cases} E = 1 \\ F = 0 \\ G = u^2 + b^2 \end{cases}$$

$$I = dudu + (u^2 + b^2)dvdv$$

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$$\mathbf{r}_x = (1, 0, \frac{\partial f}{\partial x})$$

$$\mathbf{r}_y = (0, 1, \frac{\partial f}{\partial y})$$

$$\Rightarrow \begin{cases} E = 1 + (\dfrac{\partial f}{\partial x})^2 \\ F = \dfrac{\partial f}{\partial x}\dfrac{\partial f}{\partial y} \\ G = 1 + (\dfrac{\partial f}{\partial y})^2 \end{cases}$$

$$I = [1 + (\frac{\partial f}{\partial x})^2]dxdx + 2[\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}]dxdy + [1 + (\frac{\partial f}{\partial y})^2]dydy$$

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证明:

记多项式$g(\lambda) = (\lambda - a)(\lambda - b)(\lambda - c) + x^2(\lambda - b)(\lambda - c) + y^2(\lambda - a)(\lambda - c) + z^2(\lambda - a)(\lambda - b)$

$\deg(g) = 3 \Rightarrow g = 0$的实根至多有3个，又有$g(-\infty) < 0, g(a) > 0, g(b) < 0, g(c) > 0 (a > b > c > 0)$

$\Rightarrow \exists \lambda_1, \lambda_2, \lambda_3$三个实根

分别对应3个二次曲面，三者在$P(x, y, z)$处的法向量为

$$\mathbf{n}_i = (\frac{2x}{a - \lambda_i}, \frac{2y}{b - \lambda_i}, \frac{2z}{c - \lambda_i})$$

$$\forall i \neq j, < \mathbf{n}_i, \mathbf{n}_j >= 4(\frac{x^2}{(a - \lambda_i)(a - \lambda_j)} + \frac{y^2}{(b - \lambda_i)(b - \lambda_j)} + \frac{z^2}{(c - \lambda_i)(c - \lambda_j)})$$

$$= 4\frac{1}{\lambda_i - \lambda_j}[（\frac{g(\lambda_i)}{(\lambda_i - a)(\lambda_i - b)(\lambda_i - c)} - 1） - （\frac{g(\lambda_j)}{(\lambda_j - a)(\lambda_j - b)(\lambda_j - c)} - 1）] = 0$$

（分别对每项裂项再求和）

故相互正交

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- 首先对$\mathbf{r}$的自然基底$\mathbf{r}_u,\mathbf{r}_v$做Schmidt正交化

  $\mathbf{e}_1=\frac{\mathbf{r}_u}{|\mathbf{r}_u|}$ $\mathbf{b}=\mathbf{r}_v-<\mathbf{r}_u,\mathbf{e}_1>\mathbf{e}_1$

  $\mathbf{e}_2=\frac{\mathbf{b}}{|\mathbf{b}|}$

- 接着，我们证明对任意的 p，存在 p的领域U以及U上的一个参数系(s,t)使得

  $\mathbf{r}_s//\mathbf{e}_1$,$\mathbf{r}_t//\mathbf{e}_2$

事实上，我们可以证明如下的定理

$Theorem$ 假设正则参数曲面$S:\mathbf{r}=\mathbf{r}(u,v)$上存在两个处处线性无关的连续可微切向量场$\mathbf{a}(u,v)$ $\mathbf{b}(u,v)$，则$\forall p\in S,\exists p$的领域U和U上的参数系(s,t)使得$\mathbf{r}_s//\mathbf{a}$, $\mathbf{r}_t//\mathbf{b}$

$proof$:

先对$\mathbf{a},\mathbf{b}$做如下的分解：

$$\begin{cases}\mathbf{a}(u,v)=a_1(u,v)\mathbf{r}_u+a_2(u,v)\mathbf{r}_v\\ \mathbf{b}(u,v)=b_1(u,v)\mathbf{r}_u+b_2(u,v)\mathbf{r}_v\end{cases}$$

由题设条件有

$$A=\begin{bmatrix}a_1 & a_2\\ b_1 & b_2\end{bmatrix}\neq 0$$

要使$\mathbf{r}_s//\mathbf{a},\mathbf{r}_t//\mathbf{b}$，则有$\mathbf{r}_s=\lambda(u,v)\mathbf{a},\mathbf{r}_t=\mu(u,v)\mathbf{b}$

$$\begin{cases}\mathbf{r}_s=\lambda(u,v)a_1(u,v)\mathbf{r}_u+\lambda(u,v)a_2(u,v)\mathbf{r}_v\\ \mathbf{r}_t=\mu(u,v)b_1(u,v)\mathbf{r}_u+\mu(u,v)b_2(u,v)\mathbf{r}_v\end{cases}$$

得到$J=\begin{pmatrix}\frac{\partial u}{\partial s} & \frac{\partial v}{\partial s}\\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t}\end{pmatrix}=\begin{pmatrix}\lambda a_1 & \lambda a_2\\ \mu b_1 & \mu b_2\end{pmatrix}$，$\det(J)=\lambda\mu A$

$$\begin{pmatrix}\frac{\partial s}{\partial u} & \frac{\partial t}{\partial u}\\ \frac{\partial s}{\partial v} & \frac{\partial t}{\partial v}\end{pmatrix}=\frac{1}{\lambda\mu A}\begin{pmatrix}\mu b_2 & -\lambda a_2\\ -\mu b_1 & \lambda a_1\end{pmatrix}$$

即$\begin{cases} ds = \dfrac{\partial s}{\partial u}du + \dfrac{\partial s}{\partial v}dv = \dfrac{1}{\lambda A}(b_2du - b_1dv) \\ dt = \dfrac{\partial t}{\partial u}du + \dfrac{\partial t}{\partial v}dv = \dfrac{1}{\mu A}(-a_2du + a_1dv) \end{cases}$

存在积分因子λ, μ使得这两个方程为恰当方程。

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$L =< \mathbf{r}_{uu}, \mathbf{n} >$

$$\begin{cases} L =< \mathbf{r}_{uu}, \mathbf{n} > \\ M =< \mathbf{r}_{uv}, \mathbf{n} > \\ N =< \mathbf{r}_{vv}, \mathbf{n} > \end{cases}$$

(1)

$II = \frac{f''(u)g'(u)-g''(u)f'(u)}{\sqrt{(f'(u))^2+(g'(u))^2}}dudu$

(2)

$II = -\frac{2b}{\sqrt{k^2+b^2}}dudv$

(3)

$II = -\frac{2ab}{\sqrt{a^2b^2+a^2(u-v)^2+b^2(u+v)^2}}(dudu + dvdv)$

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$\mathbf{r}(x, y) = (x, x, f(x, y))$

$r_x = (1, 0, f_x)$

$r_y = (0, 1, f_y)$

$\mathbf{n} = \frac{1}{\sqrt{1+f_x^2+f_y^2}}(-f_x, -f_y, 1)$

$\mathbf{r}_{xx} = (0, 0, f_{xx})$

$\mathbf{r}_{xy} = (0, 0, f_{xy})$

$\mathbf{r}_{yy} = (0, 0, f_{yy})$

$L = \frac{1}{\sqrt{1+f_x^2+f_y^2}} f_{xx}$

$M = \frac{1}{\sqrt{1+f_x^2+f_y^2}} f_{xy}$

$N = \frac{1}{\sqrt{1+f_x^2+f_y^2}} f_{yy}$

$II = \frac{1}{\sqrt{1+f_x^2+f_y^2}} (f_{xx} dxdx + 2f_{xy} dxdy + f_{yy} dydy)$

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局部上不妨有$F_z \neq 0 \Rightarrow \mathbf{r}(x,y) = (x, x, f(x,y))$

$F(x,y,f) = 0 \Rightarrow \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0, \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial x} = -\frac{F_x}{F_z}, \frac{\partial f}{\partial y} = -\frac{F_y}{F_z}$

结果代入到15题即可