§0.1 Gauss-Bonnet公式

这里仅讨论比较简化的情形：设曲面与圆盘微分同胚，所考虑区域上可建立测地极坐标系或者存在正交标架。

§0.1.1 测地极坐标系证明

曲面上三条测地线围成的三角形称为测地三角形。Gauss证明了测地三角形内角和与Gauss曲率积分的如下关系。

定理0.1. *(Gauss, 1827)* 设曲面上测地三角形三顶点分别为A, B, C，对应内角分别为$\angle A, \angle B, \angle C$。则

$$\angle A + \angle B + \angle C = \pi + \int_{\triangle ABC} K dA.$$

证明: 设顶点A, B, C所对测地线分别为α, β, γ。取以A为原点的测地极坐标(ρ, θ)使得连接A, B的测地射线为$\rho-$线$\theta = 0$。连接B, C的弧长参数测地线$\alpha(s)$有参数表示$\alpha(s) = r(\rho(s), \theta(s))$ 以及$\alpha(\theta) = r(\rho(\theta), \theta)$。测地三角形在测地极坐标系下对应的参数区域为

$$D = \{(\rho, \theta) | 0 < \theta < \angle A, 0 < \rho < \rho(\theta)\}.$$

利用测地极坐标系，$I = d\rho^2 + G d\theta^2$，计算Gauss曲率在测地三角形内的积分

$$\begin{aligned}
\int_{\triangle ABC} K dA &= \int_D -\frac{(\sqrt{G})_{\rho\rho}}{\sqrt{G}} \sqrt{EG - F^2} d\rho d\theta \\
&= \int_D -(\sqrt{G})_{\rho\rho} d\rho d\theta \\
&= \int_0^{\angle A} \int_0^{\rho(\theta)} -(\sqrt{G})_{\rho\rho} d\rho d\theta \\
&= \int_0^{\angle A} -(\sqrt{G})_\rho |_0^{\rho(\theta)} d\theta \\
&= \int_0^{\angle A} [1 - (\sqrt{G})_\rho(\rho(\theta), \theta)] d\theta \\
&= \angle A - \int_0^{\angle A} (\sqrt{G})_\rho(\rho(\theta), \theta) d\theta.
\end{aligned}$$

对于欧式平面上的三角形，

$$\int_0^{\angle A} (\sqrt{G})_\rho d\theta = \int_0^{\angle A} d\theta = \angle A = \pi - (\angle B + \angle C).$$

令$\varphi(s)$为$\alpha(s)$与射线$\overrightarrow{A\alpha(s)}$的夹角(射线$\overrightarrow{A\alpha(s)}$切向逆时针旋转$\varphi(s)$角度得到$\alpha(s)$的切向)。则(对曲面也成立)

$$\varphi(B) = \pi - \angle B, \quad \varphi(C) = \angle C.$$

从而

$$\int_0^{\angle A} (\sqrt{G})_\rho d\theta = \pi - (\angle B + \angle C) = \varphi(B) - \varphi(C) = \int_{s(B)}^{s(C)} -d\varphi.$$

因此只要验证曲面上如下等式成立

$$\int_0^{\angle A} (\sqrt{G})_\rho d\theta = \int_{s(B)}^{s(C)} -d\varphi.$$

记测地射线$\overrightarrow{A\alpha(s)}$(即$\rho-$线$\theta = \theta(\alpha(s))$)的切向量为$r_\rho$。沿$\alpha(s)$，定义$\alpha'(s) = \frac{d\alpha(s)}{ds}$与$r_\rho$的夹角为$\varphi(s)$，即

$$\alpha'(s) = \frac{d\alpha(s)}{ds} = \frac{d\rho}{ds} r_\rho + r_\theta \frac{d\theta}{ds} = \cos\varphi(s) r_\rho + \sin\varphi(s) \frac{r_\theta}{\sqrt{G}}.$$

因此

$$\cos\varphi(s) = \frac{d\rho}{ds} = \langle \frac{d\alpha(s)}{ds}, r_\rho \rangle,$$
$$\sin\varphi(s) = \sqrt{G}\frac{d\theta}{ds}.$$

由$\alpha(s) = r(\rho(s), \theta(s))$为弧长参数测地线，$\rho-$线为测地线，可计算

$$\begin{aligned}
\frac{d}{ds}\cos\varphi(s) &= -\sin\varphi(s)\frac{d\varphi}{ds} = \frac{d^2\rho(s)}{ds^2} = \frac{d}{ds}\langle \frac{d\alpha(s)}{ds}, r_\rho \rangle \\
&= \langle \frac{d\alpha(s)}{ds}, \frac{d}{ds} r_\rho \rangle \\
&= \langle \frac{d\alpha(s)}{ds}, \frac{d\rho}{ds} r_{\rho\rho} + \frac{d\theta}{ds} r_{\rho\theta} \rangle \\
&= \langle \frac{d\alpha(s)}{ds}, \frac{d\theta}{ds} r_{\rho\theta} \rangle \\
&= \langle \frac{d\rho}{ds} r_\rho + r_\theta \frac{d\theta}{ds}, \frac{d\theta}{ds} r_{\rho\theta} \rangle \\
&= \langle r_\theta \frac{d\theta}{ds}, \frac{d\theta}{ds} r_{\rho\theta} \rangle \\
&= \frac{1}{2}(\frac{d\theta}{ds})^2 G_\rho.
\end{aligned}$$

即

$$-\sin\varphi(s)\frac{d\varphi}{ds} = \frac{d^2\rho(s)}{ds^2} = \frac{1}{2}(\frac{d\theta}{ds})^2 G_\rho.$$

以$\sin\varphi(s)=\sqrt{G}\frac{d\theta}{ds}$代入得

$$-\frac{d\varphi}{ds}=\frac{1}{\sqrt{G}\frac{d\theta}{ds}}\frac{1}{2}(\frac{d\theta}{ds})^2G_\rho=(\sqrt{G})_\rho\frac{d\theta}{ds},$$

即沿曲线$\alpha(s)$，

$$-d\varphi=(\sqrt{G})_\rho d\theta.$$

因此有

$$\begin{aligned}\int_{\triangle ABC}KdA &= \angle A-\int_0^{\angle A}(\sqrt{G})_\rho(\rho(\theta),\theta)d\theta\\ &= \angle A+\int_B^C d\varphi=\angle A+\varphi(C)-\varphi(B)\\ &= \angle A+\angle C-(\pi-\angle B)\\ &= \angle A+\angle B+\angle C-\pi.\end{aligned}$$

□

推论：球面上测地三角形内角和大于180度，双曲空间中测地三角形内角和小于180度。

推广到曲面上测地多边形D：设n边形的内角分别为$\alpha_1,\cdots,\alpha_n$。选取多边形内部一点为测地极坐标原点，并且将多边形剖分成n个测地三角形，因此

$$\begin{aligned}\int_D KdA &= \sum_{k=1}^n[(\angle A_k+\angle B_k+\angle C_k)-\pi]\\ &= 2\pi+\sum_{k=1}^n\alpha_k-n\pi\\ &= 2\pi-\sum_{k=1}^n(\pi-\alpha_k).\end{aligned}$$

这里内角$\alpha_k\in(0,2\pi)$，$\beta_k:=\pi-\alpha_k\in(-\pi,\pi)$为顶点处的外角。计算角度时约定沿$\partial D$，$D$在左手边。

设曲面的高斯曲率$K\leq 0$，则不存在两条测地线相交于两点且它们围成一个单连通区域D。事实上若如此，记相交两点处的内角为$\alpha_k,k=1,2$，则$0<\alpha_k<\pi$，从而

$$0\geq\int_D KdA=2\pi-\sum_{k=1}^2(\pi-\alpha_k)>2\pi-(\pi+\pi)=0,$$

矛盾。

设曲面上三角形一边$\alpha(s)$为一般弧长参数曲线，β,γ为测地线，则有相应结果

定理0.2. *(Gauss-Bonnet, 1827)* 设曲面上三角形三顶点分别为A,B,C，对应内角分别为$\angle A,\angle B,\angle C$，$B,C$所对的边$\beta,\gamma$为测地线，$A$所对曲线$\alpha(s)$为一般弧长参数曲线。则

$$\int_{\triangle ABC} KdA+\int_\alpha k_g ds=\angle A+\angle B+\angle C-\pi.$$

证明：由

$$\alpha'(s)=\frac{d\alpha(s)}{ds}=\frac{d\rho}{ds}r_\rho+r_\theta\frac{d\theta}{ds}=\cos\varphi(s)r_\rho+\sin\varphi(s)\frac{r_\theta}{\sqrt{G}}.$$

及Liouville公式，$\alpha(s)$的测地曲率

$$\begin{aligned}
k_g &= \langle\frac{d}{ds}(\cos\varphi(s)r_\rho+\sin\varphi(s)\frac{r_\theta}{\sqrt{G}}),-\sin\varphi r_\rho+\cos\varphi\frac{r_\theta}{\sqrt{G}}\rangle\\
&= \frac{d\varphi}{ds}+\langle\frac{d}{ds}r_\rho,\frac{r_\theta}{\sqrt{G}}\rangle\\
&= \frac{d\varphi}{ds}+\langle\frac{d\rho}{ds}r_{\rho\rho}+\frac{d\theta}{ds}r_{\rho\theta},\frac{r_\theta}{\sqrt{G}}\rangle\\
&= \frac{d\varphi}{ds}+\langle\frac{d\theta}{ds}r_{\rho\theta},\frac{r_\theta}{\sqrt{G}}\rangle\\
&= \frac{d\varphi}{ds}+\frac{\sin\varphi}{G}\frac{1}{2}\frac{\partial}{\partial\rho}(|r_\theta|^2)\\
&= \frac{d\varphi}{ds}+\frac{1}{2}\frac{\partial\ln G}{\partial\rho}\sin\varphi.
\end{aligned}$$

因此

$$\frac{d\varphi}{ds}=k_g-\frac{1}{2}\frac{\partial\ln G}{\partial\rho}\sqrt{G}\frac{d\theta}{ds}=k_g-(\sqrt{G})_\rho\frac{d\theta}{ds}.$$

从而

$$\begin{aligned}
\int_{\triangle ABC}KdA &= \angle A-\int_0^{\angle A}(\sqrt{G})_\rho(\rho(\theta),\theta)d\theta\\
&= \angle A+\int_B^C(d\varphi-k_g ds)\\
&= \angle A+\angle C-(\pi-\angle B)-\int_\alpha k_g ds.
\end{aligned}$$

□

设曲面上一单连通区域D，其边界由分段光滑曲线$C_1,\cdots,C_n$组成，连接顶点处各内角分别为$\alpha_1,\cdots,\alpha_n$，存在其内部一点出发的测地射线将区域剖分为如上述

的三角形。则此时有Gauss-Bonnet公式

$$\begin{aligned}\int_D KdA+\int_{\partial D}k_g ds &= 2\pi+\sum_{i=1}^n \alpha_i - n\pi\\ &= \sum_{i=1}^n \alpha_i-(n-2)\pi = 2\pi-\sum_{i-1}^n(\pi-\alpha_i).\end{aligned}$$

特别，如果单连通区域D的边界为一条光滑曲线时，做类似分割可得

$$\int_D KdA+\int_{\partial D}k_g ds = 2\pi+\sum_{i=1}^n \alpha_i - n\pi = 2\pi.$$