§0.1 曲面的结构方程(外微分法)

§0.1.1 曲面的结构方程

设$\{r; e_1, e_2, e_3\}$为曲面的正交活动标架。则它的运动方程(一次微分形式向量的等式)为

$$\begin{aligned} dr &= \omega^\alpha e_\alpha, \\ de_i &= \omega_i^j e_j, \quad \omega_i^j + \omega_j^i = 0. \end{aligned}$$

对运动方程第一式求外微分(各一次微分形式分量分别求外微分)

$$\begin{aligned} 0 &= d(dr) = d(\omega^\alpha e_\alpha) = d(\omega^\alpha e_\alpha^1, \omega^\alpha e_\alpha^2, \omega^\alpha e_\alpha^3) \\ &= (d\omega^\alpha e_\alpha^1 - \omega^\alpha \wedge de_\alpha^1, d\omega^\alpha e_\alpha^2 - \omega^\alpha \wedge de_\alpha^2, d\omega^\alpha e_\alpha^3 - \omega^\alpha \wedge de_\alpha^3) \\ &= d\omega^\alpha e_\alpha - \omega^\alpha \wedge de_\alpha, \end{aligned}$$

因此利用运动方程第二式得到二次外微分形式向量的等式

$$\begin{aligned} 0 &= d(dr) = d(\omega^\alpha e_\alpha) \\ &= d\omega^\alpha e_\alpha - \omega^\alpha \wedge de_\alpha \\ &= d\omega^\alpha e_\alpha - \omega^\alpha \wedge \omega_\alpha^j e_j \\ &= (d\omega^\beta - \omega^\alpha \wedge \omega_\alpha^\beta) e_\beta - \omega^\alpha \wedge \omega_\alpha^3 e_3. \end{aligned}$$

从而有

$$d\omega^\beta - \omega^\alpha \wedge \omega_\alpha^\beta = 0, \quad \beta = 1, 2; \quad (1)$$

$$\omega^\alpha \wedge \omega_\alpha^3 = 0. \quad (2)$$

以$\omega_\alpha^3 = h_{\alpha\beta}\omega^\beta$代入(2)得

$$\omega^\alpha \wedge \omega_\alpha^3 = h_{\alpha\beta}\omega^\alpha \wedge \omega^\beta = (h_{12} - h_{21})\omega^1 \wedge \omega^2,$$

因此

$$(2) \Leftrightarrow h_{12} = h_{21},$$

即$B = (h_{\alpha\beta})$为对称方阵。

(1)式即

$$d\omega^1 - \omega^2 \wedge \omega_2^1 = 0, \quad d\omega^2 - \omega^1 \wedge \omega_1^2 = 0. \quad (1)$$

注1：由(1)式可知，$\omega_2^1 = -\omega_1^2$由$\{\omega^1, \omega^2\}$唯一确定(事实上，(1)中两式分别确定$\omega_2^1 = -\omega_1^2$的ω^1, ω^2分量)。

注2：(1)等价于$\mathbb{R}^3$中曲面的如下协变导数

$$\nabla_X e_\alpha := (Xe_\alpha)^T,$$

(其中$X \in TD$，Xe_α为e_α沿X的方向导数，T表示到TS的投影)的挠率为零。令$X_\alpha = (dr)^{-1}e_\alpha$，计算

$$\begin{aligned}
& (d\omega^1 - \omega^2 \wedge \omega_2^1)(X_1, X_2) \\
= \ & -\omega^1([X_1, X_2]) + \omega_2^1(X_1) \\
= \ & -\langle dr([X_1, X_2]), e_1\rangle + \langle de_2(X_1), e_1\rangle \\
= \ & \langle \nabla_{X_1} e_2 - dr([X_1, X_2]), e_1\rangle \\
= \ & \langle \nabla_{X_1} e_2 - \nabla_{X_2} e_1 - dr([X_1, X_2]), e_1\rangle
\end{aligned}$$

$$\begin{aligned}
& (d\omega^2 - \omega^1 \wedge \omega_1^2)(X_1, X_2) \\
= \ & -\omega^2([X_1, X_2]) - \omega_1^2(X_2) \\
= \ & -\langle dr([X_1, X_2]), e_2\rangle - \langle de_1(X_2), e_2\rangle \\
= \ & \langle -\nabla_{X_2} e_1 - dr([X_1, X_2]), e_2\rangle \\
= \ & \langle \nabla_{X_1} e_2 - \nabla_{X_2} e_1 - dr([X_1, X_2]), e_2\rangle.
\end{aligned}$$

因此(1)中两式分别等价于

$$\langle \nabla_{X_1} e_2 - \nabla_{X_2} e_1 - dr([X_1, X_2]), e_\alpha\rangle = 0, \quad \alpha = 1, 2.$$

注意到$\langle \nabla_{X_1} e_2 - \nabla_{X_2} e_1 - dr([X_1, X_2])$是曲面的切向量场，因此(1)等价于

$$\nabla_{X_1} e_2 - \nabla_{X_2} e_1 - dr([X_1, X_2]) = 0.$$

如果等同$X_\alpha = e_\alpha$，即

$$T(e_1, e_2) := \nabla_{e_1} e_2 - \nabla_{e_2} e_1 - [e_1, e_2] = 0,$$

这里$T(e_1, e_2)$为曲面协变导数∇的挠率定义。因此(1)等价于此挠率为零。也等价于自然标架下$\Gamma_{\alpha\beta}^\gamma = \Gamma_{\beta\alpha}^\gamma$，即

$$\nabla_{\frac{\partial}{\partial u^\alpha}} r_\beta = \nabla_{\frac{\partial}{\partial u^\beta}} r_\alpha.$$

对运动方程第二式

$$de_i = \omega_i^j e_j$$

求外微分(先考虑$i = \alpha = 1, 2$)得

$$\begin{aligned} 0 &= d(de_\alpha) = d(\omega_\alpha^j e_j) = d\omega_\alpha^j e_j - \omega_\alpha^j \wedge de_j \\ &= d\omega_\alpha^j e_j - \omega_\alpha^j \wedge \omega_j^k e_k \\ &= (d\omega_\alpha^k - \omega_\alpha^j \wedge \omega_j^k) e_k, \end{aligned}$$

即

$$d\omega_\alpha^k - \omega_\alpha^j \wedge \omega_j^k = 0. \quad (GC)$$

对于$i = 3$同样有

$$\begin{aligned} 0 &= d(de_3) = d(\omega_3^j e_j) = (d\omega_3^k - \omega_3^j \wedge \omega_j^k) e_k \\ &= (-d\omega_k^3 - \omega_j^3 \wedge \omega_k^j) e_k \\ &= (-d\omega_\alpha^3 - \omega_j^3 \wedge \omega_\alpha^j) e_\alpha, \end{aligned}$$

即

$$d\omega_\alpha^3 - \omega_\alpha^j \wedge \omega_j^3 = 0, \quad (C)$$

它包含在上述(GC)中，即(GC)中$k = 3$。

方程

$$d\omega_\alpha^k - \omega_\alpha^j \wedge \omega_j^k = 0 \quad (GC)$$

中所包含的独立方程为

$$d\omega_1^2 - \omega_1^3 \wedge \omega_3^2 = 0, \quad (G)$$

以及

$$d\omega_1^3 - \omega_1^2 \wedge \omega_2^3 = 0, \quad (C1)$$

$$d\omega_2^3 - \omega_2^1 \wedge \omega_1^3 = 0. \quad (C2)$$

(1)和(GC)统称为曲面正交标架的结构方程。其中(GC)包括三个独立方程(G), (C1), (C2)。它们都是二次外微分形式的等式。接下来考察它们和Gauss方程、Codazzi方程的对应。首先化简方程(G)右边：

$$\begin{aligned} d\omega_1^2 &= \omega_1^3 \wedge \omega_3^2 = -\omega_1^3 \wedge \omega_2^3 \\ &= -(h_{11}\omega^1 + h_{12}\omega^2) \wedge (h_{21}\omega^1 + h_{22}\omega^2) \\ &= -(h_{11}h_{22} - (h_{12})^2)\omega^1 \wedge \omega^2 \\ &= -K\omega^1 \wedge \omega^2 = -KdA. \end{aligned}$$

特别有Gauss绝妙定理的外微分形式版本

$$K=\frac{d\omega_2^1}{\omega^1\wedge\omega^2}.$$

另一方面，对参数空间D上任意向量场X

$$\omega_1^2(X)=\langle de_1,e_2\rangle(X)=\langle Xe_1,e_2\rangle=\langle\nabla_X e_1,e_2\rangle.$$

令$X_\alpha=(dr)^{-1}(e_\alpha)$，则

$$\begin{aligned}(d\omega_1^2)(X_1,X_2)&=X_1\omega_1^2(X_2)-X_2\omega_1^2(X_1)-\omega_1^2([X_1,X_2])\\&=X_1\langle\nabla_{X_2}e_1,e_2\rangle-X_2\langle\nabla_{X_1}e_1,e_2\rangle-\langle\nabla_{[X_1,X_2]}e_1,e_2\rangle.\end{aligned}$$

注意到$\nabla_{X_2}e_1$与e_2共线，因此

$$X_1\langle\nabla_{X_2}e_1,e_2\rangle=\langle\nabla_{X_1}(\nabla_{X_2}e_1),e_2\rangle+\langle\nabla_{X_2}e_1,\nabla_{X_1}e_2\rangle=\langle\nabla_{X_1}(\nabla_{X_2}e_1),e_2\rangle,$$

从而

$$\begin{aligned}(d\omega_1^2)(X_1,X_2)&=X_1\langle\nabla_{X_2}e_1,e_2\rangle-X_2\langle\nabla_{X_1}e_1,e_2\rangle-\langle\nabla_{[X_1,X_2]}e_1,e_2\rangle\\&=\langle\nabla_{X_1}(\nabla_{X_2}e_1),e_2\rangle-\langle\nabla_{X_2}(\nabla_{X_1}e_1),e_2\rangle-\langle\nabla_{[X_1,X_2]}e_1,e_2\rangle\\&=\langle\nabla_{X_1}\nabla_{X_2}e_1-\nabla_{X_2}\nabla_{X_1}e_1-\nabla_{[X_1,X_2]}e_1,e_2\rangle.\end{aligned}$$

将X_α与e_α等同，并采用黎曼曲率张量记号表示上面最后一行得到

$$(d\omega_1^2)(X_1,X_2)=R(e_1,e_2,e_2,e_1)=-R(e_1,e_2,e_1,e_2).$$

这里黎曼曲率同样仅依赖于第一基本形式。由上述讨论，

$$(G)\Leftrightarrow R(e_1,e_2,e_1,e_2)=K:=\det(B).$$

特别

$$d\omega_2^1=K\omega^1\wedge\omega^2=R(e_1,e_2,e_1,e_2)\omega^1\wedge\omega^2.$$

注：一般维数空间，正交标架下(黎曼)曲率(二次外微分)形式为

$$\Omega_\alpha^\beta:=d\omega_\alpha^\beta-\omega_\alpha^\gamma\wedge\omega_\gamma^\beta.$$

接下来看

$$d\omega_1^3-\omega_1^2\wedge\omega_2^3=0,\quad (C1)$$

$$d\omega_2^3-\omega_2^1\wedge\omega_1^3=0.\quad (C2)$$

利用

$$d\omega^\beta - \omega^\alpha \wedge \omega_\alpha^\beta = 0, \quad \beta = 1, 2 \quad (1)$$

(C1)等价于

$$\begin{aligned}
0 &= (d\omega_1^3 - \omega_1^2 \wedge \omega_2^3)(X_1, X_2) \\
&= d(h_{11}\omega^1 + h_{12}\omega^2)(X_1, X_2) + [(h_{21}\omega^1 + h_{22}\omega^2) \wedge \omega_1^2](X_1, X_2) \\
&= (dh_{11} \wedge \omega^1 + dh_{12} \wedge \omega^2)(X_1, X_2) + (h_{11}d\omega^1 + h_{12}d\omega^2)(X_1, X_2) \\
&\quad + [(h_{21}\omega^1 + h_{22}\omega^2) \wedge \omega_1^2](X_1, X_2) \\
&= -X_2(h_{11}) + X_1(h_{21}) + (h_{11}\omega^2 \wedge \omega_2^1 + h_{12}\omega^1 \wedge \omega_1^2)(X_1, X_2) \\
&\quad + [(h_{21}\omega^1 + h_{22}\omega^2) \wedge \omega_1^2](X_1, X_2) \\
&= X_1(h_{21}) - X_2(h_{11}) - \omega_2^1(X_1)h_{11} - \omega_1^2(X_1)h_{22} + 2\omega_1^2(X_2)h_{12},
\end{aligned}$$

类似有(C2)等价于

$$\begin{aligned}
0 &= (d\omega_2^3 - \omega_2^1 \wedge \omega_1^3)(X_1, X_2) \\
&= (dh_{21} \wedge \omega^1 + dh_{22} \wedge \omega^2)(X_1, X_2) + (h_{21}d\omega^1 + h_{22}d\omega^2 + \omega_1^3 \wedge \omega_2^1)(X_1, X_2) \\
&= X_1(h_{22}) - X_2(h_{12}) + \omega_2^1(X_2)h_{11} + \omega_1^2(X_2)h_{22} - 2\omega_2^1(X_1)h_{12}.
\end{aligned}$$

可见(C1)(C2)与自然标架下的Codazzi方程具有相同的本质，即给出$X_\alpha(h_{\beta\gamma}) - X_\beta(h_{\alpha\gamma})$的表达式。称(C1)(C2)为正交标架下的Codazzi方程。