§0.1 曲面的结构方程

利用自然标架运动方程进一步研究第一、第二基本形式系数之间的关系。关系式(Gauss方程与Codazzi方程)的导出基于一个朴素的出发点：光滑函数的二阶偏导数可交换求导次序。

回顾曲面自然标架$(r; r_1, r_2, N)$的运动方程

$$\begin{cases} \frac{\partial r}{\partial u^\alpha} = r_\alpha, & \alpha = 1,2; \quad (M_1) \\ \frac{\partial r_\alpha}{\partial u^\beta} = \Gamma^\gamma_{\beta\alpha} r_\gamma + b_{\alpha\beta} N, & \alpha,\beta = 1,2; \quad (M_2) \\ \frac{\partial N}{\partial u^\alpha} = -b^\beta_\alpha r_\beta, & \alpha = 1,2 \quad (M_3) \end{cases}$$

其中

$$g_{\alpha\beta} = \langle r_\alpha, r_\beta \rangle, \quad \alpha,\beta = 1,2;$$

$$b_{\alpha\beta} = \langle r_{\alpha\beta}, N \rangle = -\langle r_\alpha, N_\beta \rangle, \quad \alpha,\beta = 1,2;$$

$$b^\beta_\alpha = b_{\alpha\gamma} g^{\gamma\beta} = g^{\beta\gamma} b_{\gamma\alpha}, \quad \alpha,\beta = 1,2;$$

$$\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha} = \frac{1}{2} g^{\gamma\xi} \left(\frac{\partial g_{\beta\xi}}{\partial u^\alpha} + \frac{\partial g_{\alpha\xi}}{\partial u^\beta} - \frac{\partial g_{\alpha\beta}}{\partial u^\xi} \right), \quad \alpha,\beta,\gamma = 1,2.$$

(M_1)再求一次偏导数，即(M_2)，可交换次序得到

$$\Gamma^\gamma_{\beta\alpha} = \Gamma^\gamma_{\alpha\beta}, \quad b_{\alpha\beta} = b_{\beta\alpha}.$$

接下来对$(M_2), (M_3)$再求一次偏导数，并交换次序。

先看(M_3)：对(M_3)求一次偏导数并利用运动方程可得

$$\begin{aligned} N_{\alpha\gamma} &= \frac{\partial}{\partial u^\gamma} \frac{\partial N}{\partial u^\alpha} = \frac{\partial}{\partial u^\gamma} (-b^\beta_\alpha r_\beta) \\ &= -\frac{\partial b^\beta_\alpha}{\partial u^\gamma} r_\beta - b^\beta_\alpha (\Gamma^\xi_{\gamma\beta} r_\xi + b_{\beta\gamma} N). \end{aligned}$$

同样有

$$\begin{aligned} N_{\gamma\alpha} &= \frac{\partial}{\partial u^\alpha} \frac{\partial N}{\partial u^\gamma} = \frac{\partial}{\partial u^\alpha} (-b^\beta_\gamma r_\beta) \\ &= -\frac{\partial b^\beta_\gamma}{\partial u^\alpha} r_\beta - b^\beta_\gamma (\Gamma^\xi_{\alpha\beta} r_\xi + b_{\beta\alpha} N). \end{aligned}$$

其中法向分量的相等已知，即

$$-b^\beta_\alpha b_{\beta\gamma} = -b_{\alpha\eta} g^{\eta\beta} b_{\beta\gamma} = -b_{\alpha\eta} b^\eta_\gamma = -b_{\alpha\beta} b^\beta_\gamma.$$

由切向分量相等可得Codazzi方程

$$\frac{\partial b^{\xi}_{\alpha}}{\partial u^{\gamma}}+\Gamma^{\xi}_{\gamma\beta}b^{\beta}_{\alpha}=\frac{\partial b^{\xi}_{\gamma}}{\partial u^{\alpha}}+\Gamma^{\xi}_{\alpha\beta}b^{\beta}_{\gamma}.\quad \text{(Codazzi')}$$

对(M_2)求一次偏导数即坐标切向量的两阶方向导数。为分清其中只与第一基本形式有关的部分，引入如下记号：一个向量$Y\in T_P\mathbb{R}^3$在切平面和法线上的投影分别记为$Y^T,Y^{\perp}$，即

$$Y^T=Y-\langle Y,N\rangle N,\quad Y^{\perp}=\langle Y,N\rangle N.$$

设$X=X^{\beta}(u,v)r_{\beta}$为切向量场，定义

$$\nabla_{\frac{\partial}{\partial u^{\alpha}}}X=(\frac{\partial X}{\partial u^{\alpha}})^T:=\frac{\partial X}{\partial u^{\alpha}}-\langle\frac{\partial X}{\partial u^{\alpha}},N\rangle N.$$

特别有

$$\nabla_{\frac{\partial}{\partial u^{\beta}}}r_{\alpha}=(\frac{\partial r_{\alpha}}{\partial u^{\beta}})^T=\Gamma^{\xi}_{\beta\alpha}r_{\xi},$$

$$\begin{aligned}\nabla_{\frac{\partial}{\partial u^{\gamma}}}(\nabla_{\frac{\partial}{\partial u^{\beta}}}r_{\alpha}) &= [\frac{\partial}{\partial u^{\gamma}}(\Gamma^{\xi}_{\beta\alpha}r_{\xi})]^T=[\frac{\partial\Gamma^{\xi}_{\beta\alpha}}{\partial u^{\gamma}}r_{\xi}+\Gamma^{\eta}_{\beta\alpha}\frac{\partial r_{\eta}}{\partial u^{\gamma}}]^T\\ &= (\frac{\partial\Gamma^{\xi}_{\beta\alpha}}{\partial u^{\gamma}}+\Gamma^{\xi}_{\gamma\eta}\Gamma^{\eta}_{\beta\alpha})r_{\xi}.\end{aligned}$$

而

$$[\frac{\partial}{\partial u^{\gamma}}(\Gamma^{\xi}_{\beta\alpha}r_{\xi})]^{\perp}=[\frac{\partial\Gamma^{\xi}_{\beta\alpha}}{\partial u^{\gamma}}r_{\xi}+\Gamma^{\eta}_{\beta\alpha}\frac{\partial r_{\eta}}{\partial u^{\gamma}}]^{\perp}=\Gamma^{\eta}_{\beta\alpha}b_{\eta\gamma}N.$$

由(M_2)计算

$$\begin{aligned}\frac{\partial}{\partial u^{\gamma}}(\frac{\partial r_{\alpha}}{\partial u^{\beta}}) &= \frac{\partial}{\partial u^{\gamma}}(\Gamma^{\xi}_{\beta\alpha}r_{\xi}+b_{\alpha\beta}N)\\ &= [\frac{\partial}{\partial u^{\gamma}}(\Gamma^{\xi}_{\beta\alpha}r_{\xi})]^T+[\frac{\partial}{\partial u^{\gamma}}(\Gamma^{\xi}_{\beta\alpha}r_{\xi})]^{\perp}+\frac{\partial b_{\alpha\beta}}{\partial u^{\gamma}}N+b_{\alpha\beta}N_{\gamma}\\ &= \nabla_{\frac{\partial}{\partial u^{\gamma}}}(\nabla_{\frac{\partial}{\partial u^{\beta}}}r_{\alpha})+\Gamma^{\eta}_{\beta\alpha}b_{\eta\gamma}N+\frac{\partial b_{\alpha\beta}}{\partial u^{\gamma}}N+b_{\alpha\beta}(-b^{\xi}_{\gamma}r_{\xi})\\ &= [\nabla_{\frac{\partial}{\partial u^{\gamma}}}(\nabla_{\frac{\partial}{\partial u^{\beta}}}r_{\alpha})-b_{\alpha\beta}b^{\xi}_{\gamma}r_{\xi}]+(\frac{\partial b_{\beta\alpha}}{\partial u^{\gamma}}+\Gamma^{\xi}_{\beta\alpha}b_{\xi\gamma})N.\end{aligned}$$

交换β,γ的次序得

$$\frac{\partial}{\partial u^{\beta}}(\frac{\partial r_{\alpha}}{\partial u^{\gamma}}) = [\nabla_{\frac{\partial}{\partial u^{\beta}}}(\nabla_{\frac{\partial}{\partial u^{\gamma}}}r_{\alpha})-b_{\alpha\gamma}b^{\xi}_{\beta}r_{\xi}]+(\frac{\partial b_{\gamma\alpha}}{\partial u^{\beta}}+\Gamma^{\xi}_{\gamma\alpha}b_{\xi\beta})N.$$

由它们相等得到

$$\begin{aligned}
&\nabla_{\frac{\partial}{\partial u^\gamma}}(\nabla_{\frac{\partial}{\partial u^\beta}} r_\alpha) - \nabla_{\frac{\partial}{\partial u^\beta}}(\nabla_{\frac{\partial}{\partial u^\gamma}} r_\alpha) \\
= &(\frac{\partial \Gamma^\xi_{\beta\alpha}}{\partial u^\gamma} + \Gamma^\xi_{\gamma\eta}\Gamma^\eta_{\beta\alpha}) r_\xi - (\frac{\partial \Gamma^\xi_{\gamma\alpha}}{\partial u^\beta} + \Gamma^\xi_{\beta\eta}\Gamma^\eta_{\gamma\alpha}) r_\xi \\
= &b_{\alpha\beta} b^\xi_\gamma r_\xi - b_{\alpha\gamma} b^\xi_\beta r_\xi,
\end{aligned}$$

以及

$$\frac{\partial b_{\beta\alpha}}{\partial u^\gamma} + \Gamma^\xi_{\beta\alpha} b_{\xi\gamma} = \frac{\partial b_{\gamma\alpha}}{\partial u^\beta} + \Gamma^\xi_{\gamma\alpha} b_{\xi\beta}.$$

分别称为Gauss方程

$$\frac{\partial \Gamma^\xi_{\beta\alpha}}{\partial u^\gamma} - \frac{\partial \Gamma^\xi_{\gamma\alpha}}{\partial u^\beta} + \Gamma^\xi_{\gamma\eta}\Gamma^\eta_{\beta\alpha} - \Gamma^\xi_{\beta\eta}\Gamma^\eta_{\gamma\alpha} = b^\xi_\gamma b_{\beta\alpha} - b^\xi_\beta b_{\gamma\alpha}; \quad (Gauss)$$

与Codazzi方程(稍后验证它与(Codazzi')等价)

$$\frac{\partial b_{\beta\alpha}}{\partial u^\gamma} - \Gamma^\xi_{\gamma\alpha} b_{\xi\beta} = \frac{\partial b_{\gamma\alpha}}{\partial u^\beta} - \Gamma^\xi_{\beta\alpha} b_{\xi\gamma}. \quad \text{(Codazzi)}$$

合起来称为曲面的Gauss-Codazzi方程，或称为曲面的结构方程。

定理0.1. *曲面结构方程包括*

$$\frac{\partial \Gamma^\xi_{\beta\alpha}}{\partial u^\gamma} - \frac{\partial \Gamma^\xi_{\gamma\alpha}}{\partial u^\beta} + \Gamma^\xi_{\gamma\eta}\Gamma^\eta_{\beta\alpha} - \Gamma^\xi_{\beta\eta}\Gamma^\eta_{\gamma\alpha} = b^\xi_\gamma b_{\beta\alpha} - b^\xi_\beta b_{\gamma\alpha}; \quad (Gauss)$$

和

$$\frac{\partial b_{\beta\alpha}}{\partial u^\gamma} - \Gamma^\xi_{\gamma\alpha} b_{\xi\beta} = \frac{\partial b_{\gamma\alpha}}{\partial u^\beta} - \Gamma^\xi_{\beta\alpha} b_{\xi\gamma}. \quad (Codazzi)$$

Codazzi方程的两种形式

$$\frac{\partial b_{\beta\alpha}}{\partial u^\gamma} - \Gamma^\xi_{\gamma\alpha} b_{\xi\beta} = \frac{\partial b_{\gamma\alpha}}{\partial u^\beta} - \Gamma^\xi_{\beta\alpha} b_{\xi\gamma}. \quad \text{(Codazzi)}$$

$$\frac{\partial b^\xi_\alpha}{\partial u^\gamma} + \Gamma^\xi_{\gamma\beta} b^\beta_\alpha = \frac{\partial b^\xi_\gamma}{\partial u^\alpha} + \Gamma^\xi_{\alpha\beta} b^\beta_\gamma. \quad \text{(Codazzi')}$$

将验证这两种形式等价。为方便与(Codazzi)比较，将(Codazzi')改写为

$$\frac{\partial b^\eta_\beta}{\partial u^\gamma} + \Gamma^\eta_{\gamma p} b^p_\beta = \frac{\partial b^\eta_\gamma}{\partial u^\beta} + \Gamma^\eta_{\beta p} b^p_\gamma. \quad \text{(Codazzi')}$$

从而等价性由下面论断直接得到：

$$g^{\alpha\eta}(\frac{\partial b_{\beta\alpha}}{\partial u^\gamma} - \Gamma^\xi_{\gamma\alpha} b_{\xi\beta}) = \frac{\partial b^\eta_\beta}{\partial u^\gamma} + \Gamma^\eta_{\gamma p} b^p_\beta. \quad (1)$$

先证明一个常用关系式：

$$\frac{\partial g^{\alpha\beta}}{\partial u^\gamma} = -g^{\alpha p} g^{\beta q} \frac{\partial g_{pq}}{\partial u^\gamma}.$$

证明：

$$\frac{\partial (g^{\alpha p} g_{pq})}{\partial u^\gamma} = 0 = \frac{\partial g^{\alpha p}}{\partial u^\gamma} g_{pq} + g^{\alpha p} \frac{\partial g_{pq}}{\partial u^\gamma}$$

乘以$g^{\beta q}$并对q求和可得上述关系式。 □

证明(1)式：

$$\begin{aligned}
g^{\alpha\eta}(\frac{\partial b_{\beta\alpha}}{\partial u^\gamma} - \Gamma^\xi_{\gamma\alpha} b_{\xi\beta}) &= \frac{\partial b^\eta_\beta}{\partial u^\gamma} - \frac{\partial g^{\alpha\eta}}{\partial u^\gamma} b_{\beta\alpha} - g^{\alpha\eta} \frac{1}{2} g^{\xi p} (\frac{\partial g_{\alpha p}}{\partial u^\gamma} + \frac{\partial g_{\gamma p}}{\partial u^\alpha} - \frac{\partial g_{\alpha\gamma}}{\partial u^p}) b_{\xi\beta} \\
&= \frac{\partial b^\eta_\beta}{\partial u^\gamma} + g^{\alpha p} g^{\eta q} \frac{\partial g_{pq}}{\partial u^\gamma} b_{\beta\alpha} - g^{\alpha\eta} \frac{1}{2} (\frac{\partial g_{\alpha p}}{\partial u^\gamma} + \frac{\partial g_{\gamma p}}{\partial u^\alpha} - \frac{\partial g_{\alpha\gamma}}{\partial u^p}) b^p_\beta \\
&= \frac{\partial b^\eta_\beta}{\partial u^\gamma} + b^p_\beta g^{\alpha\eta} [\frac{\partial g_{p\alpha}}{\partial u^\gamma} - \frac{1}{2} (\frac{\partial g_{\alpha p}}{\partial u^\gamma} + \frac{\partial g_{\gamma p}}{\partial u^\alpha} - \frac{\partial g_{\alpha\gamma}}{\partial u^p})] \quad \text{by } q \to \alpha \\
&= \frac{\partial b^\eta_\beta}{\partial u^\gamma} + b^p_\beta \frac{1}{2} g^{\alpha\eta} [\frac{\partial g_{\alpha p}}{\partial u^\gamma} + \frac{\partial g_{\gamma\alpha}}{\partial u^p} - \frac{\partial g_{\gamma p}}{\partial u^\alpha}] \\
&= \frac{\partial b^\eta_\beta}{\partial u^\gamma} + \Gamma^\eta_{\gamma p} b^p_\beta.
\end{aligned}$$

□

Codazzi方程

$$\frac{\partial b_{\beta\alpha}}{\partial u^\gamma} - \Gamma^\xi_{\gamma\alpha} b_{\xi\beta} = \frac{\partial b_{\gamma\alpha}}{\partial u^\beta} - \Gamma^\xi_{\beta\alpha} b_{\xi\gamma} \quad \text{(Codazzi)}$$

中令$\gamma = 1, \beta = 2$，分别取$\alpha = 1, 2$得到两个独立方程：

$$\frac{\partial b_{21}}{\partial u^1} - \Gamma^\xi_{11} b_{2\xi} = \frac{\partial b_{11}}{\partial u^2} - \Gamma^\xi_{21} b_{1\xi},$$

$$\frac{\partial b_{22}}{\partial u^1} - \Gamma^\xi_{12} b_{2\xi} = \frac{\partial b_{12}}{\partial u^2} - \Gamma^\xi_{22} b_{1\xi}.$$

Codazzi方程给出第二基本形式一阶导数之间的（交换）关系。

例：如果参数(u, v)使得r_u, r_v为互相正交的主方向，即其积分曲线为曲率线。此时$F = M = 0$，即$g_{12} = b_{12} = 0$，由此可化简Codazzi方程。注：非脐点P，存在它的邻域以及参数(u, v)使得u, v-曲线为曲率线[do Carmo, § 3.4]。

由

$$\frac{\partial b_{21}}{\partial u^1} - \Gamma^\xi_{11} b_{2\xi} = \frac{\partial b_{11}}{\partial u^2} - \Gamma^\xi_{21} b_{1\xi},$$

$$(g^{\alpha\beta}) = \frac{1}{\det(g_{\alpha\beta})} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$

$$H=\frac{1}{2}tr(W)=\frac{1}{2}(k_1+k_2)=\frac{1}{2}\frac{LG-2MF+NE}{EG-F^2},$$

可知

$$\begin{aligned}\frac{\partial b_{11}}{\partial u^2} &= L_v=\Gamma_{21}^{\xi}b_{1\xi}-\Gamma_{11}^{\xi}b_{2\xi}=\Gamma_{21}^{1}L-\Gamma_{11}^{2}N\\ &= \frac{1}{2}Lg^{11}E_v-\frac{1}{2}Ng^{22}(-E_v)\\ &= \frac{1}{2}E_v(L\frac{G}{EG}+N\frac{E}{EG})\\ &= \frac{1}{2}(\frac{L}{E}+\frac{N}{G})E_v\\ &= HE_v,\end{aligned}$$

类似也有

$$N_u=HG_u.$$

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