一. 复习

1.第4章 连续型r.v.

① $X: \Omega \to R$ $F_X(x) = p(X \le x) = \int_{-\infty}^{x} f(t)dt$, $E[X] = \int_{-\infty}^{+\infty} x f(x)dx$, $var(X) = E[X^2] - E[X]^2$

②引理 X非负. $E[X] = \int_{-\infty}^{+\infty} p(X > x)dx = \int_{0}^{+\infty} (1-F(x))dx$

练习: X非负 4.14.3

$\sum_{n=1}^{+\infty} p(X \ge n) \le E[X] \le \sum_{n=1}^{+\infty} p(X \ge n) + 1$

22年期末第1题

一般. $E[X] = \int_{0}^{+\infty} (1-F(x))dx - \int_{0}^{+\infty} F(-x)dx$

★③常见分布:

(I) $X \sim U([a,b])$ $f(x) = \begin{cases} \frac{1}{b-a}, & x \in [a,b] \\ 0, & x \notin [a,b] \end{cases}$

(II) $X \sim Exp(\lambda)$ $f(x) = \begin{cases} \lambda e^{-x}, & x > 0 \\ 0, & x \le 0 \end{cases}$ $p(X > s+t \mid X > t) = p(X > s)$, $t,s > 0$

(III) $X \sim N(\mu, \sigma^2)$ $\to \frac{X-\mu}{\sigma} \sim N(0,1)$ $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $E[X] = \mu$, $Var(X) = \sigma^2$

当 $\mu=0$, $\sigma=1$ 时, $N(0,1)$ 标准正态分布 $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

(IV) Γ分布. Beta分布. cauchy分布.

④连续型随机向量(X,Y)

$F(x,y) = p(X \le x, Y \le y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)dudv$ $p(X \le x) = \int_{0}^{x}(\int_{-\infty}^{+\infty} f(u,v)dv)du$

$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)dy$. $f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)dx$

$E[X] = \iint_{R^2} x f(x,y)dxdy$, $var(X) = \iint_{R^2} (x - E[X])^2 f(x,y)dxdy$,

$cov(X,Y) = E[XY] - E[X]E[Y]$

条件密度 $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$, $f_Y(y) > 0$; 条件期望 $\psi(x) = E[Y|X=x] = \int_{-\infty}^{+\infty} y f_{Y|X}(y|x)dy$

$\psi(X) = E[Y|X]$

⑤ (X_1,X_2) 联合密度 $f(x_1,x_2)$. $Y_1 = g_1(X_1,X_2)$, $Y_2 = g_2(X_1,X_2)$ 有连续偏导.

满足 (1) $\begin{cases} y_1 = g_1(x_1,x_2) \\ y_2 = g_2(x_1,x_2) \end{cases}$ 有逆映射 $\begin{cases} x_1 = h_1(y_1,y_2) \\ x_2 = h_2(y_1,y_2) \end{cases}$ (2) $J = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix} \neq 0$

则 (Y_1,Y_2) 有联合密度 $f_Y(y_1,y_2) = f(h_1(y_1,y_2), h_2(y_1,y_2)) \cdot |J|^{-1}$

⑥ $N(0,1;0,1;\rho)$ $f(x,y)=\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)\right) \Rightarrow f_X(x),\ f_Y(y),\ \mathrm{cov}$

一般 $N(\mu_1,\sigma_1^2;\mu_2,\sigma_2^2;\rho)$ $f(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}Q(x,y)\right)$

$Q(x,y)=\frac{(x-\mu_1)^2}{\sigma_1^2}-2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}+\frac{(y-\mu_2)^2}{\sigma_2^2} \Rightarrow f_X(x),\ f_Y(y),\ \mathrm{cov},\ f_{X|Y}(x|y)$

结论: (a) $E[X|Y]=\mu_1+\rho\sigma_1(Y-\mu_2)/\sigma_2$ (b) $\mathrm{Var}(X|Y)=\sigma_1^2(1-\rho^2)$ 21年期末

多元正态分布: $f(x_1,\cdots,x_n)=\frac{1}{\sqrt{(2\pi)^n\cdot|\Sigma|}}\exp\left(-\frac{1}{2}(\vec{x}-\vec{\mu})\Sigma^{-1}(\vec{x}-\vec{\mu})^T\right)$, Σ为协方差矩阵

$E[\vec{x}]=\vec{\mu}$ $\mathrm{cov}(X_i,X_j)=\Sigma_{ij}$

- $\vec{x}\sim N(\mu,\Sigma)$ $A_{n\times m}$, $n>m$ $\mathrm{rank}(A)=m$ $\vec{Y}=\vec{x}\cdot A\sim N(\vec{\mu}A,A^T\Sigma A)$
- $Y_1,\cdots,Y_n$相互独立, 服从 $N(0,1)$, 则 $X=\sum_{i=1}^n Y_i^2\sim\chi^2(n)$

定理 $X_1,\cdots,X_n\sim N(\mu,\sigma^2)$相互独立, 则 (1) $\bar{X}=\frac{1}{n}\sum_{i=1}^n X_i\sim N(\mu,\frac{\sigma^2}{n})$

(2) $\bar{X}$与$S^2=\frac{1}{n-1}\sum_{i=1}^n(X_i-\bar{X})^2$独立 (3) $S^2\cdot\frac{n-1}{\sigma^2}\sim\chi^2(n-1)$

2.第5章 特征函数

① $k>0$, $E(|X|^k)<+\infty$, 则对$\forall 0<r<k$, $E(|X|^r)<\infty$ 且$(E|X|^r)^{\frac{1}{r}}\le(E|X|^k)^{\frac{1}{k}}$

② $\varphi(t)=E[e^{itX}]=E[\cos(tX)+i\sin(tX)]=E[\cos(tX)]+iE[\sin(tX)]$

$\varphi(0)=1$, $|\varphi(t)|\le1$, $\varphi(-t)=\overline{\varphi(t)}$, $\varphi(t)$一致收敛.

若$E(|X|^k)<\infty$, 则$\varphi^{(j)}(0)=i^jE[X^j]$, $j\le k$.

X_1,X_2独立, 则$\varphi_{X_1+X_2}(t)=\varphi_{X_1}(t)\varphi_{X_2}(t)$

③ Bernoulli 分布/二项分布/指数分布/$N(0,1)$的特征函数.

④ 反转公式: X的分布函数$F(x)$,特征函数$\varphi(t)$.

则对$a<b$, $\frac{F(b)+F(b-0)}{2}-\frac{F(a)+F(a-0)}{2}=\lim_{T\to+\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-iat}-e^{-ibt}}{it}\varphi(t)dt$

分布函数可由特征函数唯一确定. 连续性定理给出了$\varphi(t)$收敛与$F(x)$收敛的关系.

⑤ 定理 $X_1,\cdots,X_n$ i.i.d. $E[X_i]=\mu$, $S_n=X_1+\cdots+X_n$, 则$\frac{S_n}{n}\xrightarrow{D}\mu$.

定理(CLT) $X_1,\cdots,X_n$ i.i.d. $E[X_i]=\mu$, $\mathrm{Var}(X_i)=\sigma^2$, $S_n=X_1+\cdots+X_n$

则$\frac{S_n-n\mu}{\sqrt{n\sigma^2}}\xrightarrow{D}Y$, $Y\sim N(0,1)$

二. 习题

1. 设 X 是连续型 r.v.，其密度函数为 $f(x)$，特征函数为 $\varphi(t)$. 若 $\int_{-\infty}^{+\infty}|\varphi(t)|dt<+\infty$，则 $f(x)=\int_{-\infty}^{+\infty}\frac{1}{2\pi}e^{-itx}\varphi(t)dt$.

证：由反转公式，对 $a<b$

有 $$\frac{F(b)+F(b-0)}{2}-\frac{F(a)+F(a-0)}{2}=\lim_{T\to+\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-iat}-e^{-ibt}}{it}\varphi(t)dt$$

令 $a=x,\ b=x+h$. $$F(x+h)-F(x)=\frac{1}{2\pi}\lim_{T\to+\infty}\int_{-T}^{T}\frac{e^{-ixt}-e^{-i(x+h)t}}{it}\varphi(t)dt$$
$$=\frac{1}{2\pi}\lim_{T\to+\infty}\int_{-T}^{T}\frac{1-e^{-iht}}{it}\cdot e^{-ixt}\varphi(t)dt$$

$$\frac{F(x+h)-F(x)}{h}=\frac{1}{2\pi}\lim_{T\to+\infty}\int_{-T}^{T}\frac{1-e^{-iht}}{ith}\cdot e^{-ixt}\varphi(t)dt$$

令 $h\downarrow 0$ $$f(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-ixt}\varphi(t)dt$$

2. $X_1,\cdots,X_N$ 相互独立且服从 $Exp(\lambda)$ 分布. 证明 $S_N=X_1+\cdots+X_N$ 服从 $\Gamma(N,\lambda)$ 分布.

证：$X\sim\Gamma(N,\lambda)$ $\quad f(x)=\frac{\lambda e^{-\lambda x}(\lambda x)^{N-1}}{\Gamma(N)}$，$x>0$

用数学归纳法证明.

$N=1$ 时，$S_N=X_1\sim exp(\lambda)$ $\quad f_1(x)=\lambda e^{-\lambda x}$ $\Rightarrow S_1\sim\Gamma(1,\lambda)$ 成立.

假设对 $N\le k$ 均成立. 考虑 $N=k+1$ 的情况：

$$f_{k+1}(x)=\int_0^x f_k(y)f_1(x-y)dy=\int_0^x\frac{\lambda e^{-\lambda y}(\lambda y)^{k-1}}{\Gamma(k)}\cdot\lambda e^{-\lambda(x-y)}dy$$
$$=\frac{\lambda^{k+1}}{\Gamma(k)}e^{-\lambda x}\int_0^x y^{k-1}dy=\frac{\lambda^{k+1}}{\Gamma(k)}e^{-\lambda x}\cdot\frac{1}{k}x^k$$
$$=\frac{\lambda e^{-\lambda x}(\lambda x)^k}{\Gamma(k+1)}$$
注：$\Gamma(k)=(k-1)!$ $\quad k\in N_+$

思考：若 N 为离散型 r.v 如何求 S_N 密度函数？

$P(N=n)=2^{-n}$ $(n=1,2\cdots)$ 且与 $\{X_k\}$ 独立. $\qquad G_N(\varphi_S(t))$

$$P(S_N\le x)=\sum_{n=1}^{+\infty}P(S_N\le x\mid N=n)P(N=n)$$

3. $X_1,\cdots,X_n$ 相互独立且服从 $Exp(\lambda)$ 分布，令 $X_{(1)}\le X_{(2)}\le\cdots\le X_{(n)}$ 为其次序统计量. 证明：$Y_1=nX_{(1)}$，$Y_r=(n+1-r)(X_{(r)}-X_{(r-1)})$，$1<r\le n$ 相互独立且与 X_i 有相同的联合密度函数.

证：$X_1,\cdots,X_n$ 的联合密度函数为 $f(x_1,\cdots,x_n)=\lambda^n\exp(-\lambda\sum_{i=1}^n x_i)$

则 $X_{(1)},\cdots,X_{(n)}$ 的联合密度函数为 $h(x_1,\cdots,x_n)=\lambda^n\cdot n!\exp(-\lambda\sum_{i=1}^n x_i)$

$Y_1=nX_{(1)},\ Y_r=(n+1-r)(X_{(r)}-X_{(r-1)}) \Rightarrow X_{(r)}=\sum_{k=1}^r \frac{Y_k}{n-k+1},\ \sum_{i=1}^n X_{(i)}=\sum_{i=1}^n Y_i$

$|J|=\left|\frac{\partial(X_{(1)},\cdots,\partial X_{(n)})}{\partial(Y_1,\cdots,Y_r)}\right|=\frac{1}{n!}$

故 $Y_1,\cdots,Y_n$ 的联合密度函数为 $g(y_1,\cdots,y_n)=\frac{1}{n!}\cdot\lambda^n\cdot n!\exp(-\lambda\sum_{i=1}^n y_i)$

$$=\lambda^n\exp(-\lambda\sum_{i=1}^n y_i)$$

4. $\vec{X}\sim N(\vec{\mu},\Sigma)$, $\vec{X}^T=(\vec{X}^{(1)},\vec{X}^{(2)})$, $\vec{\mu}^T=(\vec{\mu}^{(1)},\vec{\mu}^{(2)})$, $\Sigma=\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}$. 求 $\vec{X}^{(2)}|\vec{X}^{(1)}$ 的条件分布及 $E[\vec{X}^{(2)}|\vec{X}^{(1)}]$, $Var(\vec{X}^{(2)}|\vec{X}^{(1)})$

解:
$$\begin{pmatrix}I_{n_1} & 0\\ -\Sigma_{21}\Sigma_{11}^{-1} & I_{n_2}\end{pmatrix}\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}\begin{pmatrix}I_{n_1} & -\Sigma_{11}^{-1}\Sigma_{12}\\ 0 & I_{n_2}\end{pmatrix}=\begin{pmatrix}\Sigma_{11} & 0\\ 0 & \Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\end{pmatrix}$$

$$\vec{Y}=\begin{pmatrix}\vec{Y}^{(1)}\\ \vec{Y}^{(2)}\end{pmatrix}=\begin{pmatrix}I_{n_1} & 0\\ -\Sigma_{21}\Sigma_{11}^{-1} & I_{n_2}\end{pmatrix}\begin{pmatrix}\vec{X}^{(1)}\\ \vec{X}^{(2)}\end{pmatrix}=\begin{pmatrix}\vec{X}^{(1)}\\ -\Sigma_{21}\Sigma_{11}^{-1}\vec{X}^{(1)}+\vec{X}^{(2)}\end{pmatrix}$$

$$\sim N\left(\begin{pmatrix}\vec{\mu}^{(1)}\\ -\Sigma_{21}\Sigma_{11}^{-1}\vec{\mu}^{(1)}+\vec{\mu}^{(2)}\end{pmatrix},\begin{pmatrix}\Sigma_{11} & 0\\ 0 & \Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\end{pmatrix}\right)$$

$\vec{Y}^{(2)}$, $\vec{Y}^{(1)}$ 独立, 则 $\vec{Y}^{(2)}|\vec{Y}^{(1)}\sim N(-\Sigma_{21}\Sigma_{11}^{-1}\vec{\mu}^{(1)}+\vec{\mu}^{(2)},\Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})$

$\Rightarrow \vec{X}^{(2)}|\vec{X}^{(1)}\sim N(\Sigma_{21}\Sigma_{11}^{-1}(\vec{X}^{(1)}-\vec{\mu}^{(1)})+\vec{\mu}^{(2)},\Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})$

$E[\vec{X}^{(2)}|\vec{X}^{(1)}]=\Sigma_{21}\Sigma_{11}^{-1}(\vec{X}^{(1)}-\vec{\mu}^{(1)})+\vec{\mu}^{(2)}$, $var(\vec{X}^{(2)}|\vec{X}^{(1)})=\Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$

5. $\vec{X}$ 服从 n 维正态分布 $N(\mu,\Sigma)$, 当且仅当对任意 n 维实向量 $\vec{t}$, $Y=\vec{t}^T\cdot\vec{X}$ 服从一维正态分布 $N(\vec{t}^T\mu,\vec{t}^T\Sigma\vec{t})$

证: 仅证 "$\Leftarrow$" $\varphi_{\vec{X}}(\vec{t})=E[e^{i\vec{t}^T\cdot\vec{X}}]=E[e^{iY}]=E[e^{iYs}]\big|_{s=1}=\varphi_Y(s)\big|_{s=1}$

又 $Y\sim N(\vec{t}^T\mu,\vec{t}^T\Sigma\vec{t})$, $\varphi_Y(s)\big|_{s=1}=e^{i\vec{t}^T\vec{\mu}s-\frac{1}{2}\vec{t}^T\Sigma\vec{t}s^2}\big|_{s=1}=e^{i\vec{t}^T\vec{\mu}-\frac{1}{2}\vec{t}^T\Sigma\vec{t}}=\varphi_X(\vec{t})$

$\Rightarrow X\sim N(\mu,\Sigma)$

6. $\vec{X}$各分量均服从一维正态分布是否能推出$\vec{X}$服从n维正态分布?

解: 不正确, 反例如下: $X, Y \sim N(0,1)$

令 $Z = \begin{cases} |Y|, & X \geq 0 \\ -|Y|, & X < 0 \end{cases}$, $Z \sim N(0,1)$ 但 $P(Y+Z=0) = \frac{1}{2}$

$\Rightarrow$ $Y+Z$不服从一维正态分布

$\overset{T5}{\Rightarrow}$ (Y,Z)不服从二维正态分布

7. 设$\{X_n\}$为正态随机变量列, $X_n \xrightarrow{D} X$, 试证明X亦服从正态分布.(可能退化为常数)

证: X_n的特征函数记为 $\varphi_n(t) = \exp(i\mu_n t - \frac{1}{2}\sigma_n^2 t^2) \longrightarrow X$的特征函数 $\varphi(t)$

证明 $\exists \mu, \sigma^2$ s.t. $\varphi(t) = \exp(i\mu t - \frac{1}{2}\sigma^2 t^2)$

$\varphi_n(t)$收敛 $\Rightarrow |\varphi_n(t)| = \exp(-\frac{1}{2}\sigma_n^2 t^2)$收敛 则 $\sigma_n^2 \longrightarrow \sigma^2$

下证μ_n收敛. $\exp(i\mu_n t) = \exp(\frac{1}{2}\sigma_n^2 t^2)\varphi_n(t) \to \exp(\frac{1}{2}\sigma^2 t^2)\varphi(t)$

$|\exp(i\mu_n t)| \leq 1$, 由DCT, $\frac{e^{i\mu_n t}-1}{i\mu_n} = \int_0^t \exp(i\mu_n t)\,dt \to \int_0^t \exp(\frac{1}{2}\sigma^2 t^2)\varphi(t) > 0$

$\mu_n = \frac{i\mu_n}{e^{i\mu_n t}-1} \cdot \frac{e^{i\mu_n t}-1}{i}$ 收敛 $\longrightarrow \mu$

Thm (5.10.(5)) $X_1, \cdots, X_n$ r.v.相互独立. $E[X_j]=0$. $Var(X_j) = \sigma_j^2$, $E|X_j^3| < +\infty$.

s.t. $\frac{1}{\sigma(n)^3}\sum_{j=1}^n E|X_j^3| \to 0$ (当$n\to\infty$) 其中 $\sigma(n)^2 = Var(\sum_{j=1}^n X_j) = \sum_{j=1}^n \sigma_j^2$, 则有 $\frac{1}{\sigma(n)}\sum_{j=1}^n X_j \xrightarrow{D} N(0,1)$

8. $X_1, \cdots, X_n$为相互独立的随机变量. $P(X_i=1) = P(X_i=-1) = \frac{1}{2}$

则 $\sqrt{\frac{3}{n^3}}\sum_{k=1}^n kX_k \xrightarrow{D} N(0,1)$, 当$n\to+\infty$时.

证: 令$Y_k = kX_k$ $E[Y_k]=0$. $Var(Y_k) = k^2$, $E|Y_k^3| = k^3$. 令 $S_n = Y_1 + \cdots + Y_n$.

$\frac{1}{Var(S_n)^{\frac{3}{2}}}\sum_{k=1}^n E|Y_k^3| \sim C \cdot \frac{n^4}{n^{\frac{9}{2}}} \to 0$, 则由Thm, $\frac{S_n}{\sqrt{Var(S_n)}} \xrightarrow{D} N(0,1)$

又 $Var(S_n) = \sum_{k=1}^n k^2 \sim \frac{1}{3}n^3$ 故 $\sqrt{\frac{3}{n^3}}\sum_{k=1}^n kX_k \xrightarrow{D} N(0,1)$

Thm (Lindeberg - Feller CLT)

$X_1, \cdots, X_n$为相互独立的随机变量，$EX_i = 0$，$S_n^2 = \sum_{i=1}^{n} E|X_i|^2$

若满足Lindeberg条件，即 $\lim_{n\to\infty} \frac{1}{S_n^2} \sum_{k=1}^{n} E|X_k|^2 I_{\{|X_k| > \varepsilon S_n\}} = 0$ 对 $\forall \varepsilon > 0$ 成立，

则 $\frac{\sum_{k=1}^{n} X_k}{S_n} \xrightarrow{D} N(0,1)$

T8另证：$|Y_k| = k \leq n$，$\frac{n}{S_n} \to 0$ as $n \to +\infty$ 故n充分大时，$I_{\{|Y_k| > \varepsilon S_n\}} = 0$

$\Rightarrow \lim_{n\to\infty} \frac{1}{S_n^2} \sum_{k=1}^{n} E|Y_k|^2 I_{\{|Y_k| > \varepsilon S_n\}} = 0$ （此处的S_n与证法一中的S_n不同）