4.3.2.

(1)° $G(K)=\{A\in GL(n,\mathbb{C}) \mid A'KA=K\}$

① $G(K)$为$GL(n,\mathbb{C})$嵌入子流形

$\forall A\in \mathbb{C}^{n\times n}$, $r:(-\varepsilon,\varepsilon)\mapsto GL(n,\mathbb{C})\subset \mathbb{C}^{n\times n}$ $X\mapsto X+tA$

$f_{*}(A)=\frac{d}{dt}(X+tA)^{T}K(X+tA)=A^{T}KX+X^{T}KA$

设$X=AY$ $A^{T}KX+X^{T}KA=KY+Y^{T}K$

$\therefore \forall A$ $Im(f_{*X})=Im(f_{*I})$ ← 維数固定 ($A\in GL(n,\mathbb{C})$ A可逆)

$\Rightarrow$ $G(K)$为$GL(n,\mathbb{C})$ C^{∞}子流形

② $G(K)$是群

① 结合律 ② 单位元 ③ 逆元

③ $J: G(K)\times G(K)\to G(K)$

$(A,B)\to A\cdot B$ $C_{ij}=\sum_{k=1}^{n} a_{ik}b_{kj}$

$G(K)\to G(K)$

$A\to A^{-1}$ $a_{ij}'=\frac{M_{ij}}{\det A}$ (M_{ij}为代数余子式)

J为C^{∞}的

$\Rightarrow$ C^{∞} Lie 子群

(2) 同理

3. 同理

5.1.3. (4°)

$$\Big(\sum_{\substack{i_1\cdots i_r=1\\ j_1\cdots j_s=1}}^{n}\theta^{i_1\cdots i_r}_{j_1\cdots j_s}\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_r}}\otimes dx^{j_1}\cdots\otimes dx^{j_s}\Big)\Big(dx^{i_1'}\cdots dx^{i_r'},\frac{\partial}{\partial x^{j_1'}}\cdots\frac{\partial}{\partial x^{j_s'}}\Big)$$

$$=\sum^{n}\theta^{i_1\cdots i_r}_{j_1\cdots j_s}\delta^{i_1'}_{i_1}\cdots\delta^{i_r'}_{i_r}\delta^{j_1}_{j_1'}\cdots\delta^{j_s}_{j_s'}=\theta^{i_1'\cdots i_r'}_{j_1'\cdots j_s'}=\theta\Big(dx^{i_1'}\cdots dx^{i_r'},\frac{\partial}{\partial x^{j_1'}}\cdots\frac{\partial}{\partial x^{j_s'}}\Big)$$

(5) 略

5.1.4

(1) $$(\theta+\eta)^{i_1\cdots i_r}_{j_1\cdots j_s}=(\theta+\eta)\Big(dx^{i_1}\cdots dx^{i_r},\frac{\partial}{\partial x^{j_1}}\cdots\frac{\partial}{\partial x^{j_s}}\Big)$$
$$=\theta\Big(dx^{i_1}\cdots dx^{i_r},\frac{\partial}{\partial x^{j_1}}\cdots\frac{\partial}{\partial x^{j_s}}\Big)+\eta\Big(dx^{i_1}\cdots dx^{i_r},\frac{\partial}{\partial x^{j_1}}\cdots\frac{\partial}{\partial x^{j_s}}\Big)$$
$$=\theta^{i_1\cdots i_r}_{j_1\cdots j_s}+\eta^{i_1\cdots i_r}_{j_1\cdots j_s}$$

(2) $$(\lambda\theta)^{i_1\cdots i_r}_{j_1\cdots j_s}=(\lambda\theta)\Big(dx^{i_1}\cdots dx^{i_r},\frac{\partial}{\partial x^{j_1}}\cdots\frac{\partial}{\partial x^{j_s}}\Big)$$
$$=\lambda\cdot\theta\Big(dx^{i_1}\cdots dx^{i_r},\frac{\partial}{\partial x^{j_1}}\cdots\frac{\partial}{\partial x^{j_s}}\Big)$$
$$=\lambda\theta^{i_1\cdots i_r}_{j_1\cdots j_s}$$

(3) $$(\theta\otimes\eta)_{j_1\cdots j_{r+s}}=(\theta\otimes\eta)\Big(\frac{\partial}{\partial x^{j_1}}\cdots\frac{\partial}{\partial x^{j_{r+s}}}\Big)=\theta\Big(\frac{\partial}{\partial x^{j_1}}\cdots\frac{\partial}{\partial x^{j_r}}\Big)\cdot\eta\Big(\frac{\partial}{\partial x^{j_{r+1}}}\cdots\frac{\partial}{\partial x^{j_{r+s}}}\Big)=\theta_{j_1\cdots j_r}\cdot\eta_{j_{r+1}\cdots j_{r+s}}$$

5.1.5 $$\bar{\theta}_{ij}=\theta(\bar{e}_i,\bar{e}_j)=\theta\Big(\sum_{l=1}^{n}c_{il}e_l,\sum_{t=1}^{n}c_{jt}e_t\Big)=\sum_{l=1}^{n}\sum_{t=1}^{n}c_{il}\theta(e_l,e_t)\cdot c_{jt}=\sum_{l,t=1}^{n}c_{il}c_{jt}\theta_{lt}$$

5.1.6 $$\theta_{j_{\pi(1)}\cdots j_{\pi(s)}}=\theta\Big(\frac{\partial}{\partial x^{j_{\pi(1)}}}\cdots\frac{\partial}{\partial x^{j_{\pi(s)}}}\Big)=\theta\Big(\frac{\partial}{\partial x^{j_1}}\cdots\frac{\partial}{\partial x^{j_s}}\Big)=\theta_{j_1\cdots j_s}$$

$\Rightarrow$ $$\theta_{\pi(1)\cdots\pi(s)}(X_{\pi(1)}\cdots X_{\pi(s)})=\theta\Big(\sum_{i_1=1}^{n}a_{\pi(1)i_1}\frac{\partial}{\partial x^{i_1}}\cdots\sum_{i_s=1}^{n}a_{\pi(s)i_s}\frac{\partial}{\partial x^{i_s}}\Big)$$

$\Leftarrow$ $$\theta(X_{\pi(1)}\cdots X_{\pi(s)})=\sum_{j_1\cdots j_s=1}^{n}a_{\pi(1)\pi(j_1)}\cdots a_{\pi(s)\pi(j_s)}\theta\Big(\frac{\partial}{\partial x^{\pi(j_1)}}\cdots\frac{\partial}{\partial x^{\pi(j_s)}}\Big)$$

$= \sum_{j_1 \cdots j_s = 1}^{n} a_{1j_1} \cdots a_{sj_s}\, \theta\left(\frac{\partial}{\partial x^{j_1}} \cdots \frac{\partial}{\partial x^{j_s}}\right) = \theta(X_1 \cdots X_s)$

5.1.12

(1) 可以

(2) 可以

5.1.13 (1) $F_p^*(\theta+\eta)(X_1 \cdots X_s) = (\theta+\eta)(F_{*p}X_1 \cdots F_{*p}X_s)$

$= \theta(F_{*p}X_1 \cdots F_{*p}X_s) + \eta(F_{*p}X_1 \cdots F_{*p}X_s)$

$= F_p^*\theta\,(X_1 \cdots X_s) + F_p^*\eta\,(X_1 \cdots X_s)$

$F_p^*(\lambda\theta)(X_1 \cdots X_s) = (\lambda\theta)(F_{*p}X_1 \cdots F_{*p}X_s)$

$= \lambda \cdot \theta\,(F_{*p}X_1 \cdots F_{*p}X_s)$

$= \lambda F_p^*\theta\,(X_1 \cdots X_s)$

(2) $\eta_{i_1 \cdots i_s} = F_p^*\theta\left(\frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_s}}\right)$

$= \theta\left(F_{*p}\frac{\partial}{\partial x^{i_1}} \cdots F_{*p}\frac{\partial}{\partial x^{i_s}}\right)$

$= \theta\left(\sum_{j_1=1}^{n} \frac{\partial y^{j_1}}{\partial x^{i_1}} \frac{\partial}{\partial y^{j_1}} \cdots \sum_{j_s=1}^{n} \frac{\partial y^{j_s}}{\partial x^{i_s}} \frac{\partial}{\partial y^{j_s}}\right) = \sum_{j_1 \cdots j_s=1}^{n} \theta\left(\frac{\partial}{\partial y^{j_1}} \cdots \frac{\partial}{\partial y^{j_s}}\right)\left(\frac{\partial y^{j_1}}{\partial x^{i_1}}\right)_p \cdots \left(\frac{\partial y^{j_s}}{\partial x^{i_s}}\right)_p$

$= \sum_{j_1 \cdots j_s=1}^{n} \left(\frac{\partial y^{j_1}}{\partial x^{i_1}}\right)_p \cdots \left(\frac{\partial y^{j_s}}{\partial x^{i_s}}\right)_p \theta_{j_1 \cdots j_s}$

(3) 同上

(4) 成立

(5) 可以

5.1.14.

(1) 略 $(UU^* = T^{r,s}(M))$ $(\varphi_1^* \cdot \varphi_2^*)^{-1}$ (C^∞)

(2) $\varphi^* \circ \theta \circ \varphi^{-1}(x^1 \cdots x^n) = (x^1 \cdots x^n\ \theta^{i_1 \cdots i_r}_{j_1 \cdots j_s})$

θ C^∞ 且 $\theta^{i_1 \cdots i_r}_{j_1 \cdots j_s} C^\infty$

$\Rightarrow \varphi^* \circ \theta \circ \varphi^{-1}$ C^∞

(3) 可以 (性质一样基本都可推广)