第五讲

Prop 10 闭集可测

Pf step 1 紧集可测

设 $F \subset \mathbb{R}^n$ 紧

$\Rightarrow m_*(F) < \infty \quad (\because \exists Q \text{ s.t. } F \subset Q)$

$$m_*(F) = \inf\{m_*(G) : G \text{ 开}, F \subset G\}$$

$\Rightarrow \forall \varepsilon > 0, \exists G$ 开 s.t. $F \subset G$ 且

$$m_*(G) < m_*(F) + \varepsilon$$

而 $G \setminus F$ 开

开集结构 $\Rightarrow G \setminus F = \biguplus_{k=1}^{\infty} Q_k$

令 $K_N \overset{def}{=} \bigcup_{k=1}^{N} Q_k, \quad N = 1, 2, \cdots$

$\Rightarrow K_N$ 紧 且 $K_N \subset G \setminus F$

$\Rightarrow \operatorname{dist}(K_N, F) > 0$

$\Rightarrow m_*(K_N \cup F) = m_*(K_N) + m_*(F)$

$\Rightarrow m_*(K_N) = m_*(K_N \cup F) - m_*(F)$

$$\le m_*(G) - m_*(F) < \varepsilon$$

$$\xRightarrow{N\to\infty} \quad m_*\left(\bigcup_{k=1}^{\infty} Q_k\right) \le \varepsilon$$

即 $\quad m_*(G\setminus F) \le \varepsilon$

Step 2 一般情形

$$F = \bigcup_{k=1}^{\infty}\left(F\cap \overline{B_k(0)}\right)$$

Prop 11 $\quad E$ 可测 $\Longrightarrow E^c$ 可测

Pf $\quad \forall k,\ \exists G_k$ 开, $E\subset G_k$ s.t.

$$m_*(G_k\setminus E) < \frac{1}{k}$$

G_k^c 闭, $\Longrightarrow G_k^c$ 可测

Prop 9 $\Longrightarrow \quad S \overset{def}{=} \bigcup_{k=1}^{\infty} G_k^c$ 可测

$$E\subset \bigcap_{k=1}^{\infty} G_k \Longrightarrow S\subset E^c$$

$$E^c\cap S^c = E^c\cap\left(\bigcap_{k=1}^{\infty} G_k\right) \subset E^c\cap G_k$$

$$\parallel \quad E^c\setminus S \subset G_k\setminus E$$

$$\Longrightarrow m_*(E^c\setminus S) \le m_*(G_k\setminus E) < \frac{1}{k}$$

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③

$\Rightarrow m_*(E^c\setminus S)=0$

$\Rightarrow E^c\setminus S$ 是零测集

$\Rightarrow E^c\setminus S$ 可测

$\Rightarrow E^c=(E^c\setminus S)\cup S$ 可测

Thm $\mathcal{L}$ 是 $\mathbb{R}^n$ 上一个σ-代数.

回忆

Borel σ-代数 $\mathcal{B} \stackrel{def}{=} \mathbb{R}^n$ 中所有开集生成的σ-代数

$\Longrightarrow \mathcal{B}\subset\mathcal{L}$

By Ex.35 $\mathcal{B}\neq\mathcal{L}$

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④

外测度 $m_*: 2^{\mathbb{R}^n} \to [0,+\infty]$

$$m_*(E) \overset{\text{def}}{=} \inf\left\{ \sum_{k=1}^{\infty} |Q_k| : \{Q_k\}_{k=1}^{\infty} 是 E 的方体覆盖 \right\}$$

称 $E \subset \mathbb{R}^n$ 可测，是指 $\forall \varepsilon > 0$，$\exists G$ 开，$E \subset G$ s.t. $m_*(G \setminus E) < \varepsilon$

$\mathcal{L} \overset{\text{def}}{=} \{ \mathbb{R}^n 中可测集 \}$

$\Rightarrow$ $\mathcal{L}$ 是 $\mathbb{R}^n$ 上一个 σ-代数

Def $m \overset{\text{def}}{=} m_*|_{\mathcal{L}}$ 称为 Lebesgue 测度

Thm（可数可加性）

设 $E_k \in \mathcal{L}$，$k=1,2,\cdots$ 互不相交，则

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k)$$

∃ 先由次可加性

LHS ≤ RHS

Pf Step 1 先假设 $\forall k$，E_k 有界

要证明 LHS ≥ RHS

$\forall \varepsilon > 0$. $\forall k$，$\exists$ 紧集 $F_k \subset E_k$ s.t.

$$m(E_k \setminus F_k) < \frac{\varepsilon}{2^k} \quad (\text{由正则性 Ex. 25})$$

$\forall N$, $F_1, \cdots, F_N$ 是互不相交的紧集

$\Longrightarrow \operatorname{dist}(F_j, F_k) > 0, \quad \forall j, k, \ j \neq k$

$$\overset{\text{Prop 4}}{\Longrightarrow} m\left(\bigcup_{k=1}^{N} F_k\right) = \sum_{k=1}^{N} m(F_k)$$

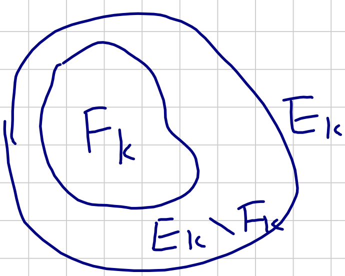
$$\Rightarrow m\left(\bigcup_{k=1}^{\infty} E_k\right) \geq m\left(\bigcup_{k=1}^{N} F_k\right)$$

$$= \sum_{k=1}^{N} m(F_k)$$

$$\geq \sum_{k=1}^{N} \left[m(E_k) - m(E_k \setminus F_k) \right]$$

$$\geq \sum_{k=1}^{N} \left[m(E_k) - \frac{\varepsilon}{2^k} \right]$$

$$\geq \sum_{k=1}^{N} m(E_k) - \varepsilon$$



$m(F_k) + m(E_k \setminus F_k) \geq m(E_k)$

$$\overset{N \to \infty}{\Longrightarrow} m\left(\bigcup_{k=1}^{\infty} E_k\right) \geq \sum_{k=1}^{\infty} m(E_k) - \varepsilon$$

$$\Longrightarrow m\left(\bigcup_{k=1}^{\infty} E_k\right) \geq \sum_{k=1}^{\infty} m(E_k)$$

Step 2 —一般情形

$$\mathbb{R}^n = \bigcup_{k=1}^{\infty} Q_k, \quad Q_k \overset{\text{def}}{=} [-k, k]^n$$

2023.3.22
⑥

令 $S_1 \overset{def}{=} Q_1$

$S_k \overset{def}{=} Q_k \setminus Q_{k-1}, \quad k \geq 2$

$\Rightarrow \mathbb{R}^n = \biguplus_{k=1}^{\infty} S_k$

不交并

令 $E_{j,k} \overset{def}{=} S_j \cap E_k$

$\Rightarrow E_k = \biguplus_{j=1}^{\infty} E_{j,k} \Rightarrow \bigcup_{k=1}^{\infty} E_k = \biguplus_{j,k} E_{j,k}$

可列

Step 1
$$\Rightarrow m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{j,k} m(E_{j,k})$$
$$= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} m(E_{j,k})$$
$$= \sum_{k=1}^{\infty} m(E_k)$$

Thm (测度的连续性). 设 $E_k \in \mathcal{L}$, $k=1,2\cdots$

(i) (向上连续)

$$E_k \nearrow E \Rightarrow m(E) = \lim_{k\to\infty} m(E_k)$$

2° (向下连续)

$$\left.\begin{array}{l} E_k \searrow E \\ \text{且} \quad m(E_{k_0}) < \infty \\ \text{for some } k_0 \end{array}\right\} \Longrightarrow m(E) = \lim_{k\to\infty} m(E_k)$$

Pf 1° 令 $\tilde{E}_1 \overset{def}{=} E_1$

$\tilde{E}_k \overset{def}{=} E_k \setminus E_{k-1}$, $k \geq 2$

$\Longrightarrow \tilde{E}_k \in \mathcal{L}$, 且

$$E = \biguplus_{k=1}^{\infty} \tilde{E}_k$$

$$\Longrightarrow m(E) = \sum_{k=1}^{\infty} m(\tilde{E}_k)$$

$$= \lim_{N\to\infty} \sum_{k=1}^{N} m(\tilde{E}_k)$$

$$= \lim_{N\to\infty} m\left(\bigcup_{k=1}^{N} \tilde{E}_k\right)$$

$$= \lim_{N\to\infty} m(E_N)$$

2° 不必对设 $m(E_1)<\infty$

令 $\tilde{E}_k = E_k \setminus E_{k+1}$, $k=1,2,\cdots$

$$\Rightarrow E_1 = \underbrace{E \cup \left(\bigcup_{k=1}^{\infty} \tilde{E}_k\right)}_{\text{不交并}}$$

$$\Rightarrow m(E_1) = m(E) + \sum_{k=1}^{\infty} m(\tilde{E}_k)$$

$$= m(E) + \lim_{N\to\infty} \sum_{k=1}^{N-1} \left[m(E_k) - m(E_{k+1})\right]$$

$$= m(E) + m(E_1) - \lim_{N\to\infty} m(E_N)$$

$$\Rightarrow m(E) = \lim_{N\to\infty} m(E_N)$$

Rmk 2° 中 "$m(E_{k_0})<\infty$ for some k_0" 不可去

反例: $E_k \overset{def}{=} (k, +\infty)$

$\Rightarrow E_k \searrow \phi$

但 $m(E_k) = +\infty$, $\forall k$.

Thm 设 $E \in \mathcal{L}$

1° $\forall \varepsilon > 0$, $\exists G$ 开 s.t. $E \subset G$ 且
$$m(G \setminus E) < \varepsilon$$

2° $\forall \varepsilon > 0$, $\exists F$ 闭 s.t. $F \subset E$ 且
$$m(E \setminus F) < \varepsilon$$

3° 若 $m(E) < \infty$, 则 $\forall \varepsilon > 0$, $\exists K$ 紧 s.t. $K \subset E$, 且 $m(E \setminus K) < \varepsilon$

4° 若 $m(E) < \infty$, 则 $\forall \varepsilon > 0$ $\exists Q_1, \cdots, Q_N$ s.t.
$$m\left(E \,\triangle\, \left(\bigcup_{k=1}^{N} Q_k\right)\right) < \varepsilon$$

其中 $E_1 \triangle E_2 \overset{\text{def}}{=} (E_1 \setminus E_2) \cup (E_2 \setminus E_1)$ 称为对称差

Pf: 1° 平凡. by defn.

2023.3.22

10

2° $E \in \mathcal{L} \Rightarrow E^c \in \mathcal{L}$

$\Rightarrow \forall \varepsilon > 0,\ \exists G$ 开 s.t. $E^c \subset G$ 且

$$m(G \setminus E^c) < \varepsilon$$

$$E \setminus F = E \cap F^c = E \cap G = G \setminus E^c$$

令 $F \overset{\text{def}}{=} G^c$ (闭)

$\Rightarrow F \subset E$ 且 $G \setminus E^c = E \setminus F$

$\Rightarrow m(E \setminus F) < \varepsilon$

3° 令 $Q_k \overset{\text{def}}{=} [-k, k]^n$, $k = 1, 2, \cdots$

$\Rightarrow E \cap Q_k \uparrow E$

$\xRightarrow{\text{测度连续性}}$ $\forall \varepsilon > 0,\ \exists k$ s.t.

$$m(E \cap Q_k) > m(E) - \frac{\varepsilon}{2}$$

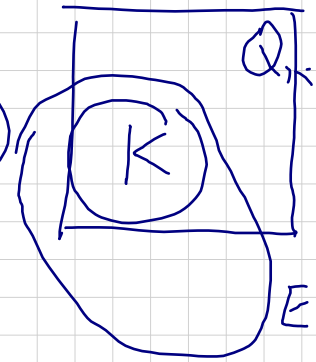
另一方面, by 2°, $\exists K$ 闭 s.t.

$K \subset E \cap Q_k$ ($\Rightarrow K$ 紧)

且

$$m((E \cap Q_k) \setminus K) < \frac{\varepsilon}{2}$$

2023.3.22 ⑪



$\Rightarrow m(E \setminus K)$

$= m(E \setminus (E \cap Q_k)) + m((E \cap Q_k) \setminus K)$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

4° 由 m_* 定义

$\forall \varepsilon > 0$, $\exists Q_k$, $k = 1, 2, \cdots$ s.t. $E \subset \bigcup_{k=1}^{\infty} Q_k$

且
$$\sum_{k=1}^{\infty} |Q_k| < m(E) + \varepsilon/2 \quad (< \infty)$$

$\Rightarrow \exists N$, s.t. $\sum_{k=N+1}^{\infty} |Q_k| < \frac{\varepsilon}{2}$

令
$$F \overset{\text{def}}{=} \bigcup_{k=1}^{N} Q_k$$

$\Rightarrow E \setminus F \subset \left(\bigcup_{k=1}^{\infty} Q_k \right) \setminus F \subset \bigcup_{k=N+1}^{\infty} Q_k$

$\Rightarrow m(E \setminus F) \leq m\left(\bigcup_{k=N+1}^{\infty} Q_k \right)$

$\leq \sum_{k=N+1}^{\infty} |Q_k| < \frac{\varepsilon}{2}$

而 $m(F\setminus E) \leq m\left(\left(\bigcup_{k=1}^{\infty} Q_k\right)\setminus E\right)$

$$\leq \sum_{k=1}^{\infty}|Q_k| - m(E) < \frac{\varepsilon}{2}$$

Thm 设 $E\subset\mathbb{R}^n$，则

E 可测 $\Longleftrightarrow$ $\exists G$ (G_δ 型)，$\exists N_1$ (零集) s.t.

$$E = G\setminus N_1$$

$\Longleftrightarrow$ $\exists F$ (F_σ 型)，$\exists N_2$ (零集) s.t.

(HW) $$E = F\cup N_2$$

Pf: "$\Longleftarrow$" 平凡

"$\Longrightarrow$" $\forall k$, $\exists G_k$ 开 s.t. $E\subset G_k$ 且

$$m(G_k\setminus E) < \frac{1}{k}$$

令 $G \overset{def}{=} \bigcap_{k=1}^{\infty} G_k$ (G_δ 型)

$\Longrightarrow$ $G\setminus E \subset G_k\setminus E$

$\Longrightarrow$ $m(G\setminus E) \leq m(G_k\setminus E) < \frac{1}{k}$

2023.3.22

13

$\Rightarrow m(G \setminus E) = 0$

$\Rightarrow$ 记 $N_1 \overset{\text{def}}{=\!=} G \setminus E$ 即可

Ex. 6–8

Hw: Ex. 16