第十四讲（2023.4.26）

Thm $L^\infty(E)$ 完备.

Pf 设 $\{f_k\}_{k=1}^{\infty} \subset L^\infty(E)$ 是 Cauchy 列

$\iff \|f_k - f_j\|_\infty \to 0 \quad as \quad k,j \to \infty$

令 $A_k \overset{def}{=} \{ |f_k| > \|f_k\|_\infty \}$

$B_{k,j} \overset{def}{=} \{ |f_k - f_j| > \|f_k - f_j\|_\infty \}$

$k,j = 1,2,\cdots$

$\implies m(A_k) = 0,$

$m(B_{k,j}) = 0,$

令 $F \overset{def}{=} \left(\bigcup_k A_k \right) \cup \left(\bigcup_{k,j} B_{k,j} \right)$

$\implies m(F) = 0$. 则

(*) $|f_k(x) - f_j(x)| \le \|f_k - f_j\|_\infty, \quad \forall x \in E \setminus F$

$\implies \{f_k(x)\}_{k=1}^{\infty}$ 是 $\mathbb{R}$ 中 Cauchy 列，故收敛.

令 $f(x) \overset{\text{def}}{=} \begin{cases} \lim\limits_{k\to\infty} f_k(x), & x \in E\setminus F, \\ 0, & x \in F. \end{cases}$

$\Rightarrow$ f 可测 且 $\exists N$, s.t.

$$\sup_{x\in E\setminus F} |f_k(x) - f_j(x)| \leq \|f_k - f_j\|_\infty \leq 1, \quad \forall k, j \geq N.$$

$$\overset{j\to\infty}{\Longrightarrow} \quad \sup_{x\in E\setminus F} |f_N(x) - f(x)| \leq 1.$$

$$\Rightarrow \quad \sup_{x\in E\setminus F} |f(x)| \leq 1 + \sup_{x\in E\setminus F} |f_N(x)| < \infty$$

$$\Rightarrow \quad f \in L^\infty(E).$$

$\forall \varepsilon > 0$, $\exists N$, s.t. $\|f_k - f_j\|_\infty < \varepsilon$,

$\forall k, j \geq N$

$\Rightarrow$ $\forall x \in E\setminus F$,

$$|f_k(x) - f(x)| = \lim_{j\to\infty} |f_k(x) - f_j(x)| \overset{(*)}{\leq} \lim_{j\to\infty} \|f_k - f_j\|_\infty \leq \varepsilon,$$

$\forall k \geq N.$

$$\Rightarrow \quad \|f_k - f\|_\infty = \inf_{\substack{Z\subset E \\ m(Z)=0}} \sup_{x\in E\setminus Z} |f_k(x) - f(x)| \leq \varepsilon,$$

$\forall k \geq N.$

Def 设 $\{f_k\}_{k=1}^{\infty} \subset L^p(E)$, $1 \le p < \infty$.

如果 $\exists f \in L^p(E)$ s.t.

$$\|f_k - f\|_p \to 0 \quad \text{as } k \to \infty$$

则称 f_k 依 L^p 范数收敛于 f. 记为 $f_k \xrightarrow{L^p} f$

或 $f_k \xrightarrow{\|\cdot\|_p} f$.

Def 依测度收敛

$$f_k \xrightarrow{m} f \overset{\text{def}}{\iff} \forall \varepsilon > 0, \ m(\{|f_k - f| \ge \varepsilon\}) \to 0 \text{ as } k \to \infty$$

Prop $f_k \xrightarrow{L^p} f \implies f_k \xrightarrow{m} f$

Pf $\forall \varepsilon > 0$,

$$\int |f_k - f|^p \, dm \ge \int_{\{|f_k - f| \ge \varepsilon\}} |f_k - f|^p \, dm$$

$$\ge \varepsilon^p \, m(\{|f_k - f| \ge \varepsilon\})$$

$$\implies m(\{|f_k - f| \ge \varepsilon\}) \le \frac{1}{\varepsilon^p} \int |f_k - f|^p \, dm$$

$$\to 0 \quad \text{as } k \to \infty.$$

Rmk: 1° $f_k \xrightarrow{m} f \not\Rightarrow f_k \to f$ a.e.

(反例: Ex. 12)

2° $f_k \to f$ a.e. $\not\Rightarrow f_k \xrightarrow{m} f$

反例: $f_k \overset{\text{def}}{=} \chi_{(-k,k)}$, $f \equiv 1$.

$f_k \to f$ pointwise, 但 $m(\{|f_k - f| \geq \frac{1}{2}\}) = \infty$

Thm (Lebesgue)

设 $m(E) < \infty$, $f, f_k, k = 1, 2, \cdots$ 在 E 上可测, 且 a.e. 有限

$$f_k \to f \text{ a.e.} \Longrightarrow f_k \xrightarrow{m} f$$

Pf 对 $k \in \mathbb{N}$, $\varepsilon > 0$, 令

$$E_k(\varepsilon) \overset{\text{def}}{=} \{|f_k - f| \geq \varepsilon\}$$

$\forall x \in \limsup\limits_{k\to\infty} E_k(\varepsilon) \overset{\text{def}}{=} \bigcap\limits_{j=1}^{\infty} \bigcup\limits_{k=j}^{\infty} E_k(\varepsilon)$

(即: $\forall j$, $\exists k_j$ s.t. $x \in E_{k_j}(\varepsilon)$)

$\Longrightarrow$ 有子列 $\{f_{k_j}\}_{j=1}^{\infty}$ s.t.

$$|f_{k_j}(x) - f(x)| \geq \varepsilon, \quad j = 1, 2, \cdots$$

$$\Longrightarrow \quad f_k(x) \not\to f(x)$$

$$\Longrightarrow \quad \limsup_{k\to\infty} E_k(\varepsilon) \subset \{ f_k \not\to f \}$$

$f_k \to f$ a.e.

$$\Longrightarrow \quad m\left(\limsup_{k\to\infty} E_k(\varepsilon)\right) = 0.$$

而

$$\Longrightarrow \quad \bigcup_{k=j}^{\infty} E_k(\varepsilon) \searrow \limsup_{k\to\infty} E_k(\varepsilon)$$

测度连续性

$m(E) < \infty$

$$\Longrightarrow \quad \lim_{j\to\infty} m\left(\bigcup_{k=j}^{\infty} E_k(\varepsilon)\right) = m\left(\limsup_{k\to\infty} E_k(\varepsilon)\right) = 0,$$

特别地

$$\Longrightarrow \quad \lim_{j\to\infty} m(E_j(\varepsilon)) = 0$$

Thm (Riesz)

$$f_k \xrightarrow{m} f \quad \Longrightarrow \quad \text{有子列 } f_{k_j} \to f \text{ a.e.}$$

Pf $\quad f_k \xrightarrow{m} f \overset{\text{def}}{\Longleftrightarrow} \forall \varepsilon > 0,\ \forall \eta > 0,\ \exists N$ s.t.

$$m(\{ |f_k - f| \geq \varepsilon \}) < \eta, \quad \forall k \geq N.$$

$$\Longrightarrow \quad \forall j,\ \exists k_j > k_{j-1}, \text{ s.t.}$$

$$m\left(\left\{ |f_k - f| \geq \frac{1}{2^j} \right\}\right) < \frac{1}{2^j}, \quad \forall k \geq k_j$$

$\Rightarrow$ ∃子列 $\{f_{k_j}\}_{j=1}^{\infty}$ s.t.

$$m\Big(\underbrace{\{|f_{k_j} - f| \geq \tfrac{1}{2^j}\}}_{\text{记为 } E_j}\Big) < \frac{1}{2^j}, \quad j = 1, 2 \cdots$$

令 $$F_N \overset{\text{def}}{=} \bigcap_{j=N}^{\infty} (E \setminus E_j)$$

$\Rightarrow$ $\forall x \in F_N$,

$$|f_{k_j}(x) - f(x)| < \frac{1}{2^j}, \quad j = N, N+1, \cdots$$

$\Rightarrow$ $f_{k_j} \to f$ on F_N

令 $$F \overset{\text{def}}{=} \bigcup_{N=1}^{\infty} F_N = \liminf_{j \to \infty} (E \setminus E_j)$$

$\Rightarrow$ $f_{k_j} \to f$ on F.

<u>Claim</u> $m(E \setminus F) = 0$

$$E \setminus F = \bigcap_{N=1}^{\infty} \underbrace{(E \setminus F_N)}_{= \bigcup_{j=N}^{\infty} E_j} = \limsup_{j \to \infty} E_j$$

而 $$\sum_{j=1}^{\infty} m(E_j) \leq \sum_{j=1}^{\infty} \frac{1}{2^j} < \infty$$

$\overset{\text{Borel-Cantelli}}{\Longrightarrow} \quad m(\limsup_{j\to\infty} E_j) = 0.$

$$
\begin{tikzcd}
 & & \text{一致收敛} \arrow[d, Rightarrow] \\
\text{依 } L^p \text{ 范数收敛} \arrow[d, Rightarrow] & & \text{点态收敛} \arrow[d, Rightarrow] \\
\text{依测度收敛} \arrow[rr, bend right, "\text{有子列}"'] & & \text{a.e. 收敛} \arrow[ll, Rightarrow, "\text{如果 } m(E)<\infty"'] \arrow[d, Rightarrow, "\text{Egorov}"] \\
 & & \text{近一致收敛.}
\end{tikzcd}
$$

Def X 一非空集合

如果函数 $d: X\times X \to \mathbb{R}$ s.t.

(i) (正定性) $d(x,y) \geq 0, \ \forall x,y \in X$

$d(x,y) = 0 \iff x = y$

(ii) (对称性) $d(x,y) = d(y,x)$

(iii) (三角不等式) $d(x,y) \leq d(x,z) + d(z,y)$

则称 d 为 X 上一个度量. (X,d) 称为度量空间.

Def (X, d)

稱 E 在 X 中稠密是指: $\forall x \in X$, $\forall \varepsilon > 0$,

$$B(x, \varepsilon) \cap E \neq \phi.$$

$$E \overset{\text{dense}}{\subset} X \iff \overline{E} = X$$

Thm 1° 簡單函數全体 $\overset{\text{dense}}{\subset} L^1$

2° 階梯 - - - $\overset{\text{dense}}{\subset} L^1$

Pf 1° 設 $f \in L^1$.

Step 1 先假設 $f \geq 0$.

$\exists\ \varphi_k \geq 0$ simple s.t. $\varphi_k \nearrow f$

$$\overset{\text{MCT}}{\Longrightarrow} \int \varphi_k \, dm \to \int f \, dm$$

$$\Longrightarrow \|f - \varphi_k\|_1 = \int (f - \varphi_k) \, dm \to 0$$

as $k \to \infty$

Step 2 一般情形

$$f = f^+ - f^-$$

$\overset{\text{step 1}}{\Longrightarrow}$ $\exists\ \varphi_k^{(1)}, \varphi_k^{(2)}$ simple, s.t.