4.6

4. Find the conditional density function and expectation of Y given X when they have joint density function:

(a) $f(x, y) = \lambda^2 e^{-\lambda y}$ for $0 \le x \le y < \infty$,

(b) $f(x, y) = xe^{-x(y+1)}$ for $x, y \ge 0$.

解: (a) $f_X(x) = \int_x^\infty \lambda^2 e^{-\lambda y}\,dy = \lambda^2 \int_x^\infty e^{-\lambda y}\,dy = \lambda^2 \cdot (-\frac{1}{\lambda} e^{-\lambda y}\big|_x^\infty) = \lambda e^{-\lambda x}$, $0 \le x < \infty$

$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{\lambda^2 e^{-\lambda y}}{\lambda e^{-\lambda x}} = \lambda e^{\lambda(x-y)}$, $0 \le x \le y < \infty$

$E[Y|X=x] = \int_x^\infty y \lambda e^{\lambda(x-y)}\,dy = x + \frac{1}{\lambda}$

(b) $f_X(x) = \int_0^\infty x e^{-x(y+1)}\,dy = x\int_0^\infty e^{-x(y+1)}\,dy = x \cdot \frac{1}{-x} e^{-x(y+1)}\big|_0^\infty = e^{-x}$

$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{xe^{-x(y+1)}}{e^{-x}} = xe^{-xy}$, $x, y \ge 0$

$E[Y|X=x] = \int_0^\infty y x e^{-xy}\,dy = \frac{1}{x}$

4.6

8. Let X, Y, Z be independent and exponential random variables with respective parameters λ, μ, ν. Find $\mathbb{P}(X < Y < Z)$.

9. Let X and Y have the joint density $f(x, y) = cx(y - x)e^{-y}$, $0 \le x \le y < \infty$.

(a) Find c.

(b) Show that:

$$f_{X|Y}(x \mid y) = 6x(y-x)y^{-3}, \qquad 0 \le x \le y,$$
$$f_{Y|X}(y \mid x) = (y-x)e^{x-y}, \qquad 0 \le x \le y < \infty.$$

(c) Deduce that $\mathbb{E}(X \mid Y) = \frac{1}{2}Y$ and $\mathbb{E}(Y \mid X) = X + 2$.

8. 解: $P(X<Y<Z) = \int_0^\infty \int_x^\infty \int_y^\infty \lambda\mu\nu e^{-\lambda x - \mu y - \nu z}\,dx\,dy\,dz$

$= \int_0^\infty \int_x^\infty (-\lambda\mu e^{-\lambda x - \mu y - \nu z}\big|_y^\infty)\,dx\,dy$

$= \int_0^\infty (\int_x^\infty \lambda\mu e^{-\lambda x - (\mu+\nu)y}\,dy)dx$

$= \int_0^\infty \frac{\lambda\mu}{\mu+\nu} e^{-(\lambda+\mu+\nu)x}\,dx$

$= \frac{\lambda\mu}{(\mu+\nu)(\lambda+\mu+\nu)}$

9. 解：(1) $f_X(x) = c\int_x^\infty x(y-x)e^{-y}dy = cx\int_x^\infty ye^{-y}dy - cx^2\int_x^\infty e^{-y}dy$

$= cx(x+1)e^{-x} - cx^2e^{-x} = cxe^{-x}$

$1 = \int_0^\infty f_X(x) = \int_0^\infty cxe^{-x} = c\cdot(-x-1)e^{-x}\Big|_0^\infty = c \Rightarrow c=1$

(2) $f_X(x) = xe^{-x}$ $\quad f(x,y) = x(y-x)e^{-y}$

$f_Y(y) = \int_0^y x(y-x)e^{-y}dx = e^{-y}\int_0^y x(y-x)dx = e^{-y}\cdot y\int_0^y xdx - e^{-y}\int_0^y x^2dx = \frac{1}{6}y^3\cdot e^{-y}$

$\Rightarrow f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = 6x(y-x)\cdot y^{-3}$ $\quad f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = (y-x)e^{x-y}$

(3) $E(X|Y=y) = \int_0^y 6x(y-x)\cdot y^{-3}\cdot x\,dx = \frac{1}{y^3}\int_0^y(6x^2y-6x^3)dx = \frac{y}{2} \Rightarrow E[X|Y] = \frac{Y}{2}$

$E(Y|X=x) = \int_x^\infty y(y-x)e^{x-y}dy = e^x\int_x^\infty y(y-x)e^{-y}dy = e^x\int_x^\infty y^2e^{-y}dy - e^x\cdot x\int_x^\infty ye^{-y}dy$

$= x+2 \Rightarrow E[Y|X] = X+2.$

4.6

10. Let $\{X_r : r \geq 0\}$ be independent and identically distributed random variables with density function f and distribution function F. Let $N = \min\{n \geq 1 : X_n > X_0\}$ and $M = \min\{n \geq 1 : X_0 \geq X_1 \geq \cdots \geq X_{n-1} < X_n\}$. Show that X_N has distribution function $F + (1-F)\log(1-F)$, and find $\mathbb{P}(M = m)$.

解：$\{N>n\}$意味着 X_0 是 $X_1,\cdots,X_n$ 中最大的 $\quad P(N>n) = \frac{1}{n+1}$

$P(N=n) = P(N>n-1) - P(N>n) = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$

$P(X_N \leq x) = \sum_{n=1}^\infty P(X_N \leq x, N=n) = \sum_{n=1}^\infty F(x)^{n+1}\cdot\frac{1}{n(n+1)}$

$= \sum_{n=1}^\infty \frac{F(x)^{n+1}}{n} - \sum_{n=1}^\infty \frac{F(x)^{n+1}}{n+1} = F(x)\sum_{n=1}^\infty \frac{F(x)^n}{n} - \sum_{n=1}^\infty \frac{F(x)^n}{n} + F(x)$

$= F(x) + (F(x)-1)\sum_{n=1}^\infty \frac{F(x)^n}{n} = F(x) + (1-F(x))\log(1-F(x))$

$P(M=m) = P(X_0 \geq X_1 \geq \cdots \geq X_{m-1}) - P(X_0 \geq X_1 \geq \cdots \geq X_m)$

$= \frac{1}{m!} - \frac{1}{(m+1)!}$

4.7

2. Let X and Y be independent exponential random variables with parameter 1. Find the joint density function of $U = X + Y$ and $V = X/(X+Y)$, and deduce that V is uniformly distributed on $[0, 1]$.

解: $\begin{cases} U = X+Y \\ V = \dfrac{X}{X+Y} \end{cases} \Rightarrow \begin{cases} X = UV \\ Y = U - UV \end{cases}$ $\quad J = \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -u$

$f_{X,Y}(x,y) = e^{-x}e^{-y} = e^{-(x+y)}$

$f_{U,V}(u,v) = e^{-u}\cdot|J| = ue^{-u} \quad 0 \le u < \infty,\ 0 \le v < 1.$

$f_V(v) = \int_0^{+\infty} ue^{-u}du = 1$ on $[0,1)$ $\Rightarrow$ V在[0,1]上均匀分布.

4.7

5. **Normal orthant probability.** Let X and Y have the bivariate normal density function

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right\}.$$

Show that X and $Z = (Y - \rho X)/\sqrt{1-\rho^2}$ are independent $N(0,1)$ variables, and deduce that

$$\mathbb{P}(X > 0,\ Y > 0) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho.$$

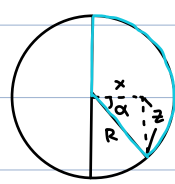
证: $\left|\dfrac{\partial(x,z)}{\partial(x,y)}\right| = \begin{vmatrix} 1 & 0 \\ -\dfrac{\rho}{\sqrt{1-\rho^2}} & \dfrac{1}{\sqrt{1-\rho^2}} \end{vmatrix} = \dfrac{1}{\sqrt{1-\rho^2}}$ $\qquad x^2 - 2\rho xy + y^2 = (x^2+z^2)(1-\rho^2)$

$f_{X,Z}(x,z) = \dfrac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\dfrac{1}{2(1-\rho^2)}(x^2+z^2)(1-\rho^2)\right)\cdot\sqrt{1-\rho^2}\,dxdz$

$= \dfrac{1}{2\pi} e^{-\frac{1}{2}(x^2+z^2)} = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\cdot \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} = f_X(x)\cdot f_Z(z)$

$\Rightarrow$ X,Z独立且同N(0,1)分布.

$\{X>0, Y>0\} = \{X>0,\ Z > -\dfrac{\rho x}{\sqrt{1-\rho^2}}\}$



$\tan\alpha = -\dfrac{\rho}{\sqrt{1-\rho^2}} \Rightarrow \alpha = -\arctan\dfrac{\rho}{\sqrt{1-\rho^2}}$

用极坐标表示:

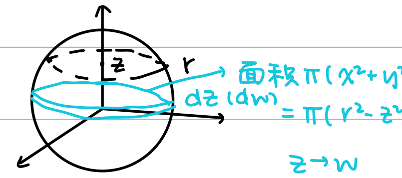
$\Rightarrow \alpha = -\sin^{-1}\rho$

$P(X>0, Y>0) = \int_\alpha^{\frac{\pi}{2}}\int_0^\infty \frac{1}{2\pi}e^{-\frac{1}{2}r^2}\cdot r\,dr\,d\theta = \int_\alpha^{\frac{\pi}{2}} \frac{1}{2\pi}\,d\theta = \frac{1}{2\pi}\left(\frac{\pi}{2} - \alpha\right) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho$

4.7

9. A point (X, Y, Z) is picked uniformly at random inside the unit ball of $\mathbb{R}^3$. Find the joint density of Z and R, where $R^2 = X^2 + Y^2 + Z^2$.

解：单位球体积 $V = \frac{4\pi}{3}$



$\{R \le r, Z \le z\}$ 内体积 $V' = \int_{-r}^{z} \pi(r^2 - w^2)\,dw$.

$P(R \le r, Z \le z) = \frac{3}{4\pi}\int_{-r}^{z} \pi(r^2 - w^2)\,dw$

求导得 $f(r, z) = \frac{3}{2} r$ as $|z| < r < 1$.

4.9

3. Let $\mathbf{X} = (X_1, X_2, \dots, X_n)$ have the $N(\boldsymbol{\mu}, \mathbf{V})$ distribution, and show that $Y = a_1X_1 + a_2X_2 + \cdots + a_nX_n$ has the (univariate) $N(\mu, \sigma^2)$ distribution where

$$\mu = \sum_{i=1}^{n} a_i \mathbb{E}(X_i), \qquad \sigma^2 = \sum_{i=1}^{n} a_i^2 \operatorname{var}(X_i) + 2\sum_{i<j} a_i a_j \operatorname{cov}(X_i, X_j).$$

证：$E[Y] = E[a_1X_1 + \cdots + a_nX_n] = a_1E[X_1] + \cdots + a_nE[X_n] = \sum_{i=1}^{n} a_iE[X_i]$

$$\operatorname{Var}(Y) = \operatorname{Var}\Big(\sum_{i=1}^{n} a_iX_i\Big) = \sum_{i=1}^{n} \operatorname{Var}(a_iX_i) + 2\sum_{i<j} \operatorname{cov}(a_iX_i, a_jX_j)$$
$$= \sum_{i=1}^{n} a_i^2 \operatorname{Var}(X_i) + 2\sum_{i<j} a_ia_j \operatorname{cov}(X_i, X_j)$$

4.9

7. Let the vector $(X_r : 1 \le r \le n)$ have a multivariate normal distribution with covariance matrix $\mathbf{V} = (v_{ij})$. Show that, conditional on the event $\sum_1^n X_r = x$, X_1 has the $N(a, b)$ distribution where $a = (\rho s/t)x$, $b = s^2(1 - \rho^2)$, and $s^2 = v_{11}$, $t^2 = \sum_{ij} v_{ij}$, $\rho = \sum_i v_{i1}/(st)$.

证：$a = E[X_1 \mid \sum_{r=1}^{n} X_r = x]$ $\qquad b = \operatorname{Var}(X_1 \mid \sum_{r=1}^{n} X_r = x)$

Lem **7.** Let X and Y have a bivariate normal density with zero means, variances σ^2, τ^2, and correlation ρ. Show that:

(a) $\mathbb{E}(X \mid Y) = \dfrac{\rho\sigma}{\tau} Y$,

(b) $\operatorname{var}(X \mid Y) = \sigma^2(1 - \rho^2)$,

有$E[X|Y] = \frac{\mathrm{cov}(X,Y)}{\mathrm{Var}(Y)} \cdot Y$, $\mathrm{Var}(X|Y) = \{1-\rho(X,Y)^2\} \cdot \mathrm{Var}(X)$

T 4.9.7 中 $\mathrm{var}\, X_1 = V_{11} = s^2$, $\mathrm{Var}(\sum_{r=1}^{n} X_r) = \sum_{i,j} V_{ij} = t^2$,

$\mathrm{cov}(X_1, \sum_{r=1}^{n} X_r) = \sum_i V_{i1} = st\cdot\rho$

代入 $a = E[X_1 | \sum_{r=1}^{n} X_r = x] = \frac{st\rho}{t^2} \cdot x = \frac{\rho s}{t} x$

$b = \mathrm{Var}(X_1 | \sum_{r=1}^{n} X_r = x) = (1 - \frac{\rho^2 s^2 t^2}{s^2 t^2}) \cdot s^2 = s^2(1-\rho^2)$