**1.** There are two roads from A to B and two roads from B to C. Each of the four roads is blocked by snow with probability $p$, independently of the others. Find the probability that there is an open road from A to B given that there is no open route from A to C.

If, in addition, there is a direct road from A to C, this road being blocked with probability $p$ independently of the others, find the required conditional probability.

**2.** Calculate the probability that a hand of 13 cards dealt from a normal shuffled pack of 52 contains exactly two kings and one ace. What is the probability that it contains exactly one ace given that it contains exactly two kings?

1.7.1. ① $P(A\leftrightarrow B \mid A \not\leftrightarrow C) = \dfrac{P(A\leftrightarrow B,\ A\not\leftrightarrow C)}{P(A\not\leftrightarrow C)}$

$= \dfrac{P(A\leftrightarrow B,\ B\not\leftrightarrow C)}{1-P(A\leftrightarrow C)}$

$= \dfrac{(1-p^2)\,p^2}{1-(1-p^2)^2}$

② $P(A\leftrightarrow B \mid A \not\leftrightarrow C) = \dfrac{P(A\leftrightarrow B,\ B\not\leftrightarrow C)\cdot P(\text{extra road be blocked})}{1-P(A\leftrightarrow C)}$

$= \dfrac{(1-p^2)\,p^3}{1-(1-p^2)^2 p-(1-p)}$

$= \dfrac{(1-p^2)\,p^2}{1-(1-p^2)^2}$

1.7.2: 令 A 为事件恰好抽到一张 Ace.

KK ···· 抽到两张 K

$P(A\cap KK) = \dfrac{C_4^1\, C_4^2\, C_{44}^{10}}{C_{52}^{13}}$, $P(KK) = \dfrac{C_4^2 \cdot C_{48}^{11}}{C_{52}^{13}}$

$\therefore P(A \mid KK) = \dfrac{C_4^1 \cdot C_{44}^{10}}{C_{48}^{11}}$

**5.** A pack contains $m$ cards, labelled $1, 2, \ldots, m$. The cards are dealt out in a random order, one by one. Given that the label of the $k$th card dealt is the largest of the first $k$ cards dealt, what is the probability that it is also the largest in the pack?

1.7.5: 记 $L_k$ 为第 $k$ 个卡的标签.

$$P(L_k > L_r \text{ for } 1 \le r < k) = \sum_{|A|=k,\, A \subseteq [m]} P(\{L_1 \cdots L_k\} = A) \cdot P(L_k \text{为}A\text{中最大} \mid \{L_1, \cdots, L_k\} = A)$$

$$= C_m^k \cdot \frac{1}{C_m^k} \cdot \frac{1}{k} = \frac{1}{k}$$

$$P(L_k = m \mid L_k > L_r \text{ for } 1 \le r < k)$$

$$= \frac{P(L_k = m)}{P(L_k > L_r \text{ for } 1 \le r < k)} = \frac{k}{m}$$

**9.** Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $B \in \mathcal{F}$ satisfies $\mathbb{P}(B) > 0$. Let $\mathbb{Q} : \mathcal{F} \to [0, 1]$ be defined by $\mathbb{Q}(A) = \mathbb{P}(A \mid B)$. Show that $(\Omega, \mathcal{F}, \mathbb{Q})$ is a probability space. If $C \in \mathcal{F}$ and $\mathbb{Q}(C) > 0$, show that $\mathbb{Q}(A \mid C) = \mathbb{P}(A \mid B \cap C)$; discuss.

1.8.9: 按定义验证 $(\Omega, \mathcal{F}, Q)$ 为 probability space.

(i) $Q(\phi) = P(\phi \mid B) = 0$, $\quad Q(\Omega) = P(\Omega \mid B) = P(B)/P(B) = 1$.

(ii) 令 $A_1, A_2, \cdots$ 为一列不交的事件.

则 $\{A_i \cap B \mid i \ge 1\} \subseteq \mathcal{F}$ 为不交事件.

$$Q\left(\bigcup_1^\infty A_i\right) = P\left(\bigcup_1^\infty A_i \mid B\right) = \frac{P(\bigcup_1^\infty (A_i \cap B))}{P(B)} = \sum_1^\infty \frac{P(A_i \cap B)}{P(B)} = \sum_1^\infty Q(A_i)$$

$$Q(A \mid C) = \frac{Q(A \cap C)}{Q(C)} = \frac{P(A \cap C \mid B)}{P(C \mid B)} = \frac{P(A \cap B \cap C)}{P(B \cap C)} = P(A \mid B \cap C)$$

**11.** Prove **Boole's inequalities**:

$$\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) \le \sum_{i=1}^n \mathbb{P}(A_i), \qquad \mathbb{P}\left(\bigcap_{i=1}^n A_i\right) \ge 1 - \sum_{i=1}^n \mathbb{P}(A_i^c).$$

1.8.11: 数学归纳法: $n=1$ 时成立.

取 $m \ge 1$, 第一个不等式对 $\forall\, n \le m$ 成立.

则 $$P\left(\bigcup_{i=1}^{m+1} A_i\right) = P\left(\bigcup_{i=1}^m A_i\right) + P(A_{m+1}) - P\left(\bigcup_{i=1}^m (A_i \cap A_{m+1})\right)$$

$$\le P\left(\bigcup_{i=1}^m A_i\right) + P(A_{m+1}) \le \sum_{i=1}^{m+1} P(A_i)$$

$$P\left(\bigcap_{i=1}^n A_i\right) = P\left(\left(\bigcup_{i=1}^n A_i^c\right)^c\right) = 1 - P\left(\bigcup_{i=1}^n A_i^c\right) \ge 1 - \sum_{i=1}^n P(A_i^c)$$

**16.** Let $A_1, A_2, \ldots$ be a sequence of events. Define

$$B_n = \bigcup_{m=n}^{\infty} A_m, \quad C_n = \bigcap_{m=n}^{\infty} A_m.$$

Clearly $C_n \subseteq A_n \subseteq B_n$. The sequences $\{B_n\}$ and $\{C_n\}$ are decreasing and increasing respectively with limits

$$\lim B_n = B = \bigcap_n B_n = \bigcap_n \bigcup_{m \geq n} A_m, \qquad \lim C_n = C = \bigcup_n C_n = \bigcup_n \bigcap_{m \geq n} A_m.$$

The events $B$ and $C$ are denoted $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$ respectively. Show that
(a) $B = \{\omega \in \Omega : \omega \in A_n$ for infinitely many values of $n\}$,
(b) $C = \{\omega \in \Omega : \omega \in A_n$ for all but finitely many values of $n\}$.
We say that the sequence $\{A_n\}$ converges to a limit $A = \lim A_n$ if $B$ and $C$ are the same set $A$. Suppose that $A_n \to A$ and show that
(c) $A$ is an event, in that $A \in \mathcal{F}$,
(d) $\mathbb{P}(A_n) \to \mathbb{P}(A)$.

1.8.16: (a) $\omega \in B \iff \forall n,\ \omega \in \bigcup_{i=n}^{\infty} A_n \iff \omega$ 属于无限个 $A_n$

(b) $\omega \in C \iff \exists n,\ \omega \in \bigcap_{i=n}^{\infty} A_n \iff \exists n,\ \omega$ 只能在 $A_1, \cdots, A_{n-1}$ 里不出现

$\iff \omega$ 只在有限个 $A_n$ 里不出现.

(c) $B$ 为可数并的可数交, $\therefore B \in \mathcal{F}$.

$A = B \Rightarrow A \in \mathcal{F}$.

(d) $C_n = \bigcap_{i=n}^{\infty} A_i \subset A_n \subseteq \bigcup_{i=n}^{\infty} A_i = B_n$.

$\therefore P(C_n) \leq P(A_n) \leq P(B_n)$.

$\because C_n \uparrow C, \quad B_n \downarrow B$

$\therefore P(C_n) \to P(C) \quad P(B_n) \to P(B)$

从而由 $B = C = A$

$\Rightarrow P(A) = P(C) \leq \lim_{n\to\infty} P(A_n) \leq P(B) = P(A)$

**2.** A random variable $X$ has distribution function $F$. What is the distribution function of $Y = aX + b$, where $a$ and $b$ are real constants?

if $a \neq 0$, $F_Y(y) = P(Y \leq y) = P(aX + b \leq y)$

$$= \begin{cases} P\left(X \leq \frac{(y-b)}{a}\right) = F\left(\frac{y-b}{a}\right), & \text{if } a > 0. \\ P\left(X \geq \frac{y-b}{a}\right) = 1 - \lim_{x \uparrow \frac{y-b}{a}} F(x), & \text{if } a < 0. \end{cases}$$

if $a = 0$, $F_Y(y) = \begin{cases} 0, & b > y \\ 1, & b \leq y \end{cases}$

**4.** Show that if $F$ and $G$ are distribution functions and $0 \le \lambda \le 1$ then $\lambda F + (1-\lambda)G$ is a distribution function. Is the product $FG$ a distribution function?

**5.** Let $F$ be a distribution function and $r$ a positive integer. Show that the following are distribution functions:
(a) $F(x)^r$,
(b) $1 - \{1 - F(x)\}^r$,
(c) $F(x) + \{1 - F(x)\}\log\{1 - F(x)\}$,
(d) $\{F(x) - 1\}e + \exp\{1 - F(x)\}$.

2.1.4: 令 $H_1 = \lambda F + (1-\lambda)G$.

$\lim_{x\to-\infty} H_1(x) = 0$, $\lim_{x\to\infty} H_1(x) = 1$. 且 $H_1$ 不减, 右连续.

从而 $H_1$ 为分布函数.

$H_2 = FG$. 则

$\lim_{x\to-\infty} H_2(x) = 0$, $\lim_{x\to+\infty} H_2(x) = 1$. 且 $H_2$ 不减, 右连续

$\therefore H_2$ 为分布函数.

2.1.5: claim: 若 $g$ 为 $[0,1]$ 上连续不减函数且 $g(0)=0$, $g(1)=1$.

则 $g(F(x))$ 为分布函数.

proof: $\lim_{x\to-\infty} g(F(x)) = g(\lim_{x\to-\infty} F(x))$

$= g(0) = 0.$

同理 $\lim_{x\to+\infty} g(F(x)) = 1$.

$g\circ F$ 不减 且右连续. $\therefore g\circ F$ 为分布函数.

(a) (b) (c) (d) 均为此形式

**2.** Let $X$ be a random variable and let $g : \mathbb{R} \to \mathbb{R}$ be continuous and strictly increasing. Show that $Y = g(X)$ is a random variable.

**3.** Let $X$ be a random variable with distribution function

$$\mathbb{P}(X \le x\} = \begin{cases} 0 & \text{if } x \le 0, \\ x & \text{if } 0 < x \le 1, \\ 1 & \text{if } x > 1. \end{cases}$$

Let $F$ be a distribution function which is continuous and strictly increasing. Show that $Y = F^{-1}(X)$ is a random variable having distribution function $F$. Is it necessary that $F$ be continuous and/or strictly increasing?

2.3.2: 对 $\forall y \in g(\mathbb{R})$ $\{Y \le y\} = \{X \le g^{-1}(y)\} \in \mathcal{F}$

2.3.3: (构造)

random variable

for any distribution function $F$. we can find a R.V. $X$,

s.t. $P(X \le x) = F(x)$

pf: first we should define $F^{-1}$,

let $F^{-1}(x) := \inf\{y : F(y) \ge x\}$, $x \in [0,1]$

then we claim: $F^{-1}(x) \le a \iff x \le F(a)$.

($\Rightarrow$) if $F^{-1}(x) \le a$, prove by contradiction.

suppose $x > F(a)$.

by right-continuous of $F$, $\exists\, \varepsilon > 0$, s.t. $x > F(a+\varepsilon) > F(a)$.

$\Rightarrow$ $a+\varepsilon \in \{y : F(y) < x\}$

$\Rightarrow$ $a+\varepsilon \le F^{-1}(x)$ . which is contradicted to $F^{-1}(x) \le a$

($\Leftarrow$) when $x \le F(a)$

note that $\{y : F(y) \ge x\} \supseteq \{y : F(y) \ge F(a)\}$

take inf on the both side.

$\Rightarrow$ $F^{-1}(x) \le F^{-1}(F(a)) \le a$

then $F^{-1}(U) \sim F$, $U \sim \text{uniform}(0,1)$