5.4

3. 令 $g(z)=f(\frac{1}{z})$，则 $g\in H(B(0,\frac{1}{R})\setminus\{0\})$

(i) ∞ 为 f 可去奇点 $\Leftrightarrow$ 0 为 g 的可去奇点

此时，$g'(z)=(f(\frac{1}{z}))'=-\frac{1}{z^2}f'(\frac{1}{z})$

而 $Res(f,\infty)=-\frac{1}{2\pi i}\int_{|z|=M}f(z)dz=-\frac{1}{2\pi i}\int_0^{2\pi}f(Me^{i\theta})Mie^{i\theta}d\theta$

$=-\frac{1}{2\pi i}\int_0^{2\pi}g(\frac{1}{M}e^{-i\theta})Mie^{i\theta}d\theta=-\frac{1}{2\pi i}\int_{|z|=\frac{1}{M}}\frac{g(z)}{z^2}dz=-g'(0)=-\lim_{z\to 0}g'(z)$

$=-\lim_{z\to 0}(-\frac{1}{z^2}f'(\frac{1}{z}))=\lim_{z\to\infty}(z^2f'(z))$

(ii) ∞ 为 f m 阶极点 $\Leftrightarrow f(z)=\sum_{k=-m}^{+\infty}a_kz^k$

$\Rightarrow Res(f,\infty)=-a_{-1}=\frac{(-1)^m}{(m+1)!}\lim_{z\to\infty}z^{m+2}f^{(m+1)}(z)$

4. a 为 $\frac{f}{g}$ 的二阶极点，由公式得：$Res(\frac{f}{g},a)=\lim_{z\to a}((z-a)^2\frac{f(z)}{g(z)})'$

$=\lim_{z\to a}\frac{2(z-a)f(z)g(z)+(z-a)^2f'(z)g(z)-(z-a)^2f(z)g'(z)}{g^2(z)}=\frac{2f'(a)}{g''(a)}-\frac{2f(a)g'''(a)}{(g''(a))^2}$

6. 由Cauchy积分公式，只需考虑单个极点或零点附近

$\frac{1}{2\pi i}\int_{|z-z_i|=\varepsilon}g(z)\frac{f'(z)}{f(z)}dz=\frac{1}{2\pi i}\int_{|z-z_i|=\varepsilon}g(z)\frac{(z-z_i)^{p_i}h'(z)+p_i(z-z_i)^{p_i-1}h(z)}{(z-z_i)^{p_i}h(z)}dz$

$(f(z)=(z-z_i)^{p_i}h(z))$ $=\frac{1}{2\pi i}\int_{|z-z_i|=\varepsilon}g(z)(\frac{h'(z)}{h(z)}+\frac{p_i}{z-z_i})dz=p_ig(z_i)$

极点附近类似

7. (2) $Res(\frac{1}{(1+z^2)^{n+1}},i)=\frac{1}{n!}\lim_{z\to i}\frac{d^n}{dz^n}(\frac{1}{(z+i)^{n+1}})=-\frac{i}{2^{2n+1}}C_{2n}^n$

(4) $e^{\frac{1}{z}}=\sum_{n=0}^{\infty}\frac{1}{n!}(\frac{1}{z})^n$，$Log\frac{1-\alpha z}{1-\beta z}=\sum_{n=1}^{\infty}(\frac{\alpha^nz^n}{n}-\frac{\beta^nz^n}{n})+C$

$\Rightarrow \frac{1}{z^2}e^{\frac{1}{z}}Log\frac{1-\alpha z}{1-\beta z}$ 中 $\frac{1}{z}$ 系数为 $\sum_{n=1}^{\infty}(\frac{\alpha^n-\beta^n}{n})\cdot\frac{1}{(n-1)!}=\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}-\sum_{n=0}^{\infty}\frac{\beta^n}{n!}$

$=e^{\alpha}-e^{\beta}$

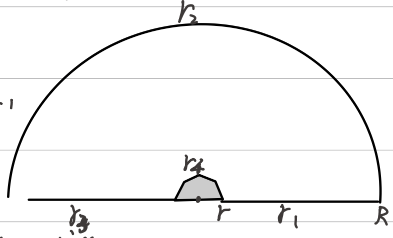
(6) $\cot z$ 的洛朗展开 $\cot z=\sum_{n=0}^{\infty}a_n z^{2n-1}$ 满足 $a_0=1$, $a_1=-\frac{1}{3}$

则 $\cot^3 z$ 中 z^{-1} 系数为 -1，$\mathrm{Res}(\cot^3 z,0)=-1$

5.5

1. (1) $\int_0^\infty \frac{x^2+1}{x^4+1}dx=\frac{1}{2}\int_{-\infty}^{\infty}\frac{z^2+1}{z^4+1}dz$，令 $f(z)=\frac{z^2+1}{z^4+1}$，则

$\lim_{z\to\infty} zf(z)=0$，由 Thm 5.5.1 $\Rightarrow \int_0^\infty \frac{x^2+1}{x^4+1}dx=\pi i\left(\mathrm{Res}(f,e^{\frac{\pi i}{4}})+\mathrm{Res}(f,e^{\frac{3\pi i}{4}})\right)=\frac{\sqrt{2}}{2}\pi$



(2) 考虑书中图5.9围道：$f(z)=\frac{e^{2iz}-1}{z^2}$

γ_1上积分为 $\int_r^R \frac{e^{2ix}-1}{x^2}dx$

γ_3上积分为 $\int_{-R}^{-r}\frac{e^{2ix}-1}{x^2}dx=\int_r^R\frac{e^{-2ix}-1}{x^2}dx$

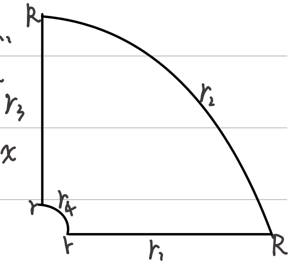
相加为：$\int_r^R\frac{2\cos 2x-2}{x^2}dx=-4\int_r^R\frac{\sin^2 x}{x^2}dx$

由引理5.5.9 γ_4上积分为 $-\pi i\lim_{z\to 0}zf(z)=-\pi i\cdot 2i=2\pi$（当 $r\to 0$ 时）

由Jordan引理 γ_2上积分趋于0（$\frac{e^{2iz}}{z^2}$ 在 γ_2 上积分 $\to 0$，$\frac{1}{z^2}$ 在 γ_2 上积分也趋于0）$\Rightarrow \lim_{\substack{r\to 0\\ R\to\infty}}4\int_r^R\frac{\sin^2 x}{x^2}dx=2\pi$，即得结果

(3) $\int_0^\infty\frac{1}{1+x^p}dx=\int_0^\infty\frac{dy^{\frac{1}{p}}}{1+y}=\frac{1}{p}\int_0^\infty\frac{y^{\frac{1}{p}-1}}{1+y}dy=\frac{\pi}{p\sin\frac{\pi}{p}}$（由例5.5.11及 $0<\frac{1}{p}<1$）

(4) $\int_0^\infty\frac{\log x}{x^2-1}dx$，令 $f(z)=\frac{\log z}{z^2-1}$，1为可去奇点，不用加围道



γ_1: $\int_r^R\frac{\log x}{x^2-1}dx$；$\gamma_3$: $\int_R^r\frac{\log x+\frac{\pi}{2}i}{-(x^2+1)}i\,dx$

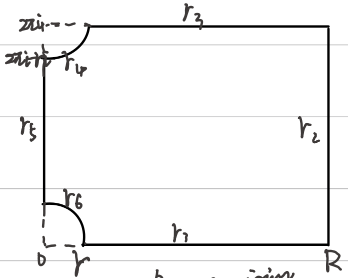
$=-\int_r^R\frac{\pi}{2(x^2+1)}dx+i\int_r^R\frac{\log x}{x^2+1}dx$

γ_4上积分 $\le\int_{\gamma_4}\frac{\log r+\frac{\pi}{2}}{1-r^2}|dz|=\frac{\frac{\pi}{2}r(\log r+\frac{\pi}{2})}{1-r^2}$

而 $r\log r\to 0$（当 $r\to 0$）$\Rightarrow \gamma_4$上趋于0，类似有 γ_2上趋于0

取实部得: $\lim\limits_{\substack{r\to 0\\ R\to\infty}} \frac{\log x}{x^2-1} = \frac{\pi}{2}\int_0^\infty \frac{1}{x^2+1}dx = \frac{\pi^2}{4}$

(4) $\int_0^\infty \frac{\sin x}{e^x-1}dx$, $f(z)=\frac{e^{iz}}{e^z-1}$



γ_1: $\int_r^R \frac{\cos x + i\sin x}{e^x-1}dx$

γ_2: $\left|\int_0^{2\pi} i\frac{e^{i(iy+R)}}{e^{iy+R}-1}dy\right| \le \int_0^{2\pi}\frac{e^y}{e^R-1}dy \to 0$

γ_3: $\int_R^r \frac{e^{i(x+2\pi i)}}{e^{x+2\pi i}-1}dx = -e^{-2\pi}\int_r^R \frac{e^{ix}}{e^x-1}dx = -e^{-2\pi}\int_r^R\frac{\cos x+i\sin x}{e^x-1}dx$

由引理5.5.9 γ_4上积分趋于 $-\frac{\pi}{2}i e^{i(2\pi i)} = -\frac{\pi}{2}e^{-2\pi}i$

γ_6上积分趋于 $-\frac{\pi}{2}ie^{i\cdot 0} = -\frac{\pi}{2}i$

γ_5上积分为 $\int_{2\pi-r}^r i\frac{e^{i(iy)}}{e^{iy}-1}dy = -i\int_r^{2\pi-r}\frac{e^{-y}}{e^{iy}-1}dy$

取虚部知 $\int_0^\infty \frac{\sin x}{e^x-1}(1-e^{-2\pi}) = \frac{\pi}{2}(1+e^{-2\pi}) + \int_0^{2\pi} e^{-y}\left(\mathrm{Re}\frac{1}{e^{iy}-1}\right)dy$

$= \frac{\pi}{2}(1+e^{-2\pi}) + \int_0^{2\pi}\frac{1}{2}e^{-y}dy$

即得所求

5.7

2. (1) $e^z-1 = e^{\frac z2}(e^{\frac z2}-e^{-\frac z2}) = e^{\frac z2}\cdot 2i\sin(-\frac{iz}{2})$

而 $\sin z = z\prod_{n=1}^\infty\left(1-\frac{z^2}{n^2\pi^2}\right) \Rightarrow e^z-1 = ze^{\frac z2}\prod_{n=1}^\infty\left(1+\frac{z^2}{4n^2\pi^2}\right)$

(2) $\cos z = e^{Az+B}\prod_{n=1}^\infty\left(1-\frac{4z^2}{(2n-1)^2\pi^2}\right)$，令 $z=0$，则 $e^B=1$

由 $\cos z=\cos(-z)$ 知 $e^{Az}=e^{-Az}$，$A=0$，即 $\cos z = \prod_{n=1}^\infty\left(1-\frac{4z^2}{(2n-1)^2\pi^2}\right)$

6.2

3. 设 $f(z)=\sum_{n=0}^{\infty}a_nz^n$，$g(z)=f(z)-\frac{Res(f,z_0)}{z-z_0}$

而 $\frac{Res(f,z_0)}{z-z_0}$ 收敛圆周与f相同，也仅有一个1阶极点 z_0

$\Rightarrow$ $g(z)$ 在 $|z|\le|z_0|$ 中全纯，且g收敛半径比 $|z_0|$ 大

$\Rightarrow g(z_0)=\sum_{n=0}^{\infty}b_nz_0^n$ 绝对收敛　而 $a_n=b_n+Res(f,z_0)z_0^{-(n+1)}$

$\Downarrow$

$\lim_{n\to\infty}|b_nz_0|^n=0$　$\Rightarrow \lim_{n\to\infty}\frac{a_n}{a_{n+1}}=\lim_{n\to\infty}\frac{b_n+Res(f,z_0)z_0^{-(n+1)}}{b_{n+1}+Res(f,z_0)z_0^{-(n+2)}}$

$=\lim_{n\to\infty}\frac{b_nz_0^{n+1}+Res(f,z_0)}{b_{n+1}z_0^{n+2}+Res(f,z_0)}\cdot z_0=z_0$

6. 1显然为奇点，$f(z)=\sum_{n=0}^{\infty}z^{2^n}$，$f(e^{2\pi i\cdot\frac{l}{2^k}}z)=\sum_{n=0}^{k-1}(ze^{2\pi i\cdot\frac{l}{2^k}})^{2^n}$

$+\sum_{n=k}^{\infty}z^{2^n}$　$\Rightarrow$ z，$ze^{2\pi i\cdot\frac{l}{2^k}}$ 同时为奇点或者同不为奇点

$\Rightarrow\{e^{2\pi i\cdot\frac{l}{2^k}}\}_{k,l\in N}$ 为奇点，由其稠密性知 $|z|=1$ 均为奇点