11.29 作业

补充题

Assignment: $X_1, \ldots, X_n$ *i.i.d.* $\sim$ *Exp*(λ). **Solve the level** $\alpha = 0.1$ **UMPT for** $H_0: \ \lambda \le 1$ **or** $\lambda \ge 2$**,** $H_1: \ 1 < \lambda < 2$**.**

Solve: Notice that the joint $p.d.f.$ is

$$f(\mathbf{x}|\lambda) = \lambda^n e^{-\lambda \sum_i x_i} I_{(0,\infty)}(x_{(1)}),$$

where λ is strictly increasing with respect to λ and $T(\mathbf{X}) := -\sum_i X_i$. Then the reject region for a level α UMPT should be $\{t_1 < T < t_2\}$. Similar to Ex. 5.48, we know that $-2\lambda T \sim \chi^2_{2n}$, thus t_1, t_2 can be solved by

$$\begin{cases} \alpha = P(t_1 < T < t_2 | \lambda = 1) = P\big(-2t_1 > \chi^2_{2n} > -2t_2\big) = \chi^2_{2n}(-2t_1) - \chi^2_{2n}(-2t_2) \\ \alpha = P(t_1 < T < t_2 | \lambda = 2) = P\big(-4t_1 > \chi^2_{2n} > -4t_2\big) = \chi^2_{2n}(-4t_1) - \chi^2_{2n}(-4t_2) \end{cases},$$

where $P\big(\chi^2_{2n} > \chi^2_{2n}(\alpha)\big) = \alpha$.

Thus, the UMPT rejects H_0 if $x_1 < \sum_i X_i < x_2$, where x_1, x_2 satisfy

$$\begin{cases} 0.1 = \chi^2_{2n}(2x_2) - \chi^2_{2n}(2x_1) \\ 0.1 = \chi^2_{2n}(4x_2) - \chi^2_{2n}(4x_1) \end{cases}$$

4.1

[Wei] 4.1. Notice that $Q(\mathbf{X}, \mu) = \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \sim N(0,1)$ is a pivot. Then a 0.95 confidence interval is $\{-z_{0.05/2} \le Q \le z_{0.05/2}\}$. With $\bar{X} = 4.7832$, $\sigma = 0.01$, $n = 5$, the confidence interval is

$$[\bar{X} - \frac{\sigma}{\sqrt{n}} z_{0.025}, \bar{X} + \frac{\sigma}{\sqrt{n}} z_{0.025}] = [4.774, 4.792].$$

4.3

[Wei] 4.3. Similar to Exercise 4.2, a 0.05 confidence interval is $[\bar{X} - \frac{S}{\sqrt{n}} t_{n-1}(\alpha/2), \bar{X} + \frac{S}{\sqrt{n}} t_{n-1}(\alpha/2)] = [1784, 2116]$. □

4.4

[Wei] 4.4. Here we have

$$\frac{\sqrt{n}(\bar{X} - \mu)}{4} \sim N(0,1).$$

and we may transform the probability as

$$P(\bar{X} - 1 < \mu < \bar{X} + 1) = P(-\frac{\sqrt{n}}{4} < Z < \frac{\sqrt{n}}{4}) \ge 0.9.$$

where $Z \sim N(0,1)$. We also note that $z_{\alpha/2} = 1.65$, which implies that $\frac{\sqrt{n}}{4} \ge 1.65$, thus we have $n \ge 44$.

4.10

[Wei] 4.10. The problem should be revised as "how many should n be at most?" Notice that $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$, and from the definition of lower confidence limit,

$$0.95 \le P\left(\sigma \ge \frac{\sqrt{(n-1)S^2}}{4}\right) = P(\chi^2_{n-1} \le 16).$$

Thus we have $n \le 9$. □

4.13

$m=5, n=7, \overline{X}=10.06, \overline{Y}=12.1857, S_1^2=0.093, S_2^2=0.1248, \overline{X}-\overline{Y}=-2.1257$

$S_\omega^2=\dfrac{(m-1)S_1^2+(n-1)S_2^2}{n+m-2}=0.1121, S_\omega=0.3348$,查表得$t_{m+n-2}(\alpha/2)=2.2281$,

算得$S_\omega t_{m+n-2}(\alpha/2)\sqrt{\dfrac{1}{n}+\dfrac{1}{m}}=0.4368$，则置信区间为

$$\left[\overline{X}-\overline{Y}-S_\omega t_{m+n-2}(\alpha/2)\sqrt{\frac{1}{n}+\frac{1}{m}}, \overline{X}-\overline{Y}+S_\omega t_{m+n-2}(\alpha/2)\sqrt{\frac{1}{n}+\frac{1}{m}}\right]$$

$=[-2.5625, -1.6889]$

4.14

[Wei] 4.14. Facts:

(1) $\frac{\bar{Y}-\bar{X}-(b-a)}{\sqrt{\sigma_1^2/m+\sigma_2^2/n}} \sim N(0,1)$.

(2) $\frac{(m-1)S_1^2}{\sigma_1^2}+\frac{(n-1)S_2^2}{\sigma_2^2} \sim \chi^2_{m+n-2}$.

Since $\sigma_2^2=\lambda\sigma_1^2$, $Q=\dfrac{\frac{\bar{Y}-\bar{X}-(b-a)}{\sqrt{\sigma_1^2/m+\sigma_2^2/n}}}{\sqrt{\frac{1}{m+n-2}\left(\frac{(m-1)S_1^2}{\sigma_1^2}+\frac{(n-1)S_2^2}{\sigma_2^2}\right)}}=\dfrac{\frac{\bar{Y}-\bar{X}-(b-a)}{\sqrt{1/m+\lambda/n}}}{\sqrt{\frac{1}{m+n-2}\left((m-1)S_1^2+(n-1)S_2^2/\lambda\right)}} \sim t_{m+n-2}$ is a pivot. Then a $1-\alpha$ confidence interval for $b-a$ is

$$\bar{Y}-\bar{X}\pm\sqrt{\frac{(1/m+\lambda/n)\left((m-1)S_1^2+(n-1)S_2^2/\lambda\right)}{(m+n-2)}}t_{m+n-2}(\alpha/2).$$

4.16

$S_1^2/S_2^2=0.893$,查表得$F_{m-1,n-1}(\alpha/2)=F_{9,9}(0.025)=4.03$,

$F_{m-1,n-1}(1-\alpha/2)=1/F_{n-1,m-1}(\alpha/2)=1/F_{9,9}(0.025)=1/4.03$,

则σ_A^2/σ_B^2的置信系数为95%的置信区间为

$$\left[\frac{S_1^2}{S_2^2}\cdot\frac{1}{F_{m-1,n-1}(\alpha/2)}, \frac{S_1^2}{S_2^2}\cdot F_{n-1,m-1}(\alpha/2)\right]=[0.222, 3.599]$$

4.19

[Wei] 4.19. We only consider the case $n>1$.

(1) Let $Y_i=\frac{X_i-\theta_1}{\theta_2-\theta_1}\sim U(0,1)$, $Z_i=\frac{\theta_2-X_i}{\theta_2-\theta_1}\sim U(0,1)$, $Q=\frac{(\theta_2-\theta_1)-(X_{(n)}-X_{(1)})}{\theta_2-\theta_1}$ then $Q=Y_{(1)}+Z_{(1)}$ and

$$f_{Y_{(1)},Z_{(1)}}(y,z)=f_{Y_{(1)},Y_{(n)}}(y,1-z)=n(n-1)(1-z-y)^{n-2}I_{(0,1)}(y,z,y+z).$$

Therefore, Q is a pivot with $p.d.f.$

$$f_Q(x)=\int f_{Y_{(1)},Z_{(1)}}(y,x-y)dy=n(n-1)x(1-x)^{n-2}I_{(0,1)}(x).$$

We can observe that $Q\sim Beta(2,n-1)$.

Suppose $P(Q \in (1-c, 1)) = 1-\alpha$ (this assumption is convenient for following analysis, you can generally suppose a confidence interval (a, b) instead), then $c = Beta_{n-1,2}(\alpha)$. Also notice that $Q = 1 - \frac{(X_{(n)} - X_{(1)})}{\theta_2 - \theta_1}$,

$$Q \in (1-c, 1) \Leftrightarrow \theta_2 - \theta_1 \in \big((X_{(n)} - X_{(1)})/c, \infty\big).$$

As a result, a $1-\alpha$ confidence interval for $\theta_2 - \theta_1$ is

$$\big((X_{(n)} - X_{(1)})/Beta_{n-1,2}(\alpha), \infty\big).$$

□

(2) Hint: the objective statistic can be rewritten as $R := \frac{Y_{(1)} + Y_{(n)} - 1}{Y_{(n)} - Y_{(1)}}$, which is auxiliary. Notice that $(X_{(1)}, X_{(n)})$ is sufficient and complete for (θ_1, θ_2). Basu's theorem tolds us $S := Y_{(n)} - Y_{(1)}$ is independent with R. Now write the $p.d.f$ of $(Y_{(1)}, Y_{(n)})$. With the relationship

$$\begin{cases} Y_{(1)} = \frac{1}{2}(RS - S + 1) \\ Y_{(n)} = \frac{1}{2}(RS + S + 1) \end{cases},$$

write the $p.d.f$ of (R, S). Consequently, $f_R = f_{(R,S)}/f_S$. The explicit form is very complicated, and should be discussed with nodes $0, \pm 1$. □

(3) Using R as the pivot. □

4.23

[Wei] 4.23.

(1) $X_{(1)} - \theta \sim Exp(n)$. □

(2) Using $Q := X_{(1)} - \theta$ in (1) as a pivot, suppose the interval as $(0, c)$ for Q. After calculation, the interval for θ is $(X_{(1)} + \log \alpha/n, X_{(1)})$. □

12.1 作业

5.20

5.20. 设 $X_i \overset{i.i.d.}{\sim} N(\mu_1, \sigma_1^2)$, $Y_j \overset{i.i.d.}{\sim} N(\mu_2, \sigma_2^2)$

$H_0: \frac{\sigma_1^2}{\sigma_2^2} = 1 \longleftrightarrow H_1: \frac{\sigma_1^2}{\sigma_2^2} \neq 1$

而 $\frac{Q_1^2/\sigma_1^2(6-1)}{Q_2^2/\sigma_2^2(9-1)} \sim F_{5,8}$，则 H_0 下，$\frac{Q_1^2}{Q_2^2} \cdot \frac{8}{5} \sim F_{5,8}$，记 $F_* := \frac{Q_1^2}{Q_2^2} \cdot \frac{8}{5}$

则水平为 α 的拒绝域为 $\{F_* > F_{5,8}(\frac{\alpha}{2})$ 或 $F_* < F_{5,8}(1-\frac{\alpha}{2})\}$

计算知 $F_* = 0.966$，$\alpha = 0.02$，查表有 $F_{5,8}(0.01) = 6.63$，$F_{5,8}(0.99) = \frac{1}{10.29} = 0.097$

$0.097 < 0.966 < 6.63$，故不能拒绝 H_0

即认为两机床方差无显著差异.

5.22

5.22. Step1. 先检验方差有无显著差异

设 I 期患者肺活量 $X_i \overset{i.i.d.}{\sim} N(\mu_1, \sigma_1^2)$，II 期肺活量 $Y_i \overset{i.i.d.}{\sim} N(\mu_2, \sigma_2^2)$

$H_0^1: \sigma_1^2 = \sigma_2^2 \longleftrightarrow H_1^1: \sigma_1^2 \neq \sigma_2^2$

考虑检验统计量 $F_* = \frac{S_1^2}{S_2^2}$

则 $\alpha = 0.1$ 时，拒绝域为 $\{F_* > F_{32,32}(0.05)$ 或 $F_* < F_{32,32}(0.95)\}$

计算知 $F_* = \left(\frac{147}{113}\right)^2 = 1.552$，$F_{32,32}(0.05) = 1.804$，$F_{32,32}(0.95) = 0.554$

水平 0.1 下不能拒绝 H_0^1

Step2. 认为方差相同，检验均值是否有显著差异

设 I 期：$X_i \sim N(\mu_1, \sigma^2)$　II 期：$Y_i \sim N(\mu_2, \sigma^2)$

$H_0^2: \mu_1 = \mu_2$　$H_1^2: \mu_1 \neq \mu_2$

考虑 $T = \frac{\bar{Y} - \bar{X}}{S_w} \cdot \sqrt{\frac{mn}{m+n}}$，$S_w^2 = \frac{1}{m+n-2}\left((m-1)S_1^2 + (n-1)S_2^2\right)$，$m = n = 33$

$T|_{H_0^2} \sim t_{m+n-2}$

则水平 $\alpha = 0.1$ 的拒绝域为 $\{|T| > t_{64}(0.05)\}$

计算知 $T = \frac{2830-2710}{\sqrt{\frac{1}{64}(32\times147^2 + 32\times113^2)}} \cdot \sqrt{\frac{32^2}{64}} = 3.601 > t_{64}(0.05) = 1.669$

拒绝 H_0^2

综上，认为 I, II 期患者肺活量有显著差异.

5.56

解：对于$H_0:\sigma_1^2=\sigma_2^2 \leftrightarrow H_1:\sigma_1^2 \neq \sigma_2^2$,令$F=\frac{S_1^2}{S_2^2}\cdot\frac{\sigma_2^2}{\sigma_1^2}$,则有接受域

检验问题$H_0:\sigma^2=\sigma_0^2 \leftrightarrow H_1:\sigma^2 \neq \sigma_0^2$的水平为$\alpha$的接受域为

$$\overline{D}=\left\{\boldsymbol{X}:\chi_{n-1}^2\left(1-\frac{\alpha}{2}\right)\leqslant\frac{(n-1)S^2}{\sigma_0^2}\leqslant\chi_{n-1}^2\left(\frac{\alpha}{2}\right)\right\}$$

H_0成立时$\sigma^2=\sigma_0^2$,故可解得置信区间为

$$\left[\frac{(n-1)S^2}{\chi_{n-1}^2\left(\frac{\alpha}{2}\right)},\frac{(n-1)S^2}{\chi_{n-1}^2\left(1-\frac{\alpha}{2}\right)}\right].$$

同理可解得置信上、下限为$\frac{(n-1)S^2}{\chi_{n-1}^2(1-\alpha)}$，$\frac{(n-1)S^2}{\chi_{n-1}^2(\alpha)}$

5.57

对于检验问题$H_0:\sigma_1^2/\sigma_2^2=c\leftrightarrow H_1:\sigma_1^2/\sigma_2^2\neq c$，$\dfrac{S_2^2}{S_1^2}\cdot c\sim F_{n-1,m-1}$,接受域为

$$\overline{D}=\left\{(\boldsymbol{X},\boldsymbol{Y}):F_{n-1,m-1}\left(1-\frac{\alpha}{2}\right)\leqslant\frac{S_2^2}{S_1^2}\cdot c\leqslant F_{n-1,m-1}\left(\frac{\alpha}{2}\right)\right\}$$

故置信区间为

$$\left[\frac{S_1^2}{S_2^2}\cdot F_{n-1,m-1}(1-\alpha/2),\frac{S_1^2}{S_2^2}\cdot F_{n-1,m-1}(\alpha/2)\right].$$

对于检验问题$H_0:\sigma_1^2/\sigma_2^2\leqslant c\leftrightarrow H_1:\sigma_1^2/\sigma_2^2>c$，接受域为

$$\overline{D}=\left\{(\boldsymbol{X},\boldsymbol{Y}):\frac{S_2^2}{S_1^2}\leqslant\frac{F_{n-1,m-1}(\alpha)}{c}\right\}\Rightarrow 1-\alpha=P\left(c\leqslant\frac{S_1^2}{S_2^2}\cdot F_{n-1,m-1}(\alpha)\right)$$

故置信上限为$\dfrac{S_1^2}{S_2^2}\cdot F_{n-1,m-1}(\alpha)$

同理可得置信下限为$\dfrac{S_1^2}{S_2^2}\cdot F_{n-1,m-1}(1-\alpha)$.

12.6 作业

作业 1

补充1：$X_1,\cdots,X_n$ i.i.d. $\sim N(\mu,\sigma^2)$，$n\geqslant4$，$\mu\in\mathbb{R}$，$\sigma^2>0$ 未知，求 μ，$\eta=\frac{1}{\sigma^2}$ 的水平为 $1-\alpha$ 的最短枢轴区间.

(1) $T=\dfrac{\sqrt{n}(\bar{X}-\mu)}{S}\sim t_{n-1}$ 为枢轴，其密度 f 单峰，对称，在0处达到极大

故考虑 $\int_{-a}^{a}f=1-\alpha\Rightarrow a=t_{n-1}(\frac{\alpha}{2})$

由 $|T|\leq t_{n-1}(\frac{\alpha}{2})$ 得置信水平 $1-\alpha$ 的最短枢轴区间为

$$\left[\bar{X}-\frac{S}{\sqrt{n}}t_{n-1}(\tfrac{\alpha}{2}),\ \bar{X}+\frac{S}{\sqrt{n}}t_{n-1}(\tfrac{\alpha}{2})\right].$$

(2) $Q=\dfrac{(n-1)S^2}{\sigma^2}\sim\chi^2_{n-1}$ 为枢轴，$n\geqslant4$时其p.d.f. g 单峰，有唯一极大值

近似地取 $a=\chi^2_{n-1}(1-\frac{\alpha}{2})$，$b=\chi^2_{n-1}(\frac{\alpha}{2})$，则 $\int_a^b g=1-\alpha$

由 $a\leq\dfrac{(n-1)S^2}{\sigma^2}\leq b$ 得 $\eta=\frac{1}{\sigma^2}$ 水平 $1-\alpha$ 的最短枢轴区间为

$$\left[\frac{\chi^2_{n-1}(1-\frac{\alpha}{2})}{(n-1)S^2},\ \frac{\chi^2_{n-1}(\frac{\alpha}{2})}{(n-1)S^2}\right]$$

□

作业 2

补充2：$X_1,\cdots,X_n$ i.i.d. $\sim f(x|\lambda)=e^{-(x-\lambda)}\mathbb{1}(x>\lambda)$，$\lambda\in\mathbb{R}$，求 λ 的水平 $1-\alpha$ 的最短枢轴区间

易见 $X_{(1)}$ 是 λ 的充分统计量

令 $Y_i=X_i-\lambda$，则其p.d.f.为 $g(y|\lambda)=e^{-y}\cdot\mathbb{1}(y>0)$

$\Rightarrow X_{(1)}-\lambda=Y_{(1)}\triangleq T$ 有p.d.f. $h(t)=ne^{-nt}\mathbb{1}(t>0)$，与$\lambda$无关

故 $T=X_{(1)}-\lambda$ 是枢轴

又因 $h(t)$ 关于 t 递减，故取 a s.t. $\int_0^a h(t)dt=1-\alpha\Rightarrow a=-\frac{1}{n}\ln\alpha$

由 $0<X_{(1)}-\lambda\leq-\frac{1}{n}\ln\alpha$ 得

λ的水平 $1-\alpha$ 的最短枢轴区间为：$[X_{(1)}+\frac{1}{n}\ln\alpha,\ X_{(1)})$

□