2. X set, algebra of sets on X: $\mathcal{A}$ is closed under finite unions and complements. σ-algebra : countably infinite. (generated algebra (σ-algebra)).

measure : X and σ-algebra $\mathcal{M}$, $\mu : \mathcal{M} \to [0, +\infty]$ s.t.

(1) $\mu(\phi) = 0$, (2) $\{E_j\}_{j=1}^{\infty}$ disjoint $\Rightarrow$ $\mu(\cup E_j) = \cup \mu(E_j)$.

algebra generated by closed intervals $\Rightarrow$

How to obtain a measure on X ?

pre measure : $\mathcal{A}$ algebra, $\mu_0 : \mathcal{A} \to [0, \infty]$, $\begin{cases} \mu_0(\phi) = 0 \\ \coprod_{j=1}^{\infty} A_j \in \mathcal{A} \Rightarrow \mu_0(\amalg A_j) = \Sigma \mu_0(A_j). \end{cases}$

$\Downarrow$

outer measure : $\mu^* : \mathcal{P}(X) \to [0, \infty]$, $\mu^*(E) = \inf \{ \sum_{j=1}^{\infty} \mu_0(A_j) \mid A_j \in \mathcal{A}, E \subset \bigcup_{j=1}^{\infty} A_j \}$.

$\Downarrow$

measure : ~~Caled~~ Carathéodory Condition : E is μ^*-measurable if $\mu^*(A) = \mu^*(E \cap A) + \mu^*(E^c \cap A)$

Eg. $\mathcal{A}$ algebra generated by $\left\{ \begin{array}{l} (a,b],\ a\in[-\infty,\infty),\ b\in(-\infty,+\infty) \\ \text{and } (a,+\infty). \end{array} \right\}$

$\mu_0\left(\coprod_{j=1}^{k}(a_j,b_j]\right) = \sum_j \mu_0((a_j,b_j]) = \sum b_j - a_j$. $\mu_0(\phi) = 0$

If $\{(a_j,b_j]\}_{j=1}^{\infty}$ disjoint, $\coprod_{j=1}^{\infty}(a_j,b_j] \notin \mathcal{A}$. $\Rightarrow \mu_0$ premeasure.

$\rightsquigarrow \mu^* \rightsquigarrow \mu$, μ is Lebesgue measure in fact.

Question: Uniqueness ?

Thm. $\mathcal{A} \subset \mathcal{P}(X)$ algebra, μ_0 premeasure on $\mathcal{A}$. whice drives from μ_0

$\mathcal{M}$ is the σ-algebra generated by $\mathcal{A}$, μ measure on $(X,\mathcal{M})$.

s.t. $\mu|_{\mathcal{A}} = \mu_0$. If ν is another measure that extends μ_0.

Then: $\mu(E) < \infty \Rightarrow \mu(E) = \nu(E)$.

for $E \in \mathcal{M}$

$\nu(E) \le \sum \mu_0(A_j) \xrightarrow{\inf} \mu(E)$

$\mu(E) < \infty$, $\exists A_j$, $\mu(\bigcup_{j=1}^{\infty} A_j) < \mu(E) + \varepsilon$, $E \subset \bigcup_{j=1}^{\infty} A_j$

$\mu(A \setminus E) < \varepsilon$. $\mu(E) \le \mu(A) = \lim_{n\to\infty} \mu(\bigcup_1^n A_j) = \lim_{n\to\infty} \nu(\bigcup_1^n A_j)$

$= \nu(A) = \nu(E) + \nu(A\setminus E) \le \nu(E) + \mu(A \setminus E) \le \nu(E) + \varepsilon$

Since ε is arbitrary, $\mu(E) = \nu(E)$

In fact, $\mathcal{M}$ is "larger" than the σ-algebra generated by $\mathcal{A}$ because of the theorem below:

Caiathéodory's Theorem.

μ^* outer measure on X, $\mathcal{M} = \{E \mid \forall A \in \mathcal{P}(X),\ \mu^*(A) = \mu^*(A\cap E) + \mu^*(A \cap E^c)\}$

Then $(X, \mathcal{M}, \mu^*)$ is a <u>complete</u> measure.

Rmk: Borel set $\subsetneqq$ Lebesgue measurable set.

$g(x) = x + f(x) : [0,1] \to [0,2]$ increase strictly, and is continuous.

$m(g(C)) = 1$, choose a non Borel set $S \subset g(C)$, $g^{-1}(S) \subset C$, therefore, is measurable

But it is NOT Borel measurable.

1. $C = [0,1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right)$.

$\downarrow$

$\left((b_1 \cdots b_k, 0, 2 \cdots 2 \cdots), (b_1 \cdots b_k, 2, 0, \cdots) \right) = \{ (a_1 \cdots a_k, 1, a_{k+2}, \cdots) \mid a_{k+2}, \cdots$ 不全为0且不全为2. $\}$

$(b_1 \cdots b_k, 1, 0 \cdots)$

$x \in C \longleftrightarrow \sum_{k=1}^{\infty} \frac{2a_k}{3^k} \longleftrightarrow (a_1 \cdots a_n \cdots) \in \{0,1\}^{\mathbb{N}} \longrightarrow [0,1] \ni \sum \frac{a_k}{2^k}$.

$a_k \in \{0,1\}$

$\downarrow$

$[0,1]^{n} \ni \left(\sum_{k=1}^{\infty} \frac{a_{nk-n+1}}{2^k}, \cdots \sum \frac{a_{nk}}{2^k} \right)$.

~~Two facts:~~

~~1) the disjoint open sets, f continuous~~

$f: C \to [0,1]$ cts $\xrightarrow{\text{extend}} f: [0,1] \to [0,1]$. ~~$(a_1' \cdots a_n' \cdots) \to (a_1 \cdots a_n \cdots)$~~

$x \in [0,1] \setminus C$, f cts at x. $x \in C$, $\exists \varepsilon$, $y \in B(x,\varepsilon) \cap C$, $\Rightarrow |f(y) - f(x)| < \varepsilon$.

$z \in B(x,\varepsilon)$, $\exists y \in B(x,\varepsilon) \cap C$, $z \in (x,y)$ (or (y,x)).

$|f(z) - f(x)| < |f(y) - f(x)| < \varepsilon$

3. $\forall \varepsilon$, $\exists O_\varepsilon$ open, $\supset E$, $m^*(O_\varepsilon \setminus E) < \varepsilon$, $\Longleftrightarrow$ $\forall A$, $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$.

"$\Leftarrow$" $\exists A$ open, $A \supset E$, $m^*(A) - m^*(E) < \varepsilon$. $m^*(A \setminus E) = m^*(A) - m^*(E) < \varepsilon$.

"$\Rightarrow$"

$$
\begin{aligned}
m^*(A) &= m^*(A \cap O_\varepsilon) + m^*(A \cap O_\varepsilon^c) \\
&\cancel{\le m^*(A \cap E) + \varepsilon + m^*(A \cap E^c)} \\
&\ge m^*(A \cap E) + m^*(A \cap E^c) - \varepsilon .
\end{aligned}
$$

$$
\begin{aligned}
O_\varepsilon^c &= E^c \setminus (O_\varepsilon \setminus E). \\
&= E^c \cap (O_\varepsilon \setminus E)^c .
\end{aligned}
$$

$$
\begin{aligned}
m^*(A \cap E^c) &\le m^*(A \cap O_\varepsilon^c) + m^*(A \cap (O_\varepsilon \setminus E)) \\
&\le m^*(A \cap O_\varepsilon^c) - \varepsilon .
\end{aligned}
$$

4. Lebesgue $\to$ Lebesgue Stieltjes

$\mu_0(\bigcup_1^n (a_j, b_j]) = \sum \mu_0(F(b_j)) - \mu_0(F(a_j))$ F increasing, right cts.

$\mu(E) = \inf\left[\sum_1^\infty [F(b_j) - F(a_j)] : E \subset \bigcup_1^\infty (a_j, b_j]\right\}$

$= \inf\{\sum_1^\infty \mu(a_j, b_j] : E \subset \bigcup_1^\infty (a_j, b_j]\}$

(Lemma) $= \inf\{\sum_1^\infty \mu(a_j, b_j) : E \subset \bigcup_1^\infty (a_j, b_j)\}$.

Thm. $\mu \in M_\mu$

$\mu(E) = \inf\{\mu(U) \mid U \supset E, U \text{ open}\}$

$= \sup\{\mu(K) \mid K \subset E, K \text{ cpt}\}$.

$\mu_F \longleftrightarrow F = \begin{cases} \mu(0, x], & x > 0 \\ 0 & x = 0 \\ -\mu(-x, 0], & x < 0. \end{cases}$