X概率密度 $f_X(x)=\int_{-\infty}^{+\infty} f(x,y)\,dy$

Y概率密度 $f_Y(y)=\int_{-\infty}^{+\infty} f(x,y)\,dx$

例 $(X,Y)$在区域$D=\{(x,y)\mid x^2+y^2\le R^2\}$上均匀分布.

$$f(x,y)=\begin{cases}\frac{1}{\pi R^2}, & (x,y)\in D\\ 0, & (x,y)\notin D\end{cases}$$

(1) 求边缘分布的概率密度 (2) $\rho=\sqrt{X^2+Y^2}$ 求$E(\rho)$.

解:(1) $f_X(x)=\int_{-\infty}^{+\infty} f(x,y)\,dy=\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\frac{1}{\pi R^2}\,dy=\frac{2}{\pi R^2}\sqrt{R^2-x^2}\quad -R\le x\le R.$

$f_Y(y)=\frac{2}{\pi R^2}\sqrt{R^2-y^2}\quad -R\le y\le R$

(2) $P(\rho\le x)=\frac{\pi x^2}{\pi R^2}=\frac{x^2}{R^2}\qquad \rho$密度 $f_\rho(x)=\frac{2x}{R^2}$

$E(\rho)=\int_0^R x\cdot\frac{2x}{R^2}\,dx=\frac{2}{3}R$

二. 期望. 协方差

定理: $g:\mathbb{R}^2\to\mathbb{R}$ Borel可测函数. $(X,Y)$连续型随机变量.

$g(X,Y)$是连续型r.v. 期望存在. 则$E(g(X,Y))=\iint_{\mathbb{R}^2} g(x,y)f(x,y)\,dx\,dy$.

$f(x,y)$为$(X,Y)$的联合概率函数.

特别地, $g(X,Y)=aX+bY$.

$E(aX+bY)=aE(X)+bE(Y)$, $\mathrm{cov}(X,Y)=E[XY]-E[X]E[Y]$

hw: 4.4.3, 4.4.5, 4.5.4, 4.5.6, 4.5.7, 4.5.9

$E[X]=\iint_{\mathbb{R}^2} x f(x,y)\,dx\,dy$ (绝对收敛时)

$\mathrm{Var}(X)=\iint_{\mathbb{R}^2}(x-E[X])^2 f(x,y)\,dx\,dy\qquad \mathrm{cov}(X,Y)=E[XY]-E[X]E[Y]$

例 $f(x,y)=\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)\right)\quad N(0,1;0,1;\rho)$

$f_X(x)=\int_{-\infty}^{+\infty}\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)\right)dy$

$$= \int_{-\infty}^{+\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}(y-\rho x)^2 - \frac{x^2}{2}\right) dy$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(y-\rho x)^2}{2(1-\rho^2)}\right) dy \qquad u = \frac{y-\rho x}{\sqrt{1-\rho^2}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du = 1$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

$$Cov(X,Y) = E[XY] - E[X]E[Y] = E[XY]$$

$$= \iint_{R^2} xy \cdot \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right) dxdy$$

$$= \iint_{R^2} xy \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(y-\rho x)^2}{2(1-\rho^2)}\right) dxdy$$

$$= \iint_{R^2} [x(y-x\rho) + \rho x^2] f(x,y)\, dxdy$$

$$= \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} x\underbrace{(y-\rho x)}_{\text{奇}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(y-\rho x)^2}{2(1-\rho^2)}\right) dy + \rho \iint_{R^2} x^2 f(x,y)\, dxdy$$

$$= \rho \cdot E(X^2) = \rho\,(Var(X) + E(X)^2) = \rho$$

$$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

一般 $N(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2, \rho)$

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} Q(x,y)\right)$$

$$Q(x,y) = \frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}, \quad u = \frac{x-\mu_1}{\sigma_1}, \quad v = \frac{y-\mu_2}{\sigma_2}$$

(1) $f_X(x_1) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}$, $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$

(2) $\frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \rho$

(3) $\rho = 0$ 时 $f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\frac{1}{2}\cdot\left(\frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right)\right)$

不相关

$$= \frac{1}{\sqrt{2\pi}\sigma_1} \exp^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma_2} \exp^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}}$$

$$= f_X(x) \cdot f_Y(y)$$ 独立.

相关系数 $\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$

1° $|\rho| \le 1$ $\quad E[XY] \le (E(X^2))^{\frac{1}{2}} \cdot (E(Y^2))^{\frac{1}{2}}$

2° $|\rho| = 1 \iff \exists a, b \in R$, s.t. $P(Y = aX + b) = 1$

## 三.连续型随机向量条件分布.

$f_Y(y)>0$ , $f_Y(y)\Delta y>0$

$$\lim_{\Delta y\to 0} P(X\le x \mid y\le Y\le y+\Delta y) = \lim_{\Delta y\to 0} P(X\le x\mid y<Y\le y+\Delta y)$$

$$= \lim_{\Delta y\to 0}\frac{P(X\le x,\ y<Y\le y+\Delta y)}{P(y<Y\le y+\Delta y)}$$

$$= \lim_{\Delta y\to 0}\frac{\int_{-\infty}^{x}du\int_{y}^{y+\Delta y}f(u,v)dv\cdot\frac{1}{\Delta y}}{\int_{-\infty}^{+\infty}du\int_{y}^{y+\Delta y}f(u,v)dv\cdot\frac{1}{\Delta y}}$$

$$= \frac{\int_{-\infty}^{x}f(u,y)du}{\int_{-\infty}^{+\infty}f(u,y)du} = \int_{-\infty}^{x}\frac{f(u,y)}{f_Y(y)}du$$

定义

(1)给定 $Y=y$ 条件下, $X$ 的条件密度 $f_{X|Y}(x|y)=\dfrac{f(x,y)}{f_Y(y)}$ $(f_Y(y)>0)$

$f_X(x)>0$ 可定义 $f_{Y|X}(y|x)=\dfrac{f(x,y)}{f_X(x)}$

(2) $F_{X|Y}(x|y)=\int_{-\infty}^{x}f_{X|Y}(x|y)du$ $\leftarrow P(X\le x\mid Y=y)$ 给定 $Y=y$ 条件下, $X$ 的条件分布函数.

$F_{Y|X}(y|x)=\int_{-\infty}^{y}f_{Y|X}(v|x)dv$ 给定 $X=x$ 条件下, $Y$ 的条件分布函数.

(3) 条件期望

$\psi(x)=E[Y|X=x]=\int_{-\infty}^{+\infty}y\,f_{Y|X}(y|x)dy$ $\quad \psi(X)=E[Y|X]$

例 $(X,Y)\sim N(\mu_1,\sigma_1^2,\mu_2,\sigma_2^2,\rho)$ 求 $f_{X|Y}(x|y)$

解: $f_{X|Y}(x|y)=\dfrac{f(x,y)}{f_Y(y)}$

$$=\frac{\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_1}{\sigma_1}\right)^2-2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)+\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right)\right]}{\frac{1}{\sqrt{2\pi}\sigma_2}\exp\left(-\frac{(y-\mu_2)^2}{2\sigma_2^2}\right)}$$

$$=\frac{1}{\sqrt{2\pi}\sigma_1\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)\sigma_1^2}\left((x-\mu_1)-\rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)\right)^2\right)$$

给定 $Y=y$ 条件下, $X\sim N\left(\mu_1+\rho\cdot\frac{\sigma_1}{\sigma_2}(y-\mu_2),\ \sigma_1^2(1-\rho^2)\right)$

例 $(X,Y)\sim U(D)$　　$D=\{(x,y)\mid x^2+y^2\le r^2\}$

$$f(x,y)=\begin{cases}\frac{1}{\pi r^2}, & (x,y)\in D\\ 0, & (x,y)\notin D\end{cases}$$　　求 $f_{X|Y}(x|y)$

解：$f_Y(y)=\int_{-\infty}^{+\infty}f(x,y)dx=\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}}\frac{1}{\pi r^2}dx=\frac{2}{\pi r^2}\sqrt{r^2-y^2}\quad(-r\le y\le r)$

$-r\le y\le r$ 时，$f_{X|Y}(x|y)=\begin{cases}\frac{1}{\pi r^2}\cdot\frac{\pi r^2}{2\sqrt{r^2-y^2}}=\frac{1}{2\sqrt{r^2-y^2}} & -\sqrt{r^2-y^2}\le x\le\sqrt{r^2-y^2}\\ 0 & 其他\end{cases}$

$Y=y$ 条件下，$X$ 服从 $[-\sqrt{r^2-y^2},\sqrt{r^2-y^2}]$ 上的均匀分布.

例 $(X,Y)$ 联合密度 $f(x,y)=\begin{cases}\frac{1}{x}, & 0<y\le x<1\\ 0, & 其它\end{cases}$　　求 $f_{Y|X}(y|x)$

$P(X^2+Y^2\le 1\mid X=x)$

解：$0<x<1$ 时，$f_X(x)=\int_{-\infty}^{+\infty}f(x,y)dy=\int_0^x\frac{1}{x}dy=1$

$$f_{Y|X}(y|x)=\frac{f(x,y)}{f_X(x)}=\frac{1}{x}\qquad 0<y<x$$

$$P(X^2+Y^2\le 1\mid X=x)=P(-\sqrt{1-x^2}\le Y\le\sqrt{1-x^2}\mid X=x)$$

$$=\begin{cases}P(0\le Y\le x\mid X=x) & 0<x\le\frac{\sqrt2}{2}\\ P(0\le Y\le\sqrt{1-x^2}\mid X=x) & \frac{\sqrt2}{2}<x<1\end{cases}$$

$$=\begin{cases}\int_0^x\frac{1}{x}dy=1 & 0<x\le\frac{\sqrt2}{2}\\ \int_0^{\sqrt{1-x^2}}\frac{1}{x}dy=\frac{\sqrt{1-x^2}}{x} & \frac{\sqrt2}{2}<x<1\end{cases}$$

例 设 $X_1,\cdots,X_n$ 独立，且均服从 $(0,1)$ 上均匀分布.

$N=\min\{n\mid\sum_{i=1}^n X_i>1\}$　求 $E[N]$.

解：$N(x)=\min\{n\mid\sum_{i=1}^n X_i>x\}$　　$m(x)=E(N(x))=E(E(N(x)\mid X_1))$

$$E(N(x)\mid X_1=y)=\begin{cases}1, & y>x\\ 1+m(x-y), & y\le x\end{cases}$$

$$m(x) = E(E(N(x) \mid X_1)) = \int_0^1 E(N(x) \mid X_1 = y) f_{X_1}(y)\, dy$$

$$= \int_0^x 1 + m(x-y)\, dy + \int_x^1 1\, dy = 1 + \int_0^x m(t)\, dt \qquad \begin{cases} m'(x) = m(x) \\ m(0) = 1 \end{cases}$$