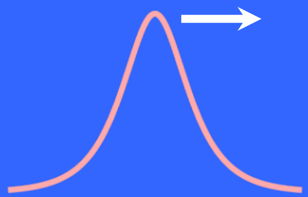
微分方程

孤立子与反散射变换

Solitons and Inverse Scattering Transform

What are Solitons?



Stable solitary wave

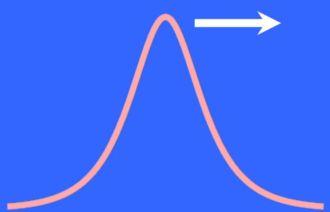
In 1834 **John Scott Russell**, an engineer, was riding along a canal and observed a horse-drawn boat that *suddenly stopped,* causing a *violent agitation,* giving rise to a lump of water that rolled forward with great velocity *without change of form or diminution of speed.*

Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

Russell's ***Wave of Translation***

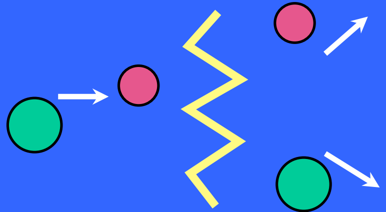
- Experiments showed that the solitary wave speed was proportional to height.
- Data conflicted with contemporary fluid dynamics (by big deals like Newton)

Soliton on an Aqueduct

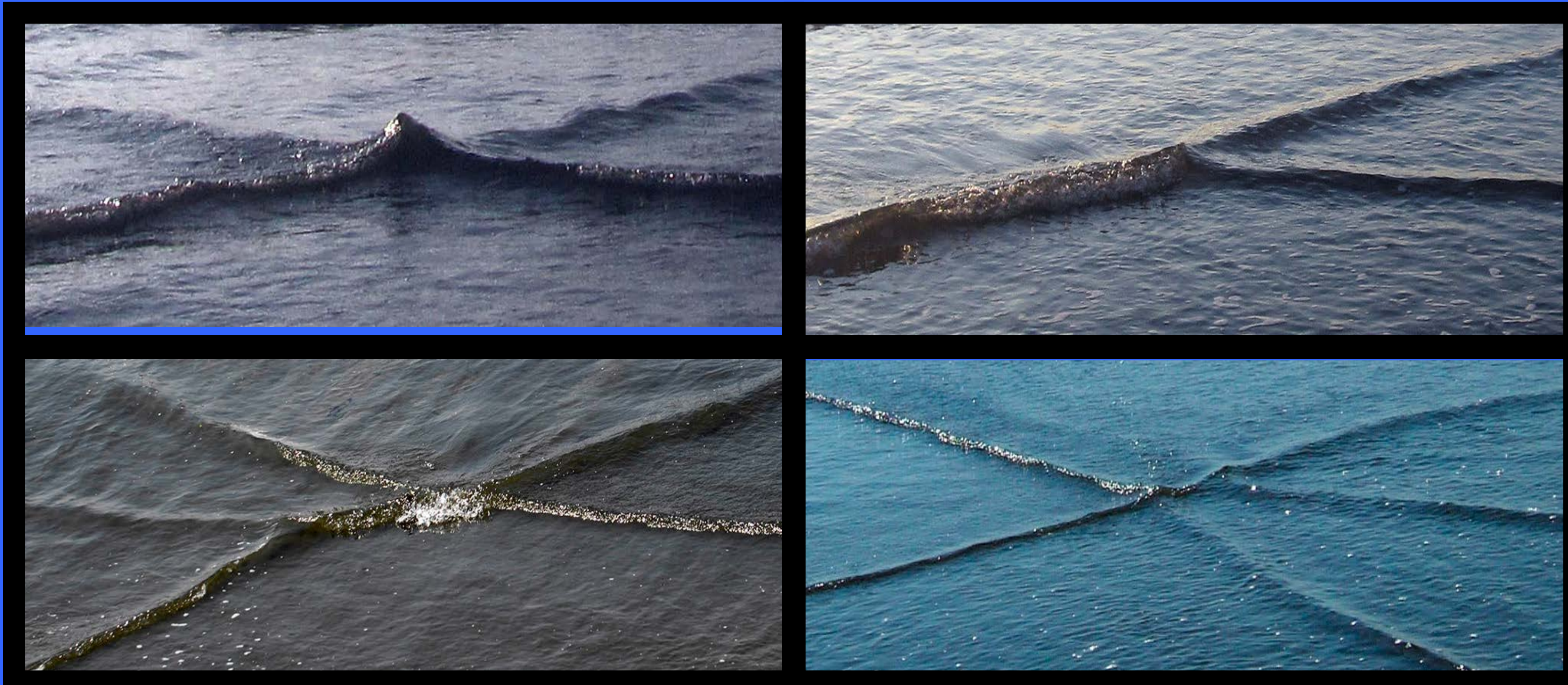


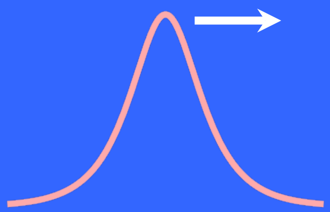
Solitons in Nature

Alphabet Waves

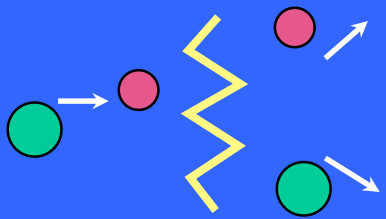


- *Not as unusual as once thought*
- *May play a role in tsunami and rogue wave formation*





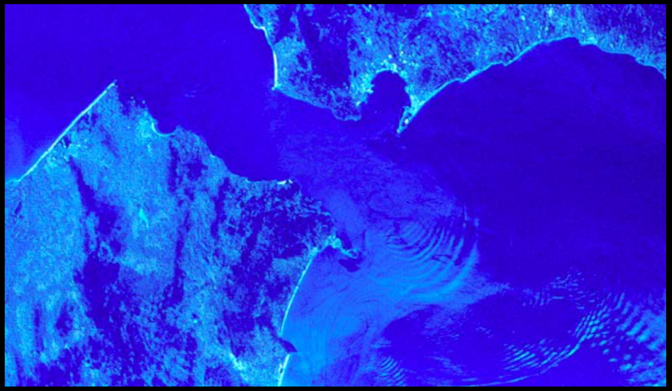
Solitons in Nature



Morning Glory Clouds

Jupiter’s Red Spot

Strait of Gibraltar



Deep and shallow water waves, plasmas, particle interactions, optical systems, neuroscience, Earth’s magnetosphere...

Nonlinear PDEs and solitons

- **John Scott Russell (1834); Boussinesq (1871 - 1877); Korteweg & de Vries (1895)**: solitary waves on shallow water, $u = a\,\mathrm{sech}^2[\beta(x - ct)]$, $u_t + 6uu_x + u_{xxx} = 0$.
- **Zabusky** & **Kruskal (1965)**: numerical simulation of the continuum limit of the **Fermi-Pasta-Ulam (1955)** problem. The KdV equation

$$u_t + uu_x + \delta^2 u_{xxx} = 0\,, \qquad \delta = 0.022$$

with initial conditions $u(x, 0) = \cos \pi x$, $0 \le x \le 2$ and u, u_x, u_{xx} periodic on $[0, 2]$ for all t.

Generation of solitary waves elastically interacting with each other; wave-particle duality: solitons

- IST: **Gardner, Green, Kruskal & Miura (1967)** – KdV; **Zakharov & Shabat (1972)** – NLS; **AKNS (1974)** – many other equations.

1967: KdV & Inverse Scattering Transform

In 1967 Gardner, Greene, Miura e Kruskal, in order to solve the initial value problem for the KdV equation, introduced a method known as Inverse Scattering Transform (which could be considered the analogous of the Fourier transform for linear ODE).
The IST is not a direct method...it works by associating the Schroedinger equation to the Cauchy problem of the KdV:

$$-\psi_{xx} + u(x,0)\psi = \lambda^2\psi, \; x \in \mathbb{R}.$$

$$\begin{tikzcd}
\boxed{\text{given } u(x,0)} \arrow[rr, "\substack{\text{direct scattering problem} \\ \text{with potential } u(x,0)}"] \arrow[d, "\text{Solution NPDE}"] & & \boxed{S(\lambda,0)} \arrow[d, "\substack{\text{time evolution of} \\ \text{scattering data}}"'] \\
\boxed{u(x,t)} & & \boxed{S(\lambda,t)} \arrow[ll, "\substack{\text{inverse scattering problem} \\ \text{with time evolved scattering data}}"]
\end{tikzcd}$$

1967: KdV & Inverse Scattering Transform

GGKM were also lucky because three years before their work, Faddeev in 1964 had had success in solving the inverse problem for the Schroedinger equation.

1. **Direct Scattering** consists of: Find the so-called scattering data $\{R(k), \{\kappa_j, N_j\}_{j=1}^{N}\}$ of the Schroedinger equation with potential $u(x,0)$.

$$f_r(k,x) = \begin{cases} \frac{1}{T(k)} e^{-ikx} + \frac{R(k)}{T(k)} e^{ikx} + o(1), & x \to +\infty, \\ e^{-ikx}[1 + o(1)], & x \to -\infty. \end{cases}$$

2. **Propagation of scattering data**: the scattering data evolve in time following the equations

$$\{R(k), \{\kappa_j, N_j\}_{j=1}^{N}\} \mapsto \{R(k)e^{8ik^3 t}, \{\kappa_j, N_j e^{8\kappa_j^3 t}\}_{j=1}^{N}\}.$$

3. **Inverse scattering** consists of: (Re)-construct the potential. To do that: 1) Solve this Marchenko equation

$$K(x,y) + \Omega(x+y) + \int_x^\infty dz\, K(x,z)\Omega(z+y) = 0,$$

where $\Omega(x) = \sum_{j=1}^{N} N_j e^{-\kappa_j x} + \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx} R(k)$ and 2) get $u(x,t)$ from the relation $u(x,0) = 2\frac{d}{dx}K(x,x)$.

IST: The Big Picture

Consider an IVP for a nonlinear evolution equation

$$u_t = F(u, u_x, u_{xx}, \dots), \quad u(x,0) = u_0(x). \qquad (*)$$

Assume that ($*$) can be represented as a compatibility condition for two linear equations

$$\begin{aligned} \mathcal{L}\phi &= \lambda\phi, \\ \phi_t &= \mathcal{A}\phi. \end{aligned} \qquad (**)$$

Let $\{S(\lambda, t)\}$ be spectral (scattering) data for $u(x,t)$ in $(**)$: the discrete eigenvalues, the norming coefficients of eigenfunctions, and the reflection and transmission coefficients.

Then the IST steps are (cf. solution via the Fourier Transform):

$$u_0(x) \mapsto \{S(\lambda, 0)\} \mapsto \{S(\lambda, t)\} \mapsto u(x,t)$$

At each step we have to solve a linear problem!

Inverse Scattering Transform: Examples

Few years later, it became clear that many other nonlinear evolution equation could be solved by the IST:

- Nonlinear Schroedinger equation (1972, Zakharov and Shabat)
- sine-Gordon (1973-74, Ablowitz, Kaup, Newell, Segur or Zakharov)
- Manakov system (1973, Manakov)
- AKNS system (1974, Ablowitz, Kaup, Newell, Segur)
- Camassa-Holm equation (1992, Camassa e Holm)
- Degasperis-Procesi equation (1999, Degasperis e Procesi)
- the list is not complete...

Examples of integrable equation

$u_t + 6uu_x + u_{xxx} = 0,$ *Korteweg-de Vries (KdV) equation*

$u_{xt} = \sin u,$ *sine-Gordon equation*

$i\, u_t + u_{xx} \pm 2uu^\dagger u = 0,$ *Nonlinear Schrödinger (NLS) equation,*

$u_t - u_{xxt} + 2\omega u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0,$ *Camassa-Holm (CH) equation,*

$u_{zz} + \dfrac{\partial}{\partial x}(u_t + 6uu_x + u_{xxx}) = 0,$ *Kadomtsev-Petviashvily (KP) equation.*