§0.1 Gauss-Bonnet公式

§0.1.1 正交标架证明

设D为曲面S上一单连通区域，$\{e_1,e_2\}$为曲面S切平面的单位正交基，$\{\omega^1,\omega^2\}$为$(X_1,X_2), X_i = (dr)^{-1}(e_i)$，的对偶基。则曲面第一基本形式$I = \omega^1\omega^1+\omega^2\omega^2$。设$\omega_1^2$为联络形式，则Gauss方程为

$$d\omega_1^2 = -K\omega^1\wedge\omega^2 = -KdA.$$

则由Green公式

$$\int_{\partial D}(fdu+gdv) = \int_D(\frac{\partial g}{\partial u}-\frac{\partial f}{\partial v})du\wedge dv = \int_D d(fdu+gdv)$$

可得

$$\int_D KdA = -\int_D d\omega_1^2 = -\int_{\partial D}\omega_1^2.$$

其中ω_1^2为光滑一次微分形式。接下来分析$\int_{\partial D}\omega_1^2$的几何意义。

设D的边界∂D为分段光滑闭曲线，即

$$\partial D = C_1\cup C_2\cup\cdots\cup C_n.$$

设其中光滑曲线C_i的弧长参数表示为$r(s)$，

$$\bar{e}_1 = \frac{dr(s)}{ds} = \cos\alpha(s)e_1+\sin\alpha(s)e_2.$$

$\bar{e}_1$在顶点处不连续。令

$$\bar{e}_2 = -\sin\alpha e_1+\cos\alpha e_2,$$

则

$$\begin{cases} e_1 = \cos\alpha\bar{e}_1-\sin\alpha\bar{e}_2, \\ e_2 = \sin\alpha\bar{e}_1+\cos\alpha\bar{e}_2. \end{cases}$$

从而

$$\begin{aligned}
\int_{C_i}\omega_1^2 &= \int_{C_i}\omega_1^2(\frac{d}{ds})ds = \int_{C_i}\langle\frac{de_1}{ds},e_2\rangle ds \\
&= \int_{C_i}\langle\frac{d}{ds}(\cos\alpha\bar{e}_1-\sin\alpha\bar{e}_2),\sin\alpha\bar{e}_1+\cos\alpha\bar{e}_2\rangle ds \\
&= \int_{C_i}[-\frac{d\alpha}{ds}+\langle\frac{d\bar{e}_1}{ds},\bar{e}_2\rangle]ds \\
&= \int_{C_i}(-\frac{d\alpha}{ds}+k_g)ds.
\end{aligned}$$

因此沿曲线C_i也有

$$(d\alpha+\omega_1^2)(\frac{d}{ds})=k_g.$$

代入得(其中沿∂D积分为分段积分之后求和)

$$\int_D KdA=-\int_{\partial D}\omega_1^2=\int_{\partial D}(\frac{d\alpha}{ds}-k_g)ds,$$

即

$$\int_D KdA+\int_{\partial D}k_g ds=\int_{\partial D}d\alpha.$$

接下来考虑边界积分$\int_{\partial D}d\alpha$得到如下Gauss-Bonnet公式。

定理0.1. 设D是曲面S上一单连通区域，∂D为分段光滑闭曲线，β_i为∂D的顶点的外角。则

$$\int_D KdA+\int_{\partial D}k_g ds=2\pi-\sum\beta_i.$$

证明: 首先设∂D为光滑闭曲线情形。设∂D的参数表示为$r(s),s\in[0,l]$。则$r(0)=r(l)$，$\frac{dr}{ds}(0)=\frac{dr}{ds}(l)$，因此

$$\alpha(l)=\alpha(0)+2k\pi,\quad k\in\mathbb{Z}.$$

∂D光滑形变到一个小邻域内的圆周的过程中$\int d\alpha$也连续变化，因此$\int d\alpha$保持不变，从而$\int_{\partial D}d\alpha=2\pi$。此时

$$\int_D KdA+\int_{\partial D}k_g ds=2\pi.$$

当∂D为分段光滑曲线时，由于

$$\int_D KdA+\int_{\partial D}k_g ds=\int_{\partial D}d\alpha,$$

只需求$\int_{\partial D}d\alpha$，其中α由下式定义

$$\frac{dr(s)}{ds}=\cos\alpha(s)e_1+\sin\alpha(s)e_2.$$

在各顶点两边各取∂D上一点并用光滑曲线连接，所得曲线记作$\partial\widetilde{D}$。则相应有

$$\int_{\partial\widetilde{D}}d\alpha=2\pi.$$

当顶点附近两点趋于顶点取极限得

$$2\pi=\lim\int_{\partial\widetilde{D}}d\alpha=\int_{\partial D}d\alpha+\sum\beta_i.$$

□

注：Gauss-Bonnet公式的整体形式参见教材第七章。它的高维情形由H. Hopf, C. B. Allendoerfer, Fenchel, C. B. Allendoerfer - A. Weil, S. S. Chern建立。

应用：向量沿曲线平移一周后的角度差与Gauss曲率积分的联系。设曲面上弧长参数光滑闭曲线$r(s), 0 \leq s \leq l$围成一单连通区域D。设$v(s)$为沿$r(s)$的平行切向量场。取S的正交标架$\{e_1, e_2\}$，令

$$\frac{dr}{ds} = \cos\alpha e_1 + \sin\alpha e_2,$$

$$v(s) = \cos\beta e_1 + \sin\beta e_2.$$

由$v(s)$沿$r(s)$平行，

$$\frac{Dv}{ds} = (-\sin\beta e_1 + \cos\beta e_2)\frac{d\beta}{ds} + (\cos\beta e_2 - \sin\beta e_1)\omega_1^2(\frac{d}{ds}) = 0,$$

因此

$$\frac{d\beta}{ds} = -\omega_1^2(\frac{d}{ds}).$$

另一方面有

$$(d\alpha + \omega_1^2)(\frac{d}{ds}) = k_g.$$

因此沿$r(s)$有

$$d\beta - d\alpha = -k_g ds,$$

其中$\beta(s), \alpha(s)$分别为平移向量$v(s)$与$\frac{dr}{ds}$和e_1的夹角。

$v(s)$沿曲线$r(s)$平移一周后角度差

$$\begin{aligned}
\beta(l) - \beta(0) &= \int_{\partial D} d\beta \\
&= \int_{\partial D} (d\alpha - k_g ds) \\
&= 2\pi - \int_{\partial D} k_g ds \\
&= \int_D K dA.
\end{aligned}$$

例：考虑半径为$a > 0$的球面，在球坐标下

$$r(\theta, \varphi) = (a\sin\theta\cos\varphi, a\sin\theta\sin\varphi, a\cos\theta), \quad \theta \in [0, \pi], \varphi \in [0, 2\pi].$$

则有

$$I = a^2(d\theta d\theta + \sin^2\theta d\varphi d\varphi),$$

$$K = \frac{1}{a^2},$$

$$dA = a^2 \sin\theta d\theta d\varphi.$$

令$\theta_0 \in (0,\pi)$。考虑

$$D = \{(\theta,\varphi) | 0 \le \theta < \theta_0, \quad \varphi \in [0,2\pi)\}$$

$$\partial D = \{\theta = \theta_0\}.$$

则有

$$Area(D) = 2a^2\pi(1-\cos\theta_0).$$

∂D为纬线圈，有弧长参数表示

$$r(s) = a(\sin\theta_0\cos(cs), \sin\theta_0\sin(cs), \cos\theta_0), \quad c = \frac{1}{a\sin\theta_0}.$$

事实上

$$r'(s) = (-\sin(cs), \cos(cs), 0).$$

沿$r(s) = \partial D$取

$$e_1 := r'(s) = (-\sin(cs), \cos(cs), 0),$$

$$\begin{aligned} e_2 := n \wedge e_1 &= (\sin\theta_0\cos(cs), \sin\theta_0\sin(cs), \cos\theta_0) \\ &\wedge (-\sin(cs), \cos(cs), 0) \\ &= (-\cos\theta_0\cos(cs), -\cos\theta_0\sin(cs), \sin\theta_0), \end{aligned}$$

从而$r(s) = \partial D$有

$$k_g = \langle \frac{de_1}{ds}, e_2 \rangle = c\cos\theta_0 = \frac{\cos\theta_0}{a\sin\theta_0}.$$

可验证Gauss-Bonnet公式

$$\int_D K dA + \int_{\partial D} k_g ds = 2\pi(1-\cos\theta_0) + \frac{\cos\theta_0}{a\sin\theta_0} 2\pi a\sin\theta_0 = 2\pi.$$

设

$$v(s) = \cos\gamma e_1 + \sin\gamma e_2$$

沿$r(s)$平行，即

$$\frac{Dv}{ds} = (\frac{dv}{ds})^T = 0,$$

当且仅当

$$\begin{aligned}\langle\frac{dv}{ds},-\sin\gamma e_1+\cos\gamma e_1\rangle &= \langle\frac{d}{ds}(\cos\gamma e_1+\sin\gamma e_2),-\sin\gamma e_1+\cos\gamma e_1\rangle\\ &= \frac{d\gamma}{ds}+\langle\frac{de_1}{ds},e_2\rangle\\ &= \frac{d\gamma}{ds}+\omega_1^2(\frac{d}{ds})=\frac{d\gamma}{ds}+k_g\\ &= 0.\end{aligned}$$

从而，

$$\frac{d\gamma}{ds}=-k_g=-\frac{\cos\theta_0}{a\sin\theta_0}.$$

曲线∂D长度为$2\pi a\sin\theta_0$，平行移动相对$r'(s)$转过的角度为

$$\begin{aligned}\gamma(2\pi a\sin\theta_0)-\gamma(0) &= -\int_{\partial D}k_g ds=-2\pi\cos\theta_0\\ &= -\int_{\partial D}k_g ds+2\pi-2\pi\\ &= \int_D KdA-2\pi\\ &= -2\pi\cos\theta_0.\end{aligned}$$

练习（不用交）：21，22