$$E(X)=\sum_{k=0}^{n} k\cdot\frac{n!}{k!(n-k)!}p^k q^{n-k} = \sum_{k=1}^{n}\frac{np\cdot(n-1)!}{(k-1)!(n-k)!}p^{k-1}q^{n-k} = np\cdot(p+q)^{n-1} = np$$

$$E(X^2)=\sum_{k=0}^{n} k^2\cdot\frac{n!}{k!(n-k)!}p^k q^{n-k} = \sum_{k=0}^{n} k(k-1)\cdot\frac{n!}{k!(n-k)!}p^k q^{n-k}+\sum_{k=0}^{n} k\cdot C_n^k p^k q^{n-k}$$

$$= n(n-1)p^2\cdot(p+q)^{n-2} + E(X) = (n^2-n)p^2+np$$

$$Var(X) = n^2p^2 - np^2 + np - n^2p^2 = npq$$

(2) $X\sim G(p)$ $\quad P(X=k)=q^{k-1}\cdot p$

$$E(X)=\sum_{k=1}^{\infty} kq^{k-1}\cdot p = p(\sum_{k=1}^{\infty} q^k)' = p\cdot(\frac{q}{1-q})' = p\cdot\frac{1-q+q}{(1-q)^2} = \frac{1}{p}$$

$$E(X^2)=\sum_{k=1}^{\infty} k^2\cdot q^{k-1}\cdot p = \sum_{k=1}^{\infty} k(k+1)q^{k-1}\cdot p - \sum_{k=1}^{\infty} k\cdot q^{k-1}\cdot p = p\cdot(\frac{q^2}{1-q})'' - \frac{1}{p} = \frac{2-p}{p^2}$$

$$var(X) = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

(3) $X\sim f(r,p)$ $\quad P(X=k) = C_{k-1}^{r-1}p^r q^{k-r}$ $\quad k=r,r+1,\cdots$

$X = X_1+X_2+\cdots+X_r$ $\quad X_i$表示第$i-1$次成功到第i次成功所需次数.

$X_1, X_2, \cdots, X_r$独立. $\quad X_i\sim G(p)$

$$EX = E(X_1+\cdots+X_r) = \frac{r}{p} \qquad var(X) = \sum_{i=1}^{r} var(X_i) = \frac{r(1-p)}{p^2}$$

(4) Poisson分布 $\quad P(X=k) = e^{-\lambda}\cdot\frac{\lambda^k}{k!}$ $\quad k=0,1,2\cdots$

$$E(X)=\sum_{k=1}^{\infty} k\cdot e^{-\lambda}\cdot\frac{\lambda^k}{k!} = \lambda e^{-\lambda}\cdot\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!} = \lambda$$

$$E(X^2)=\sum_{k=1}^{\infty} k^2\cdot e^{-\lambda}\cdot\frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} k(k-1)e^{-\lambda}\cdot\frac{\lambda^k}{k!} + \sum_{k=1}^{\infty} ke^{-\lambda}\cdot\frac{\lambda^k}{k!} = \lambda^2+\lambda.$$

$$var(X) = \lambda$$

hw: 3.2.5, 3.3.2, 3.3.3

§3.4 示性函数举例

$A, B\in F$ $\quad I_A(w)=\begin{cases}1, & w\in A\\ 0, & w\in A\end{cases}$ $\qquad I_{A^c}(w) = 1-I_A(w)$, $I_{AB}(w) = I_A(w)I_B(w)$

$E[I_A] = P(A)$

例 $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_iA_j) + \sum_{i<j<k} P(A_iA_jA_k) - \cdots + (-1)^{n-1}P(\bigcap_{i=1}^{n} A_i)$

证: $(\bigcup_{i=1}^{n} A_i)^c = \bigcap_{i=1}^{n} A_i^c$

$$P(\bigcap_{i=1}^{n} A_i^c) = E(I_{\bigcap_{i=1}^{n} A_i^c}) = E(\prod_{i=1}^{n} I_{A_i^c}) = E((1-I_{A_1})(1-I_{A_2})\cdots(1-I_{A_n}))$$

$$= E(1+\sum_{k=1}^{n}(-1)^k\sum I_{A_{i_1}}I_{A_{i_2}}\cdots I_{A_{i_k}})$$

$$= 1+\sum_{k=1}^{n}(-1)^k \sum_{i_1<i_2<\cdots<i_k} P(A_{i_1}\cap A_{i_2}\cap\cdots A_{i_k})$$

$$P(\bigcup_{i=1}^{n}A_i)= 1-P(\bigcap_{i=1}^{n}A_i^c)= \sum_{k=1}^{n}(-1)^{k-1}\sum_{i_1<\cdots<i_k} P(A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_k})$$

例 围绕花坛有52棵树，15只小鸟随机在树上建窝，每棵树上至多1只小鸟。

证明：∃连续7棵树上至少生活3只小鸟。

证：$\Omega=\{1,2,\cdots,52\}$ $\quad I_{\{k\}}=\begin{cases}1, & \text{第}k\text{棵树上有小鸟}\\ 0, & \text{否则}\end{cases}$

$$X(k)=I_{(k)}+I_{(k+1)\bmod 52}+\cdots+I_{(k+6)\bmod 52}$$

$$E(X)=\sum_{k=1}^{52}X(k)\underbrace{P(W=k)}_{\frac{1}{52}} = \frac{1}{52}\times 7\times 15=\frac{105}{52}>2$$

$\therefore P(X>2)>0 \Rightarrow \exists k\in\Omega \quad k\in\{X>2\}\Rightarrow \quad X(k)\geq 3$

例 n把伞放在1个箱中，每个人随机拿1把，N表示拿对自己伞的人数，讨论N的分布列及期望，方差。

解：$\Omega=\{i_1,i_2,\cdots,i_n \mid 1,\cdots,n\text{的一个排列}\}$

A_k表示第k个人拿到自己伞。 $I_{A_k}=\begin{cases}1, & W\in A_k\\ 0, & W\notin A_k\end{cases}$ $\qquad N=\sum_{k=1}^{n}I_{A_k}$

$$P(N=i)=C_n^i E(I_{A_1}I_{A_2}\cdots I_{A_i}I_{A_{i+1}^c}\cdots I_{A_n^c})$$

$$=C_n^i E(I_{A_1}I_{A_2}\cdots I_{A_i}(1-I_{A_{i+1}})\cdots(1-I_{A_n}))$$

$$=C_n^i\cdot\sum_{j=0}^{n-i}(-1)^j C_{n-i}^{j} E(I_{A_1}\cdots I_{A_i}I_{A_{i+1}}\cdots I_{A_{i+j}})$$

$$=C_n^i\cdot\sum_{j=0}^{n-i}(-1)^j C_{n-i}^{j}\cdot\frac{(n-i-j)!}{n!}$$

$$=\frac{1}{i!}\sum_{j=0}^{n-i}\frac{(-1)^j}{j!} \qquad j=0,1,\cdots,n$$

$$E(N)=\sum_{i=0}^{n}E(I_{A_i})=n\cdot\frac{1}{n}=1$$

$$\mathrm{var}(N)=E(N^2)-E(N)^2=E((\sum_{i=1}^{n}I_{A_i})^2)-1=E(\sum_{i=1}^{n}I_{A_i}^2+2\sum_{i<j}I_{A_i}I_{A_j})-1$$

$$=1+2\cdot\frac{(n-2)!}{n!}C_n^2-1=1$$

§3.5 条件分布与条件期望

(X, Y) 2-dim r.v.

分布列 $p_{ij} = P(X = x_i, Y = y_j)$ $\sum_{i,j} p_{ij} = 1$

定义 若 $E[(X-E(X))(Y-E(Y))]$ 存在，称之为 $\overset{r.v.}{X.Y}$ 的协方差记为 $\mathrm{cov}(X, Y)$

$\rho(X, Y) = \dfrac{\mathrm{cov}(X, Y)}{\sqrt{\mathrm{Var}(X)} \cdot \sqrt{\mathrm{Var}(Y)}}$ 称为 X.Y 的相关系数.

性质：

(1) $\mathrm{cov}(X, X) = \mathrm{Var}(X)$

(2) $E[(X-E(X))(Y-E(Y))] = E[XY - E(X)Y - XE(Y) + E(X)E(Y)]$

$= E(XY) - E(X)E(Y) - E(Y)E(X) + E(X)E(Y) = E(XY) - E(X)E(Y)$

(3) $\mathrm{cov}(X, a) = 0, \ \forall a \in R$

(4) $\mathrm{cov}(aX, bY) = ab\,\mathrm{cov}(X, Y) \quad \forall a, b \in R$

(5) $\mathrm{cov}(X+Y, Z) = \mathrm{cov}(X, Z) + \mathrm{cov}(Y, Z)$

(6) $\mathrm{Var}(X+Y) = E((X+Y)^2) - (E(X+Y))^2 = E(X^2) + 2E(XY) + E(Y^2) - (E(X))^2 - (E(Y))^2 - 2E(X)E(Y)$

$= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{cov}(X, Y)$

相关系数性质：

(1) $|\rho(X, Y)| \le 1$

(2) $|\rho(X, Y)| = 1 \Leftrightarrow \exists a, b \in R, s.t. \ P(Y = aX + b) = 1$

证：(1) $(E((X-EX)(Y-EY)))^2 \le E((X-EX)^2) \cdot E((Y-EY)^2)$

即 $(\mathrm{cov}(X, Y))^2 \le \mathrm{var}(X)\,\mathrm{var}(Y) \Rightarrow |\rho(X, Y)| \le 1$

(2) $\mathrm{var}(X), \mathrm{var}(Y) > 0$, $E(X^2) > 0$

$|\rho(X, Y)| = 1 \Leftrightarrow$ cauchy 不等式取 "=" $\Leftrightarrow E[(t(X-EX) + (Y-E(Y)))^2]$ 关于 t 判别式 $= 0$

$\Leftrightarrow \exists t_0, \ P(t_0(X-EX) + Y - EY = 0) = 1$ 即 $P(Y + t_0X - (t_0EX - EY) = 0) = 1$

令 $a = -t_0$, $b = t_0EX - EY$ 即可.

X	-1	0	1
Y	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$Y=X^2$

$cov(X,Y)=E(XY)-E(X)E(Y)=E(X^3)-E(X)E(Y)=0$

X,Y相互独立 $\Rightarrow$ $cov(X,Y)$ 反之不对

$X_1,X_2,\cdots,X_n$ 协方差矩阵 $\Sigma=(cov(X_i,X_j))_{n\times n}$

$$=\begin{pmatrix} cov(X_1,X_1) & \cdots & cov(X_1,X_n) \\ cov(X_2,X_1) & & \vdots \\ \vdots & & \vdots \\ cov(X_n,X_1) & \cdots & cov(X_n,X_n) \end{pmatrix}$$

$var(\sum_{i=1}^{n}X_i)=E((\sum_{i=1}^{n}(X_i-EX_i))^2)=\sum_{i=1}^{n}\sum_{j=1}^{n}cov(X_i,X_j)$

例 独立重复实验.实验有r种可能的结果. X_k表示n次实验中第k个结果出现次数.

$(X_1,\cdots,X_r)$

$a_1+\cdots+a_r=n$时, $P(X_1=a_1,X_2=a_2,\cdots,X_r=a_r)=C_n^{a_1}C_{n-a_1}^{a_2}\cdots C_{a_r}^{a_r}\cdot p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$

$$=\frac{n!}{a_1!\cdots a_r!}p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$$

多项分布

$X_1\sim B(n,p_1)$ $(X_1,X_2)\sim(n,(p_1,p_2,1-p_1-p_2))$

$X_1+X_2\sim B(n,p_1+p_2)$

计算 $Cov(X_i,X_j)$ $\rho(X_i,X_j)$

$Var(X_i+X_j)=varX_i+varX_j+2cov(X_i,X_j)$ $X_i+X_j\sim B(n,p_i+p_j)$

$var(X_i+X_j)=n(p_i+p_j)(1-(p_i+p_j))$

$cov(X_i,X_j)=\frac{1}{2}(var(X_i+X_j)-varX_i-varX_j)=\frac{1}{2}(n(p_i+p_j)(1-p_i-p_j)-np_i(1-p_i)-np_j(1-p_j))$

$=-np_ip_j$

$\rho(X_i,X_j)=\frac{-np_ip_j}{\sqrt{np_i(1-p_i)\cdot np_j(1-p_j)}}=-\sqrt{\frac{p_ip_j}{(1-p_i)(1-p_j)}}$

例

X \ Y	-1	0	2	f_X
1	$\frac{1}{18}$	$\frac{3}{18}$	$\frac{2}{18}$	$\frac{6}{18}$

$cov(X,Y)=E(XY)-E(X)E(Y)$

$=\sum_{x,y}xy\,P(X=x,Y=y)-\sum_x x\cdot P(X=x)\cdot\sum_y y\cdot P(Y=y)$

2	$\frac{2}{18}$	0	[illegible]/18	$\frac{5}{18}$
3	0	$\frac{4}{18}$	$\frac{3}{18}$	$\frac{7}{18}$
f_Y	$\frac{3}{18}$	$\frac{7}{18}$	$\frac{8}{18}$	

$= \frac{41}{324}$

条件期望

$\Omega = \bigcup_{i=1}^{n} B_i \quad (B_i \cap B_j = \phi)$

$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i) \quad E(I_A) = \sum_{i=1}^{n} E(I_A|B_i)P(B_i) \quad B_i = \{w \mid Y(w) = y_i\}$

定义 $(X, Y) \quad P(Y=y) > 0$

$f_{X|Y}(x|y) = P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$ 在 $Y=y$ 条件下，X 的条件分布列

$F_{X|Y}(x|y) = P(X \le x \mid Y=y)$ 条件分布函数.

$E[X|Y=y] = \sum_x x f_{X|Y}(x|y)$ 称 $Y=y$ 时 X 的条件期望.

记 $\psi(y) = E(X|Y=y) \quad \psi(Y(w))$ 是一个 r.v.

例 重复射击. 命中可能性为 p. S_i: 第 i 次射中的射击次数.

(1) $P(S_1=i \mid S_2=j) = \frac{P(S_1=i, S_2=j)}{P(S_2=j)} = \frac{p^2 \cdot (1-p)^{j-2}}{C_{j-1}^{1} \cdot p^2 \cdot (1-p)^{j-2}} = \frac{1}{j-1}$ （均匀分布） $\quad i=1,2,\cdots,j-1$

(2) $E[S_1 \mid S_2=j] = \sum_{i=1}^{j-1} i \cdot \frac{1}{j-1} = \frac{j(j-1)}{2} \cdot \frac{1}{j-1} = \frac{j}{2}$ $\quad \to \psi(S_2) = \frac{S_2}{2}$ r.v.

$P(\psi(S_2) = j) = P(S_2 = 2j) = C_{2j-1}^{1} p^2 (1-p)^{2j-2}$

命题 $\psi(X) = E[Y|X]$，则 $E(\psi(X)) = E(Y)$

证: $E[\psi(X)] = \sum_x \psi(x) P(X=x) = \sum_x E(Y|X=x) \cdot P(X=x)$

$= \sum_x (\sum_y y P(Y=y \mid X=x)) \cdot P(X=x)$

$= \sum_x (\sum_y y \cdot \frac{P(X=x, Y=y)}{P(X=x)} \cdot P(X=x))$

$= \sum_x \sum_y y \cdot P(X=x, Y=y)$

$= \sum_y y P(Y=y) = E(Y)$

$E(Y) = \sum_x E(Y|X=x) P(X=x)$

定理 X.Y相互独立时，$E(Y|X)=E(Y)$

$$\psi(x)=E(Y|X=x)=\sum_y y\cdot P(Y=y|X=x)=\sum_y y\cdot P(Y=y)=E(Y)$$

$$\frac{P(Y=y,X=x)}{P(X=x)}=\frac{P(Y=y)P(\cancel{X=x})}{P(\cancel{X=x})}$$

hw: 3.4.2, 3.4.4, 3.6.3, 3.6.4, 3.6.5