**4.** Let $X$ and $Y$ have the bivariate normal distribution with zero means, unit variances, and correlation $\rho$. Find the joint density function of $X+Y$ and $X-Y$, and their marginal density functions.

4.9.4: $\begin{cases} u = x+y \\ v = x-y \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \end{cases}$ $|J| = \frac{1}{2}$

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right\}$$

$$\Rightarrow f(u,v) = \frac{1}{4\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{u^2}{4(1+\rho)} - \frac{v^2}{4(1-\rho)}\right\}$$

从 $U, V$ 为独立的正态分布 $N(0, 2(1+\rho)), N(0, 2(1-\rho))$

**6.** Let $\{Y_r : 1 \le r \le n\}$ be independent $N(0,1)$ random variables, and define $X_j = \sum_{r=1}^n c_{jr} Y_r$, $1 \le r \le n$, for constants $c_{jr}$. Show that

$$\mathbb{E}(X_j \mid X_k) = \left(\frac{\sum_r c_{jr} c_{kr}}{\sum_r c_{kr}^2}\right) X_k.$$

What is $\text{var}(X_j \mid X_k)$?

Ex 4.8.7: for any pair of centred normal R.V.

$$\mathbb{E}(X|Y) = \frac{\text{cov}(X,Y)}{\text{var}\,Y} Y, \quad \text{var}(X|Y) = \{1 - \rho(X,Y)^2\} \text{var}\,X.$$

4.9.6: 由 Ex 4.8.7 即得 $\mathbb{E}(X_j|X_k) = \left(\frac{\sum_r c_{jr} c_{kr}}{\sum_r c_{kr}^2}\right) X_k$

$$\text{Var}(X_j|X_k) = \{1 - \rho(X_j, X_k)^2\} \text{var}\,X_j$$

$$= \left\{1 - \frac{(\sum_r c_{jr} c_{kr})^2}{\sum_r c_{jr}^2 \sum_r c_{kr}^2}\right\} \sum_r c_{jr}^2$$

**1.** Let $X_1$ and $X_2$ be independent variables with the $\chi^2(m)$ and $\chi^2(n)$ distributions respectively. Show that $X_1 + X_2$ has the $\chi^2(m+n)$ distribution.

**2.** Show that the mean of the $t(r)$ distribution is 0, and that the mean of the $F(r,s)$ distribution is $s/(s-2)$ if $s > 2$. What happens if $s \le 2$?

4.10.1: $\chi^2(m)$ 密度 $f_m(x) = \frac{1}{\Gamma(\frac{m}{2})} 2^{-\frac{m}{2}} x^{\frac{1}{2}m-1} e^{-\frac{1}{2}x}, \ x \geq 0.$

$Z = X_1 + X_2.$

$Z$ 密度 $g(z) = c\int_0^z x^{\frac{m}{2}-1} e^{-\frac{1}{2}x} (z-x)^{\frac{1}{2}n-1} e^{-\frac{1}{2}(z-x)} dx$

$$= c z^{\frac{1}{2}(m+n)-1} e^{-\frac{1}{2}z} \int_0^1 u^{\frac{1}{2}m-1}(1-u)^{\frac{1}{2}n-1} du \quad (u = \tfrac{x}{z})$$

$$\sim \chi^2(m+n)$$

法二，取 $Z_1, \cdots, Z_{m+n}$ 为 i.i.d. $\sim N(0,1)$.

则 $X_1$ 与 $Z_1^2+\cdots+Z_m^2$ 同分布

$X_2$ 与 $Z_{m+1}^2+Z_{m+2}^2+\cdots+Z_{m+n}^2$ 同分布

$\Rightarrow X_1+X_2$ 与 $Z_1^2+\cdots+Z_{m+n}^2$ 同分布 $\sim \chi^2(n+m)$

**2.** Let $X_1, X_2, \ldots$ be random variables satisfying $\mathbb{E}(\sum_{i=1}^{\infty}|X_i|) < \infty$. Show that

$$\mathbb{E}\left(\sum_{i=1}^{\infty} X_i\right) = \sum_{i=1}^{\infty} \mathbb{E}(X_i).$$

5.6.2: 令 $Z_n = \sum_{i=1}^{n} X_i$, $Z = \sum_{i=1}^{\infty} |X_i|$

则 $|Z_n| \leq Z$. 由控制收敛定理即得

**4.** Suppose that $\mathbb{E}|X^r| < \infty$ where $r > 0$. Deduce that $x^r\mathbb{P}(|X| \geq x) \to 0$ as $x \to \infty$. Conversely, suppose that $x^r\mathbb{P}(|X| \geq x) \to 0$ as $x \to \infty$ where $r \geq 0$, and show that $\mathbb{E}|X^s| < \infty$ for $0 \leq s < r$.

5.6.4: 若 $\mathbb{E}|X^r| < \infty$, $r > 0$

则 $x^r P(|X| \geqslant x) \leqslant \int_{[x,\infty)} u^r dF(u) \to 0$. $F$ 为 $|X|$ 的分布.

反之, $x^r P(|X| \geqslant x) \to 0$, 令 $0 \leqslant s < r$.

$\mathbb{E}|X^s| = \lim_{M\to\infty} \int_0^M u^s dF(u)$.

分部积分 $\int_0^M u^s dF(u) = [-u^s(1-F(u))]_0^M + \int_0^M s u^{s-1}(1-F(u))\,du$

第一项为负数.

当 $u$ 足够大时. $P(|X|>u) \leqslant u^{-r}$ 即 $1-F(u) \leqslant u^{-r}$ $\therefore s u^{s-1}(1-F(u)) \leqslant s u^{s-r-1}$

$\therefore \int_0^M s u^{s-1}(1-F(u))du$ 对于 $M$ 一致有界.

$\therefore \mathbb{E}|X^s| < \infty$

**2.** If $\phi$ is a characteristic function, show that $\mathrm{Re}\{1-\phi(t)\} \geq \frac{1}{4}\mathrm{Re}\{1-\phi(2t)\}$, and deduce that $1-|\phi(2t)| \leq 8\{1-|\phi(t)|\}$.

**3.** The **cumulant generating function** $K_X(\theta)$ of the random variable $X$ is defined by $K_X(\theta) = \log\mathbb{E}(e^{\theta X})$, the logarithm of the moment generating function of $X$. If the latter is finite in a neighbourhood of the origin, then $K_X$ has a convergent Taylor expansion:

$$K_X(\theta) = \sum_{n=1}^{\infty} \frac{1}{n!} k_n(X)\theta^n$$

(a) Express $k_1(X)$. $k_2(X)$ $k_3(X)$

(b) $X, Y$ independent, then $k_n(X+Y) = k_n(X) + k_n(Y)$

5.7.2: (i) $Re(\phi(t)) = E(\cos tX)$

$$\therefore Re\{1-\phi(2t)\} = \int_{-\infty}^{\infty} \{1-\cos(2tx)\} dF(x)$$
$$= 2\int_{-\infty}^{\infty} \{1-\cos(tx)\}\{1+\cos(tx)\} dF(x)$$
$$\le 4\int_{-\infty}^{\infty} \{1-\cos(tx)\} dF(x) = 4Re\{1-\phi(t)\}$$

(ii) 若 $X, Y$ 独立 同有特征函数 $\phi$.

则 $X-Y$ 的特征函数

$$\psi(t) = E(e^{itX}) E(e^{-itY})$$
$$= \phi(t)\phi(-t)$$
$$= \phi(t)\overline{\phi(t)} = |\phi(t)|^2$$

由(i)得 $1-\psi(2t) \le 4(1-\psi(t))$

$$\Rightarrow 1-|\phi(2t)|^2 \le 4(1-|\phi(t)|^2)$$

然而 $|\phi(t)| \le 1$.

$$\therefore 1-|\phi(2t)| \le 1-|\phi(2t)|^2 \le 4(1-|\phi(t)|^2) \le 8(1-|\phi(t)|)$$

5.7.3: (a) 令 $E(X^k) = m_k$.

则 $$E(e^{\theta X}) = E\left(1+\sum_{k=1}^{\infty}\frac{1}{k!}\theta^k X^k\right)$$
$$= 1+\sum_{k=1}^{\infty}\frac{1}{k!} m_k \theta^k \triangleq 1+S(\theta)$$

$$K_X(\theta) = \log(1+S(\theta))$$
$$= \sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r} S(\theta)^r$$ , 将 $S(\theta)^r$ 展开.

计算得 $k_1(X) = m_1$,

$$k_2(X) = m_2 - m_1^2$$
$$k_3(X) = m_3 - 3m_1m_2 + 2m_1^3$$

(b) $K_{X+Y}(\theta) = \log\{E(e^{\theta X})E(e^{\theta Y})\} = K_X(\theta) + K_Y(\theta)$.