(2) 证明 $Y_n(w) \xrightarrow{a.s.} Y(w)$.

$F(x)<y \Leftrightarrow x<F^{-1}(y)$ $\quad$ $F(x)\ge y \Leftrightarrow x\ge F^{-1}(y)$

对 $\forall \varepsilon>0, w\in\Omega'$, $\exists x\in C_F$. $Y(w)-\varepsilon<x<Y(w)$

$x<Y(w) \Leftrightarrow F(x)<w$

$F_n(x) \xrightarrow{w} F(x)$. $\lim\limits_{n\to\infty} F_n(x)=F(x)<w$ 当n充分大时, $F_n(x)<w$

$\therefore x<F_n^{-1}(w)$ $\quad$ $Y(w)-\varepsilon<x<F_n^{-1}(w)=Y_n(w)$ $\quad$ $n\to+\infty$, $\varepsilon\to 0^+$. $\liminf\limits_{n\to\infty} Y_n(w)>Y(w)$

取 $w<w'<1$, $\exists x\in C_F$, $Y(w')<x<Y(w')+\varepsilon$ $\quad$ $w<w'\le F(x)=\lim\limits_{n\to\infty} F_n(x)$

n充分大时, $w<w'\le F_n(x)$

由逆映射定义, $Y_n(x)\le x<Y(w')+\varepsilon$, $\limsup\limits_{n\to\infty} Y_n(w)\le Y(w')$

若w是$Y(w)$的连续点. $w'\downarrow w$. 故 $Y(w)\le \liminf\limits_{n\to\infty} Y_n(w)\le \limsup\limits_{n\to\infty} Y_n(w)\le Y(w)$

$Y(w)$单调增加, 不连续点至多可数个.

$P'(\lim\limits_{n\to\infty} Y_n(w)=Y(w))=P'(Y(w)\text{连续点})=1$

定理 $X_n \xrightarrow{D} X \Leftrightarrow \forall$有界连续函数 $g(x)$, s.t. $\lim\limits_{n\to\infty} E[g(X_n)]=E[g(X)]$

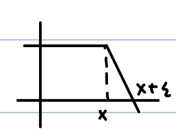
证: "$\Rightarrow$" 由表示定理 $\exists Y_n, Y$. $Y_n \xrightarrow{a.s.} Y$ $\quad$ $F_{Y_n}(x)=F_{X_n}(x)$

$$E[g(X_n)]=\int_{-\infty}^{+\infty} g(x)\,dF_{X_n}(x)=E[g(Y_n)] \quad g(Y_n)\xrightarrow{a.s.} g(Y) \quad g\text{有界}$$

由DCT. $\lim\limits_{n\to\infty} E[g(Y_n)]=E[g(Y)] \Rightarrow \lim\limits_{n\to\infty} E[g(X_n)]=E[g(X)]$

"$\Leftarrow$" 要证 $F_{X_n}(x)\xrightarrow{w} F_X(x)$ 即 $P(X_n\le x)\xrightarrow{w} P(X\le x)$ 即 $E(I_{\{X_n\le x\}})\longrightarrow E(I_{\{X\le x\}})$

定义: $g_{x,\varepsilon}(t)=\begin{cases}1, & t\le x\\ 0, & t>x+\varepsilon\\ \dfrac{-t+x}{\varepsilon}+1, & x<t\le x+\varepsilon\end{cases}$ 为有界连续函数.



$P(X_n\le x)\le E(g_{x,\varepsilon}(X_n))\longrightarrow E[g_{x,\varepsilon}(X)]\le E[I_{\{X\le x+\varepsilon\}}]=P(X\le x+\varepsilon)$

另一方面, $P(X_n\le x)\ge E[g_{x-\varepsilon,\varepsilon}(X_n)]\longrightarrow E[g_{x-\varepsilon,\varepsilon}(X)]\ge E[I_{\{X\le x-\varepsilon\}}]\ge P(X\le x-\varepsilon)$

当n充分大时, $P(X\le x-\varepsilon)\le P(X_n\le x)\le P(X\le x+\varepsilon)$

若 x 是$F(x)$连续点. 令 $\varepsilon\to 0$. $F(x)\le \liminf\limits_{n\to\infty} F_n(x)\le \limsup\limits_{n\to\infty} F_n(x)\le F(x)$

$\therefore \lim F_n(x)=F(x)$, $x\in C_F$

§7.3 辅助结论

一. 不等式

1. $h(x)$ 非负可测函数. $a>0$ X r.v. 则 $P(h(x)\geq a) \leq \frac{E[h(x)]}{a}$.

证: $E(h(x))<+\infty$ $E[h(x)] \geq E[I_{\{h(x)\geq a\}}\cdot a] = a P(h(x)\geq a)$

(1) $h(x)=|x|^r$ $P(|x|^r \geq a) \leq \frac{E[|x|^r]}{a}$ Markov

(2) $P(|X-E(x)| \geq a) \leq \frac{E(|X-E(X)|^2)}{a^2} = \frac{Var(X)}{a^2}$ Chebyshev

2. Hölder $p,q>1$ $\frac{1}{p}+\frac{1}{q}=1$ $E(|XY|) \leq (E|X|^p)^{\frac{1}{p}}\cdot(E|Y|^q)^{\frac{1}{q}}$

3. Minkovski $(E|X+Y|^p)^{\frac{1}{p}} \leq (E|X|^p)^{\frac{1}{p}} + (E|Y|^p)^{\frac{1}{p}}$

4. $E(|X+Y|^r) \leq C_r(E|X|^r+E|Y|^r)$

证: $r>1$ $g(x)=x^r$ 在 $(0,+\infty)$ 凸函数. $E\left(\frac{|X+Y|}{2}\right)^r \leq E\left(\frac{|X|+|Y|}{2}\right)^r \leq \frac{1}{2}(E|X|^r+E|Y|^r)$

$E(|X+Y|^r) \leq 2^{r-1}(E|X|^r+E|Y|^r)$

$0<r\leq 1$ 时, $|X+Y|^r \leq (|X|+|Y|)^r \leq \frac{|X|+|Y|}{(|X|+|Y|)^{1-r}} = \frac{|X|}{(|X|+|Y|)^{1-r}} + \frac{|Y|}{(|X|+|Y|)^{1-r}} \leq |X|^r+|Y|^r$

$C_r = \begin{cases} 2^{r-1}, & r>1 \\ 1, & 0<r\leq 1 \end{cases}$

二. 运算性质

定理: $X_n \xrightarrow{*} X$, $Y_n \xrightarrow{*} Y$ $*$表示 a.s., P., r.

则 (1) $X_n \xrightarrow{*} X$, $X_n \xrightarrow{*} Y$, $P(X=Y)=1$

(2) $X_n+Y_n \xrightarrow{*} X+Y$, $X_nY_n \xrightarrow{*} XY$.

(3) 依分布收敛时, (1)(2) 不成立.

证: (1) $X_n \xrightarrow{P} X$, $X_n \xrightarrow{P} Y$. $|X-Y| \leq |X-X_n| + |X_n-Y|$

任取 $\varepsilon>0$. $\{|X-Y|>\varepsilon\} \subset \{|X-X_n|>\frac{\varepsilon}{2}\} \cup \{|X_n-Y|>\frac{\varepsilon}{2}\}$.

$w \in \{|X-X_n|>\frac{\varepsilon}{2}\} \cap \{|X_n-Y|>\frac{\varepsilon}{2}\}$, 则 $w \in \{|X-Y|\leq\varepsilon\}$

$P(|X-Y|>\varepsilon) \leq P(|X-X_n|>\frac{\varepsilon}{2}) + P(|Y-X_n|>\frac{\varepsilon}{2}) \to 0$ $(n\to\infty)$

由 ε 任意性, $P(|X-Y|>0)=0$.

$X_n \xrightarrow{r} X$, $X_n \xrightarrow{r} Y$.

$E[|X-Y|^r] \le E[|X-X_n|^r + |Y-X_n|^r] \longrightarrow 0 \quad \therefore P(|X-Y|=0)=1$

(2) $X_n \xrightarrow{P} X,\ Y_n \xrightarrow{P} Y.$

$\forall \varepsilon>0,\ P(|X_n+Y_n-(X+Y)|>\varepsilon) = P(|(X_n-X)+(Y_n-Y)|>\varepsilon) \le P(|X_n-X|>\frac{\varepsilon}{2})+P(|Y_n-Y|>\frac{\varepsilon}{2})$

$\longrightarrow 0$

$X_n \xrightarrow{P} X,\ Y_n \xrightarrow{P} Y,\ X_nY_n \xrightarrow{P} XY$

$X_nY_n - XY = (X_n-X)(Y_n-Y) + X(Y_n-Y) + Y(X_n-X)$

对 $\forall \varepsilon>0,\ P(|X_n-X|\cdot|Y_n-Y|>\varepsilon) \le P(|X_n-X|\ge\sqrt{\varepsilon})+P(|Y_n-Y|\ge\sqrt{\varepsilon}) \to 0$

$\lim\limits_{k\to\infty} P(|X|<k)=1$，对上述 $\varepsilon>0$，$\exists M$. s.t. $P(|X|\ge M)<\frac{\varepsilon}{4}$

$P(|X(Y_n-Y)|>\varepsilon) = P(|X(Y_n-Y)|>\varepsilon,\ |X|\le M) + P(|X(Y_n-Y)|>\varepsilon,\ |X|>M)$

$\le P(|Y_n-Y|>\frac{\varepsilon}{M}) + P(|X|>M) \longrightarrow P(|X|>M) \quad n\to\infty$

再令 $M\to+\infty$，$P(|X(Y_n-Y)|>\varepsilon)\longrightarrow 0$

(3) X

1	-1
$\frac{1}{2}$	$-\frac{1}{2}$

$X_n=(-1)^nX \xrightarrow{D} X \qquad X_n+Y_n=0 \stackrel{D}{\not\rightarrow} X+X$

$Y_n=(-1)^{n+1}X \xrightarrow{D} X$

定理 $X_n \xrightarrow{D} X,\ Y_n \xrightarrow{P} c$（常数），则 $X_n+Y_n \xrightarrow{D} X+c$.

证：$F_n(t)=P(X_n+Y_n\le t) = P(X_n+Y_n\le t,\ |Y_n-c|\le\varepsilon)+P(X_n+Y_n\le t,\ |Y_n-c|>\varepsilon)$

$\le P(X_n\le t-c+\varepsilon)+P(|Y_n-c|>\varepsilon)$

$P(X_n\le t-c+\varepsilon) = P(X_n\le t-c+\varepsilon,\ |Y_n-c|\le\varepsilon) + P(X_n\le t-c+\varepsilon,\ |Y_n-c|>\varepsilon)$

$\le P(X_n+Y_n\le t)+P(|Y_n-c|>\varepsilon)$

$P(X_n+Y_n\le t) \ge P(X_n\le t-c-\varepsilon)+P(|Y_n-c|>\varepsilon)$ → 取等号

三. Borel-Cantelli 引理

$\{A_n\}$ 事件 $\quad \bigcap\limits^{\infty} \bigcup\limits^{\infty} A_m = \limsup A_n \triangleq \{A_n \text{ i.o.}\}$ infinitely often

$$\bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty} A_m = \liminf_{n\to\infty} A_n = \{A_n^c \text{ i.o.}\}^c$$

定理：$\{A_n\}$ 事件列

(1) 若 $\sum_{n=1}^{\infty} P(A_n) < \infty$，则 $P(A_n \text{ i.o.}) = 0$

(2) 若 $\{A_n\}$ 相互独立，$\sum_{n=1}^{\infty} P(A_n) = \infty \iff P(A_n \text{ i.o.}) = 1$

证：(1) $P(\bigcup_{m=n}^{\infty} A_m) \le \sum_{m=n}^{\infty} P(A_m) \to 0 \quad n\to\infty$

$$P(A_n \text{ i.o.}) = \lim_{n\to\infty} P(\bigcup_{m=n}^{\infty} A_m) = 0$$

(2) "$\Leftarrow$" 显然

"$\Rightarrow$" $P(\bigcup_{m=n}^{+\infty} A_m) = 1 - P(\bigcap_{m=n}^{\infty} A_m^c)$. $P(\bigcap_{m=n}^{\infty} A_m^c) = \lim_{r\to\infty} P(\bigcap_{m=n}^{r} A_m^c) = \lim_{r\to\infty} \prod_{m=n}^{r} (1-P(A_m))$

$$\le \lim_{r\to\infty} \prod_{m=n}^{r} e^{-P(A_m)} = \lim_{r\to\infty} e^{-\sum_{m=n}^{r} P(A_m)} \to 0$$

$$P(\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m) = \lim_{n\to\infty} P(\bigcup_{m=n}^{\infty} A_m) = 1$$

若不加独立条件：$\Omega = (0,1)$ $\quad A_n = (0, \frac{1}{n})$ $\quad \sum_{n=1}^{\infty} P(A_n) = \infty$ 而 $\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m = \phi$

$X_n \xrightarrow{a.s.} X$ 对 $\forall \varepsilon > 0$，$P(\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} |X_n - X| > \varepsilon) = 0 \iff P(|X_n - X| > \varepsilon \text{ i.o.}) = 0 \Leftarrow \sum_{n=1}^{\infty} P(|X_n - X| > \varepsilon) < \infty$.

例 $\{X_n\}$ 独立同分布 $E(X_i) = \mu$ $\quad E(X_i^4) < \infty$. $S_n = \sum_{k=1}^{n} X_k$，则 $\frac{S_n}{n} \xrightarrow{a.s.} \mu$

证：不妨设 $\mu = 0$ （$\mu \ne 0$，$S_n' = \frac{S_n - n\mu}{n} \xrightarrow{a.s.} 0$）

$$E[S_n^4] = \sum_{i=1}^{n} E[X_i]^4 + \sum_{i\ne j} E(X_i^2 X_j^2) + \sum_{i\ne j\ne k} E(X_i^2 X_j X_k) + \sum_{i\ne j} E(X_i^3 X_j) + \sum_{i\ne j\ne k\ne l} E(X_i X_j X_k X_l)$$

$$= nE(X^4) + \frac{n(n-1)}{2} \cdot \frac{4!}{2\cdot 2} (E[X_i^2])^2 \le (n + 3n(n-1)) E(X^4) \le c\cdot n^2 E(X^4)$$

$$\sum_{n=1}^{\infty} P(|\frac{S_n}{n}| > \varepsilon) \le \sum_{n=1}^{\infty} \frac{E((\frac{S_n}{n})^4)}{\varepsilon^4} = \sum_{n=1}^{\infty} \frac{c\cdot n^2 E(X^4)}{n^4 \varepsilon^4} < \infty$$

$P(\frac{S_n}{n} > \varepsilon \text{ i.o.}) = 0 \qquad \frac{S_n}{n} \xrightarrow{a.s.} 0$.

hw: 7.11.2 (2)，7.11.4，7.11.7，7.11.8