7.4.1

1. Let $X_2, X_3, \ldots$ be independent random variables such that

$$\mathbb{P}(X_n = n) = \mathbb{P}(X_n = -n) = \frac{1}{2n\log n}, \quad \mathbb{P}(X_n = 0) = 1 - \frac{1}{n\log n}.$$

Show that this sequence obeys the weak law but not the strong law, in the sense that $n^{-1}\sum_1^n X_i$ converges to 0 in probability but not almost surely.

证：先证 $\frac{\sum_{i=1}^n X_i}{n} \xrightarrow{P} 0$. $S_n = X_2 + \cdots + X_n$ $\qquad f(x) = \frac{x}{\log x} \uparrow$

$E[S_n] = 0$, $E[S_n^2] = Var(S_n) \overset{独立}{=} \sum_{k=2}^n Var(X_k) = \sum_{k=2}^n k^2 \cdot \frac{1}{k\log k} = \sum_{k=2}^n \frac{k}{\log k} \leq \frac{n^2}{\log n}$

$\Rightarrow \frac{1}{n^2} Var(\sum_{k=1}^n X_k) \to 0$ as $n \to \infty$ 由 Markov LLN, $\frac{S_n - ES_n}{n} \xrightarrow{P} 0$.

即 $\frac{\sum_{i=1}^n X_i}{n} \xrightarrow{P} 0$. 再证 $\frac{\sum_{i=1}^n X_i}{n} \overset{a.s}{\not\to} 0$

$\sum_{k=2}^{+\infty} P(|X_k| \geq k) = \sum_{k=2}^{+\infty} \frac{1}{k\log k} = +\infty$

由 Borel-Cantelli 引理，$P(|X_k| \geq k \text{ i.o.}) = 1$ 即 $P(|S_k - S_{k-1}| \geq k \text{ i.o.}) = 1$

$\therefore \frac{S_k}{k} = \frac{\sum_{i=1}^k X_i}{k} \overset{a.s}{\not\to} 0$

7.11.17

17. Let $g : \mathbb{R} \to \mathbb{R}$ be bounded and continuous. Show that

$$\sum_{k=0}^{\infty} g(k/n)\frac{(n\lambda)^k}{k!}e^{-n\lambda} \to g(\lambda) \quad \text{as } n \to \infty.$$

证：令 r.v. $X_1, \cdots, X_n \overset{i.i.d.}{\sim} Poi(\lambda)$ 则 $S_n = X_1 + \cdots + X_n \sim Poi(n\lambda)$.

由 WLLN, $\frac{S_n}{n} \xrightarrow{D} E[X_i] = \lambda$

对于有界连续函数 g 有 $E[g(\frac{S_n}{n})] \longrightarrow E[g(\lambda)] = g(\lambda)$

$E[g(\frac{S_n}{n})] = \sum_{k=0}^{+\infty} g(\frac{k}{n}) \cdot \frac{(n\lambda)^k}{k!} \cdot e^{-n\lambda} \longrightarrow g(\lambda)$ as $n \to +\infty$

7.11.20

20. Let $X_1, X_2, \ldots$ be random variables satisfying $\text{var}(X_n) < c$ for all n and some constant c. Show that the sequence obeys the weak law, in the sense that $n^{-1}\sum_1^n (X_i - \mathbb{E}X_i)$ converges in probability to 0, if the correlation coefficients satisfy either of the following:

(i) $\rho(X_i, X_j) \leq 0$ for all $i \neq j$,

(ii) $\rho(X_i, X_j) \to 0$ as $|i - j| \to \infty$.

证：(i) 令 $Y_i = X_i - EX_i$. $T_n = \sum_{i=1}^{n} Y_i$. 要证 $\frac{T_n}{n} \xrightarrow{P} 0$.

$E[T_n]=0$. $E[T_n^2] = Var(T_n) = \sum_{i=1}^{n} Var(Y_i) + 2\sum_{1\le i<j\le n} cov(Y_i, Y_j)$

$= \sum_{i=1}^{n} Var(X_i) + 2\sum_{1\le i<j\le n} cov(X_i, X_j) \le nc$

$\therefore \frac{Var(\sum_{i=1}^{n} Y_i)}{n^2} \to 0$ 由 Markov LLN. $\frac{T_n - ET_n}{n} \xrightarrow{P} 0$ 即 $\frac{T_n}{n} \xrightarrow{P} 0$.

(ii) 同理只需证 $\frac{E[T_n^2]}{n^2} \to 0$. $\forall \varepsilon>0$, $\exists I$, 对 $\forall |i-j|\ge I$, 都有 $|\rho(X_i, X_j)| \le \varepsilon$

$E[T_n^2] = \sum_{i,j=1}^{n} cov(X_i, X_j)$

$= \sum_{|i-j|\le I} cov(X_i, X_j) + \sum_{|i-j|>I} cov(X_i, X_j) \le 2nI\cdot c + n^2\varepsilon\cdot c$

$cov(X_i, X_j) = \rho(X_i, X_j)\cdot\sqrt{Var(X_i)Var(X_j)} \le c\cdot|\rho(X_i, X_j)|$

$\frac{E[T_n^2]}{n^2} \le \frac{2Ic}{n} + \varepsilon\cdot c \to \varepsilon\cdot c$ 当 $n\to\infty$ 时 对 $\forall\varepsilon>0$ 成立.

$\therefore \frac{E[T_n^2]}{n^2} \longrightarrow 0$ 得证!