$$m(x)=E(E(N(x)|X_1))=\int_0^1 E(N(x)|X_1=y)f_{X_1}(y)dy$$

$$=\int_0^x 1+m(x-y)dy+\int_x^1 1dy = 1+\int_0^x m(t)dt \qquad \begin{cases} m'(x)=m(x) \\ m(0)=1 \end{cases}$$

hw: 4.6.4, 4.6.8, 4.6.9, 4.6.10

§4.5 随机变量的函数.

一. X的密度$f(x)$

$g(x)$是连续函数. $g(x)$是否为连续型.

$X\sim U[0,2]$

$Y=g(X)$ $\qquad g(x)=\begin{cases} x, & 0\le x\le 1 \\ 1, & 1<x\le 2 \end{cases}$

则 $P(Y\le y)=\begin{cases} 0, & y<0 \\ \frac{y}{2}, & 0\le y<1 \\ 1, & y>1 \end{cases}$

定理 (1) X概率密度为$f(x)$. $Y=g(X)$严格单调连续函数.有连续导数.

则$Y=g(x)$的概率密度 $f_Y(y)=f(g^{-1}(y))\cdot|g^{-1}(y)'|$

(2) $g(x)$在不重叠区间段$I_1, I_2, \cdots, I_n$上严格单调,每小段上确定反函数$x=h_i(y)$,

$i=1,2,\cdots,n$, $h_i(y)$有连续导数,则 $f_Y(y)=\sum_i f(h_i(y))|h_i'(y)|$

证: (1) $P(Y\le a)=P(X\in\{x|g(x)\le a\})=\int_{\{x|g(x)\le a\}} f(x)dx \overset{y=g(x)}{=\!=} \int_{-\infty}^a f(g^{-1}(y))|g^{-1}(y)'|dy$

$f_Y(a)=f(g^{-1}(a))\cdot|g^{-1}(a)'|$

(2) $E_i(a)=\{x|x\in I_i, g(x)\le a\}$

$P(Y\le a)=P(X\in\bigcup_i E_i(a))=\sum_i P(X\in E_i(a))=\sum_i\int_{E_i(a)} f(x)dx$

$\overset{y=g(x)}{=\!=}\sum_i\int_{-\infty}^a f(h_i(y))|h_i'(y)|dy$

$Y\sim f_Y(y)=\sum_i f(h_i(y))|h_i'(y)|$

例 r.v. X 分布函数 $F(x)$ 严格增，连续函数，则 $Y=F(X)\sim U([0,1])$

证：$P(F(X)\le y)=P(X\le F^{-1}(y))=\begin{cases} 0, & y<0 \\ F(F^{-1}(y))=y, & 0\le y<1 \\ 1, & y\ge 1 \end{cases}$

$F(X)$ 密度 $f_Y(y)=\begin{cases} 1, & y\in[0,1) \\ 0, & \text{其它} \end{cases}$

注：$\theta\sim U([0,1])$，对 $\forall$ 严格增分布函数 $F(x)$，可定义一个 r.v. 服从 $F(x)$.

$X=F^{-1}(\theta)$　　$P(X\le x)=P(F^{-1}(\theta)\le x)=P(\theta\le F(x))=F(x)$

$X\sim N(\mu,\sigma^2)$ 求 $Y=e^X$ 的概率密度. $f_Y(y)=\dfrac{1}{\sqrt{2\pi}\sigma y}e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}}$

二. (X_1,X_2) 联合密度 $f(x_1,x_2)$

$Y_1=g_1(X_1,X_2)$，$Y_2=g_2(X_1,X_2)$

满足 (1) $\begin{cases} y_1=g_1(x_1,x_2) \\ y_2=g_2(x_1,x_2) \end{cases}$　可以确定逆映射 $\begin{cases} x_1=h_1(y_1,y_2) \\ x_2=h_2(y_1,y_2) \end{cases}$

(2) $J=\begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix}\ne 0$　　g_1,g_2 有连续偏导数.

则 (Y_1,Y_2) 有联合密度 $f_Y(y_1,y_2)=f(h_1(y_1,y_2),h_2(y_1,y_2))\cdot|J|^{-1}$

证：$P(Y_1\le y_1,Y_2\le y_2)=P(g_1(X_1,X_2)\le y_1,g_2(X_1,X_2)\le y_2)$

$$=\iint\limits_{\substack{g_1(x_1,x_2)\le y_1\\ g_2(x_1,x_2)\le y_2}} f(x_1,x_2)\,dx_1dx_2$$

$$\xlongequal[x_2=h_2(u,v)]{x_1=h_1(u,v)}\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}f(h_1(u,v),h_2(u,v))\left|\frac{\partial(x_1,x_2)}{\partial(u,v)}\right|du\,dv$$

$f_Y(y_1,y_2)=f(h_1(y_1,y_2),h_2(y_1,y_2))\cdot|J|^{-1}$

例 (X,Y)　　X,Y 相互独立，服从 $N(0,1)$

令 $R=\sqrt{X^2+Y^2}$　　　$\theta=\{\arctan\frac{Y}{X}$，$X>0,Y>0$

求 R, Θ 分布. $\begin{cases} \pi+\arctan\frac{Y}{X}, & X<0 \\ 2\pi+\arctan\frac{Y}{X}, & X>0, Y<0 \end{cases}$

解: (X, Y) 联合密度 $\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

(R, Θ) 联合密度 $\frac{r}{2\pi}e^{-\frac{r^2}{2}} = \frac{1}{2\pi}\cdot r\cdot e^{-\frac{r^2}{2}}$ R, Θ 独立.

$\Theta \sim U([0,2\pi])$ $R = f_R(r) = r\cdot e^{-\frac{r^2}{2}} \ (r\geq 0)$ $F_R(r) = 1-e^{-\frac{r^2}{2}} = u_2 \Rightarrow r = \sqrt{-2\ln(1-u_2)}$

注: $u_1, u_2 \sim U([0,1])$ 相互独立.

$\Theta = 2\pi u_1$ $X = \sqrt{-2\ln u_2}\cos(2\pi u_1)$

独立 独立, 服从 $N(0,1)$

$R = \sqrt{-2\ln u_2}$ $Y = \sqrt{-2\ln u_2}\sin(2\pi u_1)$

例 $(X,Y) \sim f(x,y)$ 联合密度. 求 $Z = X+Y$ 分布.

解法1 $(X+Y, X)$ 联合分布 $\Rightarrow$ $X+Y$ 边缘分布.

解法2 $P(X+Y\leq a) = \iint\limits_{x+y\leq a} f(x,y)\,dx\,dy = \int_{-\infty}^{+\infty}dx\int_{-\infty}^{a-x}f(x,y)\,dy$

若 X, Y 独立. $f(x,y) = f_X(x)f_Y(y)$

$$P(X+Y\leq a) = \int_{-\infty}^{+\infty}dx\int_{-\infty}^{a-x}f_X(x)f_Y(y)\,dy$$
$$= \int_{-\infty}^{+\infty}dx\int_{-\infty}^{a}f_X(x)f_Y(t-x)\,dt$$
$$= \int_{-\infty}^{a}\left(\int_{-\infty}^{+\infty}f_X(x)f_Y(t-x)\,dx\right)dt$$

X+Y 概率密度 $\int_{-\infty}^{+\infty}f_X(x)f_Y(t-x)\,dx = f_X * f_Y(t)$

例 顺序统计量

$X_1, X_2, \cdots, X_n$ 独立同分布. 分布函数 $F(x)$.

$X_{i_1}(w) < X_{i_2}(w) < \cdots < X_{i_n}(w)$

$X_{i_k}(w) = X_k^*(w)$ $X_1^* = \min\{X_1, X_2, \cdots, X_n\}$ $X_n^* = \max\{X_1, \cdots, X_n\}$

求 X_k^* 的概率密度.

解: $w\in(X_k^*\le x)\Leftrightarrow X_1(w),\cdots,X_n(w)$中至少有k个$\le x$.

$A_m=\{w\mid X_1(w),\cdots,X_n(w)$中恰好m个$\le x\}$

$P(A_m)=C_n^m(F(x))^m(1-F(x))^{n-m}$

$P(X_k^*\le x)=\sum_{i=k}^{n}P(A_i)=\sum_{i=k}^{n}C_n^i(F(x))^i(1-F(x))^{n-i}$

X_k^*密度 $f_k(x)=\frac{d}{dx}\left(\sum_{i=k}^{n}C_n^i(F(x))^i(1-F(x))^{n-i}\right)$

$$=\sum_{i=k}^{n}\left(C_n^i\cdot i\cdot F(x)^{i-1}(1-F(x))^{n-i}\cdot f(x)-C_n^i(n-i)F(x)^i(1-F(x))^{n-i-1}f(x)\right)$$

$$=\frac{n!}{(k-1)!(n-k)!}F(x)^{k-1}(1-F(x))^{n-k}f(x)$$

§4.6 多元正态分布.

二元正态 $N(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,\rho)$

$$f(x_1,x_2)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2-2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)+\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right)\right)$$

$$\frac{1}{\sqrt{(2\pi)^2|\Sigma|}}\exp\left(-\frac{1}{2}(\vec{x}-\vec{\mu})\Sigma^{-1}(\vec{x}-\vec{\mu})^T\right)$$

$\vec{x}=(x_1,x_2)$ $\vec{\mu}=(\mu_1,\mu_2)$

$$\Sigma=\begin{pmatrix}\mathrm{cov}(x_1,x_1) & \mathrm{cov}(x_1,x_2)\\ \mathrm{cov}(x_2,x_1) & \mathrm{cov}(x_2,x_2)\end{pmatrix}=\begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}$$

$\det\Sigma\triangleq|\Sigma|=\sigma_1^2\sigma_2^2(1-\rho^2)>0$

正定对称矩阵.

$$\Sigma^{-1}=\frac{1}{1-\rho^2}\cdot\begin{pmatrix}\frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1\sigma_2}\\ \frac{-\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2}\end{pmatrix}$$

$$\frac{1}{1-\rho^2}\left(\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2-2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)+\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right)=(\vec{x}-\vec{\mu})\Sigma^{-1}(\vec{x}-\vec{\mu})^T$$

推广 $\vec{x}=(x_1,x_2,\cdots,x_n)$ Σ n阶正定对称矩阵 $\vec{\mu}=(\mu_1,\cdots,\mu_n)$

$f(x_1,\cdots,x_n)=\frac{1}{\sqrt{(2\pi)^n\cdot|\Sigma|}}\exp\left(-\frac{1}{2}(\vec{x}-\vec{\mu})\Sigma^{-1}(\vec{x}-\vec{\mu})^T\right)$ 是n元随机向量$(x_1,\cdots,x_n)$密度函数.

验证: $\int_{R}f(x_1,\cdots,x_n)dx_1\cdots dx_n=1$ Σ 正定. 对称矩阵

$\exists$正定矩阵 B $(B^TB=I_n)$ s.t. $\Sigma=B^T\Lambda B$. $\Lambda=\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$

$\Sigma^{-1}=B^{-1}\Lambda^{-1}(B^T)^{-1}=B^T\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}B$ 作变量代换 $\vec{Y}=(\vec{x}-\vec{\mu})B^{-1}$. $\vec{x}=\vec{Y}B+\vec{\mu}$

$$\int_{R^n}\frac{1}{\sqrt{(2\pi)^n\cdot|\Sigma|}}\exp\left(-\frac{1}{2}(\vec{x}-\vec{\mu})\Sigma^{-1}(\vec{x}-\vec{\mu})^T\right)dx_1\cdots dx_n$$

$$= \int_{R^n} \frac{1}{\sqrt{(2\pi)^n \cdot |\Sigma|}} \cdot \exp(-\frac{1}{2}(\vec{y}B)B^T \Lambda B B^T \vec{y}^T)\, dx_1 \cdots dx_n$$

$$= \int_{R^n} \frac{1}{\sqrt{(2\pi)^n \cdot |\Sigma|}} \exp(-\frac{1}{2}\sum_{k=1}^{n} \frac{y_k^2}{\lambda_k})\, dy_1 \cdots dy_n = \frac{1}{\sqrt{(2\pi)^n \cdot \underset{\lambda_1 \cdots \lambda_n}{|\Sigma|}}} \cdot \prod_{k=1}^{n} \int_{-\infty}^{+\infty} e^{-\frac{y_k^2}{2\lambda_k}}\, dy_k = 1$$

$(X_1, \cdots, X_n) = (Y_1, \cdots, Y_n)B + \vec{\mu}$，$X_i = \mu_i + \sum_{j=1}^{n} y_j B_{ji}$

$\vec{Y} = (\vec{X} - \vec{\mu})B^{-1}$ $\quad f_Y(y_1, \cdots, y_n) = \frac{1}{\sqrt{(2\pi)^n |\Sigma|}} \exp(-\frac{1}{2}\sum_{k=1}^{n} \frac{y_k^2}{\lambda_k})$

$Y_1, \cdots, Y_n$ 独立1元标准正态

定义 $(Y_1, \cdots, Y_n)$ 服从n元标准正态，则 $\vec{X} = \vec{Y}A + \mu$ 服从参数为 $\mu, \Sigma = A^T A$ 的n元正态分布

定理 $\vec{X} = (X_1, \cdots, X_n) \sim N(\vec{\mu}, \Sigma)$

则 (1) $E[\vec{X}] = \vec{\mu}$ (2) Σ为协方差矩阵

证: (1) $E[X_i] = \int_{R^n} x_i \cdot \frac{1}{\sqrt{(2\pi)^n \cdot |\Sigma|}} \exp(-\frac{1}{2}(\vec{X}-\vec{\mu})\Sigma^{-1}(\vec{X}-\vec{\mu})^T)\, dx_1 \cdots dx_n$

$\vec{Y} = (\vec{X} - \vec{\mu})B^{-1}$

$$= \int_{R^n} (\mu_i + \sum_{k=1}^{n} b_{ki} y_k) \exp(-\frac{1}{2}\sum_{k=1}^{n} \frac{y_k^2}{\lambda_k}) \cdot \frac{1}{\sqrt{(2\pi)^n \cdot |\Sigma|}}\, dy_1 \cdots dy_n$$

$= \mu_i$

...

hw: 4.7.2, 4.7.5, 4.7.9, 4.9.3, 4.9.7