§1. 特征函数复习

$\boxed{\text{Thm}}$ 特征函数可导性与矩关系

(1) R.V. X s.t. $\mathbb{E}|X|^n < +\infty \Rightarrow \phi_X(t) = \sum_{j=0}^{k} \frac{(it)^j}{j!} \mathbb{E}[X^j] + o(|t|^k)$ ← 上课讲过

进一步 $\phi(t)$ k阶可导且 $\phi^{(k)}(t) = \int_{-\infty}^{+\infty} (iX)^k e^{itx} dF(x)$

(2) $\phi_X(t)$ 在0处 2k阶导数存在 $\Rightarrow \mathbb{E}|X|^{2k} < +\infty$.

pf: (1) 对 n 归纳，$\phi_X(t) = \int e^{itx} dF(x)$

$n=1$ 时 $\phi_X'(t) = \lim_{h\to 0} \frac{\phi_X(t+h) - \phi_X(t)}{h} = \lim_{h\to 0} \int e^{itx} \frac{e^{ihx} - 1}{h} dF(x)$

注意到 $\frac{e^{ihx}-1}{h} \to iX$ as $h \to 0$.

由 DCT，$\phi_X'(t) = \int e^{itx} \cdot ix \, dF(x) = \mathbb{E}[iX \cdot e^{itX}]$

$n=k$ 时 ✓

$n=k+1$ 时 由 $\phi_X^{(k+1)}(t) = \lim_{h\to 0} \frac{\phi_X^{(k)}(t+h) - \phi^{(k)}(t)}{h}$

$= \lim_{h\to 0} \int (ix)^k e^{itX} \frac{e^{ihX}-1}{h} dF(x)$

同理由 DCT 得到结论

(2) $k=2$ 时 设 ϕ'' 在0附近存在

$$\phi''(0) = \lim_{h\to 0} \frac{\phi(h) + \phi(-h) - 2\phi(0)}{h^2}$$

$$= \lim_{h\to 0} \int \frac{e^{ixh} + e^{-ixh} - 2}{h^2} dF(x)$$

$$= -2 \lim_{h\to 0} \int \frac{1 - \cos hx}{h^2} dF(x)$$

$\lim_{h\to 0} \frac{1-\cos hx}{h^2} = \frac{1}{2}x^2$

$$\therefore \int x^2 dF(x) = \int \lim_{h\to 0} \frac{2(1-\cos hx)}{h^2} dF(x)$$

$$\overset{\text{Fatou}}{\leq} \varliminf_{h\to 0} \int \frac{2(1-\cos hx)}{h^2} dF(x)$$

$$= -f''(0)$$

对一般 k，用归纳

对 $k-2$ ✓ $\Rightarrow X$ 有 $k-2$ 阶矩 $\Rightarrow \phi^{(k-2)}(t) = (-1)^{\frac{k-2}{2}} \int e^{itx} \cdot x^{k-2} dF(x)$

k 时 由于 $\phi^{(k)}(0)$ 存在 $\Rightarrow \phi^{(k-2)}(0)$ 在 0附近存在且连续.

由(1)结论 $(-1)^{\frac{k-2}{2}} \int x^{k-2} dP = \phi^{(k-2)}(0)$

构造另一个分布函数：$G(x) \triangleq \int_{-\infty}^{x} y^{k-2} dF(y)$

则 $\frac{G(x)}{G(\infty)}$ 为分布函数. $\Rightarrow$ 对应特征函数 $\psi(t)=\frac{1}{G(\infty)}\int e^{itx}x^{k-2}dF(x)$

$$=\frac{\varphi^{(2k-2)}(t)\cdot(-1)^{\frac{k}{2}-1}}{G(\infty)}$$

G 2阶矩存在 $\Rightarrow$ ψ 在0处 2阶导存在

§2. 几种收敛.

$$\begin{tikzcd}
\text{a.s. 收敛} \arrow[dr, Rightarrow] & & \\
 & \mathbb{P}\text{ 收敛} \arrow[r, Rightarrow] \arrow[ul, bend left, Rightarrow, "\text{子列}"] & \text{依分布收敛} \arrow[l, bend left, Rightarrow, "\text{极限为常数 } C"] \arrow[ull, bend right, Rightarrow, "\text{Skorohod thm}"'] \\
L^r\text{ 收敛} \arrow[ur, Rightarrow] & &
\end{tikzcd}$$

$\boxed{\text{Prop}}$ (依概率收敛的度量化)

$X_n \xrightarrow{P} X \iff \mathbb{E}(|X_n-X|\wedge 1)\to 0$

pf: $(\Rightarrow)$ $\mathbb{E}(|X_n-X|\wedge 1)=\mathbb{E}(|X_n-X|\wedge 1;|X_n-X|>\varepsilon)+\mathbb{E}(|X_n-X|\wedge 1;|X_n-X|\le\varepsilon)$

$$\le P(|X_n-X|>\varepsilon)+\varepsilon.$$

$n\to\infty$, 由 ε 任意性 ✓

$(\Leftarrow)$ $P(|X_n-X|\wedge 1>\varepsilon)\le\frac{\mathbb{E}(|X_n-X|\wedge 1)}{\varepsilon}\to 0$, $n\to\infty$.

取 $\varepsilon<1$ 时, $P(|X_n-X|\wedge 1>\varepsilon)\to 0$.

$\boxed{\text{Prop}}$ (其他刻画依概率收敛)

$X_n\xrightarrow{P}X \iff \forall$ 子列 X_{n_k}, $\exists$ 子列 $X_{n_{k_j}}$ a.s. 收敛.

$\boxed{\text{应用}}$ 1. $X_n\xrightarrow{P}X$. $Y_n\xrightarrow{P}Y$. 则 $X_n\pm Y_n\xrightarrow{P}X+Y$

$X_nY_n\xrightarrow{P}XY$

2. $\forall f$ cts, $X_n\xrightarrow{P}X$, 证明 $f(X_n)\xrightarrow{P}f(X)$ (习题)

(Slutsky 定理): 设 $X_n\xrightarrow{d}X$, $Y_n\xrightarrow{P}b$, $Z_n\xrightarrow{P}c$. 则

$X_nY_n+Z_n\xrightarrow{d}bX+c$ [Grimmett 7.2.5(a)]

例：设 X_n i.i.d. 均值为 μ，二阶矩有限. 令 $\overline{X}=\frac{1}{n}\sum_{k=1}^{n}X_k$. 证明

$$\sum_{k=1}^{n}(X_k-\mu)\Big/\sqrt{\sum_{k=1}^{n}(X_k-\overline{X})^2}\xrightarrow{D}N(0,1)$$

pf：即证 $\displaystyle\sum_{k=1}^{n}\frac{(X_k-\mu)}{\sqrt{n}\,\sigma}\Big/\sqrt{\sum_{k=1}^{n}\frac{(X_k-\overline{X})^2}{n\sigma^2}}\xrightarrow{D}N(0,1)$

注意到 $\displaystyle\sum_{k=1}^{n}\frac{(X_k-\overline{X})^2}{n\sigma^2}=\frac{1}{n\sigma^2}\sum_{k=1}^{n}(X_k-\mu)^2-\frac{2}{n\sigma^2}(\overline{X}-\mu)\sum_{k=1}^{n}(X_k-\mu)+\frac{1}{\sigma^2}(\overline{X}-\mu)^2$

由 CLT. $\displaystyle\sum_{k=1}^{n}\frac{(X_k-\mu)}{\sqrt{n}\,\sigma}\xrightarrow{D}N(0,1)$

由 WLLN $\displaystyle\frac{1}{n\sigma^2}\sum_{k=1}^{n}(X_k-\mu)^2\xrightarrow{P}1$ $\displaystyle\frac{1}{n}\sum_{k=1}^{n}(X_k-\mu)\xrightarrow{P}0$, $\overline{X}-\mu\xrightarrow{P}0$

结合 Slutsky 引理 得证.

强收敛：(a.s. 收敛) 常用 Borel-Cantelli 引理：

(i) $\displaystyle\sum_{n=1}^{\infty}P(A_n)<\infty\Rightarrow P(A_n\ i.o.)=0$

(ii) $\{A_n\}$ 独立. 则 $P(A_n\ i.o.)=1\Leftrightarrow\displaystyle\sum_{n=1}^{\infty}P(A_n)=\infty$.

- 要证 a.s. 收敛，只需证 $\forall\varepsilon>0$, $\sum\mathbb{P}(|X_n-X|>\varepsilon)<\infty$ (不等价)

  利用 $\mathbb{P}(|X_n-X|>\varepsilon)\leq\dfrac{\mathbb{E}[|X_n-X|^k]}{\varepsilon^k}$ (k 常常选 2)

证明 a.s. 收敛常用套路：

① 先找子列：$\forall\varepsilon>0$. $\sum\mathbb{P}(|X_{n_k}-X|>\varepsilon)<\infty$.

$\Rightarrow X_{n_k}\xrightarrow{a.s.}X$.

② 考虑 $\sup|X_n-X_{n_k}|\xrightarrow{a.s.}0$ $(n_k<n<n_{k+1})$

[Durr] 2.3.19. Let X_n be independent Poisson r.v. with $\mathbb{E}X_n=\lambda_n$ and let $S_n=X_1+\cdots+X_n$. Show that if $\sum\lambda_n=\infty$, then $\frac{S_n}{\mathbb{E}S_n}\xrightarrow{a.s.}1$

proof: (1) $\frac{S_n}{\mathbb{E}S_n}\xrightarrow{a.s.}1$

(2) 考虑子列 n_k, $n_k \triangleq \inf\{n: \sum_{i=1}^{n}\lambda_i \geq k^2\}$

then $\sum_{k=1}^{\infty} P\left(\left|\frac{S_{n_k}}{\mathbb{E}[S_{n_k}]}-1\right|>\varepsilon\right) \leq \frac{1}{\varepsilon^2}\sum_{k=1}^{\infty}\frac{1}{\sum_{i=1}^{n_k}\lambda_i} \leq \frac{1}{\varepsilon^2}\sum_k \frac{1}{k^2}<\infty$

(3) $P\left(\left|\frac{S_{n_k}}{\mathbb{E}[S_{n_k}]}-1\right|>\varepsilon \text{ i.o.}\right)=0. \quad \forall\varepsilon>0$

$$\Rightarrow \frac{S_{n_k}}{\mathbb{E}[S_{n_k}]}\xrightarrow{a.s.}1$$

(4) Notice that $S_{n_k}\leq S_n\leq S_{n_{k+1}}$

$$\Rightarrow \frac{S_{n_k}}{\mathbb{E}[S_{n_k}]}\frac{\mathbb{E}[S_{n_k}]}{\mathbb{E}[S_{n_{k+1}}]} \leq \frac{S_n}{\mathbb{E}[S_n]} \leq \frac{S_{n_{k+1}}}{\mathbb{E}[S_{n_{k+1}}]}\frac{\mathbb{E}[S_{n_{k+1}}]}{\mathbb{E}[S_{n_k}]}$$

only to show $\frac{\mathbb{E}[S_{n_{k+1}}]}{\mathbb{E}[S_{n_k}]}\to 1$

$$\frac{\mathbb{E}[S_{n_{k+1}}]}{\mathbb{E}[S_{n_k}]} \leq \frac{(k+1)^2}{k^2}+\frac{\lambda_{n_{k+1}}}{k^2}$$

we assume all variances are bounded by decompose X_n into independent r.v.'s with smaller λ.

$\boxed{\text{Thm.}}$ $X_n \xrightarrow{D} X \iff \forall g\in C_b(\mathbb{R})$ (有界连续函数), 则 $\mathbb{E}[g(X_n)]\to\mathbb{E}[g(X)]$

[Yau21] $X_n\xrightarrow{D}X$, $\exists\, r, C>0$ s.t. $\mathbb{E}|X_n|^r\leq C$ for all n.
show that $\lim_{n\to\infty}\mathbb{E}|X_n|^s=E|X|^s$

proof: $\forall f\in C_b(\mathbb{R})$ $\quad \mathbb{E}f(X_n)\to\mathbb{E}f(X)$

$$f_M(x)=\begin{cases}|x|^s & |x|<M\\ M^s & |x|\geq M\end{cases}$$

then $\lim_{n\to\infty}\mathbb{E}f_M(X_n)=\mathbb{E}f_M(X)$

$0\leq \mathbb{E}|X_n|^s-\mathbb{E}f_M(X_n)\leq \mathbb{E}[|X_n|^s; |X_n|\geq M]\leq \frac{1}{M^{r-s}}\mathbb{E}[|X_n|^r; |X_n|\geq M]$
$\leq \frac{C}{M^{r-s}}$

先 $n\to\infty$, 再 $M\to\infty$. $\quad \lim_{M\to\infty}\mathbb{E}[g_M(X)]=\mathbb{E}[|X|^s]$ (单调收敛)

随机变量的截尾：① $Y_n = X_n 1_{\{|X_n| \le M\}}$

② $Y_n = X_n 1_{\{|X_n| \le n\}}$

$\boxed{\text{Prop:}}$ 设 X_n 非负 i.i.d. R.V. ，$\mathbb{E}[X_1] < \infty$.

令 $Y_n = X_n I_{\{|X_n| \le n\}}$ 则有 $\frac{1}{n}\sum_{k=1}^{\infty}(X_k - Y_k) \xrightarrow{a.s.} 0$

从而 $\frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{a.s.} a \Leftrightarrow \frac{1}{n}\sum_{k=1}^{n} Y_k \xrightarrow{a.s.} a$.

pf: $\sum_{m=1}^{\infty} P(X \ge m) \le \mathbb{E}[X] \le 1 + \sum_{m=1}^{\infty} P(X \ge m)$

有 $\sum_{n=1}^{\infty} P(Y_n \ne X_n) = \sum_{n=1}^{\infty} P(X_n \ge n) = \sum_{n=1}^{\infty} P(X_1 \ge n) \le \mathbb{E}[X_1] < \infty$.

由 B-C . $P(X_n \ne Y_n \text{ i.o.}) = 0$.

$\therefore \frac{1}{n}\sum_{i=1}^{n}(X_i - Y_i) \xrightarrow{a.s.} 0$.

用截尾证 WLLN.

Thm: $X_1, \cdots, X_n$, i.i.d. $\mu = \mathbb{E}[X_1]$ 且 $\mathbb{E}[|X_1|] < +\infty$. 记 $S_n = \sum_{i=1}^{n} X_i$.

则 $\frac{S_n}{n} \xrightarrow{P} \mu$

pf: $X_n^{(1)} = X_n I_{\{|X_n| \le M\}}$, $X_n^{(2)} = X_n I_{\{|X_n| > M\}}$

$X_n = X_n^{(1)} + X_n^{(2)}$. $M > 0$. 设 $S_n^{(1)} = \sum_{i=1}^{n} X_i^{(1)}$, $S_n^{(2)} = \sum_{i=1}^{n} X_i^{(2)}$

则对 $\forall \varepsilon > 0$. $P(|\frac{S_n}{n} - \mu| > \varepsilon) = \underbrace{P(|\frac{S_n^{(1)} - \mathbb{E}S_n^{(1)}}{n}| > \frac{\varepsilon}{2})}_{①} + \underbrace{P(|\frac{S_n^{(2)} - \mathbb{E}S_n^{(2)}}{n}| > \frac{\varepsilon}{2})}_{②}$

① $\le \frac{Var(S_n^{(1)})}{(\frac{1}{2}\varepsilon n)^2} = \frac{4Var(X_1^{(1)})}{\varepsilon^2 n} \le \frac{4\mathbb{E}[(X_1^{(1)})^2]}{\varepsilon^2 n} \le \frac{4M^2}{\varepsilon^2 n} \longrightarrow 0$

② $\le \frac{\mathbb{E}|S_n^{(2)} - \mathbb{E}[S_n^{(2)}]|}{\frac{\varepsilon}{2} n} \le \frac{4\mathbb{E}[|X_1| I_{\{|X_1| > M\}}]}{\varepsilon} \xrightarrow{M \to \infty} 0$

其中 $\mathbb{E}|S_n^{(2)} - \mathbb{E}S_n^{(2)}| \le \mathbb{E}[|S_n^{(2)}|] + |\mathbb{E}[S_n^{(2)}]| \le 2\mathbb{E}[|S_n^{(2)}|]$

$\le 2\sum_{k=1}^{n}\mathbb{E}[|X_k^{(2)}|]$

$= 2n\mathbb{E}[|X_1^{(2)}|]$

$\therefore P(|\frac{S_n}{n} - \mu| > \varepsilon) \to 0$. $\frac{S_n}{n} \xrightarrow{P} \mu$