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1.1. 证明：∵ f 在 $x=\bar{x}$ 处二次可微，在 $x=\bar{x}$ 的邻域内 Taylor 展开

$$f(x) = f(\bar{x}+\Delta x)$$
$$= f(\bar{x}) + \nabla f(\bar{x})\cdot \Delta x + \frac{1}{2}\Delta x^T \nabla^2 f(\bar{x})\Delta x + o(\|\Delta x\|^2)$$

∵ $\nabla f(\bar{x}) = 0$ 且 $\nabla^2 f(\bar{x}) > 0$

∴ $\exists$ 一个邻域 U, s.t. $\forall x \in U$, $f(x) \geq f(\bar{x})$ #.

1.2 证明：可行域 $S = \{(0,0)\}$ 凸
目标函数 $f(x,y) = y$ 凸 } 是凸问题

∵ $\bar{x} = (0,0)$

∴ $\nabla C_1(\bar{x}) = (2(x-1), 2y)|_{(0,0)} = (-2, 0)$
$\nabla C_2(\bar{x}) = (2(x+1), 2y)|_{(0,0)} = (2, 0)$

显然是线性相关的，故 LICQ 不成立

考察 KKT 中的稳定条件，$\nabla_x L(x,\lambda,\mu) = 0$ 得：

$$\begin{cases} \mu_1 - \mu_2 = 0 \\ 1 = 0 \end{cases}$$ 无解 ∴ KKT 条件不成立 #.

2.1 证明：$\mathrm{rank}(A) = k < m$ 设 $\tilde{A} = \begin{pmatrix} a_{i_1}^T \\ \vdots \\ a_{i_k}^T \end{pmatrix}$ 线性无关

由行变换，$\exists$ 可逆阵 P, s.t. $PA = \begin{pmatrix} \tilde{A} \\ 0 \end{pmatrix}$

∵ $Ax = b$ ∴ $PAx = Pb$ $\Rightarrow$ $\begin{pmatrix} \tilde{A} \\ 0 \end{pmatrix} x = \begin{pmatrix} \tilde{b} \\ 0 \end{pmatrix}$

∵ 行变换是同解变形

∴ $P = Q$

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2.2. 证明：起作用集 $I := \{ i \mid a_i^T x = b_i \}$，构造 $c = \sum_{i\in I} a_i$

∵ x 是可行基解，∴ $\forall y \in P\setminus\{x\}$ $\exists j \in I$, s.t. $a_j^T y > b_j$

∴ ~~$c^T y / \alpha$~~ $c^T y = \sum_{i\in I} a_i^T y > \sum_{i\in I} b_i = \sum_{i\in I} a_i^T x = c^T x$

∴ x 是顶点

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2.3 解：通过两两联立可得：

(0,4)，(8,0)，(0,0)，(0,2)，(4,2) 是基解

代入约束条件，知除(0,4)外都是极点

注：本题变量是2维的，故也可通过画图求解 #.

2.4. 解：引入松弛变量将问题转化为标准形

$$\min \quad -4x_1 - x_2$$
$$\text{s.t.} \quad -x_1 + 2x_2 + x_3 = 4$$
$$2x_1 + 3x_2 + x_4 = 12$$
$$x_1 - x_2 + x_5 = 3$$
$$x_i \geq 0, \quad i = 1, \dots, 5$$

设 $c = (-4, -1, 0, 0, 0)^T$

$$A = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 \\ 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix}$$

$b = (4, 12, 3)^T$

自然的基矩阵 $B = I$，$B^{-1} = I$

此时，~~$c_N - c_B^T$~~ $c_1 - c_B^T B^{-1} A_1 = -4 < 0$

$\tilde{x}_B = B^{-1}b - B^{-1}A_1 \cdot \Delta \geq 0 \quad \Rightarrow \quad \Delta = 3$

∴ 新可行基解 $x = (3, 0, 7, 6, 0)^T$

对应 $B^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}$，此时 $c_2 - c_B^T B^{-1} A_2 = -5 < 0$

$\tilde{x}_B = B^{-1}b - B^{-1}A_2 \cdot \Delta \geq 0 \quad \Rightarrow \quad \Delta = \frac{6}{5}$

∴ 新可行基解 $x = (\frac{21}{5}, \frac{6}{5}, \frac{29}{5}, 0, 0)^T$

对应 $B^{-1} = \begin{pmatrix} 0 & 1/5 & 3/5 \\ 0 & 1/5 & -2/5 \\ 1 & -1/5 & 7/5 \end{pmatrix}$，此时 $c_N^T - c_B^T B^{-1} A_N = (1, 1)^* > 0$

∴ 最优解 $(\frac{21}{5}, \frac{6}{5})^T$，最优值 -18 #.

2.5 解：(1) 对偶问题 $\max\limits_{\lambda \geq 0} \min\limits_{x} \; c^T x - \lambda^T (Ax - b)$

$$= \max_{\lambda \geq 0} \min_{x} \; (c^T - \lambda^T A) x + \lambda^T b$$

$$= \begin{array}{ll} \max & b^T \lambda \\ \text{s.t.} & A^T \lambda = c \\ & \lambda \geq 0 \end{array}$$

(2) 对偶问题 $\max\limits_{\lambda \geq 0} \min\limits_{x} \; c^T x - \lambda^T (b - Ax)$

$$= \max_{\lambda \geq 0} \min_{x} \; (c^T + \lambda^T A) x - \lambda^T b$$

$$= \begin{array}{ll} \max & -b^T \lambda \\ \text{s.t.} & -A^T \lambda = c \\ & \lambda \geq 0 \end{array}$$

(3) 以标准形为例

$$(DP) \quad \begin{array}{ll} \max & b^T w \\ \text{s.t.} & A^T w \leq c \end{array} \quad = \quad \begin{array}{ll} \min & -b^T w \\ \text{s.t.} & A^T w \leq c \end{array}$$

$\therefore$ (DDP) $\max\limits_{x \geq 0} \min\limits_{w} \; -b^T w - x^T (c - A^T w)$

$$= \max_{x \geq 0} \min_{w} \; (-b^T + \cancel{A}\, x^T A^T) w - x^T c$$

$$= \begin{array}{ll} \max & -c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{array} \quad = \quad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \quad (2P)$$

i.e. $(DDP) = (2P)$

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