Introduction to Algorithms

Proof by Induction

Definition

Suppose we have a formula $F(n)$ which we wish to show is true for all values $n \geq n_0$

- Usually $n_0 = 0$ or $n_0 = 1$

For example, we may wish to show that

$$F(n) = \sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

for all $n \geq 0$

Definition

We then proceed by:

- Demonstrating that $F(n_0)$ is true
- Assuming that the formula $F(n)$ is true for an arbitrary n
- If we are able to demonstrate that this assumption allows us to also show that the formula is true for $F(n + 1)$, the *inductive principle* allows us to conclude that the formula is true for all $n \geq n_0$

Definition

Thus, if $F(n_0)$ is true, $F(n_0 + 1)$ is true

and, if $F(n_0 + 1)$ is true, $F(n_0 + 2)$ is true

and, if $F(n_0 + 2)$ is true, $F(n_0 + 3)$ is true

and so on, and so on, for all $n \geq n_0$

Formulation

Often $F(n)$ is an equation:

- For example, $F(n)$ may be an equation such as:

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2} \quad \text{for } n \geq 0$$

$$\sum_{k=1}^{n} 2k - 1 = n^2 \quad \text{for } n \geq 1$$

$$\sum_{k=0}^{n} 2^k = 2^{n+1} - 1 \quad \text{for } n \geq 0$$

It may also be a statement:

- The integer $n^3 - n$ is divisible by 3 for all $n \geq 1$

Example 1

Prove that $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$ is true for $n \geq 0$

- When $n = 0$: $\sum_{k=0}^{0} k = 0 = \frac{0(0+1)}{2}$
- Assume that the statement is true for a given n: $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$
- We now show:

$$\sum_{k=0}^{n+1} k = (n+1) + \sum_{k=0}^{n} k$$

Example 1

Prove that $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$ is true for $n \geq 0$

- When $n = 0$: $\sum_{k=0}^{0} k = 0 = \frac{0(0+1)}{2}$
- Assume that the statement is true for a given n: $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$
- We now show:

$$\begin{aligned}
\sum_{k=0}^{n+1} k &= (n+1) + \sum_{k=0}^{n} k \\
&= (n+1) + \frac{n(n+1)}{2} \\
&= \frac{2(n+1) + n(n+1)}{2} \\
&= \frac{(n+2)(n+1)}{2} = \frac{(n+1)(n+2)}{2}
\end{aligned}$$

Example 2

Prove that the sum of the first n odd integers is n^2:

$$\sum_{k=1}^{n} 2k-1 = n^2 \quad \text{for } n \geq 1$$

- When $n = 1$: $\sum_{k=1}^{1} 2k-1 = 1 = 1^2$
- Assume that the statement is true for a given n: $\sum_{k=1}^{n} 2k-1 = n^2$
- We now show:

$$\sum_{k=1}^{n+1} 2k-1 = 2(n+1)-1+\sum_{k=0}^{n} 2k-1$$

Example 2

Prove that the sum of the first n odd integers is n^2:

$$\sum_{k=1}^{n} 2k-1 = n^2 \quad \text{for } n \geq 1$$

- When $n = 1$: $\sum_{k=1}^{1} 2k-1 = 1 = 1^2$
- Assume that the statement is true for a given n: $\sum_{k=1}^{n} 2k-1 = n^2$
- We now show:

$$\begin{aligned}
\sum_{k=1}^{n+1} 2k-1 &= 2(n+1)-1+\sum_{k=1}^{n} 2k-1 \\
&= 2(n+1)-1+n^2 \\
&= 2n+2-1+n^2 \\
&= n^2+2n+1 \\
&= (n+1)^2
\end{aligned}$$

Example 3

Prove that $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$ for $n \geq 0$

- When $n = 0$: $\sum_{k=0}^{0} 2^k = 2^0 = 1 = 2^{0+1} - 1$
- Assume that the statement is true for a given n: $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$
- We now show:

$$\sum_{k=0}^{n+1} 2^k = 2^{n+1} + \sum_{k=0}^{n} 2^k$$

Example 3

Prove that $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$ for $n \geq 0$

- When $n = 0$: $\sum_{k=0}^{0} 2^k = 2^0 = 1 = 2^{0+1} - 1$
- Assume that the statement is true for a given n: $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$
- We now show:

$$
\begin{aligned}
\sum_{k=0}^{n+1} 2^k &= 2^{n+1} + \sum_{k=0}^{n} 2^k \\
&= 2^{n+1} + 2^{n+1} - 1 \\
&= 2 \cdot 2^{n+1} - 1 \\
&= 2^{n+2} - 1
\end{aligned}
$$

Example 4

Prove that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

- When $n = 0$: $\sum_{k=0}^{0} \binom{n}{k} = \binom{0}{0} = 1 = 2^0$

- Assume that the statement is true for a given n: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$
- We now show:

$$\sum_{k=0}^{n+1} \binom{n+1}{k} = \binom{n+1}{0} + \left[\sum_{k=1}^{n} \binom{n+1}{k} \right] + \binom{n+1}{n+1}$$

Example 4

Prove that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

- When $n = 0$: $\sum_{k=0}^{0} \binom{n}{k} = \binom{0}{0} = 1 = 2^0$

- Assume that the statement is true for a given n: $\sum_{k=0}^{n} \binom{n}{k} = 2^n$
- We now show:

$$\sum_{k=0}^{n+1} \binom{n+1}{k} = \binom{n+1}{0} + \left[\sum_{k=1}^{n} \binom{n+1}{k} \right] + \binom{n+1}{n+1}$$

$$= 1 + \left[\sum_{k=1}^{n} \binom{n}{k} + \binom{n}{k-1} \right] + 1$$

$$= 1 + \sum_{k=1}^{n} \binom{n}{k} + \sum_{k=1}^{n} \binom{n}{k-1} + 1 = \left[\binom{n}{0} + \sum_{k=1}^{n} \binom{n}{k} \right] + \left[\sum_{k=0}^{n-1} \binom{n}{k} + \binom{n}{n} \right]$$

Example 4

Prove that $\sum_{k=0}^{n}\binom{n}{k}=2^n$

- When $n = 0$: $\sum_{k=0}^{0}\binom{n}{k}=\binom{0}{0}=1=2^0$
- Assume that the statement is true for a given n: $\sum_{k=0}^{n}\binom{n}{k}=2^n$
- We now show:

$$\sum_{k=0}^{n+1}\binom{n+1}{k}=\binom{n+1}{0}+\left[\sum_{k=1}^{n}\binom{n+1}{k}\right]+\binom{n+1}{n+1}$$

$$=1+\left[\sum_{k=1}^{n}\binom{n}{k}+\binom{n}{k-1}\right]+1$$

$$=1+\sum_{k=1}^{n}\binom{n}{k}+\sum_{k=1}^{n}\binom{n}{k-1}+1=\left[\binom{n}{0}+\sum_{k=1}^{n}\binom{n}{k}\right]+\left[\sum_{k=0}^{n-1}\binom{n}{k}+\binom{n}{n}\right]$$

$$=2\sum_{k=0}^{n}\binom{n}{k}$$

Example 4

Prove that $\sum_{k=0}^{n}\binom{n}{k}=2^n$

- When $n=0$: $\sum_{k=0}^{0}\binom{n}{k}=\binom{0}{0}=1=2^0$
- Assume that the statement is true for a given n: $\sum_{k=0}^{n}\binom{n}{k}=2^n$
- We now show:

$$\sum_{k=0}^{n+1}\binom{n+1}{k}=\binom{n+1}{0}+\left[\sum_{k=1}^{n}\binom{n+1}{k}\right]+\binom{n+1}{n+1}$$

$$=1+\left[\sum_{k=1}^{n}\binom{n}{k}+\binom{n}{k-1}\right]+1$$

$$=1+\sum_{k=1}^{n}\binom{n}{k}+\sum_{k=1}^{n}\binom{n}{k-1}+1=\left[\binom{n}{0}+\sum_{k=1}^{n}\binom{n}{k}\right]+\left[\sum_{k=0}^{n-1}\binom{n}{k}+\binom{n}{n}\right]$$

$$=2\sum_{k=0}^{n}\binom{n}{k}=2\cdot 2^n=2^{n+1}$$

Strong Induction

A similar technique is *strong induction* where we replace the statement

- Assume that $F(n)$ true

with

- Assume that $F(n_0), F(n_0+1), F(n_0+2), \ldots, F(n)$ are all true

For example:

Prove that with 3 and 7 cent coins, it is possible to make exact change for any amount greater than or equal to 12 cents

https://math.stackexchange.com/questions/1415475/mathematical-induction-using-3-cent-and-7-cent-stamps

References

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Donald E. Knuth, *The Art of Computer Programming,Vol 1*, *Fundamental Algorithms*, 3rd Ed., Addison Wesley, 1997