1. 分别在 $\mathbb{C}$, $\mathbb{R}$ 上因式分解

$$f(x) = x^{2n} - C_{2n}^{2} x^{2n-2} + C_{2n}^{4} x^{2n-4} + \cdots + (-1)^n C_{2n}^{2n}$$

解: $f(x) = \dfrac{(x+i)^{2n} + (x-i)^{2n}}{2}$

令 $f(x)=0$ 解得 $(x+i)^{2n}\left(1+\left(\frac{x-i}{x+i}\right)^{2n}\right) = 0$.

$-i$ 不是 f 的根 $\Rightarrow$ $\dfrac{x_k - i}{x_k + i} = \omega_{4n}^{2k-1}$. $\left(\omega_{4n} = e^{i\frac{\pi}{2n}}\right)$ $k=1,\cdots,2n$

$$x_k = -\frac{i\left(\omega_{4n}^{2k-1}+1\right)}{\omega_{4n}^{2k-1}-1} \qquad k=1,\cdots,2n$$

$\forall\ \theta \in \mathbb{R}$.

$$\frac{e^{i\theta}+1}{e^{i\theta}-1} = \frac{\cos\theta + i\sin\theta + 1}{\cos\theta + i\sin\theta - 1}$$

$$= \frac{2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{-2\sin^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \frac{\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}{\sin\frac{\theta}{2}\left(-\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}\right)}$$

$$= -i\cot\frac{\theta}{2}.$$

则 $x_k = -\cot\dfrac{2k-1}{4n}\pi \in \mathbb{R}$.

则 $f(x) = \prod\limits_{k=1}^{2n}\left(x + \cot\left(\frac{2k-1}{4n}\pi\right)\right)$ $\square$

2. 判别 $f(x)=x^4-x^3+2x+1$ 在 $\mathbb{Z}$ 上是否可约.

法一: $f(x+1)=(x+1)^4-(x+1)^3+2x+1$

$= x^4+3x^3+3x^2+3x+3$

取 $p=3$. 则由 Eisenstein 判别法 知不可约.

注: 有些同学只证了没有一次因式, 这显然不够.

法二: 硬做. (讨论一下即可).

3. 设 x, y, z 满足 $\begin{cases} x+y+z=3 \\ x^2+y^2+z^2=4 \\ x^3+y^3+z^3=6 \end{cases}$ 求 $x^4+y^4+z^4$.

$\sigma_1=3$.

$\sigma_2=xy+yz+zx=\frac{1}{2}(\sigma_1^2-x^2-y^2-z^2)=\frac{5}{2}$.

$\sigma_3=xyz=\frac{1}{3}(S_3-\sigma_1 S_2+\sigma_2 S_1)=\frac{1}{2}$

$S_4=\sigma_1 S_3-\sigma_2 S_2+\sigma_3 S_1=\frac{19}{2}$.