Introduction to Algorithms

Linear-Time Sorting Algorithms

Sorting So Far

- Insertion sort:
  - Easy to code
  - Fast on small inputs (less than ~50 elements)
  - Fast on nearly-sorted inputs
  - $O(n^2)$ worst case
  - $O(n^2)$ average (equally-likely inputs) case
  - $O(n^2)$ reverse-sorted case

Sorting So Far

- Merge sort:
  - Divide-and-conquer:
    - Split array in half
    - Recursively sort subarrays
    - Linear-time merge step
  - O(n lg n) worst case
  - Doesn’t sort in place

Sorting So Far

- Heap sort:
  - Uses the very useful heap data structure
    - Complete binary tree
    - Heap property: parent key > children's keys
  - O(n lg n) worst case
  - Sorts in place
  - Fair amount of shuffling memory around

Sorting So Far

- Quick sort:
  - Divide-and-conquer:
    - Partition array into two subarrays, recursively sort
    - All of first subarray < all of second subarray
    - No merge step needed!
  - O(n lg n) average case
  - Fast in practice
  - $O(n^2)$ worst case
    - Naïve implementation: worst case on sorted input
    - Address this with randomized quicksort

How Fast Can We Sort?

- We will provide a lower bound, then beat it
  - *How do you suppose we'll beat it?*
- First, an observation: all of the sorting algorithms so far are *comparison sorts*
  - The only operation used to gain ordering information about a sequence is the pairwise comparison of two elements
  - Theorem: all comparison sorts are Ω(n lg n)
    - A comparison sort must do O(n) comparisons (*why?*)
    - What about the gap between O(n) and O(n lg n)

Decision Trees

- *Decision trees* provide an abstraction of comparison sorts
  - A decision tree represents the comparisons made by a comparison sort. Every thing else ignored
  - (Draw examples on board)
- *What do the leaves represent?*
- *How many leaves must there be?*

Decision Trees

- Decision trees can model comparison sorts. For a given algorithm:
  - One tree for each n
  - Tree paths are all possible execution traces
  - *What's the longest path in a decision tree for insertion sort? For merge sort?*
- *What is the asymptotic height of any decision tree for sorting n elements?*
- Answer: $\Omega(n \lg n)$ (now let's prove it…)

Lower Bound For Comparison Sorting

- Thm: Any decision tree that sorts n elements has height $\Omega(n \lg n)$
- *What's the minimum # of leaves?*
- *What's the maximum # of leaves of a binary tree of height h?*
- Clearly the minimum # of leaves is less than or equal to the maximum # of leaves

Lower Bound For Comparison Sorting

- So we have…

  $n! \le 2^h$

- Taking logarithms:

  $\lg(n!) \le h$

- Stirling's approximation tells us:

  $$n! > \left(\frac{n}{e}\right)^n$$

- Thus: $h \ge \lg\left(\frac{n}{e}\right)^n$

Lower Bound For Comparison Sorting

- So we have

$$h \geq \lg\left(\frac{n}{e}\right)^n$$

$$= n \lg n - n \lg e$$

$$= \Omega(n \lg n)$$

- Thus the minimum height of a decision tree is $\Omega(n \lg n)$

Lower Bound For Comparison Sorts

- Thus the time to comparison sort n elements is $\Omega(n \lg n)$
- Corollary: Heapsort and Mergesort are asymptotically optimal comparison sorts
- But the name of this lecture is "Sorting in linear time"!
  - *How can we do better than $\Omega(n \lg n)$?*

Sorting In Linear Time

- Counting sort
  - No comparisons between elements!
  - ***But***…depends on assumption about the numbers being sorted
    - We assume numbers are in the range *1.. k*
  - The algorithm:
    - Input: A[1..n], where A[j] $\in$ {1, 2, 3, …, k}
    - Output: B[1..n], sorted (notice: not sorting in place)
    - Also: Array C[1..k] for auxiliary storage

Counting Sort

```
CountingSort(A, B, k)
    for i=1 to k
        C[i]= 0;
    for j=1 to n
        C[A[j]] += 1;
    for i=2 to k
        C[i] = C[i] + C[i-1];
    for j=n downto 1
        B[C[A[j]]] = A[j];
        C[A[j]] -= 1;
```

Work through example: A={4 1 3 4 3}, k = 4

Counting Sort

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              B[C[A[j]]] = A[j];
              C[A[j]] -= 1;
```

Takes time O(k)

Takes time O(n)

What will be the running time?

Counting Sort

- Total time: $O(n + k)$
  - Usually, $k = O(n)$
  - Thus counting sort runs in $O(n)$ time
- But sorting is $\Omega(n \lg n)$!
  - No contradiction--this is not a comparison sort (in fact, there are *no* comparisons at all!)
  - Notice that this algorithm is *stable*

Counting Sort

- Cool! *Why don't we always use counting sort?*
- Because it depends on range k of elements
- *Could we use counting sort to sort 32 bit integers? Why or why not?*
- Answer: no, k too large ($2^{32} = 4,294,967,296$)

Radix Sort

- Intuitively, you might sort on the most significant digit, then the second msd, etc.
- Problem: lots of intermediate piles of cards (read: scratch arrays) to keep track of
- Key idea: sort the *least* significant digit first

```
RadixSort(A, d)
    for i=1 to d
        StableSort(A) on digit i
```

Radix Sort

- *Can we prove it will work?*
- Sketch of an inductive argument (induction on the number of passes):
  - Assume lower-order digits $\{j: j<i\}$ are sorted
  - Show that sorting next digit i leaves array correctly sorted
    - If two digits at position i are different, ordering numbers by that digit is correct (lower-order digits irrelevant)
    - If they are the same, numbers are already sorted on the lower-order digits. Since we use a stable sort, the numbers stay in the right order

Radix Sort

- *What sort will we use to sort on digits?*
- Counting sort is obvious choice:
  - Sort n numbers on digits that range from $1..k$
  - Time: $O(n + k)$
- Each pass over n numbers with d digits takes time $O(n+k)$, so total time $O(dn+dk)$
  - When d is constant and $k=O(n)$, takes $O(n)$ time

Radix Sort

- Problem: sort 1 million 64-bit numbers
  - Treat as four-digit radix 2^{16} numbers
  - Can sort in just four passes with radix sort!
- Compares well with typical O(n lg n) comparison sort
  - Requires approx lg n = 20 operations per number being sorted
- *So why would we ever use anything but radix sort?*

Radix Sort

- In general, radix sort based on counting sort is
  - Fast
  - Asymptotically fast (i.e., O(n))
  - Simple to code
  - A good choice

Summary: Radix Sort

- Radix sort:
  - Assumption: input has d digits ranging from 0 to k
  - Basic idea:
    - Sort elements by digit starting with *least* significant
    - Use a stable sort (like counting sort) for each stage
  - Each pass over n numbers with d digits takes time $O(n+k)$, so total time $O(dn+dk)$
    - When d is constant and $k=O(n)$, takes $O(n)$ time
  - Fast! Stable! Simple!

Bucket Sort

- Bucket sort
  - Assumption: input is n reals from [0, 1)
  - Basic idea:
    - Create n linked lists (*buckets*) to divide interval [0,1) into subintervals of size $1/n$
    - Add each input element to appropriate bucket and sort buckets with insertion sort
  - Uniform input distribution → O(1) bucket size
    - Therefore the expected total time is O(n)
  - These ideas will return when we study *hash tables*

Order Statistics

- The i-th *order statistic* in a set of n elements is the i-th smallest element
- The *minimum* is thus the 1-st order statistic
- The *maximum* is (duh) the n-th order statistic
- The *median* is the $n/2$ order statistic
  - If n is even, there are 2 medians
- *How can we calculate order statistics?*
- *What is the running time?*

Order Statistics

- *How many comparisons are needed to find the minimum element in a set? The maximum?*
- *Can we find the minimum and maximum with less than twice the cost?*
- Yes:
  - Walk through elements by pairs
    - Compare each element in pair to the other
    - Compare the largest to maximum, smallest to minimum
  - Total cost: 3 comparisons per 2 elements = O(3n/2)

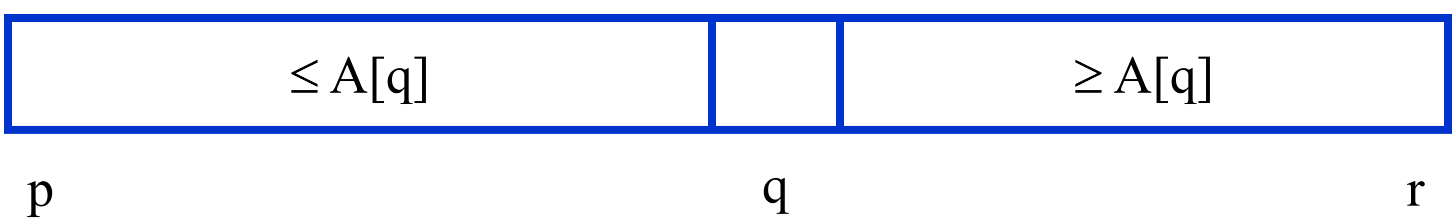
Finding Order Statistics: The Selection Problem

- A more interesting problem is *selection*: finding the *i*-th smallest element of a set
- We will show:
  - A practical randomized algorithm with O(n) expected running time
  - A cool algorithm of theoretical interest only with O(n) worst-case running time

Randomized Selection

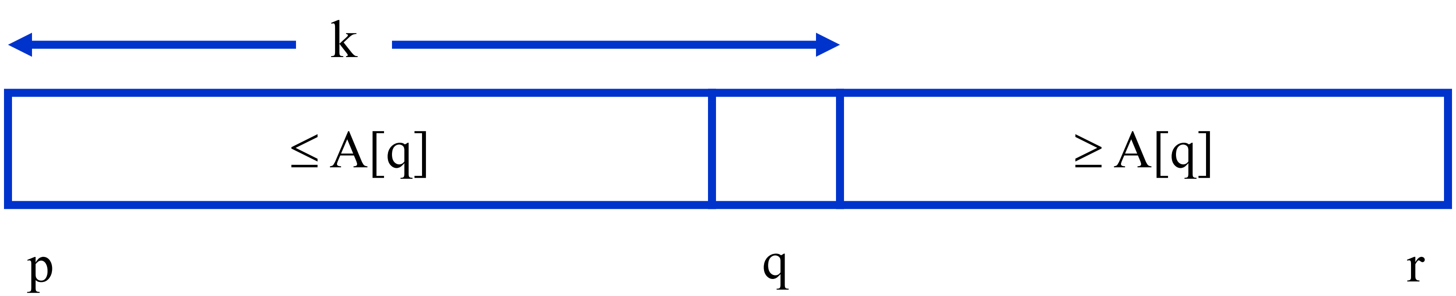
- Key idea: use partition() from quicksort
  - But, only need to examine one subarray
  - This savings shows up in running time: O(n)
- We will again use a slightly different partition than the book:

  q = RandomizedPartition(A, p, r)



Randomized Selection

```
RandomizedSelect(A, p, r, i)
    if (p == r) then return A[p];
    q = RandomizedPartition(A, p, r)
    k = q - p + 1;
    if (i == k) then return A[q];    // not in book
    if (i < k) then
        return RandomizedSelect(A, p, q-1, i);
    else
        return RandomizedSelect(A, q+1, r, i-k);
```



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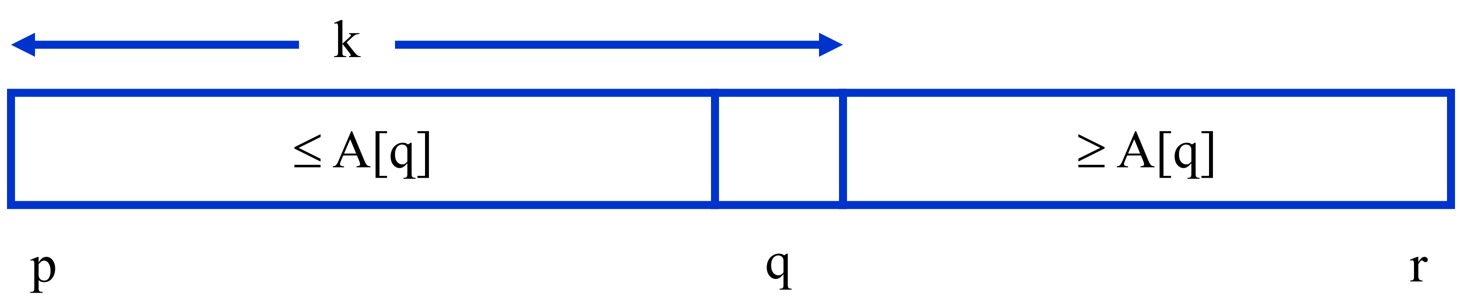

Randomized Selection

- Average case
  - For upper bound, assume *i*-th element always falls in larger side of partition:

$$T(n) \le \frac{1}{n}\sum_{k=0}^{n-1} T(\max(k, n-k-1)) + \Theta(n)$$

$$\le \frac{2}{n}\sum_{k=n/2}^{n-1} T(k) + \Theta(n)$$

What happened here?

  - Let's show that T(*n*) = O(*n*) by substitution

Randomized Selection

- Assume T(n) ≤ cn for sufficiently large c:

$$T(n) \le \frac{2}{n}\sum_{k=n/2}^{n-1} T(k) + \Theta(n)$$

The recurrence we started with

$$\le \frac{2}{n}\sum_{k=n/2}^{n-1} ck + \Theta(n)$$

Substitute T(n) ≤ cn for T(k)

$$= \frac{2c}{n}\left(\sum_{k=1}^{n-1} k - \sum_{k=1}^{n/2-1} k\right) + \Theta(n)$$

"Split" the recurrence

$$= \frac{2c}{n}\left(\frac{1}{2}(n-1)n - \frac{1}{2}\left(\frac{n}{2}-1\right)\frac{n}{2}\right) + \Theta(n)$$

Expand arithmetic series

$$= c(n-1) - \frac{c}{2}\left(\frac{n}{2}-1\right) + \Theta(n)$$

Multiply it out

Randomized Selection

- Assume T(n) $\leq cn$ for sufficiently large c:

$$
\begin{aligned}
T(n) &\leq c(n-1) - \frac{c}{2}\left(\frac{n}{2}-1\right) + \Theta(n) && \textit{The recurrence so far} \\
&= cn - c - \frac{cn}{4} + \frac{c}{2} + \Theta(n) && \textit{Multiply it out} \\
&= cn - \frac{cn}{4} - \frac{c}{2} + \Theta(n) && \textit{Subtract c/2} \\
&= cn - \left(\frac{cn}{4} + \frac{c}{2} - \Theta(n)\right) && \textit{Rearrange the arithmetic} \\
&\leq cn \quad \text{(if c is big enough)} && \textit{What we set out to prove}
\end{aligned}
$$

Worst-Case Linear-Time Selection

- Randomized algorithm works well in practice
- What follows is a worst-case linear time algorithm, really of theoretical interest only
- Basic idea:
  - Generate a good partitioning element
  - Call this element x

Worst-Case Linear-Time Selection

- The algorithm in words:
  1. Divide n elements into groups of 5
  2. Find median of each group (*How? How long?*)
  3. Use Select() recursively to find median x of the $\lfloor n/5 \rfloor$ medians
  4. Partition the n elements around x. Let k = rank(x)
  5. **if** (i == k) **then** return x

     if (i < k) **then** use Select() recursively to find ith smallest element in first partition

     else (i > k) use Select() recursively to find (i-k)th smallest element in last partition

Worst-Case Linear-Time Selection

- (Sketch situation on the board)
- *How many of the 5-element medians are $\leq x$?*
  - At least 1/2 of the medians = $\lfloor \lfloor n/5 \rfloor / 2 \rfloor = \lfloor n/10 \rfloor$
- *How many elements are $\leq x$?*
  - At least $3 \lfloor n/10 \rfloor$ elements
- For large n, $3 \lfloor n/10 \rfloor \geq n/4$ *(How large?)*
- So at least $n/4$ elements $\leq x$
- Similarly: at least $n/4$ elements $\geq x$

Worst-Case Linear-Time Selection

- Thus after partitioning around x, step 5 will call Select() on at most $3n/4$ elements
- The recurrence is therefore:

$$
\begin{aligned}
T(n) &\le T(\lfloor n/5 \rfloor) + T(3n/4) + \Theta(n) \\
&\le T(n/5) + T(3n/4) + \Theta(n) && \lfloor n/5 \rfloor \le n/5 \\
&\le cn/5 + 3cn/4 + \Theta(n) && \textit{Substitute } T(n) = cn \\
&= 19cn/20 + \Theta(n) && \textit{Combine fractions} \\
&= cn - (cn/20 - \Theta(n)) && \textit{Express in desired form} \\
&\le cn \quad \text{if } c \text{ is big enough} && \textit{What we set out to prove}
\end{aligned}
$$

Worst-Case Linear-Time Selection

- Intuitively:
  - Work at each level is a constant fraction (19/20) smaller
    - Geometric progression!
  - Thus the O(n) work at the root dominates

Linear-Time Median Selection

- Given a "black box" O(n) median algorithm, what can we do?
  - *i*th order statistic:
    - Find median x
    - Partition input around x
    - if ($i \leq (n+1)/2$) recursively find ith element of first half
    - else find ($i - (n+1)/2$)th element in second half
    - $T(n) = T(n/2) + O(n) = O(n)$
  - *Can you think of an application to sorting?*

Linear-Time Median Selection

- Worst-case O(n lg n) quicksort
  - Find median *x* and partition around it
  - Recursively quicksort two halves
  - T(n) = 2T(n/2) + O(n) = O(n lg n)