[Wei] 2.10. Define the survival function $S(x) = P(X \geq x) = 1 - F(x)$, then

$$S_X(x) = \begin{cases} e^{-(x/\beta)^\alpha} & , x \geq 0 \\ 1 & , x < 0 \end{cases}.$$

$$\begin{aligned} \therefore \quad S_Y(x) &= P(X_1, \ldots, X_n \geq x) = (S_X(x))^n \\ &= \begin{cases} e^{-n(x/\beta)^\alpha} = e^{-(x/(\beta/n^{1/\alpha}))^\alpha} & , x \geq 0 \\ 1 & , x < 0 \end{cases}. \end{aligned}$$

$$\therefore \quad Y \text{ is also a Weibull distribution with parameters}$$

$$\alpha_Y = \alpha, \quad \beta_Y = \frac{\beta}{n^{1/\alpha}}.$$

$\square$

[Wei] 2.26.

$$\because \quad Exp(\lambda) = \Gamma\left(1, \frac{1}{\lambda}\right), \quad \chi^2_p = \Gamma\left(\frac{p}{2}, \frac{1}{2}\right).$$

$$\therefore \quad \frac{2}{\lambda} Exp(\lambda) = \chi^2_2. \tag{1}$$

Let $Y_1 = X_{(1)}$, $Y_i = X_{(i)} - X_{(i-1)}$, $i = 2, \ldots, n$, then $\left|\frac{\partial(Y_1,\ldots,Y_n)}{\partial(X_{(1)},\ldots,X_{(n)})}\right| = 1$.

$$\begin{aligned} \therefore \quad f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) &= f_{X_{(1)},\ldots,X_{(n)}}(y_1, y_1 + y_2, \ldots, y_1 + \ldots + y_n) \left|\frac{\partial(Y_1, \ldots, Y_n)}{\partial(X_{(1)}, \ldots, X_{(n)})}\right| \\ &= n! \prod_{i=1}^n f_X(y_i) \\ &= \prod_{i=1}^n \left(\frac{n+1-i}{\lambda} e^{\frac{n+1-i}{\lambda} y_i} I_{(0,+\infty)}(y_i)\right). \end{aligned}$$

Separated the joint $p.d.f$, we conclude that $Y_i, i = 1, \ldots, n$ are independently distributed as $\text{Exp}(\frac{\lambda}{n+1-i})$. From (1),

$$\frac{2(n+1-i)}{\lambda} Y_i \overset{i.i.d.}{\sim} \chi^2_2. \tag{2}$$

$$\begin{aligned} \therefore \quad \frac{2T}{\lambda} &= \frac{2}{\lambda}\left(nX_{(1)} - (n-1)X_{(1)} + (n-1)X_{(2)} - (n-2)X_{(2)} + \ldots + (n+1-r)X_{(r)}\right) \\ &= \frac{2}{\lambda}\left(nY_1 + (n-1)Y_2 + \ldots + (n+1-r)Y_r\right) \\ &= \sum_{i=1}^r \frac{2(n+1-i)}{\lambda} Y_i \sim \chi^2_{2r}. \end{aligned}$$

$\square$

Remark: Intuitively speaking, the *memoryless* property of exponential distributions makes $X_{(i)} - X_{(i-1)}$ "forget" the information before (to say, $X_{(i-1)} - X_{(i-2)}, \ldots, X_{(2)} - X_{(1)}$) and therefore be independent with them.

[Wei] 2.27. Since $\frac{2(n-i+1)}{\sigma}\left(X_{(i)} - X_{(i-1)}\right) = \frac{2(n-i+1)}{\sigma}\left((X_{(i)} - \mu) - (X_{(i-1)} - \mu)\right)$, we can assume $\mu = 0$ without loss of generality. Then it's the case in 2.26 with $\lambda = \sigma$. According to (2), we have

$$\frac{2(n+1-i)}{\sigma}(X_{(i)} - X_{(i-1)}) \overset{i.i.d.}{\sim} \chi_2^2.$$

□

2021/09/29, 10/08

[Wei] 2.39. Here we denote $Negbin(r, p)$ for negative binomial distribution with only one parameter $0 < p < 1$ (fix r). $Exp(\lambda)$ has the same $p.d.f.$ as in **2.26**.

- $Negbin(r, p)$:
$$\begin{aligned} f(n; p) =& \binom{n-1}{r-1} p^r (1-p)^{n-r} \overset{\theta := \log(1-p)}{=} \left(\frac{1-e^\theta}{e^\theta}\right)^r \exp\{\theta n\} \binom{n-1}{r-1} \\ :=& C(\theta) \exp\{\theta n\} h(n), \quad n \in \mathbb{Z}_{\geq r}, \end{aligned}$$
where $C(\theta) := \left(\frac{1-e^\theta}{e^\theta}\right)^r$, $h(n) := \binom{n-1}{r-1}$.
Its natural parametric space is $\{\theta : \theta \in (-\infty, 0)\}$.
- $Exp(\lambda)$:
$$\begin{aligned} f(x; \lambda) =& \frac{1}{\lambda} e^{-\frac{x}{\lambda}} I_{(0,+\infty)}(x) \overset{\theta := -\frac{1}{\lambda}}{=} -\theta \exp\{\theta x\} I_{(0,+\infty)}(x) \\ :=& C(\theta) \exp\{\theta x\} h(x), \end{aligned}$$
where $C(\theta) := -\theta$, $h(x) := I_{(0,+\infty)}(x)$.
Its natural parametric space is $\{\theta : \theta \in (-\infty, 0)\}$.

□

Remarks: (i) The answer can be various.
(ii) For negative binomial, some may think that $r = \sum X_i$ which depends on the sample. In this case, construct another $\tilde{\theta} := \log\left(\frac{p}{1-p}\right)$ to obtain a natural form.

2.40 由 $\int f(x,\theta)\,dx = 1$，知

$$C(\theta) = \frac{1}{\int \exp\{\sum_{j=1}^{k}\theta_j T_j(x)\}\, h(x)} := \frac{1}{p(\theta)}$$

则 $$\frac{\partial C(\theta)}{\partial \theta_j} = -\frac{1}{p^2(\theta)} \int T_j(x) \exp\{\sum_{i=1}^{k}\theta_i T_i(x)\} h(x)\,dx = -\frac{1}{C(\theta)} E_\theta T_j(x)$$

$$\Rightarrow -\frac{\partial \log C(\theta)}{\partial \theta_j} = -\frac{1}{C(\theta)}\frac{\partial C(\theta)}{\partial \theta_j} = E_\theta(T_j(x)) \qquad ①$$

$$\frac{\partial^2 \log C(\theta)}{\partial \theta_j \partial \theta_s} = -\frac{1}{C^2(\theta)}\frac{\partial C}{\partial \theta_s}\frac{\partial C}{\partial \theta_j} + \frac{1}{C(\theta)}\frac{\partial^2 C}{\partial \theta_j \partial \theta_s} \qquad ②$$

$$\frac{\partial^2 C}{\partial \theta_j \partial \theta_s} = \frac{\partial}{\partial \theta_j}\left(-\frac{1}{p^2(\theta)}\int T_s(x)\exp\{\sum_{i=1}^{k}\theta_i T_i(x)\} h(x)\,dx\right)$$

$$= \frac{2}{p^3(\theta)}\int T_j(x)\exp\{\sum_{i=1}^{k}\theta_i T_i(x)\} h(x)\,dx \int T_s(x)\exp\{\sum_{i=1}^{k}\theta_i T_i(x)\} h(x)\,dx$$

$$-\frac{1}{p^2(\theta)}\int T_j(x)\, T_s(x)\exp\{\sum_{i=1}^{k}\theta_i T_i(x)\} h(x)\,dx$$

$$= 2C(\theta)\, E_\theta T_j(x)\, E_\theta T_s(x) - C(\theta)\, E_\theta T_j(x) T_s(x) \qquad ③$$

联立①-③ $\Rightarrow \dfrac{\partial^2 \log C(\theta)}{\partial \theta_j \partial \theta_s} = -\mathrm{Cov}_\theta(T_j(x), T_s(x))$ $\square$

至于ppt上那道题，看个乐呵就行

假设 $|x-\mu| = Q(\mu)T(x) + A(\mu) + B(x)$

$$Q(\mu)T'(x) + B'(x) = \begin{cases} 1, & x>\mu \\ -1, & x<\mu \end{cases}$$

易见 $\forall \Delta\mu > 0$，$Q(\mu+\Delta\mu) \neq Q(\mu)$，否则 $1=-1$

$\forall x_0 < \mu$，有 $Q(\mu)T'(x_0) + B'(x_0) = -1$

$Q(\mu+\Delta\mu)T'(x_0) + B'(x_0) = -1$

$\therefore (Q(\mu) - Q(\mu+\Delta\mu))T'(x_0) = 0 \quad \therefore T'(x_0) = 0$

$\therefore T'(x) = 0$，$x<\mu$，$\forall \mu$ 令 $\mu \to \infty \Rightarrow T'(x) \equiv 0$

$$\therefore B'(x) = \begin{cases} 1, & x>\mu \\ -1, & x<\mu \end{cases}$$ 与 μ 有关，矛盾