§0.1 正交活动标架

本章前一部分讨论了曲面自然标架的运动方程、通过该标架运动方程导出了曲面结构方程(Gauss-Codazzi方程)以及曲面基本定理。接下来将介绍曲面的正交活动标架，讨论它的运动方程、以及相应结构方程。

曲面$r(u,v)$上各点处选取单位正交切向量$\{e_1,e_2\}$，且$e_1(u,v),e_2(u,v)$光滑。(例如由$\{r_u,r_v\}$作Schmidt正交化得到这样一组$\{e_1,e_2\}$。) 再令$e_3=e_1\wedge e_2=N$。从而有沿曲面的一个(右手系)正交标架(或称规范标架)

$$\{r;e_1,e_2,e_3\}.$$

令

$$\begin{pmatrix} r_u \\ r_v \end{pmatrix} = A\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = (a_\alpha^\beta)\begin{pmatrix} e_1 \\ e_2 \end{pmatrix},$$

即

$$\begin{cases} r_u = a_1^1 e_1 + a_1^2 e_2, \\ r_v = a_2^1 e_1 + a_2^2 e_2. \end{cases}$$

则有曲面映射的微分

$$\begin{aligned} dr &= r_u du + r_v dv = (du,dv)\begin{pmatrix} r_u \\ r_v \end{pmatrix} = (du^1,du^2)A\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \\ &= du^\beta a_\beta^\alpha e_\alpha. \end{aligned}$$

令

$$\omega^\alpha := du^\beta a_\beta^\alpha, \quad \alpha=1,2,$$

即

$$(\omega^1 \quad \omega^2) = (du \quad dv)A,$$

$$\omega^1 = a_1^1 du + a_2^1 dv, \quad \omega^2 = a_1^2 du + a_2^2 dv.$$

则有

$$dr = \omega^1 e_1 + \omega^2 e_2 = \omega^\alpha e_\alpha, \quad \omega^\gamma = \langle dr, e_\gamma\rangle. \quad (1)$$

因此当确定正交标架之后，可以不再借助参数坐标而是直接定义$\omega^\gamma=\langle dr,e_\gamma\rangle$。

利用一次微分形式ω^1,ω^2，曲面的第一基本形式

$$I = \langle dr,dr\rangle = \langle \omega^\alpha e_\alpha, \omega^\beta e_\beta\rangle = \omega^\alpha\otimes\omega^\beta\delta_{\alpha\beta} = \omega^\alpha\otimes\omega^\alpha = \omega^1\otimes\omega^1+\omega^2\otimes\omega^2.$$

注：上述一次微分形式ω^1,ω^2与(e_1,e_2)有更直接的联系。令

$$X_\alpha=(dr)^{-1}e_\alpha,\quad \alpha=1,2.$$

则由$dr=\omega^\alpha e_\alpha$可得

$$dr(X_\beta)=e_\beta=\omega^\alpha(X_\beta)e_\alpha,$$

因此

$$\omega^\alpha(X_\beta)=\delta^\alpha_\beta.$$

即ω^α为X_α的对偶基。

注：与自然标架方程类似，要得到第二基本形式需要对e_i求微分。对标架求微分(例如dr,de_i)即求一个向量值函数的全微分，其结果为向量值的一次微分形式。设有向量值函数$F:D\to\mathbb{R}^3$，即$F(u,v)=(F^1(u,v),F^2(u,v),F^3(u,v))$。则

$$dF:=\frac{\partial F}{\partial u^\alpha}du^\alpha=(\frac{\partial F^1}{\partial u^\alpha},\frac{\partial F^2}{\partial u^\alpha},\frac{\partial F^3}{\partial u^\alpha})du^\alpha.$$

其中$du,dv\in T^*D$，(du,dv)为$(\frac{\partial}{\partial u},\frac{\partial}{\partial v})$的对偶基。令

$$X=X^1(u,v)\frac{\partial}{\partial u}+X^2(u,v)\frac{\partial}{\partial v}=X^\beta\frac{\partial}{\partial u^\beta}\in TD,$$

则

$$\begin{aligned}
dF(X) &= (\frac{\partial F^1}{\partial u^\alpha},\frac{\partial F^2}{\partial u^\alpha},\frac{\partial F^3}{\partial u^\alpha})du^\alpha(X^\beta\frac{\partial}{\partial u^\beta})\\
&= (\frac{\partial F^1}{\partial u^\alpha},\frac{\partial F^2}{\partial u^\alpha},\frac{\partial F^3}{\partial u^\alpha})X^\beta\delta^\alpha_\beta\\
&= X^\alpha(\frac{\partial F^1}{\partial u^\alpha},\frac{\partial F^2}{\partial u^\alpha},\frac{\partial F^3}{\partial u^\alpha})\\
&= (X(F^1),X(F^2),X(F^3))\\
&= X(F).
\end{aligned}$$

特别

$$de_i=\frac{\partial e_i}{\partial u^\alpha}du^\alpha,$$

$$de_i(X^\gamma\frac{\partial}{\partial u^\gamma})=X(e_i)=X^\gamma\frac{\partial e_i}{\partial u^\gamma}.$$

对正交标架e_1,e_2,e_3(作为D上向量值函数)求微分得到向量值的一次微分形式

$$de_i=\omega_i^je_j=\omega_i^1e_1+\omega_i^2e_2+\omega_i^3e_3,\quad i=1,2,3,$$

其中

$$\omega_i^j = \langle de_i, e_j \rangle = \langle \frac{\partial e_i}{\partial u^\alpha}, e_j \rangle du^\alpha, \quad i, j = 1, 2, 3$$

都是一次微分形式。注意到

$$\omega_i^j = \langle de_i, e_j \rangle = -\langle e_i, de_j \rangle = -\omega_j^i,$$

即(ω_i^j)为反对称矩阵，其元素为一次微分形式。特别

$$\omega_1^1 = \omega_2^2 = \omega_3^3 = 0.$$

曲面第二基本形式

$$\begin{aligned} II &= -\langle dr, de_3 \rangle = -\langle \omega^\alpha e_\alpha, \omega_3^\beta e_\beta \rangle \\ &= -\omega^\alpha \otimes \omega_3^\alpha = -\omega^1 \otimes \omega_3^1 - \omega^2 \otimes \omega_3^2 \\ &= \omega^1 \otimes \omega_1^3 + \omega^2 \otimes \omega_2^3. \end{aligned}$$

综上讨论：

Proposition 0.1. 设$\{r; e_1, e_2, e_3 = N\}$为参数曲面$r(u, v)$的一个正交标架。则有标架运动方程：

$$\begin{aligned} dr &= \omega^\alpha e_\alpha, \quad (\omega^\alpha = \langle dr, e_\alpha \rangle) \\ de_i &= \omega_i^j e_j, \quad \omega_i^j + \omega_j^i = 0. \quad (\omega_i^j = \langle de_i, e_j \rangle) \end{aligned}$$

曲面的第一、第二基本形式分别为

$$I = \omega^\alpha \otimes \omega^\alpha, \quad II = \omega^\alpha \otimes \omega_\alpha^3.$$

可计算

$$\begin{aligned} \omega_\alpha^3(X_\beta) &= \langle de_\alpha, N \rangle (X_\beta) = \langle X_\beta(e_\alpha), N \rangle \\ &= -\langle e_\alpha, X_\beta(N) \rangle = -\langle dr(X_\alpha), dN(X_\beta) \rangle \\ &= II(X_\alpha, X_\beta) = II(X_\beta, X_\alpha). \end{aligned}$$

由$\{\omega^\alpha\}$为$\{X_\alpha\}$的对偶基可得

$$\omega_\alpha^3 = II(X_\alpha, X_\beta)\omega^\beta, \quad II = \omega^\alpha \otimes \omega_\alpha^3 = II(X_\alpha, X_\beta)\omega^\alpha \otimes \omega^\beta.$$

特别II为对称二次微分形式。令

$$h_{\alpha\beta} = II(X_\alpha, X_\beta), \quad B = (h_{\alpha\beta})$$

则

$$\omega_\alpha^3 = h_{\alpha\beta}\omega^\beta,$$

$$II = (\omega^1, \omega^2)\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}.$$

而

$$\omega_\alpha^\beta(X_\gamma) = \langle de_\alpha, e_\beta\rangle(X_\gamma) = \langle de_\alpha(X_\gamma), e_\beta\rangle = \langle X_\gamma e_\alpha, e_\beta\rangle = \langle \nabla_{X_\gamma} e_\alpha, e_\beta\rangle.$$

接下来考察Weingarten变换在切平面的单位正交基(e_1, e_2)之下的系数矩阵。可计算

$$\langle W(e_\alpha), e_\gamma\rangle = \langle -dN(X_\alpha), dr(X_\gamma)\rangle = II(X_\gamma, X_\alpha) = h_{\gamma\alpha} = h_{\alpha\gamma}.$$

即

$$W(e_\alpha) = h_{\alpha\beta}e_\beta,$$

或者

$$\begin{pmatrix} W(e_1) \\ W(e_2) \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = B\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

即Weingarten变换在(e_1, e_2)之下的系数矩阵为B。从而平均曲率和Gauss曲率分别为

$$H = \frac{1}{2}tr(B) = \frac{1}{2}(h_{11} + h_{22}), \quad K = \det(B) = h_{11}h_{22} - (h_{12})^2.$$

这里的定义不依赖于同定向的正交基(e_1, e_2)的选取。

当曲面没有脐点时，可以取e_1, e_2为曲面的主方向，此时

$$\langle W(e_1), e_1\rangle = \langle k_1e_1, e_1\rangle = k_1 = h_{11}, \quad \langle W(e_2), e_2\rangle = \langle k_2e_2, e_2\rangle = k_2 = h_{22},$$

$$\langle W(e_1), e_2\rangle = \langle k_1e_1, e_2\rangle = 0 = h_{12}, \quad \langle W(e_2), e_1\rangle = \langle k_2e_2, e_1\rangle = 0 = h_{21}.$$

即B为对角矩阵$diag(k_1, k_2)$。从而

$$H = \frac{1}{2}(k_1 + k_2), \quad K = k_1k_2.$$

由$\omega_\alpha^3 = h_{\alpha\beta}\omega^\beta$得

$$\omega_1^3 = k_1\omega^1, \quad \omega_2^3 = k_2\omega^2,$$

从而曲面的第二基本形式为

$$II = \omega^\alpha \otimes \omega_\alpha^3 = k_1\omega^1 \otimes \omega^1 + k_2\omega^2 \otimes \omega^2.$$

给定曲面及其定向，p处的正交标架选取可以相差一个转动，即$SO(2)=U(1)$作用。接下来考察D上一次微分形式$\omega^\alpha, \omega_i^j$在同定向的不同正交标架$\{r;\bar{e}_1,\bar{e}_2,\bar{e}_3=e_3=N\}$下的变换，并验证第一、第二基本形式与正交标架选取无关。设

$$\begin{cases} \bar{e}_1=\cos\theta e_1+\sin\theta e_2, \\ \bar{e}_2=-\sin\theta e_1+\cos\theta e_2. \end{cases}$$

即$(\bar{e}_1,\bar{e}_2)$由(e_1,e_2)逆时针转动角度θ得到。由$dr=\bar{\omega}^\alpha\bar{e}_\alpha$可得

$$\bar{\omega}^1=\langle dr,\bar{e}_1\rangle=\langle\omega^\alpha e_\alpha,\cos\theta e_1+\sin\theta e_2\rangle=\cos\theta\omega^1+\sin\theta\omega^2,$$

$$\bar{\omega}^2=\langle dr,\bar{e}_2\rangle=\langle\omega^\alpha e_\alpha,-\sin\theta e_1+\cos\theta e_2\rangle=-\sin\theta\omega^1+\cos\theta\omega^2.$$

从而

$$\begin{aligned} \bar{I} &= \bar{\omega}^1\otimes\bar{\omega}^1+\bar{\omega}^2\otimes\bar{\omega}^2 \\ &= (\cos\theta\omega^1+\sin\theta\omega^2)\otimes(\cos\theta\omega^1+\sin\theta\omega^2) \\ &\quad +(-\sin\theta\omega^1+\cos\theta\omega^2)\otimes(-\sin\theta\omega^1+\cos\theta\omega^2) \\ &= \omega^1\otimes\omega^1+\omega^2\otimes\omega^2=I. \end{aligned}$$

由$\bar{\omega}_i^j=\langle d\bar{e}_i,\bar{e}_j\rangle$可得

$$\bar{\omega}_1^3=\langle d\bar{e}_1,\bar{e}_3\rangle=\langle d(\cos\theta e_1+\sin\theta e_2),e_3\rangle=\cos\theta\omega_1^3+\sin\theta\omega_2^3,$$

$$\bar{\omega}_2^3=\langle d\bar{e}_2,\bar{e}_3\rangle=\langle d(-\sin\theta e_1+\cos\theta e_2),e_3\rangle=-\sin\theta\omega_1^3+\cos\theta\omega_2^3.$$

从而

$$\begin{aligned} \overline{II} &= \bar{\omega}^1\otimes\bar{\omega}_1^3+\bar{\omega}^2\otimes\bar{\omega}_2^3 \\ &= (\cos\theta\omega^1+\sin\theta\omega^2)\otimes(\cos\theta\omega_1^3+\sin\theta\omega_2^3) \\ &\quad +(-\sin\theta\omega^1+\cos\theta\omega^2)\otimes(-\sin\theta\omega_1^3+\cos\theta\omega_2^3) \\ &= \omega^1\otimes\omega_1^3+\omega^2\otimes\omega_2^3=II. \end{aligned}$$

最后计算

$$\begin{aligned} \bar{\omega}_1^2 &= \langle d\bar{e}_1,\bar{e}_2\rangle=\langle d(\cos\theta e_1+\sin\theta e_2),(-\sin\theta e_1+\cos\theta e_2)\rangle \\ &= d\theta+\cos^2\theta\langle de_1,e_2\rangle-\sin^2\theta\langle e_1,de_2\rangle \\ &= d\theta+\omega_1^2. \end{aligned}$$

曲面第一、第二基本形式可以通过参数坐标给出，若其矩阵表示的基选取为(du,dv)，则

$$I=(du,dv)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}, \quad II=(du,dv)\begin{pmatrix} L & M \\ M & N \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}.$$

另一方面若

$$\begin{pmatrix} r_u \\ r_v \end{pmatrix} = A\begin{pmatrix} e_1 \\ e_2 \end{pmatrix},$$

则由

$$dr=(du,dv)\begin{pmatrix} r_u \\ r_v \end{pmatrix}=(du,dv)A\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}=(\omega^1,\omega^2)\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

可知

$$(\omega^1,\omega^2)=(du,dv)A. \quad (*)$$

在基(ω^1,ω^2)之下，已知第一、第二基本形式有矩阵表示

$$I=(\omega^1,\omega^2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix},$$

$$II=(\omega^1,\omega^2)\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix},$$

其中矩阵B中元素

$$h_{\alpha\beta}=II(X_\alpha,X_\beta)=II((dr)^{-1}(e_\alpha),(dr)^{-1}(e_\beta)).$$

以$(*)$代入得

$$AA^T=\begin{pmatrix} E & F \\ F & G \end{pmatrix}=(I),$$

$$A\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}A^T=\begin{pmatrix} L & M \\ M & N \end{pmatrix}=(II).$$

由此，

$$\begin{aligned} tr(B) &= tr[A^{-1}(II)(A^T)^{-1}]=tr[(II)(A^T)^{-1}A^{-1}]=tr[(II)(AA^T)^{-1}] \\ &= tr[(II)(I)^{-1}]=2H, \end{aligned}$$

同样

$$\det(B) = \det[(II)(AA^T)^{-1}]=\det[(II)(I)^{-1}]=K.$$

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