第十六讲 (2023.5.5)

Thm (Fubini)

设 $f \in L^1(\mathbb{R}^{n_1+n_2})$.

(F1) 对 a.e. $y \in \mathbb{R}^{n_2}$, $f^y \in L^1(\mathbb{R}^{n_1})$.

对 a.e. $x \in \mathbb{R}^{n_1}$, $f_x \in L^1(\mathbb{R}^{n_2})$.

(F2) $y \mapsto \int_{\mathbb{R}^{n_1}} f^y \, dx \in L^1(\mathbb{R}^{n_2})$

$x \mapsto \int_{\mathbb{R}^{n_2}} f_x \, dy \in L^1(\mathbb{R}^{n_1})$.

(F3)

$$\int_{\mathbb{R}^{n_1+n_2}} f \, dm = \int_{\mathbb{R}^{n_2}} \left[\int_{\mathbb{R}^{n_1}} f(x,y) \, dx \right] dy = \int_{\mathbb{R}^{n_1}} \left[\int_{\mathbb{R}^{n_2}} f(x,y) \, dy \right] dx$$

Thm (Tonelli)

设 $f \in L^+(\mathbb{R}^{n_1+n_2})$.

(T1) 对 a.e. $y \in \mathbb{R}^{n_2}$, $f^y \in L^+(\mathbb{R}^{n_1})$.

对 a.e. $x \in \mathbb{R}^{n_1}$, $f_x \in L^+(\mathbb{R}^{n_2})$.

(T2) $y \mapsto \int_{\mathbb{R}^{n_1}} f^y \, dx \in L^+(\mathbb{R}^{n_2})$,

$x \mapsto \int_{\mathbb{R}^{n_2}} f_x \, dy \in L^+(\mathbb{R}^{n_1})$.

(T3) 同 (F3).

Tonelli $\Longrightarrow$ Fubini

Idea of Pf of Tonelli

$\hat{\imath}$ $\quad L^+ \overset{\text{def}}{=} L^+(\mathbb{R}^{n_1+n_2})$.

$\mathcal{F} \overset{\text{def}}{=} \{ f \in L^+ : f \text{ satisfies } (T1)-(T3) \}$

Tonelli $\Longleftrightarrow$ $\mathcal{F} = L^+$

1° $\mathcal{F}$ 对加法和非负数乘封闭

2° $\mathcal{F}$ 对单调增序列极限封闭

$\Longrightarrow$ 约化为:

<u>Claim</u> $\forall E \in \mathcal{L}$, $\chi_E \in \mathcal{F}$

Indeed,

非负简单函数

$$\left.\begin{array}{l}\left.\begin{array}{c}\text{Claim}\\ 1^\circ\end{array}\right\} \Longrightarrow S^+ \subset \mathcal{F} \\ \qquad\qquad 2^\circ\end{array}\right\} \Longrightarrow L^+ \subset \mathcal{F}$$

$\forall f \in L^+$, $\exists \varphi_k \nearrow f$

$\overset{2^\circ}{\Longrightarrow} f \in \mathcal{F}$

而证明 claim，

3° $\forall F$ (F_σ 集)，$\chi_F \in \mathcal{F}$

4° $\forall Z$ (零集)，$\chi_Z \in \mathcal{F}$

然后，$\forall E \in \mathcal{L}$，$\exists F, Z$，s.t. $E = F \uplus Z$

$$\Longrightarrow \chi_E = \chi_F + \chi_Z \in \mathcal{F}$$

<u>Lem 1</u> $\mathcal{F}$ 对加法和非负数乘封闭.

平凡

<u>Lem 2</u> $\forall f, g \in \mathcal{F}$ with $f - g \geq 0$，$g \in L^1$

$$\Longrightarrow f - g \in \mathcal{F}$$

<u>Pf</u> $g \in \mathcal{F} \cap L^1$

$$\Longrightarrow +\infty > \int_{\mathbb{R}^{n_1+n_2}} g \, dm = \int_{\mathbb{R}^{n_2}} \left[\int_{\mathbb{R}^{n_1}} g^y dx \right] dy$$

$\Longrightarrow y \mapsto \int_{\mathbb{R}^{n_1}} g^y dx$ 在 $\mathbb{R}^{n_2}$ 上 a.e. 有限

$\Longrightarrow$ 对这些使得 $\int_{\mathbb{R}^{n_1}} g^y dx$ 有限的 y，g^y 在 $\mathbb{R}^{n_1}$ 上 a.e. 有限.

$\Longrightarrow$ $f^y - g^y$ a.e. 有意义，则

$$(f-g)^y = f^y - g^y \quad \text{a.e. on } \mathbb{R}^{n_1}$$

同理，$(f-g)_x = f_x - g_x$ a.e. on $\mathbb{R}^{n_2}$

$\Longrightarrow$

$$\int_{\mathbb{R}^{n_2}} \left[\int_{\mathbb{R}^{n_1}} f^y \, dx \right] dy \overset{(T3)}{=} \int_{\mathbb{R}^{n_1+n_2}} f \, dm$$

$$= \int_{\mathbb{R}^{n_1+n_2}} (f-g) \, dm + \int_{\mathbb{R}^{n_1+n_2}} g \, dm$$

$$\overset{(T3)}{=} \int_{\mathbb{R}^{n_1+n_2}} (f-g) \, dm + \int_{\mathbb{R}^{n_2}} \left[\int_{\mathbb{R}^{n_1}} g^y \, dx \right] dy$$

$$\Longrightarrow \int_{\mathbb{R}^{n_1+n_2}} (f-g) \, dm = \int_{\mathbb{R}^{n_2}} \left[\int_{\mathbb{R}^{n_1}} f^y \, dx \right] dy - \int_{\mathbb{R}^{n_2}} \underbrace{\left[\int_{\mathbb{R}^{n_1}} g^y \, dx \right]}_{\text{a.e. 有限}} dy$$

$$= \int_{\mathbb{R}^{n_2}} \left[\int_{\mathbb{R}^{n_1}} f^y \, dx - \int_{\mathbb{R}^{n_1}} g^y \, dx \right] dy$$

$$= \int_{\mathbb{R}^{n_2}} \left[\int_{\mathbb{R}^{n_1}} (f-g)^y \, dx \right] dy$$

$\Longrightarrow$ $f - g$ 满足 (T3).

Lem 3 $\mathcal{F} \ni f_k \nearrow f \implies f \in \mathcal{F}$

Pf 对任 k, $\exists A_k \subset \mathbb{R}^{n_2}$ with $m_{n_2}(A_k) = 0$,

s. t.

$$\forall y \in \mathbb{R}^{n_2} \setminus A_k, \quad (f_k)^y \in L^+(\mathbb{R}^{n_1})$$

令

$$A \overset{\text{def}}{=} \bigcup_{k=1}^{\infty} A_k$$

$\implies m_{n_2}(A) = 0$, 则

$$\forall y \in \mathbb{R}^{n_2} \setminus A, \quad (f_k)^y \in L^+(\mathbb{R}^{n_1})$$

则 $(f_k)^y \nearrow f^y$

$$\implies f^y \in L^+(\mathbb{R}^{n_1}) \qquad ((T1) \checkmark)$$

则

$$y \longmapsto \int_{\mathbb{R}^{n_1}} f^y \, dx \overset{\text{MCT}}{=} \lim_{k \to \infty} \int_{\mathbb{R}^{n_1}} (f_k)^y \, dx$$

$$\in L^+(\mathbb{R}^{n_2}). \qquad ((T2) \checkmark)$$

$$\int_{\mathbb{R}^{n_1+n_2}} f \, dm \overset{\text{MCT}}{=} \lim_{k \to \infty} \int_{\mathbb{R}^{n_1+n_2}} f_k \, dm$$

$$\overset{(T3) \text{ for } f_k}{=} \lim_{k \to \infty} \int_{\mathbb{R}^{n_2}} \left[\int_{\mathbb{R}^{n_1}} (f_k)^y \, dx \right] dy$$

$$\overset{MCT}{=} \int_{\mathbb{R}^{n_2}} \left[\int_{\mathbb{R}^{n_1}} f^y \, dx \right] dy. \qquad ((T3) \checkmark)$$

<u>Lem 4</u> $\quad f_k \in \mathcal{F} \cap L^1, \ f_k \searrow f \implies f \in \mathcal{F}$

<u>Pf</u> $\quad$ 不 $\quad g_k \overset{def}{=} f_1 - f_k$

$$\overset{\text{Lem 2}}{\implies} \quad \mathcal{F} \ni g_k \nearrow f_1 - f$$

$$\overset{\text{Lem 3}}{\implies} \quad f_1 - f \in \mathcal{F}$$

$$\overset{\text{Lem 2}}{\implies} \quad f = f_1 - \underbrace{(f_1 - f)}_{\in L^1} \in \mathcal{F}$$

<u>Pf of Tonelli</u>

<u>Case 1</u> $\quad E = Q' \times Q''$ with $Q' \subset \mathbb{R}^{n_1}$, $Q'' \subset \mathbb{R}^{n_2}$

$$E^y = \begin{cases} Q', & \text{if } y \in Q'' \\ \phi, & \text{otherwise} \end{cases}$$

$$\implies \quad \forall y \in \mathbb{R}^{n_2}, \ E^y \in \mathcal{L}_{\mathbb{R}^{n_1}}$$

$$\implies \quad (\chi_E)^y = \chi_{E^y} \in L^+(\mathbb{R}^{n_1})$$

$$\Longrightarrow \int_{\mathbb{R}^{n_1}} (\chi_E)^y \, dx = m_{n_1}(E^y)$$

$$= \begin{cases} |Q'|, & \text{if } y \in Q'' \\ 0, & \text{otherwise} \end{cases}$$

$$= |Q'| \, \chi_{Q''}(y)$$

t.f

$$\int_{\mathbb{R}^{n_2}} \left[\int_{\mathbb{R}^{n_1}} (\chi_E)^y \, dx \right] dy = |Q'| \int_{\mathbb{R}^{n_2}} \chi_{Q''}(y) \, dy$$

$$= |Q'| \, |Q''|$$

$$= m(E)$$

$$= \int_{\mathbb{R}^{n_1+n_2}} \chi_E \, dm$$

$$\Longrightarrow \chi_E \in \mathcal{F}$$

<u>Case 2</u> E 开.

$$E = \biguplus_{k=1}^{\infty} Q_k \qquad \left(\Longrightarrow \chi_E = \sum_{k=1}^{\infty} \chi_{Q_k} \text{ a.e.} \right.$$

$$\Longrightarrow E^y = \biguplus_{k=1}^{\infty} Q_k^y$$ [illegible] (by Case 1)

$$\Longrightarrow (\chi_E)^y = \sum_{k=1}^{\infty} \chi_{Q_k^y} \in L^+(\mathbb{R}^{n_1}).$$

$$\Longrightarrow \int_{\mathbb{R}^{n_1}} (\chi_E)^y \, dx = m_{n_1}(E^y)$$

$$= \sum_{k=1}^{\infty} m_{n_1}(Q_k^y) = \sum_{k=1}^{\infty} \underbrace{\int_{\mathbb{R}^{n_1}} \chi_{Q_k^y} \, dx}_{\in L^+(\mathbb{R}^{n_2})}$$

$$\Longrightarrow y \longmapsto \int_{\mathbb{R}^{n_1}} (\chi_E)^y \, dx \in L^+(\mathbb{R}^{n_2})$$

$$\int_{\mathbb{R}^{n_2}} \left[\int_{\mathbb{R}^{n_1}} (\chi_E)^y \, dx \right] dy \overset{\substack{\text{MCT} \\ \downarrow}}{=} \sum_{k=1}^{\infty} \int_{\mathbb{R}^{n_2}} \left[\int_{\mathbb{R}^{n_1}} (\chi_{Q_k})^y \, dx \right] dy$$

$$\overset{\text{Case 1}}{=} \sum_{k=1}^{\infty} m(Q_k)$$

$$= m(E)$$

$$= \int_{\mathbb{R}^{n_1+n_2}} \chi_E \, dm$$

<u>Case 3</u> E 紧

$\exists R > 0$ s.t. $E \subset B_R(0)$.

令 $G \overset{\text{def}}{=} B_R(0) \setminus E$ (开)

$\Longrightarrow E = B_R(0) \setminus G$

$\overset{\text{Lem 2}}{\Longrightarrow} \chi_E = \chi_{B_R(0)} - \chi_G \in \mathcal{F}$

Case 4 E 是 F_σ 集

$\Rightarrow E = \bigcup_{k=1}^{\infty} F_k$，$F_k$ 闭

不妨设：$\forall k$，F_k 紧（代以 $F_k \cap \overline{B_k(0)}$）

$\mathcal{F} \ni \chi_{\bigcup_{k=1}^{N} F_k} \nearrow \chi_E$

by Case 3

$\overset{\text{Lem. 3}}{\Longrightarrow} \chi_E \in \mathcal{F}$

Case 5 $\underbrace{E \text{ 是零集}}$

$\Downarrow$

$\forall k$，$\exists G_k$ 开，s.t. $E \subset G_k$ 且

$m(G_k) < \frac{1}{k}$

令 $G \overset{\text{def}}{=} \bigcap_{k=1}^{\infty} G_k$

$\Rightarrow G$ 是 G_δ 集，且 $m(G) = 0$

（称为 E 之零测包）

$\bigcap_{k=1}^{N} G_k \searrow G \Rightarrow \chi_{\bigcap_{k=1}^{N} G_k} \searrow \chi_G$

Lem 3

$$\Longrightarrow \quad \chi_G \in \mathcal{F}$$

(T3) for χ_G

$$\Longrightarrow \int_{\mathbb{R}^{n_2}} \left[\int_{\mathbb{R}^{n_1}} (\chi_G)^y \, dx \right] dy = \int_{\mathbb{R}^{n_1+n_2}} \chi_G \, dm$$

$$= m(G) = 0.$$

$$\Longrightarrow \underbrace{\int_{\mathbb{R}^{n_1}} (\chi_G)^y \, dx}_{= m_{n_1}(G^y)} = 0 \quad \text{for a.e. } y \in \mathbb{R}^{n_2}$$

$E^y \subset G^y$

$$\Longrightarrow m_{n_1}(E^y) = 0 \quad \text{for a.e. } y \in \mathbb{R}^{n_2}$$

$$\Longrightarrow (\chi_E)^y = \chi_{E^y} \in L^+(\mathbb{R}^{n_1})$$

$$\int_{\mathbb{R}^{n_1}} (\chi_E)^y \, dx = m_{n_1}(E^y) = 0$$

$$\Longrightarrow \int_{\mathbb{R}^{n_2}} \left[\int_{\mathbb{R}^{n_1}} (\chi_E)^y \, dx \right] dy = 0 = \int_{\mathbb{R}^{n_1+n_2}} \chi_E \, dm$$

故 $\chi_E \in \mathcal{F}$

HW: 14, 17, 20. 并把 Ex. 19 之结论推广到 L^p $(1 \le p < \infty)$ 版本