$$= \int_{R^n} \frac{1}{\sqrt{(2\pi)^n\cdot|\Sigma|}}\cdot \exp(-\tfrac{1}{2}(\vec{y}B)B^T\Lambda B B^T \vec{y}^T)\, dx_1\cdots dx_n$$

$$= \int_{R^n} \frac{1}{\sqrt{(2\pi)^n\cdot|\Sigma|}} \exp(-\tfrac{1}{2}\sum_{k=1}^n \frac{y_k^2}{\lambda_k})\, dy_1\cdots dy_n = \frac{1}{\sqrt{(2\pi)^n\cdot\underset{\lambda_1\cdots\lambda_n}{|\Sigma|}}}\cdot \prod_{k=1}^n \int_{-\infty}^{+\infty} e^{-\frac{y_k^2}{2\lambda_k}}\, dy_k = 1$$

$(x_1,\cdots,x_n)=(y_1,\cdots,y_n)B+\vec{\mu}$ , $x_i=\mu_i+\sum_{j=1}^n y_j B_{ji}$

$\vec{Y}=(\vec{X}-\vec{\mu})B^{-1}$ $\quad f_Y(y_1,\cdots,y_n)=\frac{1}{\sqrt{(2\pi)^n|\Sigma|}}\exp(-\frac{1}{2}\sum_{k=1}^n \frac{y_k^2}{\lambda_k})$

$Y_1,\cdots,Y_n$ 独立1元标准正态

$\exists A$ 非退化, s.t. $\vec{X}=\vec{Y}A+\mu$.

定义 $(Y_1,\cdots,Y_n)$ 服从n元标准正态, 则 $\vec{X}=\vec{Y}A+\mu$ 服从参数为 $\mu$, $\Sigma=A^TA$ 的n元正态分布

定理 $\vec{X}=(X_1,\cdots,X_n)\sim N(\vec{\mu},\Sigma)$

则 (1) $E[\vec{X}]=\vec{\mu}$ (2) $\Sigma$ 为协方差矩阵 (3) $\vec{X}$ 的各分量相互独立 $\Leftrightarrow \operatorname{cov}(X_i,X_j)=\begin{cases}0, & i\neq j\\ \Sigma_{ii}, & i=j\end{cases}$

证: (1) $E[X_i]=\int_{R^n} x_i\cdot\frac{1}{\sqrt{(2\pi)^n\cdot|\Sigma|}}\exp(-\frac{1}{2}(\vec{X}-\vec{\mu})\Sigma^{-1}(\vec{X}-\vec{\mu})^T)\, dx_1\cdots dx_n$

$\vec{Y}=(\vec{X}-\vec{\mu})B^{-1}$

$$= \int_{R^n}(\mu_i+\sum_{k=1}^n b_{ki}y_k)\exp(-\frac{1}{2}\sum_{k=1}^n\frac{y_k^2}{\lambda_k})\cdot\frac{1}{\sqrt{(2\pi)^n\cdot|\Sigma|}}\, dy_1\cdots dy_n$$

$$= \mu_i$$

(2) $\operatorname{cov}(X_i,X_j)=\int_{R^n}(x_i-\mu_i)(x_j-\mu_j)\cdot\frac{1}{\sqrt{(2\pi)^n\cdot|\Sigma|}}\exp(-\frac{1}{2}(\vec{X}-\vec{\mu})\Sigma^{-1}(\vec{X}-\vec{\mu})^T)\, dx_1\cdots dx_n$

$$= \int_{R^n}(\sum_{k=1}^n y_k B_{ki})(\sum_{l=1}^n y_l B_{lj})\cdot\frac{1}{\sqrt{(2\pi)^n\cdot|\Sigma|}}\exp(-\frac{1}{2}\sum_{i=1}^n\frac{y_i^2}{\lambda_i})\, dy_1\cdots dy_n$$

$$= \sum_{k=1}^n B_{ki}B_{kj}\int_{R^n} y_k^2\cdot\frac{1}{\sqrt{(2\pi)^n\cdot|\Sigma|}}\exp(-\frac{1}{2}\sum_{i=1}^n\frac{y_i^2}{\lambda_i})\, dy_1\cdots dy_n = \sum_{k=1}^n B_{ki}B_{kj}\lambda_k$$

$$= (B^T\Lambda B)_{ij}$$

hw: 4.7.2, 4.7.5, 4.7.9, 4.9.3, 4.9.7

(3) "$\Leftarrow$" $\Sigma=\begin{pmatrix}\Sigma_{11} & & 0\\ & \ddots & \\ 0 & & \Sigma_{nn}\end{pmatrix}$ $\quad |\Sigma|=\prod_{i=1}^n\Sigma_{ii}$ $\quad \Sigma^{-1}=\begin{pmatrix}\Sigma_{11}^{-1} & & 0\\ & \ddots & \\ 0 & & \Sigma_{nn}^{-1}\end{pmatrix}$

$$(\vec{X}-\vec{\mu})\Sigma^{-1}(\vec{X}-\vec{\mu})^T=\sum_{i=1}^n\frac{(x_i-\mu_i)^2}{\Sigma_{ii}}$$

$$f(x_1,\cdots,x_n)=\prod_{i=1}^n\frac{1}{\sqrt{(2\pi)\Sigma_{ii}}}e^{-\frac{1}{2}\cdot\frac{(x_i-\mu_i)^2}{\Sigma_{ii}}}\triangleq f_1(x_1)\cdots f_n(x_n)$$

说明 $x_1,\cdots,x_n$ 独立

定理2 $\vec{X}\sim N(\vec{\mu},\Sigma)$ D n阶非退化矩阵，则 $\vec{Y}=\vec{X}\cdot D\sim N(\vec{\mu}D, D^T\Sigma D)$

$\vec{X}=\vec{Y}\cdot D^{-1}$

证：$f_{\vec{Y}}(y_1,\cdots,y_n)=\frac{1}{\sqrt{(2\pi)^n\cdot|\Sigma|}}\exp(-\frac{1}{2}(\vec{Y}D^{-1}-\vec{\mu})\Sigma^{-1}(\vec{Y}D^{-1}-\vec{\mu})^T)\cdot|D^{-1}|$

$=\frac{1}{\sqrt{(2\pi)^n\cdot|D^T\Sigma D|}}\exp(-\frac{1}{2}(\vec{Y}-\vec{\mu}D)D^{-1}\Sigma^{-1}(D^{-1})^T(\vec{Y}-\vec{\mu}D)^T)$

$=\frac{1}{\sqrt{(2\pi)^n\cdot|D^T\Sigma D|}}\exp(-\frac{1}{2}(\vec{Y}-\vec{\mu}D)(D^T\Sigma D)^{-1}(\vec{Y}-\vec{\mu}D)^T)$

$\vec{Y}\sim N(\vec{\mu}D, D^T\Sigma D)$

定理3 $\vec{X}\sim N(\vec{\mu},\Sigma)$

$\Sigma=\begin{pmatrix}\Sigma_{11} & 0\\ 0 & \Sigma_{22}\end{pmatrix}$ $\Sigma_{11}$ l阶正定矩阵 $\vec{X}=(\vec{X}^{(1)},\vec{X}^{(2)})$ $\vec{X}^{(1)}=x_1,\cdots,x_l$

$\Sigma_{22}$ n-l 阶正定矩阵 $\vec{\mu}=(\vec{\mu}^{(1)},\vec{\mu}^{(2)})$

则 $\vec{X}^{(1)}\sim N(\vec{\mu}^{(1)},\Sigma_{11})$ l元正态正布，$\vec{X}^{(2)}\sim N(\vec{\mu}^{(2)},\Sigma_{22})$ n-l元正态分布.

证：$|\Sigma|=|\Sigma_{11}|\cdot|\Sigma_{22}|$ $\Sigma^{-1}=\begin{pmatrix}\Sigma_{11}^{-1} & 0\\ 0 & \Sigma_{22}^{-1}\end{pmatrix}$

$f(x_1,\cdots,x_n)=\frac{1}{\sqrt{(2\pi)^n\cdot|\Sigma_{11}|\cdot|\Sigma_{22}|}}\exp[-\frac{1}{2}(\vec{X}^{(1)}-\vec{\mu}^{(1)},\vec{X}^{(2)}-\vec{\mu}^{(2)})\begin{pmatrix}\Sigma_{11}^{-1} & 0\\ 0 & \Sigma_{22}^{-1}\end{pmatrix}\begin{pmatrix}\vec{X}^{(1)}-\vec{\mu}^{(1)}\\ \vec{X}^{(2)}-\vec{\mu}^{(2)}\end{pmatrix}$

$=\frac{1}{\sqrt{(2\pi)^n\cdot|\Sigma_{11}|\cdot|\Sigma_{22}|}}\exp[-\frac{1}{2}((\vec{X}^{(1)}-\vec{\mu}^{(1)})\Sigma_{11}^{-1}(\vec{X}^{(1)}-\vec{\mu}^{(1)})^T+(\vec{X}^{(2)}-\vec{\mu}^{(2)})\Sigma_{22}^{-1}(\vec{X}^{(2)}-\vec{\mu}^{(2)})^T)]$

$=\frac{1}{\sqrt{(2\pi)^l\cdot|\Sigma_{11}|}}\exp(-\frac{1}{2}(\vec{X}^{(1)}-\vec{\mu}^{(1)})\Sigma_{11}^{-1}(\vec{X}^{(1)}-\vec{\mu}^{(1)})^T)\cdot$

$\frac{1}{\sqrt{(2\pi)^{n-l}\cdot|\Sigma_{22}|}}\exp(-\frac{1}{2}(\vec{X}^{(2)}-\vec{\mu}^{(2)})\Sigma_{22}^{-1}(\vec{X}^{(2)}-\vec{\mu}^{(2)})^T)$

$f_1(x_1,\cdots,x_l)=\int_{R^{n-l}}f(x_1,\cdots,x_n)dx_{l+1}\cdots dx_n$

注：$(X_1,X_2,\cdots,X_l)$ 与 $(X_{l+1},\cdots,X_n)$ 独立.

$p(X_1\le x_1,\cdots,X_n\le x_n)=p(X_1\le x_1,\cdots,X_l\le x_l)\,p(X_{l+1}\le x_{l+1},\cdots,X_n\le x_n)$

定理4. $\vec{X}\sim N(\vec{\mu},\Sigma)$

$\vec{X}=(\vec{X}^{(1)},\vec{X}^{(2)})$, $\vec{\mu}=(\vec{\mu}^{(1)},\vec{\mu}^{(2)})$, $\Sigma=\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}$，则 $\vec{X}^{(1)}\sim N(\vec{\mu}^{(1)},\Sigma_{11})$

证：$\begin{pmatrix}I_l & 0\\ -\Sigma_{21}\Sigma_{11}^{-1} & I_{n-l}\end{pmatrix}\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}\begin{pmatrix}I_l & -\Sigma_{11}^{-1}\Sigma_{12}\\ 0 & I_{n-l}\end{pmatrix}=\begin{pmatrix}\Sigma_{11} & 0\\ 0 & \Sigma_{22}-\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\end{pmatrix}$

令 $\vec{Y}=(\vec{Y}^{(1)},\vec{Y}^{(2)})=\vec{X}\begin{pmatrix}I_l & -\Sigma_{11}^{-1}\Sigma_{12}\\ 0 & I_{n-l}\end{pmatrix}\triangleq\vec{X}D\sim N(\vec{\mu}D,D^T\Sigma D)$

$\therefore \vec{Y}^{(1)}\sim N(\vec{\mu}^{(1)},\Sigma_{11})$，即 $\vec{X}^{(1)}\sim(\vec{\mu}^{(1)},\Sigma_{11})$

定理5 $\vec{X}\sim N(\mu,\Sigma)$ $A_{n\times m}$矩阵 $n>m$ $\mathrm{rank}(A)=m$ $\vec{Y}=\vec{X}A\sim N(\vec{\mu}A,A^T\Sigma A)$

证：$D=(A,B)$ 补充 $B$ s.t. $|D|\neq0$.

$Z=\vec{X}\cdot D=(\vec{X}A,\vec{X}B)\sim N(\vec{\mu}D,D^T\Sigma D)$，$\vec{\mu}D=(\vec{\mu}A\ \ \vec{\mu}B)$ $D^T\Sigma D=\begin{pmatrix}A^T\\ B^T\end{pmatrix}\Sigma(A\ \ B)$

$=\begin{pmatrix}A^T\Sigma A & A^T\Sigma B\\ B^T\Sigma A & B^T\Sigma B\end{pmatrix}$

由定理4. $\vec{X}A\sim N(\vec{\mu}A,A^T\Sigma A)$

特别地，$m=1$. $\vec{X}\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}=\sum_{i=1}^n a_iX_i\sim N(\sum_{i=1}^n a_i\mu_i,\ (a_1\cdots a_n)\Sigma\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix})$

$\chi^2$分布.

$X_1,\cdots,X_n\sim N(\mu,\sigma^2)$ 相互独立. $E(\frac{1}{n}\sum_{i=1}^n X_i)=\mu$ 记 $\bar{X}=\frac{1}{n}\sum_{i=1}^n X_i$ 样本均值.

$S^2=\frac{1}{n-1}\sum_{i=1}^n(X_i-\bar{X})^2$ 样本方差 $E(S^2)=\sigma^2$

密度函数 $f(x)=\frac{1}{\Gamma(\frac{d}{2})2^{\frac{d}{2}}}x^{\frac{d}{2}-1}e^{-\frac{x}{2}}$，$x>0$. 自由度为d的$\chi^2$分布. $\chi^2(d)$

引理 $Y_1,\cdots,Y_n$相互独立，服从$N(0,1)$，$X=\sum_{i=1}^n Y_i^2\sim\chi^2(n)$

证：$P(Y_1^2\le x)=P(-\sqrt{x}\le Y_1\le\sqrt{x})=\Phi(\sqrt{x})-\Phi(-\sqrt{x})$

$f_1(x)=\varphi(\sqrt{x})\cdot\frac{1}{2\sqrt{x}}+\varphi(-\sqrt{x})\cdot\frac{1}{2\sqrt{x}}=\frac{1}{\sqrt{x}}\cdot\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{x}{2}}=\frac{1}{\Gamma(\frac{1}{2})2^{\frac{1}{2}}}x^{\frac{1}{2}-1}e^{-\frac{x}{2}}$ $\Rightarrow Y_i^2\sim\chi^2(1)$

归纳法，假设 $Y_1^2+\cdots+Y_k^2\sim\chi^2(k)$，记密度函数为$f_k(x)$

$f_{k+1}(x)=\int_{-\infty}^{+\infty}f_k(u)f_1(x-u)\,du$ $u>0$，$x-u>0$

$=\int_0^x\frac{1}{\Gamma(\frac{k}{2})2^{\frac{k}{2}}\Gamma(\frac{1}{2})2^{\frac{1}{2}}}u^{\frac{k}{2}-1}e^{-\frac{u}{2}}(x-u)^{-\frac{1}{2}}e^{-\frac{x-u}{2}}du$

$$= \frac{1}{\Gamma(\frac{k}{2})\Gamma(\frac{1}{2})2^{\frac{k+1}{2}}} e^{-\frac{x}{2}} \int_0^x u^{\frac{k}{2}-1}(x-u)^{-\frac{1}{2}} du$$

$$= \frac{1}{\Gamma(\frac{k}{2})\Gamma(\frac{1}{2})2^{\frac{k+1}{2}}} e^{-\frac{x}{2}} x^{\frac{k+1}{2}-1} B(\tfrac{k}{2},\tfrac{1}{2}) = \frac{1}{\Gamma(\frac{k+1}{2})2^{\frac{k+1}{2}}} e^{-\frac{x}{2}} x^{\frac{k+1}{2}-1} \quad \therefore Y_1^2+\cdots+Y_{k+1}^2 \sim \chi^2(k+1)$$

定理 $X_1,\cdots,X_n \sim N(\mu,\sigma^2)$ 相互独立.

则 (1) $\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i \sim N(\mu, \frac{\sigma^2}{n})$ (2) $\overline{X}$ 与 $S^2$ 相互独立 (3) $S^2 \cdot \frac{n-1}{\sigma^2} \sim \chi^2(n-1)$

证: (2) 令 $Y_i = \frac{X_i-\mu_i}{\sigma}$, $i=1,\cdots,n$ $\quad Y_i \quad i=1,\cdots,n$ 相互独立. 服从 $N(0,1)$

$$\overline{Y} = \frac{1}{n}\sum_{i=1}^n \frac{X_i-\mu}{\sigma} = \frac{1}{\sigma}(\overline{X}-\mu), \quad \sum_{i=1}^n (Y_i-\overline{Y})^2 = \sum_{i=1}^n \left(\frac{X_i-\mu}{\sigma} - \frac{\overline{X}-\mu}{\sigma}\right)^2 = \frac{1}{\sigma^2}\sum_{i=1}^n (X_i-\overline{X})^2 = \frac{n-1}{\sigma^2}S^2$$

$$\overline{Y} = \frac{1}{n}\sum_{i=1}^n Y_i = \frac{1}{\sqrt{n}}\sum_{i=1}^n \frac{Y_i}{\sqrt{n}}$$

引入正交阵 $A = \begin{pmatrix} \frac{1}{\sqrt{n}} & \\ \vdots & * \\ \frac{1}{\sqrt{n}} & \end{pmatrix} \triangleq (\vec{a_1}^T, \vec{a_2}^T, \cdots, \vec{a_n}^T)$ $\quad \vec{a_i}\cdot\vec{a_j} = 0,\ i\neq j.\ |\vec{a_i}|=1$

$\vec{Z} = \vec{Y}\cdot A \quad \vec{Y} \sim N(\vec{0}, I_n) \quad \vec{Z} \sim N(\vec{0}A, A^T I_n A) = N(\vec{0}, I_n)$

$\underline{Z_1 = \sqrt{n}\,\overline{Y}} \quad Z_1^2+\cdots+Z_n^2 = \vec{Z}\cdot\vec{Z}^T = \vec{Y}AA^T\vec{Y}^T = \vec{Y}\vec{Y}^T = Y_1^2+\cdots+Y_n^2$

$$\underline{\sum_{i=2}^n Z_i^2} = \sum_{i=1}^n Y_i^2 - Z_1^2 = \sum_{i=1}^n (Y_i-\overline{Y}+\overline{Y})^2 - n\overline{Y}^2 = \sum_{i=1}^n (Y_i-\overline{Y})^2 + 2\underbrace{\sum_{i=1}^n (Y_i-\overline{Y})}_{=0}\overline{Y} + n\overline{Y}^2 - n\overline{Y}^2$$

$$= \sum_{i=1}^n (Y_i-\overline{Y})^2 = \underline{\frac{(n-1)S^2}{\sigma^2}} \sim \chi_{n-1}^2 \quad (3)$$

$Z_1$ 与 $Z_2,\cdots,Z_n$ 独立. $\Rightarrow \overline{Y}.\ S^2$ 独立.

## §5 特征函数及应用

### §5.1 数学期望.

离散型 $\sum_i x_i p(X=x_i)$ 绝对收敛

连续型 $\int_{-\infty}^{+\infty} x f(x) dx$.

一般 $\begin{cases} \text{Riemann-Stieltjes积分} \\ (\Omega,F,P)\ \text{Lebesgue积分.} \end{cases}$

一. Riemann-Stieltjes积分

$[a,b]$ $\quad T:\ a=x_0<x_1<\cdots<x_n=b \quad \xi_i \in [x_{i-1},x_i)$

$S_T = \sum_{i=1}^n f(\xi_i)(g(x_i)-g(x_{i-1})) \qquad \|T\| = \max_i |x_i - x_{i-1}|$

若 $\lim_{\|T\|\to 0} S_T$ 有与分割取点方式无关的极限. 称 $f(x)$ 关于 $g(x)$ 在 $[a,b]$ R-S可积.

$\int_a^b f(x)dg(x)$ $g(x)$单调有界 $f(x)\in C[a,b)$ R-S积分存在

$\int_{-\infty}^{+\infty} f(x)dg(x) = \lim\limits_{\substack{a\to+\infty \\ b\to+\infty}} \int_a^b f(x)dg(x)$

若$\int_{-\infty}^{+\infty}|x|\,dF(x)<\infty$，其中$F(x)$是r.v. $X$的分布函数，称$E(X)$存在。$E(X)=\int_{-\infty}^{+\infty} x\,dF(x)$

连续型：$dF(x)=F(x)-F(x-0)$　离散型：$dF(x)=f(x)dx$

二、$(\Omega,F,P)$　$X:\Omega\to R$ 可测函数.

抽象积分.

定义 1° 简单随机变量（只取有限个值）

$A_i=\{w\,|\,X(w)=x_i\}$　$X=\sum_{i=1}^n x_i I_{A_i}$　定义$E(X)=\sum_{i=1}^n x_i P(A_i)$

2° 对非负随机变量 $X$.

$A_n=\{w\,|\,X>n\}$　$A_{ni}=\{w\,|\,\frac{i-1}{2^n}\le X<\frac{i}{2^n}\}$

$X_n=\sum_{i=1}^{n\cdot 2^n}\frac{i-1}{2^n}I_{ni}+nI_{A_n}$　$X_n\uparrow$　$|X_n-X|<\frac{1}{2^n}\to 0$

定义 $E(X)=\lim\limits_{n\to\infty}E(X_n)$

3° 一般随机变量$X$.　$X=X^+-X^-$.　$X^+=\max\{X,0\}$　$X^-=\max\{-X,0\}$

若$E(X^+)$, $E(X^-)$都存在，定义$E(X)=E(X^+)-E(X^-)$

记号 $\int_\Omega X(w)\,dp=E(X)$

hw: 4.9.4, 4.9.6, 4.10.1, 4.10.2