2.3 主要存在问题：① - ④

5. Which of the following are density functions? Find c and the corresponding distribution function F for those that are.

(a) $f(x) = \begin{cases} cx^{-d} & x > 1, \\ 0 & \text{otherwise.} \end{cases}$

(b) $f(x) = ce^x(1+e^x)^{-2}$, $x \in \mathbb{R}$.

5. 解：(a) $\int_1^{+\infty} f(x)\,dx = \int_1^{+\infty} cx^{-d}\,dx = c\int_1^{+\infty} x^{-d}\,dx$

① $d>1$ $\quad \int_1^{+\infty} f(x)\,dx = c\cdot \frac{1}{1-d} x^{1-d}\Big|_1^{+\infty} = \frac{c}{d-1} = 1 \Rightarrow c = d-1$

② $d=1$ $\quad \int_1^{+\infty} f(x)\,dx = c\cdot \ln x\Big|_1^{+\infty} = 1 \Rightarrow c$ 不存在

③ $d<1$ $\quad \int_1^{+\infty} f(x)\,dx = c\cdot \frac{1}{1-d} x^{1-d}\Big|_1^{+\infty} = 1 \Rightarrow c$ 不存在

(b) $\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{+\infty} ce^x(1+e^x)^{-2}\,dx = c\int_{-\infty}^{+\infty} e^x(1+e^x)^{-2}\,dx$

$= c\cdot \frac{e^x}{1+e^x}\Big|_{-\infty}^{+\infty} = c = 1$

2.4

2. Truncation. Let X be a random variable with distribution function F, and let $a < b$. Sketch the distribution functions of the 'truncated' random variables Y and Z given by

$$Y = \begin{cases} a & \text{if } X < a, \\ X & \text{if } a \le X \le b, \\ b & \text{if } X > b, \end{cases} \qquad Z = \begin{cases} X & \text{if } |X| \le b, \\ 0 & \text{if } |X| > b. \end{cases}$$

Indicate how these distribution functions behave as $a \to -\infty$, $b \to \infty$.

证：$a \to -\infty$. $b \to \infty$. $\quad Y, Z \longrightarrow X$

$F_Y(x), F_Z(x) \longrightarrow F(x)$.

3.1

1. For what values of the constant C do the following define mass functions on the positive integers $1, 2, \ldots$?

(a) Geometric: $f(x) = C2^{-x}$.

(b) Logarithmic: $f(x) = C2^{-x}/x$.

(c) Inverse square: $f(x) = Cx^{-2}$.

(d) 'Modified' Poisson: $f(x) = C2^{x}/x!$.

2. For a random variable X having (in turn) each of the four mass functions of Exercise (3.1.1), find:

(i) $\mathbb{P}(X > 1)$,

(ii) the most probable value of X,

(iii) the probability that X is even.

1.解：(b) $\sum_{k=1}^{+\infty} f(k) = C\sum_{k=1}^{+\infty} \frac{2^{-k}}{k} = C\cdot(-\ln(1-\frac{1}{2})) = \ln 2\cdot C = 1 \Rightarrow C = \frac{1}{\ln 2}$

(利用 $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots \Rightarrow -\ln(1-\frac{1}{x}) = \sum_{k=1}^{+\infty} \frac{x^{-k}}{k}$) ①

(d) $\sum_{k=1}^{+\infty} f(k) = C\sum_{k=1}^{+\infty} \frac{2^k}{k!} = C\cdot(e^2-1) = 1 \Rightarrow C = \frac{1}{e^2-1}$

2.解：对(b)，$f(x) = \frac{1}{\ln 2}\cdot\frac{2^{-x}}{x}$，$x = 1, 2, \cdots$

① Taylor展开

(i) $P(X>1) = 1 - P(X=1) = 1 - \frac{1}{2\ln 2}$

(ii) 令 $g(x) = 2^x\cdot x$，$x = 1, 2\cdots$ 求 $\min g(x)$，则 $X = 1$

(iii) $\sum_{k=1}^{\infty} P(X=2k) = \sum_{k=1}^{\infty} \frac{1}{\ln 2}\cdot\frac{2^{-2k}}{2k} = \frac{1}{2\ln 2}\sum_{k=1}^{\infty}\frac{2^{-2k}}{k} = \frac{1}{2\ln 2}\cdot(-\ln(1-\frac{1}{2^2})) = \frac{\ln 4 - \ln 3}{\ln 4}$

对(d)，$f(x) = \frac{1}{e^2-1}\cdot\frac{2^x}{x!}$，$x = 1, 2, \cdots$

(i) $P(X>1) = 1 - P(X=1) = 1 - \frac{2}{e^2-1}$

(ii) 令 $g(x) = \frac{2^x}{x!}$，求 $\max g(x)$。

当 $x>1$ 时，$\frac{2^{x+1}}{(x+1)!} \Big/ \frac{2^x}{x!} = \frac{2}{x+1} < 1$，而 $g(1) = g(2) = 2 > g(3) > \cdots$ $X = 1$ or 2

(iii) $\sum_{k=1}^{\infty} P(X=2k) = \frac{1}{e^2-1}\sum_{k=1}^{\infty}\frac{2^{(2k)}}{(2k)!} = \frac{1}{e^2-1}\cdot(\cosh 2 - 1) = \frac{1}{e^2-1}\cdot(\frac{e^2+e^{-2}}{2} - 1) = \frac{e^2-1}{2e^2}$ ①

3.2

2. Let X and Y be independent random variables taking values in the positive integers and having the same mass function $f(x) = 2^{-x}$ for $x = 1, 2, \ldots$. Find:

(a) $\mathbb{P}(\min\{X, Y\} \le x)$, (b) $\mathbb{P}(Y > X)$,
(c) $\mathbb{P}(X = Y)$, (d) $\mathbb{P}(X \ge kY)$, for a given positive integer k,
(e) $\mathbb{P}(X \text{ divides } Y)$, (f) $\mathbb{P}(X = rY)$, for a given positive rational r.

2. 解：(a) $F(x) = 1 - 2^{-x}$. $P(X > x) = 2^{-x}$

② 计算补集更方便

$$P(\min\{X,Y\} \le x) = 1 - P(X > x, Y > x) = 1 - P(X > x)P(Y > x) = 1 - 2^{-x} \cdot 2^{-x} = 1 - 4^{-x}$$

(b) 易知 $P(Y > X) = P(X > Y)$

$$P(X = Y) = \sum_{x=1}^{+\infty} P(X = Y = x) = \sum_{x=1}^{+\infty} P(X = x)P(Y = x) = \sum_{x=1}^{+\infty} 2^{-x} \cdot 2^{-x} = \frac{1}{3}$$

$$\Rightarrow P(Y > X) = \frac{1}{3}$$

(c) 由(b)，$P(X = Y) = \frac{1}{3}$

(d) $P(X \ge x) = 2^{-x+1}$

$$P(X \ge kY) = \sum_{y=1}^{+\infty} P(X \ge ky, Y = y) = \sum_{y=1}^{+\infty} P(X \ge ky)P(Y = y)$$

$$= \sum_{y=1}^{+\infty} 2^{-ky+1} \cdot 2^{-y} = \sum_{y=1}^{+\infty} 2^{-(k+1)y+1} = \frac{2}{2^{k+1} - 1}$$

(e) $P(X \text{ divides } Y) = P(X \mid Y) = \sum_{k=1}^{+\infty} P(Y = kX)$

$$= \sum_{k=1}^{+\infty} \sum_{x=1}^{+\infty} P(Y = kx, X = x) = \sum_{k=1}^{+\infty} \sum_{x=1}^{+\infty} P(X = x)P(Y = kx)$$

$$= \sum_{k=1}^{+\infty} \sum_{x=1}^{+\infty} 2^{-x} \cdot 2^{-kx} = \sum_{k=1}^{+\infty} \frac{1}{2^{k+1} - 1} \quad / \sum_{x=1}^{+\infty} \frac{1}{2^x(2^x - 1)}$$

(f) 令 $r = \frac{m}{n}$, $m, n \in N_+$ ③ $r \in R$，注意区分(e)(f)

$$P(X = rY) = \sum_{k=1}^{+\infty} P(X = mk, Y = nk) = \sum_{k=1}^{+\infty} 2^{-(m+n)k} = \frac{1}{2^{m+n} - 1}$$

3. Let X_1, X_2, X_3 be independent random variables taking values in the positive integers and having mass functions given by $\mathbb{P}(X_i = x) = (1 - p_i)p_i^{x-1}$ for $x = 1, 2, \ldots$, and $i = 1, 2, 3$.

(a) Show that

$$\mathbb{P}(X_1 < X_2 < X_3) = \frac{(1 - p_1)(1 - p_2)p_2 p_3^2}{(1 - p_2 p_3)(1 - p_1 p_2 p_3)}.$$

(b) Find $\mathbb{P}(X_1 \le X_2 \le X_3)$.

3. 解：(a) $P(X_1 < X_2 < X_3) = \sum_{i<j<k} P(X_1 = i, X_2 = j, X_3 = k)$ ④ 很多同学做复杂了！

$$= \sum_{i<j<k} (1-p_1)(1-p_2)(1-p_3) p_1^{i-1} p_2^{j-1} p_3^{k-1}$$

$$= \sum (1-p_1)(1-p_2) p_1^{i-1} p_2^{j-1} p_3^{j}$$

$i<j$

$$= \sum_{i} \frac{(1-p_1)(1-p_2)p_1^{i-1}(p_2p_3)^{i}p_3}{1-p_2p_3}$$

$$= \frac{(1-p_1)(1-p_2)p_2p_3^{2}}{(1-p_2p_3)(1-p_1p_2p_3)}$$

(b) $$P(X_1 \le X_2 \le X_3) = \sum_{i \le j \le k} (1-p_1)(1-p_2)(1-p_3)p_1^{i-1}p_2^{j-1}p_3^{k-1}$$

$$= \sum_{i \le j} (1-p_1)(1-p_2)\, p_1^{i-1}p_2^{j-1}p_3^{j-1}$$

$$= \sum_{i} \frac{(1-p_1)(1-p_2)p_1^{i-1}(p_2p_3)^{i-1}}{1-p_2p_3}$$

$$= \frac{(1-p_1)(1-p_2)}{(1-p_2p_3)(1-p_1p_2p_3)}$$