1 设V中的向量组$\alpha_1,\cdots,\alpha_m$线性无关，并且可以被向量组$\beta_1,\cdots,\beta_n$线性表示. 证明:
(1) $m\le n$; (2) 可以用$\alpha_1,\cdots,\alpha_m$替换$\beta_1,\cdots,\beta_n$中的m个向量，不妨设为$\beta_1,\cdots,\beta_m$，使得$\alpha_1,\cdots,\alpha_m,\beta_{m+1},\cdots,\beta_n$与$\beta_1,\cdots,\beta_n$等价.

(1) 证明:

$$m = \dim\langle \alpha_1,\cdots\alpha_m\rangle \le \dim\langle\beta_1\cdots\beta_n\rangle \le n.$$

(2) 对 m 归纳.

① $m=1$. 不妨 $\alpha_1 = \lambda_1\beta_1+\cdots+\lambda_n\beta_n$, $\lambda_1\neq 0$.

则 $\beta_1 = \frac{1}{\lambda_1}(\alpha_1-\lambda_2\beta_2-\cdots-\lambda_n\beta_n)$.

则 $\langle\alpha_1,\beta_2,\cdots,\beta_n\rangle = \langle\beta_1,\cdots,\beta_n\rangle$.

② 设结论对 $m=k-1$ 成立.

③ $m=k$:

由归纳假设, 存在 $\beta_1\cdots,\beta_{k-1}$, 使得

$\langle\beta_1\cdots\beta_n\rangle = \langle\alpha_1\cdots\alpha_{k-1},\beta_k,\cdots,\beta_n\rangle$.

此时 $\alpha_k = \mu_1\alpha_1+\cdots+\mu_{k-1}\alpha_{k-1}+\mu_k\beta_k+\cdots+\mu_n\beta_n$.

又由 $\alpha_1\cdots\alpha_k$ 线性无关. 知 $\mu_k,\cdots,\mu_n$ 不全为零,

不妨设 $\mu_k\neq 0$.

则 $\beta_k = \frac{1}{\mu_k}(\alpha_k-\mu_1\alpha_1-\cdots-\mu_{k-1}\alpha_{k-1}-\mu_{k+1}\beta_{k+1}-\cdots-\mu_n\beta_n)$.

则有 $\langle\alpha_1\cdots\alpha_k,\beta_{k+1},\cdots\beta_n\rangle = \langle\beta_1\cdots\beta_n\rangle$.

归纳成立. $\square$

Rk: $\langle \alpha_1 \cdots \alpha_m \rangle \subseteq \langle \beta_1 \cdots \beta_n \rangle$.

$\exists$ $i_1 \cdots i_m$, s.t

$$\langle \alpha_1 \cdots \alpha_n \rangle = \langle \beta_{i_1} \cdots \beta_{i_m} \rangle.$$

反例: $\alpha_1 = \mathbb{R}(1,1)$ $\quad$ $\beta_1 = \mathbb{R}(1,0)$

$\beta_2 = \mathbb{R}(0,1)$.

2 设$V=F^{2n}$,
$V_1=\{(a_1,\ldots,a_{2n})\,|\,a_i=a_{i+n},1\le i\le n\}$,
$V_2=\{(a_1,\ldots,a_{2n})\,|\,a_i=-a_{i+n},1\le i\le n\}$.
证明:V_1,V_2是V的子空间, 且$V=V_1\oplus V_2$.

证明: ① $\forall\alpha,\beta\in V_1$, $\lambda\in K$.

易见 $\alpha+\beta\in V_1$, $\lambda\alpha\in V_1$.

则 V_1 为 V 子空间.

同理, V_2 为 V 子空间.

② $V=V_1\oplus V_2$:

• $V=V_1+V_2$.

$\forall\ x\in V$, $x=(x_1\cdots x_{2n})$.

则 $x=\left(\frac{x_1+x_{n+1}}{2},\cdots,\frac{x_n+x_{2n}}{2},\frac{x_1+x_{n+1}}{2},\cdots,\frac{x_n+x_{2n}}{2}\right)$

$+\left(\frac{x_1-x_{n+1}}{2},\cdots,\frac{x_n-x_{2n}}{2},\frac{x_{n+1}-x_1}{2},\cdots,\frac{x_{2n}-x_n}{2}\right)$.

• $\forall\ y\in V_1\cap V_2$. ⟵ 很多人没写.

$y_i=y_{n+i}=-y_{n+i}\Rightarrow y=0$.

□

Rk: ● : 不论是否显然, 都至少要提一下!

• 直和"$\oplus$"不要乱写! 要先说明是直和.

很多人没写 $V_1\cap V_2=\{0\}$ 就写出了"$V_1\oplus V_2$"..
(不是φ).

(若有 $V_1+V_2=V$, $\dim V_1+\dim V_2=\dim V$
可得 $V_1\oplus V_2=V$)

$\Big(\forall\, x \in V_1 + V_2 = V,$

$x = x_1 + x_2$. 若此分解唯一，则 $V = V_1 \oplus V_2 \Big)$.

3 设V_1, V_2, V_3是线性空间V的子空间. 证明:
$V_1 \cap (V_2 + V_1 \cap V_3) = V_1 \cap V_2 + V_1 \cap V_3$.
并举例说明等式$V_1 \cap (V_2 + V_3) = V_1 \cap V_2 + V_1 \cap V_3$不一定成立.

证明: 法一:

已知: $(V_1' + V_2') \cap (V_1' + V_3')$

$= V_1' + (V_1' + V_2') \cap V_3'$. (例题)

回到本题, 取 $V_1' = V_1 \cap V_3$

$V_2' = V_1$

$V_3' = V_2$.

则 $V_1 \cap (V_2 + V_1 \cap V_3)$

$= (V_1 + V_1 \cap V_3) \cap (V_2 + V_1 \cap V_3)$

$= V_1 \cap V_3 + (V_1 + V_1 \cap V_3) \cap V_2$

$= V_1 \cap V_3 + V_1 \cap V_2$. $\square$

法二: "$\supseteq$" 显然. 仅在 dim $< \infty$ 时成立.

$\dim(V_1 \cap (V_2 + V_1 \cap V_3))$

$= \dim(V_1) + \dim(V_2 + V_1 \cap V_3) - \dim(V_1 + V_2 + V_1 \cap V_3)$

$= \dim(V_1) + \dim(V_2) + \dim(V_1 \cap V_3)$

$- \dim(V_1 \cap V_2 \cap V_3) - \dim(V_1 + V_2)$

$= \dim(V_1 \cap V_2) + \dim(V_1 \cap V_3)$

$- \dim(V_1 \cap V_2 \cap V_3)$

$= \dim(V_1 \cap V_2 + V_1 \cap V_3)$.

法二："$\supseteq$" 显然.

$\forall \alpha \in V_1 \cap (V_2 + V_1 \cap V_3) = \beta + \gamma$.

$\beta \in V_2$, $\gamma \in V_1 \cap V_3$, $\Rightarrow \beta = \alpha - \gamma \in V_1$

$\Rightarrow \beta \in V_1 \cap V_2 \Rightarrow \alpha \in V_1 \cap V_2 + V_1 \cap V_3$.

$\square$

反例：$\mathbb{R}^2$. $V_1 = \mathbb{R}(1,1)$

$V_2 = \mathbb{R}(1,0)$

$V_3 = \mathbb{R}(0,1)$

则 $V_1 \cap (V_2 + V_3) = V_1 \cap \mathbb{R}^2 = V_1$

$V_1 \cap V_2 = (0,0) = V_1 \cap V_3$.

则 $V_1 \cap V_2 + V_1 \cap V_3 = 0 \neq V_1 \cap (V_2 + V_3)$

$\square$

· $\mathbb{Z}$ 不是线性空间.