$$\bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty} A_m = \liminf_{n\to\infty} A_n = \{A_n^c \ \text{i.o.}\}^c$$

定理:$\{A_n\}$事件列

(1) 若 $\sum_{n=1}^{\infty} P(A_n) < \infty$, 则 $P(A_n \ \text{i.o.}) = 0$

(2) 若 $\{A_n\}$相互独立, $\sum_{n=1}^{\infty} P(A_n) = \infty \iff P(A_n \ \text{i.o.}) = 1$

证: (1) $P(\bigcup_{m=n}^{\infty} A_m) \le \sum_{m=n}^{\infty} P(A_m) \to 0 \quad n\to\infty$

$P(A_n \ \text{i.o.}) = \lim_{n\to\infty} P(\bigcup_{m=n}^{\infty} A_m) = 0$

(2) "$\Leftarrow$" 显然

"$\Rightarrow$" $P(\bigcup_{m=n}^{+\infty} A_m) = 1 - P(\bigcap_{m=n}^{\infty} A_m^c)$. $P(\bigcap_{m=n}^{\infty} A_m^c) = \lim_{r\to\infty} P(\bigcap_{m=n}^{r} A_m^c) = \lim_{r\to\infty} \prod_{m=n}^{r} (1-P(A_m))$

$\le \lim_{r\to\infty} \prod_{m=n}^{r} e^{-P(A_m)} = \lim_{r\to\infty} e^{-\sum_{m=n}^{r} P(A_m)} \to 0$

$P(\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m) = \lim_{n\to\infty} P(\bigcup_{m=n}^{\infty} A_m) = 1$

若不加独立条件: $\Omega = (0,1)$ $\quad A_n = (0, \frac{1}{n})$ $\quad \sum_{n=1}^{\infty} P(A_n) = \infty$ 而 $\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m = \phi$

$X_n \xrightarrow{a.s.} X$ 对$\forall \varepsilon > 0$, $P(\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} |X_n - X| > \varepsilon) = 0 \iff P(|X_n - X| > \varepsilon \ \text{i.o.}) = 0 \Leftarrow \sum_{n=1}^{\infty} P(|X_n - X| > \varepsilon) < \infty$.

例 $\{X_n\}$独立同分布 $E(X_i) = \mu$ $\quad E(X_i^4) < \infty$. $S_n = \sum_{k=1}^{n} X_k$, 则 $\frac{S_n}{n} \xrightarrow{a.s.} \mu$

证: 不妨设$\mu = 0$ ($\mu \ne 0$, $S_n' = \frac{S_n - n\mu}{n} \xrightarrow{a.s.} 0$)

$$E[S_n^4] = \sum_{i=1}^{n} E[X_i]^4 + \sum_{i\ne j} E(X_i^2 X_j^2) + \sum_{i\ne j\ne k} E(X_i^2 X_j X_k) + \sum_{i\ne j} E(X_i^3 X_j) + \sum_{i\ne j\ne k\ne l} E(X_i X_j X_k X_l)$$

$$= nE(X_i^4) + \frac{n(n-1)}{2}\cdot\frac{4!}{2\cdot 2}(E[X_i^2])^2 \le (n + 3n(n-1))E(X_i^4) \le c\cdot n^2 E(X_i^4)$$

$$\sum_{n=1}^{\infty} P(|\frac{S_n}{n}| > \varepsilon) \le \sum_{n=1}^{\infty} \frac{E((\frac{S_n}{n})^4)}{\varepsilon^4} = \sum_{n=1}^{\infty} \frac{c\cdot n^2 E(X_i^4)}{n^4 \varepsilon^4} < \infty$$

$P(\frac{S_n}{n} > \varepsilon \ \text{i.o.}) = 0 \qquad \frac{S_n}{n} \xrightarrow{a.s.} 0$.

hw: 7.11.2 (2), 7.11.4, 7.11.7, 7.11.8

§7.3 大数定律

Khinchin LLN $\{X_n\}$独立同分布. $E[X_n] = \mu$, $S_n = X_1 + \cdots + X_n$, $\frac{S_n}{n} - \mu = \frac{S_n - E(S_n)}{n} \xrightarrow{D} 0$ (常数)

则 $\frac{S_n}{n} - \mu \xrightarrow{P} 0$.

Bernoulli LLN

$X_n \begin{array}{c|c} 1 & 0 \\ \hline p & 1-p \end{array}$ $\quad \dfrac{\sum_{k=1}^{n} X_k}{n} \xrightarrow{P} p(X_n=1)$

Chebyshev LLN

$\{X_n\}$两两不相关. $Var(X_n)<K$, 则 $\dfrac{\sum_{i=1}^{n} X_i - E(\sum_{i=1}^{n} X_i)}{n} \xrightarrow{P} 0$

证: 记 $S_n=\sum_{i=1}^{n} X_i$. 对$\forall \varepsilon>0$, $P(|\dfrac{S_n-ES_n}{n}|>\varepsilon) \le \dfrac{1}{\varepsilon^2} E[(\dfrac{S_n-ES_n}{n})^2] = \dfrac{1}{n^2\varepsilon^2}\sum_{i=1}^{n} E[(X_i-EX_i)^2]$

$\le \dfrac{k\cdot n}{n^2\varepsilon^2} \to 0$

Markov LLN

$\dfrac{1}{n^2} Var(\sum_{k=1}^{n} X_k) \to 0$, 则 $\dfrac{S_n-ES_n}{n} \xrightarrow{P} 0$

定理 $\{X_n\}$独立.

$Y_{n,k} = \begin{cases} X_k, & |X_k| \le n \\ 0, & |X_k|>n \end{cases}$ $\quad$ 记 $a_n=\sum_{k=1}^{n} E(Y_{n,k})$, $b_n=n$

$S_n = X_1+\cdots+X_n$

满足 (1) $\sum_{k=1}^{n} P(|X_k|>n) \to 0$ (2) $\dfrac{1}{n^2}\sum_{k=1}^{n} E(Y_{n,k}^2) \to 0$ 则 $\dfrac{S_n-a_n}{b_n} \xrightarrow{P} 0$

证: 记 $S_n^*=\sum_{k=1}^{n} Y_{n,k}$. 对$\forall\varepsilon>0$, $P(|S_n-S_n^*|>\varepsilon) \le P(S_n\ne S_n^*) \le P(\bigcup_{k=1}^{n}\{X_k\ne Y_{n,k}\})$

$\le \sum_{k=1}^{n} P(X_k\ne Y_{n,k}) = \sum_{k=1}^{n} P(|X_k|>n) \to 0$

$\therefore S_n \xrightarrow{P} S_n^*$ $\quad P(|\dfrac{S_n^*-a_n}{b_n}|>\varepsilon) = P(|\dfrac{S_n^*-E(S_n^*)}{n}|>\varepsilon) \le \dfrac{1}{n^2\varepsilon^2} Var(S_n^*) \to 0$

$\dfrac{S_n-a_n}{n} = \dfrac{S_n-S_n^*}{n} + \dfrac{S_n^*-a_n}{n} \xrightarrow{P} 0$

二. 强大数律 SLLN

$\{X_n\}$ $\quad S_n=X_1+\cdots+X_n$ $\quad \{a_n\}$ $\quad b_n>0$ $\quad b_n\uparrow+\infty$ 若 $\dfrac{S_n-a_n}{b_n} \xrightarrow{a.s.} 0$ 称$\{X_n\}$服从强大数律.

B-C引理. 不等式 *

定理 $\{X_n\}$独立同分布(即 i.i.d.) $E(X_i^2)<\infty$, $E[X_i]=\mu$

则 (1) $\dfrac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{2} \mu$ (2) $\dfrac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{a.s.} \mu$

证：(1) $E(|\frac{1}{n}\sum_{i=1}^{n}X_i-\mu|^2)=\frac{1}{n^2}E((\sum_{i=1}^{n}(X_i-\mu))^2)\overset{独立}{=}\frac{1}{n^2}Var(X_1)\cdot n\to 0$

(2) 取 $n_k=k^2$

$$P(|\frac{S_{n_k}}{n_k}-\mu|>\varepsilon)\le\frac{1}{(n_k\varepsilon)^2}E((S_{n_k}-n_k\mu)^2)=\frac{1}{(n_k\varepsilon)^2}n_k Var(X_1)=\frac{1}{k^2\varepsilon^2}Var(X_1)$$

$$\sum_{k=1}^{\infty}P(|\frac{S_{n_k}}{n_k}-\mu|>\varepsilon)\le\sum_{k=1}^{\infty}\frac{Var(X_1)}{k^2\varepsilon^2}<\infty\Rightarrow P(\{|\frac{S_{n_k}}{n_k}-\mu|>\varepsilon\}\ i.o)=0\Rightarrow\frac{S_{n_k}}{n_k}-\mu\xrightarrow{a.s.}0$$

分类讨论 若 $X_1\ge0$. $\forall n,\exists k,k^2\le n<(k+1)^2$

$\mu\xleftarrow{a.s}\frac{S_{k^2}}{(k+1)^2}\le\frac{S_n}{n}\le\frac{S_{(k+1)^2}}{k^2}=\frac{S_{(k+1)^2}}{(k+1)^2}\cdot\frac{(k+1)^2}{k^2}\xrightarrow{a.s.}\mu$ 则 $\frac{S_n}{n}\xrightarrow{a.s.}\mu$

X_1 不定号 $X_1=X_1^+-X_1^-$

$X_1^+=\begin{cases}X_1, & X_1\ge0\\ 0, & X_1<0\end{cases}$, $X_1^-=\begin{cases}-X_1, & X_1<0\\ 0, & X_1\ge0\end{cases}$

$E[|X_1|]<\infty$. $E[X_1^+]$. $E[X_1^-]<\infty$. $\frac{\sum_{k=1}^{n}X_k^+}{n}\xrightarrow{a.s.}E[X_1^+]$, $\frac{\sum_{k=1}^{n}X_k^-}{n}\xrightarrow{a.s.}E[X_1^-]$

相减可得 $\frac{S_n}{n}\xrightarrow{a.s.}E(X_1^+-X_1^-)$

独立情形

定理 $\{X_n\}$ 相互独立. $\sum_{n=1}^{\infty}\frac{Var(X_n)}{n^2}<\infty$. $S_n=X_1+\cdots+X_n$. 则 $\frac{S_n-E(S_n)}{n}\xrightarrow{a.s.}0$

Kolmogrov 强大数律.

(1) 先证 Kolmogrov 不等式. $\{X_n\}$ 独立. $Var(X_i)<\infty.\forall i.$

对 $\forall\varepsilon>0$. $P(\max_{1\le m\le n}|\sum_{k=1}^{m}(X_k-EX_k)|\ge\varepsilon)\le\frac{1}{\varepsilon^2}\sum_{i=1}^{n}Var(X_i)$

记 $\tilde{S}_n=\sum_{i=1}^{n}(X_i-EX_i)$

$\{\max_{1\le m\le n}\sum_{k=1}^{m}|X_k-EX_k|>\varepsilon\}=\bigcup_{m=1}^{n}\{|\tilde{S}_1|<\varepsilon,|\tilde{S}_2|<\varepsilon,\cdots,|\tilde{S}_{m-1}|<\varepsilon,|\tilde{S}_m|\ge\varepsilon\}\triangleq\bigcup_{m=1}^{n}A_m$

$P(\max_{1\le m\le n}|\sum_{k=1}^{m}(X_k-EX_k)|\ge\varepsilon)=\sum_{m=1}^{n}P(A_m)$

$E[\tilde{S}_n^2]=E[(\sum_{i=1}^{n}(X_i-EX_i))^2]=E(\sum_{i=1}^{n}(X_i-EX_i)^2)=Var(X_i)$

$E[\tilde{S}_n^2]\ge E(\tilde{S}_n^2\cdot\sum_{m=1}^{n}I_{A_m}]=\sum_{m=1}^{n}E(\tilde{S}_n^2\cdot I_{A_m})$

$=\sum_{m=1}^{n}E(\tilde{S}_m^2\cdot I_{A_m}+\sum_{i=m+1}^{n}(X_i-EX_i)^2I_{A_m})+2E[\underline{\tilde{S}_m\cdot I_{A_m}}\cdot\underline{\sum_{i=m+1}(X_i-EX_i)}]\underset{独立}{=0}$

$\ge\sum_{m=1}^{n}E(\tilde{S}_m^2\cdot I_{A_m})\ge\sum_{m=1}^{n}\varepsilon^2P(A_m)\Rightarrow\sum_{m=1}^{n}P(A_m)\le\frac{1}{\varepsilon^2}E[\tilde{S}_n^2]$ 得证.

(2) 证 Kolmogrov 强大数律. (SLLN)

对 $\forall \varepsilon>0$. 令 $Y_n = \frac{\tilde{S}_n}{n}$.

$$P\left(\max_{2^m\le n<2^{m+1}} |Y_n|>\varepsilon\right) \le P\left(\max_{2^m\le n<2^{m+1}} |\tilde{S}_n| > 2^m\cdot\varepsilon\right) \le P\left(\max_{1\le n<2^{m+1}} |\tilde{S}_n| > 2^m\cdot\varepsilon\right)$$

$$\le \frac{1}{(2^m\varepsilon)^2}\sum_{i=1}^{2^{m+1}-1} Var(X_i)$$

$$\sum_{m=1}^{+\infty} P\left(\max_{2^m\le n<2^{m+1}} |Y_n|>\varepsilon\right) \le \sum_{m=1}^{+\infty} \frac{1}{(2^m\varepsilon)^2}\sum_{i=1}^{2^{m+1}-1} Var(X_i) = \sum_{i=1}^{\infty} \frac{1}{\varepsilon^2} Var(X_i) \sum_{m=m(i)}^{\infty} \frac{1}{2^{2m}} < \infty$$

$m(i) = \min\{m : i < 2^{m+1}\}$

hw: 7.4.1, 7.11.17, 7.11.20