§0.1 Gauss方程与Gauss绝妙定理

回顾

$$\frac{\partial\Gamma^{\xi}_{\beta\alpha}}{\partial u^{\gamma}}-\frac{\partial\Gamma^{\xi}_{\gamma\alpha}}{\partial u^{\beta}}+\Gamma^{\xi}_{\gamma\eta}\Gamma^{\eta}_{\beta\alpha}-\Gamma^{\xi}_{\beta\eta}\Gamma^{\eta}_{\gamma\alpha}=b^{\xi}_{\gamma}b_{\beta\alpha}-b^{\xi}_{\beta}b_{\gamma\alpha}.\quad(Gauss)$$

在Gauss方程中，左边只与第一基本形式的系数及其偏导数有关。按照Riemann的记号，定义黎曼曲率张量

$$R_{\gamma\beta}{}^{\xi}{}_{\alpha}\ :=\ \frac{\partial\Gamma^{\xi}_{\beta\alpha}}{\partial u^{\gamma}}-\frac{\partial\Gamma^{\xi}_{\gamma\alpha}}{\partial u^{\beta}}+\Gamma^{\xi}_{\gamma\eta}\Gamma^{\eta}_{\beta\alpha}-\Gamma^{\xi}_{\beta\eta}\Gamma^{\eta}_{\gamma\alpha},$$

特别

$$\nabla_{\frac{\partial}{\partial u^{\beta}}}r_{\alpha}:=[r_{\alpha\beta}]^{T}=\Gamma^{\xi}_{\beta\alpha}r_{\xi},$$

$$\begin{aligned}&\nabla_{\frac{\partial}{\partial u^{\gamma}}}(\nabla_{\frac{\partial}{\partial u^{\beta}}}r_{\alpha})-\nabla_{\frac{\partial}{\partial u^{\beta}}}(\nabla_{\frac{\partial}{\partial u^{\gamma}}}r_{\alpha})\\ =\ &(\frac{\partial\Gamma^{\xi}_{\beta\alpha}}{\partial u^{\gamma}}-\frac{\partial\Gamma^{\xi}_{\gamma\alpha}}{\partial u^{\beta}}+\Gamma^{\xi}_{\gamma\eta}\Gamma^{\eta}_{\beta\alpha}-\Gamma^{\xi}_{\beta\eta}\Gamma^{\eta}_{\gamma\alpha})r_{\xi}\\ =\ &R_{\gamma\beta}{}^{\xi}{}_{\alpha}r_{\xi}.\end{aligned}$$

利用第一基本形式降指标定义黎曼曲率张量的另一种形式(可以更方便讨论其对称性质)，即

$$\begin{aligned}R_{\gamma\beta\delta\alpha}:=g_{\delta\xi}R_{\gamma\beta}{}^{\xi}{}_{\alpha}\ &=\ g_{\delta\xi}(\frac{\partial\Gamma^{\xi}_{\beta\alpha}}{\partial u^{\gamma}}-\frac{\partial\Gamma^{\xi}_{\gamma\alpha}}{\partial u^{\beta}}+\Gamma^{\xi}_{\gamma\eta}\Gamma^{\eta}_{\beta\alpha}-\Gamma^{\xi}_{\beta\eta}\Gamma^{\eta}_{\gamma\alpha})\\ &=\ \langle R_{\gamma\beta}{}^{\xi}{}_{\alpha}r_{\xi},r_{\delta}\rangle\\ &=\ \langle\nabla_{\frac{\partial}{\partial u^{\gamma}}}\nabla_{\frac{\partial}{\partial u^{\beta}}}r_{\alpha}-\nabla_{\frac{\partial}{\partial u^{\beta}}}\nabla_{\frac{\partial}{\partial u^{\gamma}}}r_{\alpha},r_{\delta}\rangle.\end{aligned}$$

从而有Gauss方程的等价形式

$$R_{\gamma\beta\delta\alpha}=b_{\gamma\delta}b_{\beta\alpha}-b_{\gamma\alpha}b_{\beta\delta}.\quad(Gauss)$$

记号：

$$\frac{\partial^{k}g_{\alpha\beta}}{\partial u^{\alpha_k}\cdots\partial u^{\alpha_1}}:=g_{\alpha\beta,\alpha_1\cdots\alpha_k}.$$

Proposition 0.1.

$$R_{\alpha\beta\gamma\delta}=-\frac{1}{2}(\partial_{\alpha}\partial_{\gamma}g_{\beta\delta}+\partial_{\beta}\partial_{\delta}g_{\alpha\gamma}-\partial_{\alpha}\partial_{\delta}g_{\beta\gamma}-\partial_{\beta}\partial_{\gamma}g_{\alpha\delta})-g^{\xi\eta}\Gamma_{\xi\alpha\gamma}\Gamma_{\eta\beta\delta}+g^{\xi\eta}\Gamma_{\xi\alpha\delta}\Gamma_{\eta\beta\gamma}.$$

证明：由定义，直接计算

$$
\begin{aligned}
R_{\alpha\beta\gamma\delta} &= g_{\xi\gamma} R_{\alpha\beta}{}^{\xi}{}_{\delta} = g_{\xi\gamma}\left(\frac{\partial\Gamma^{\xi}_{\beta\delta}}{\partial u^{\alpha}} - \frac{\partial\Gamma^{\xi}_{\alpha\delta}}{\partial u^{\beta}} + \Gamma^{\xi}_{\alpha\eta}\Gamma^{\eta}_{\beta\delta} - \Gamma^{\xi}_{\beta\eta}\Gamma^{\eta}_{\alpha\delta}\right) \\
&= \left(\frac{\partial\Gamma_{\gamma\beta\delta}}{\partial u^{\alpha}} - \Gamma^{\xi}_{\beta\delta}\frac{\partial g_{\xi\gamma}}{\partial u^{\alpha}}\right) - \left(\frac{\partial\Gamma_{\gamma\alpha\delta}}{\partial u^{\beta}} - \Gamma^{\xi}_{\alpha\delta}\frac{\partial g_{\xi\gamma}}{\partial u^{\beta}}\right) + \Gamma_{\gamma\alpha\eta}\Gamma^{\eta}_{\beta\delta} - \Gamma_{\gamma\beta\eta}\Gamma^{\eta}_{\alpha\delta} \\
&= \frac{\partial\Gamma_{\gamma\beta\delta}}{\partial u^{\alpha}} - \frac{\partial\Gamma_{\gamma\alpha\delta}}{\partial u^{\beta}} + \Gamma^{\eta}_{\beta\delta}(\Gamma_{\gamma\alpha\eta} - g_{\eta\gamma,\alpha}) - \Gamma^{\eta}_{\alpha\delta}(\Gamma_{\gamma\beta\eta} - g_{\eta\gamma,\beta}).
\end{aligned}
$$

其中

$$
\begin{aligned}
\Gamma_{\gamma\beta\delta} &= g_{\xi\gamma}\Gamma^{\xi}_{\beta\delta} = g_{\xi\gamma}\frac{1}{2}g^{\xi\eta}(g_{\eta\beta,\delta} + g_{\eta\delta,\beta} - g_{\beta\delta,\eta}) \\
&= \frac{1}{2}(g_{\gamma\beta,\delta} + g_{\gamma\delta,\beta} - g_{\beta\delta,\gamma}).
\end{aligned}
$$

从而

$$
\begin{aligned}
\frac{\partial\Gamma_{\gamma\beta\delta}}{\partial u^{\alpha}} - \frac{\partial\Gamma_{\gamma\alpha\delta}}{\partial u^{\beta}} &= \frac{1}{2}(g_{\gamma\beta,\delta\alpha} + g_{\gamma\delta,\beta\alpha} - g_{\beta\delta,\gamma\alpha}) - \frac{1}{2}(g_{\gamma\alpha,\delta\beta} + g_{\gamma\delta,\beta\alpha} - g_{\alpha\delta,\gamma\beta}) \\
&= -\frac{1}{2}(\partial_\alpha\partial_\gamma g_{\beta\delta} + \partial_\beta\partial_\delta g_{\alpha\gamma} - \partial_\alpha\partial_\delta g_{\beta\gamma} - \partial_\beta\partial_\gamma g_{\alpha\delta})
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{\gamma\alpha\eta} - g_{\eta\gamma,\alpha} &= \frac{1}{2}(g_{\gamma\alpha,\eta} + g_{\gamma\eta,\alpha} - g_{\alpha\eta,\gamma} - 2g_{\eta\gamma,\alpha}) \\
&= -\Gamma_{\eta\alpha\gamma}.
\end{aligned}
$$

因此有

$$
\begin{aligned}
R_{\alpha\beta\gamma\delta} &= -\frac{1}{2}(\partial_\alpha\partial_\gamma g_{\beta\delta} + \partial_\beta\partial_\delta g_{\alpha\gamma} - \partial_\alpha\partial_\delta g_{\beta\gamma} - \partial_\beta\partial_\gamma g_{\alpha\delta}) \\
&\quad -g^{\xi\eta}\Gamma_{\xi\alpha\gamma}\Gamma_{\eta\beta\delta} + g^{\xi\eta}\Gamma_{\xi\alpha\delta}\Gamma_{\eta\beta\gamma}.
\end{aligned}
$$

□

由上述表达式，黎曼曲率张量$R_{\alpha\beta\gamma\delta}$具有如下对称性质。

Proposition 0.2.

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = R_{\gamma\delta\alpha\beta} = -R_{\alpha\beta\delta\gamma}.$$

证明：第一个等号通过互换α与β得到；第二等号通过互换α与γ、β与δ得到；第三个等号通过互换γ与δ得到。 □

在Gauss方程

$$R_{\alpha\beta\gamma\delta} = b_{\alpha\gamma}b_{\beta\delta} - b_{\alpha\delta}b_{\beta\gamma} \quad (Gauss)$$

中右边具有同样的对称性、反对称性。由Riemann曲率张量的对称性质，Gauss方程中只有一个独立方程

$$R_{1212} = b_{11}b_{22} - (b_{12})^2. \quad \text{(Gauss)}$$

定理0.3. *Gauss绝妙定理(Theorem Egregium, 1827年)*:

$$K = \frac{R_{1212}}{EG - F^2} = \frac{1}{2} g^{\alpha\gamma} g^{\beta\delta} R_{\alpha\beta\gamma\delta}.$$

Gauss绝妙定理说明虽然Gauss曲率通过第一、第二基本形式来定义，但它只依赖于第一基本形式。它揭示了曲面的内蕴几何(即只与第一基本形式有关)。在此基础上，Riemann于1854年创立了黎曼几何。他在高维空间中引入正定的对称二次微分形式(即黎曼度量)和黎曼曲率张量。

证明：第一个等号直接由Gauss曲率的定义和Gauss公式可得。验证第二个等号如下：

$$g^{\alpha\gamma} g^{\beta\delta} R_{\alpha\beta\gamma\delta} = g^{\alpha\gamma} g^{\beta\delta} (b_{\alpha\gamma} b_{\beta\delta} - b_{\alpha\delta} b_{\beta\gamma}) = b_\alpha^\alpha b_\beta^\beta - b_\alpha^\beta b_\beta^\alpha,$$

其中

$$b_\alpha^\alpha b_\beta^\beta = tr(W) tr(W) = (k_1 + k_2)^2,$$

$$b_\alpha^\beta b_\beta^\alpha = tr(W^2) = k_1^2 + k_2^2.$$

从而

$$g^{\alpha\gamma} g^{\beta\delta} R_{\alpha\beta\gamma\delta} = (k_1 + k_2)^2 - (k_1^2 + k_2^2) = 2k_1k_2 = 2K.$$

□

Gauss绝妙定理的直接证明：Gauss曲率表达式为

$$K = \frac{LN - M^2}{EG - F^2},$$

其中

$$L = \langle r_{uu}, N \rangle = \frac{1}{|r_u \wedge r_v|} (r_{uu}, r_u, r_v) = \frac{1}{\sqrt{EG - F^2}} \det(r_{uu}, r_u, r_v),$$

$$M = \langle r_{uv}, N \rangle = \frac{1}{|r_u \wedge r_v|} (r_{uv}, r_u, r_v) = \frac{1}{\sqrt{EG - F^2}} \det(r_{uv}, r_u, r_v),$$

$$N = \langle r_{vv}, N \rangle = \frac{1}{|r_u \wedge r_v|} (r_{vv}, r_u, r_v) = \frac{1}{\sqrt{EG - F^2}} \det(r_{vv}, r_u, r_v),$$

从而

$$
\begin{aligned}
(EG-F^2)^2K &= \det(r_{uu},r_u,r_v)\det(r_{vv},r_u,r_v)-\det(r_{uv},r_u,r_v)^2\\
&= \det\begin{pmatrix} r_{uu}^t\\ r_u^t\\ r_v^t\end{pmatrix}\det(r_{vv},r_u,r_v)-\det\begin{pmatrix} r_{uv}^t\\ r_u^t\\ r_v^t\end{pmatrix}\det(r_{uv},r_u,r_v)\\
&= \det\begin{pmatrix} \langle r_{uu},r_{vv}\rangle & \langle r_{uu},r_u\rangle & \langle r_{uu},r_v\rangle\\ \langle r_u,r_{vv}\rangle & \langle r_u,r_u\rangle & \langle r_u,r_v\rangle\\ \langle r_v,r_{vv}\rangle & \langle r_v,r_u\rangle & \langle r_v,r_v\rangle\end{pmatrix}-\det\begin{pmatrix} \langle r_{uv},r_{uv}\rangle & \langle r_{uv},r_u\rangle & \langle r_{uv},r_v\rangle\\ \langle r_u,r_{uv}\rangle & \langle r_u,r_u\rangle & \langle r_u,r_v\rangle\\ \langle r_v,r_{uv}\rangle & \langle r_v,r_u\rangle & \langle r_v,r_v\rangle\end{pmatrix}\\
&= \det\begin{pmatrix} \langle r_{uu},r_{vv}\rangle & \frac12 E_u & \langle r_{uu},r_v\rangle\\ \langle r_u,r_{vv}\rangle & E & F\\ \frac12 G_v & F & G\end{pmatrix}-\det\begin{pmatrix} \langle r_{uv},r_{uv}\rangle & \frac12 E_v & \frac12 G_u\\ \frac12 E_v & E & F\\ \frac12 G_u & F & G\end{pmatrix}
\end{aligned}
$$

考虑两个行列式对第一行展开可知

$$
(EG-F^2)^2K = \det\begin{pmatrix} \langle r_{uu},r_{vv}\rangle-\langle r_{uv},r_{uv}\rangle & \frac12 E_u & \langle r_{uu},r_v\rangle\\ \langle r_u,r_{vv}\rangle & E & F\\ \frac12 G_v & F & G\end{pmatrix}-\det\begin{pmatrix} 0 & \frac12 E_v & \frac12 G_u\\ \frac12 E_v & E & F\\ \frac12 G_u & F & G\end{pmatrix}
$$

其中

$$
\langle r_{uu},r_v\rangle = \frac{\partial}{\partial u}\langle r_u,r_v\rangle-\langle r_u,r_{uv}\rangle = F_u-\frac12 E_v,
$$
$$
\langle r_u,r_{vv}\rangle = \frac{\partial}{\partial v}\langle r_u,r_v\rangle-\langle r_{uv},r_v\rangle = F_v-\frac12 G_u,
$$

$$
\begin{aligned}
\langle r_{uu},r_{vv}\rangle-\langle r_{uv},r_{uv}\rangle &= \frac{\partial}{\partial u}\langle r_u,r_{vv}\rangle-\langle r_u,r_{uvv}\rangle-\frac{\partial}{\partial v}\langle r_u,r_{uv}\rangle+\langle r_u,r_{uvv}\rangle\\
&= \frac{\partial}{\partial u}(F_v-\frac12 G_u)-\frac{\partial}{\partial v}(\frac12 E_v)\\
&= -\frac12 E_{vv}+F_{uv}-\frac12 G_{uu}.
\end{aligned}
$$

因此有

$$
\begin{aligned}
(EG-F^2)^2K &= \det\begin{pmatrix} -\frac12 E_{vv}+F_{uv}-\frac12 G_{uu} & \frac12 E_u & F_u-\frac12 E_v\\ F_v-\frac12 G_u & E & F\\ \frac12 G_v & F & G\end{pmatrix}\\
&\quad -\det\begin{pmatrix} 0 & \frac12 E_v & \frac12 G_u\\ \frac12 E_v & E & F\\ \frac12 G_u & F & G\end{pmatrix}\\
&= (EG-F^2)R_{1212}.
\end{aligned}
$$

□

§0.1.1 Gauss-Codazzi方程在曲面特殊参数坐标系下的化简

(1) (u,v)为正交参数系，即$F=0$。这样的局部坐标系存在。此时

$$R_{1212}=-\sqrt{EG}\{(\frac{(\sqrt{E})_v}{\sqrt{G}})_v+(\frac{(\sqrt{G})_u}{\sqrt{E}})_u\}.$$

从而由

$$R_{1212}=b_{11}b_{22}-(b_{12})^2=LN-M^2$$

可得Gauss方程为

$$K=\frac{R_{1212}}{EG}=-\frac{1}{\sqrt{EG}}\{(\frac{(\sqrt{E})_v}{\sqrt{G}})_v+(\frac{(\sqrt{G})_u}{\sqrt{E}})_u\}=\frac{LN-M^2}{EG}.$$

(2) (u,v)为等温坐标系，即$F=0,E=G$。这样的局部坐标系也存在。令

$$E=G=e^{2f(u,v)},\quad I=e^{2f}(dudu+dvdv)=E(dudu+dvdv)$$

代入得

$$K=-\frac{1}{E}\triangle\log\sqrt{E}=-e^{-2f}(f_{vv}+f_{uu})=e^{-2f}(-\triangle f).$$

注：旋转曲面的自然坐标系为正交坐标系。球面的球坐标为正交坐标系，但不是等温坐标系。球极投影给出的坐标系为等温坐标系。

(3) 无脐点曲面可以通过曲率线给出正交坐标系，在此类坐标系下$F=M=0$(第三章习题29)，如之前讨论(第四章习题6)此时Codazzi方程化简为

$$\begin{cases} L_v=HE_v,\\ N_u=HG_u.\end{cases}$$

例：设曲面S无脐点，Gauss曲率为零。证明S为可展曲面。

证明：由于曲面无脐点，它在每一点确定了两个互相正交的主方向。可选取局部坐标(u,v)使得u,v-线为曲率线(参见do Carmo书)，即r_u,r_v为主方向。从而$F=M=0$(第三章习题29)。进一步(第四章习题6)

$$L_v=HE_v,\quad N_u=HG_u.$$

由假设，曲面无脐点$(k_1\neq k_2)$

$$K=\frac{LN}{EG}=0,\quad H=\frac{1}{2}\frac{LG+NE}{EG}=\frac{1}{2}(\frac{L}{E}+\frac{N}{G}),$$

因此不妨设

$$k_1 = \frac{L}{E} \neq 0, \quad k_2 = \frac{N}{G} = 0.$$

从而$N \equiv 0, H = \frac{1}{2}k_1 \neq 0$，因此由$N_u = HG_u$可得

$$G_u = 0.$$

因为

$$N = \langle r_{vv}, N \rangle = 0,$$

而直纹面$a(u) + vb(u)$同样满足此条件。故接下来判断r_v是否为直纹面的直线方向？
计算

$$\begin{aligned} r_{vv} &= \Gamma^1_{22} r_u + \Gamma^2_{22} r_v \\ &= \frac{1}{2} g^{11} (-\partial_u G) r_u + \Gamma^2_{22} r_v \\ &= \Gamma^2_{22} r_v, \end{aligned}$$

因此

$$r_{vv} \wedge r_v \equiv 0,$$
$$\frac{\partial}{\partial v}\left(\frac{r_v}{|r_v|}\right) \wedge \frac{r_v}{|r_v|} = 0,$$
$$\frac{\partial}{\partial v}\left(\frac{r_v}{|r_v|}\right) = 0,$$

从而r_v沿v-曲线同方向，因此v-曲线为直线。 □

作业：8