# AMS 572 Data Analysis I
# Nonparametric Statistical Methods

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# Nonparametric inference for two independent samples

# Wilcoxon (Mann-Whitney) Rank Sum Test

1. Assume:
   $Y_{11}, \ldots, Y_{1n_1}$ iid $\sim F_1(y)$
   $Y_{21}, \ldots, Y_{2n_2}$ iid $\sim F_2(y)$

   $$H_0 : F_1(y) = F_2(y)$$

   $$H_a : F_1(y) = F_2(y + \Delta)$$

   where $\Delta$ is a constant
2. Pool the two samples
3. Rank them from smallest to largest
4. Compute the sum of the ranks, $W_1$, in group 1

# Wilcoxon Rank Sum Test

- There are $N = n_1 + n_2$ subjects in our study
- Thus there are $\binom{N}{n_1}$ possible outcomes
- Under $H_0$, each is equally likely
- We compute the distribution of $W_1$ by enumeration

# Wilcoxon Rank Sum Test: Example

- A new drug is being test in humans for the first time to assess effect on CD4+ T cells in patients with HIV
- 7 individuals are randomized to 2 groups: control ($n_1 = 3$) or drug ($n_2 = 4$)
- Endpoint is percent change in CD4+ count from baseline
- Null hypothesis is the drug has no effect

$$H_0 : \Delta = 0; H_a : \Delta \neq 0$$

# Wilcoxon Rank Sum Test: Example

- Data: control (65, 73, 69); drug (89, 70, 92, 88)
- There are $\binom{7}{3} = 35$ possible outcomes

# Wilcoxon Rank Sum Test: $n_1 = 3, n_2 = 4$

| Ranks | $W_1$ | Ranks | $W_1$ | Ranks | $W_1$ |
|---|---|---|---|---|---|
| 1,2,3 | 6 | 1,5,6 | 12 | 2,6,7 | 15 |
| 1,2,4 | 7 | 1,5,7 | 13 | 3,4,5 | 12 |
| 1,2,5 | 8 | 1,6,7 | 14 | 3,4,6 | 13 |
| 1,2,6 | 9 | 2,3,4 | 9 | 3,4,7 | 14 |
| 1,2,7 | 10 | 2,3,5 | 10 | 3,5,6 | 14 |
| 1,3,4 | 8 | 2,3,6 | 11 | 3,5,7 | 15 |
| 1,3,5 | 9 | 2,3,7 | 12 | 3,6,7 | 16 |
| 1,3,6 | 10 | 2,4,5 | 11 | 4,5,6 | 15 |
| 1,3,7 | 11 | 2,4,6 | 12 | 4,5,7 | 16 |
| 1,4,5 | 10 | 2,4,7 | 13 | 4,6,7 | 17 |
| 1,4,6 | 11 | 2,5,6 | 13 | 5,6,7 | 18 |
| 1,4,7 | 12 | 2,5,7 | 14 | | |

# Wilcoxon Rank Sum Test: $n_1 = 3, n_2 = 4$

| w | Freq | $F(w)$ | w | Freq | $F(w)$ |
|---|---|---|---|---|---|
| 6 | 1 | 0.0286 | 13 | 5 | 0.6857 |
| 7 | 1 | 0.0571 | 14 | 4 | 0.8000 |
| 8 | 2 | 0.1143 | 15 | 3 | 0.8857 |
| 9 | 3 | 0.2000 | 16 | 2 | 0.9429 |
| 10 | 4 | 0.3142 | 17 | 1 | 0.9714 |
| 11 | 4 | 0.4286 | 18 | 1 | 1 |
| 12 | 4 | 0.5714 | | | |

# Wilcoxon Rank Sum Test: $n_1 = 3, n_2 = 4$

- Note it is impossible to reject $H_0$ for a two-sided alternative when $\alpha = 0.05$.
- Observed $W_1 = 1 + 2 + 4 = 7$; do not reject $H_0$

$$p = 2(0.05714) = 0.1143$$

# Wilcoxon Rank Sum Test

- It can be shown that

$$E(W_1) = \frac{n_1}{N}\frac{N(N+1)}{2} = \frac{n_1(N+1)}{2}$$

- Similarly

$$Var(W_1) = \frac{n_1 n_2(N+1)}{12}$$

# Wilcoxon Rank Sum Test: Large Sample Approx

- If $n_1$ and $n_2$ are large

  $$Z = \frac{W_1 - E(W_1) - 0.5}{\sqrt{Var(W_1)}}$$

  will be approx $N(0,1)$.
- Approximation is good for $n_1, n_2 \geq 12$
- If there are ties

  $$Var(W_1) = \frac{n_1 n_2 (N+1)}{12} - \frac{n_1 n_2}{12N(N-1)} \sum_{i=1}^{q} t_i(t_i - 1)(t_i + 1)$$

  where $q$ equals the number of sets of ties and $t_i$ is the number of observations in the $i$th set

# Mann-Whitney Test

- Consider all $n_1 n_2$ possible pairs

$$(Y_{1i}, Y_{2j}); i = 1, 2, \ldots, n_1; j = 1, 2, \ldots, n_2$$

- Let $U_1$ equal the number of pairs with $Y_{1i} > Y_{2j}$ and $U_2$ equal the number of pairs with $Y_{1i} < Y_{2j}$
- It can be shown that

$$U_1 = W_1 - \frac{n_1(n_1+1)}{2} \qquad U_2 = W_2 - \frac{n_2(n_2+1)}{2}$$

- Mann Whitney test and Wilcoxon rank sum test are equivalent

# Wilcoxon Rank Sum Test: Example

Test whether drug group has higher median than placebo group

| Drug | Rank | Placebo | Rank |
|---|---|---|---|
| 6.9 | **18** | 6.4 | 11 |
| 7.6 | **25.5** | 6.7 | 13 |
| 7.3 | **23.5** | 5.4 | 3 |
| 7.6 | **25.5** | 8.2 | **28.5** |
| 6.8 | **15** | 5.3 | 2 |
| 7.2 | 22 | 6.6 | 12 |
| 8.0 | 27 | 5.8 | **8.5** |
| 5.5 | 4 | 5.7 | **6.5** |
| 5.8 | **8.5** | 6.2 | 10 |
| 7.3 | **23.5** | 7.1 | 21 |
| 8.2 | **28.5** | 7.0 | 20 |
| 6.9 | **18** | 6.9 | **18** |
| 6.8 | **15** | 5.6 | 5 |
| 5.7 | **6.5** | 4.2 | 1 |
| 8.6 | 30 | 6.8 | **15** |

# Wilcoxon Rank Sum Test: Example

- $H_0 : \Delta = 0$; $H_a : \Delta > 0$
- $C_{.05} = \{z : z > 1.645\}$
- $E(W_1) = \frac{15(31)}{2} = 232.5$

# Wilcoxon Rank Sum Test: Example

- Tie correction:
  $q = 7;\ t_1 = t_2 = 2;\ t_3 = t_4 = 3;\ t_5 = t_6 = t_7 = 2$

$$\sum_{i=1}^{q} t_i(t_i - 1)(t_i + 1) = 78$$

$$Var(W_1) = 581.25 - \frac{78(15)^2}{12(30)(29)} = 579.57$$

# Wilcoxon Rank Sum Test: Example

- $w_1 = 290.5$

$$z = \frac{290.5 - 232.5 - 0.5}{\sqrt{579.57}} = 2.388;$$

- Reject $H_0$
- $p = 1 - \Phi(2.388) = 0.00846$

# SAS Code

```
data drug;
input trt $ bp @@;
datalines;
drug 6.9  drug 7.6  drug 7.3  drug 7.6  drug 6.8  drug 7.2  drug 8  dru
;
run;

proc npar1way wilcoxon correct=yes data=drug;
class trt;
var bp;
run;
```

# SAS Output

```
                    The NPAR1WAY Procedure

         Wilcoxon Scores (Rank Sums) for Variable bp
                 Classified by Variable trt

                        Sum of     Expected      Std Dev         Mean
trt            N        Scores     Under H0     Under H0        Score

drug          15        290.50       232.50    24.074239    19.366667
placebo       15        174.50       232.50    24.074239    11.633333

               Average scores were used for ties.


                    Wilcoxon Two-Sample Test

                 Statistic            290.5000

                 Normal Approximation
                 Z                      2.3884
                 One-Sided Pr >  Z      0.0085
                 Two-Sided Pr > |Z|     0.0169

                 t Approximation
                 One-Sided Pr >  Z      0.0118
                 Two-Sided Pr > |Z|     0.0237

           Z includes a continuity correction of 0.5.
```

# R Code and Output

```
> drug <- c(6.9,7.6,7.3,7.6,6.8,7.2,8.0,5.5,5.8,
7.3,8.2,6.9,6.8,5.7,8.6)
> placebo <- c(6.4,6.7,5.4,8.2,5.3,6.6,5.8,5.7,6.2,
7.1,7.0,6.9,5.6,4.2,6.8)
>
> wilcox.test(drug,placebo,exact=FALSE,correct=TRUE,
alternative='greater')

Wilcoxon rank sum test with continuity correction

data:  drug and placebo
W = 170.5, p-value = 0.00846
alternative hypothesis: true location shift is greater than 0
```