AMS 572 Data Analysis I
Power and sample size for two population means

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Based on Z test

Power calculation

$$H_0 : \mu_1 - \mu_2 = \Delta_1$$
$$H_a : \mu_1 - \mu_2 = \Delta_2 \neq \Delta_1$$

$$\text{power} = P(\text{reject } H_0 | H_a)$$

$$
\begin{aligned}
&= P(Z_0 \geq z_{\alpha/2} | \mu_1 - \mu_2 = \Delta_2) + P(Z_0 \leq -z_{\alpha/2} | \mu_1 - \mu_2 = \Delta_2) \\
&= P(\tfrac{\bar{X} - \bar{Y} - \Delta_1}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \geq z_{\alpha/2} | \mu_1 - \mu_2 = \Delta_2) \\
&+ P(\tfrac{\bar{X} - \bar{Y} - \Delta_1}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \leq -z_{\alpha/2} | \mu_1 - \mu_2 = \Delta_2) \\
&= P(\tfrac{\bar{X} - \bar{Y} - \Delta_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \geq z_{\alpha/2} - \tfrac{\Delta_2 - \Delta_1}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} | \mu_1 - \mu_2 = \Delta_2) \\
&+ P(\tfrac{\bar{X} - \bar{Y} - \Delta_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \leq -z_{\alpha/2} - \tfrac{\Delta_2 - \Delta_1}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} | \mu_1 - \mu_2 = \Delta_2) \\
&= P(Z \geq z_{\alpha/2} - \tfrac{\Delta_2 - \Delta_1}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}) + P(Z \leq -z_{\alpha/2} - \tfrac{\Delta_2 - \Delta_1}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}})
\end{aligned}
$$

Sample size determination for a given margin of error E

Based on exact or the large sample approximate z-test ;

$n_1 = n_2 = n$

Suppose the margin of error is E with probability $(1 - \alpha)$

$$P(|(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)| \leq E) = 1 - \alpha$$

$$\frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \dot{\sim} N(0, 1)$$

$$P(-E \leq \bar{X} - \bar{Y} - (\mu_1 - \mu_2) \leq E) = 1 - \alpha$$

$$P(\frac{-E}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \leq \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \leq \frac{E}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}) = 1 - \alpha$$

Sample size determination for a given CI length L

Sample size determination for a given power

$$H_0 : \mu_1 - \mu_2 = \Delta_1$$
$$H_a : \mu_1 - \mu_2 = \Delta_2 > (\text{or} <)\Delta_1$$

$$1 - \beta = \text{power} = P(\text{reject } H_0 | H_a)$$

$$\begin{aligned}
&= P(Z_0 \geq z_\alpha | \mu_1 - \mu_2 = \Delta_2) \\
&= P(\frac{\bar{X} - \bar{Y} - \Delta_1}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n}}} \geq z_\alpha | \mu_1 - \mu_2 = \Delta_2) \\
&= P(\frac{\bar{X} - \bar{Y} - \Delta_2}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n}}} \geq z_\alpha - \frac{\Delta_2 - \Delta_1}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n}}} | \mu_1 - \mu_2 = \Delta_2)
\end{aligned}$$

$$z_\alpha - \frac{\Delta_2 - \Delta_1}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n}}} = -z_\beta$$

Sample size determination for a given power

One-tailed test, exact Z

$$n = \frac{(z_\alpha + z_\beta)^2 (\sigma_1^2 + \sigma_2^2)}{(\Delta_2 - \Delta_1)^2}$$

One-tailed test, approximate Z

$$n \doteq \frac{(z_\alpha + z_\beta)^2 (\sigma_1^2 + \sigma_2^2)}{(\Delta_2 - \Delta_1)^2}$$

Two-tailed test, exact or approximate Z

$$n \doteq \frac{(z_{\alpha/2} + z_\beta)^2 (\sigma_1^2 + \sigma_2^2)}{(\Delta_2 - \Delta_1)^2}$$

Example: A new method of making concrete blocks has been proposed. To test whether or not the new method increases the compressive strength, 5 sample blocks are made by each method.

New Method	14	15	13	15	16
Old Method	13	15	13	12	14

(a) Get a 95% CI for the mean difference of the 2 methods.

(b) At $\alpha = 0.05$, can you conclude the new method is better?

Write a SAS and R program for part (b).

SAS Code

```
data block ;
input method $ strength ;
datalines ;
new 14
new 15
new 13
new 15
new 16
old 13
old 15
old 13
old 12
old 14
;
run ;
```

SAS Code

```
proc univariate data=block normal plot ;
class method ;
var strength ;
run ;

proc ttest data=block sides=U ;
class method ;
var strength ;
run ;
```

The SAS System

Method	Variances	DF	t Value	Pr > t
Pooled	Equal	8	1.66	0.0673
Satterthwaite	Unequal	8	1.66	0.0673

Equality of Variances

Method	Num DF	Den DF	F Value	Pr > F
Folded F	4	4	1.00	1.0000

R Code and Output

```
> new <- c(14,15,13,15,16)
> old <- c(13,15,13,12,14)
> var.test(new,old)

F test to compare two variances

data:  new and old
F = 1, num df = 4, denom df = 4, p-value = 1
alternative hypothesis: true ratio of variances is not equal to
95 percent confidence interval:
 0.1041175 9.6045299
sample estimates:
ratio of variances
                 1
```

R Code and Output

```
> shapiro.test(new)

Shapiro-Wilk normality test

data:  new
W = 0.96086, p-value = 0.814

> shapiro.test(old)

Shapiro-Wilk normality test

data:  old
W = 0.96086, p-value = 0.814
```

R Code and Output

```
> t.test(new,old,var.equal=TRUE,alternative='greater')

Two Sample t-test

data:  new and old
t = 1.6641, df = 8, p-value = 0.06733
alternative hypothesis: true difference in means is greater than
95 percent confidence interval:
 -0.1409392        Inf
sample estimates:
mean of x mean of y
     14.6      13.4
```

Example: An experiment was done to determine the effect on dairy cattle of a diet supplement with liquid whey. While no differences were noted in milk production between the group with a standard diet (hay + grain + water) and the experimental group with whey supplement (hay + grain + whey), a considerable difference was noted in the amount of hay ingested. For a 2-tailed test with α=0.05, determine the approximate number of cattle that should be included in each group if we want $\beta \leq 0.1$ for $|\mu_1 - \mu_2| \geq 0.5$. Previous study has shown $\sigma \approx 0.8$.

Example: Do fraternities help or hurt your academic progress at college? To investigate this question, 5 students who joined fraternities in 1998 were randomly selected. It was shown that their GPA before and after they joined the fraternities are as follows.

Student	1	2	3	4	5
Before	3	4	3	3	2
After	2	3	3	2	1
Diff	1	1	0	1	1

Test the hypothesis at α=0.05

$$H_0 : \mu_d = 0$$
$$H_a : \mu_d \neq 0$$

Assumption : the difference follows a normal distribution.

$$\bar{X}_d = 0.8, s_d = 0.447, n = 5, \alpha = 0.05$$

Test statistic : $T_0 = \frac{\bar{X}_d - 0}{s_d / \sqrt{n}} \overset{H_0}{\sim} t_{n-1}$

$$|T_0| = 4.02 > t_{4,0.025} = 2.776$$

We reject H_0 at α=0.05 and conclude fraternities does hurt

SAS Code

```
data frat ;
input before after ;
diff = before - after ;
datalines ;
3 2
4 3
3 3
3 2
2 1
;
run ;

proc univariate data=frat normal ;
var diff ;
run ;
```

R Code and Output

```
> before <- c(3,4,3,3,2)
> after <- c(2,3,3,2,1)
> shapiro.test(before-after)

Shapiro-Wilk normality test

data:  before - after
W = 0.55218, p-value = 0.000131

### normality assumption is violated,
### thus the appropriate approach is to
### use non-parametric approach from Chapter 14.
```

R Code and Output

```
> t.test(before,after,paired=TRUE)

Paired t-test

data:  before and after
t = 4, df = 4, p-value = 0.01613
alternative hypothesis: true difference in means is not equal to
95 percent confidence interval:
 0.244711 1.355289
sample estimates:
mean of the differences
                    0.8
```

R Code and Output

```
> d <- c(1,1,0,1,1)
> t.test(d)

One Sample t-test

data:  d
t = 4, df = 4, p-value = 0.01613
alternative hypothesis: true mean is not equal to 0
95 percent confidence interval:
 0.244711 1.355289
sample estimates:
mean of x
      0.8
```