Assignment No 5

R Sumedhan

March 24, 2021

1 Overview

This assignment is for solving currents in a resistor. By doing so, we find the parts of the resistor likely to be the hottest since the currents depend on the shape of the conductor.

A wire is soldered to the middle of a copper plate and its voltage is held at 1 Volt. One side of the plate is grounded, while the remaining are floating. The plate is 1 cm by 1 cm in size.

Ohm's law yields :

$$\vec{j} = \sigma \vec{E}$$

Now the potential is given as :

$$\vec{E} = -\nabla \phi$$

The continuity equation is given by :

$$\nabla . \vec{j} = -\frac{\partial \rho}{\partial t}$$

Assuming that our resistor contains a material of constant conductivity, the equation becomes :

$$\nabla^2 \phi = \frac{1}{\sigma} \frac{\partial \rho}{\partial t}$$

For DC currents, right side is zero and we obtain :

$$\nabla^2 \phi = 0$$

2 Numerical Solutions in two-dimensions

To solve the differential equation, we convert it to a difference equation. The differential equation is given as :

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

We write the differentials as :

$$\frac{\partial \phi}{\partial x} \left.\right|_{(x_i, y_j)} = \frac{\phi(x_{i+1/2}, y_j) - \phi(x_{i-1/2}, y_j)}{\Delta x}$$

$$\frac{\partial^2 \phi}{\partial x^2} \left.\right|_{(x_i, y_j)} = \frac{\phi(x_{i+1}, y_j) - 2\phi(x_i, y_j) + \phi(x_{i-1}, y_j)}{(\Delta x)^2}$$

Combining this with the corresponding equation for the y derivatives, we obtain :

$$\phi_{i,j} = \frac{\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1}}{4}$$

All we do is keep iterating the above difference equation to get the final values of the potential in the metal plate. For the boundaries, we make sure that the potential does not have a gradient normally since the current at the boundaries is tangential.

3 Solution to the difference equation

Solving the equation

We first define the initial arguments required to calculate ϕ, those being the no. of rows, no. of columns, radius and no. of iterations. The potential is then defined as an array and using the condition

$$X^2 + Y^2 \leq radius^2$$

where X and Y are the meshgrids created based on the x and y column vectors, we initialize the ϕ array.
We then used vectorized code in a `for loop` and iterate it a particular no. of times by using the difference equation and also applying the boundary conditions to obtain the calculated array of the potential ϕ.

Stopping Condition

By plotting the error obtained in each iteration, we find that the maximum error scales as :

$$error = Ae^{Bk}$$

where k is the iteration number. For one fit, we write the equation as :

$$error_k = 0.0014exp(-0.00226k)$$

Summing up the terms, we have :

$$\begin{aligned} Error &= \sum_{k=N+1}^{\infty} error_k \\ &< \sum_{k=N+1}^{\infty} Ae^{Bk} \\ &\approx \int_{N+0.5}^{\infty} Ae^{Bk}dk \\ &= -\frac{A}{B}exp(B(N+0.5)) \end{aligned}$$

We look for a worst case result and sum up the absolute values. For our values, this expression evaluates to :

Error ¡ 0.63*0.03 = 0.02

4 Plots

Error Plot

Using a log-log plot, we find that the error values vary as a straight line upto the 500^{th} iteration but beyond that, it varies as an exponential. Whereas, the semi-log plot varies completely as a straight line.
We make four plots in total. For the first two plots, the error is plotted against Niter as semilogy and loglog. The last two plots are also plotted as semilogy and loglog but with the following features -

- Error plot every 50^{th} point as dots
- Error plot for values after 500^{th} iteration
- Error plot for all values

We see that all the values are in a straight line.

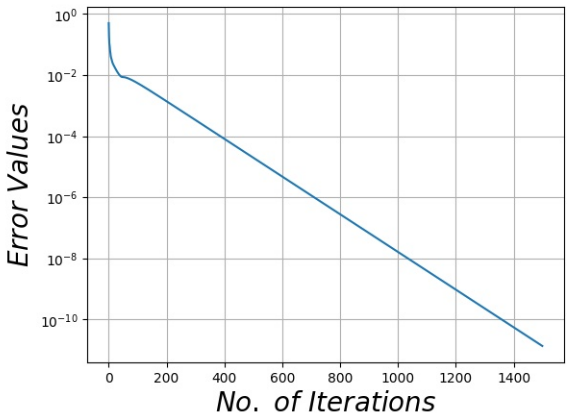

Figure 1: Error plot every 50^{th} point

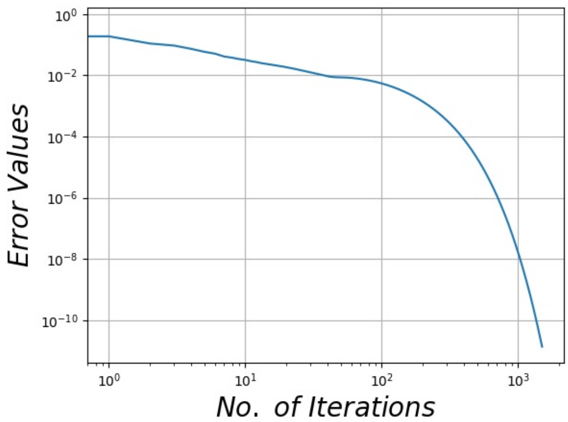

Figure 2: Error plots for a set of iterations

Surface Plot of Potential

We then make a 3-D plot of the potential based on the x and y vectors. For this, we use the `plot surface` function in `Axes3D`.

Contour Plot of Potential

We use the `contour` function to plot the contour lines of the potential, marking the electrodes in the centre as red dots.

Vector Plot of Currents

To compute the currents, we need the gradient of the potential, which is given by the equations :

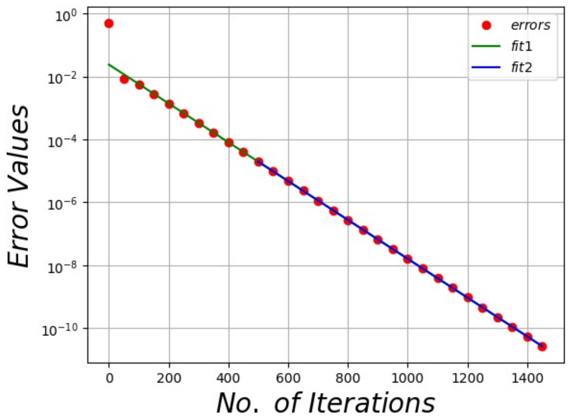

Figure 3: Error plot every 50^{th} point

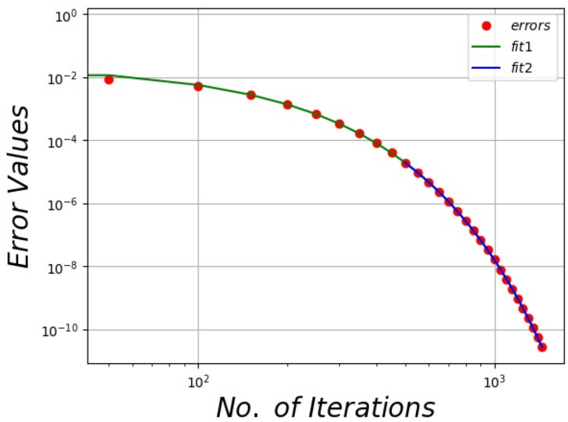

Figure 4: Error plot every 50^{th} point

$$j_x = -\partial\phi/\partial x$$
$$j_y = -\partial\phi/\partial y$$

This translates to:

$$J_{x,ij} = \frac{1}{2}(\phi_{i,j-1} - \phi_{i,j+1})$$
$$J_{y,ij} = \frac{1}{2}(\phi_{i-1,j} - \phi_{i+1,j})$$

We then create arrays Jx and Jy and then use the quiver command to plot the currents as arrows.

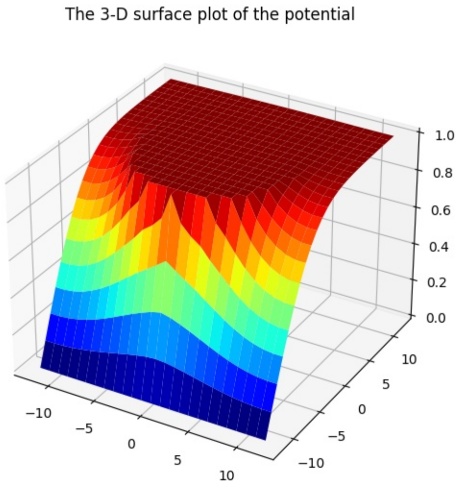

Figure 5: Surface plot of Potential

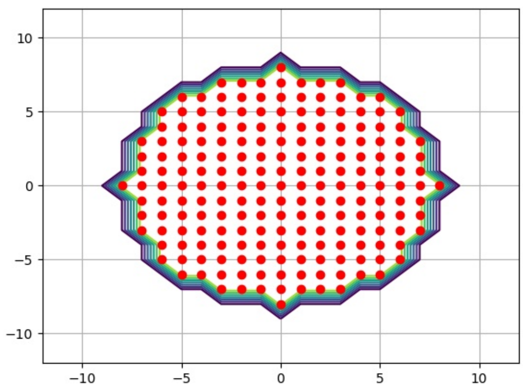

Figure 6: Contour plot of Potential

5 Conclusion

The remaining python code can be referenced in the other file containing the code.

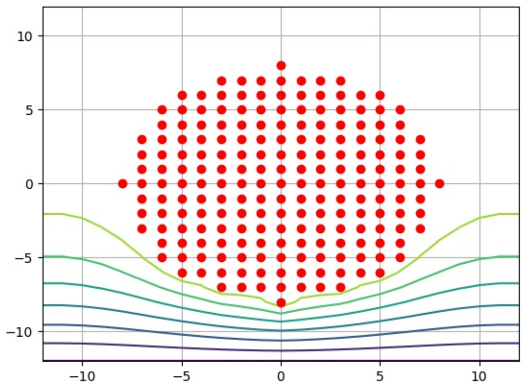

Figure 7: Contour plot of Potential

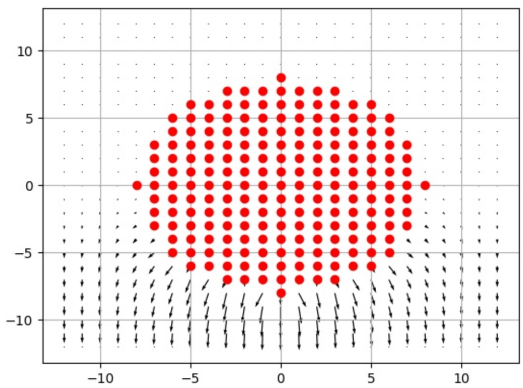

Figure 8: Vector plot of Currents