Q.1. $x_1 \sim N(\mu, \Sigma_1)$

$x_2 \sim N(\mu, \Sigma_2)$

$$P(\omega_i/x) = P(x/\omega_i)\,P(\omega_i)$$

$$\ln(P(\omega_i/x)) = \ln(P(\omega_i) - \frac{1}{2}\ln|\Sigma_i| - \frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu) - \frac{d}{2}\ln(2\pi)$$

$$\Sigma_i = \sigma_i^2[(1-\rho_i)I + \rho_i 11^T]$$

$$\Sigma_i^{-1} = \frac{I}{\sigma_i^2(1-\rho_i)} - \frac{\rho_i 11^T}{\sigma_i^2(1-\rho_i)(1+(d-1)\rho_i)}$$

$$(x-\mu)^T\Sigma^{-1}(x-\mu) = \frac{(x-\mu)^T(x-\mu)}{\sigma_i^2(1-\rho_i)} - \frac{\rho_i(x-\mu)^T 11^T(x-\mu)}{\sigma_i^2(1-\rho_i)(1+(d-1)\rho_i)}$$

$$= \frac{(x-\mu)^T(x-\mu)}{\sigma_i^2(1-\rho_i)} - \frac{\rho_i(1^T(x-\mu))^2}{\sigma_i^2(1-\rho_i)(1+(d-1)\rho_i)}$$

Bayes discriminant function is given by,

$$g(x) = P(\omega_1/x) - P(\omega_2/x).$$

$$g(x) = \ln\left(P\left(\frac{\omega_1}{\omega_2}\right)\right) - \frac{1}{2}\ln\frac{|\Sigma_1|}{|\Sigma_2|} - \frac{1}{2}(x-\mu)^T(x-\mu)\left[\frac{1}{\sigma_1^2(1-\rho_1)} - \frac{1}{\sigma_2^2(1-\rho_2)}\right] +$$

$$\frac{1}{2}(1^T(x-\mu))^2\left[\frac{\rho_1}{\sigma_1^2(1-\rho_1)(1+(d-1)\rho_1)} - \frac{\rho_2}{\sigma_2^2(1-\rho_2)(1+(d-1)\rho_2)}\right]$$

$$\boxed{g(x) = -\frac{1}{2}(C_{11}-C_{12})b_1 + \frac{1}{2}(C_{21}-C_{22})b_2 + \text{constant}}$$

where

$$b_1 = (x-\mu)^T (x-\mu)$$

$$b_2 = (1^T (x-\mu))^2$$

$$c_{1j} = [\sigma_j^2 (1-\rho_j)]^{-1}$$

$$c_{2j} = \rho_j [\sigma_j^2 (1-\rho_j)(1+(d-1)\rho_j)]^{-1}.$$

Q. 2. $$P(x) = \sum_{j=1}^{g} \pi_j \, p(x/\mu_j, m_j)$$

Assuming independence model for each mixture component,

$$p(x/\mu_j, m_j) = \prod_{k=1}^{g} p(x_k/\mu_{jk}, m_{jk}). \pi_k$$

$$p(x_s / \theta_{js}) = \frac{m_{js}}{(m_{js}-1)! \, \mu_{js}} \left( \frac{m_{js} x_s}{\mu_{js}} \right)^{m_{js}-1} . e^{-\frac{m_{js} x_s}{\mu_{js}}}$$

$$L(\theta) = \prod_{i=1}^{n} \sum_{j=1}^{g} \pi_j \, p(x_i / \theta_j)$$

$$\log L = \sum_{i=1}^{n} \sum_{j=1}^{g} \ln(\pi_j) + \ln(p(x_i/\theta_j))$$

At E-step

$$\boxed{\omega_{ij} = \frac{\pi_j \, p(x_i/\theta_j)}{\sum_{k=1}^{g} \pi_k \, p(x_i/\theta_k)}}$$

At M-step, where $\lambda$ is Lagrange multiplier.

$$F = \log L - \lambda\left(\sum_{j=1}^{g} \pi_j - 1\right)$$

$$\frac{\partial F}{\partial \pi_j} = 0.$$

$$\frac{\sum_{i=1}^{n} \omega_{ij}}{\pi_j} - \lambda = 0.$$

since, $\lambda = \sum_{j=1}^{g}\sum_{i=1}^{n} \omega_{ij} = n.$

$$\pi_j = \frac{1}{n}\sum_{i=1}^{n} \omega_{ij}$$

$$\frac{\partial \log L}{\partial \mu_j} = 0.$$

For $\mu_j$,

$$\frac{\partial\left[-\sum_{j=1}^{g}\sum_{i=1}^{n} \omega_{ij} \log \mu - \sum_{j=1}^{g}\sum_{i=1}^{n} \omega_{ij}.(m-1)\log \mu - \frac{m}{\mu}\sum_{j=1}^{g}\sum_{i=1}^{n} \omega_{ij} x_i\right]}{\partial \mu_j} = 0$$

$$-\sum_{i=1}^{n} \omega_{ij}\frac{m}{\mu} + \frac{m}{\mu^2}\sum_{i=1}^{n} \omega_{ij}.x_i = 0.$$

$$\boxed{\mu = \frac{\sum_{i=1}^{n} \omega_{ij} x_i}{\sum_{i=1}^{n} \omega_{ij}}}$$

For $m_j$,

$$\frac{\partial \log L}{\partial m_j} = 0$$

$$\frac{\partial\left[\sum_{j=1}^{g}\sum_{i=1}^{n}\omega_{ij}\log m-\sum_{j=1}^{g}\sum_{i=1}^{n}\omega_{ij}\log(m-1)!+\sum_{j=1}^{g}\sum_{i=1}^{n}\omega_{ij}(m-1)\log\frac{mx_i}{\mu_j}-\sum_{j=1}^{g}\sum_{i=1}^{n}\omega_{ij}\frac{mx_i}{\mu_j}\right]}{\partial m_j}=0$$

$$\sum_{i=1}^{n}\omega_{ij}\frac{1}{m}-\sum_{i=1}^{n}\omega_{ij}\frac{\partial\log(m-1)!}{\partial m_j}+\sum_{i=1}^{n}\omega_{ij}\log\frac{mx_i}{\mu_j}$$

$$+\sum_{i=1}^{n}\omega_{ij}\frac{m-1}{m}-\sum_{i=1}^{n}\omega_{ij}\frac{x_i}{\mu_j}=0.$$

$$\sum_{i=1}^{n}\omega_{ij}-\frac{\sum_{i=1}^{n}\omega_{ij}x_i}{\mu_j}+\sum_{i=1}^{n}\omega_{ij}\log m+\sum_{i=1}^{n}\omega_{ij}\log\frac{x_i}{\mu_j}$$

$$-\sum_{i=1}^{n}\omega_{ij}\psi_m=0$$

$$\sum_{i=1}^{n}\omega_{ij}(\log m-\psi(m))-\sum_{i=1}^{n}\omega_{ij}\log\frac{x_i}{\mu_j}$$

$$\boxed{\log m_{js}-\psi(m_{js})=\frac{-\sum_{i=1}^{n}\omega_{ij}\log\frac{x_i}{\mu_j}}{\sum_{i=1}^{n}\omega_{ij}}}$$

where $\psi_m=\frac{\partial\log(m-1)!}{\partial m.}$