Assignment

sumeeth kumar
AI21BTECH11008

Contents

***Abstract*—This manual provides solutions to the Assignment of Random Numbers**

I. Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

I.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
wget https://github.com/sumeethkumar12/
    random−variables/blob/main/1/unidat.c
wget https://github.com/sumeethkumar12/
    random−variables/blob/main/1/coeffs.h
```

Download the above files and execute the following commands

a) $ gcc unidat.c -lm
b) $ ./a.out

I.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1}$$

Solution: The following code plots Fig. I.2

```
wget https://github.com/sumeethkumar12/
    random−variables/blob/main/1/1_2.py
```

Download the above files and execute the following commands to produce Fig.I.2

a) $ python3 1_2.py

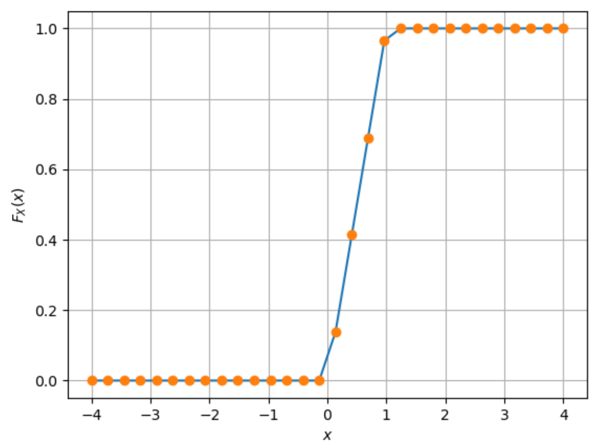

Fig. I.2. The CDF of U

I.3 Find a theoretical expression for $F_U(x)$.
Solution: Given U is a uniform Random Variable

$$p_U(x) = 1 \text{ for} \tag{2}$$

$$F_U(x) = \int_{-\infty}^{\infty} p_U(x)dx \tag{3}$$

$$\implies F_U(x) = x \tag{4}$$

I.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{5}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{6}$$

Write a C program to find the mean and variance of U.
Solution: Download the following files and execute the C program.

```
wget ttps://github.com/sumeethkumar12/
    random−variables/blob/main/1/1_4.c
wget https://github.com/sumeethkumar12/
    random−variables/blob/main/1/coeffs.h
```

Download the above files and execute the following commands

a) $ gcc 1_4.c
b) $ ./a.out

I.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{7}$$

Solution:

$$\text{var}\left[U\right] = E\left[U - E\left[U\right]\right]^2 \tag{8}$$

$$\implies \text{var}\left[U\right] = E\left[U^2\right] - E\left[U\right]^2 \tag{9}$$

$$E\left[U\right] = \int_{-\infty}^{\infty} x dF_U(x) \tag{10}$$

$$E\left[U\right] = \int_{0}^{1} x \tag{11}$$

$$\implies \boxed{E\left[U\right] = \frac{1}{2}} \tag{12}$$

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{13}$$

$$E\left[U^2\right] = \int_{0}^{1} x^2 dF_U(x) \tag{14}$$

$$\implies E\left[U^2\right] = \frac{1}{3} \tag{15}$$

$$\implies \boxed{\text{var}\left[U\right] = \frac{1}{12} = 0.0833} \tag{16}$$

II. Central Limit Theorem

II.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{17}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Download the following files and execute the C program.

```
wget https://github.com/sumeethkumar12/
    random−variables/blob/main/2/2_1.c
wget ttps://github.com/sumeethkumar12/
    random−variables/blob/main/1/coeffs.h
```

Download the above files and execute the following commands

a) $ gcc 2_1.c
b) $ ./a.out

II.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. II.2 using the code below

```
wget https://github.com/sumeethkumar12/
    random−variables/blob/main/2/2_2.py
```

Download the above files and execute the following commands to produce Fig.II.2

a) $ python3 2.2.py

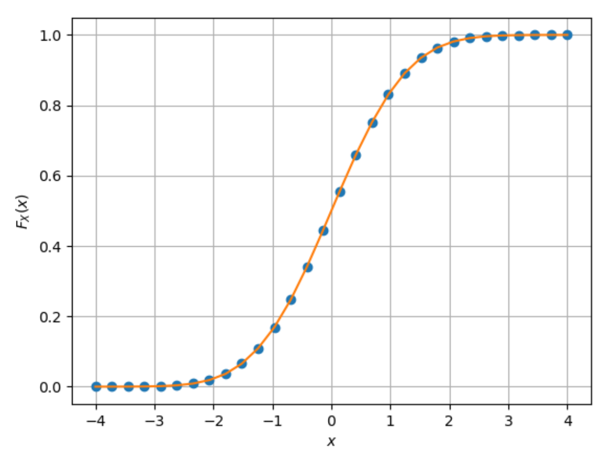

Fig. II.2. The CDF of X

Some of the properties of CDF

a) $F_X(x)$ is non decreasing function.
b) Symmetric about one point.
c) The Q function is defined as,
$Q(X) = Pr(X > x)$

d) Hence we can use it to calculate $F_x(x)$,
$F_x(x) = 1 - Q(x)$

II.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{18}$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. II.3 using the code below

```
wget https://github.com/sumeethkumar12/
    random−variables/blob/main/2/2_3.py
```

Download the above files and execute the following commands to produce Fig.II.3

a) $ python3 2_3.py

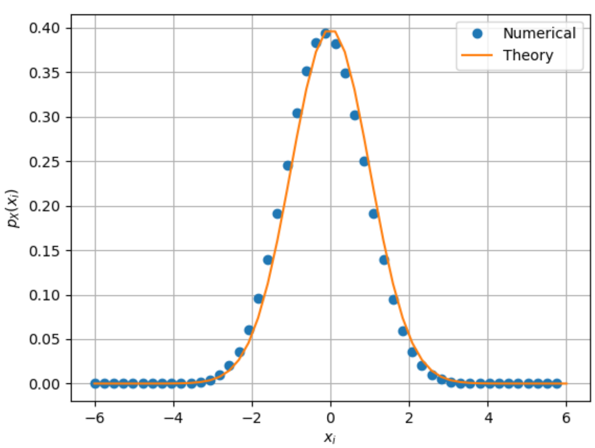

Fig. II.3. The PDF of X

Some of the properties of the PDF:

a) Symmetric about $x = \mu$
b) decreasing function for $x < \mu$ and increasing for $x > \mu$ and attains maximum at $x = \mu$
c) Area under the curve is unity.

II.4 Find the mean and variance of X by writing a C program.

Solution: Download the following files and execute the C program.

```
wget https://github.com/sumeethkumar12/
    random−variables/blob/main/2/2_4.c
wget https://github.com/sumeethkumar12/
    random−variables/blob/main/2/coeffs.h
```

Download the above files and execute the following commands

a) $ gcc 2_4.c
b) $ ./a.out

II.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{19}$$

repeat the above exercise theoretically.

Solution:

1) CDF is given by

$$F_X(x) = \int_{-\infty}^{\infty} p_X(x)dx \tag{20}$$

$$\boxed{F_X(x) = 1} \tag{21}$$

2) Mean is given by

$$E(x) = \int_{-\infty}^{\infty} xp_X(x)dx$$

$$\implies \boxed{E(x) = 0}$$

3) Variance is given by To solve the above integral, we will use integration by parts, i.e,

$$\int uvdx = u\int vdx - \int u'\left(\int vdx\right)dx \tag{22}$$

$$E\left[x^2\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x\left(xe^{-\frac{x^2}{2}}\right)dx \tag{23}$$

$$= \frac{1}{\sqrt{2\pi}}\left(x\int xe^{-\frac{x^2}{2}}dx - \int\left(\int xe^{-\frac{x^2}{2}}dx\right)\right) \tag{24}$$

For the integral $\int x\exp\left(-\frac{x^2}{2}\right)dx$ let us take,

$$t = \frac{x^2}{2} \tag{25}$$

$$dt = xdx \tag{26}$$

$$\int x\exp\left(-\frac{x^2}{2}\right)dx = \int \exp(-t)\,dt \tag{27}$$

$$= -\exp(-t) + c \tag{28}$$

$$\implies = -\exp\left(-\frac{x^2}{2}\right) + c \tag{29}$$

Using (29),we can write

$$E\left[x^2\right] = \frac{1}{\sqrt{2\pi}}\left(-xe^{-\frac{x^2}{2}} + \int e^{\frac{-x^2}{2}}dx\right) \tag{30}$$

And we know that,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} = 1 \tag{31}$$

Now putting limits and using (**??**),(31),

$$E\left[x^2\right] = 1 \tag{32}$$

$var(x) = E[x^2] - E[x] = 1 - 0 = 1$

III. From Uniform to Other

III.1 Generate samples of

$$V = -2\ln(1-U) \tag{33}$$

and plot its CDF.

Solution: Download the following files and execute the C program.

```
wget https://github.com/sumeethkumar12/
    random−variables/blob/main/3/3_1.c
wget https://github.com/sumeethkumar12/
    random−variables/blob/main/3/coeffs.h
```

Download the above files and execute the following commands

a) $ gcc 3_1.c -lm
b) $ ./a.out

The CDF of V is plotted in Fig. III.1 using the code below

```
wget https://github.com/sumeethkumar12/
    random−variables/blob/main/3/3_1.py
```

Download the above files and execute the following commands to produce Fig.III.1

a) $ python3 3_1.py

III.2 Find a theoretical expression for $F_V(x)$.

Solution: If Y = g(X), we know that $F_Y(y) = F_X(g^{-1}(y))$, here

$$V = -2\ln(1-U) \tag{34}$$

$$1 - U = e^{\frac{-V}{2}} \tag{35}$$

$$U = 1 - e^{\frac{-V}{2}} \tag{36}$$

$$F_V(X) = F_U(1 - e^{\frac{-X}{2}}) \tag{37}$$

when , $0 \leq 1 - e^{\frac{-X}{2}} \leq 1$

$$0 \leq e^{\frac{-X}{2}} \leq 1 \tag{38}$$

$$X \geq 0, \text{So}, \tag{39}$$

$$F_V(X) = 1 - e^{\frac{-X}{2}}, X \geq 0 \tag{40}$$

$$F_V(X) = 0, X < 0 \tag{41}$$

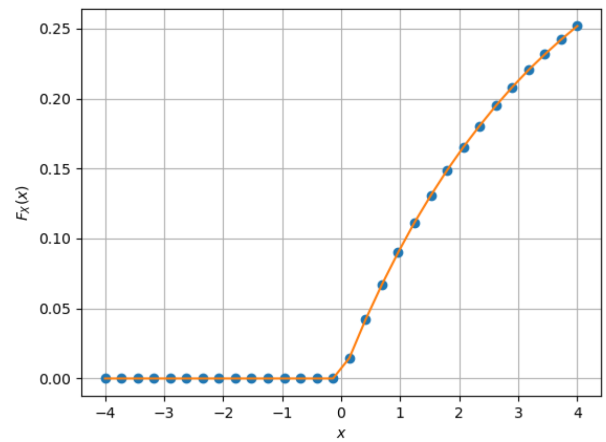

Fig. III.1. The PDF of X

IV. Triangular Distribution

IV.1 Generate

$$T = U_1 + U_2 \tag{42}$$

Solution: Download the below code,

```
wget https://github.com/sumeethkumar12/
    random−variables/blob/main/4/coeffs.h
wget https://github.com/sumeethkumar12/
    random−variables/blob/main/4/4_1.c
```

and run the following command,

```
cc triangular.c −lm
./a.out
```

You will get required generated random numbers in tri.dat file.

IV.2 Find the CDF of T.

Solution: Download the below files,

```
wget https://github.com/
    sumeethkumar12/random−variables/
    blob/main/4/tri.dat
wget https://github.com/
    sumeethkumar12/random−variables/
    blob/main/4/tri_cdf.png
```

Run the following command,

```
python3 tri_cdf_plot.py
```

IV.3 Find the PDF of T.

Solution: Download the below files,

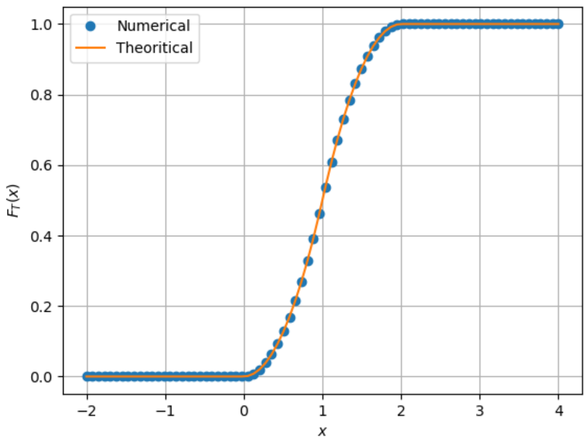

Fig. IV.2. The CDF of T

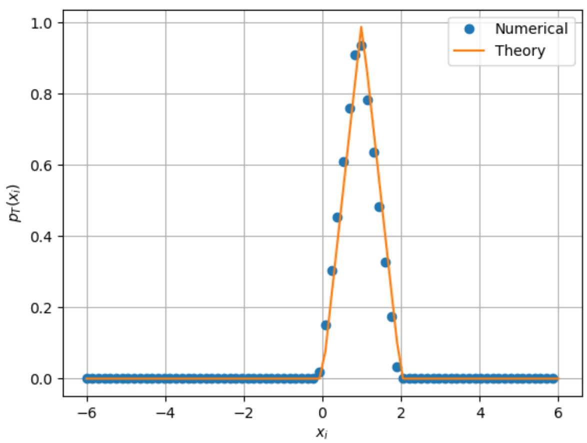

Fig. IV.2. The PDF of T

```
wget https://github.com/
    sumeethkumar12/random−variables/
    blob/main/4/tri.dat
wget https://github.com/
    sumeethkumar12/random−variables/
    blob/main/4/tri_pdf.png
```

Run the following command,

```
python3 tri_pdf_plot.py
```

IV.4 Find the theoretical expressions for the PDF and CDF of T.
Solution: Given that,

$$T = U1 + U2 \tag{43}$$

where$U1, U2$ are uniform random variables $\in (0, 1)$.
Calculation of CDF The CDF of T is defined as,

$$F_T(t) = \Pr(T \leq t) \tag{44}$$

Now from (43) we can write,

$$F_T(t) = \Pr(U1 + U2 \leq t) \tag{45}$$

Case -1 : For $t > 2$.
The $\Pr(U1 + U2 \leq t) = 1$, because for every $U1 = u1$ and $U2 = u2$, $u1 + u2 < 2$,

$$\implies F_T(t) = 1 \tag{46}$$

Case -2 : For $t < 0$.
The $\Pr(U1 + U2 \leq t) = 0$ because for every $U1 = u1$ and $U2 = u2$, $u1 + u2 > 0$,

$$\implies F_T(t) = 0 \tag{47}$$

Case - 3: For, $t \in (0, 2)$.
We cannot eliminate the inequality like we did before, so in this case we will operate the inequality by fixing $U1 = x$ where $x \in (0, t)$. So in this case CDF will be,

$$F_T(t) = \Pr(U1 + U2 \leq t) \tag{48}$$
$$= \Pr(U1 = x, U2 \leq t - x) \tag{49}$$
$$= \Pr(U1 = x)\Pr(U2 \leq t - x) \tag{50}$$

Since U1,U2 are i.i.d.
Now note that x is a variable and varies in $(0, t)$, so we have to take integral over x to evaluate the $\Pr(U1 = x)$,

$$F_T(t) = \int_0^t f_U(x)\Pr(U2 \leq t - x)\,dx \tag{51}$$
$$= \int_0^t f_U(x)F_U(t - x)\,dx \tag{52}$$

Case - 1 For $t \in (0, 1)$,we know $f_U(x) = 1$ so,

$$F_T(t) = \int_0^t 1.F_U(t - x)\,dx \tag{53}$$

As, $x < t$, $0 < t - x < t < 1$ using (**??**),we can write

$$F_T(t) = \int_0^t (t-x)\,dx \tag{54}$$

$$= \left\{tx - \frac{x^2}{2}\right\}_0^t \tag{55}$$

$$= \frac{t^2}{2} \tag{56}$$

Case -2 For $t \in (1,2)$, we know $f_U(x) = 0$ at $x > 1$,so the integral solves down to,

$$F_T(t) = \int_0^1 f_U(x)\,F_U(t-x)\,dx \tag{57}$$

$$= \int_0^1 1.F_U(t-x)\,dx \tag{58}$$

$$\tag{59}$$

To solve the above integral we will use integration by substitution,

$$k = t - x \tag{60}$$

$$dk = -dx \tag{61}$$

$$F_T(t) = \int_t^{t-1} F_U(k)\,(-dk) \tag{62}$$

$$= \int_{t-1}^t F_U(k)\,dk \tag{63}$$

As $1 \leq t \leq 2, 0 \leq t-1 \leq 1$ we will break integral at 1 because $F_U(k)$ changes at 1.Using (**??**),

$$F_T(t) = \int_{t-1}^1 F_U(k)\,dk + \int_1^t F_U(k)\,dk \tag{64}$$

$$= \int_{t-1}^1 k dk + \int_1^t 1 dk \tag{65}$$

$$= \left\{\frac{k^2}{2}\right\}_{t-1}^1 + t - 1 \tag{66}$$

$$= \frac{1}{2} - \left(\frac{(t-1)^2}{2}\right) + t - 1 \tag{67}$$

$$= 2t - \frac{t^2}{2} - 1 \tag{68}$$

Overall we can write the CDF of $F_T(x)$ as,

$$F_T(x) = \begin{cases} 0 & , x < 0 \\ \frac{x^2}{2} & , 0 \leq x \leq 1 \\ 2t - \frac{t^2}{2} - 1 & , 1 \leq x \leq 2 \\ 1 & , x > 2 \end{cases} \tag{69}$$

Calculation of PDF Now we will find PDF of T,
As,

$$T = U1 + U2 \tag{70}$$

We will use method of convolution to get PDF of T as $U1$ and $U2$ are i.i.d.

$$f_T(t) = \int_{-\infty}^{\infty} f_{U1}(x)\,f_{U2}(t-x)\,dx \tag{71}$$

Since $U1, U2$ are of same distribution we can write,

$$f_{U1}(x) = f_{U2}(x) = f_U(x) \tag{72}$$

$$\implies f_T(t) = \int_{-\infty}^{\infty} f_U(x)\,f_U(t-x)\,dx \tag{73}$$

From the PDF of U,we can write

$$f_T(t) = \int_0^1 f_U(x)\,f_U(t-x)\,dx \tag{74}$$

$$= \int_0^1 1.f_U(t-x)\,dx \tag{75}$$

$$\tag{76}$$

we will solve the above integral using substitution.

$$z = t - x \tag{77}$$

$$dz = -dx \tag{78}$$

$$\implies f_T(t) = \int_t^{t-1} f_U(z)\,(-dz) \tag{79}$$

$$= \int_{t-1}^t f_U(z)\,dz \tag{80}$$

Case -1 For $t < 0$ as $z < t$,the PDF $f_U(z) = 0$. So,

$$f_T(t) = 0 \tag{81}$$

Case -2 For $0 \leq t \leq 1$, we will break the integral at $z = 0$, since $f_U(z)$ changes at 0.

$$f_T(t) = \int_{t-1}^0 f_U(z)\,dz + \int_0^t f_U(z)\,dz \tag{82}$$

$$= 0 + \int_0^t 1dz \tag{83}$$

$$= t \tag{84}$$

Case-3 Similarly for $1 \leq t \leq 2$, we will break the integral at $z = 1$,

$$f_T(t) = \int_{t-1}^{1} f_U(z)\, dz + \int_{1}^{t} f_U(z) \tag{85}$$

$$= \int_{t-1}^{1} 1.dz + 0 \tag{86}$$

$$= 2 - t \tag{87}$$

Case-4 For $t > 2$,as $z > t - 1 > 1$,the PDF $f_U(z) = 0$. So,

$$f_T(z) = 0 \tag{88}$$

Overall, the PDF of T will be,

$$f_T(x) = \begin{cases} 0 & , x < 0 \\ x & , 0 \leq x \leq 1 \\ 2 - x & , 1 \leq x \leq 2 \\ 0 & , x > 2 \end{cases} \tag{89}$$

IV.5 Verify your results through a plot.
Solution: This is already done in IV.2 ,IV.2.

V. Maximum Likelihood

V.1 Generate equiprobable $X \in \{1, -1\}$.
Solution: The generating X or bernoulie random variable (X) is done by using uni.dat file. Download the below files

```
wget https://github.com/
    sumeethkumar12/random−
    variables/blob/main/4/coeffs.h
wget https://github.com/
    sumeethkumar12/random−
    variables/blob/main/4/4_1.c
```

Run the following command

```
cc bernoulie.c −lm
./a.out
```

V.2 Generate

$$Y = AX + N, \tag{90}$$

where $A = 5$ dB, and $N \sim 01$.
Solution: To generate distribution of Y random variable we will need previously generated bernoulie distribution and gaussian distribution.Download the below files

```
wget https://github.com/Charanyash/
    Random−Numbers−/blob/main/
    codes/Q5/coeffs.h
wget https://github.com/Charanyash/
    Random−Numbers−/blob/main/
    codes/Q5/Y.c
```

Then run the following command,

```
cc Y.c −lm
./a.out
```

V.3 Plot Y using a scatter plot.
Solution: Download the below files

```
wget https://github.com/Charanyash/
    Random−Numbers−/blob/main/
    codes/Q5/Y.py
wget https://github.com/Charanyash/
    Random−Numbers−/blob/main/
    codes/Q5/Y.dat
```

Then run the following command,

```
python3 Y.py
```

V.4 Guess how to estimate X from Y. **Solution:** When $Y > 0$, we can more probably say that $X = 1$ as X can take values from $[-1, 1]$.As A increases the signal contribution will increase compared to noise.The scatter plot will not be intermixed as A increases. So in this case, the scatter plot of Y is seperated with decision boundary as 0. So we can more probably say that,

$$X = \begin{cases} 1 & , Y > 0 \\ -1 & , Y < 0 \end{cases} \tag{91}$$

V.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{92}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{93}$$

Solution: The $\hat{X}$ is defined as,

$$\hat{X} = \begin{cases} 1 & , Y > 0 \\ 0 & , Y \leq 0 \end{cases} \tag{94}$$

The error probability, when the actual signal is $X = 1$ but transmitted as $\hat{X} = -1$ is,

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{95}$$
$$= \Pr\left(Y \leq 0|X = 1\right) \tag{96}$$
$$= \Pr\left(AX + N \leq 0|X = 1\right) \tag{97}$$
$$= \Pr\left(A + N \leq 0\right) \tag{98}$$
$$= \Pr\left(N \leq -A\right) \tag{99}$$
$$= F_N\left(-A\right) \tag{100}$$
$$= 0 \tag{101}$$

And for the case when actual signal is $X = -1$ but transmitted as $\hat{X} = 1$ the error probability will be,

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{102}$$
$$= \Pr\left(Y > 0|X = -1\right) \tag{103}$$
$$= \Pr\left(AX + N > 0|X = 1\right) \tag{104}$$
$$= \Pr\left(N - A > 0\right) \tag{105}$$
$$= \Pr\left(N > A\right) \tag{106}$$
$$= 1 - F_N\left(A\right) \tag{107}$$
$$= F_N\left(-A\right) \tag{108}$$
$$= 0 \tag{109}$$

The above calculations are coded in below python file,

```
wget https://github.com/Charanyash/
    Random−Numbers−/tree/main/
    codes/Q5/5.5.py
```

Run the following command

```
python3 5.5.py
```

V.6 Find P_e assuming that X has equiprobable symbols.

V.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.

V.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.

V.9 Repeat the above exercise when

$$p_X(0) = p \tag{110}$$

V.10 Repeat the above exercise using the MAP criterion.

VI. Gaussian to Other

VI.1 Let $X_1 \sim 01$ and $X_2 \sim 01$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{111}$$

VI.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{112}$$

find α.

VI.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{113}$$

VII. Conditional Probability

VII.1 Plot

$$P_e = \Pr\left(\hat{X} = -1|X = 1\right) \tag{114}$$

for

$$Y = AX + N, \tag{115}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim 01, X \in (-1, 1)$ for $0 \leq \gamma \leq 10$ dB.

VII.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

VII.3 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{116}$$

Find $P_e = E\left[P_e(N)\right]$.

VII.4 Plot P_e in problems VII.VII.2 and VII.VII.2 on the same graph w.r.t γ. Comment.

VIII. Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{117}$$

where

$$x \in \left(\mathbf{s}_0, \mathbf{s}_1\right), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{118}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim 01. \tag{119}$$

VIII.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{120}$$

on the same graph using a scatter plot.

VIII.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

VIII.3 Plot

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{121}$$

with respect to the SNR from 0 to 10 dB.

VIII.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.