Digital Signal Processing

sumeeth kumar

Contents

Abstract—**This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands (commands may change depending on Linux distro)

```
$ sudo apt update && sudo apt upgrade
$ sudo apt install libffi−dev libsndfile1 python3−
    scipy python3−numpy python3−matplotlib
$ pip3 install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file using

```
$https://github.com/sumeethkumar12/signal−
    processing/blob/main/
    Sound_With_ReducedNoise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem **??** in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution: Download the source code using

```
$ https://github.com/sumeethkumar12/signal−
    processing/blob/main/codes/cancel_noise.
    py
```

and execute it using

```
$ python3 cancel_noise.py
```

2.4 The output of the python script in Problem **??** is the audio file `Sound_With_ReducedNoise.wav`. Play the file in the spectrogram in Problem **??**. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.

3.2 Let

$$y(n) + \frac{1}{2} y(n-1) = x(n) + x(n-2), \qquad y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The following code yields Fig. (**??**).

```
$ https://github.com/sumeethkumar12/signal−
    processing/blob/main/codes/xnyn.py
```

and execute it using

```
$ python3 xnyn.py
```

3.3 Repeat the above exercise using a C code.

```
$ https://github.com/sumeethkumar12/signal−
    processing/blob/main/codes/xnyn.c
```

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

X(z)=Z{x(n)}=$\sum_{n=-\infty}^{\infty} x(n) z^{-n}$(4.1)Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1} X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

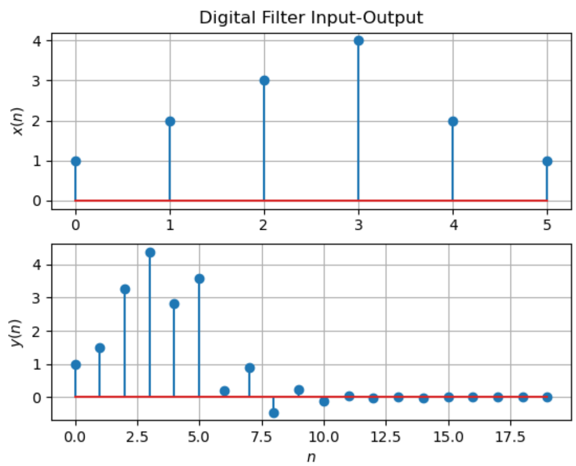

Fig. 3.2: Plot of $x(n)$ and $y(n)$

Solution: Given that,

$$X(z) = \mathcal{Z}\{x(n)\} \tag{4.4}$$
$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{4.5}$$

So,

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1) z^{-n} \tag{4.6}$$

Take $k = n - 1$,

$$= \sum_{k=-\infty}^{\infty} x(k) z^{-(k+1)} \tag{4.7}$$
$$= z^{-1} \sum_{k=-\infty}^{\infty} x(k) z^{-k} \tag{4.8}$$
$$= z^{-1} \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{4.9}$$
$$= z^{-1} X(z) \tag{4.10}$$

resulting in and similarly following the above steps you will get,

$$\mathcal{Z}\{x(n-k)\} = z^{-k} X(n) \tag{4.11}$$

4.2 Now we will find Z transform of the signal

$$\mathcal{Z}\{x(n)\} = \sum_{n=0}^{5} x(n) z^{-n} \tag{4.12}$$
$$= 1z^{0} + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + 1z^{-5} \tag{4.13}$$
$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \tag{4.14}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.15}$$

from (**??**) assuming that the Z-transform is a linear operation.

Solution:

$$Y(z) + \frac{1}{2} z^{-1} Y(z) = X(z) + z^{-2} X(z) \tag{4.16}$$
$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2} z^{-1}} \tag{4.17}$$

Solution: Now we will rewrite ,

$$y(n) + \frac{1}{2} y(n-1) = x(n) + x(n-2) \tag{4.18}$$

Now since Z-transform is a linear operator we can write that,

$$Y(n) + \frac{1}{2} Y(n-1) = X(n) + X(n-2) \tag{4.19}$$

$$Y(n) + \frac{z^{-1}}{2} Y(n) = X(n) + z^{-2} X(n) \tag{4.20}$$
$$\implies \frac{Y(n)}{X(n)} = \frac{1 + z^{-2}}{1 + \frac{z^{-1}}{2}} \tag{4.21}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.22}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.23}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.24}$$

Solution: The Z-transform of δn is,

$$\mathcal{Z}\{\delta n\} = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n} \quad (4.25)$$
$$= \delta(0) z^0 + 0 \ \text{(Using (??))} \quad (4.26)$$
$$= 1 \quad (4.27)$$

and the Z-transform of unit-step function $u(n)$ is,

$$U(n) = \sum_{n=-\infty}^{\infty} u(n) z^{-n} \quad (4.28)$$
$$= 0 + \sum_{n=0}^{\infty} 1.z^{-n} \quad (4.29)$$
$$= 1 + z^{-1} + z^{-2} + \dots \quad (4.30)$$

Above is a infinite geometric series with z^{-1} as common ratio , so we can write it as

$$U(n) = \frac{1}{1 - z^{-1}} \because |z| > 1 \quad (4.31)$$

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \quad (4.32)$$

Solution: The Z- transform will be

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=0}^{\infty} a^n z^{-n} \quad (4.33)$$
$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \dots. \quad (4.34)$$

Above is a infinite geometric series with first term 1 and common ratio as $\frac{a}{z}$ and it can be written as,

$$\mathcal{Z}\{a^n u(n)\} = \frac{1}{1 - \frac{a}{z}} \because |a| < |z| \quad (4.35)$$

Therefore,

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \quad (4.36)$$

4.6 let

$$H(e^{\jmath\omega}) = H(z = e^{\jmath\omega}). \quad (4.37)$$

Plot $|H(e^{\jmath\omega})|$. Comment. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $x(n)$. **Solution:** The following code plots Fig. .

```
$ https://github.com/sumeethkumar12/signal−
    processing/blob/main/codes/dtft.py
```

The figure can be generated using

```
$ python3 dtft.py
```

We observe that $\left|H\left(e^{j\omega}\right)\right|$ is periodic with fundamental period 2π. Now using we will find

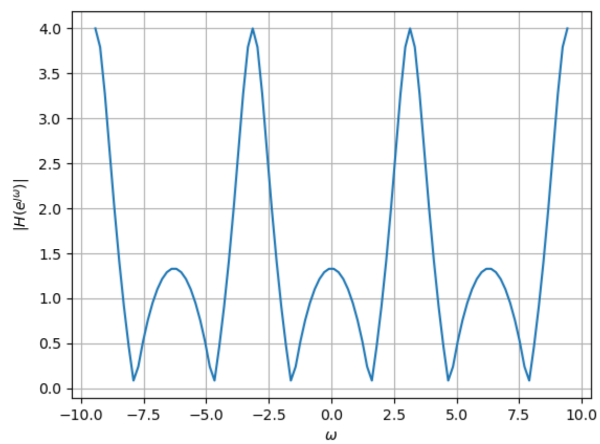

Fig. 4.6: Plot of $\left|H\left(e^{j\omega}\right)\right|$ against ω

$\left|H\left(e^{j\omega}\right)\right|$,

$$H\left(e^{j\omega}\right) = \frac{1 + e^{-2j\omega}}{1 + \frac{e^{-j\omega}}{2}} \quad (4.38)$$
$$\implies \left|H\left(e^{j\omega}\right)\right| = \frac{\left|1 + e^{-2j\omega}\right|}{\left|1 + \frac{e^{-j\omega}}{2}\right|} \quad (4.39)$$
$$= \frac{\left|1 + e^{2j\omega}\right|}{\left|e^{2j\omega} + \frac{e^{j\omega}}{2}\right|} \quad (4.40)$$
$$= \frac{|1 + \cos 2\omega + j \sin 2\omega|}{\left|e^{j\omega} + \frac{1}{2}\right|} \quad (4.41)$$
$$= \frac{\left|4\cos^2(\omega) + 4j\sin(\omega)\cos(\omega)\right|}{|2e^{j\omega} + 1|} \quad (4.42)$$
$$= \frac{|4\cos(\omega)|\,|\cos(\omega) + j\sin(\omega)|}{|2\cos(\omega) + 1 + 2j\sin(\omega)|} \quad (4.43)$$
$$\therefore \left|H\left(e^{j\omega}\right)\right| = \frac{|4\cos(\omega)|}{\sqrt{5 + 4\cos(\omega)}} \quad (4.44)$$

Using the above expression we can say that graph is symmetric about origin and has a period of 2π.

4.7 Express $h(n)$ in terms of $H\left(e^{j\omega}\right)$.

Solution:

$$\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}\omega \tag{4.45}$$

$$= \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} h(k)e^{-\jmath\omega k}e^{\jmath\omega n}\omega \tag{4.46}$$

$$= \sum_{k=-\infty}^{\infty} h(k) \int_{-\pi}^{\pi} e^{\jmath\omega(n-k)}\omega \tag{4.47}$$

Now,

$$\int_{-\pi}^{\pi} e^{\jmath\omega(n-k)}\omega = \begin{cases} \int_{-\pi}^{\pi}\omega & n-k=0 \\ \left.\frac{\exp\left(\jmath\omega(n-k)\right)}{\jmath(n-k)}\right|_{-\pi}^{\pi} & n-k\neq 0 \end{cases} \tag{4.48}$$

$$= \begin{cases} 2\pi & n-k=0 \\ 0 & n-k\neq 0 \end{cases} \tag{4.49}$$

$$= 2\pi\delta(n-k) \tag{4.50}$$

Thus,

$$\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}\omega = 2\pi \sum_{k=-\infty}^{\infty} h(k)\delta(n-k) \tag{4.51}$$

$$= 2\pi h(n) * \delta(n) \tag{4.52}$$

$$= 2\pi h(n) \tag{4.53}$$

Therefore, $h(n)$ is given by the inverse DTFT (IDTFT) of $H\left(e^{\jmath\omega}\right)$

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}\omega \tag{4.54}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in .

Solution: we can write

$$H(z) = \frac{1+z^{-2}}{1+\frac{z^{-1}}{2}} \tag{5.2}$$

$$\begin{array}{r|lll} & 2z^{-1} & -4 & \\ \hline 1+z^{-1}/2 & 1 & +z^{-2} & \\ & 2z^{-1} & +z^{-2} & \\ \hline & & 1 & -2z^{-1} \\ & & -4 & -2z^{-1} \\ \hline & & & 5 \end{array}$$

So we can replace as,

$$\frac{1+z^{-2}}{1+\frac{z^{-1}}{2}} = 2z^{-1} - 4 + \frac{5}{1+z^{-1}/2} \tag{5.3}$$

Now we can expand the second term of above expression as an infinite geometric series,

$$\frac{5}{1+z^{-1}/2} = 5\left(1 + \left(\frac{-1}{2z}\right) + \left(\frac{-1}{2z}\right)^2 + \dots\right) \tag{5.4}$$

where we assume $\left|\frac{1}{2z}\right| < 1$.
ROC is $|z| > \frac{1}{2}$.
So will become,

$$= 2z^{-1} - 4 + 5 + \frac{-5}{2}z^{-1} + \frac{5}{4}z^{-2} + \frac{-5}{8}z^{-3} + \frac{5}{16}z^{-4} + \dots \tag{5.5}$$

$$= 1z^0 + \frac{-1}{2}z^{-1} + \frac{5}{4}z^{-2} + \frac{-5}{8}z^{-3} + \frac{5}{16}z^{-4} + \dots \tag{5.6}$$

Now to get $h(n)$ for $n < 5$ we will compare with the below equation,

$$H(z) = \sum_{n=-\infty}^{n=\infty} h(n)z^{-n} \tag{5.7}$$

$h(n)$ will be the coefficient of z^{-n}.
Using this, we can write,

$$h(0) = 1 \tag{5.8}$$

$$h(1) = \frac{-1}{2} \tag{5.9}$$

$$h(2) = \frac{5}{4} \tag{5.10}$$

$$h(3) = \frac{-5}{8} \tag{5.11}$$

$$h(4) = \frac{5}{16} \tag{5.12}$$

And for $n < 0$ $h(n) = 0$.
For $n > 5$, we can get h(n) from the geometric series,

$$h(n) = 5\left(\frac{-1}{2}\right)^n \tag{5.13}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

h(n) $\overset{\mathcal{Z}}{\rightleftharpoons}$ $H(z)$(5.14)and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is

known as the *impulse response* of the system defined .
Solution: The $H(z)$ can be written as,

$$H(z) = \frac{1}{1+\frac{z^{-1}}{2}} + \frac{z^{-2}}{1+\frac{z^{-1}}{2}} \tag{5.15}$$

we can write it as,

$$h(n) = \left(\frac{-1}{2}\right)^n u(n) + \left(\frac{-1}{2}\right)^{n-2} u(n-2) \tag{5.16}$$

5.3 Sketch $h(n)$. Is it bounded? Justify Theoritically.
Solution: Download the code for the plot from the below link,

```
wget https://github.com/
    sumeethkumar12Signal_Processing/blob/
    master/Sound%201/Codes/hn.py
```

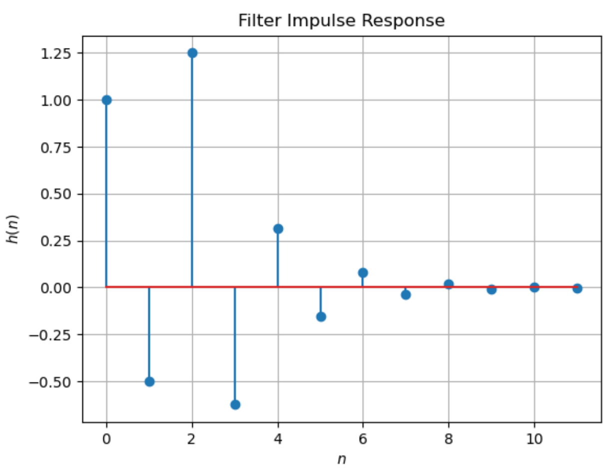

Fig. 5.3: $h(n)$ as inverse of $H(n)$

From the plot it seems like $h(n)$ is bounded and becomes smaller in magnitude as n increases. Using , we can get theoritical expression as,

$$h(n) = \begin{cases} 0 & ,n<0 \\ \left(\frac{-1}{2}\right)^n & ,0 \le n < 2 \\ 5\left(\frac{-1}{2}\right)^n & ,n \ge 2 \end{cases} \tag{5.17}$$

A sequence $\{x_n\}$ is said to be bounded if and only if there exist a positive real number M such that,

$$|x_n| \le M, \forall n \in \mathcal{N} \tag{5.18}$$

So to say $h(n)$ is bounded we should able to find the M which satisfies .

For $n < 0$,

$$|h(n)| \le 0 \tag{5.19}$$

For $0 \le n < 2$,

$$|h(n)| = \left|\frac{-1}{2}\right|^n \tag{5.20}$$

$$= \left(\frac{1}{2}\right)^n \le 1 \tag{5.21}$$

And for $n \ge 2$,

$$|h(n)| = \left|5\left(\frac{-1}{2}\right)\right|^n \tag{5.22}$$

$$= \left(\frac{5}{2}\right)^n \le \frac{5}{4} \tag{5.23}$$

From above three cases, we can get M as,

$$M = max\left\{0, 1, \frac{5}{4}\right\} \tag{5.24}$$

$$= \frac{5}{4} \tag{5.25}$$

Therefore, $h(n)$ is bounded with $M = \frac{5}{4}$ i.e.,

$$|h(n)| \le \frac{5}{4} \forall n \in \mathcal{N} \tag{5.26}$$

5.4 Convergent? Justify using the ratio test.
Solution: We can say a given real sequence $\{x_n\}$ is convergent if

$$\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| < 1 \tag{5.27}$$

This is known as Ratio test.
In this case the limit will become,

$$\lim_{n\to\infty} \left|\frac{h(n+1)}{h(n)}\right| = \lim_{n\to\infty} \left|\frac{5\left(\frac{-1}{2}\right)^{n+1}}{5\left(\frac{-1}{2}\right)^n}\right| \tag{5.28}$$

$$= \lim_{n\to\infty} \left|\frac{-1}{2}\right| \tag{5.29}$$

$$= \frac{1}{2} \tag{5.30}$$

As $\frac{1}{2} < 1$, from root test we can say that $h(n)$ is convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.31}$$

Is the system defined stable for the impulse

response ?
Solution: ,

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty} \left(\left(\frac{-1}{2} \right)^n u(n) + \left(\frac{-1}{2} \right)^{n-2} u(n-2) \right) \tag{5.32}$$

$$= 2 \left(\frac{1}{1+\frac{1}{2}} \right) \tag{5.33}$$

$$= \frac{4}{3} \tag{5.34}$$

$\therefore$ the system is stable.

5.6 Verify the above result using a python code.
Solution: Download the python code from the below link

```
wget https://github.com/sumeethkumar12/
    Signal_Processing/blob/master/Sound
    %201/Codes/hnstable.py
```

Then run the following command,

```
python3 hn_stable.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.35}$$

This is the definition of $h(n)$.
Solution: Download the code for the plot **??** from the below link,

```
wget https://github.com/sumeethkumar12/
    Signal_Processing Codes/hndef.py
```

Note that this is same as **??**.
For $n < 0$, $h(n) = 0$ and,

$$h(0) = \delta(0) \tag{5.36}$$

$$= 1 \tag{5.37}$$

For $n = 1$,

$$h(1) + \frac{1}{2}h(0) = \delta(1) + \delta(-1) \tag{5.38}$$

$$\implies h(1) = -\frac{1}{2}h(0) \tag{5.39}$$

$$= -\frac{1}{2} \tag{5.40}$$

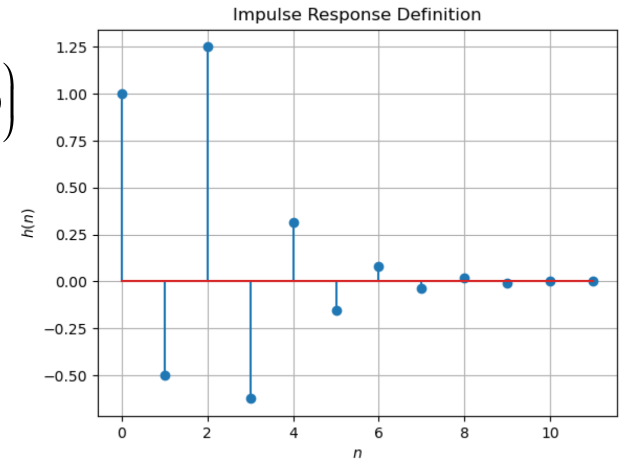

Fig. 5.7: From the definition of $h(n)$

$n = 2$,

$$h(2) + \frac{1}{2}h(1) = 0 + \delta(0) \tag{5.41}$$

$$h(2) = 1 + \frac{1}{4} \tag{5.42}$$

$$= \frac{5}{4} \tag{5.43}$$

And for $n > 2$ RHS will be 0 so,

$$h(n) = -\frac{1}{2}h(n-1) \tag{5.44}$$

Overall

$$h(n) = \begin{cases} 0 & ,n < 0 \\ 1 & ,n = 0 \\ -\frac{1}{2} & ,n = 1 \\ \frac{5}{4} & ,n = 2 \\ -\frac{1}{2}h(n-1) & ,n > 2 \end{cases} \tag{5.45}$$

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.46}$$

Comment. The operation is known as *convolution*.
Solution: Download the code for plot from the below link

```
wget https://github.com/
    sumeethkumar12Signal_Processing/blob/
    master/Sound%201/Codes/ynconv.py
```

Note that the plot is same that as in .

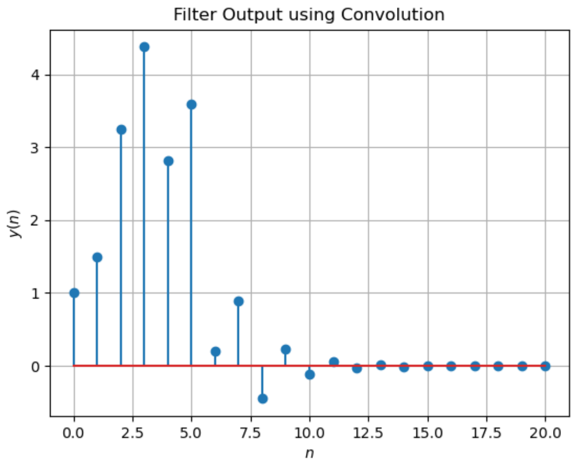

Fig. 5.8: $y(n)$ using the convolution definition

5.9 Express the above convolution using a Toeplitz matrix.
Solution: Download the python code from the below link for the plot,

```
wget https://github.com/
    sumeethkumar12Signal_Processing/blob/
    master/Sound%201/Codes/
    ynconv_toeplitz.py
```

Then run the following command,

```
python3 ynconv_toeplitz.py
```

we express $y(n)$ as

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \tag{5.47}$$

To understand how we can use a Toeplitz matrix, we will see what we are doing in

$$y(0) = x(0)\, h(0) \tag{5.48}$$

$$y(1) = x(0)\, h(1) + x(1)\, h(0) \tag{5.49}$$

$$y(2) = x(0)\, h(2) + x(1)\, h(1) + x(2)\, h(0) \tag{5.50}$$

.

.

The same thing can be written as,

$$y(0) = \begin{pmatrix} h(0) & 0 & 0 & . & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.51}$$

$$y(1) = \begin{pmatrix} h(1) & h(0) & 0 & 0 & . & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.52}$$

$$y(2) = \begin{pmatrix} h(2) & h(1) & h(0) & 0 & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.53}$$

.

.

Using Toeplitz matrix of $h(n)$ we can simplify it as,

$$y(n) = \begin{pmatrix} h(0) & 0 & 0 & . & . & .0 \\ h(1) & h(0) & 0 & . & . & .0 \\ h(2) & h(1) & h(0) & . & . & .0 \\ & & .. & & & \\ & & .. & & & \\ 0 & 0 & 0 & . & . & .h(m-1) \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.54}$$

Now we will take n

$$x(n) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{5.55}$$

And ,

$$h(n) = \begin{pmatrix} 1 \\ -0.5 \\ 1.25 \\ . \\ . \end{pmatrix} \tag{5.56}$$

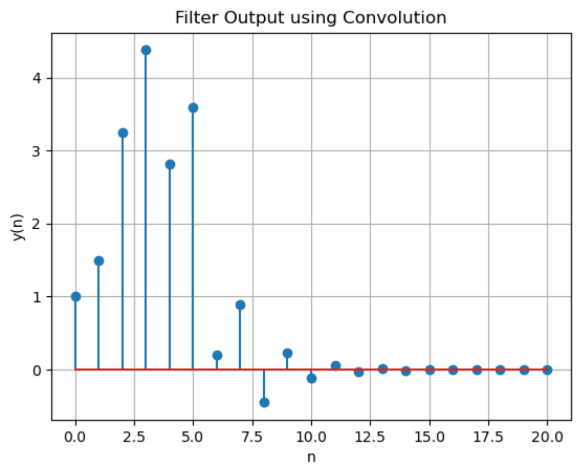

Fig. 5.9: Convolution of $x(n)$ and $h(n)$ using toeplitz matrix

Now ,

$$y(n) = x(n) * h(n) \tag{5.57}$$

$$= \begin{pmatrix} 1 & 0 & 0 & . & . & .0 \\ -0.5 & 1 & 0 & . & . & .0 \\ 1.25 & -0.5 & 1 & . & . & .0 \\ & & \vdots & & & \\ & & \vdots & & & \\ 0 & 0 & 0 & . & . & . \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.58}$$

$$= \begin{pmatrix} 1 \\ 1.5 \\ 3.25 \\ . \\ . \\ . \end{pmatrix} \tag{5.59}$$

5.10 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.60}$$

Solution: Substitute $k := n - k$, we will get

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.61}$$

$$= \sum_{n-k=-\infty}^{\infty} x(n-k)h(k) \tag{5.62}$$

$$= \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.63}$$

6 DFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.
Solution: Run the following codes to compute $X(k)$ which is plotted in fig:6.1.

```
wget https://github.com/sumeethkumar12/
    signal-processing/blob/main/codes/6.1.py
```

Run the following codes to compute $H(k)$.

```
wget https://github.com/sumeethkumar12/
    signal-processing/blob/main/codes/6.12.
    py
```

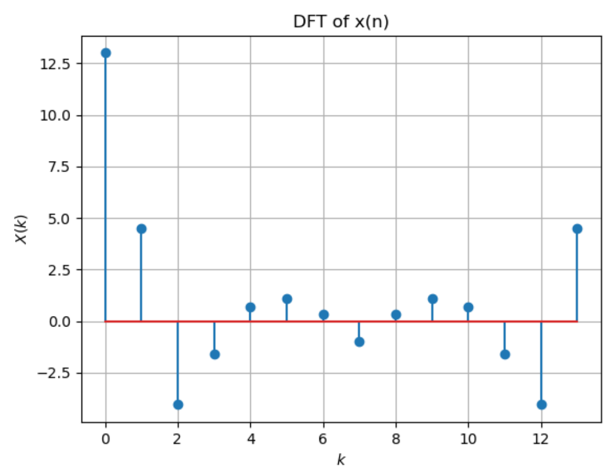

Fig. 6.1: DFT of $x(k)$

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution: Run the following codes to compute $Y(k)$ respectively.

```
wget https://github.com/sumeethkumar12/
    signal-processing/blob/main/codes/6.2.py
```

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\cdot e^{\jmath 2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \tag{6.3}$$

Solution: The following code plots Fig. fig:6.3. Note that this is the same as $y(n)$ in Fig. fig:3.1

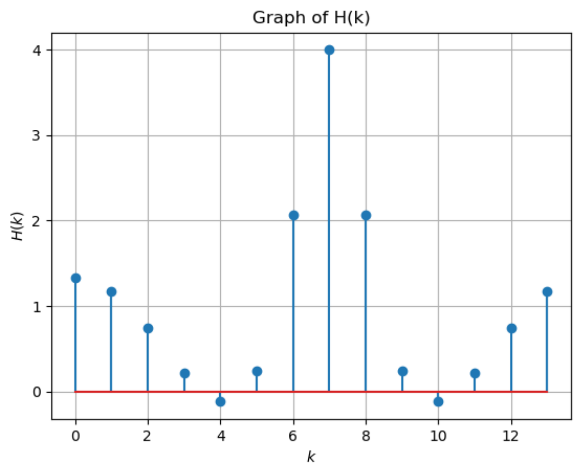

Fig. 6.1: $H(n)$

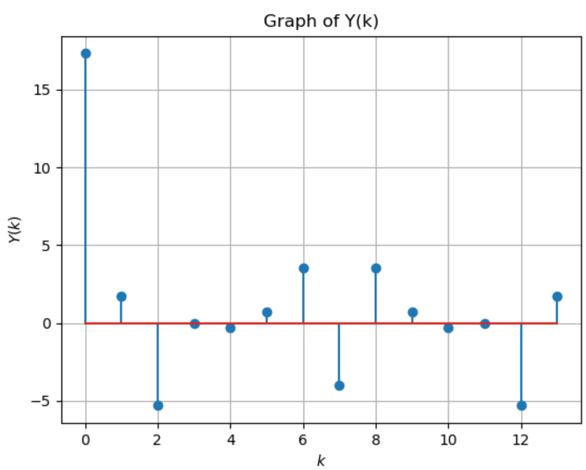

Fig. 6.2: DFT of $xyn)$

```
wget https://github.com/sumeethkumar12/
    signal−processing/blob/main/codes/6.3.py
```

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
Solution: Download the below python code for the plot ,

```
wget https://github.com/sumeethkumar12/
    signal−processing/blob/main/codes/6.4.py
```

Then run the following command,

```
python3 yn_ifft.py
```

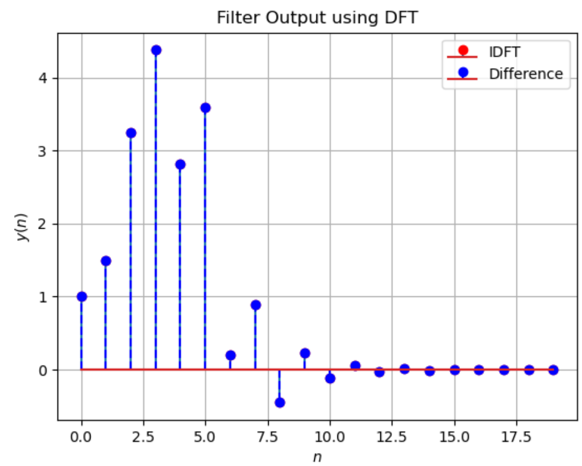

Fig. 6.3: $y(n)$

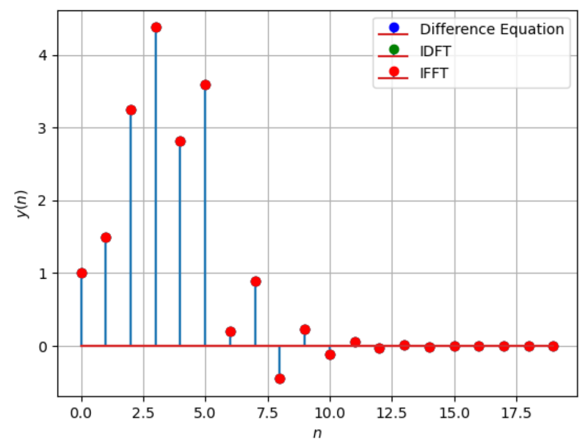

Fig. 6.4: The plot of $y(n)$ using IFFT

7 FFT

1. The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{7.1}$$

2. Let

$$W_N = e^{-j2\pi/N} \tag{7.2}$$

Then the N-point $DFTmatrix$ is defined as

$$\vec{F}_N = \left[W_N^{mn}\right], \quad 0 \le m, n \le N-1 \tag{7.3}$$

where W_N^{mn} are the elements of $\vec{F}_N$.

3. Let
$$\vec{I}_4 = \begin{pmatrix} \vec{e}_4^1 & \vec{e}_4^2 & \vec{e}_4^3 & \vec{e}_4^4 \end{pmatrix} \tag{7.4}$$
be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as
$$\vec{P}_4 = \begin{pmatrix} \vec{e}_4^1 & \vec{e}_4^3 & \vec{e}_4^2 & \vec{e}_4^4 \end{pmatrix} \tag{7.5}$$
4. The 4 point $DFT diagonalmatrix$ is defined as
$$\vec{D}_4 = diag\begin{pmatrix} W_8^0 & W_8^1 & W_8^2 & W_8^3 \end{pmatrix} \tag{7.6}$$
5. Show that
$$W_N^2 = W_{N/2} \tag{7.7}$$
Solution: From (**??**),
$$W_N = e^{-j2\pi/N} \tag{7.8}$$
Consider,
$$W_N^2 = \left(e^{-j2\pi/N}\right)^2 \tag{7.9}$$
$$= e^{-j2\pi/(N/2)} \tag{7.10}$$
$$= W_{N/2} \tag{7.11}$$
Hence proved.
6. Show that
$$\vec{F}_4 = \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \vec{P}_4 \tag{7.12}$$
Solution: From the given eq,
$$\vec{P}_4 = \begin{pmatrix} \vec{e}_4^1 & \vec{e}_4^3 & \vec{e}_4^2 & \vec{e}_4^4 \end{pmatrix} \tag{7.13}$$
Clearly $\vec{P}_4$ is an elementary matrix of $\vec{I}_4$, so on multiplication with a matrix it will interchange the rows/columns of matrix depending on positions of unit vectors.
Generalising the condition ,
$$\vec{P}_N^2 = \vec{I}_N \tag{7.14}$$
So it is similar to prove that,
$$\vec{F}_4\vec{P}_4 = \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \tag{7.15}$$
Now ,
$$\vec{F}_2 = \begin{bmatrix} W_2^{0.0} & W_2^{0.1} \\ W_2^{1.0} & W_2^{1.1} \end{bmatrix} \tag{7.16}$$
$$= \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} \tag{7.17}$$
we can write
$$\vec{F}_2 = \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{bmatrix} \tag{7.18}$$
And $\vec{D}_2$ is a diagonal matrix,
$$\vec{D}_2 = diag\left(W_4^0, W_4^1\right) \tag{7.19}$$
$$= diag\left(1, W_4\right) \tag{7.20}$$
Then,
$$\vec{D}_2\vec{F}_2 = \begin{bmatrix} 1 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{bmatrix} \tag{7.21}$$
$$= \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^1 & W_4^3 \end{bmatrix} \tag{7.22}$$
And for $k \in \mathcal{N}$ and N be a even integer we know that,
$$W_N^{Nk} = 1 \tag{7.23}$$
$$W_N^{Nk+N/2} = -1 \tag{7.24}$$
Using that we can write,
$$-\vec{D}_2\vec{F}_2 = \begin{bmatrix} W_4^2 & W_4^6 \\ W_4^3 & W_4^9 \end{bmatrix} \tag{7.25}$$
And ,
$$\vec{F}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \tag{7.26}$$
And
$$\vec{F}_4\vec{P}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^2 & W_4^1 & W_4^3 \\ W_4^0 & W_4^4 & W_4^2 & W_4^6 \\ W_4^0 & W_4^6 & W_4^3 & W_4^9 \end{bmatrix} \tag{7.27}$$
This is same as,
$$\begin{bmatrix} \vec{F}_2 & \vec{D}_2\vec{F}_2 \\ \vec{F}_2 & -\vec{D}_2\vec{F}_2 \end{bmatrix} \tag{7.28}$$
$$\implies \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \tag{7.29}$$
Hence proved.
7. Show that
$$\vec{F}_N = \begin{bmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{bmatrix} \begin{bmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{bmatrix} \vec{P}_N \tag{7.30}$$

Solution: For N even ; We already know ;

$$\vec{F}_N = [W_N^{mn}], \quad 0 \le m, n \le N-1 \tag{7.31}$$

$$\vec{D}_N\vec{F}_N = \left[W_N^{m.(2k+1)}\right], \quad 0 \le m, k \le \frac{N}{2}-1 \tag{7.32}$$

$$\vec{F}_N\vec{P}_N = \begin{bmatrix} W_N^{2mk} & W_N^{m.(2k+1)} \\ W_N^{2mk+Nk} & W_N^{m.(2k+1)+\frac{N}{2}.(2k+1)} \end{bmatrix}$$
$$0 \le m, k \le \frac{N}{2}-1$$

$$\vec{F}_N\vec{P}_N = \begin{bmatrix} W_N^{2mk} & W_N^{m.(2k+1)} \\ W_N^{2mk} & -W_N^{m.(2k+1)} \end{bmatrix} \tag{7.33}$$

$$\vec{F}_N\vec{P}_N = \begin{bmatrix} W_{N/2}^{mk} & W_{N/2}^{m.(k+1/2)} \\ W_{N/2}^{mk} & -W_{N/2}^{m.(k+1/2)} \end{bmatrix} \tag{7.34}$$

$$\vec{F}_N\vec{P}_N = \begin{bmatrix} \vec{F}_{N/2} & \vec{D}_{N/2}\vec{F}_{N/2} \\ \vec{F}_{N/2} & -\vec{D}_{N/2}\vec{F}_{N/2} \end{bmatrix} \tag{7.35}$$

Following;

$$\vec{F}_N = \begin{bmatrix} \vec{F}_{N/2} & \vec{D}_{N/2}\vec{F}_{N/2} \\ \vec{F}_{N/2} & -\vec{D}_{N/2}\vec{F}_{N/2} \end{bmatrix}\vec{P}_N \tag{7.36}$$

From above it follows ;

$$\vec{F}_N = \begin{bmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{bmatrix}\begin{bmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{bmatrix}\vec{P}_N \tag{7.37}$$

8. Find

$$\vec{P}_4\vec{x} \tag{7.38}$$

Solution: From ,

$$\vec{P}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.39}$$

$$\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.40}$$

After proper zero padding of $\vec{P}_4$,

$$\vec{P}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.41}$$

$$\vec{P}_4\vec{x} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.42}$$

$$= \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 0 \\ 0 \end{pmatrix} \tag{7.43}$$

9. Show that

$$\vec{X} = \vec{F}_N\vec{x} \tag{7.44}$$

where $\vec{x}, \vec{X}$ are the vector representations of $x(n), X(k)$ respectively.
Solution: Given $\vec{x}, \vec{X}$ are the vector representations of $x(n), X(k)$ respectively.

$$\vec{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \tag{7.45}$$

$$\vec{X} = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} \tag{7.46}$$

$$\vec{F}_N = \begin{bmatrix} 1 & 1 & ! & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \tag{7.47}$$

As

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \tag{7.48}$$

Upon linear transformation over k,

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{N-1} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \tag{7.49}$$

$\therefore \vec{X} = \vec{F}_N \vec{x}$

10. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.50}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.51}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.52}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.53}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.54}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.55}$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{7.56}$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{7.57}$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{7.58}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.59}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.60}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.61}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.62}$$

11. For

$$\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.63}$$

compte the DFT using 7.44 **Solution:**

$$\vec{F}_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-j\pi/3} & e^{-j2\pi/3} & e^{-j\pi} & e^{-j4\pi/3} & e^{-j5\pi/3} \\ 1 & e^{-j2\pi/3} & e^{-j4\pi/3} & e^{-j2\pi} & e^{-j8\pi/3} & e^{-j10\pi/3} \\ 1 & e^{-j\pi} & e^{-j2\pi} & e^{-j3\pi} & e^{-j4\pi} & e^{-j5\pi} \\ 1 & e^{-j4\pi/3} & e^{-j8\pi/3} & e^{-j4\pi} & e^{-j16\pi/3} & e^{-j20\pi/3} \\ 1 & e^{-j5\pi/3} & e^{-j10\pi/3} & e^{-j5\pi} & e^{-j20\pi/3} & e^{-j25\pi/3} \end{bmatrix} \tag{7.64}$$

$$\vec{X} = \vec{F}_6 \vec{x} \tag{7.65}$$

$$\vec{X} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-j\pi/3} & e^{-j2\pi/3} & e^{-j\pi} & e^{-j4\pi/3} & e^{-j5\pi/3} \\ 1 & e^{-j2\pi/3} & e^{-j4\pi/3} & e^{-j2\pi} & e^{-j8\pi/3} & e^{-j10\pi/3} \\ 1 & e^{-j\pi} & e^{-j2\pi} & e^{-j3\pi} & e^{-j4\pi} & e^{-j5\pi} \\ 1 & e^{-j4\pi/3} & e^{-j8\pi/3} & e^{-j4\pi} & e^{-j16\pi/3} & e^{-j20\pi/3} \\ 1 & e^{-j5\pi/3} & e^{-j10\pi/3} & e^{-j5\pi} & e^{-j20\pi/3} & e^{-j25\pi/3} \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.66}$$

$$= \begin{bmatrix} 13 \\ -3.12 - 6.53j \\ 1j \\ 1.12 - 0.53j \\ -1j \\ 1.12 + 0.53j \end{bmatrix} \quad (7.67)$$

12. Repeat the above exercise using the FFT after zero padding $\vec{x}$.
Solution: $\vec{x}$ after padding is

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad (7.68)$$

Using 8-point fft ,

$$\vec{F}_8 = \begin{bmatrix} \vec{I}_4 & \vec{D}_4 \\ \vec{I}_4 & -\vec{D}_4 \end{bmatrix} \begin{bmatrix} \vec{F}_4 & 0 \\ 0 & \vec{F}_4 \end{bmatrix} \vec{P}_8 \quad (7.69)$$

$$\vec{F}_4 = \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \vec{P}_4 \quad (7.70)$$

$$\vec{F}_2 = \begin{bmatrix} \vec{I}_1 & \vec{D}_1 \\ \vec{I}_1 & -\vec{D}_1 \end{bmatrix} \begin{bmatrix} \vec{F}_1 & 0 \\ 0 & \vec{F}_1 \end{bmatrix} \vec{P}_2 \quad (7.71)$$

$$\vec{F}_1 = [1] \quad (7.72)$$

Calculating $\vec{F}_2$,

$$\vec{F}_2 = \begin{bmatrix} \vec{F}_1 & D_1\vec{F}_1 \\ \vec{F}_1 & -D_1\vec{F}_1 \end{bmatrix} \vec{P}_2 \quad (7.73)$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (7.74)$$

Calculating $\vec{F}_4$,

$$\vec{D}_2 = diag(1, W_4) = \begin{bmatrix} 1 & 0 \\ 0 & -j \end{bmatrix} \quad (7.75)$$

$$D_2\vec{F}_2 = \begin{bmatrix} 1 & 0 \\ 0 & -j \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (7.76)$$

$$= \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} \quad (7.77)$$

$$\vec{F}_4 = \begin{bmatrix} \vec{F}_2 & D_2\vec{F}_2 \\ \vec{F}_2 & -D_2\vec{F}_2 \end{bmatrix} \vec{P}_4 \quad (7.78)$$

$$\vec{F}_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -j & j \\ 1 & 0 & -1 & -1 \\ 0 & 1 & j & -j \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (7.79)$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -j & 1 & j \\ 1 & -1 & 0 & j \\ 0 & j & 1 & -j \end{bmatrix} \quad (7.80)$$

Calculating $\vec{F}_8$,

$$\vec{D}_4 = diag\left(1, W_8, W_8^2, W_8^3\right) \quad (7.81)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1-j}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{-1-j}{\sqrt{2}} \end{bmatrix} \quad (7.82)$$

$$D_4\vec{F}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1-j}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{-1-j}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -j & 1 & j \\ 1 & -1 & 0 & j \\ 0 & j & 1 & -j \end{bmatrix} \quad (7.83)$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & \frac{-1-j}{\sqrt{2}} & \frac{1-j}{\sqrt{2}} & \frac{1+j}{\sqrt{2}} \\ -1 & 1 & 0 & -j \\ 0 & \frac{1-j}{\sqrt{2}} & \frac{-1-j}{\sqrt{2}} & \frac{-1+j}{\sqrt{2}} \end{bmatrix} \quad (7.84)$$

$F_8 = A\vec{B}P_8$ where

$$\vec{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{-1-j}{\sqrt{2}} & \frac{1-j}{\sqrt{2}} & \frac{1+j}{\sqrt{2}} \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -j \\ 0 & 0 & 0 & 1 & 0 & \frac{1-j}{\sqrt{2}} & \frac{-1-j}{\sqrt{2}} & \frac{-1+j}{\sqrt{2}} \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1+j}{\sqrt{2}} & \frac{-1+j}{\sqrt{2}} & \frac{-1-j}{\sqrt{2}} \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & j \\ 0 & 0 & 0 & 1 & 0 & \frac{-1+j}{\sqrt{2}} & \frac{1+j}{\sqrt{2}} & \frac{1-j}{\sqrt{2}} \end{bmatrix} \tag{7.85}$$

$$\vec{B} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -j & 1 & j & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & j & 0 & 0 & 0 & 0 \\ 0 & j & 1 & -j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & j & -1 & j \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -j \\ 0 & 0 & 0 & 0 & 0 & -j & -1 & j \end{bmatrix} \tag{7.86}$$

And $\vec{P_8}$ is a permutation matrix.

$$\vec{X} = \begin{bmatrix} 13 \\ -3.12 - 6.53j \\ 1j \\ 1.12 - 0.53j \\ -1 \\ 1.12 + 0.53j \\ -1j \\ -3.12 + 6.53j \end{bmatrix} \tag{7.87}$$

13. Write a C program to compute the 8-point FFT. **Solution:**

```
wget https://github.com/
    sumeethkumar12/signal−processing/
    blob/main/codes/713.c
gcc 713.c −lm
```

8 Exercises

Answer the following questions by looking at the python code.

8.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

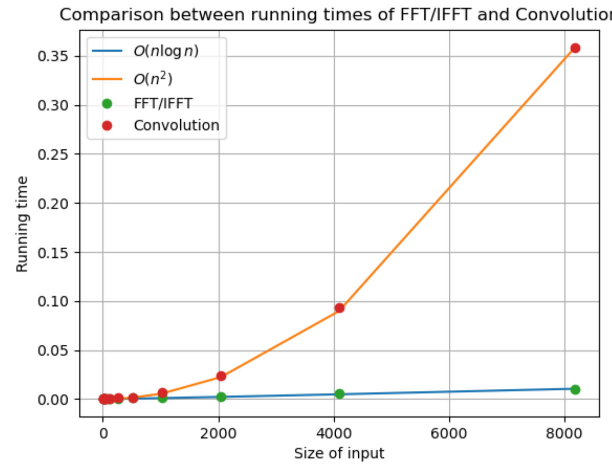

Fig. 7.4: Plot of the running times of FFT/IFFT and convolution

in solution is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \tag{8.1}$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.

Solution: On taking the Z-transform on both sides of the difference equation

$$\sum_{m=0}^{M} a(m)\, z^{-m} Y(z) = \sum_{k=0}^{N} b(k)\, z^{-k} X(z) \tag{8.2}$$

$$\implies H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b(k)\, z^{-k}}{\sum_{m=0}^{M} a(m)\, z^{-m}} \tag{8.3}$$

For obtaining the discrete Fourier transform, put $z = \jmath\frac{2\pi i}{I}$ where I is the length of the input signal and $i = 0, 1, \ldots, I-1$

Download the following Python code that does the above

```
wget
```

Run the code by executing

```
python3 8_1.py
```

8.2 Repeat all the exercises in the previous sections for the above a and b

Solution: The polynomial coefficients obtained are

$$\vec{a} = \begin{pmatrix} 1.000 \\ -2.519 \\ 2.561 \\ -1.206 \\ 0.220 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 0.003 \\ 0.014 \\ 0.021 \\ 0.014 \\ 0.003 \end{pmatrix} \tag{8.4}$$

The difference equation is then given by

$$\vec{a}^{\top}\vec{y} = \vec{b}^{\top}\vec{x} \tag{8.5}$$

where

$$\vec{y} = \begin{pmatrix} y(n) \\ y(n-1) \\ y(n-2) \\ y(n-3) \\ y(n-4) \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} x(n) \\ x(n-1) \\ x(n-2) \\ x(n-3) \\ x(n-4) \end{pmatrix} \tag{8.6}$$

We have

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b\left(k\right) z^{-k}}{\sum_{m=0}^{M} a\left(m\right) z^{-m}} \tag{8.7}$$

By using partial fraction decomposition, we can write this as

$$H(z) = \sum_i \frac{r(i)}{1 - p(i)z^{-1}} + \sum_j k(j)z^{-j} \tag{8.8}$$

On taking the inverse Z-transform on both sides by using

$$H(z) \overset{\mathcal{Z}}{\rightleftharpoons} h(n) \tag{8.9}$$

$$\frac{1}{1 - p(i)z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} \left(p(i)\right)^n u(n) \tag{8.10}$$

$$z^{-j} \overset{\mathcal{Z}}{\rightleftharpoons} \delta(n-j) \tag{8.11}$$

Thus

$$h(n) = \sum_i r(i)\left(p(i)\right)^n u(n) + \sum_j k(j)\delta(n-j) \tag{8.12}$$

Download the following Python code

```
wget https://github.com/karthik6281/
Signal−Processing/tree/main/
Assignment1/codes/8_2.py
```

Run the code by executing

```
python3 8_2.py
```

The above code outputs the values of $r(i), p(i), k(i)$

$$h(n) = \Re\left((0.24 - 0.71\jmath)(0.56 + 0.14\jmath)^n\right) u(n) + \Re\left((0.24 + 0.71\jmath)(0.56 - 0.14\jmath)^n\right) u(n) + 0.016\delta(n) \tag{8.13}$$

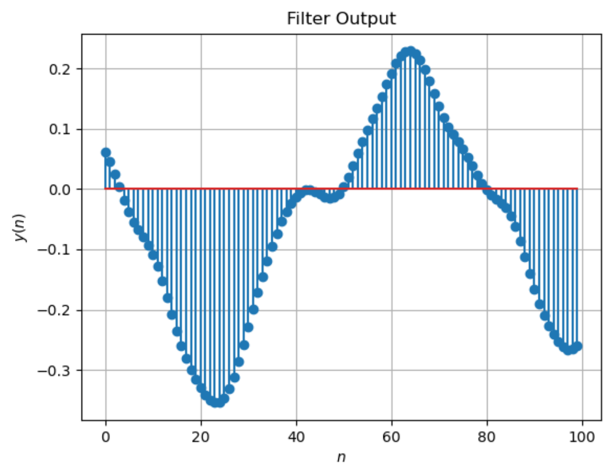

Fig. 8.4: Plot of $y(n)$

./figs/8_2_2.png

Fig. 8.4: Plot of $\left|H(e^{j\omega})\right|$

8.3 What is the sampling frequency of the input signal?
Solution: Sampling frequency(fs)=44.1kHZ.

8.4 What is type, order and cutoff-frequency of the above butterworth filter

./figs/8_2_3.png

Fig. 8.4: Plot of $h(n)$

Solution: The given butterworth filter is low pass with order=4 and cutoff-frequency=4kHz.

8.5 Modifying the code with different input parameters and to get the best possible output.
Solution: Order: 10 Cutoff frequency: 3000 Hz