Digital Signal Processing

sumeeth kumar

Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands (commands may change depending on Linux distro)

```
$ sudo apt update && sudo apt upgrade
$ sudo apt install libffi−dev libsndfile1 python3−
    scipy python3−numpy python3−matplotlib
$ pip3 install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file using

```
$https://github.com/sumeethkumar12/signal−
    processing/blob/main/
    Sound_With_ReducedNoise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution: Download the source code using

```
$ https://github.com/sumeethkumar12/signal−
    processing/blob/main/codes/cancel_noise.
    py
```

and execute it using

```
$ python3 cancel_noise.py
```

2.4 The output of the python script in Problem 2.3 is the audio file `Sound_With_ReducedNoise.wav`. Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{\underset{\uparrow}{1}, 2, 3, 4, 2, 1\right\} \tag{3.1}$$

Sketch $x(n)$.

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The following code yields Fig. (3.2).

```
$ https://github.com/sumeethkumar12/signal−
    processing/blob/main/codes/xnyn.py
```

and execute it using

```
$ python3 xnyn.py
```

3.3 Repeat the above exercise using a C code.

```
$ https://github.com/sumeethkumar12/signal−
    processing/blob/main/codes/xnyn.c
```

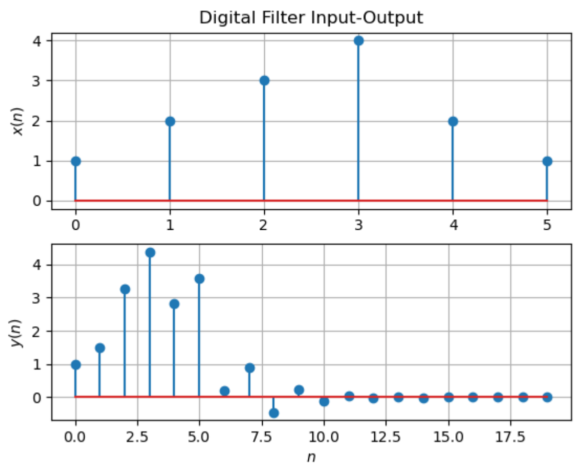

Fig. 3.2: Plot of $x(n)$ and $y(n)$

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: Given that,

$$X(z) = \mathcal{Z}\{x(n)\} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{4.5}$$

So,

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1) z^{-n} \tag{4.6}$$

Take $k = n - 1$,

$$= \sum_{k=-\infty}^{\infty} x(k) z^{-(k+1)} \tag{4.7}$$

$$= z^{-1} \sum_{k=-\infty}^{\infty} x(k) z^{-k} \tag{4.8}$$

$$= z^{-1} \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{4.9}$$

$$= z^{-1} X(z) \tag{4.10}$$

resulting in (4.2) and similarly following the above steps you will get,

$$\mathcal{Z}\{x(n-k)\} = z^{-k} X(n) \tag{4.11}$$

4.2 Now we will find Z transform of the signal $x(n)$,from (**??**),

$$\mathcal{Z}\{x(n)\} = \sum_{n=0}^{5} x(n) z^{-n} \tag{4.12}$$

$$= 1z^0 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + 1z^{-5} \tag{4.13}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \tag{4.14}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.15}$$

from (3.2) assuming that the Z-transform is a linear operation.

Solution: Applying (4.11) in (3.2),

$$Y(z) + \frac{1}{2} z^{-1} Y(z) = X(z) + z^{-2} X(z) \tag{4.16}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2} z^{-1}} \tag{4.17}$$

Solution: Now we will rewrite (3.2),

$$y(n) + \frac{1}{2} y(n-1) = x(n) + x(n-2) \tag{4.18}$$

Now since Z-transform is a linear operator we can write that,

$$Y(n) + \frac{1}{2} Y(n-1) = X(n) + X(n-2) \tag{4.19}$$

From (4.11),

$$Y(n) + \frac{z^{-1}}{2} Y(n) = X(n) + z^{-2} X(n) \tag{4.20}$$

$$\implies \frac{Y(n)}{X(n)} = \frac{1 + z^{-2}}{1 + \frac{z^{-1}}{2}} \tag{4.21}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.22}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.23}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.24}$$

Solution: The Z-transform of δn is,

$$\mathcal{Z}\{\delta n\} = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n} \tag{4.25}$$

$$= \delta(0) z^0 + 0 \ \ (\text{Using (4.22)}) \tag{4.26}$$

$$= 1 \tag{4.27}$$

and the Z-transform of unit-step function $u(n)$ is,

$$U(n) = \sum_{n=-\infty}^{\infty} u(n) z^{-n} \tag{4.28}$$

$$= 0 + \sum_{n=0}^{\infty} 1.z^{-n} \tag{4.29}$$

$$= 1 + z^{-1} + z^{-2} + \dots \tag{4.30}$$

Above is a infinite geometric series with z^{-1} as common ratio , so we can write it as

$$U(n) = \frac{1}{1 - z^{-1}} \because |z| > 1 \tag{4.31}$$

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.32}$$

Solution: The Z- transform will be

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=0}^{\infty} a^n z^{-n} \tag{4.33}$$

$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \dots. \tag{4.34}$$

Above is a infinite geometric series with first term 1 and common ratio as $\frac{a}{z}$ and it can be written as,

$$\mathcal{Z}\{a^n u(n)\} = \frac{1}{1 - \frac{a}{z}} \because |a| < |z| \tag{4.35}$$

Therefore,

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.36}$$

4.6 let

$$H(e^{j\omega}) = H(z = e^{j\omega}). \tag{4.37}$$

Plot $|H(e^{j\omega})|$. Comment. $H(e^{j\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $x(n)$. **Solution:** The following code plots Fig. (4.6).

```
$ https://github.com/sumeethkumar12/signal-processing/blob/main/codes/dtft.py
```

The figure can be generated using

```
$ python3 dtft.py
```

We observe that $\left|H\left(e^{j\omega}\right)\right|$ is periodic with fundamental period 2π. Now using (4.17), we will

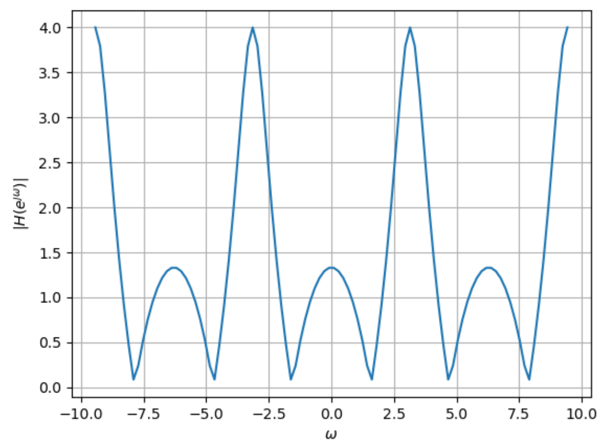

Fig. 4.6: Plot of $\left|H\left(e^{j\omega}\right)\right|$ against ω

find $\left|H\left(e^{j\omega}\right)\right|$,

$$H\left(e^{j\omega}\right) = \frac{1 + e^{-2j\omega}}{1 + \frac{e^{-j\omega}}{2}} \tag{4.38}$$

$$\implies \left|H\left(e^{j\omega}\right)\right| = \frac{\left|1 + e^{-2j\omega}\right|}{\left|1 + \frac{e^{-j\omega}}{2}\right|} \tag{4.39}$$

$$= \frac{\left|1 + e^{2j\omega}\right|}{\left|e^{2j\omega} + \frac{e^{j\omega}}{2}\right|} \tag{4.40}$$

$$= \frac{|1 + \cos 2\omega + j \sin 2\omega|}{\left|e^{j\omega} + \frac{1}{2}\right|} \tag{4.41}$$

$$= \frac{\left|4\cos^2(\omega) + 4j\sin(\omega)\cos(\omega)\right|}{|2e^{j\omega} + 1|} \tag{4.42}$$

$$= \frac{|4\cos(\omega)|\,|\cos(\omega) + j\sin(\omega)|}{|2\cos(\omega) + 1 + 2j\sin(\omega)|} \tag{4.43}$$

$$\therefore \left|H\left(e^{j\omega}\right)\right| = \frac{|4\cos(\omega)|}{\sqrt{5 + 4\cos(\omega)}} \tag{4.44}$$

Using the above expression we can say that graph is symmetric about origin and has a period of 2π.

4.7 Express $h(n)$ in terms of $H\left(e^{j\omega}\right)$.

Solution:

$$\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}\omega \tag{4.45}$$

$$= \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} h(k)e^{-\jmath\omega k}e^{\jmath\omega n}\omega \tag{4.46}$$

$$= \sum_{k=-\infty}^{\infty} h(k) \int_{-\pi}^{\pi} e^{\jmath\omega(n-k)}\omega \tag{4.47}$$

Now,

$$\int_{-\pi}^{\pi} e^{\jmath\omega(n-k)}\omega = \begin{cases} \int_{-\pi}^{\pi}\omega & n-k=0 \\ \left.\frac{\exp\left(\jmath\omega(n-k)\right)}{\jmath(n-k)}\right|_{-\pi}^{\pi} & n-k\neq 0 \end{cases} \tag{4.48}$$

$$= \begin{cases} 2\pi & n-k=0 \\ 0 & n-k\neq 0 \end{cases} \tag{4.49}$$

$$= 2\pi\delta(n-k) \tag{4.50}$$

Thus,

$$\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}\omega = 2\pi \sum_{k=-\infty}^{\infty} h(k)\delta(n-k) \tag{4.51}$$

$$= 2\pi h(n) * \delta(n) \tag{4.52}$$

$$= 2\pi h(n) \tag{4.53}$$

Therefore, $h(n)$ is given by the inverse DTFT (IDTFT) of $H\left(e^{\jmath\omega}\right)$

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}\omega \tag{4.54}$$

5 Impulse Response

5.1 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.1}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).

Solution: From (4.17),

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.2}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^{n} u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.3}$$

using (4.32) and (**??**).

5.2 Sketch $h(n)$. Is it bounded? Convergent?
Solution: The following code plots Fig. **??**.

```
wget https://github.com/sumeethkumar12/
    signal-processing/blob/main/figs/h_n.png
```

Use the following command in the terminal to run the code

```
python3 h_n.py
```

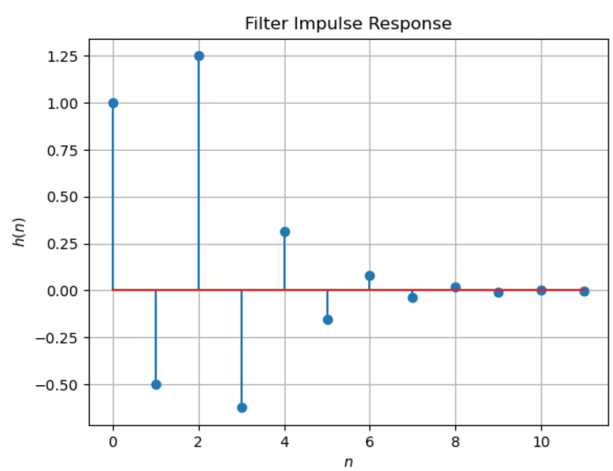

Fig. 5.2: $h(n)$ as the inverse of $H(z)$

we can say it is bounded and convergent

5.3 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.4}$$

Is the system defined by (3.2) stable for the impulse response in (5.1)?

Solution:

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{5.5}$$

$$u(n-2) = \begin{cases} 1 & n \geq 2 \\ 0 & n < 2 \end{cases} \tag{5.6}$$

$$\therefore h(n) = \begin{cases} 0 & n < 0 \\ \left(\frac{-1}{2}\right)^n & 0 \leq n < 2 \\ \left(\frac{-1}{2}\right)^n + \left(\frac{-1}{2}\right)^{(n-2)} & n \geq 2 \end{cases} \tag{5.7}$$

$$\therefore \sum_{n=-\infty}^{\infty} h(n) = 0 + 1 + \frac{-1}{2} + \sum_{n=2}^{\infty} \left[\left(\frac{-1}{2}\right)^n + \left(\frac{-1}{2}\right)^{(n-} \right] \tag{5.8}$$

$$= \frac{1}{2} + \frac{5}{4} * \left(\frac{2}{3}\right) = \frac{4}{3} < \infty \tag{5.9}$$

$$\tag{5.10}$$

$\therefore$ system defined is stable

5.4 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.11}$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. 5.4. Note that this is the same as Fig. **??**.

```
wget https://github.com/sumeethkumar12/
    signal-processing/blob/main/codes/hndef.
    py
```

Use the following command in the terminal to run the code

```
python3 hndef.py
```

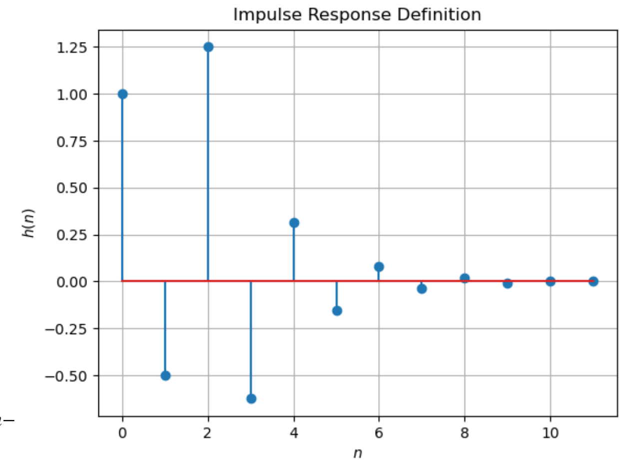

Fig. 5.4: $h(n)$ from the definition

```
python3 ynconv.py
```

5.5 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.12}$$

Comment. The operation in (5.12) is known as *convolution*.
Solution: The following code plots Fig. 5.5. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://github.com/sumeethkumar12/
    signal-processing/blob/main/codes/
    ynconv.py
```

Use the following command in the terminal to run the code

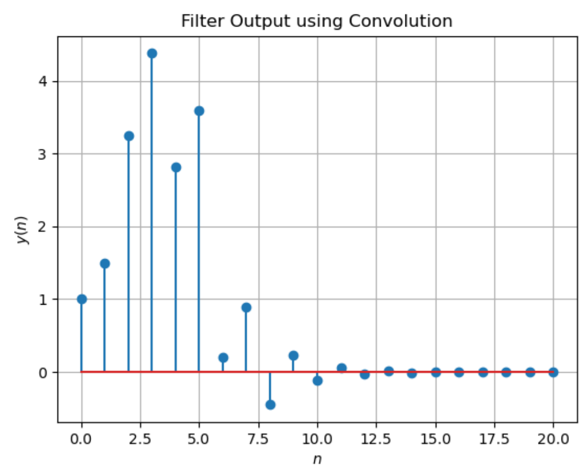

Fig. 5.5: $y(n)$ from the definition of convolution

5.6 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.13}$$

Solution: wkt

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.14}$$

Replacing k with $n-k$

$$= \sum_{n-k=-\infty}^{\infty} x(n-k)h(k) \therefore y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.15}$$