Duality : given (primal)

$$\begin{cases} \max \ c^T x \\ Ax \le b \\ 0 \le x \end{cases}$$

Dual:

$$\begin{cases} \min \ b^T y \\ \text{s.t.} \ A^T y \geq c \\ y \geq 0 \end{cases}$$

"weak duality":

value of any primal feasible soln ≤

" " " dual feasible soln

How about duality for:

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax = b \\ & x \geqslant 0 \end{array} \quad ?$$

$$\begin{array}{ll} \max & 2x_1 - 2x_2 + \quad - 10x_4 \\ 3 & 2x_1 + x_2 + 3x_3 - x_4 = 9 \\ -1 & 3x_1 + 4x_2 - x_3 + 2x_4 = 16 \\ & x \geqslant 0 \end{array}$$

$$\begin{array}{ll} (1,1) & 5x_1 + 5x_2 + 2x_3 + x_4 = 25 \\ (3,-1) & 3x_1 - x_2 + 10x_3 - 5x_4 = 11 \end{array}$$

$$(y^T A - c^T) x = 0$$

primal $\quad \max \; c^T x$

s.t. $Ax = b$

$x \geq 0$

dual: $\quad \min \; \boxed{b^T y}$

s.t. $A^T y \geq c$

y unrestricted in sign

weak duality: if x primal feasible and y dual feasible

then $\quad c^T x \leq b^T y$

Proof: $\quad b^T y = (Ax)^T y = x^T A^T y \geq x^T c = c^T x$

strong duality: If the primal is feasible and bounded then the primal and the dual LPs have the same value.

Look at the tableau at the optimal basis

Suppose B is the optimal basis

$$
\begin{aligned}
z \quad - c_B^T x_B \quad - c_N^T x_N &= 0 \\
B x_B + A_N x_N &= b
\end{aligned}
$$

$$
\begin{aligned}
z \quad - c_B^T x_B \quad - c_N^T x_N &= 0 \\
x_B + B^{-1} A_N x_N &= B^{-1} b
\end{aligned}
\qquad
\begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} B^{-1} b \\ 0 \end{pmatrix}
$$

$$
\begin{aligned}
z \quad + (c_B^T B^{-1} A_N - c_N^T) x_N &= c_B^T B^{-1} b \\
x_B + B^{-1} A_N x_N &= B^{-1} b
\end{aligned}
$$

$$z - c_B^T x_B - c_N^T x_N = 0$$
$$B x_B + A_N x_N = b$$

$$z + (c_B^T B^{-1} A_N - c_N^T) x_N = c_B^T B^{-1} b$$
$$x_B + B^{-1} A_N x_N = B^{-1} b$$

Optimal tableau: so value of primal LP = $\boxed{c_B^T B^{-1} b}$

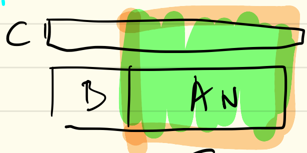
optimal primal solution = $\begin{array}{cc} \begin{pmatrix} B^{-1} b \\ 0 \end{pmatrix} & \begin{array}{c} B \\ N \end{array} \end{array}$

$$z = c_B^T B^{-1} b + (c_N^T - c_B^T B^{-1} A_N) x_N$$

$$c_B^T B^{-1} A_N \geq c_N^T$$

$$c_B^T B^{-1} A \geq c^T$$

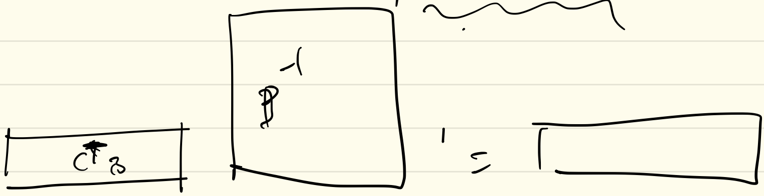
$$A^T y \geq c$$



$$y^T = c_B^T B^{-1} \qquad y^T A \geq c^T$$

Def: $y = (c_B^T B^{-1})^T$

$$b^T y = \underbrace{c_B^T B^{-1}}\, b = \text{value of primal}$$



$$\max \; c^T x \quad \text{s.t.} \quad Ax = b, \quad x \geq 0$$

$$\min \; b^T y \quad \text{s.t.} \quad A^T y \geq c$$

Complementary slackness: If x and y are primal and dual feasible (resp.) then x and y are optimal if: $(y^T A - c^T) x = 0$

$$y^T A x - c^T x = y^T b - c^T x = 0$$

primal: $\max\ c^T x$

s.t. $Ax \le b$

$x \geq 0$

dual: $\min\ b^T y$

s.t. $A^T y \geq c$

$y \geq 0$

$\max\ c^T y$

s.t. $Ax + Is = b$

$x \geq 0 \quad s \geq 0$

$\min\ b^T y$

$\begin{pmatrix} A^T \\ I \end{pmatrix} y \geq \begin{pmatrix} c \\ 0 \end{pmatrix}$

y unrestricted