IEOR 4004
Lecture 1 - Introduction to Optimization

Example 1. Cash flows
We have \$100 to invest. There are three investment vehicles.

a. for every \$1 invested now, we get 0.1 one year from now, and \$1.3 three years from now.

b. for every \$1 invested now, we get 0.2 one year from now, and \$1.1 two years from now.

c. for every \$1 invested a year from now, we get 1.5 three years from now.

d. money-market account, 2%1 per year.

Goal: to maximize value in three years.
First: we model the decisions to be made using numerical variables:

- x_a, x_b, x_c, x_d: quantities invested in each option

Next: what other numerical quantities are useful?

- y_1, y_2, y_3: cash available at the end of each period.

Next describe logical relationships using equations.

$$x_a + x_b + x_d \;=\; 100.$$

What else?

$$\begin{aligned} y_1 &= 0.1x_a + 0.2x_b + 1.02x_d && (1)\\ x_c &\leq y_1 \qquad \text{why?} && (2)\\ y_2 &= 1.1x_b + 1.02(y_1 - x_c) && (3)\\ y_3 &= 1.02y_2 + 1.3x_a + 1.5x_c && (4) \end{aligned}$$

What else?

$$x_a, x_b, x_c, x_d \geq 0$$

What else?

$$\max \; y_3$$

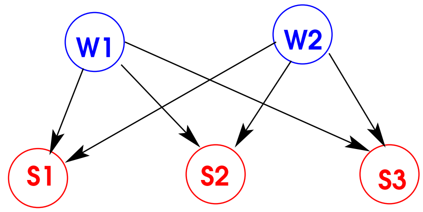
Let's solve it using Gurobi.

Example 2. (Transportation.) A paper company wants to ship truckloads of paper from warehouses to stores.

- There are two warehouses with supplies of 100 and 200 units resp.
- There are three stores with demands 50, 100, 150 units resp.
- Per-unit transportation costs:

	S1	S2	S3
W1	10	12	14
W2	11	12	13

- Problem: minimize total transportation cost.



Formulation as an LP:

$$
\begin{aligned}
&\min\ 10x_{11} + 12x_{12} + 14x_{13} + 11x_{21} + 12x_{22} + 13x_{23} \\
&\text{Subject to} \\
&\quad x_{11} + x_{12} + x_{13} \ \le\ 100 \\
&\quad x_{21} + x_{22} + x_{23} \ \le\ 200 \\
&\quad x_{11} + x_{21} \ \ge\ 50 \\
&\quad x_{12} + x_{22} \ \ge\ 100 \\
&\quad x_{13} + x_{23} \ \ge\ 150 \\
&\quad x \ge 0
\end{aligned} \tag{5}
$$

Example 3. A familiar example: data mining / linear regression

- We are given vectors $x_1, x_2, \ldots, x_N$ all in n-vectors.
- We are given scalar values $y_1, y_2, \ldots, y_N$.
- We want to come up with an "approximate" **linear** model for the data. We want to find n-vectors a and b so that

$$y_i \ \approx\ a^T x_i + b, \quad 1 \le i \le N$$

Method 1. The traditional method: linear regression.

$$\min_{a,b} \left\{ \sum_{i=1}^{N} (a^T x_i + b - y_i)^2 \right\}.$$

What kind of problem is this? What does it try to do?
Method 2. Non-traditional method.

$$\min_{a,b}\ \max_{1\le i\le N} |a^T x_i + b - y_i|.$$

What kind of problem is this? What does it try to do?
Actually, this is another linear program: let's do it in two steps.

$$\min\ V \tag{6}$$
$$\text{Subject to} \tag{7}$$
$$V \ \ge\ |a^T x_i + b - y_i|, \qquad 1 \le i \le N. \tag{8}$$

Here the **variables** are V, a (both n-vectors) and b. But is this really linear? No: (8) is not linear.

Second attempt:

$$\min\ V \tag{9}$$
$$\text{Subject to} \tag{10}$$
$$V \ \ge\ a^T x_i + b - y_i, \qquad 1 \le i \le N, \tag{11}$$
$$V \ \ge\ -a^T x_i - b + y_i, \qquad 1 \le i \le N. \tag{12}$$

This is linear!

Broad modeling issues:

- Linear or nonlinear?

- Continuous variables or discrete?

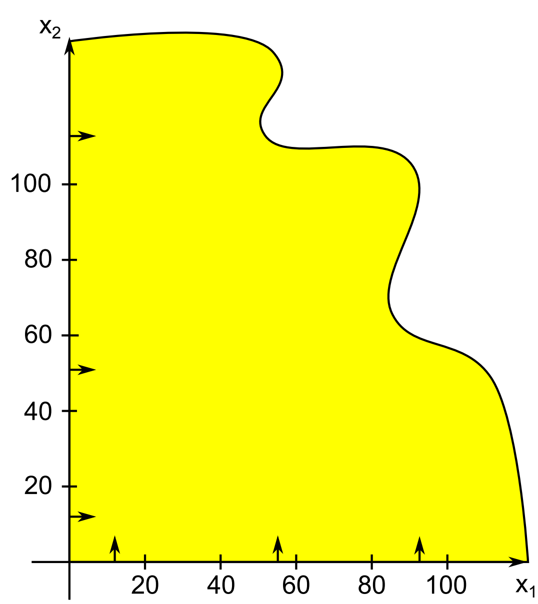
- Convex model or not?

0.1 Graphical method

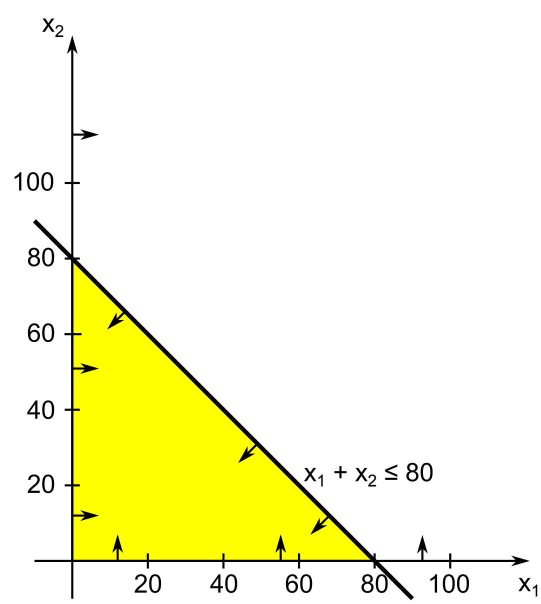
$$\begin{array}{lrcrcl} \text{Max} & 3x_1 & + & 2x_2 & & \\ & x_1 & + & x_2 & \leq & 80 \\ & 2x_1 & + & x_2 & \leq & 100 \\ & x_1 & & & \leq & 40 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

1. Find the feasible region.

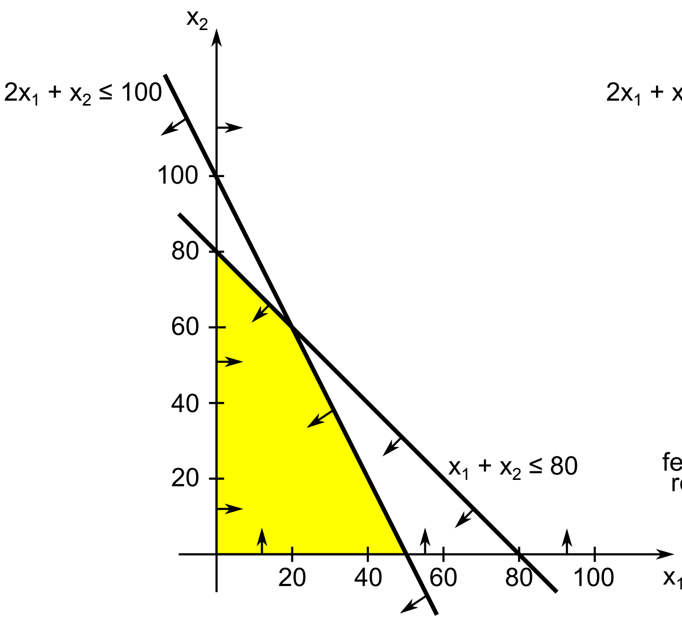
   - Plot each constraint as an equation $\equiv$ line in the plane
   - Feasible points on one side of the line – plug in (0,0) to find out which



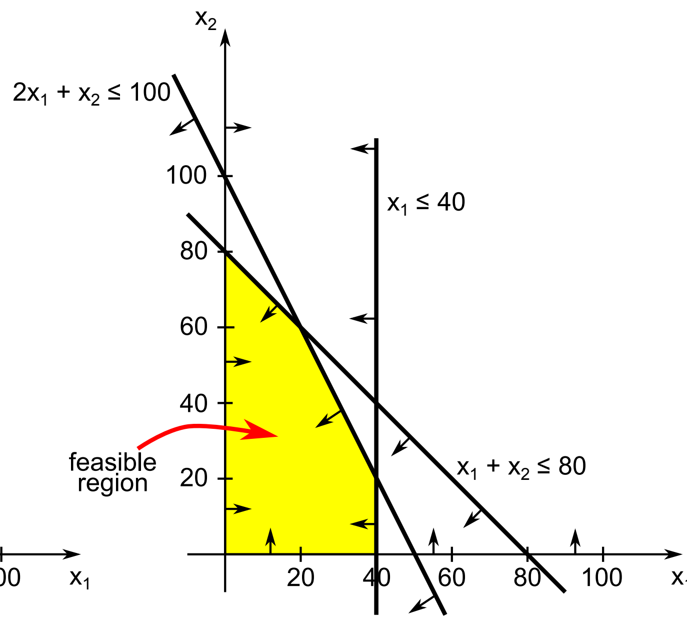
Start with $x_1 \geq 0$ and $x_2 \geq 0$



add $x_1 + x_2 \leq 80$



add $2x_1 + x_2 \leq 100$

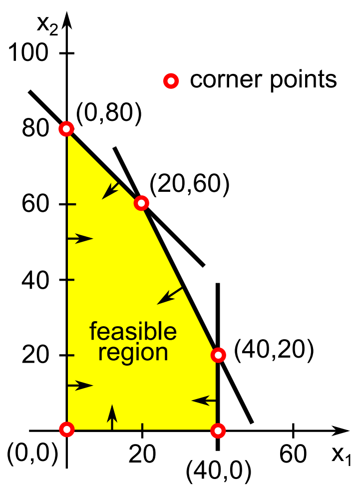


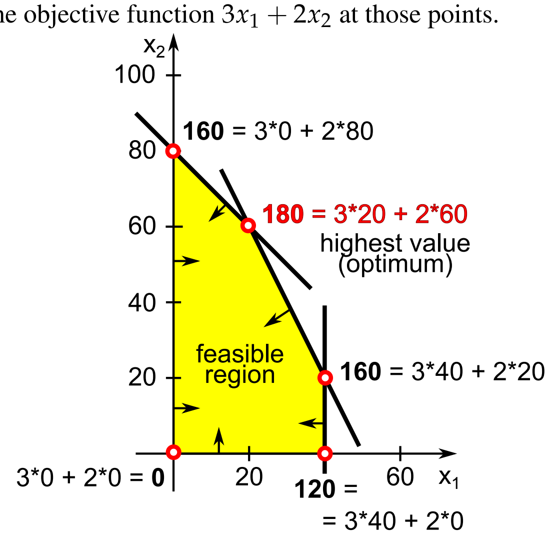
add $x_1 \leq 40$

A **corner** (extreme) point X of the region $R \equiv$ every line through X intersects R in a segment whose one endpoint is X. Solving a linear program amounts to finding a best corner point by the following theorem.

Theorem 1. *If a linear program with nonnegative variables has an* **optimal** *solution, then it also has an* **optimal solution** *that is a* **corner point** *of the feasible region.*

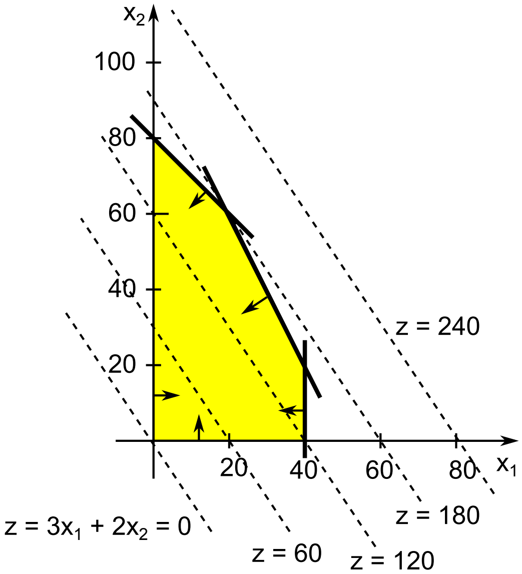
Exercise. Try to find all corner points. Evaluate the objective function $3x_1 + 2x_2$ at those points.

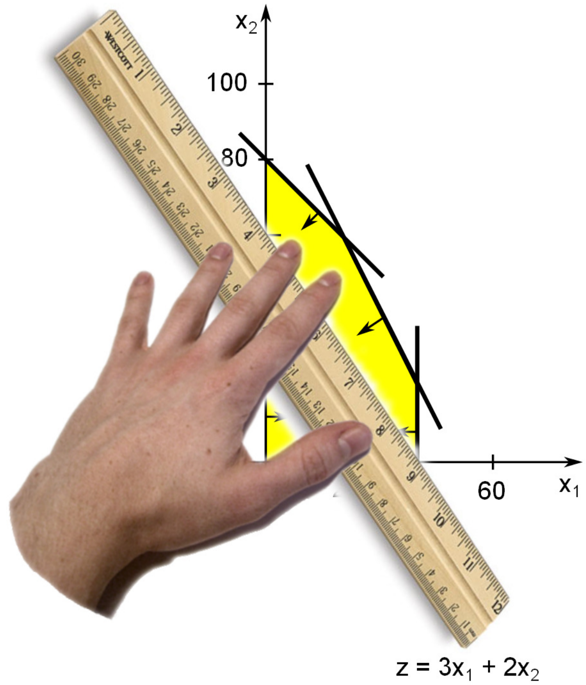




Problem: there may be too many corner points to check. There's a better way.
Iso-value line $\equiv$ in all points on this line the objective function has the same value
For our objective $3x_1 + 2x_2$ an iso-value line consists of points satisfying $3x_1 + 2x_2 = z$ where z is some number.
Graphical Method (main steps):

1. Find the feasible region.
2. Plot an iso-value (isoprofit, isocost) line for some value.
3. Slide the line in the direction of increasing value until it only touches the region.
4. Read-off an optimal solution.





Optimal solution is $(x_1, x_2) = (20, 60)$.

Observe that this point is the intersection of two lines forming the boundary of the feasible region. Recall that lines we use to construct the feasible region come from inequalities (the points on the line satisfy the particular inequality with equality). The next figure highlights the situation at the optimum.