IEOR 4004
Lecture 3 - Intro to Linear Programming Algorithms

1 Motivation

We assume an LP in standard form:

$$\begin{aligned} \textbf{(LP):} \quad & \max w^T x && \text{(1a)} \\ \text{Subject to} \quad & && \\ & Ax = b && \text{(1b)} \\ & x \geq 0. && \text{(1c)} \end{aligned}$$

Here we are assuming that A has m rows and n columns, and so $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. The algorithmic ideas we will discuss are based on the idea of finding optimal solutions with *simple* structure: as few nonzero variables as we can manage.

Definition. A vector $\hat{x} \in \mathbb{R}^n$ is called a **basic solution** if the set of columns j of A where $\hat{x}_j \neq 0$ is linearly independent.

Example:

$$\begin{aligned} & \max 5x_1 + 3x_2 + 4x_3 - 7x_4 && \text{(2a)} \\ \text{Subject to} \quad & && \\ & x_1 + x_2 + 5x_3 + x_4 = 7 && \\ & x_1 + 3x_2 - x_3 = 3 && \\ & x \geq 0. && \text{(2b)} \end{aligned}$$

The vector $(1, 2, 3, 0)^T$ is **not** basic. Why?

Definition A vector $\hat{x}$ is called feasible if $A\hat{x} = b$ and $\hat{x} \geq 0$, and is called **a basic feasible solution** if it is basic and feasible.

Fact: Basic feasible solutions are "the same" as extreme points.

Why do we care about basic feasible solutions? A preview of the **simplex** method.

Consider the vector $\hat{x} = (1, 1, 1, 0)^T$. Feasible? Yes. Basic? No. Objective value is **12**.

But even if optimal, it is not "simple" enough. Why? Because for $|\epsilon| > 0$ small enough,

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \epsilon \begin{pmatrix} 1 \\ -.375 \\ -.125 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 + \epsilon \\ 1 - .375\epsilon \\ 1 - .125\epsilon \\ 0 \end{pmatrix} \tag{3}$$

is feasible. Why? Nonnegative if $|\epsilon|$ is small enough, and also

$$\begin{aligned} 1 - .375 - 5 \times .125 &= 0 \\ 1 - 3 \times .375 + .125 &= 0. \end{aligned}$$

But the objective value is

$$5(1+\epsilon) + 3(1 - .375\epsilon) + 4(1 - .125\epsilon) \;=\; 12 \;+\; 3.375\,\epsilon.$$

SO IF WE CHOOSE $\epsilon > 0$ WE IMPROVE!

- How much can we improve? Make ϵ as large as we can. How large? $1/.375 = 2.6666$. Why?
- If we use this value for ϵ we get the vector $(3.6666, 0, 0.6666, 0)$.
- Check: feasible? Objective: 20.999.

What is the general principle?

We have a feasible vector $\hat{x} \in \mathbb{R}^n$ that is not basic. In other words, if we write

$$K \;\doteq\; \{\text{indices } j \,:\, \hat{x}_j > 0\},$$

then the set of columns of the constraint matrix A corresponding to K is *not* linearly independent. So there is a vector $y \in \mathbb{R}^n$ with

$$Ay \;=\; 0, \qquad \text{and} \quad y_j = 0 \text{ when } j \notin K.$$

So for $|\epsilon|$ small enough, $\hat{x} + \epsilon y$ is feasible. (Because $Ay = 0$).

And its objective value is $w^T\hat{x} \;+\; \epsilon w^T y$. So?

- if $w^T y \neq 0$ we can improve on $\hat{x}$ and stay feasible.
- if $w^T y = 0$ then we stay feasible, keep objective value, and reduce the number of positive variables.

Same example as above but different objective:

$$\max\ x_1 + 2x_2 + 2x_3 - 7x_4 \tag{5a}$$

Subject to

$$\begin{aligned} x_1 + x_2 + 5x_3 + x_4 &= 7 \\ x_1 + 3x_2 - x_3 &= 3 \\ x &\geq 0. \end{aligned} \tag{5b}$$

With this objective, $\hat{x} = (1,1,1,0)^T$ has objective 5. And we have $y = (1, -.375, -.125, 0)$. Now we have

$$w^T y \;=\; 0.$$

So the new vector, $(3.6666, 0, 0.6666, 0)^T$ also has objective 5. It's not better than $\hat{x}$ but it has one more zero element.

At this point we have basically proved a theorem. Consider a linear program (LP) **in standard form**.

- Not feasible? Bad luck.
- Feasible, but unbounded? Could happen, probably data is a little loose.
- Feasible, and bounded. Then optimum is attained. So using the above technique, we get that optimum is attained at a basic feasible solution. This the theorem mentioned before: if an LP (with nonnegative variables) attains its optimum value, then it does so at a basic feasible solution.

Consider the following **example** in standard form:

$$
\begin{array}{ll}
 & \max\ 10x_1 + 5x_2 + 2x_3 - \quad 10x_5 \\
\text{Subject to} & \\
 & x_1 + x_2 + x_3 + 2x_4 \quad = \ 10 \\
 & -x_1 + 5x_2 + \qquad\quad x_5 \quad = \ 8 \\
 & 2x_1 + x_2 + \qquad 4x_4 \quad = \ 17 \\
 & x \geq 0.
\end{array}
$$

Consider the solution $x^* = (8.5, 0, 1.5, 0, 16.5)$. It is feasible. Is it a basic feasible solution? And how about $x^* = (7, 3, 0, 0, 0)$?

===

Review concept: basic and nonbasic variables (= variables that are not basic)

Review concept: basic feasible solutions.

Fact: The number of basic feasible solutions, while possibly very large, is FINITE. Can you see why?

To understand unbounded cases, we need to introduce a new concept.

Definition: A *ray* or *direction of unboundedness* is a vector $d \in \mathbb{R}^n$ such that

$$Ad = 0, \quad \text{and} \quad d \geq 0.$$

So if x is feasible, so is $x + d$, and $x + 1.5d$, in fact $x + td$ for any $t > 0$.

Theorem. Problem **LP** (1) is unbounded if, and only if, there is a ray d such that $w^T d > 0$.

Example in non-standard form:

$$\max\ x_1 + 4x_2 \tag{7a}$$

Subject to

$$\begin{aligned} x_1 + x_2 &\geq 7 \\ -5x_1 + x_2 &\leq 8 \\ x_2 &\geq 4. \end{aligned} \tag{7b}$$

$$x \geq 0. \tag{7c}$$

The feasible region is nonempty: for example, for $M \geq 4$ if we set $x_1 = x_2 = M$ then we obtain a feasible solution. And its objective value is $5M$ so we can see that the LP is unbounded.

Convert to standard form:

$$\begin{array}{rrrrrrrr} \max & x_1 & +4x_2 & & & & & \\ \text{Subject to} & & & & & & & \\ & -x_1 & -x_2 & +s_1 & & & = & -7 \\ & -5x_1 & +x_2 & & +s_2 & & = & 8 \\ & & -x_2 & & & +s_3 & = & -4 \\ & x \geq 0 & s \geq 0 & & & & & \end{array}$$

So a ray is given by $(x_1, x_2, s_1, s_2, s_3)^T = (1, 1, 2, 4, 1)^T$.

2 Pivoting

Consider the following example:

$$\begin{array}{rrrcr} \max & 3x_1 & +2x_2 & & \\ \text{Subject to} & & & & \\ & x_1 & +x_2 & \leq & 80 \\ & 2x_1 & +x_2 & \leq & 100 \\ & x_1 & & \leq & 40 \\ & & x \geq 0 & & \end{array}$$

Convert to standard form:

$$\begin{array}{rrrrrrrr} \max & 3x_1 & +2x_2 & & & & & \\ \text{Subject to} & & & & & & & \\ & x_1 & +x_2 & +x_3 & & & = & 80 \\ & 2x_1 & +x_2 & & +x_4 & & = & 100 \\ & x_1 & & & & +x_5 & = & 40 \\ & & x \geq 0 & & & & & \end{array}$$

One further elaboration: **0.** Initial bfs (basic feasible solution) can be read in from the above

$$
\begin{array}{lllllllll}
\max & z & = & 3x_1 & +2x_2 & & & & \\
\text{Subject to} & & & & & & & & \\
 & & & x_1 & +x_2 & +x_3 & & & = & 80 \\
 & & & 2x_1 & +x_2 & & +x_4 & & = & 100 \\
 & & & x_1 & & & & +x_5 & = & 40 \\
 & x \geq 0 & & & & & & & &
\end{array}
$$

equations: $x = (0, 0, 80, 100, 40)^T$. This vector attains $z = 0$.

Why is this a bfs?

1. We will **improve** on this bfs. We will do so by **increasing one** of $\boldsymbol{x_1, x_2}$ from zero to a positive value. What are our choices?

1. Increase x_1. Then z increases by 3 per unit of increase.
2. Increase x_2. Then z increases by 2 per unit of increase.

Here we will be greedy and choose x_1. But if we increase x_1 and leave all other variables unchanged, we become infeasible. Let's say that we increase x_1 from zero to some value $\boldsymbol{\delta}$.

1. Decrease x_3 from 80 to $80 - \delta$.
2. Decrease x_4 from 100 to $100 - 2\delta$.
3. Decrease x_5 from 40 to $40 - \delta$.

How big can δ be? Answer: 40. We are now at the vector

$$(40, 0, 40, 20, 0)$$

and $z = 120$. Is this a bfs? It is if we can verify that $\{1, 3, 4\}$ form a basis, i.e. they are linearly independent columns of the constraint matrix. To see this is the case, let us write the above equations one more time:

$$
\begin{array}{ccccccccc}
z & -3x_1 & -2x_2 & & & & = & 0 \\
\hline
 & x_1 & +x_2 & +x_3 & & & = & 80 \\
 & 2x_1 & +x_2 & & +x_4 & & = & 100 \\
 & \boldsymbol{x_1} & & & & +x_5 & = & 40
\end{array}
$$

Here we have highlighted the coefficient of x_1 (the variable we have chosen to increase) on the row where δ was critical. We will use this row to zero out all other entries of x_1 in all the equations.
First, we subtract the critical row from the first constraint:
Next we subtract 2 times the last row from the second constraint: Note that from this set of equations we can read-off the solution that we described above: $x_1 = 40$, $x_3 = 40$, $x_4 = 20$, all other variables at zero. And how about z? Here we use the last row to zero-out the coefficient of x_1 in the row defining z:

$$
\begin{array}{cccccccc}
z & -3x_1 & -2x_2 & & & & = & 0 \\
\hline
 & & +x_2 & +x_3 & & \boldsymbol{-x_5} & = & \mathbf{40} \\
 & 2x_1 & +x_2 & & +x_4 & & = & 100 \\
 & \boldsymbol{x_1} & & & & +x_5 & = & 40
\end{array}
$$

$$
\begin{array}{cccccccc}
z & -3x_1 & -2x_2 & & & & = & 0 \\
\hline
 & & +x_2 & +x_3 & & \boldsymbol{-x_5} & = & \mathbf{40} \\
 & & +x_2 & & +x_4 & -2x_5 & = & 20 \\
 & \boldsymbol{x_1} & & & & +x_5 & = & 40
\end{array}
$$

$$
\begin{array}{cccccccc}
z & & -2x_2 & & & \boldsymbol{+3x_5} & = & 120 \\
\hline
 & & +x_2 & +x_3 & & -x_5 & = & \mathbf{40} \\
 & & +x_2 & & +x_4 & -2x_5 & = & 20 \\
 & x_1 & & & & +x_5 & = & 40
\end{array}
$$

In other words, $z = 120 + 2x_2 - 3x_5$ which at the current vector means $z = 120$.

This procedure simply rewrote the original equations by adding multiples of the last equation to the others. So the rank of any submatrix does not change. We can see that indeed the new vector is a bfs.

2. The current basis is $\{1, 3, 4\}$. Can we improve on it? From

$$z = 120 + 2x_2 - 3x_5$$

we see that increasing x_2 from zero to something positive, while keeping x_5 fixed at zero, will indeed increase the value of z. But how big can x_2 be made? Now we repeat the above reasoning.

1. From $x_3 = 40 - x_2$ we see that $x_2 \leq 40$.

2. From $x_4 = 20 - x_2$ we see that $x_2 \leq 20$.

So $x_2 = 20$, and x_3 decreases to 20 while x_4 goes to zero. Let us once more use linear combinations of rows to get rid of all entries of x_2 in the equations other than the critical equation. We get, first:

$$
\begin{array}{cccccccc}
z & & -2x_2 & & & +3x_5 & = & 120 \\
\hline
 & & & +x_3 & \boldsymbol{-x_4} & \boldsymbol{+x_5} & = & \mathbf{20} \\
 & & \boldsymbol{x_2} & & +x_4 & -2x_5 & = & 20 \\
 & x_1 & & & & +x_5 & = & 40
\end{array}
$$

And then:

3. The current basis is $\{1, 2, 3\}$ with $z = 160$. Can we improve on it? Looks like increasing x_5 will help. How big can it get if we keep $x_4 = 0$?

1. From $x_3 = 20 - x_5$, we have $x_5 \leq 20$.

$$
\begin{array}{cccccccc}
z & & & & \boldsymbol{+2x_4} & \boldsymbol{-x_5} & = & 160 \\
\hline
 & & & +x_3 & -x_4 & +x_5 & = & 20 \\
 & & x_2 & & +x_4 & -2x_5 & = & 20 \\
 & x_1 & & & & +x_5 & = & 40
\end{array}
$$

2. From $x_2 = 20 + 2x_5$, we have no limit on x_5.

3. From $x_1 = 40 - x_5$ we have that $x_5 \leq 40$.

So $x_5 \to 20$ and $x_3 \to 0$ and the first constraint provides the critical row. Then we use the x_5 column to zero out all entries of x_5 outside the critical row.

$$
\begin{array}{cccccccc}
z & & & & \boldsymbol{+2x_4} & \boldsymbol{-x_5} & = & 160 \\
\hline
 & & & +x_3 & -x_4 & \color{red}{\boldsymbol{+x_5}} & = & 20 \\
 & & x_2 & +2x_3 & -x_4 & & = & 60 \\
 & x_1 & & -x_3 & +x_4 & & = & 20
\end{array}
$$

And finally:

$$
\begin{array}{cccccccc}
z & & & +x_3 & \boldsymbol{+x_4} & & = & 180 \\
\hline
 & & & +x_3 & -x_4 & \color{red}{\boldsymbol{+x_5}} & = & 20 \\
 & & x_2 & +2x_3 & -x_4 & & = & 60 \\
 & x_1 & & -x_3 & +x_4 & & = & 20
\end{array}
$$

So in other words $x_1 = 20$, $x_2 = 60$, $x_5 = 20$, all other $x_j = 0$ and $z = 180$. What else?

$$z = 180 - x_3 - x_4.$$

This means: no feasible change in any variable can improve on $z = 180$. In other words, we are optimal.

Linear algebra mechanics

Consider the very first pivot in the above sequence. Let us first write the matrix of coefficients for all constraints, plus an additional column column for the RHS. This matrix, at the very beginning, was

$$
\begin{pmatrix}
1 & 1 & 1 & & & 80 \\
2 & 1 & & 1 & & 100 \\
1 & & & & 1 & 40
\end{pmatrix}
$$

Here we have left blanks in cases where the coefficient is zero. To pivot, we used the coefficient of x_1 (the second column in this matrix) in the bottom row to "zero out" all other coefficients of x_1. This sequence of steps produced the following matrix:

$$
\begin{pmatrix}
 & 1 & 1 & & -1 & 40 \\
 & 1 & & 1 & -2 & 20 \\
1 & & & & 1 & 100
\end{pmatrix}
$$

In fact, by looking at the first matrix we can see what the zeroing out steps are: First, subtract the last row from the first, and then subtract 2 times the last row from the second.

Exercise: verify that these actions are accomplished by multiplying the first matrix, **from the left** by the following matrix:

$$\begin{pmatrix} 1 & & \\ & 1 & -2 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & -1 \\ & 1 & \\ & & 1 \end{pmatrix}$$

[In fact, we can also account for the transformation to the z-row in this manner].

Note: this matrix is invertible. Why?

Taking a step back

LP in standard form:

$$\begin{array}{rl} \textbf{(LP):} & \max w^T x \\ \text{Subject to} & \\ & Ax \;=\; b \\ & x \geq 0. \end{array}$$

where A is $m \times n$.

1. Assumption. The matrix A has rank m (full row rank). Why? This implies $m \leq n$. Why?

2. Definition. A *basis* of A is an $m \times m$ submatrix that is *invertible.*

3. Fact. Suppose that x^* is a b.f.s. (basic feasible solution). Then, there is a basis B such that the columns of B include all indices j where $x_j^* > 0$. Why?

Proof. Without loss of generality (reorder columns) suppose that $x_j^* > 0$ for $j = 1, \ldots, k$.

1. $k \leq m$. Why?
2. There are $m - k$ columns of A, which, together with the first k columns, form a basis. Why?
3. Without loss of generality these columns are $k + 1, \ldots, m$. Let B be the submatrix of A made up of the first m columns. So B is a basis.

4. There is only **one** solution to the linear system of equations

$$By = b$$

Why? We must have

$$y = B^{-1}b$$

5. So $(y_1, \ldots, y_m) = (x_1^*, \ldots, x_m^*) = (x_1^* \ldots, x_k^*, \underbrace{0, \ldots, 0}_{m-k})$.

3 Examples with multiple optimal solutions and unbounded LPs

The above was an example of linear program with *unique* optimal solution. Now, let us consider an example of multiple optimal solutions and an example of unbounded linear program.

Multiple optimal solutions

$$\begin{array}{rrcrcl} \max & x_1 & + & \frac{1}{2}x_2 & & \\ \text{s.t.} & 2x_1 & + & x_2 & \leq & 4 \\ & x_1 & + & 2x_2 & \leq & 3 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

Convert to Standard form, and use variable z record the objective:

$$\begin{array}{rllllll} \max z = & x_1 & +\frac{1}{2}x_2 & & & \\ & 2x_1 & +x_2 & +x_3 & & = 4 \\ & x_1 & +2x_2 & & +x_4 & = 3 \\ & x_1, & x_2, & x_3, & x_4 & \geq 0 \end{array}$$

Writing in "tableau" form

$$\begin{array}{llllll} z & -x_1 & -\frac{1}{2}x_2 & & & = 0 \\ \hline & 2x_1 & +x_2 & +x_3 & & = 4 \\ & x_1 & +2x_2 & & +x_4 & = 3 \end{array}$$

Basic feasible solution: $x_1 = 0, x_2 = 0$ (non-basic variables), $x_3 = 4, x_4 = 3$ (basic variables).

Objective:

$$z = x_1 + \frac{1}{2}x_2$$

Increase variable x_1 by δ.
How big δ can be? From the two constraints: $\delta \leq 2$, $\delta \leq 3$. Critical constraint is constraint

1.
Pivoting: zero out the coefficient of x_1 in objective and row 2.

$$\begin{array}{ccccc}
z & & +\frac{1}{2}x_3 & & = 2 \\
\hline
x_1 & +\frac{1}{2}x_2 & +\frac{1}{2}x_3 & & = 2 \\
 & \frac{3}{2}x_2 & -\frac{1}{2}x_3 & +x_4 & = 1
\end{array}$$

Basic Feasible solution: $x_2 = 0, x_3 = 0$ (non-basic variables), $x_1 = 2$, $x_4 = 1$. Objective value $z = 2$.

Objective

$$z = 2 - \frac{1}{2}x_3$$

Cannot be improved. But note that x_2 appears with zero coefficient in the expression for z

$\rightarrow$ increasing x_2 is possible, but does not affect the value of z.
we pivot again: increase x_2, critical row is the second row.
New tableau:

$$\begin{array}{ccccc}
z & & +\frac{1}{2}x_3 & & = 2 \\
\hline
x_2 & & +\frac{2}{3}x_3 & -\frac{1}{3}x_4 & = \frac{5}{3} \\
 & +x_2 & -\frac{1}{3}x_3 & +\frac{2}{3}x_4 & = \frac{2}{3}
\end{array}$$

Again an optimal solution $x_1 = \frac{5}{3}$, $x_2 = \frac{2}{3}$, $x_3 = 0$, $x_4 = 0$, $z = 2 \rightarrow$ same value
What if we pivot again (increase x_4) ?

Unbounded LPs

Recall
Ray: or *direction of unboundedness* is a vector $d \in \mathbb{R}^n$ such that

$$Ad = 0, \quad \text{and} \quad d \geq 0.$$

Fact. An LP is unbounded if, and only if, there is a ray d such that $c^T d > 0$.

Now consider the following example:

$$\begin{array}{cccc}
max & 2x_1 & +x_2 & \\
s.t. & -x_1 & +x_2 & \leq 1 \\
 & x_1 & -2x_2 & \leq 2 \\
 & x_1, & x_2 & \geq 0
\end{array}$$

Standard form

$$\begin{array}{cccccc}
\max z = & 2x_1 & +x_2 & & & \\
 & -x_1 & +x_2 & +x_3 & & = 1 \\
 & x_1 & -2x_2 & & +x_4 & = 2 \\
 & x_1, & x_2 & & & \geq 0
\end{array}$$

Tableau

$$\begin{array}{ccccccc} z & -2x_1 & -x_2 & & & = 0 \\ \hline & -x_1 & +x_2 & +x_3 & & = 1 \\ & x_1 & -2x_2 & & +x_4 & = 2 \end{array}$$

Initial bfs: $x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 2$.

Objective $z = 2x_1 + x_2$. Increase x_1.
Critical row is the second row.
Pivoting: using x_1 in the second row as pivot, zero out the coefficients of x_1 in all other rows.

$$\begin{array}{cccccc} z & & -5x_2 & & +2x_4 & = 4 \\ \hline & & -x_2 & +x_3 & +x_4 & = 3 \\ & x_1 & -2x_2 & & +x_4 & = 2 \end{array}$$

Basic Feasible Solution: $x_2 = x_4 = 0$ (non-basic variables), $x_1 = 2$, $x_3 = 3$, $z = 4$.

Objective

$$z = 4 + 5x_2 - 2x_4$$

What if we now increase x_2 by δ, while keeping $x_4 = 0$? How big can δ be? From the two constraints:

- $x_3 = 3 + \delta \to \delta \geq -3$
- $x_1 = 2 + 2\delta \to \delta \geq -2$

That is, no positive value of δ makes x_1 or x_3 negative; there is no critical row. Therefore, we can make x_2 arbitrarily large and thus make z arbitrarily large $\longrightarrow$ **unbounded** LP

direction of unboundedness: setting $x_2 = \delta$, $x_4 = 0 \longrightarrow x_1 = 2 + 2\delta$, $x_3 = 3 + \delta$, $z = 4 + 5\delta$ for increasing $\delta \longrightarrow$ gives a sequence of feasible solution of increasing value.

Ray: can be found from coefficients of δ

$$d = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$Ad = \begin{bmatrix} 0 & -1 & +1 & 1 \\ 1 & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$c^T d = \begin{bmatrix} 0 & 5 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} = 5 > 0$$

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