IEOR 4004
Simplex Method: duality

1 Pricing interpretation

Consider a manufacturing problem with two resources, *blocks of wood* and *cans of paint*, and two products, *toy soldiers* and *toy trains*.

Manufacturer

$$\begin{array}{rrcrcrl} \text{Max} & 3x_1 & + & 2x_2 & & & \\ & x_1 & + & x_2 & \leq & 80 & \text{[wood]} \\ & 2x_1 & + & x_2 & \leq & 100 & \text{[paint]} \\ & & & x_1, x_2 & \geq & 0 & \end{array}$$

Market

Prices:
y_1 = price (in \$) of one block of wood
y_2 = price (in \$) of one can of paint

The manufacturer owns 80 blocks of wood and 100 cans of paint. He can sell these materials at market prices or buy additional materials at market prices. He can also produce and sell goods (toys) using the available materials.

What is his best strategy (assuming everything produced will be sold)?

- ⋆ Selling material generates a profit of $80y_1 + 100y_2$.
- ⋆ If the cost (in market prices) of producing x_1 *toy soldiers* is strictly less than the sale price, i.e. if
  $$y_1 + 2y_2 < 3$$
  then there is **no limit** on the profit of manufacturer. He can generate arbitrarily large profit by buying additional materials to produce toy soldiers in arbitrary amounts.

  Why? The manufacturer can produce x_1 toy soldiers by purchasing x_1 blocks of wood, and $2x_1$ additional cans of paint. He pays $x_1(y_1 + 2y_2)$ and makes $3x_1$ in sales. Net profit is then $x_1(3 - y_1 - 2y_2)$. Now, if $y_1 + 2y_2 < 3$, say if $y_1 + 2y_2 \leq 2.9$, then the net profit is then $x_1(3 - y_1 - 2y_2) \geq (3 - 2.9) = 0.1x_1$. So making arbitrarily many x_1 toy soldiers generates a profit of $0.1x_1$ (arbitrarily high).
- ⋆ Similarly, **no limit** on the profit if the cost of producing x_2 *toy trains* is less than the sale price, i.e. if
  $$y_1 + y_2 < 2$$
- ⋆ Market prices are **non-negative**.

Market (the competition) will not allow the manufacturer to make arbitrarily large profit. It will set its prices so that the manufacturer makes as little as possible. The market is thus solving the following:

$$\left.\begin{array}{rrcrcrl} \text{Min} & 80y_1 & + & 100y_2 & & & \\ & y_1 & + & 2y_2 & \geq & 3 & \text{[toy soldiers]} \\ & y_1 & + & y_2 & \geq & 2 & \text{[toy trains]} \\ & & & y_1, y_2 & \geq & 0 & \end{array}\right\} \textbf{Dual} \text{ of the manufacturing problem}$$

Estimating the optimal value

$$\begin{array}{rrcrcrl} \text{Max} & 3x_1 & + & 2x_2 & & & \\ & x_1 & + & x_2 & \leq & 80 & \text{[wood]} \\ & 2x_1 & + & x_2 & \leq & 100 & \text{[paint]} \\ & & & x_1, x_2 & \geq & 0 & \end{array}$$

Before solving the LP, the manufacturer wishes to get a quick rough estimate (upper bound) on the value of the optimal solution. For instance, the objective function is $3x_1 + 2x_2$ which is certainly less than $3x_1 + \mathbf{3}x_2$, since the variables x_1, x_2 are non-negative. We can rewrite this as $3(x_1 + x_2)$ and we notice that $x_1 + x_2 \leq 80$ by the first constraint. Together we have:

$$z \;=\; 3x_1 + 2x_2 \;\leq\; 3x_1 + 3x_2 \;\leq\; 3(x_1 + x_2) \;\leq\; 3 \times 80 \;=\; \$240$$

As a result every production plan will generate no more than \$240, i.e., the value of any feasible solution (including the optimal one) is not more than 240. Likewise we can write:

$$z \;=\; 3x_1 + 2x_2 \;\le\; 4x_1 + 2x_2 \;\le\; 2(2x_1 + x_2) \;\le\; 2 \times 100 \;=\; \$200$$

since $2x_1 + x_2 \le 100$ by the 2nd constraint. We can also combine constraints for an even better estimate:

$$z \;=\; 3x_1 + 2x_2 \;\le\; (x_1 + x_2) + (2x_1 + x_2) \;\le\; 80 \;+\; 100 \;=\; \$180$$

In general, we consider $y_1 \ge 0$, $y_2 \ge 0$ and take y_1 times the 1st contraint + y_2 times the 2nd constraint.

$$y_1(x_1 + x_2) + y_2(2x_1 + x_2) \;\le\; 80y_1 \;+\; 100y_2$$

We can rewrite this expression by collecting coefficients of x_1 and x_2:

$$(y_1 + 2y_2)x_1 + (y_1 + y_2)x_2 \;\le\; 80y_1 \;+\; 100y_2$$

In this expression, if the **coefficient** of x_1 is **at least** 3 and the coefficient of x_2 is **at least** 2, i.e., if

$$\begin{aligned} y_1 + 2y_2 &\ge 3 \\ y_1 + y_2 &\ge 2 \end{aligned}$$

then, just like before, we obtain an upper bound on the value of $z = 3x_1 + 2x_2$:

$$z \;=\; 3x_1 + 2x_2 \;\le\; (y_1 + 2y_2)x_1 + (y_1 + y_2)x_2 \;=\; y_1(x_1 + x_2) + y_2(2x_1 + x_2) \;\le\; 80y_1 \;+\; 100y_2$$

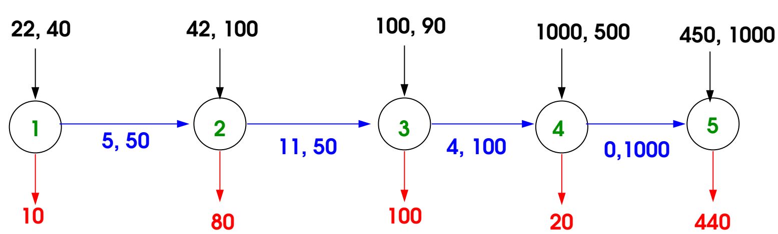
If we want the best possible upper bound, we want this expression be as small as possible.

$$\left.\begin{array}{lrcrcl} \text{Min} & 80y_1 & + & 100y_2 & & \\ & y_1 & + & 2y_2 & \ge & 3 \\ & y_1 & + & y_2 & \ge & 2 \\ & & & y_1, y_2 & \ge & 0 \end{array}\right\} \text{The } \textbf{Dual} \text{ problem}$$

The original problem is then called the **Primal** problem.

$$\textbf{Primal} \qquad \begin{array}{lrcrcr} \text{Max} & 3x_1 & + & 2x_2 & & \\ & x_1 & + & x_2 & \le & 80 \\ & 2x_1 & + & x_2 & \le & 100 \\ & & & x_1, x_2 & \ge & 0 \end{array} \qquad\qquad \begin{array}{lrcrcl} \text{Min} & 80y_1 & + & 100y_2 & & \\ & y_1 & + & 2y_2 & \ge & 3 \\ & y_1 & + & y_2 & \ge & 2 \\ & & & y_1, y_2 & \ge & 0 \end{array} \qquad \textbf{Dual}$$

As another example, consider the 5-period inventory problem given in the following figure:



Here demands are shown in red, and per-unit inventory costs and capacities are shown in blue. Finally, per-unit production costs and capacities are shown in black. Here is a formulation:

```
Minimize
 + 22.0 x1 + 5.0 i1,2 + 42.0 x2 + 11.0 i2,3 + 100.0 x3 + 4.0 i3,4
 + 1000.0 x4 + 450.0 x5
Subject To
Demand1: x1 - i1,2 = 10.0
Demand2: i1,2 + x2 - i2,3 = 80.0
Demand3: i2,3 + x3 - i3,4 = 100.0
Demand4: i3,4 + x4 - i4,5 = 20.0
Demand5: i4,5 + x5 = 440.0
ProdCap1: x1 <= 40.0
ProdCap2: x2 <= 100.0
ProdCap3: x3 <= 90.0
ProdCap4: x4 <= 500.0
ProdCap5: x5 <= 1000.0
InvCap1: i1,2 <= 50.0
InvCap2: i2,3 <= 50.0
InvCap3: i3,4 <= 100.0
InvCap4: i4,5 <= 1000.0
END
```

with all variables assumed nonnegative.

Exercise: Can you state some of the constraints for the dual of this problem?

Matrix formulation

In general, for maximization problem with $\leq$ inequalities, the dual is obtained simply by

- transposing (flipping around the diagonal) the matrix $\mathbf{A}$,
- swapping vectors $\mathbf{b}$ and $\mathbf{c}$,
- switching the inequalities to $\geq$, and
- changing max to min.

$$\begin{array}{rl} \max & \mathbf{c}^T\mathbf{x} \\ & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array} \qquad\qquad \begin{array}{rl} \min & \mathbf{b}^T\mathbf{y} \\ & \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq 0 \end{array}$$

2 Duality Theorems and Feasibility

Theorem 2.1 (Weak Duality Theorem) *If* $\mathbf{x}$ *is any feasible solution of the primal and* $\mathbf{y}$ *is any feasible solution of the dual, then*

$$\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$$

In other words, the value of **any** feasible solution to the **dual** yields an upper bound on the value of any feasible solution (including the optimal) to the **primal**.

$$\mathbf{c}^T\mathbf{x} \leq (\mathbf{A}^T\mathbf{y})^T\mathbf{x} = (\mathbf{y}^T\mathbf{A})\mathbf{x} = \mathbf{y}^\mathbf{T}(\mathbf{A}\mathbf{x}) \leq \mathbf{y}^T\mathbf{b} = \mathbf{b}^T\mathbf{y}$$

Consequently, if **primal** is *unbounded*, then **dual** must be *infeasible* and likewise, if **dual** is *unbounded*, then **primal** must be *infeasible*. Note that is it possible that both **primal** and **dual** are *infeasible*. But if both are *feasible*, then neither of them is *unbounded*.

dual \ **primal**	infeasible	feasible bounded	unbounded
infeasible	✓	✗	✓
feasible bounded	✗	✓	✗
unbounded	✓	✗	✗

✓ possible

✗ impossible

Strong Duality and Complementary Slackness

Theorem 2.2 (Strong Duality Theorem) *If* $\mathbf{x}$ *is an* **optimal** *solution to the primal and* $\mathbf{y}$ *is an* **optimal** *solution to the dual, then*

$$\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$$

This is a fundamentally important theorem that we will study in more detail later. As a consequence, we have the following: if $\mathbf{x}$ is an **optimal** solution to the primal and $\mathbf{y}$ is an **optimal** solution to the dual, then $\mathbf{y}^T(\underbrace{\mathbf{b}-\mathbf{A}\mathbf{x}}_{\substack{\text{slack in}\\ \text{primal}}}) = 0$ and $\mathbf{x}^T(\underbrace{\mathbf{A}^T\mathbf{y}-\mathbf{c}}_{\substack{\text{slack in}\\ \text{dual}}}) = 0$.

In simple terms: whenever a constraint is **not tight** (has a positive slack) in the **primal**, then the **dual** variable corresponding to this constraint must be 0. Conversely, if a **primal** variable is strictly positive, then the corresponding **dual** constraint must be tight (slack is zero).

A proof of this fact can be seen as follows (note that $\mathbf{x} \geq 0$, $\mathbf{y} \geq 0$, $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ and $\mathbf{A}^T\mathbf{y} \geq \mathbf{c}$)

$$0 \leq \mathbf{y}^T(\mathbf{b}-\mathbf{A}\mathbf{x}) = \mathbf{y}^T\mathbf{b} - \mathbf{y}^T\mathbf{A}\mathbf{x} = \mathbf{b}^T\mathbf{y} - (\mathbf{A}^T\mathbf{y})^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y} - \mathbf{c}^T\mathbf{x} \;=\; 0$$
$$0 \leq \mathbf{x}^T(\mathbf{A}^T\mathbf{y}-\mathbf{c}) = (\mathbf{y}^T\mathbf{A} - \mathbf{c}^T)\mathbf{x} = \mathbf{y}^T(\mathbf{A}\mathbf{x}) - \mathbf{c}^T\mathbf{x} \leq \mathbf{y}^T\mathbf{b} - \mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y} - \mathbf{c}^T\mathbf{x} \;=\; 0.$$

We will return to these properties in Section 4 below.

3 General LPs

If LP contains equalities or free variables (i.e. variables that are *unrestricted in sign*, these can be also handled with ease. In particular,

equality constraint corresponds to an **free** variable, and **vice-versa**.

Why? Notice that when we produced an upper bound, we considered only non-negative $y_1 \geq 0$, since multiplying the $\leq$ constraint by a negative value changes the sign to $\geq$ and thus the upper bound becomes a lower bound instead. However, if the constraint was an equality (i.e. if we had $x_1 + x_2 = 80$ instead), we could allow negative y_1 as well and still produce an upper bound. For instance, we could write

$$3x_1 + 2x_2 \leq 5x_1 + 2x_2 = (-1) \times (\underbrace{x_1 + x_2}_{=80}) + 3 \times (\underbrace{2x_1 + x_2}_{\leq 100}) \leq -80 + 3 \times 100 = \$220$$

So we would make y_1 free.

Conversely, if some variable in our problem, say x_1, were free in sign (could be negative as well), then we could **not** conclude that $3x_1 + 2x_2 \leq 4x_1 + 2x_2$ holds for all feasible solutions, as we did in our 2nd estimate; namely if x_1 is **negative**, then this is **not true** (it is actually $>$ rather than $\leq$). However, if x_1 is free but $x_2 \geq 0$, we could still conclude that $3x_1 + 2x_2 \leq 3x_1 + 2x_2$, since the coefficient of x_1 is not changing in this expression. In our general expression, we had $(y_1 + 2y_2)x_1$ and we demanded that the coefficient $y_1 + 2y_2$ of x_1 is at least 3 for the upper bound to work. If x_1 is **free**, we can simply insist that the coefficient $y_1 + 2y_2$ **equals** 3 to make the upper bound work.

The same way we can conclude that

a $\geq$ (primal or dual) constraint corresponds to an **non-positive** (dual or primal) variable, and **vice-versa**.

Primal (Max)	Dual (Min)
i-th constraint $\leq$	variable $y_i \geq 0$
i-th constraint $\geq$	variable $y_i \leq 0$
i-th constraint $=$	variable y_i free
$x_i \geq 0$	i-th constraint $\geq$
$x_i \leq 0$	i-th constraint $\leq$
x_i free	i-th constraint $=$

$$
\begin{array}{rrrrrrcl}
\text{Max} & 3x_1 & + & 2x_2 & + & x_3 & & \\
& x_1 & + & x_2 & + & \frac{1}{2}x_3 & \leq & 80 \\
& 2x_1 & + & x_2 & + & x_3 & = & 100 \\
& x_1 & & & + & x_3 & \geq & 40 \\
& & & & & x_1 & \text{free} & \\
& & & & & x_2 & \leq & 0 \\
& & & & & x_3 & \geq & 0
\end{array}
\qquad
\begin{array}{rrrrrrcl}
\text{Min} & 80y_1 & + & 100x_2 & + & 40x_3 & & \\
& y_1 & + & 2y_2 & + & y_3 & = & 3 \\
& y_1 & + & y_2 & & & \leq & 2 \\
& \frac{1}{2}y_1 & + & y_2 & + & y_3 & \geq & 1 \\
& & & & & y_1 & \geq & 0 \\
& & & & & y_2 & \text{free} & \\
& & & & & y_3 & \leq & 0
\end{array}
$$

Primal **Dual**

4 Complementary slackness

$$
\begin{array}{rrrrrrrrcl}
\max & 6x_1 & + & x_2 & - & x_3 & - & x_4 & & \\
& x_1 & + & 2x_2 & + & x_3 & + & x_4 & \leq & 5 \\
& 3x_1 & + & x_2 & - & x_3 & & & \leq & 8 \\
& & & x_2 & + & x_3 & + & x_4 & = & 1 \\
& & & & & & & x_2, x_3, x_4 & \geq & 0 \\
& & & & & & & & & x_1 \text{ free}
\end{array}
$$

We wish to check if one of the following assignments is an optimal solution.

a) $x_1 = 2$, $x_2 = 1$, $x_3 = 0$, $x_4 = 0$

b) $x_1 = 3$, $x_2 = 0$, $x_3 = 1$, $x_4 = 0$

To this end, we use **Complementary Slackness**. Let us discuss the theory first.

Theory

As usual, let $\mathbf{x}$ denote the vector of variables, let $\mathbf{c}$ be the vector of coefficients of variables of the objective function, let $\mathbf{A}$ be the coefficient matrix of the left-hand side of our constraints, and let $\mathbf{b}$ be the vector of the right-hand side of the constraints. Let $\mathbf{y}$ be the variables of the dual.

$$
\begin{array}{rcl}
\max & \mathbf{c}^{\mathrm{T}}\mathbf{x} & \\
\mathbf{Ax} & \leq & \mathbf{b} \quad \text{PRIMAL} \\
\mathbf{x} & \geq & 0
\end{array}
\qquad\qquad
\begin{array}{rcl}
\min & \mathbf{b}^{\mathrm{T}}\mathbf{y} & \\
\mathbf{A}^{\mathrm{T}}\mathbf{y} & \geq & \mathbf{c} \quad \text{DUAL} \\
\mathbf{y} & \geq & 0
\end{array}
$$

We say that vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ are **complementary** if

$$
\mathbf{y}^T(\underbrace{\mathbf{b} - \mathbf{Ax}}_{\substack{\text{slack in} \\ \text{primal}}}) = 0 \text{ and } \mathbf{x}^T(\underbrace{\mathbf{A}^T\mathbf{y} - \mathbf{c}}_{\substack{\text{slack in} \\ \text{dual}}}) = 0
$$

In other words,

- whenever $y_i > 0$, then $\mathbf{x}$ satisfies the i-th constraint with equality ("the constraint is **tight**")
- whenever $x_i > 0$, then $\mathbf{y}$ satisfies the i-th constraint of the dual with equality

We can restate these properties in terms of LPs in standard form:

$$
\begin{array}{rcl}
\max & \mathbf{c}^{\mathrm{T}}\mathbf{x} & \\
\mathbf{Ax} & = & \mathbf{b} \quad \text{PRIMAL} \\
\mathbf{x} & \geq & 0
\end{array}
\qquad\qquad
\begin{array}{rcl}
\min & \mathbf{b}^{\mathrm{T}}\mathbf{y} & \\
\mathbf{A}^{\mathrm{T}}\mathbf{y} & \geq & \mathbf{c} \quad \text{DUAL} \\
\mathbf{y} & & \text{free}
\end{array}
$$

At this point it is useful to recall what a simplex tableau (or dictionary) looked like. Given a basis B, we can group variables to write the LP as:

$$
\begin{array}{ccccc}
z & -c_B^T x_B & -c_N^T x_N & = & 0 \\
\hline
& Bx_B & +Nx_N & = & b
\end{array}
$$

The simplex method got us to the basis B by performing elementary row operations, i.e. by premultiplying the constraints by "eta" matrices.

$$\begin{array}{cccccc} z & -c_B^T x_B & & -c_N^T x_N & = & 0 \\ \hline & x_B & + & B^{-1} N x_N & = & B^{-1} b \end{array}$$

We then zero-out the coefficients of the basic variables in the objective: we add c_B^T times the constraints in this last dictionary to the z-row, obtaining

$$\begin{array}{cccc} z & -(c_N^T - c_B^T B^{-1} N) x_N & = & c_B^T B^{-1} b \\ \hline x_B & + B^{-1} N x_N & = & B^{-1} b \end{array}$$

At this point we notice that the z-row in this set of equations can be directly obtained from the **initial** LP (or the initial dictionary). Namely, from

$$z = c^T x, \quad \text{and} \quad Ax = b$$

we obtain

$$z + c_B^T B^{-1} A x = c^T x + c_B^T B^{-1} b,$$

or,

$$z - (c^T - c_B^T B^{-1} A) x = c_B^T B^{-1} b,$$

which is what the last dictionary above says.
The quantities

$$y \doteq c_B^T B^{-1}$$

are called the **shadow prices** corresponding to the current basic solution.

> **Exercise.** Show that the shadow prices y defined by (a basic solution) $\mathbf{x}$ are always complementary to $\mathbf{x}$.

Recall that **Strong Duality** this says that if $\mathbf{x}$ is an optimal solution to the primal and $\mathbf{y}$ is an optimal solution to the dual, then $\mathbf{c}^{\mathrm{T}}\mathbf{x} = \mathbf{b}^{\mathrm{T}}\mathbf{y}$. In fact, more is true.

> **Complementary Slackness** (and some consequences): Assume that $\mathbf{x}$ is an optimal solution to the primal.
> - If $\mathbf{y}$ is an **optimal** solution to the dual, then $\mathbf{x}$ and $\mathbf{y}$ are **complementary**.
> - If $\mathbf{y}$ is a **feasible** solution in the dual and is **complementary** to $\mathbf{x}$, then $\mathbf{y}$ is **optimal** in the dual.
> - There **exists** a **feasible** solution $\mathbf{y}$ to the dual such that $\mathbf{x}$ and $\mathbf{y}$ are **complementary**.

Notice that the last bullet follows from our observation about shadow prices. Another consequence of this is:

> If $\mathbf{x}$ is a basic feasible primal solution and π are the corresponding shadow prices, then
> $\mathbf{x}$ is **optimal** if and only if π is a **feasible** solution of the dual

If we have equalities, $\geq$-inequalities, free or non-positive variables, everything works just the same.

Back to example

To check if the provided solutions are optimal, we need the dual.

$$\begin{array}{rrcrcrcrcl} \max & 6x_1 & + & x_2 & - & x_3 & - & x_4 & & \\ & x_1 & + & 2x_2 & + & x_3 & + & x_4 & \leq & 5 \\ & 3x_1 & + & x_2 & - & x_3 & & & \leq & 8 \\ & & & x_2 & + & x_3 & + & x_4 & = & 1 \\ & & & & & & x_2, & x_3, x_4 & \geq & 0 \\ & & & & & & & & x_1 & \text{free} \end{array}$$

$$\begin{array}{rrcrcrcr} \min & 5y_1 & + & 8y_2 & + & y_3 & & \\ & y_1 & + & 3y_2 & & & = & 6 \\ & 2y_1 & + & y_2 & + & y_3 & \geq & 1 \\ & y_1 & - & y_2 & + & y_3 & \geq & -1 \\ & y_1 & & & + & y_3 & \geq & -1 \\ & & & & & y_1, y_2 & \geq & 0 \\ & & & & & y_3 & \text{free} & \end{array} \qquad \text{DUAL}$$

a) $x_1 = 2$, $x_2 = 1$, $x_3 = 0$, $x_4 = 0$ $\quad\rightarrow\quad$ assume $\mathbf{x}$ is optimal, then

$\rightarrow$ there are y_1, y_2, y_3 such that $\mathbf{y} = (y_1, y_2, y_3)$ is feasible in the dual and complementary to $\mathbf{x}$

check 1st primal constraint: $x_1 + 2x_2 + x_3 + x_4 = 2 + 2 + 0 + 0 = 4 < 5$ **not tight**

$\rightarrow$ therefore y_1 must be 0 because $\mathbf{y}$ is complementary to $\mathbf{x}$

check 2nd primal constraint: $3x_1 + x_2 - x_3 = 6 + 1 - 0 = 7 < 8$ **not tight**

$\rightarrow$ therefore y_2 must be 0 because $\mathbf{y}$ is complementary to $\mathbf{x}$

Knowing this, check the 1st dual constraint: $y_1 + 3y_2 = 0 + 0 = 0 \neq 6$

$\rightarrow$ this shows that (y_1, y_2, y_3) **not feasible** in the dual, but we assume that it is.

This means that our assumptions were wrong and so (x_1, x_2, x_3, x_4) is **not optimal**.

b) $x_1 = 3$, $x_2 = 0$, $x_3 = 1$, $x_4 = 0$ $\quad\rightarrow$ again assume that $\mathbf{x}$ is optimal, then

$\rightarrow$ there are y_1, y_2, y_3 such that $\mathbf{y} = (y_1, y_2, y_3)$ is feasible in the dual and complementary to $\mathbf{x}$

check 1st primal constraint: $x_1 + 2x_2 + x_3 + x_4 = 3 + 0 + 1 + 0 = 4 < 5$ **not tight** $\quad\rightarrow\quad y_1 = 0$

check 2nd primal constraint: $3x_1 + x_2 - x_3 = 9 + 0 - 1 = 8$ **tight**

check 3rd primal constraint: $x_1 + x_2 + x_3 = 1$ **tight**

check sign constraints: $x_2, x_3, x_4 \geq 0 \rightarrow$ we conclude that (x_1, x_2, x_3, x_4) is **feasible** in the primal

Now we look at values in $\mathbf{x}$ with respect to the dual constraints:

x_1 is free $\quad\rightarrow\quad$ 1st dual constraint $y_1 + 3y_2 = 6$ is (always) **tight**

$x_3 > 0$ we deduce $\quad\rightarrow\quad$ 3rd dual constraint must be **tight**: $y_1 - y_2 + y_3 = -1$

Together we have

$$\begin{array}{rcrcrcr} y_1 & & & & & = & 0 \\ y_1 & + & 3y_2 & & & = & 6 \\ y_1 & - & y_2 & + & y_3 & = & -1 \end{array}$$

This has a unique solution $y_1 = 0$, $y_2 = 2$, $y_3 = 1$. By construction, this solution is **complementary** to $\mathbf{x}$.

The last step is to **check** if $\mathbf{y}$ is also **feasible** in the dual. We already checked 1st and 3rd dual constraint.

check 2nd dual constraint: $2y_1 + y_2 + y_3 = 0 + 2 + 1 = 3 \geq 1 \rightarrow$ the constraint is **satisfied**

check 4th dual constraint: $y_1 + y_3 = 0 + 1 \geq -1$ the constraint is **satisfied**

check sign restrictions: $y_1 = 0 \geq 0$ and $y_2 = 2 \geq 0 \rightarrow$ sign restrictions **satisfied**

$\rightarrow$ this shows that (y_1, y_2, y_3) indeed a **feasible** solution to the dual.

From this we can conclude that (x_1, x_2, x_3, x_4) is indeed **optimal**.

Summary

- given $\mathbf{x}$, check if $\mathbf{x}$ is feasible
- then find which variables y_i should be 0
- then find which dual constraints should be tight
- this yields a system of equations
- solve the system
- verify that the solution is feasible in the dual

If all these steps succeed, then the given $\mathbf{x}$ is indeed optimal; otherwise, it is not.
Question: what happens if $\mathbf{x}$ is feasible but not a basic solution ?

Review

$$\begin{array}{rrcrcr} \text{Max} & 3x_1 & + & 2x_2 & & \\ & x_1 & + & x_2 & \leq & 80 \\ & 2x_1 & + & x_2 & \leq & 100 \\ & x_1 & & & \leq & 40 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

Original problem

$$\begin{array}{rcrcrcr} x_3 & = & 80 & - & x_1 & - & x_2 \\ x_4 & = & 100 & - & 2x_1 & - & x_2 \\ x_5 & = & 40 & - & x_1 & & \\ \hline z & = & 0 & + & 3x_1 & + & 2x_2 \end{array}$$

Initial dictionary

$$\begin{array}{rcrcrcr} x_1 & = & 20 & + & x_3 & - & x_4 \\ x_2 & = & 60 & - & 2x_3 & + & x_4 \\ x_5 & = & 20 & - & x_3 & + & x_4 \\ \hline z & = & 180 & - & x_3 & - & x_4 \end{array}$$

Optimal dictionary

Add a new activity: *toy cars*, $\frac{1}{2}$h carving, 1h finishing, 1 unit towards demand limit, \$1 price $\rightarrow x_6 =$ #cars

$$\begin{array}{rrrrrrrl} \text{Max} & 3x_1 & + & 2x_2 & + & x_6 & & \\ & x_1 & + & x_2 & + & \frac{1}{2}x_6 & \leq & 80 \\ & 2x_1 & + & x_2 & + & x_6 & \leq & 100 \\ & x_1 & & & + & x_6 & \leq & 40 \\ & & & & & x_1, x_2, x_6 & \geq & 0 \end{array}$$

Original problem

$$\begin{array}{rcrcrcrcr} x_3 & = & 80 & - & x_1 & - & x_2 & - & \frac{1}{2}x_6 \\ x_4 & = & 100 & - & 2x_1 & - & x_2 & - & x_6 \\ x_5 & = & 40 & - & x_1 & & & - & x_6 \\ \hline z & = & 0 & + & 3x_1 & + & 2x_2 & + & x_6 \end{array}$$

Initial dictionary

?

Optimal dictionary

in the optimal dictionary $\rightarrow$ make x_6 *non-basic* (no production of cars) $\rightarrow$ **feasible** modified dictionary

$$\begin{array}{rcrcrcr} x_3 + \frac{1}{2}x_6 & = & 80 & - & x_1 & - & x_2 \\ x_4 + x_6 & = & 100 & - & 2x_1 & - & x_2 \\ x_5 + x_6 & = & 40 & - & x_1 & & \\ \hline z - x_6 & = & 0 & + & 3x_1 & + & 2x_2 \end{array} \quad \begin{array}{rcrcrcr} x'_3 & = & 80 & - & x_1 & - & x_2 \\ x'_4 & = & 100 & - & 2x_1 & - & x_2 \\ x'_5 & = & 40 & - & x_1 & & \\ \hline z' & = & 0 & + & 3x_1 & + & 2x_2 \end{array} \quad \rightarrow \quad \begin{array}{rcrcrcr} x_1 & = & 20 & + & x'_3 & - & x'_4 \\ x_2 & = & 60 & - & 2x'_3 & + & x'_4 \\ x'_5 & = & 20 & - & x'_3 & + & x'_4 \\ \hline z' & = & 180 & - & x'_3 & - & x'_4 \end{array}$$

substitute: $x'_3 = x_3 + \frac{1}{2}x_6$, $x'_4 = x_4 + x_6$, $x'_5 = x_5 + x_6$, $z' = z - x_6$

$$\begin{array}{rcrcrcr} x_1 & = & 20 & + & (x_3 + \frac{1}{2}x_6) & - & (x_4 + x_6) \\ x_2 & = & 60 & - & 2(x_3 + \frac{1}{2}x_6) & + & (x_4 + x_6) \\ x_5 + x_6 & = & 20 & - & (x_3 + \frac{1}{2}x_6) & + & (x_4 + x_6) \\ \hline z - x_6 & = & 180 & - & (x_3 + \frac{1}{2}x_6) & - & (x_4 + x_6) \end{array} \quad \begin{array}{rcrcrcrcr} x_1 & = & 20 & + & x_3 & - & x_4 & - & \frac{1}{2}x_6 \\ x_2 & = & 60 & - & 2x_3 & + & x_4 & & \\ x_5 & = & 20 & - & x_3 & + & x_4 & - & \frac{1}{2}x_6 \\ \hline z & = & 180 & - & x_3 & - & x_4 & - & \frac{1}{2}x_6 \end{array}$$

Add a new constraint: *packaging*, 250 units of cardboard, 3/soldier, 4/train, 1/car $\rightarrow x_7 =$ slack

$$\begin{array}{rrrrrrrl} \text{Max} & 3x_1 & + & 2x_2 & + & x_6 & & \\ & x_1 & + & x_2 & + & \frac{1}{2}x_6 & \leq & 80 \\ & 2x_1 & + & x_2 & + & x_6 & \leq & 100 \\ & x_1 & & & + & x_6 & \leq & 40 \\ & 3x_1 & + & 4x_2 & + & x_6 & \leq & 250 \\ & & & & & x_1, x_2, x_6 & \geq & 0 \end{array} \quad \begin{array}{rcrcrcrcr} x_3 & = & 80 & - & x_1 & - & x_2 & - & \frac{1}{2}x_6 \\ x_4 & = & 100 & - & 2x_1 & - & x_2 & - & x_6 \\ x_5 & = & 40 & - & x_1 & & & - & x_6 \\ x_7 & = & 250 & - & 3x_1 & - & 4x_2 & - & x_6 \\ \hline z & = & 0 & + & 3x_1 & + & 2x_2 & + & x_6 \end{array}$$

in the optimal dictionary $\rightarrow$ make x_7 **basic** $\rightarrow$ express x_7 using the dictionary $\rightarrow$ new dictionary

$x_7 = 250 - 3x_1 - 4x_2 - x_6 = 250 - 3(20 + x_3 - x_4 - \frac{1}{2}x_6) - 4(60 - 2x_3 + x_4) - x_6 = -50 + 5x_3 - x_4 + \frac{1}{2}x_6$

$$\begin{array}{rcrcrcrcr} x_1 & = & 20 & + & x_3 & - & x_4 & - & \frac{1}{2}x_6 \\ x_2 & = & 60 & - & 2x_3 & + & x_4 & & \\ x_5 & = & 20 & - & x_3 & + & x_4 & - & \frac{1}{2}x_6 \\ \hline z & = & 180 & - & x_3 & - & x_4 & - & \frac{1}{2}x_6 \end{array} \quad \rightarrow \quad \begin{array}{rcrcrcrcr} x_1 & = & 20 & + & x_3 & - & x_4 & - & \frac{1}{2}x_6 \\ x_2 & = & 60 & - & 2x_3 & + & x_4 & & \\ x_5 & = & 20 & - & x_3 & + & x_4 & - & \frac{1}{2}x_6 \\ x_7 & = & -50 & + & 5x_3 & - & x_4 & + & \frac{1}{2}x_6 \\ \hline z & = & 180 & - & x_3 & - & x_4 & - & \frac{1}{2}x_6 \end{array}$$

If the resulting dictionary is feasible, then it is also optimal (we don't change z, all coeffs still non-positive)

However, the resulting dictionary may be **infeasible** if some basic variable is negative (here $x_7 < 0$)

Sun.Sep.25.121858.2016@rabbitchaser