IEOR 4004
Lecture 2 - Basic Linear Programming Formulations

0.1 From last lecture

Example: maximum throughput supply chain. A food company owns three warehouses with limited space for storage of a perishable good (fish, say). Warehouse i ($i = 1, 2, 3$) can hold up to s_i truckloads per day. The company also owns four stores. Store j ($j = 1, 2, 3, 4$) can sell up to d_j truckloads/day. Transportation costs are the same between all warehouses and stores, but we have a limit of U_{ij} truckloads that can be shipped from each warehouse i to each store j. Problem: what is the maximum number of truckloads that can be sold on one day?

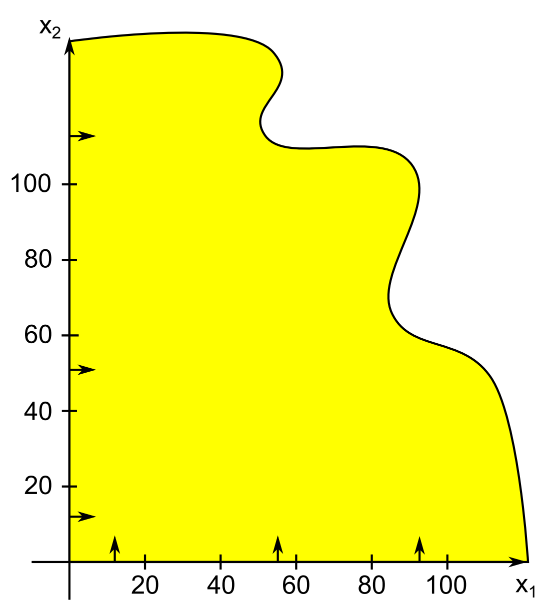
Answer. We can begin to answer the question by choosing the appropriate variables. In this case, let x_{ij} be the number of truckloads shipped from i to j. What are the constraints?

A graphical look at LPs

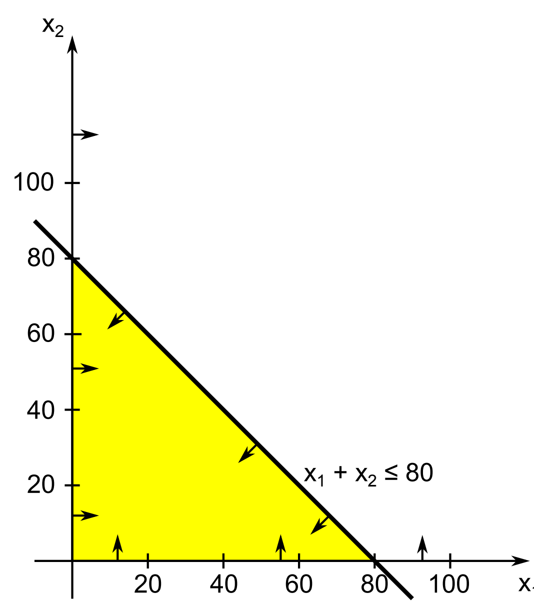
$$
\begin{array}{rrcrcl}
\text{Max} & 3x_1 & + & 2x_2 & & \\
& x_1 & + & x_2 & \leq & 80 \\
& 2x_1 & + & x_2 & \leq & 100 \\
& x_1 & & & \leq & 40 \\
& & & x_1, x_2 & \geq & 0
\end{array}
$$

1. Find the feasible region.

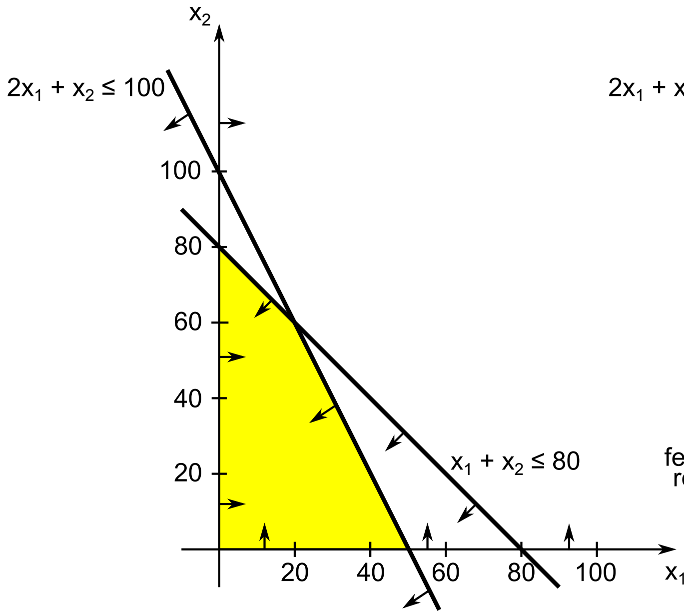
   - Plot each constraint as an equation $\equiv$ line in the plane
   - Feasible points on one side of the line – plug in (0,0) to find out which



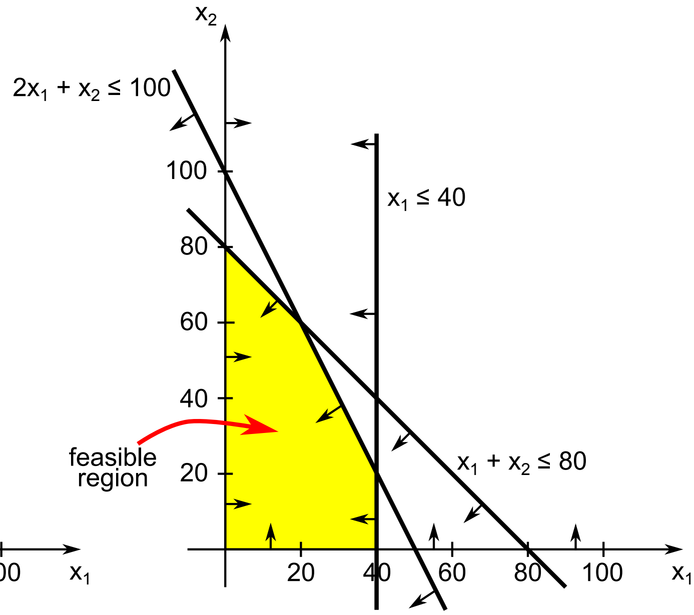
Start with $x_1 \geq 0$ and $x_2 \geq 0$



add $x_1 + x_2 \leq 80$



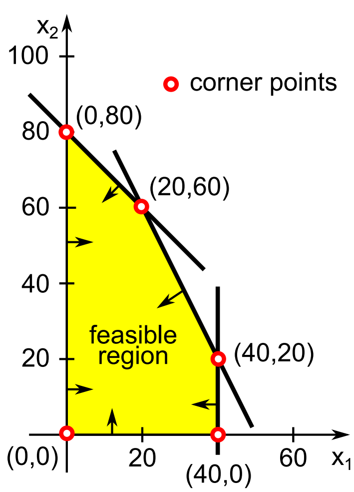
add $2x_1 + x_2 \leq 100$

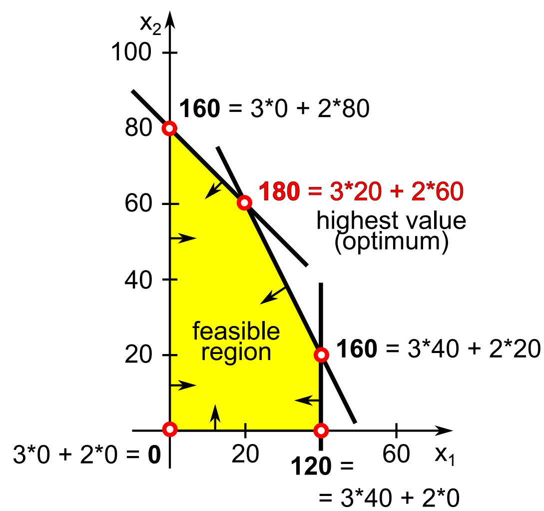


add $x_1 \leq 40$

A **corner** (extreme) point X of the region $R \equiv$ every line through X intersects R in a segment whose one endpoint is X. Solving a linear program amounts to finding a best corner point by the following theorem.

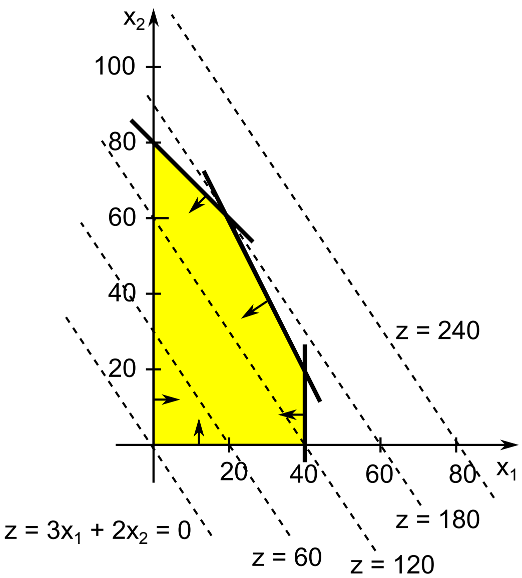
Theorem 1. *If a linear program over nonnegative variables has an* **optimal** *solution, then it also has an* **optimal solution** *that is a* **corner point** *of the feasible region.*

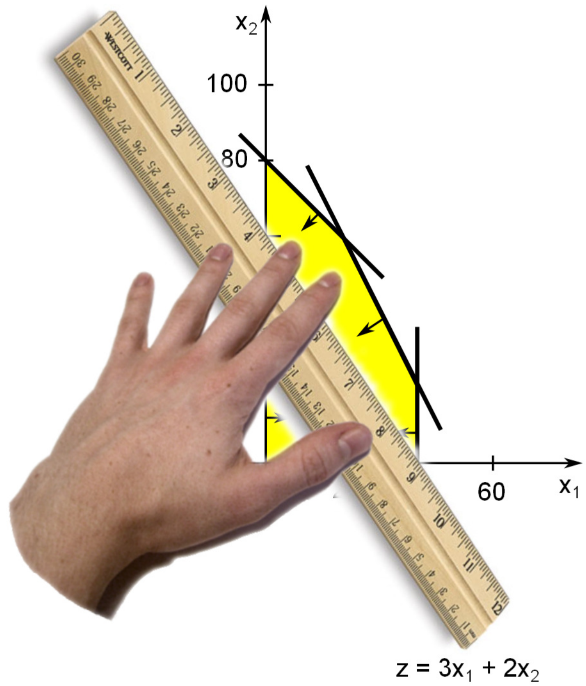




Problem: there may be too many corner points to check. There's a better way.
Iso-value line $\equiv$ in all points on this line the objective function has the same value
For our objective $3x_1 + 2x_2$ an iso-value line consists of points satisfying $3x_1 + 2x_2 = z$ where z is some number.
Graphical Method (main steps):

1. Find the feasible region.
2. Plot an iso-value (isoprofit, isocost) line for some value.
3. Slide the line in the direction of increasing value until it only touches the region.
4. Read-off an optimal solution.





Optimal solution is $(x_1, x_2) = (20, 60)$.

Observe that this point is the intersection of two lines forming the boundary of the feasible region. Recall that lines we use to construct the feasible region come from inequalities (the points on the line satisfy the particular inequality with equality).

0.2 Basic concepts related to linear programming

- Feasible region, infeasible problems
- Convexity
- Bounded, unbounded feasible regions
- Extreme points = corner solutions
- Unique optimum vs multiple optima
- Fact: any feasible LP attains its optimum at an extreme point

0.3 Software review and practice

0.4 Standard form for LPs

There are many kinds of forms for linear programs. We could have a constraint like

$$2x_1 - 5x_2 + x_4 \ = \ 10$$

or an inequality like

$$-8x_1 + 10x_2 - x_3 \ \leq \ 20$$

or

$$33x_1 + 17x_2 \ \geq \ 30.$$

Also the objective could be "max" or "min". Finally we could have variables that are nonnegative, or variables that don't have any constraint in terms of their sign. It turns out that we can just use one standard description. In generic, linear algebra notation, this is:

$$\max\ c^T x \tag{1a}$$

Subject to

$$Ax = b \tag{1b}$$

$$x \geq 0. \tag{1c}$$

Here A is an $m \times n$ matrix, c is an n-vector and b is an m-vector. In longhand notation (1) is the same as

$$\max \sum_{j=1}^{n} c_j x_j \tag{2a}$$

Subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \qquad i = 1, 2, \ldots, m \tag{2b}$$

$$x_j \geq 0, \qquad j = 1, 2, \ldots, n. \tag{2c}$$

Important: We will use the standard form when discussing algorithms for solving linear programs. From the perspective of practical applications, all (valid) formulations are good, and the more transparent the better.

Converting other LP forms into standard form.
1. Suppose we have

$$\min\ c^T x \tag{3a}$$

Subject to

$$Ax = b \tag{3b}$$

$$x \geq 0. \tag{3c}$$

This is "min" rather than "max". But (3) is the same problem, i.e. **it has the same solution vector as:**

$$\max\ (-c)^T x \tag{4a}$$

Subject to

$$Ax = b \tag{4b}$$

$$x \geq 0. \tag{4c}$$

2. Suppose we have a free variable, i.e. a variable that is not required to be nonnegative (for example a profit or loss variable).

Suppose x_j is one such variable. Then we change variables, using the substitution

$$x_j = x_j^+ - x_j^-, \quad x_j^+ \geq 0,\ x_j^- \geq 0.$$

Here, x_j^+ and x_j^- are new variables, both nonnegative. Can you see how this is an equivalent formulation.
3. How about inequality constraints? For example, if we have

$$x_1 - 4x_2 + x_3 \leq 10,$$

then we add a **new** variable $s \geq 0$ and we replace the inequality with

$$x_1 - 4x_2 + x_3 + s = 10.$$

Can you see how to handle $\geq$ inequalities?
4. How about nonpositive variables? Example $x_8 \leq 5$.