An Analysis of Optimal Balance in a Dynamic System

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Abstract

Liquid spills in classrooms, workplaces, and households often wreak havoc in terms of damaging electronics and stainable surfaces. Containers spill when they reach a specific angle - called the critical angle of a container. This critical angle is determined by the center of mass of the container, and is affected by the dimensions of the container, as well as the density and volume of the liquid inside the container. Determining a function for the critical angle can allow for the maximization of the angle for any given constraints.

Mathematical models and computer science programs have the potential to allow for the calculation of this critical angle. While it would have been possible to measure the angle by tipping over the bottle and estimating by eye the point at which it tipped in a series of trials, the critical angle would have to be calculated for each individually sized container, and again for each liquid with a different density or volume. Additionally, the measurement would not be completely precise.

To find a function for determining the critical angle, first, the center of mass of the system was calculated, while accounting for the fact that the center of mass of the system changes as the bottle is tipped. To calculate mass amounts of data, a program was coded in Java which uses the mathematical model to immediately determine the critical angle with various parameters. The accuracy of this system was proven by statistically comparing theoretical and experimental critical angle values for various sized containers.

It was determined that decreasing the volume of the liquid in the container, decreasing the height of the container, and increasing the base area of the container all increased the critical angle. Therefore, the optimal container size is flat and wide. The density of the liquid inside the container showed no change in critical angle due to minimal effect on the containers center of mass.

Introduction

Common spills in the classroom, workplace, and around the house that cause damage to electronics and stainable surfaces give need for finding a method of preventing common drink containers from tipping over.

It is possible to measure the angle at which containers tip by trials of estimation by eye, but each differently sized container with varying amounts of liquid would need to be tested individually.

By determining a function to measure the angle at which a bottle tips over, it is possible to maximize the angle at which the bottle tips over given a constraint.

Research Goals:

1. To create a mathematical model that will immediately determine a critical angle based on desired parameters.
2. To determine the effects of density of liquid in the container, volume of liquid in the container, height of the container, and base area of the container on the critical angle.
3. To maximize the critical angle based given constraint function on parameters.

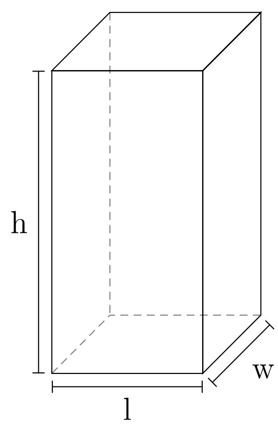
Hypothesis:

1. If the container is tipped, then its critical angle will be determined by its center of mass.
2. If we increase the density of the liquid in the container, no change in critical angle will be measurable.
3. If we increase the volume of liquid in the container, increase the height of the container, or decrease the area of the base of the container, the critical angle will decrease.

Methods

To determine the maximum angle we can tilt the water bottle so that it returns to its original position when released, we can find the angle at which the bottle would balance, and then slightly decrease the value by some epsilon. In order to find this angle, we will find the coordinates of the center of mass, and then find θ by using the fact that the x- coordinate of the center of mass must be greater than 0 in order for the bottle not to tip.

We can define our water bottle as a rectangular prism with dimensions as follows:



One key observation is that the coordinate of the center of mass in the w direction doesn't affect θ because the prism is symmetric with respect to the hw plane. The vertex connecting edges labeled h and l is (0,0) on the lh plane. The mass of the frame of the water bottle is:

$m_f = \rho(lwh - (l-2t)(w-2t)(h-2t))$, where t is the thickness of the frames.

The un-rotated coordinates of the center of mass are then $\left(\frac{l}{2}, \frac{h}{2}\right)$

Using the matrix for rotation around the origin, namely :

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Therefore the (x', y') coordinates are

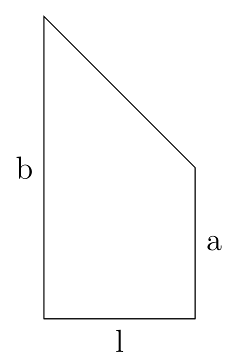
$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

Hence the rotated coordinates for the center of mass of the frame are:

$$\left(\frac{l}{2}\cos\theta - \frac{h}{2}\sin\theta, \frac{l}{2}\sin\theta + \frac{h}{2}\cos\theta\right)$$

We can easily find the mass of the water (or any liquid for that matter) if given the volume: $m_w = \rho_w v$ Finding the center of mass for the water is slightly more complicated.

One key observation is that the water will fill a trapezoidal shape once the bottle is tilted. We can find the center of mass of the following shape where the bottom left vertex is $(0, 0)$.



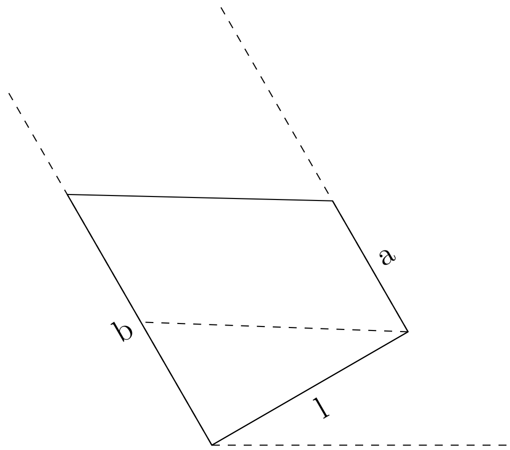
This will be the shape of the water inside the bottle if it keeps its shape once it returns back to the ground.

However, using area equality, and some basic trigonometry it is easy to find a and b in terms of θ.

Note that

$$A = \frac{v}{w} = \frac{l}{2}(a + b)$$

because the volume of the water doesn't change as it is rotated.



Since the two dashed lines are parallel, we know that the rotated angle and the interior opposite angles are congruent. Also the triangle formed by the the dashed line inside the trapezoid is right, meaning we can use trigonometric ratios to solve for a and b. Namely,

$$a + b = \frac{2A}{l}$$

$$b - a = l \tan\theta$$

Solving this system of equations by adding we see that:

$$2b = \frac{2A}{l} + l \tan\theta$$

$$b = \frac{A}{l} + \frac{l \tan\theta}{2}$$

$$b = \frac{l^2 \tan\theta + 2A}{2l}$$

and

$$a = \frac{2A - l^2 \tan\theta}{2l}$$

Using these values of a and b we can derive an equation for the coordinates of the center of mass of water. We begin by finding the un-rotated coordinates.

The top edge can be defined by the equation

$$f(x) = b + \frac{x}{l}(a - b)$$

Now,

$$A\bar{x} = \int_0^l xf(x)dx \implies \int_0^l x(b + \frac{x}{l}(a-b))dx$$

This evaluates to

$$\frac{1}{2}bx^2 + \frac{1}{3l}(a-b)x^3\Big|_0^l$$

then,

$$\frac{l^2}{3}(a-b) + \frac{l^2}{2}b \implies \frac{l^2}{6}(2a+b)$$

Hence,

$$\bar{x} = \frac{\frac{l^2}{6}(2a+b)}{\frac{h}{2}(a+b)} \implies \frac{l}{2} + \frac{l}{6}\left(\frac{a-b}{a+b}\right)$$

We can find the y- coordinate in a similar fashion,

$$A\bar{y} = \int_0^l \frac{1}{2}f^2(x)dx \implies \int_0^l \frac{1}{2}\left(b + \frac{x}{l}(a-b)\right)^2 dx$$

$$\int_0^l \left(\frac{1}{2}b^2 + b\frac{x}{l}a - \frac{x}{l}b^2 + \frac{1}{2}\frac{x^2}{l^2}a^2 - \frac{x^2}{l^2}ab + \frac{1}{2}\frac{x^2}{l^2}b^2\right) dx$$

This evaluates to

$$\frac{1}{2}b^2x + \frac{1}{2}b\frac{x^2}{l}b^2 + \frac{1}{6}\frac{x^3}{l^2}a^2 - \frac{1}{3}\frac{x^3}{l^2}ab + \frac{1}{6}\frac{x^3}{l^2}b^2\Big|_0^l$$

then,

$$\frac{l}{6}(b^2 + ab + a^2)$$

Hence,

$$\bar{y} = \frac{\frac{l}{6}(b^2+ab+a^2)}{\frac{l}{2}(a+b)} \implies \frac{1}{3}\frac{b^3-a^3}{b^2-a^2}$$

Putting $\bar{x}$ and $\bar{y}$ in terms of θ gives:

For $\bar{x}$:

$$\bar{x} = \frac{l}{2} + \frac{l}{6}\left(\frac{\dfrac{\frac{2v}{w} - l^2\tan\theta}{2l} - \left(\dfrac{\frac{2v}{w} + l^2\tan\theta}{2l}\right)}{\dfrac{\frac{2v}{w} - l^2\tan\theta}{2l} + \dfrac{\frac{2v}{w} + l^2\tan\theta}{2l}}\right)$$

Simplifying,

$$\bar{x} = \frac{l}{2} + \frac{l}{6}\left(\frac{\dfrac{-2l^2\tan\theta}{2l}}{\dfrac{\frac{4v}{w}}{2l}}\right) \implies \frac{l}{2} + \frac{l}{6}\left(\frac{\dfrac{-2l^2\tan\theta}{2l}}{\dfrac{4v}{2lw}}\right)$$

Then,

$$\bar{x} = \frac{l}{2} + \frac{l}{6}\left(\frac{-2l^2 w\tan\theta}{4v}\right) \implies \frac{l}{2} + \frac{-2l^3 w\tan\theta}{24v}$$

Finding a common denominator give us,

$$\bar{x} = \frac{12vl - 2l^3 w\tan\theta}{24v}$$

For $\bar{y}$:

$$\bar{y} = \frac{1}{3}\frac{\left(\dfrac{l^2\tan\theta + \frac{2v}{w}}{2l}\right)^3 - \left(\dfrac{\frac{2v}{w} - l^2\tan\theta}{2l}\right)^3}{\left(\dfrac{l^2\tan\theta + \frac{2v}{w}}{2l}\right)^2 - \left(\dfrac{\frac{2v}{w} - l^2\tan\theta}{2l}\right)^2}$$

Expanding both the numerator and denominator gives us:

$$\bar{y} = \frac{1}{3}\left(\frac{\dfrac{l^4w^2\tan\theta + 6l^2\tan\theta 4v^2}{4lw^2}}{\dfrac{2v\tan\theta}{w}}\right)$$

This simplifies to

$$\bar{y} = \frac{l^4w^2\tan^2\theta + 12v^2}{24lvw}$$

Now we once again will rotate the co-ordinates $(\bar{x}, \bar{y}$ counterclockwise around the origin by using the rotational matrix:

$$\begin{pmatrix} x_w \\ y_w \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

Therefore the (x_w, y_w) coordinates are

$$x_w = \bar{x}\cos\theta - \bar{y}\sin\theta$$

$$y_w = \bar{x}\sin\theta + \bar{y}\cos\theta$$

For notation's sake, we will refer to the rotated co-ordinates as (x_p, y_p). Now the center of mass of the entire system (x_c, y_c) is as follows:

$$x_c = \frac{m_wx_w + m_f\left(\dfrac{l}{2}\cos\theta - \dfrac{h}{2}\sin\theta\right)}{m_w + m_f}$$

$$y_c = \frac{m_wy_w + m_f\left(\dfrac{l}{2}\sin\theta + \dfrac{h}{2}\cos\theta\right)}{m_w + m_f}$$

In order for the bottle to balance, the x- coordinate must lie on the line $x = 0$,where (0,0) is the bottom left corner of the water bottle.

$\therefore$ setting the x- coordinate of the center of mass to 0 gives us the following equation:

$$0 = \frac{m_w x_w + m_f \left(\frac{l}{2} \cos\theta - \frac{h}{2} \sin\theta \right)}{m_w + m_f} \implies m_w x_w + m_f \left(\frac{l}{2} \cos\theta - \frac{h}{2} \sin\theta \right)$$

Now plugging back in our expression for x_p we arrive at

$$0 = \frac{m_w x_w + m_f \left(\frac{l}{2} \cos\theta - \frac{h}{2} \sin\theta \right)}{m_w + m_f}$$

so,

$$0 = m_w x_w + m_f \left(\frac{l}{2} \cos\theta - \frac{h}{2} \sin\theta \right)$$

where

$$x_w = \left(\frac{12vl - 2l^3 \tan\theta}{24v} \right) \cos\theta - \left(\frac{l^4 w^2 \tan^2\theta + 12v^2}{24lvw} \right) \sin\theta$$

$$m_f = \rho(lwh - (l - 2t)(w - 2t)(h - 2t))$$

$$m_w = \rho_w v$$

In this equation it is very difficult to isolate θ to solve for it. Instead, we can binary search for the answer with a lower bound of 0, and an upper bound of $\frac{\pi}{2}$. Since we are searching for our answer in a continuous variable, our search space contains an infinite number of values. On average, a computer is able to do 10^8 operations in a second. Since, our algorithm's run-time is a very efficient $O(\log n)$, we can let $n \approx 2^{10^8}$ and still achieve this mark. Therefore we will be able to theoretically be able to find our answer to approximately 2^{10^8} decimal points. Accounting for double inaccuracies, and other such errors, we can effectively find our answer to fifteen decimal places, comparing our answers by some epsilon $\epsilon = 10^{-7}$. However, for practical purposes we can round our answer to four decimal places.

As inputs our program will take:
The density of the liquid ρ_w.
The volume of the liquid v.

The density of the container ρ_f.
The length, width, and height of the container as described in the earlier figures l, w, h.
And the thickness of the frame t.

One other path that can be explored is maximizing the angle θ that the water bottle can be tilted to, given some constraint function in terms of the other variable. For instance, if you only had a certain amount of material, and a certain amount of liquid to put inside of the water bottle. In order to do this, we can use the Lagrange theorem, which relates the partial derivatives of the function to be optimized, and the constraint function by some λ.

For our case, we can define our function as:

$$F(l, w, h, t, v) = m_w \left(\left(\frac{12vl - 2l^3 \tan\theta}{24v} \right) \cos\theta - \left(\frac{l^4 w^2 \tan^2\theta + 12v^2}{24lvw} \right) \sin\theta \right) +$$

$$m_f \left(\frac{l}{2} \cos\theta - \frac{h}{2} \sin\theta \right)$$

We can now optimize this function with respect to θ with respect to some constraint function $H(l, w, h, t, v)$. By Lagrange's theorem we know that

$$\nabla F(l, w, h, t, v) = \lambda \nabla H(l, w, h, t, v)$$

Setting each partial derivative equal to each other, we can solve the resulting system of equations to arrive at the desired result.

Results and Data Analysis

In order to test how accurate my formulas are, I compared the angle calculated by them, to the angle measured by running tests experimentally. I did this for three different container sizes, and four different amounts of water for each container. I allowed for a two-degree difference to help account for experimental error. The following tables include both the calculated and observed angles.

Proving Model Accuracy: Data:

Container 1

Dimensions: 13.7 by 13.7 by 20.7 cm   Mass: 1.14 kg

	Volume	Theta Observed	Theta Expected
Volume 1	0.000528	21.8497	18.50707054
Volume 2	0.000887	22.3237	21.33544922
Volume 3	0.001417	26.7588	24.25493717
Volume 4	0.001789	27.2271	25.48819542

Container 2

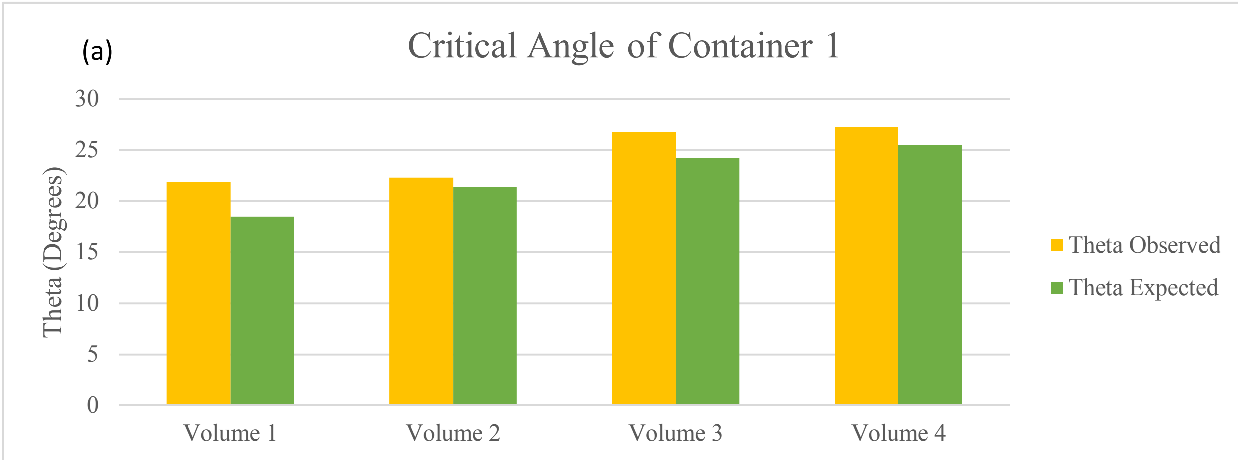
Dimensions: 11 by 11 by 25 cm   Mass: 1.499 kg

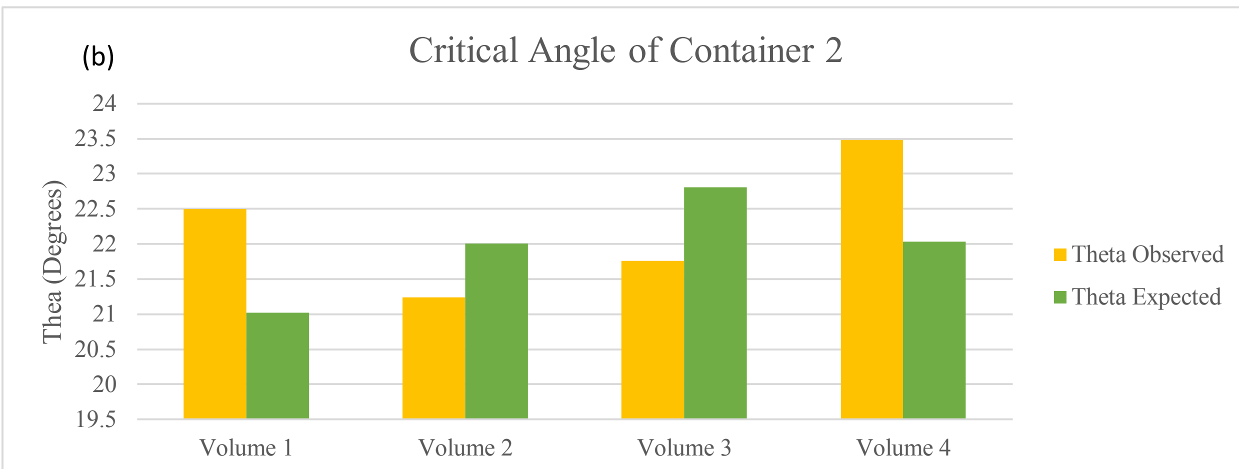
	Volume	Theta Observed	Theta Expected
Volume 1	0.000756	22.4982	21.0168457
Volume 2	0.00101	21.2371	22.00630188
Volume 3	0.00151	21.7621	22.80555725
Volume 4	0.00227	23.4776	22.03308105

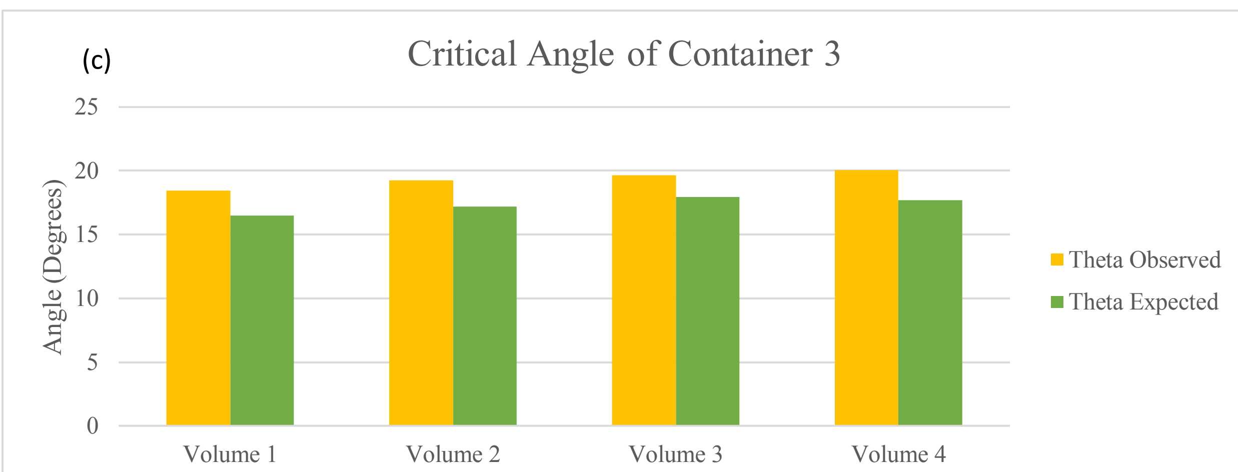
Container 3

Dimensions: 5.4 by 5.4 by 14.5 cm   Mass: .288 kg

	Volume	Theta Observed	Theta Expected
Volume 1	0.000106	18.4561	16.47949219
Volume 2	0.000141	19.2327	17.18811035
Volume 3	0.000211	19.6579	17.91595459
Volume 4	0.000317	20.0214	17.67253876







Figures (a), (b), and (c) each compare the theoretical and experimental critical angles at four different liquid volumes of the three differently sized containers.

Next, a hypothesis test was done to prove that the differences between theoretical and experimental critical angles were not statistically significant. Since the observed angle, and expected angle measures are paired, for each container I ran the following Paired Data Difference in Means Hypothesis Tests.

Container 1:
Dimensions: 13.7 by 13.7 by 20.7 cm
Mass: 1.14 kg

Hypothesis
$H_0 : d = 2$
$H_1 : d \neq 2$
$\alpha = 0.05$

Check Assumptions
Simple Random Sample: Yes. The amounts of water were randomly chosen.
Independence: $n \leq .1N$; $40 \leq N$. Yes, there are more than 40 angles.
Normality: We cannot establish normality, therefore we will use a t hypothesis test.

Calculations
Paired Difference in Means t Hypothesis Test:
t : 1.1151
PVAL: .3461
df: 3

Conclusion $\therefore$ Since the PVAL $> \alpha$ we fail to reject the H_0. There is no significant difference between expected and observed measures of θ.

Container 2:
Dimensions: 11 by 11 by 25 cm
Mass: 1.499 kg

Hypothesis
$H_0 : d = 2$
$H_1 : d \neq 2$
$\alpha = 0.05$

Check Assumptions
Simple Random Sample: Yes. The amounts of water were randomly chosen.
Independence: $n \leq .1N$; $40 \leq N$. Yes, there are more than 40 angles.
Normality: We cannot establish normality, therefore we will use a t hypothesis test.

Calculations
Paired Difference in Means t Hypothesis Test:
$t : -2.5087$
PVAL: .0870
df: 3

Conclusion $\therefore$ Since the PVAL $> \alpha$ we fail to reject the H_0. There is no significant difference between expected and observed measures of θ.

Container 3:
Dimensions: 5.4 by 5.4 by 14.5 cm
Mass: 0.288 kg

Hypothesis
$H_0 : d = 2$
$H_1 : d \neq 2$
$\alpha = 0.05$

Check Assumptions
Simple Random Sample: Yes. The amounts of water were randomly chosen.
Independence: $n \leq .1N$; $40 \leq N$. Yes, there are more than 40 angles.
Normality: We cannot establish normality, therefore we will use a t hypothesis test.

Calculations
Paired Difference in Means t Hypothesis Test:
t : .2243
PVAL: .8369
df: 3

Conclusion $\therefore$ Since the PVAL $>$ α we fail to reject the H_0. There is no significant difference between expected and observed measures of θ.

We have shown by three different hypothesis tests, that the difference in θ calculated and measured was not significantly different. Therefore, we can establish that this mathematical model may be a good fit.

Effects of Parameter Changes on the Critical Angles:

The Effect of Density of Liquid on the Critical Angle
Dimensions: 1 by 1 by 5 m Mass: 1 kg
Volume: 3 m cubed

Table 1	Density	Theta
Trial 1 (Water)	1000	18.1018424
Trial 2 (Milk)	1030	18.10194433
Trial 3 (Beer)	1060	18.10204089
Trial 4 (Coca Cola)	1200	18.10243249

The Effect of Volume on the Critical Angle
Dimensions: 1 by 1 by 5 m Mass: 1 kg
Density: 1000 kg/m^3

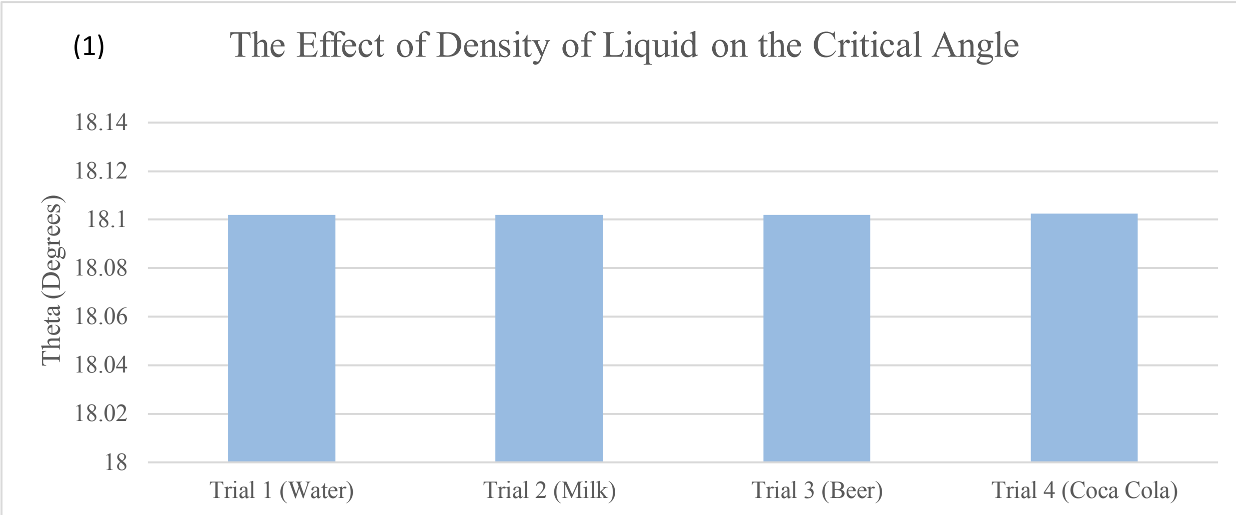
Table 2	Volume	Theta
1/4 of the Container	1.25	35.16760647
1/2 of the Container	2.5	21.24113202
3/4 of the Container	3.75	14.75727797

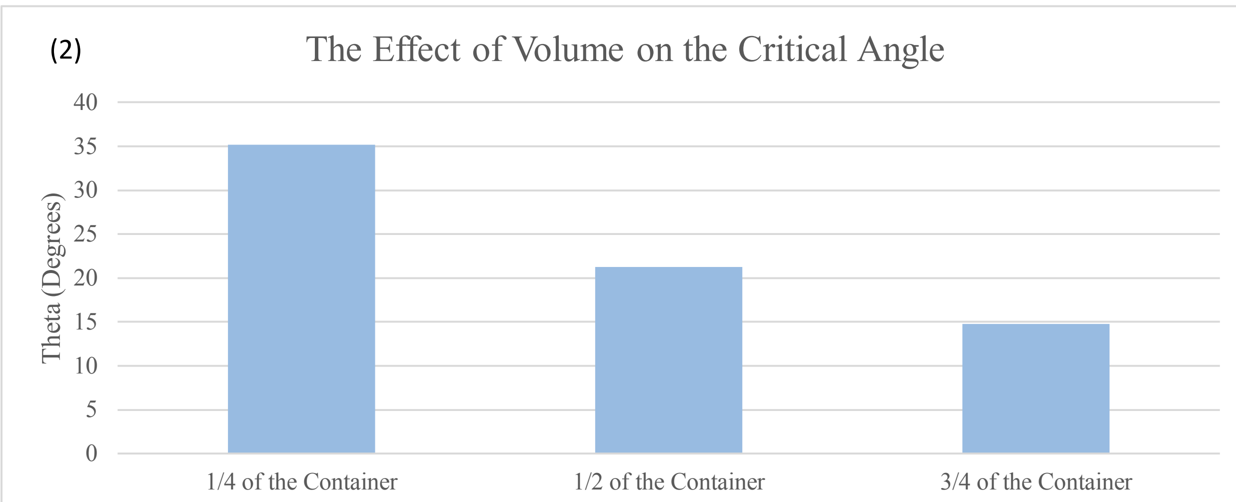
The Effect of Height of the Container on the Critical Angle
Dimensions: 1 by 1 by x m, t= .1, Density 900 kg/m^3
Volume: 1 m^3 Density (liquid): 1000 kg/m^3

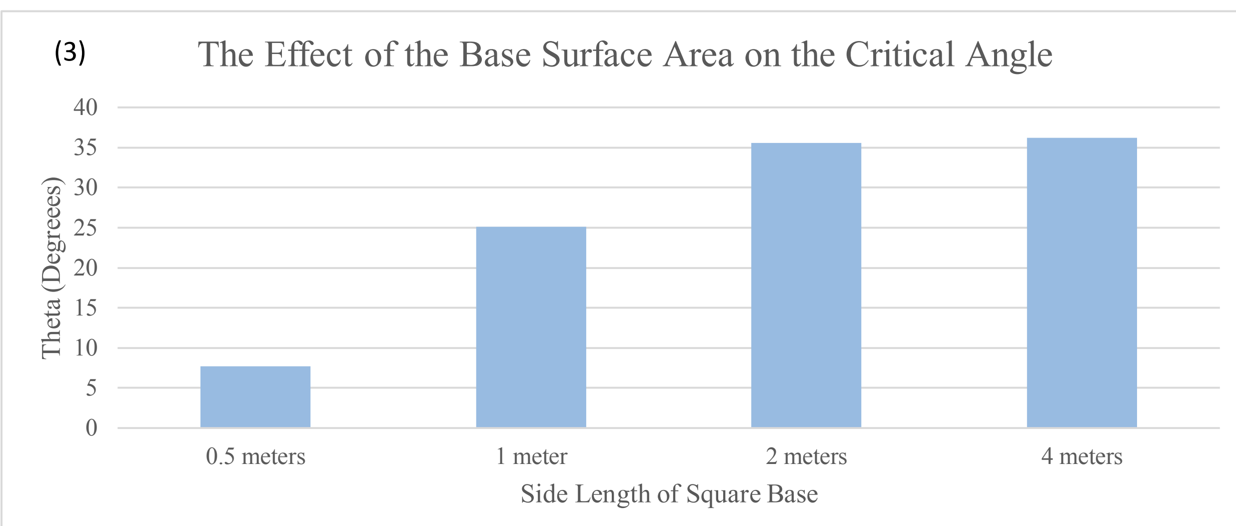
Table 3	Heights	Theta
2 meters	2	32.8771019
3 meters	3	25.14486194
4 meters	4	19.46600854
5 meters	5	15.52054882

The Effect of the Base Surface Area on the Critical Angle
Dimensions: X by X by 3 m, t= .1, Density 900 kg/m^3
Volume: 1 Density (liquid): 1000 kg/m^3

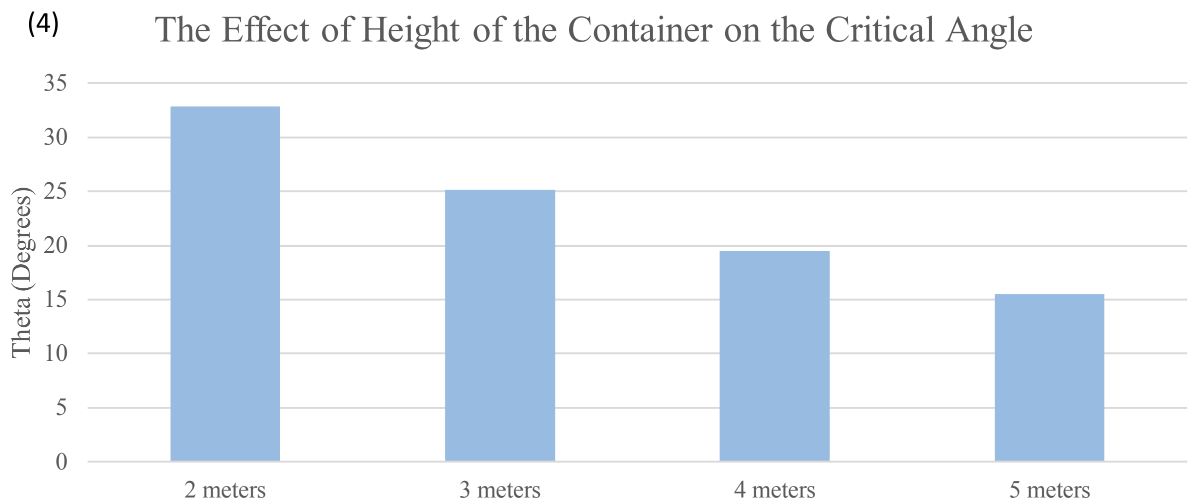
Table 4	X	Theta
0.5 meters	0.5	7.685864568
1 meter	1	25.14486194
2 meters	2	35.55965424
4 meters	4	36.24182582


(1)
The Effect of Density of Liquid on the Critical Angle
Theta (Degrees)
18.14
18.12
18.1
18.08
18.06
18.04
18.02
18
Trial 1 (Water)
Trial 2 (Milk)
Trial 3 (Beer)
Trial 4 (Coca Cola)


(2)
The Effect of Volume on the Critical Angle
Theta (Degrees)
40
35
30
25
20
15
10
5
0
1/4 of the Container
1/2 of the Container
3/4 of the Container


(3)
The Effect of the Base Surface Area on the Critical Angle
Theta (Degreees)
40
35
30
25
20
15
10
5
0
0.5 meters
1 meter
2 meters
4 meters
Side Length of Square Base

(4)



Figures (1), (2), (3), and (4) show the effects of changes in one parameter while holding the others constant. Each figure correlates to the hypotheses. Figure 1 shows that density of the liquid in the container has minimal effect on the critical angle. Figure 2 shows that increasing the volume of the liquid in the container decreases the critical angle. Figure 3 shows that increasing the height of the container decrease the critical angle. Figure 4 shows that increasing the base area of the container increases the critical angle.

Conclusion

It was determined that the mathematical model created produced results not significantly different to experimental values. Then, by using this model, it was determined that decreasing the volume of the liquid in the container, decreasing the height of the container, and increasing the base area of the container all increased the critical angle. Therefore, the optimal container size is flat and wide. One key property that was observed was that the changes in density of the liquid did not have much of an impact on the center of mass, therefore not affecting the critical angle as much as other variable do. This means that regardless of the liquid, an optimal shape of a container can be designed to maximize the critical angle.

Applications and Further Study

Mathematical models for different shapes of containers can also be created. The models can be compared to each other to see if there is a certain design of container that has the maximum critical angle given some constraint function. This optimization can also be found by using Lagrange's theorem. Additionally, we can explore the idea of using a container with a non-uniform density. We can find the center of mass, moments of inertia, and radius of Gyration by using double integrals and a density function. This non uniform density function will allow us to model a container that is weighted differently at different locations, allowing for the design of different structures that are more apt to balancing at a higher critical angle.

Additionally, we can explore finding the maximum volume in a certain container such that the critical is at its greatest. As we saw from our results, increasing volume doesn't necessarily guarantee an increase in the critical angle. Therefore, knowing this value can allow companies to mark their bottles at a certain level to show users what height the container should be filled to for maximum safety.

Furthermore, this model can be used for containers of all sizes. Thus, it can be used to help determine what the best way to ship containers with fluid contents is, such that the critical angle is maximal.

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