MA3209 Revision

Sum Hung Yee

December 5, 2024

1 Section 22

Question 2a.

Let $p : X \to Y$ be a continuous map. If $f : Y \to X$ is continuous with $p \circ f$ the identity, then p is a quotient map.

Proof. Since p has a right inverse, it is surjective. Since p is continuous, for open $U \subset Y$, $p^{-1}(U)$ is open. Now suppose $p^{-1}(V) \subset X$ is open. Then $f^{-1}(p^{-1}(V)) \subset Y$ is open. But $V = \{x \in X \mid p \circ f(x) \in V\}$ $\square$

Question 5. Let $p : X \to Y$ be an open map. Show that if A is open in X, then the map $q : A \to p(A)$ obtained by restricting p is an open map.

Proof. Clearly, $p(A) \subset Y$ is open. Consider any open $U \subset A$. Then $A \subset X$ being open implies $U \subset X$ is open. Then $p(U) \subset Y$ is open. Since $p(U) \subset p(A)$, $p(U) \cap p(A) \subset p(A)$ is open. $\square$

2 Section 23

Question 1.

Let X be a set, and let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on X. If $\mathcal{T}' \supset \mathcal{T}$, then the connectedness of X in one topology implies connectedness in the other.

Proof. Suppose $(X, \mathcal{T}')$ is connected. Then $(X, \mathcal{T})$ is connected. $\square$

Question 2.

Let $\{A_n\}$ be a sequence of connected subspaces of X, such that $A_n \cap A_{n+1} \neq \emptyset$ for all n. Show that $\bigcup_n A_n$ is connected.

Proof. Suppose $\bigcup_n A_n$ is disconnected. Then there exists open disjoint U, V whose union is $\bigcup_n A_n$. Since each A_n is connected, then each A_n either belongs to U, or V. Suppose $A_1 \subset U$. Then $A_2 \subset U$ and by induction, all $A_n \subset U$. Then, $V = \emptyset$, a contradiction. $\square$

Question 3.

Let $\{A_\alpha\}$ be a collection of connected subspaces of X; let A be a connected subspace of X. Show that if $A \cap A_\alpha \neq \emptyset$ for all α, then $A \cup (\bigcup_\alpha A_\alpha)$ is connected.

Proof. Assume $B = A \cup (\bigcup_\alpha A_\alpha)$ is disconnected. Then there exists U, V, nonempty, open and disjoint whose union is B. Since A is connected, assume $A \subset U$. Then for each A_α, since $A \cap A_\alpha \neq \emptyset$, $A_\alpha \subset U$, leaving V empty, a contradiction. □

Question 4.

Show that if X is an infinite set, it is connected in the cofinite topology.

Proof. Let U, V be nonempty open sets in X such that $U \cup V = X$. Then $U = X \setminus F_1, V = X \setminus F_2$ for finite sets F_1, F_2. Since X is infinite, these sets cannot be disjoint. Alternatively, if $U = X \setminus F$ is open, then F is finite and hence cannot also be open. Then, the only open and closed sets are $\emptyset$ and X. □

Question 5.

A space is totally disconnected if its only connected subspaces are one-point sets. Show that if X has the discrete topology, then X is totally disconnected. Does the converse hold?

Proof. In the discrete topology, all singletons are open and closed. Let $U \subset X$ be open and $|U| > 1$. Then $U \setminus \{x\}$ and $\{x\}$ form a separation of U. The converse is not true, since $\mathbb{Q} \subset \mathbb{R}$ is totally disconnected.

For any $(a, b) \in \mathbb{Q}$ with $a < b$, there exists $c \in \mathbb{R} \setminus \mathbb{Q}$ with $a < c < b$. Then, $(a, c) \cup (c, b)$ form a separation of (a, b) □

Question 6.

Let $A \subset X$. Show that if C is a connected subspace of X that intersects both A and $X \setminus A$, then C intersects δA.

Proof. Suppose otherwise that C does not intersect δA. Note that $\overline{A} = \mathring{A} \cup \delta A$ and $\delta A = \overline{A} \setminus \mathring{A}$. such that δA and $\mathring{A}$ are disjoint. Then $C \cap \delta A$ and $C \cap \mathring{A}$ form a separation of C. □

Question 8.

$\mathbb{R}^\omega$ is not connected in the uniform topology.

Proof. Consider the set A of all bounded sequences and B of all unbounded sequences. Clearly, they are disjoint and nonempty. Furthermore $A \cup B = \mathbb{R}^\omega$. Let $a = \{a_n\}_{n=1}^\infty$. Suppose $a \in A$. To show A is open, consider the open set containing a, $U = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times ....$ (This is the open ball $B^{\overline{\rho}}_{1/2}(a)$). Since a is bounded, U only contains bounded sequences and so it is contained in A. This shows A is open. Suppose $a \in B$. Then a is unbounded, and U only contains unbounded sequences and is contained in B. This shows B is open.

□

Remark 1. *The set $\mathcal{C}(X,Y) = \{f \in Y^X : f$ is continuous$\}$ is closed because, for any sequence $f_n \in (\mathcal{C}(X,Y), \overline{\rho})$ that converges in the uniform topology, it converges uniformly. This sequence exists due to the metrizability of the set. Then, by the uniform limit theorem, $f_n \to f$ and $f \in \mathcal{C}(X,Y)$. Then $\mathcal{C}(X,Y) = \overline{\mathcal{C}(X,Y)}$ and it is closed.*

- *$\mathcal{B}(X,Y)$ is closed. $\mathcal{B}(X,Y)$ is metrizable and we can find a sequence $f_n \in \mathcal{B}(X,Y)$ such that $f_n \to f \in \overline{\mathcal{B}(X,Y)}$. Let $N \in \mathbb{Z}^+$ such that $\overline{\rho}(f_N, f) < \frac{1}{2}$, then $\frac{1}{2}d(f_N(x), f(x)) \leq \rho(f_N(x), f(x)) < \frac{1}{2}$ for all $x \in X$ and therefore $d(f_N(x), f(x)) < 1$. Then for any $a, b \in X$,*

$$d(f(a), f(b)) \leq d(f(a), f_N(a)) + d(f_N(a), f_N(b)) + d(f(b), f_N(b))$$

  By taking sup over $a, b \in X$, $\operatorname{diam}(f) \leq 1 + \operatorname{diam}(f_N) + 1 < \infty$. Then $f \in \mathcal{B}(X,Y)$ and thus it is closed.

- *$\mathcal{B}(X,Y)$ is open. Let $g \in \mathcal{B}(X,Y)$. Take a ball $B_{1/2}(g) = \{f \in Y^X \mid \overline{\rho}(f,g) < 1/2\}$. Then for all $f \in B_{1/2}(g)$, for all $x \in X$, $\frac{1}{2}d(f(x), g(x)) \leq \rho(f(x), g(x)) < \frac{1}{2}$ and $d(f(x), g(x)) < 1$. Then by the previous point, f is bounded and $B_{1/2}(g) \subset \mathcal{B}(X,Y)$.*

Question 9.

Let A be a proper subset of X, and let B be a proper subset of Y. If X and Y are connected, show that $(X \times Y) - (A \times B)$ is connected.

Proof. Suppose on the contrary there exists a separation in $(X \times Y) \setminus (A \times B)$, $U, V \subset (X \times Y) \setminus (A \times B)$. Under the subspace topology, let $C \times D, E \times F$ be open in $X \times Y$ and $U = (C \times D) \setminus (A \times B)$, $V = (E \times F) \setminus (A \times B)$. Then $U \sqcup V = (X \times Y) \setminus (A \times B)$ implies $E \times F \sqcup C \times D = X \times Y$, a contradiction, because $X \times Y$ is connected.

□

Question 11.

Let $p : X \to Y$ be a quotient map. Show that if each set $p^{-1}(\{y\})$ is connected, and if Y is connected, then X is connected.

Proof. Suppose U, V form a separation of X. Since $p^{-1}(\{y\}) \subset X$ is connected, for all $y \in Y$, $p^{-1}(\{y\})$ is either in U or V. For some subset $S, T \subset Y$, $U = p^{-1}(S)$, $V = p^{-1}(T)$. Then V, U are saturated open sets w.r.t p. Then p maps U and V to open, sets, $W, A \subset Y$. Since U, V are disjoint, W, A are disjoint, a contradiction. □

3 Section 24

Question 1a.

Show that no two of the spaces $(0,1), (0,1]$, and $[0,1]$ are homeomorphic.

Proof. Suppose there exists a homeomorphism from $(0,1]$ to $[0,1]$. $[0,1] \subset \mathbb{R}$ is compact by the Heine-Borel theorem, however $(0,1]$ is not, since it is not closed. However $h^{-1}([0,1])$ is a homeomorphism that preserves compactness. A similar proof can be shown between $(0,1)$ and $[0,1]$. Lastly, consider $(0,1)$ and $(0,1]$. Let $x \in (0,1)$ such that $h(x) = 1 \in (0,1]$. Then $(0,1) \subset (0,1]$ is connected, implying $h^{-1}((0,1)) = (0,1) \setminus \{x\}$ should be connected (by continuity of h^{-1}). However, $(0,x) \cup (x,1)$ is not connected. $\square$

Question 1b.

Suppose that there exist imbeddings $f : X \to Y$ and $g : Y \to X$. Show by means of an example that X and Y need not be homeomorphic. f is a topological imbedding if f is a homeomorphism onto its image.

Proof. $f : (0,1) \to (0,1]$, $x \mapsto x$ is a homeomorphism to its image. $g : (0,1] \to (0,1)$, $y \mapsto y/2$ is a homeomorphism onto its image, $g((0,1]) = (0,1/2]$. These spaces are not homeomorphic. $\square$

Remark 2. *If $f : X \to Y$ is a homeomorphism, then $f|_U : U \to f(U)$ is a homeomorphism for any $U \subset X$.*

The restriction $g : U \to Y$, $x \mapsto f|_U(x)$ is continuous. By definition f^{-1} is a homeomorphism so $g^{-1} = f^{-1}|_{f(U)} : f(U) \to X$ is continuous. $g^{-1} \circ g$ and $g \circ g^{-1}$ are the identity maps.

Question 1c.

Show $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic with $n > 1$.

Proof. Suppose there exists a homeomorphism $h : \mathbb{R} \to \mathbb{R}^n$. Then $f : \mathbb{R} \setminus \{x\} \to \mathbb{R}^n \setminus \{h(x)\}$ is a homeomorphism. However, $\mathbb{R} \setminus \{x\}$ is disconnected, while $\mathbb{R}^n \setminus \{h(x)\}$ is connected, because it is path connected. Let p_1, p_2, where $p_i : [0,1] \to \mathbb{R}^n$ be paths from a to b, $p_i(0) = a$, $p_i(1) = b$. We may define $p_1(t) = a + t(b-a)$ We may define

$$p_2(t) = \begin{cases} a + 2t((0,0,...,b_n - a_n)) & t \leq 0.5 \\ a + 2(t-0.5)(b_1 - a_1, ..., 0) & t > 0.5 \end{cases}$$

Suppose $h(x)$ is on path p_1. Then p_2 is a path, ensuring that $\mathbb{R}^n \setminus \{h(x)\}$ is path connected and thus connected. Then, this contradicts the assumption that there exists a homeomorphism h. $\square$

Question 8a.

Is a product of path-connected spaces necessarily path connected?

Proof. Suppose X_α is path connected for all $\alpha \in \Lambda$. Let $X = \prod_{\alpha\in\Lambda} X_\alpha$. Get two points $\mathbf{p}, \mathbf{q} \in X$. Let $t \in [0,1]$ and let $f_\alpha(0) = \mathbf{p}_\alpha$, $f_\alpha(1) = \mathbf{q}_\alpha$ be a continuous function $f_\alpha : [0,1] \to X_\alpha$, since X_α is path-connected. Under the product topology, $f(t) = (f_\alpha(t))_{\alpha\in\Lambda}$ is continuous and a path from $\mathbf{p}$ to $\mathbf{q}$. $\square$

Remark 3. *Suppose each coordinate function $\alpha \in \Lambda$ is continuous. Under $f = (f_\alpha)_{\alpha\in\Lambda}$ with $f_\alpha = \pi_\alpha \circ f$ (this is continuous), the inverse image of the subbasis element $\pi_\alpha^{-1}(U_\alpha)$ given by $f^{-1}(\pi_\alpha^{-1}(U_\alpha))$ is $(\pi_\alpha \circ f)^{-1}(U_\alpha)$, which is open.*

Question 8b.

If $A \subset X$ and A is path connected, is $\overline{A}$ necessarily path connected?

Proof. No. Refer to topologist's sine curve, where S is path connected and $\overline{S}$ is not path connected. □

Question 8c.

If $f : X \to Y$ is continuous and X is path connected, is $f(X)$ necessarily path connected?

Proof. Let $p : [0,1] \to X$ be a path in X. Then $f \circ p : [0,1] \to f(X)$ is a path in $f(X)$. It is continuous by a composition of continuous functions. □

Question 8d.

If $\{A_\alpha\}$ is a collection of path-connected subspaces of X and if $\bigcap A_\alpha \neq \emptyset$, $\bigcup A_\alpha$ is also necessarily path connected.

Proof. Let $x \in \bigcap A_\alpha$ and choose a point $y_\alpha \in A_\alpha$. Let p_α be a path, $p_\alpha : [0,1] \to A_\alpha$ from y_α to x with $p_\alpha(0) = y_\alpha$ and $p_\alpha(1) = x$. There also exists a path $p_\beta : [1,2] \to A_\beta$ with $p_\beta(1) = x$ and $p_\beta(2) = y_\beta \in A_\beta$. By pasting lemma, there exists the path $p : [0,2] \to \bigcup_\alpha A_\alpha$, where

$$p(t) = \begin{cases} p_\alpha(t) & t \leq 1 \\ p_\beta(t) & t > 1 \end{cases}$$

□

Question 9. Assume that $\mathbb{R}$ is uncountable. Show that if A is a countable subset of $\mathbb{R}^2$, then $\mathbb{R}^2 \setminus A$ is path-connected. **Hint:** How many lines are there passing through a given point of $\mathbb{R}^2$?

Proof. If $A \subset \mathbb{R}^2$ is countable, let $(x,y) \in \mathbb{R}^2$. There is an uncountably infinite number of lines passing through (x,y). Then that means there are uncountable number of lines not intersecting A. Also, for any pair of points, there is a pair of lines that intersect each other but do not intersect A, showing that the lines are connected at the intersection. □

Question 11.

If A is a connected subspace of X, does it follow that $\overset{\circ}{A}$ and δA are connected? Does the converse hold? Justify your answers.

Proof. No. $(0,1) \subset \mathbb{R}$ is connected but its boundary, $\{0,1\}$ is disconnected. The open balls $B_1(0,0) \cup B_1(2,0)$ are disconnected but the boundary is connected. For the converse, let $\mathbb{Q}$ be a disconnected subspace of $\mathbb{R}$. The closure of $\mathbb{Q}$ is $\mathbb{R}$. The interior of $\mathbb{Q}$ is $\emptyset$. Consider any open $(a,b) \subset \mathbb{R}$ with $b > a$. Then (a,b) contains an irrational number. So

$$\bigcup_{U \in \mathcal{T}, U \subset \mathbb{Q}} U$$

is empty and hence connected. Then the boundary of $\mathbb{Q}$ is $\mathbb{R}$ itself, which is connected. $\square$

Remark 4. *Let $(X, \mathcal{T})$ be a topological space and $A \subset X$.*

1. *The interior of A is $\mathring{A} = \bigcup_{U \in \mathcal{T}, U \subset A} U$.*
2. *The closure of A is $\overline{A} = \bigcap_{G \in \mathcal{T}, G \supset A} G$.*
3. *The boundary of A is $\partial A = \overline{A} - \mathring{A}$.*

4 Section 25

Question 2a.

The connected components and path components of $\mathbb{R}^\omega$ (in the product topology) are itself.

Proof. $\mathbb{R}^\omega$ is connected so it is the connected component. (Munkres p. 151 Example 7) $\mathbb{R}^\omega$ is also path connected. Using $\mathbb{R}$ the path connectedness of $\mathbb{R}$, let $a, b \in \mathbb{R}^\omega$. Then there exists a path $p_n : [0,1] \to \mathbb{R}$ from a_n to b_n. Since each p_n is continuous for $n \in \mathbb{N}$, let $p : [0,1] \to \mathbb{R}^\omega$ be defined as $x \mapsto (p_n(x))_{n \in \mathbb{N}}$. Under the product topology, each p_n is continuous (as a path) implies p is continuous and is a path from a to b.

$\square$

Question 2b.

Consider $\mathbb{R}^\omega$ under the uniform topology. Show that $\mathbf{x}$ and $\mathbf{y}$ lie in the same connected component of $\mathbb{R}^\omega$ iff the sequence $\mathbf{x} - \mathbf{y}$ is bounded. Hint: It suffices to consider the case where $\mathbf{y} = 0$

Proof. Suppose $\mathbf{x} - \mathbf{y}$ is bounded. This direction is clear. Then there exists some path $p_n : [0,1] \to \mathbb{R}$ from $\mathbf{x}_n$ to $\mathbf{y}_n$. Since the projection map π_n is continuous (due to being finer than the product topology), let $p = (p_n)_{n \in \mathbb{N}}$, then let $U_n \subset \mathbb{R}(= X_n)$ if $\pi_n^{-1}(U_n) \subset \mathbb{R}^\omega$ is an open subbasis element, $p^{-1}(\pi_n^{-1}(U_n)) = (\pi_n \circ p)^{-1}(U_n) = p_n^{-1}(U_n)$ is open, and hence p is continuous and a path $[0,1] \to \mathbb{R}^\omega$. Then they lie in some path connected component, which lies in some connected component. Suppose $\mathbf{x}, \mathbf{y}$ lie in the same connected component. Consider the case where $\mathbf{y} = 0$ and $\mathbf{x}$ diverges. Note that $\mathbf{y} \in \mathcal{B}(\mathbb{N}, \mathbb{R})$, but $\mathbf{x} \notin \mathcal{B}(\mathbb{N}, \mathbb{R})$. Since $\mathcal{B}(\mathbb{N}, \mathbb{R})$ is both closed and open, $\mathcal{B}(\mathbb{N}, \mathbb{R})$ and its complement are a separation of $\mathbb{R}^\omega$ and hence $\mathbf{y}$ and $\mathbf{x}$ cannot be part of the same connected component, which is a contradiction. $\square$

Question 4.

Let X be locally path connected. Show that every connected open set in X is path connected.

Proof. If X is locally path connected, each path component is a connected component and vice versa. Then every connected open set in X lies in a connected component C, which is a path connected component. Them X is also path connected as it lies in a path connected component. □

Question 8.

Let $p : X \to Y$ be a quotient map. Show that if X is locally connected, then Y is locally connected. [Hint: If C is a component of the open set U of Y, show that $p^{-1}(C)$ is a union of components of $p^{-1}(U)$.]

Proof. Since X is locally connected, for every open $V \subset X$, all connected components of V are open in X. Let C be a connected component of open set $U \subset Y$, with $C \subset U \subset Y$. Then $p^{-1}(C) \subset p^{-1}(U)$ and $p^{-1}(U) \subset X$ is open. Consider a connected component containing x $S \subset p^{-1}(U)$ noting that $p(S)$ is also connected by continuity of p. Then $x \in p^{-1}(C) \cap S$ and so $p(x) \in C \cap p(S)$. Since $p(S)$ must lie in a connected component, it lies in C. □

5 Section 26

Question 1a.

Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the set X, and suppose that $\mathcal{T}' \supset \mathcal{T}$. What does the compactness of X under one of these topologies imply about compactness under the other?

Proof. If $(X, \mathcal{T}')$ is compact, then $(X, \mathcal{T})$ is compact. □

Question 1b.

Show that if X is compact Hausdorff under both $\mathcal{T}$ and $\mathcal{T}'$, then either $\mathcal{T} = \mathcal{T}'$ or $\mathcal{T}$ and $\mathcal{T}'$ are not comparable.

Proof. If $(X, \mathcal{T})$ is Hausdorff, then $(X, \mathcal{T}')$ is Hausdorff. □

Question 6.

Show that if $f : X \to Y$ is continuous, where X is compact and Y is Hausdorff, then f is a closed map (that is, f carries closed sets to closed sets).

Proof. Let $B \subset X$ be closed. Since X is compact, B is also compact. Then $f(B) \subset Y$ is compact. As a result, Since Y is Hausdorff, $f(B)$ is also closed. □

Question 7.

Show that if Y is compact, then the projection map $\pi_1 : X \times Y \to X$ is a closed map.

Proof. Let $B \subset X \times Y$ be closed. Let $A = \pi_1(B) \subset X$. Let $x_0 \in X \setminus A \subset X$ where this set is open. Then for any $(x_0, y) \in X \times Y$, there exists an open set in the product space $U_y \times V_y \ni (x_0, y)$. Clearly, $V_y \supset Y$. By compactness of Y, we may choose finitely many V_y covering Y. Then take $U_{y_i} \times V_{y_i}$, $i \in \{1, 2, ..., n\}$ to cover $\{x_0\} \times Y$. We know from the openness of each $U_y \times V_y$ that $U_y \times V_y \subset X \setminus A$. Take $U = \bigcap_{i=1}^n U_{y_i}$ and $V = \bigcup_{i=1}^n V_{y_i} = Y$. Then $U \subset X$ is open. Let $U_{x_0} = U$. Then $\bigcup_{x \in X \setminus A} U_x \subset X$ is open and $\bigcup_{x \in X \setminus A} U_x = X \setminus A$, so $A \subset X$ is closed.

□

Question 8.

Closed graph theorem. If $f : X \to Y$ where Y is compact Hausdorff. Then f is continuous if and only if

$$\Gamma_f = \{(x, f(x)) \mid x \in X\} \subset X \times Y \text{ is closed.}$$

Hint: If Γ_f is closed and V is a neighbourhood of $f(x_0)$ then the intersection of G_f and $X \times (Y \setminus V)$ is closed. Apply Question 7.

Proof. First, we show Γ_f is closed $\implies$ f is continuous, then the converse.

- Suppose $\Gamma_f \subset X \times Y$ is closed. Let $B \subset Y$ be closed. Then $\Gamma_f \cap (X \times B) = \{(x, f(x)) \mid f(x) \in B\} \subset Y$ is closed, since $(X \times B) = X \times (Y \setminus V) = (X \times Y) \setminus (X \times V)$ for some open V.
- Note that $C = \Gamma_f \cap (X \times B) = \{(x, f(x)) \mid f(x) \in B\}$. Then $\pi_X(C) = \{x \mid f(x) \in B\} = f^{-1}(B)$ is closed, by the previous question.

This proves $\implies$. Now to show the converse $\impliedby$.

- Suppose f is continuous. Let $(x, y) \in \overline{\Gamma_f}$ and $(x, f(x)) \in \Gamma_f$ where $y \neq f(x)$, which is the assumption that Γ_f is not closed. By Hausdorffness, there exists U, V disjoint and open containing y and $f(x)$ respectively.
- By continuity of f, there exists an open $W \ni x$ with $f(W) \subset V$.
- Then $W \times U$ and $W \times V$ are disjoint open sets containing (x, y) and $(x, f(x))$.
- By continuity of f, since (x, y) is in the closure of Γ_f, $W \times U$ intersects Γ_f.
- Then there exists $(w, f(w))$ in the intersection, $(w, f(w)) \in W \times U$, where $w \in W$.
- But $f(w) \in V$, a contradiction.

□

6 Section 28

Question 1.

Give $[0,1]^\omega$ the uniform topology. Find an infinite subset that has no limit point.

Proof. $X = [0,1]^\omega$ is not limit point compact. Let $A = \{\mathbf{e}_i \mid i \in \mathbb{N}\} \subset X$. This is an infinite set. Note that any $x \in X$ is not a limit point of A, since the open ball containing x, with radius $r/2$, contains all y with

$$\overline{\rho}(y,x) < r/2 \implies \forall n, d(x_n, y_n) < r$$

Then we can choose r small enough, if $x \in X \setminus A$, and if $r < 1$ if $x \in A$, these balls do not intersect $A \setminus \{x\}$. □

Question 3.

Let X be limit point compact. If $A \subset X$ is closed, is A limit point compact?

Proof. If $A = A \cup A'$ (it is closed), any infinite subset $B \subset A$ will have limit points $B' \subset A' \subset A$. (Any limit point of B is also a limit point of A) □

Remark 5. *Let $x \in B'$ be a limit point of $B \subset A$. Then for all open $U \subset X$, $U \cap B \setminus \{x\} \neq \emptyset$. This implies x is a limit point of A, since $B \subset A$.*

7 Section 29

Question 1.

Show the rationals are not locally compact

Proof. Since the rationals are dense in $\mathbb{R}$, let $(a,b) \cap \mathbb{Q}$ contain $x \in \mathbb{Q}$. Let $C \supset (a,b) \cap \mathbb{Q}$. It is not compact because it is not sequentially compact. We may construct a sequence in $(a,b) \cap \mathbb{Q}$ (which is in C) converging to $p \in \mathbb{R} \setminus \mathbb{Q}$. Then the subsequence also converges to $p \in \mathbb{R} \setminus \mathbb{Q}$. For instance, let c be an irrational number in (a,b). $x_n \in (c - 1/n, c) \cap \mathbb{Q}$ and the sequence converges to $c \in \mathbb{R} \setminus \mathbb{Q}$.

□

Remark 6. *$\mathbb{R}^\omega$ equipped with the product topology is not locally compact. The product topology is metrizable. Let Λ be a finite set. Take any basic open element*

$$U = \prod_{n \in \Lambda} (a_n, b_n) \times \prod_{n \in \mathbb{N} \setminus \Lambda} \mathbb{R}$$

Let $C \supset U$. We show that C is not sequentially compact and therefore is not compact. Consider the sequence $(x_i)_{i=1}^\infty \in U \subset C$, where $\pi_k(x_i) \in (a_i, b_i)$ for all $k \in \Lambda$ and $\pi_k(x_i) = i$ if $k \notin \Lambda$. Then, each x_i is in $U \subset C$, but the sequence x_n does not have a convergent subseqeuence.

Question 2a.

Let $\{X_\alpha\}$ be an indexed family of nonempty spaces.

(a) Show that if $\prod_{\alpha \in \Lambda} X_\alpha$ is locally compact, then each X_α is locally compact and X_α is compact for all but finitely many values of α.

(b) Prove the converse, assuming the Tychonoff theorem.

Proof. (a). Let $X = \prod_{\alpha \in \Lambda} X_\alpha$ be locally compact. Then for any basic open element

$$U = \prod_{\alpha \in F} U_\alpha \times \prod_{\alpha \in \Lambda \setminus F} X_\alpha$$

containing $\mathbf{x} = (x_\alpha)_{\alpha \in \Lambda}$ there exists a compact $C \supset U$ containing $\mathbf{x}$. Let $U_\alpha \subset X_\alpha$ be open and contain x_α. Then $\pi_\alpha(C) \supset U_\alpha$ and is compact. Compactness is preserved due to the continuity of the projection map, which is preserved in the product topology. Furthermore, for all $\alpha \in \Lambda \setminus F$, $\pi_\alpha(C) = X_\alpha$ and therefore X_α is compact for $\alpha \in \Lambda \setminus F$.

(b). Let X_α be compact for all $\alpha \in \Lambda \setminus F$ and each X_α locally compact. Again, F is an arbitrary finite set. Let $Y = \prod_{\alpha \in \Lambda \setminus F} X_\alpha$. Then Y is compact by Tychonoff's theorem. $X = \prod_{\alpha \in F} X_\alpha \times Y$. Now consider any basic open element $U \subset X$, as seen in the proof in (a). By local compactness of each X_α, there exists a compact $C_\alpha \subset X_\alpha$ containing U_α. $\prod_{\alpha \in F} C_\alpha \times Y$ (finite product of compact sets) is compact and contains U. □

Remark 7. *A topological space X is compactly generated if it satisfies either of the following equivalent conditions*

1. *$A \subset X$ is open $\iff$ $A \cap C$ is open in C for every compact $C \subset X$*

2. *$B \subset X$ is closed $\iff$ $B \cap C$ is closed in C for every compact $C \subset X$*

For any closed set $B \subset X$, $A = X \setminus B$ is open. For any closed set B, $A = X \setminus B$ is open. Then, $A = X \setminus B$ is open $\iff$ $A \cap C \subset C$ is open and $(X \setminus B) \cap C$ is open $\iff$ $B \cap C \subset C$ is closed $\iff$ B is closed.

Question 3.

Let X be locally compact. If $f : X \to Y$ is continuous, is $f(X)$ locally compact?

Proof. Let $y \in f(X)$ and $U \subset f(X)$ be open. By continuity of f, $f^{-1}(U)$ is open in X and contains x, and hence there exists a compact C such that $x \in f^{-1}(U) \subset C$. Then, $f(x) \in U \subset f(C)$, where $f(C)$ is compact by continuity of f.

□

Question 5.

If $f : X_1 \to X_2$ is a homeomorphism of locally compact Hausdorff spaces, then show f extends to a homeomorphism of their one point compactifications.

Proof. Note that since f is a homeomorphism from X_1 to X_2, consider the topologies of their one point compactifications, $\mathcal{A}_1 \cup \mathcal{A}_2$ of $X_1 \cup \{\infty_1\}$ and $\mathcal{B}_1 \cup \mathcal{B}_2$ of $X_1 \cup \{\infty_2\}$.

$$\mathcal{A}_1 = \{U \subset Y_1 \mid U \subset X_1 \text{ is open}\}$$

$$\mathcal{A}_2 = \{Y_1 \setminus C \mid C \subset X_1 \text{ is compact}\}$$

Extend f by letting $f(\infty_1) = \infty_2$. Then, since f is a homeomorphism, $U \subset X_1$ is open $\iff$ $f(U) \subset X_2$ is open, and $C \subset X_1$ is compact $\iff$ $f(C) \subset X_2$ is compact. The same could be said about its inverse. Then, $\mathcal{A}_1 \cup \mathcal{A}_2$ is homeomorphic to $\mathcal{B}_1 \cup \mathcal{B}_2$ under f.

□

8 Section 29

Question 7.

Every locally compact Hausdorff space is completely regular.

Proof. Let X be locally compact Hausdorff. Then there exists a compact Hausdorff Y, which is a one point compactification of X. Compact Hausdorff implies normality, which implies complete regularity.

□

Question 8.

Let X be completely regular, A, B, closed disjoint subsets of X. If A is compact, there is a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$, $f(B) = \{1\}$.

Proof. X is regular, let U, V separate x and B. By compactness of A, since U is open, there exist open sets $U_x \subset U$ where U_x contain x and V_x open, disjoint from U_x and containing B. Therefore this collection covers A. By compactness of A, we reduce this to a finite subcover of A, consisting of $U_{x_1}, ..., U_{x_n}$ The finite intersection of V_{x_i} is still open and contains x. Therefore, $U = \bigcup_{i=1}^n U_{x_i}$ contain x and $V = \bigcap_{i=1}^n V_{x_i}$ contain B. This shows X is normal and by Urysohn's lemma, completely normal.

□

9 Biglist

Prove that $\mathbb{R}$ with the co-countable topology is not locally compact.

Proof. Let $U = \mathbb{R} \setminus \mathbb{Z}^+$ and $x \in U$. Suppose $C \supset U$. Then $C = \mathbb{R} \setminus G$ with $G \subset \mathbb{Z}^+$. Then C is not compact. Since G is countable, $\mathbb{R} \setminus G = C$ is uncountably infinite and thus cannot be compact, since the compact sets are finite.

□

Remark 8. *Let X be uncountable, equipped with the co-countable topology. Suppose A is an infinite set. Then A is not compact. Let A' be a countably infinite subset of A. Then $U_0 = X \setminus A'$ is open. In particular, let $\{U_0, U_1, ...\}$ be an open cover of A, where*

$$U_i = X \setminus (A' \setminus \{a_i\})$$

and $U_0 = X \setminus A'$. Then U_0 covers $A \setminus A'$ and U_i covers each $\{a_i\} \subset A'$. There is no finite subcover.

Let $D \subset \mathbb{R}^2$ be countable. Then $\mathbb{R}^2 \setminus D$ is connected.

Proof. $\mathbb{R}^2 \setminus D$ is connected because it is path connected. Choose any $(p, q) \in \mathbb{R}^2 \setminus D$. There are uncountably infinite lines passing through (p, q) and not intersecting D. Choose these two lines L_1, L_2 not passing through a point in D, and passing through (p_1, q_1) and (p_2, q_2) respectively. They intersect at some other point $(p_3, q_3) \in \mathbb{R}^2 \setminus D$. Then there exists a path $p : [0, 1] \to \mathbb{R}^2$ from (p_1, q_1) to (p_2, q_2). Clearly, p is continuous, such that $\mathbb{R}^2 \setminus D$ is path connected and hence connected. □

Show that $\mathbb{R}^\omega$ in the box topology is disconnected. What about $\mathbb{R}^\omega$ under the uniform topology?

Proof. Consider A, the set of all bounded sequences and B, the set of all unbounded sequences. Clearly, $A \sqcup B$. It remains to see if they are open. Consider an open set in the box topology that contains the sequence $a = \{a_n\}_{n\in\mathbb{N}}$, given by $U_a = \prod_{n\in\mathbb{N}}(a_n - 1, a_n + 1)$. a is bounded if and only if every sequence in U_a is bounded. By considering $\bigcup_{a\in A} U_a = A$, we obtain the fact that A is open. Similarly, B is open. Then A and B form a separation of $\mathbb{R}^\omega$.

Remark 9. *Under the uniform topology, $U_a = \prod_{n\in\mathbb{N}}(a_n - 1, a_n + 1)$ is also open (it is a uniform ball), so we may use the same argument.*

□

10 Tutorials

10.1 Problem 3

Let X be equipped with the metric d. The subspace topology on $A \subset X$ is the discrete topology

Proof. A basic open element in A is

$$B_r(x) = \{y \in A \mid d_A(x, y) < r\}$$

Since A is finite, each $d(x, y) > \epsilon$ where $\epsilon = \min_y d(x, y)/2$. Then the singleton set $\{x\}$ is open. □

Let $\mathbb{R}$ be equipped with the standard topology. The subspace topology on $A = \{\frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{R}$ is the discrete topology. Show that the subspace topology on $A' = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{R}$ is NOT the discrete topology.

Proof. All singleton sets are open if and only if it is the discrete topology.

- Take an open ball $A \cap (\frac{1}{n} - \epsilon, \frac{1}{n} + \epsilon)$ where $\epsilon < 1/2(\frac{1}{n+1} - \frac{1}{n})$. Then the singleton set is open.
- Take any open ball $(-\epsilon, \epsilon) \cap A'$ containing 0. Then it always contains something else by the archimedean property. This implies $\{0\}$ is not open.

$\square$