General Topologies

Definition: $\{x_n\}_{n\in\mathbb{N}}$ converges if for all open $U \ni x$, $\exists N$ st $x_k \in U\ \forall k > N$.

Let $(X, \mathcal{T})$ be a topological space and $A \subset X$.

$\mathring{A} = \bigcup_{U\in\mathcal{T},\, U\subset A} U$ and $\bar{A} = \bigcap_{G\in\mathcal{T},\, G\supset A} G$.

Cofinite/Cocountable Topology

Let X be equipped with the cofinite topology.

- X is T_1. X is T_2 iff it is finite (discrete).
- If X is infinite, X is NOT metrizable and NOT first countable (and not 2nd countable)
- $\mathbb{R}$ with the cocountable topology is not compactly generated. Only finite sets are compact. Intersect with a non-closed infinite set to form a finite and hence countable and hence closed set.

Discrete and Trivial Topologies

Let X be equipped with the discrete topology. Then X is T_2 and all $A \subset X$ have NO limit points.

Product, Box Topologies

- The product of compact spaces, equipped with the product topo, is compact (Tychonoff). Not true for box topo.
- Closed subset of R^w (product space) NOT sequentially compact (let the infinite part diverge). Let $\overline{B} = \prod_{i\in\Lambda}[a_i, b_i] \times \prod_{i\in\mathbb{Z}^+\setminus\Lambda}\mathbb{R}$ for finite Λ
- Box topo not first countable (diagonalisation), not metrizable.
- Product topo of metric spaces are equipped with metric $D(x, y) = \sup\{\frac{1}{i}\rho(\pi_i(x), \pi_i(y)) : i \in I\}$
- $S(x, U) = \{f \mid f \in Y^X \text{ and } f(x) \in U\}$ is subbasis that generates product topo where $U \subset Y$ is open, $x \in X$
- $\mathbb{R}^\omega$ in the box topo is not connected: Let $a = \{a_n\}_{n\in\mathbb{N}} \in \mathbb{R}^\omega$. The open set $U = \prod_{i=1}^{\infty}(a_i - 1, a_i + 1)$ contains bounded sequences iff a is a bounded sequence. Same idea for uniform topo.
- $\mathbb{R}^\omega$ in the product topo is connected and complete

Continuous Functions & the Uniform Topology

Definition: (f_n) converges uniformly to f if, given $\epsilon > 0$, $\exists N$ such that for all $n > N$, $x \in X$, $d(f_n(x), f(x)) < \epsilon$

Definition: (f_n) converges pointwisely to f if, given $\epsilon > 0$, for all $x \in X$, $\exists N_x$ such that $d(f_n(x), f(x)) < \epsilon$ for $n > N_x$

- $f : X \to (Y, d)$ is cont. iff for any ϵ-ball $W \ni f(x)$, exists open U, $x \in U \subset X$, $f(U) \subset W$
- Let (Y, d) be a metric space. $\rho = \frac{d}{1+d}$. The uniform metric on Y^Λ is $\bar{\rho}(x, y) = \sup\{\rho(\pi_\alpha(x), \pi_\alpha(y)) \mid \alpha \in \Lambda\}$ and generates the uniform topo on Y^X
- If (Y, d) is complete (iff (Y, ρ) is complete due to bilipschitzness), then $(Y^\Lambda, \overline{\rho})$ is complete.
- Alternatively, $Y^X = \{\text{maps from } X \to Y\}$ with $\overline{\rho}(f, g) = \sup\{\rho(f(\alpha), g(\alpha)), \alpha \in X\}$
- $\mathcal{C}(X, Y) = \{f \in Y^X \mid f \text{ is continuous}\}$
- $\mathcal{B}(X, Y) = \{f \in Y^X \mid f(X) \subset Y \text{ bounded diam}\}$
- On $\mathcal{B}(X, Y)$, we use this metric: $d_{\text{sup}} = \sup\{d(f(x), g(x)) : x \in X\}$
- (Y, d) complete $\implies$ $(\mathcal{B}(X, Y), d_{\text{sup}})$ complete
- (Refer to Complete Metric Spaces). $(\mathcal{C}(X, Y), \overline{\rho}) \subset Y^X$ and $(\mathcal{B}(X, Y), \overline{\rho}) \subset Y^X$ are closed in uni topo on Y^X. If (Y, d) is complete $\implies$ $\mathcal{B}(X, Y)$ & $\mathcal{C}(X, Y)$ complete.
- Under uni topo, convergence of $f_n \to f \in Y^X \Leftrightarrow f_n \to f$ uniformly.
- Uniform limit theorem: If $f_n : X \to Y$ is a sequence of continuous functions from X (topo space) to Y (metric space) and converges uniformly to f, then f is continuous.
- If $f : X \to Y$ is a homeomorphism, then $f|_U : U \to f(U)$ is a homeomorphism for any $U \subset X$.
- Homeomorphisms preserve T_2, compactness, connectedness).

Summary

- The uniform topology is finer than the product topology but coarser than the box topology. (π_α is continuous)
- $U = \{(x_n)_{n\in\mathbb{N}} \mid |x_n| < 2^{-n} \forall n \in \mathbb{N}\}$ is open in box but not uniform. Assume it contains a uniform ball with radius ϵ. But $\forall n \in \mathbb{N}, x_n < 2^{-n} \implies \epsilon = 0$
- $V = \{x \in \mathbb{R}^\mathbb{N} \mid \rho(x, 0) < 0.01\}$ is open in uniform but not in product. Assume it contains a basic element $\prod_{i\in\Lambda} U_i \times \prod_{i\in\mathbb{N}\setminus\Lambda}\mathbb{R}$. Then it contains sets of the form $\{(x_n) \mid n \in \mathbb{N}, x_{n_1} = \ldots = x_{n_k} = 0\}$. Contradiction.

Quotient Topologies

Definition of Quotient map: $p : X \to Y$ is surjective AND $V \subset Y$ is open $\iff$ $p^{-1}(V) \subset X$ is open.

Definition: Let $f : X \to Y$ be surjective continuous and $A \subset X$. A is saturated w.r.t f if $A = f^{-1}(f(A))$ or $A = f^{-1}(S)$ for $S \subset Y$

- If X is a topo space, let X^* be the cells of a partition of X. Let $p : X \to X^*$ be surjective s.t. $x \mapsto [x]$. X^* is a quotient space of X.
- Let X be a space, A is a set, and $p : X \to A$ a surjective map. The quotient topo on A is the unique space s.t p is a quotient map, i.e. $\mathcal{T} = \{U \subset A : p^{-1}(U) \subset X \text{ is open}\}$
- A surjective continuous map f is a quotient map $\iff$ f maps every saturated open/closed sets to open/closed sets.
- If a surjective continuous map f is a quotient map, and $A \subset X$ is saturated and open/closed, then $f|_A : A \to f(A)$ is a quotient map. (Use prev statement creating $B \subset A$ saturated w.r.t $f|_A$ and hence saturated with f)

Metrizable Topologies

Let X be metrizable for this section.

Definition: X is totally bounded if $\forall\epsilon > 0$, $\exists$ a finite cover of X by balls of radius ϵ.

Definition: A number $\delta > 0$ is a Lebesgue number for an open cover $\mathcal{U}$ if for all $S \subset X$ such that $diam(S) < \delta$, $\exists U \in \mathcal{U}, S \subset U$

- Every metrizable space is T_4
- Finite sets in metric spaces are closed.
- X is first countable with countable basis $\{B_{1/i}(x), i \in \mathbb{Z}^+\}$ and Hausdorff
- X is compact $\iff$ X is sequentially compact $\iff$ X is limit point compact $\iff$ X is complete and totally bounded.
- If X is sequentially compact (and metrizable), X is totally bounded.
- If X is totally bounded, it has finite diameter.
- If X is a sequentially compact metric space, every open cover of X has a Lebesgue number (Pf by contradiction, create S_n, construct $x_n \in S_n$).

Almost $\iff$

- Let $A \subset X$, X is a topo space. If exists $(x_i)_{i=1}^{\infty} \subset A$ such that $x_i \to x$, then $x \in \overline{A}$. The converse is true if X is first countable.
- If $f : X \to Y$ is continuous, then for any convergent $(x_i)_{i=1}^{\infty} \subset X$, $f(x_i) \to f(x)$. The converse is true if X is first countable.
- Definition: X is Lindelof if every open cover has a countable subcover. Suppose X is 2nd countable (see countability). Then,
  - X is Lindelof
  - There exists countable subset $A \subset X$ that is dense ($\overline{A} = X$)

  Converse is true if X is metrizable.

Complete Metric Spaces

- A Metric Space is complete if every Cauchy sequence converges.
- A cauchy sequence converges $\iff$ it has a convergent subsequence.
- A subspace of a complete metric space is complete $\iff$ the subspace is closed.
- If $(X, d), (X', d')$ are bi-lipschitz, then (X, d) totally bounded iff (X', d') totally bounded, and if $X = X'$, a sequence is cauchy in (X, d) iff it is cauchy in (X, d').
- (X, d) compact $\iff$ (X, d) complete & totally bounded.
- Completeness not preserved by homeomorphisms: $(-1, 1)$ is not complete (use a sequence converging to 1) and $\mathbb{R}$ is complete (but homeomorphic)

Isometry

Definition: Let X, Y be two metric spaces. $f : (X, d_X) \to (Y, d_Y)$ is an isometric imbedding if for all $a, b \in X$, $d_X(a, b) = d_Y(f(a), f(b))$. If f is also surjective, then it is an isometry.

Definition: If (X, d_X) is a metric space, a metric completion of X is a complete metric space (Y, d_Y) and an isometric imbedding $\phi : X \to Y$ such that $\overline{\phi(X)} = Y$

- Every metric space (X, d) has a unique metric completion: There is an isometric imbedding ϕ of X into a complete metric space Y such that $\overline{\phi(X)} = Y$ (dense). Furthermore, if there is another metric completion Y' where ϕ' is the isometric imbedding, then there exists an isometry $f : Y \to Y'$ such that $f|_{\phi(X)} = \phi' \circ \phi^{-1}$

Compactness

- Compactness implies limit point compactness (converse false)
- Sequential compactness implies limit point compactness. (converse false)
- (Countereg to both converses) Consider $\mathbb{R}$ with the discrete topology and $Y = \{1, 2\}$ the trivial topology, then $\mathbb{R} \times Y$ is limit point compact but not sequentially compact/compact. The product topology on X has open sets $A \times Y$ with $A \subset X$.
- X is compact is equivalent to: Let $\mathcal{G}$ be a collection of closed sets in X with fip (applies for all finite subcollection of $\mathcal{G}$). Then $\bigcap_{G\in\mathcal{G}} \neq \varnothing$
- X is compact & $\{G_i\}_i$ is nested sequence of closed sets in X $\implies$ $\bigcap_{i=1}^{\infty} G_i = \varnothing$

Sequential and Limit Point Compactness

Definition: X is sequentially compact if every sequence HAS A convergent subsequence (in X).

Definition: X is limit point compact if every infinite $A \subset X$ has a limit point $x \in X$. ($x \in X$ is a limit point of A iff every open $U \subset X$ containing x intersects $A\backslash\{x\}$)

Local compactness

Definition: X is locally compact if for all $x \in X$, exists compact $C \subset X$, open set $U \subset X$, $x \in U \subset C$.

If X is Hausdorff, X is locally compact iff: For any $x \in X$, for any open $U \subset X$ containing x, there exists open $V \subset X$ such that $x \in V$, $\overline{V} \subset U$. with $\overline{V}$ compact.

- $\mathbb{R}^n$ is locally compact, take the closure of an open ball.
- $\mathbb{Q} \subset \mathbb{R}$ not locally compact, use denseness to construct a sequence (see metric spaces)
- Let X be locally compact. If $A \subset X$ is closed, A is locally compact.
- Let X be locally compact. If $A \subset X$ open and X Hausdorff, A is locally compact.

NOT locally compact list

- $\mathbb{R}^\omega$ (infinite products): Pf by contradiction, see product topology.
- $[0,1] \cap \mathbb{Q}$ (infinitely locally disconnected)
- $[0,1] \subset \mathbb{R}_l$ (strictly finer than compact Hausdorff topology)
- $\mathbb{R}$ with the cocountable topology.

Compactly Generated

- Definition: $A \subset X$ is open $\Longleftarrow$ $A \cap C \subset C$ is open for every compact $C \subset X$.
- If X is locally compact or first countable, then it is compactly generated.
- Let X be compactly generated. $f : X \to Y$ is continuous iff for all compact $C \subset X, f|_C$ is continuous.

Compactification and friends

X is a topo space. X is locally compact and Hausdorff iff there exists a compact Hausdorff space Y and a map $h_Y : X \to Y$ where h_Y is a homeomorphism onto $h_Y(X)$ and $Y \backslash h_Y(X)$ is a single point.

- If f is continuous (applies for homeomorphisms) then C is compact $\implies f(C)$ is compact.
- Let $X^* = X \cup \{p\}$. Note that $\mathcal{T}_{X^*} = \{U \subset X^* : U \subset X \text{ open}\} \cup \{X^* \backslash C : C \subset X \text{ compact}\}$
- General case of $\mathbb{R}^n$ to S^n (1 pt compactification): $N = (0,0,\dots,1)$. Define $h : S^n \backslash \{N\} \to \mathbb{R}^n$ where $(x,t) \mapsto \frac{1}{1-t}x$. It has inverse $h^{-1}(y) = \frac{1}{\|y\|^2}(2y, \|y\|^2 - 1)$.
- S^n is the one point compactification of $S^n \backslash \{N\}$ since S^n is compact Hausdorff, and $h(S^n \backslash \{N\})$ is dense in S^n

Hausdorff = T_2

- Every closed subspace of a compact space is compact and every compact subspace of a T_2 space is closed.
- If X is compact T_2 with no isolated points (so $\{x\}$ is not open and $U \ni y \neq x$) then X is uncountable.
- If $U \subset X$ is nonempty and open, $x \in X$ not isolated, there exists a nonempty open $V \subset U$ with $x \notin \overline{V}$.
- **Any** product of T_2 space is T_2. Subspace of T_2 space is T_2.
- X is locally compact and Hausdorff $\iff$ for any $x \in X$ and open $U \subset X$ containing x, exists $V \subset X$ with $x \in V$, $\overline{V} \subset U$, $\overline{V}$ compact.
- X is homeomorphic to open subset of a compact T_2 space $\iff$ X is locally compact and Hausdorff.

Connectedness

Definition: A separation of X is a pair of disjoint, open, **nonempty** subsets of X, U, V whose union is X.

- X is connected iff the only sets in X that are open and closed are $\varnothing$ and X.
- If $U, V \subset X$ is a separation of X and $Y \subset X$ is a connected subspace then $Y \subset U$ or $Y \subset V$.
- If $\{A_\alpha\}_{\alpha \in \Lambda}$ is a collection of connected subsets of X such that $\bigcap_{\alpha \in \Lambda} A_\alpha \neq \varnothing$, then $\bigcup_{\alpha \in \Lambda} A_\alpha \subset X$ is connected.
- If $A \subset X$ is connected and $A \subset B \subset \overline{A}$, then B is connected.
- If $f : X \to Y$ is continuous and $A \subset X$ is connected, then $f(A) \subset Y$ is connected.
- If X, Y are connected, then $X \times Y$ is connected.

Definition: Given $x, y \in X$, a path is a continuous map $f : [a,b] \to X$ with $f(a) = x, f(b) = y$. X is path connected if for all $x, y \in X$, there is a path from x to y.

Definition: The equivalence class of $\sim$ (resp. $\sim^p$), where $x \sim y$ iff there exists a connected set $C \subset X$ such that $x, y \in C$ (resp. if there exists a path in X from x to y), are the connnected (resp. path connected) components of X.

- Path connected implies connected. Every path component is path connected and every path component of X lies in a connected component of X.

Local Connectedness

Definition: X is locally (resp. path) connected at x if for all open $U \subset X$ containing x, there exists a (resp. path) connected open $V \subset X$ with $x \in V \subset U$

- X is locally (resp. path) connected $\iff$ for all open $U \subset X$, each (resp. path) connected component of U is open in X.
- Let X is locally path connected. Then:
  - The quotient topology on $\tilde{X} = \{$Connected components of X$\}$ is discrete.
  - The connected and path components are the same. (Recall all path components $P \subset C$ for some connected component C)
- All connected components are a disjoint union of path components.

Topologist's Sine Curve

The topologist's sine curve $\overline{S}$ is connected, because $S = f((0,1])$ is connected and $S \subset \overline{S} \subset \overline{S}$. S is path connected as well. However, $\overline{S}$ is not path connected. Suppose on the contrary that a path (continuous!) p exists with $p(0) = (0,0)$, $p(1)$ on the curve. Use continuity of p with closed $\{0\} \times [-1,1] \subset \mathbb{R}^2$ and $p^{-1}(B) \subset \mathbb{R}$ is closed with a maximum. Consider $p(b) \in B$, $p((b,1]) \subset S$. Take a sequence $t_i \to b$ and show $p(t_i)$ does not converge (see Almost $\iff$)

The topologist's sine curve is not locally connected and not locally path connected. Begin by taking a ball $B_1((0,0))$.

Examples

$\mathbb{Q}$ is neither connected nor locally connected. $(0,1) \cup (1,2)$ is not connected but locally connected. See product.

Countability

Definition: A topological space X is second countable if every open subset of X is a union of elements in some countable collection $\mathcal{B}$

- A topological space X is first countable if it has a countable basis at every $x \in X$
- A countable basis of X at x is a countable collection $\mathcal{B}$ of open sets containing x s.t. any open $U \subset X$ containing x also contains some $B \in \mathcal{B}$.
- 2nd countable implies 1st countable, pick sets in the countable basis containing x

Examples

- $\mathbb{R}^n$ is 2nd countable: $\{B_r(x) : r \in \mathbb{Q}, x \in \mathbb{Q}^n\}$
- $\mathbb{R}^\omega$ is 2nd countable: $\{\prod_{n \in \Lambda}(a_n, b_n) \times \prod_{n \in \mathbb{Z} \backslash \Lambda} \mathbb{R} \mid a_n, b_n \in \mathbb{Q}\}$
- $\mathbb{R}^\omega$ under uniform topology is not 2nd countable.
- Uncountable set in the discrete metric not second countable.
- Countable product and subspaces of first and second countable spaces preserve first and second countability.

Separation Axioms

Definition: Regular: A T_1 space X is T_3 if for every closed $B \subset X$ and $x \in X$ where $x \notin B$, $\exists$ open disjoint $U, V \subset X$, $x \in U, B \subset V$

Definition: Normal: A T_1 space X is T_4 if for every closed, disjoint $A, B \subset X$, $\exists$ open disjoint $U, V \subset X$, $A \subset U, B \subset V$

Definition: Separated: A and B (sometimes $\{x\}$) are separated by a continuous function if there exists cont. $f : X \to [0,1]$ st $f(A) = 0, f(B) = 1$.

Definition: Completely: ..., ... are separated by a continuous function

Defition: f is a topo. embedding if f is a homeomorphism between X and $f(X)$ (f need to be injective to its range)

- If X is compact, $T_2 \iff T_3 \iff T_4$
- $T_3 \iff \forall x \in X$, open $U \subset X$ containing x, $\exists V \subset X$ open, containing x, such that $\overline{V} \subset U$
- $T_4 \iff \forall$ closed $A \subset X$, open $U \supset A$, $\exists$ open $V \supset A$ such that $\overline{V} \subset U$
- If X is T_3 with countable basis, then X is T_4.
- Urysohn's Lemma: X is normal $\iff$ X is completely normal
- Urysohn's Theorem: X is regular with countable basis, then it is metrizable.
- Let X be T_1 space, $\{f_{\alpha \in \Lambda}$ is a family of cont. functions from X to $\mathbb{R}$ satisfying: $\forall x \in X, \forall U \overset{\text{open}}{\subset} X$ s.t. $x \in U$, there exists $\alpha \in \Lambda$ s.t. $f_\alpha(x) > 0$ & $f_\alpha(X \backslash U) = \{0\}$. Then map $F : X \to \mathbb{R}^\Lambda, x \mapsto (f_\alpha(x))_{\alpha \in \Lambda}$ is an embedding of X into $\mathbb{R}^\Lambda$

Function Spaces & Equicontinuity

Let (Y,d) be a metric space and X a topo space. The topology of compact convergence is defined over Y^X. Given a compact $C \subset X$, $\epsilon > 0$, $B(C, f, \epsilon) = \{g \in Y^X \mid \sup_{x \in C} d(f(x), g(x)) < \epsilon\}$ is a basis element.

- A sequence of functions in Y^X converges to f in the topology of compact convergence $\iff$ for every compact $C \subset X$, $f_n|_C \to f|_C$ uniformly.

The compact-open topology is defined over $C(X,Y)$. Let $C \subset X$ be compact and $U \subset Y$ open, $S(C,U) = \{g \in C(X,Y) \mid g(C) \subset U\}$ is a subbasis element.

- Let X be a compactly generated topo space and (Y,d) a metric space. $C(X,Y) \subset Y^X$ is closed in the topology of compact convergence (generated by basis elements).

Definition: Let (Y,d) be a metric space and $\mathcal{F} \subset C(X,Y)$. Fix $x_0 \in X$, then $\mathcal{F}$ is **equicontinuous** at x_0 if $\forall \epsilon > 0, \exists U \subset X$ open, $U \ni x_0$ st $\forall x \in U, \forall f \in \mathcal{F}$, $d(f(x), f(x_0)) < \epsilon$.

Let X be topo space, (Y,d) be metric space, $\bar{\rho}$ the uniform metric on $C(X,Y)$. $\mathcal{F} \subset C(X,Y)$ is totally bounded wrt $\bar{\rho}$ $\implies$ $\mathcal{F}$ is equicontinuous.

If $\mathcal{G} \subset C(X,Y)$ is equicontinuous, the topology of compact convergence and the topology of pointwise convergence on $\mathcal{G}$ agrees.

Arzela-Ascoli Theorem: Let X be a topo space, (Y,d) be a metric space. Endow $C(X,Y)$ with the compact-open topology and let $\mathcal{F} \subset C(X,Y)$.

i. If $\mathcal{F}$ is equicontinuous under d and $\mathcal{F}_a = \{f(a) : f \in \mathcal{F}\}$ has a compact closure for each $a \in X$, then $\overline{\mathcal{F}} \subset C(X,Y)$ is compact wrt uniform topology.

ii. The converse holds if X is locally compact and Hausdorff.