

MA3209 Revision

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1 Section 22

Question 2a.

Let $p : X \rightarrow Y$ be a continuous map. If $f : Y \rightarrow X$ is continuous with $p \circ f$ the identity, then p is a quotient map.

Proof. Since p has a right inverse, it is surjective. Since p is continuous, for open $U \subset Y$, $p^{-1}(U)$ is open. Now suppose $p^{-1}(V) \subset X$ is open. Then $f^{-1}(p^{-1}(V)) \subset Y$ is open. But $V = \{x \in X \mid p \circ f(x) \in V\}$ \square

Question 5.

Let $p : X \rightarrow Y$ be an open map. Show that if A is open in X , then the map $q : A \rightarrow p(A)$ obtained by restricting p is an open map.

Proof. Clearly, $p(A) \subset Y$ is open. Consider any open $U \subset A$. Then $A \subset X$ being open implies $U \subset X$ is open. Then $p(U) \subset Y$ is open. Since $p(U) \subset p(A)$, $p(U) \cap p(A) \subset p(A)$ is open. \square

2 Section 23

Question 1.

Let X be a set, and let \mathcal{T} and \mathcal{T}' be two topologies on X . If $\mathcal{T}' \supset \mathcal{T}$, then the connectedness of X in one topology implies connectedness in the other.

Proof. Suppose (X, \mathcal{T}') is connected. Then (X, \mathcal{T}) is connected. \square

Question 2.

Let $\{A_n\}$ be a sequence of connected subspaces of X , such that $A_n \cap A_{n+1} \neq \emptyset$ for all n . Show that $\bigcup_n A_n$ is connected.

Proof. Suppose $\bigcup_n A_n$ is disconnected. Then there exists open disjoint U, V whose union is $\bigcup_n A_n$. Since each A_n is connected, then each A_n either belongs to U , or V . Suppose $A_1 \subset U$. Then $A_2 \subset U$ and by induction, all $A_n \subset U$. Then, $V = \emptyset$, a contradiction. \square

Question 3.

Let $\{A_\alpha\}$ be a collection of connected subspaces of X ; let A be a connected subspace of X . Show that if $A \cap A_\alpha \neq \emptyset$ for all α , then $A \cup (\bigcup_\alpha A_\alpha)$ is connected.

Proof. Assume $B = A \cup (\bigcup_\alpha A_\alpha)$ is disconnected. Then there exists U, V , nonempty, open and disjoint whose union is B . Since A is connected, assume $A \subset U$. Then for each A_α , since $A \cap A_\alpha \neq \emptyset$, $A_\alpha \subset U$, leaving V empty, a contradiction. \square

Question 4.

Show that if X is an infinite set, it is connected in the cofinite topology.

Proof. Let U, V be nonempty open sets in X such that $U \cup V = X$. Then $U = X \setminus F_1, V = X \setminus F_2$ for finite sets F_1, F_2 . Since X is infinite, these sets cannot be disjoint. Alternatively, if $U = X \setminus F$ is open, then F is finite and hence cannot also be open. Then, the only open and closed sets are \emptyset and X . \square

Question 5.

A space is totally disconnected if its only connected subspaces are one-point sets. Show that if X has the discrete topology, then X is totally disconnected. Does the converse hold?

Proof. In the discrete topology, all singletons are open and closed. Let $U \subset X$ be open and $|U| > 1$. Then $U \setminus \{x\}$ and $\{x\}$ form a separation of U . The converse is not true, since $\mathbb{Q} \subset \mathbb{R}$ is totally disconnected.

For any $(a, b) \in \mathbb{Q}$ with $a < b$, there exists $c \in \mathbb{R} \setminus \mathbb{Q}$ with $a < c < b$. Then, $(a, c) \cup (c, b)$ form a separation of (a, b) \square

Question 6.

Let $A \subset X$. Show that if C is a connected subspace of X that intersects both A and $X \setminus A$, then C intersects δA .

Proof. Suppose otherwise that C does not intersect δA . Note that $\overline{A} = \overset{\circ}{A} \cup \delta A$ and $\delta A = \overline{A} \setminus \overset{\circ}{A}$. such that δA and $\overset{\circ}{A}$ are disjoint. Then $C \cap \delta A$ and $C \cap \overset{\circ}{A}$ form a separation of C . \square

Question 8.

\mathbb{R}^ω is not connected in the uniform topology.

Proof. Consider the set A of all bounded sequences and B of all unbounded sequences. Clearly, they are disjoint and nonempty. Furthermore $A \cup B = \mathbb{R}^\omega$. Let $a = \{a_n\}_{n=1}^\infty$. Suppose $a \in A$. To show A is open, consider the open set containing a , $U = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \dots$ (This is the open ball $B_{1/2}^{\bar{p}}(a)$). Since a is bounded, U only contains bounded sequences and so it is contained in A . This shows A is open. Suppose $a \in B$. Then a is unbounded, and U only contains unbounded sequences and is contained in B . This shows B is open.

□

Remark 1. The set $\mathcal{C}(X, Y) = \{f \in Y^X : f \text{ is continuous}\}$ is closed because, for any sequence $f_n \in (\mathcal{C}(X, Y), \bar{\rho})$ that converges in the uniform topology, it converges uniformly. This sequence exists due to the metrizable of the set. Then, by the uniform limit theorem, $f_n \rightarrow f$ and $f \in \mathcal{C}(X, Y)$. Then $\mathcal{C}(X, Y) = \overline{\mathcal{C}(X, Y)}$ and it is closed.

- $\mathcal{B}(X, Y)$ is closed. $\mathcal{B}(X, Y)$ is metrizable and we can find a sequence $f_n \in \mathcal{B}(X, Y)$ such that $f_n \rightarrow f \in \overline{\mathcal{B}(X, Y)}$. Let $N \in \mathbb{Z}^+$ such that $\bar{\rho}(f_N, f) < \frac{1}{2}$, then $\frac{1}{2}d(f_N(x), f(x)) \leq \rho(f_N(x), f(x)) < \frac{1}{2}$ for all $x \in X$ and therefore $d(f_N(x), f(x)) < 1$. Then for any $a, b \in X$,

$$d(f(a), f(b)) \leq d(f(a), f_N(a)) + d(f_N(a), f_N(b)) + d(f(b), f_N(b))$$

By taking sup over $a, b \in X$, $\text{diam}(f) \leq 1 + \text{diam}(f_N) + 1 < \infty$. Then $f \in \mathcal{B}(X, Y)$ and thus it is closed.

- $\mathcal{B}(X, Y)$ is open. Let $g \in \mathcal{B}(X, Y)$. Take a ball $B_{1/2}(g) = \{f \in Y^X \mid \bar{\rho}(f, g) < 1/2\}$. Then for all $f \in B_{1/2}(g)$, for all $x \in X$, $\frac{1}{2}d(f(x), g(x)) \leq \rho(f(x), g(x)) < \frac{1}{2}$ and $d(f(x), g(x)) < 1$. Then by the previous point, f is bounded and $B_{1/2}(g) \subset \mathcal{B}(X, Y)$.

Question 9.

Let A be a proper subset of X , and let B be a proper subset of Y . If X and Y are connected, show that $(X \times Y) - (A \times B)$ is connected.

Proof. Suppose on the contrary there exists a separation in $(X \times Y) \setminus (A \times B)$, $U, V \subset (X \times Y) \setminus (A \times B)$. Under the subspace topology, let $C \times D, E \times F$ be open in $X \times Y$ and $U = (C \times D) \setminus (A \times B)$, $V = (E \times F) \setminus (A \times B)$. Then $U \sqcup V = (X \times Y) \setminus (A \times B)$ implies $E \times F \sqcup C \times D = X \times Y$, a contradiction, because $X \times Y$ is connected.

□

Question 11.

Let $p : X \rightarrow Y$ be a quotient map. Show that if each set $p^{-1}(\{y\})$ is connected, and if Y is connected, then X is connected.

Proof. Suppose U, V form a separation of X . Since $p^{-1}(\{y\}) \subset X$ is connected, for all $y \in Y$, $p^{-1}(\{y\})$ is either in U or V . For some subset $S, T \subset Y$, $U = p^{-1}(S)$, $V = p^{-1}(T)$. Then V, U are saturated open sets w.r.t p . Then p maps U and V to open, sets, $W, A \subset Y$. Since U, V are disjoint, W, A are disjoint, a contradiction. □

3 Section 24

Question 1a.

Show that no two of the spaces $(0, 1)$, $(0, 1]$, and $[0, 1]$ are homeomorphic.

Proof. Suppose there exists a homeomorphism from $(0, 1]$ to $[0, 1]$. $[0, 1] \subset \mathbb{R}$ is compact by the Heine-Borel theorem, however $(0, 1]$ is not, since it is not closed. However $h^{-1}([0, 1])$ is a homeomorphism that preserves compactness. A similar proof can be shown between $(0, 1)$ and $[0, 1]$. Lastly, consider $(0, 1)$ and $(0, 1]$. Let $x \in (0, 1)$ such that $h(x) = 1 \in (0, 1]$. Then $(0, 1) \subset (0, 1]$ is connected, implying $h^{-1}((0, 1)) = (0, 1) \setminus \{x\}$ should be connected (by continuity of h^{-1}). However, $(0, x) \cup (x, 1)$ is not connected. \square

Question 1b.

Suppose that there exist imbeddings $f : X \rightarrow Y$ and $g : Y \rightarrow X$. Show by means of an example that X and Y need not be homeomorphic. f is a topological imbedding if f is a homeomorphism onto its image.

Proof. $f : (0, 1) \rightarrow (0, 1]$, $x \mapsto x$ is a homeomorphism to its image. $g : (0, 1] \rightarrow (0, 1)$, $y \mapsto y/2$ is a homeomorphism onto its image, $g((0, 1]) = (0, 1/2]$. These spaces are not homeomorphic. \square

Remark 2. If $f : X \rightarrow Y$ is a homeomorphism, then $f|_U : U \rightarrow f(U)$ is a homeomorphism for any $U \subset X$.

The restriction $g : U \rightarrow Y$, $x \mapsto f|_U(x)$ is continuous. By definition f^{-1} is a homeomorphism so $g^{-1} = f^{-1}|_{f(U)} : f(U) \rightarrow X$ is continuous. $g^{-1} \circ g$ and $g \circ g^{-1}$ are the identity maps.

Question 1c.

Show \mathbb{R}^n and \mathbb{R} are not homeomorphic with $n > 1$.

Proof. Suppose there exists a homeomorphism $h : \mathbb{R} \rightarrow \mathbb{R}^n$. Then $f : \mathbb{R} \setminus \{x\} \rightarrow \mathbb{R}^n \setminus \{h(x)\}$ is a homeomorphism. However, $\mathbb{R} \setminus \{x\}$ is disconnected, while $\mathbb{R}^n \setminus \{h(x)\}$ is connected, because it is path connected. Let p_1, p_2 , where $p_i : [0, 1] \rightarrow \mathbb{R}^n$ be paths from a to b , $p_i(0) = a$, $p_i(1) = b$. We may define $p_1(t) = a + t(b - a)$. We may define

$$p_2(t) = \begin{cases} a + 2t((0, 0, \dots, b_n - a_n)) & t \leq 0.5 \\ a + 2(t - 0.5)(b_1 - a_1, \dots, 0) & t > 0.5 \end{cases}$$

Suppose $h(x)$ is on path p_1 . Then p_2 is a path, ensuring that $\mathbb{R}^n \setminus \{h(x)\}$ is path connected and thus connected. Then, this contradicts the assumption that there exists a homeomorphism h . \square

Question 8a.

Is a product of path-connected spaces necessarily path connected?

Proof. Suppose X_α is path connected for all $\alpha \in \Lambda$. Let $X = \prod_{\alpha \in \Lambda} X_\alpha$. Get two points $\mathbf{p}, \mathbf{q} \in X$. Let $t \in [0, 1]$ and let $f_\alpha(0) = \mathbf{p}_\alpha$, $f_\alpha(1) = \mathbf{q}_\alpha$ be a continuous function $f_\alpha : [0, 1] \rightarrow X_\alpha$, since X_α is path-connected. Under the product topology, $f(t) = (f_\alpha(t))_{\alpha \in \Lambda}$ is continuous and a path from \mathbf{p} to \mathbf{q} . \square

Remark 3. Suppose each coordinate function $\alpha \in \Lambda$ is continuous. Under $f = (f_\alpha)_{\alpha \in \Lambda}$ with $f_\alpha = \pi_\alpha \circ f$ (this is continuous), the inverse image of the subbasis element $\pi_\alpha^{-1}(U_\alpha)$ given by $f^{-1}(\pi_\alpha^{-1}(U_\alpha))$ is $(\pi_\alpha \circ f)^{-1}(U_\alpha)$, which is open.

Question 8b.

If $A \subset X$ and A is path connected, is \overline{A} necessarily path connected?

Proof. No. Refer to topologist's sine curve, where S is path connected and \overline{S} is not path connected. \square

Question 8c.

If $f : X \rightarrow Y$ is continuous and X is path connected, is $f(X)$ necessarily path connected?

Proof. Let $p : [0, 1] \rightarrow X$ be a path in X . Then $f \circ p : [0, 1] \rightarrow f(X)$ is a path in $f(X)$. It is continuous by a composition of continuous functions. \square

Question 8d.

If $\{A_\alpha\}$ is a collection of path-connected subspaces of X and if $\bigcap A_\alpha \neq \emptyset$, $\bigcup A_\alpha$ is also necessarily path connected.

Proof. Let $x \in \bigcap A_\alpha$ and choose a point $y_\alpha \in A_\alpha$. Let p_α be a path, $p_\alpha : [0, 1] \rightarrow A_\alpha$ from y_α to x with $p_\alpha(0) = y_\alpha$ and $p_\alpha(1) = x$. There also exists a path $p_\beta : [1, 2] \rightarrow A_\beta$ with $p_\beta(1) = x$ and $p_\beta(2) = y_\beta \in A_\beta$. By pasting lemma, there exists the path $p : [0, 2] \rightarrow \bigcup_\alpha A_\alpha$, where

$$p(t) = \begin{cases} p_\alpha(t) & t \leq 1 \\ p_\beta(t) & t > 1 \end{cases}$$

\square

Question 9. Assume that \mathbb{R} is uncountable. Show that if A is a countable subset of \mathbb{R}^2 , then $\mathbb{R}^2 \setminus A$ is path-connected. **Hint:** How many lines are there passing through a given point of \mathbb{R}^2 ?

Proof. If $A \subset \mathbb{R}^2$ is countable, let $(x, y) \in \mathbb{R}^2$. There is an uncountably infinite number of lines passing through (x, y) . Then that means there are uncountable number of lines not intersecting A . Also, for any pair of points, there is a pair of lines that intersect each other but do not intersect A , showing that the lines are connected at the intersection. \square

Question 11.

If A is a connected subspace of X , does it follow that $\overset{\circ}{A}$ and δA are connected? Does the converse hold? Justify your answers.

Proof. No. $(0, 1) \subset \mathbb{R}$ is connected but its boundary, $\{0, 1\}$ is disconnected. The open balls $B_1(0, 0) \cup B_1(2, 0)$ are disconnected but the boundary is connected. For the converse, let \mathbb{Q} be a disconnected subspace of \mathbb{R} . The closure of \mathbb{Q} is \mathbb{R} . The interior of \mathbb{Q} is \emptyset . Consider any open $(a, b) \subset \mathbb{R}$ with $b > a$. Then (a, b) contains an irrational number. So

$$\bigcup_{U \in \mathcal{T}, U \subset \mathbb{Q}} U$$

is empty and hence connected. Then the boundary of \mathbb{Q} is \mathbb{R} itself, which is connected. \square

Remark 4. Let (X, \mathcal{T}) be a topological space and $A \subset X$.

1. The interior of A is $\mathring{A} = \bigcup_{U \in \mathcal{T}, U \subset A} U$.
2. The closure of A is $\overline{A} = \bigcap_{G \in \mathcal{T}, G \supset A} G$.
3. The boundary of A is $\partial A = \overline{A} - \mathring{A}$.

4 Section 25

Question 2a.

The connected components and path components of \mathbb{R}^ω (in the product topology) are itself.

Proof. \mathbb{R}^ω is connected so it is the connected component. (Munkres p. 151 Example 7) \mathbb{R}^ω is also path connected. Using \mathbb{R} the path connectedness of \mathbb{R} , let $a, b \in \mathbb{R}^\omega$. Then there exists a path $p_n : [0, 1] \rightarrow \mathbb{R}$ from a_n to b_n . Since each p_n is continuous for $n \in \mathbb{N}$, let $p : [0, 1] \rightarrow \mathbb{R}^\omega$ be defined as $x \mapsto (p_n(x))_{n \in \mathbb{N}}$. Under the product topology, each p_n is continuous (as a path) implies p is continuous and is a path from a to b . \square

Question 2b.

Consider \mathbb{R}^ω under the uniform topology. Show that \mathbf{x} and \mathbf{y} lie in the same connected component of \mathbb{R}^ω iff the sequence $\mathbf{x} - \mathbf{y}$ is bounded. Hint: It suffices to consider the case where $\mathbf{y} = 0$

Proof. Suppose $\mathbf{x} - \mathbf{y}$ is bounded. This direction is clear. Then there exists some path $p_n : [0, 1] \rightarrow \mathbb{R}$ from \mathbf{x}_n to \mathbf{y}_n . Since the projection map π_n is continuous (due to being finer than the product topology), let $p = (p_n)_{n \in \mathbb{N}}$, then let $U_n \subset \mathbb{R} (= X_n)$ if $\pi_n^{-1}(U_n) \subset \mathbb{R}^\omega$ is an open subbasis element, $p^{-1}(\pi_n^{-1}(U_n)) = (\pi_n \circ p)^{-1}(U_n) = p_n^{-1}(U_n)$ is open, and hence p is continuous and a path $[0, 1] \rightarrow \mathbb{R}^\omega$. Then they lie in some path connected component, which lies in some connected component. Suppose \mathbf{x}, \mathbf{y} lie in the same connected component. Consider the case where $\mathbf{y} = 0$ and \mathbf{x} diverges. Note that $\mathbf{y} \in \mathcal{B}(\mathbb{N}, \mathbb{R})$, but $\mathbf{x} \notin \mathcal{B}(\mathbb{N}, \mathbb{R})$. Since $\mathcal{B}(\mathbb{N}, \mathbb{R})$ is both closed and open, $\mathcal{B}(\mathbb{N}, \mathbb{R})$ and its complement are a separation of \mathbb{R}^ω and hence \mathbf{y} and \mathbf{x} cannot be part of the same connected component, which is a contradiction. \square

Question 4.

Let X be locally path connected. Show that every connected open set in X is path connected.

Proof. If X is locally path connected, each path component is a connected component and vice versa. Then every connected open set in X lies in a connected component C , which is a path connected component. Then X is also path connected as it lies in a path connected component. □

Question 8.

Let $p : X \rightarrow Y$ be a quotient map. Show that if X is locally connected, then Y is locally connected. [Hint: If C is a component of the open set U of Y , show that $p^{-1}(C)$ is a union of components of $p^{-1}(U)$.]

Proof. Since X is locally connected, for every open $V \subset X$, all connected components of V are open in X . Let C be a connected component of open set $U \subset Y$, with $C \subset U \subset Y$. Then $p^{-1}(C) \subset p^{-1}(U)$ and $p^{-1}(U) \subset X$ is open. Consider a connected component containing $x \in p^{-1}(C)$ noting that $p(S)$ is also connected by continuity of p . Then $x \in p^{-1}(C) \cap S$ and so $p(x) \in C \cap p(S)$. Since $p(S)$ must lie in a connected component, it lies in C . □

5 Section 26

Question 1a.

Let \mathcal{T} and \mathcal{T}' be two topologies on the set X , and suppose that $\mathcal{T}' \supset \mathcal{T}$. What does the compactness of X under one of these topologies imply about compactness under the other?

Proof. If (X, \mathcal{T}') is compact, then (X, \mathcal{T}) is compact. □

Question 1b.

Show that if X is compact Hausdorff under both \mathcal{T} and \mathcal{T}' , then either $\mathcal{T} = \mathcal{T}'$ or \mathcal{T} and \mathcal{T}' are not comparable.

Proof. If (X, \mathcal{T}) is Hausdorff, then (X, \mathcal{T}') is Hausdorff. □

Question 6.

Show that if $f : X \rightarrow Y$ is continuous, where X is compact and Y is Hausdorff, then f is a closed map (that is, f carries closed sets to closed sets).

Proof. Let $B \subset X$ be closed. Since X is compact, B is also compact. Then $f(B) \subset Y$ is compact. As a result, Since Y is Hausdorff, $f(B)$ is also closed. □

Question 7.

Show that if Y is compact, then the projection map $\pi_1 : X \times Y \rightarrow X$ is a closed map.

Proof. Let $B \subset X \times Y$ be closed. Let $A = \pi_1(B) \subset X$. Let $x_0 \in X \setminus A \subset X$ where this set is open. Then for any $(x_0, y) \in X \times Y$, there exists an open set in the product space $U_y \times V_y \ni (x_0, y)$. Clearly, $V_y \supset Y$. By compactness of Y , we may choose finitely many V_y covering Y . Then take $U_{y_i} \times V_{y_i}$, $i \in \{1, 2, \dots, n\}$ to cover $\{x_0\} \times Y$. We know from the openness of each $U_y \times V_y$ that $U_y \times V_y \subset X \setminus A$. Take $U = \bigcap_{i=1}^n U_{y_i}$ and $V = \bigcup_{i=1}^n V_{y_i} = Y$. Then $U \subset X$ is open. Let $U_{x_0} = U$. Then $\bigcup_{x \in X \setminus A} U_x \subset X$ is open and $\bigcup_{x \in X \setminus A} U_x = X \setminus A$, so $A \subset X$ is closed. □

Question 8.

Closed graph theorem. If $f : X \rightarrow Y$ where Y is compact Hausdorff. Then f is continuous if and only if

$$\Gamma_f = \{(x, f(x)) \mid x \in X\} \subset X \times Y \text{ is closed.}$$

Hint: If Γ_f is closed and V is a neighbourhood of $f(x_0)$ then the intersection of Γ_f and $X \times (Y \setminus V)$ is closed. Apply Question 7.

Proof. First, we show Γ_f is closed $\implies f$ is continuous, then the converse.

- Suppose $\Gamma_f \subset X \times Y$ is closed. Let $B \subset Y$ be closed. Then $\Gamma_f \cap (X \times B) = \{(x, f(x)) \mid f(x) \in B\} \subset Y$ is closed, since $(X \times B) = X \times (Y \setminus V) = (X \times Y) \setminus (X \times V)$ for some open V .
- Note that $C = \Gamma_f \cap (X \times B) = \{(x, f(x)) \mid f(x) \in B\}$. Then $\pi_X(C) = \{x \mid f(x) \in B\} = f^{-1}(B)$ is closed, by the previous question.

This proves \implies . Now to show the converse \impliedby .

- Suppose f is continuous. Let $(x, y) \in \overline{\Gamma_f}$ and $(x, f(x)) \in \Gamma_f$ where $y \neq f(x)$, which is the assumption that Γ_f is not closed. By Hausdorffness, there exists U, V disjoint and open containing y and $f(x)$ respectively.
 - By continuity of f , there exists an open $W \ni x$ with $f(W) \subset V$.
 - Then $W \times U$ and $W \times V$ are disjoint open sets containing (x, y) and $(x, f(x))$.
 - By continuity of f , since (x, y) is in the closure of Γ_f , $W \times U$ intersects Γ_f .
 - Then there exists $(w, f(w))$ in the intersection, $(w, f(w)) \in W \times U$, where $w \in W$.
 - But $f(w) \in V$, a contradiction.
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6 Section 28

Question 1.

Give $[0, 1]^\omega$ the uniform topology. Find an infinite subset that has no limit point.

Proof. $X = [0, 1]^\omega$ is not limit point compact. Let $A = \{\mathbf{e}_i \mid i \in \mathbb{N}\} \subset X$. This is an infinite set. Note that any $x \in X$ is not a limit point of A , since the open ball containing x , with radius $r/2$, contains all y with

$$\bar{\rho}(y, x) < r/2 \implies \forall n, d(x_n, y_n) < r$$

Then we can choose r small enough, if $x \in X \setminus A$, and if $r < 1$ if $x \in A$, these balls do not intersect $A \setminus \{x\}$. \square

Question 3.

Let X be limit point compact. If $A \subset X$ is closed, is A limit point compact?

Proof. If $A = A \cup A'$ (it is closed), any infinite subset $B \subset A$ will have limit points $B' \subset A' \subset A$. (Any limit point of B is also a limit point of A) \square

Remark 5. Let $x \in B'$ be a limit point of $B \subset A$. Then for all open $U \subset X$, $U \cap B \setminus \{x\} \neq \emptyset$. This implies x is a limit point of A , since $B \subset A$.

7 Section 29

Question 1.

Show the rationals are not locally compact

Proof. Since the rationals are dense in \mathbb{R} , let $(a, b) \cap \mathbb{Q}$ contain $x \in \mathbb{Q}$. Let $C \supset (a, b) \cap \mathbb{Q}$. It is not compact because it is not sequentially compact. We may construct a sequence in $(a, b) \cap \mathbb{Q}$ (which is in C) converging to $p \in \mathbb{R} \setminus \mathbb{Q}$. Then the subsequence also converges to $p \in \mathbb{R} \setminus \mathbb{Q}$. For instance, let c be an irrational number in (a, b) . $x_n \in (c - 1/n, c) \cap \mathbb{Q}$ and the sequence converges to $c \in \mathbb{R} \setminus \mathbb{Q}$. \square

Remark 6. \mathbb{R}^ω equipped with the product topology is not locally compact. The product topology is metrizable. Let Λ be a finite set. Take any basic open element

$$U = \prod_{n \in \Lambda} (a_n, b_n) \times \prod_{n \in \mathbb{N} \setminus \Lambda} \mathbb{R}$$

Let $C \supset U$. We show that C is not sequentially compact and therefore is not compact. Consider the sequence $(x_i)_{i=1}^\infty \in U \subset C$, where $\pi_k(x_i) \in (a_i, b_i)$ for all $k \in \Lambda$ and $\pi_k(x_i) = i$ if $k \notin \Lambda$. Then, each x_i is in $U \subset C$, but the sequence x_n does not have a convergent subsequence.

Question 2a.

Let $\{X_\alpha\}$ be an indexed family of nonempty spaces.

- (a) Show that if $\prod_{\alpha \in \Lambda} X_\alpha$ is locally compact, then each X_α is locally compact and X_α is compact for all but finitely many values of α .
- (b) Prove the converse, assuming the Tychonoff theorem.

Proof. (a). Let $X = \prod_{\alpha \in \Lambda} X_\alpha$ be locally compact. Then for any basic open element

$$U = \prod_{\alpha \in F} U_\alpha \times \prod_{\alpha \in \Lambda \setminus F} X_\alpha$$

containing $\mathbf{x} = (x_\alpha)_{\alpha \in \Lambda}$ there exists a compact $C \supset U$ containing \mathbf{x} . Let $U_\alpha \subset X_\alpha$ be open and contain x_α . Then $\pi_\alpha(C) \supset U_\alpha$ and is compact. Compactness is preserved due to the continuity of the projection map, which is preserved in the product topology. Furthermore, for all $\alpha \in \Lambda \setminus F$, $\pi_\alpha(C) = X_\alpha$ and therefore X_α is compact for $\alpha \in \Lambda \setminus F$.

(b). Let X_α be compact for all $\alpha \in \Lambda \setminus F$ and each X_α locally compact. Again, F is an arbitrary finite set. Let $Y = \prod_{\alpha \in \Lambda \setminus F} X_\alpha$. Then Y is compact by Tychonoff's theorem. $X = \prod_{\alpha \in F} X_\alpha \times Y$. Now consider any basic open element $U \subset X$, as seen in the proof in (a). By local compactness of each X_α , there exists a compact $C_\alpha \subset X_\alpha$ containing U_α . $\prod_{\alpha \in F} C_\alpha \times Y$ (finite product of compact sets) is compact and contains U . \square

Remark 7. A topological space X is compactly generated if it satisfies either of the following equivalent conditions

1. $A \subset X$ is open $\iff A \cap C$ is open in C for every compact $C \subset X$
2. $B \subset X$ is closed $\iff B \cap C$ is closed in C for every compact $C \subset X$

For any closed set $B \subset X$, $A = X \setminus B$ is open. For any closed set B , $A = X \setminus B$ is open. Then, $A = X \setminus B$ is open $\iff A \cap C \subset C$ is open and $(X \setminus B) \cap C$ is open $\iff B \cap C \subset C$ is closed $\iff B$ is closed.

Question 3.

Let X be locally compact. If $f : X \rightarrow Y$ is continuous, is $f(X)$ locally compact?

Proof. Let $y \in f(X)$ and $U \subset f(X)$ be open. By continuity of f , $f^{-1}(U)$ is open in X and contains x , and hence there exists a compact C such that $x \in f^{-1}(U) \subset C$. Then, $f(x) \in U \subset f(C)$, where $f(C)$ is compact by continuity of f . \square

Question 5.

If $f : X_1 \rightarrow X_2$ is a homeomorphism of locally compact Hausdorff spaces, then show f extends to a homeomorphism of their one point compactifications.

Proof. Note that since f is a homeomorphism from X_1 to X_2 , consider the topologies of their one point compactifications, $\mathcal{A}_1 \cup \mathcal{A}_2$ of $X_1 \cup \{\infty_1\}$ and $\mathcal{B}_1 \cup \mathcal{B}_2$ of $X_1 \cup \{\infty_2\}$.

$$\mathcal{A}_1 = \{U \subset Y_1 \mid U \subset X_1 \text{ is open}\}$$

$$\mathcal{A}_2 = \{Y_1 \setminus C \mid C \subset X_1 \text{ is compact}\}$$

Extend f by letting $f(\infty_1) = \infty_2$. Then, since f is a homeomorphism, $U \subset X_1$ is open $\iff f(U) \subset X_2$ is open, and $C \subset X_1$ is compact $\iff f(C) \subset X_2$ is compact. The same could be said about its inverse. Then, $\mathcal{A}_1 \cup \mathcal{A}_2$ is homeomorphic to $\mathcal{B}_1 \cup \mathcal{B}_2$ under f . □

Question 7.

Every locally compact Hausdorff space is completely regular.

Proof. Let X be locally compact Hausdorff. Then there exists a compact Hausdorff Y , which is a one point compactification of X . Compact Hausdorff implies normality, which implies complete regularity. Y is completely regular, so X , as a subspace of Y is completely regular too. □

Remark 8. *A subspace of a completely regular space is completely regular. A product of completely regular spaces is completely regular. (Theorem 33.2, Munkres)*

Question 8.

Let X be completely regular, A, B , closed disjoint subsets of X . If A is compact, there is a continuous function $f : X \rightarrow [0, 1]$ such that $f(A) = \{0\}$, $f(B) = \{1\}$.

Proof. X is regular, let U, V separate x and B . By compactness of A , since U is open, there exist open sets $U_x \subset U$ where U_x contain x and V_x open, disjoint from U_x and containing B . Therefore this collection covers A . By compactness of A , we reduce this to a finite subcover of A , consisting of U_{x_1}, \dots, U_{x_n} . The finite intersection of V_{x_i} is still open and contains x . Therefore, $U = \bigcup_{i=1}^n U_{x_i}$ contain x and $V = \bigcap_{i=1}^n V_{x_i}$ contain B . This shows X is normal and by Urysohn's lemma, completely normal. □

8 Biglist

Prove that \mathbb{R} with the co-countable topology is not locally compact.

Proof. Let $U = \mathbb{R} \setminus \mathbb{Z}^+$ and $x \in U$. Suppose $C \supset U$. Then $C = \mathbb{R} \setminus G$ with $G \subset \mathbb{Z}^+$. Then C is not compact. Since G is countable, $\mathbb{R} \setminus G = C$ is uncountably infinite and thus cannot be compact, since the compact sets are finite. \square

Remark 9. Let X be uncountable, equipped with the co-countable topology. Suppose A is an infinite set. Then A is not compact. Let A' be a countably infinite subset of A . Then $U_0 = X \setminus A'$ is open. In particular, let $\{U_0, U_1, \dots\}$ be an open cover of A , where

$$U_i = X \setminus (A' \setminus \{a_i\})$$

and $U_0 = X \setminus A'$. Then U_0 covers $A \setminus A'$ and U_i covers each $\{a_i\} \subset A'$. There is no finite subcover.

Let $D \subset \mathbb{R}^2$ be countable. Then $\mathbb{R}^2 \setminus D$ is connected.

Proof. $\mathbb{R}^2 \setminus D$ is connected because it is path connected. Choose any $(p, q) \in \mathbb{R}^2 \setminus D$. There are uncountably infinite lines passing through (p, q) and not intersecting D . Choose these two lines L_1, L_2 not passing through a point in D , and passing through (p_1, q_1) and (p_2, q_2) respectively. They intersect at some other point $(p_3, q_3) \in \mathbb{R}^2 \setminus D$. Then there exists a path $p : [0, 1] \rightarrow \mathbb{R}^2$ from (p_1, q_1) to (p_2, q_2) . Clearly, p is continuous, such that $\mathbb{R}^2 \setminus D$ is path connected and hence connected. \square

Show that \mathbb{R}^ω in the box topology is disconnected. What about \mathbb{R}^ω under the uniform topology?

Proof. Consider A , the set of all bounded sequences and B , the set of all unbounded sequences. Clearly, $A \sqcup B$. It remains to see if they are open. Consider an open set in the box topology that contains the sequence $a = \{a_n\}_{n \in \mathbb{N}}$, given by $U_a = \prod_{n \in \mathbb{N}} (a_n - 1, a_n + 1)$. a is bounded if and only if every sequence in U_a is bounded. By considering $\bigcup_{a \in A} U_a = A$, we obtain the fact that A is open. Similarly, B is open. Then A and B form a separation of \mathbb{R}^ω .

Remark 10. Under the uniform topology, $U_a = \prod_{n \in \mathbb{N}} (a_n - 1, a_n + 1)$ is also open (it is a uniform ball), so we may use the same argument. \square

9 Tutorials

9.1 Problem 3

Let X be equipped with the metric d . The subspace topology on $A \subset X$ is the discrete topology

Proof. A basic open element in A is

$$B_r(x) = \{y \in A \mid d_A(x, y) < r\}$$

Since A is finite, each $d(x, y) > \epsilon$ where $\epsilon = \min_y d(x, y)/2$. Then the singleton set $\{x\}$ is open. \square

Let \mathbb{R} be equipped with the standard topology. The subspace topology on $A = \{\frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{R}$ is the discrete topology. Show that the subspace topology on $A' = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{R}$ is NOT the discrete topology.

Proof. All singleton sets are open if and only if it is the discrete topology.

- Take an open ball $A \cap (\frac{1}{n} - \epsilon, \frac{1}{n} + \epsilon)$ where $\epsilon < 1/2(\frac{1}{n+1} - \frac{1}{n})$. Then the singleton set is open.
- Take any open ball $(-\epsilon, \epsilon) \cap A'$ containing 0. Then it always contains something else by the archimedean property. This implies $\{0\}$ is not open.

□