Delayed Feedback Control of Periodic Orbits in Autonomous Systems

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For controlling periodic orbits with delayed feedback methods the periodicity has to be known *a priori*. We propose a simple scheme, how to detect the period of orbits from properties of the control signal, at least if a periodic but nonvanishing signal is observed. We analytically derive a simple expression relating the delay, the control amplitude, and the unknown period. Thus, the latter can be computed from experimentally accessible quantities. Our findings are confirmed by numerical simulations and electronic circuit experiments. [S0031-9007(98)06640-X]

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Control techniques using time-delayed output signals are a very well established field and known for at least half a century in the engineering and mathematical context (e.g., [1] and references therein). Delayed feedback control methods, which have for the physicists' purpose been rediscovered in [2], are very useful since neither special knowledge of the system under consideration nor sophisticated reconstruction techniques are required, and the method is easily implemented in experiments [3]. As a certain kind of drawback, the success of delayed feedback methods is difficult to predict, and the stability analysis of the corresponding delay systems shows a rich behavior (e.g., [4]). Only recently some progress in the understanding of general features has been made in the physical context [5]. Since control of actual periodic orbits with delayed feedback methods requires a delay time which is an integer multiple of the period, one runs into principle difficulties whenever the period is not known *a priori*. Some empirical schemes have been reported to circumvent such problems [6]. They work quite well for special cases but no theoretical foundation has been proposed. Here we address the problem that the period of the unstable periodic orbit is unknown. A systematic strategy is developed to obtain the desired period, whenever a periodic control signal is observed.

Theoretical approach.—To keep our approach as general as possible the theoretical considerations are based on a fairly arbitrary equation of motion:

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}(t), K\{g[\boldsymbol{x}(t)] - g[\boldsymbol{x}(t-\tau)]\}). \tag{1}$$

Here $\boldsymbol{x}$ denotes the phase space variables, $g[\boldsymbol{x}]$ is the measured scalar quantity, τ is the delay time, and K is the control amplitude. We do not specify the functional dependence of the systems on the control signal $g[\boldsymbol{x}(t)] - g[\boldsymbol{x}(t-\tau)]$, since this dependence is, in general, difficult to estimate from the experimental point of view. Without control, $K = 0$, the system should admit an unstable periodic orbit $\boldsymbol{\xi}(t)$ with period T and Floquet exponent $\lambda + i\omega$, $\lambda > 0$. We intend to stabilize this orbit.

Whenever the delay differs from the period, $\tau \neq T$, the orbit $\boldsymbol{\xi}$ does not yield a solution of the system subjected to control. However, the system admits a periodic solution $\boldsymbol{\eta}$ with period Θ. Such a statement can even be proven rigorously [7] provided that the delay mismatch $\tau - T$ is not too large. In addition, the fictitious solution $\boldsymbol{\eta}$ tends towards the unstable orbit $\boldsymbol{\xi}$ in the limit $\tau \to T$. Of course, the period of this fictitious orbit depends on the parameters of the system, in particular, on the delay time and the control amplitude, $\Theta = \Theta(K, \tau)$. We remind the reader that the quantity Θ can be observed from the period of the control signal, whenever the orbit $\boldsymbol{\eta}$ is stable. In what follows we assume that the system parameters are adjusted in such a way; i.e., we can observe the period Θ for different values of the control amplitude K and the delay time τ.

The strategy for the determination of the desired period T is quite simple. Since the orbit $\boldsymbol{\xi}$ yields a periodic orbit of the controlled system for $\tau = T$, the measured period of the control signal obeys $\Theta(K, T) = T$. Hence, we simply have to look for zeros of the function $\Theta(K, \tau) - \tau$. The latter can be measured, in principle, provided we meet the assumption made above. Nevertheless, it would be helpful if some analytical result about the dependence of Θ on the delay and the control amplitude would be available. We show that up to second order in the mismatch $\tau - T$ the relation

$$\Theta(K, \tau) = T + \frac{K}{K - \kappa}(\tau - T) + \mathcal{O}((\tau - T)^2) \tag{2}$$

holds. Here κ denotes a system parameter which captures all of the details concerning the coupling of the control force to the system. Since the parameters τ and K are adjustable in experiments and Θ is a measurable quantity, the desired period can be computed from Eq. (2) using two data points.

In order to derive expression (2) we rewrite Eq. (1) for the periodic orbit $\boldsymbol{\eta}$ in terms of the dimensionless time $s = t/\Theta$ as

$$\boldsymbol{\eta}'(s) = \Theta \boldsymbol{F}(\boldsymbol{\eta}(s), K\{g[\boldsymbol{\eta}(s)] - g[\boldsymbol{\eta}(s - \tau/\Theta)]\}), \quad (3)$$

and

$$\boldsymbol{\eta}(s) = \boldsymbol{\eta}(s + 1). \quad (4)$$

Since Eq. (2) represents a Taylor expansion we are looking for the derivative $\partial_\tau \Theta|_{\tau=T}$. For that reason one takes the derivative of Eq. (3) with respect to τ, keeping in mind that the periodic solution $\boldsymbol{\eta}$ depends explicitly on τ,

$$\begin{aligned}(\partial_\tau \boldsymbol{\eta})' - \Theta D_1 \boldsymbol{F}(\cdots)\partial_\tau \boldsymbol{\eta}(s) - \Theta K d_2 \boldsymbol{F}(\cdots)\{Dg[\boldsymbol{\eta}(s)]\partial_\tau \boldsymbol{\eta}(s) - Dg[\boldsymbol{\eta}(s - \tau/\Theta)]\partial_\tau \boldsymbol{\eta}(s - \tau/\Theta)\} \\ = (\partial_\tau \Theta)\boldsymbol{F}(\cdots) + \Theta K d_2 \boldsymbol{F}(\cdots)\{Dg[\boldsymbol{\eta}(s - \tau/\Theta)]\boldsymbol{\eta}'(s - \tau/\Theta)\}\partial_\tau(\tau/\Theta). \quad (5)\end{aligned}$$

Here D_1 and d_2 denote the derivative with respect to the first/second argument of $\boldsymbol{F}$, and the arguments abbreviated by $(\cdots)$ coincide with those from Eq. (3). The contributions involving the derivative of the orbit with respect to the explicit τ dependence, $\partial_\tau \boldsymbol{\eta}$, have been collected on the left-hand side. The boundary value problem [(5), (4)] determines both $\partial_\tau \Theta$ as well as $\partial_\tau \boldsymbol{\eta}$. In order to separate the former quantity we trace back to the fact that the linear operator on the left-hand side of Eq. (5) admits a vanishing eigenvalue. The corresponding Goldstone mode is related to the translation invariance in time of the original system. In fact, taking the derivative of Eq. (3) with respect to s, one obtains

$$\begin{aligned}0 = (\boldsymbol{\eta}')' - \Theta D_1 \boldsymbol{F}(\cdots)\boldsymbol{\eta}'(s) - \Theta K d_2 \boldsymbol{F}(\cdots) \\ \times \{Dg[\boldsymbol{\eta}(s)]\boldsymbol{\eta}'(s) - Dg[\boldsymbol{\eta}(s - \tau/\Theta)] \\ \times \eta' s - \tau/\Theta)\}. \quad (6)\end{aligned}$$

Equation (6) just states that $\boldsymbol{\eta}'$ yields the right-null eigenfunction. Within the canonical scalar product $\int_0^1 \boldsymbol{v}(s)\boldsymbol{u}(s)ds$ we denote the corresponding left-null eigenfunction by $\boldsymbol{\zeta}(s)$. All of the terms on the left-hand side of Eq. (5), which involve $\partial_\tau \eta$, vanish identically after multiplication with ζ. Hence, we are left with

$$\begin{aligned}0 = \partial_\tau \Theta \int_0^1 \boldsymbol{\zeta}(s)\boldsymbol{F}(\cdots)ds + \Theta K \partial_\tau(\tau/\Theta) \\ \times \int_0^1 \boldsymbol{\zeta}(s) d_2 \boldsymbol{F}(\cdots)\{Dg[\boldsymbol{\eta}(s - \tau/\Theta)] \\ \times \boldsymbol{\eta}'(s - \tau/\Theta)\} ds. \quad (7)\end{aligned}$$

The details of the system, which are only contained in the integrals, are now condensed to simple numbers. But, in general, the integrals depend on the delay τ and, in particular, on the control amplitude K through the left eigenfunction $\boldsymbol{\zeta}$ [cf. Eq. (6)]. For that reason we evaluate Eq. (7) at $\tau = T$. Then $\Theta = \tau$ holds and the delay in the arguments of $\boldsymbol{\eta}$ drops by virtue of the boundary condition (4). Because of the same argument, the linear operator (6) and therefore the eigenfunction $\boldsymbol{\zeta}$ become independent of K. Hence, the integrals become constant real numbers and Eq. (7) yields

$$0 = \kappa \partial_\tau \Theta|_{\tau=T} + TK \partial_\tau(\tau/\Theta)|_{\tau=T}. \quad (8)$$

Here κ denotes the ratio of the integrals occurring in Eq. (7). We solve for $\partial_\tau \Theta|_{\tau=T}$, and obtain Eq. (2) from a simple Taylor series expansion.

Numerical simulations.—We demonstrate the applicability of our analytical results by numerical simulations in an autonomous system. First of all stabilization of periodic orbits by delay methods requires a finite torsion, i.e., a finite frequency in the Floquet exponent of the controlled orbit [5]. Since autonomous equations always admit a vanishing exponent a finite frequency can be realized in dissipative three-dimensional models only by a complete flip of the neighborhood of the orbit. For that reason certain equations like the Lorenz model cannot be stabilized at all by delay methods, apart from the fixed points for which the reasoning given above does not apply. Therefore we concentrate here on the Rössler equations as a certain minimal model for our purpose,

$$\begin{aligned}\dot{x}_1 &= -x_2 - x_3 - K\{g[\boldsymbol{x}(t)] - g[\boldsymbol{x}(t - \tau)]\}, \\ \dot{x}_2 &= x_1 + a x_2 - K\{g[\boldsymbol{x}(t)] - g[\boldsymbol{x}(t - \tau)]\}, \quad (9) \\ \dot{x}_3 &= b + x_1 x_3 - c x_3.\end{aligned}$$

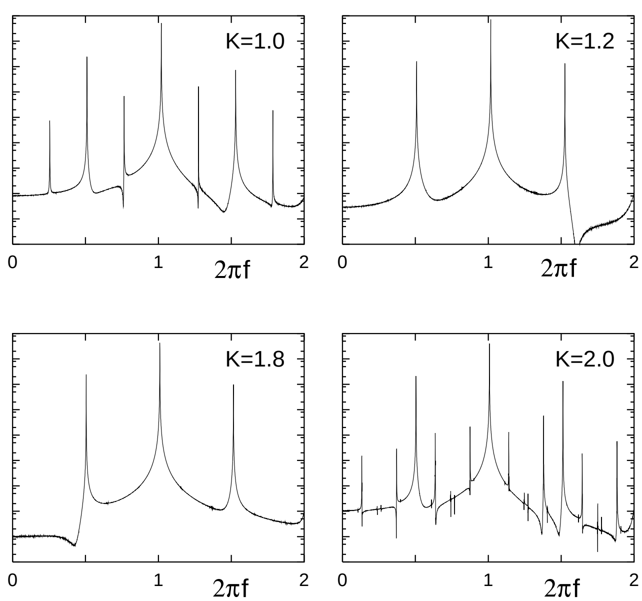

FIG. 1. Absolute square value of the Fourier transform of the scalar quantity $g[\boldsymbol{x}]$ for $\tau - T = 1.0$ in the vicinity of the lower and upper stability threshold. The transform has been performed for a series of length $1024 \times \tau$ discarding a transient of $100 \times \tau$. The spectrum has not been properly normalized and the abscissa extends over nine decades.

Our results do not seem to depend significantly on the coupling of the control force to the original equations of motion and on the particular choice of the scalar quantity $g[\boldsymbol{x}]$. We have used a bounded quantity in order to avoid diverging solutions. The results presented here correspond to the choice $g[\boldsymbol{x}] = \tanh[(x_1 + x_2)/10]$. In addition, the system parameters have been fixed to the values $a = b = 0.2$, $c = 5.7$ to ensure chaotic dynamics in the absence of control. For our control purpose we concentrate on the period-two orbit in the canonical Poincaré map with $T = 11.758\ldots$, $\lambda T = 1.256\ldots$, and $\omega T = \pi$. Numerical simulations have been performed by means of an adaptive stepsize Runge-Kutta method of order 4, together with a cubic spline for the delay from the numerical recipes library [8].

For a quite large range of delay times τ, one observes two critical values of the control amplitude which limit an interval where a stable periodic orbit $\boldsymbol{\eta}$ can be observed. From the Fourier transform of the scalar quantity $g[\boldsymbol{x}]$ it is evident (cf. Fig. 1) that at the lower critical value the orbit loses stability via a flip bifurcation, whereas at the upper critical value a Hopf bifurcation occurs. In order to check the accuracy of Eq. (2), the period Θ of the fictitious orbit has been extracted from the peaks in the Fourier spectra of the control signal. The dependence of Θ on the control amplitude for several delay times is summarized in Fig. 2 and compared with our analytical expression. The apparent systematic deviation of the analytical result just comes from the fact that the latter is a first-order approximation to the curved manifold $\Theta(K, T)$ in the three-dimensional K-T-Θ space. In summary, Eq. (2) describes the observed periods quite accurately.

Finally, we have checked, whether Eq. (2) successfully predicts the period of the unstable periodic orbit $\boldsymbol{\xi}$ whenever a few data points are accessible. To this end we evaluated the K dependence of the power spectrum of the control signal within a regime where a periodic signal can be observed. Starting from $\tau = 14.0$, which differs tremendously from the true period, we evaluate Θ for $K = 0.8$, 0.9, and 1.0 to obtain $\kappa = -0.8 \pm 0.01$ and $T = 11.745 \pm 0.015$ from Eq. (2). The accuracy of T is in fact of the order of the numerical resolution of the power spectra. In that sense the result is striking.

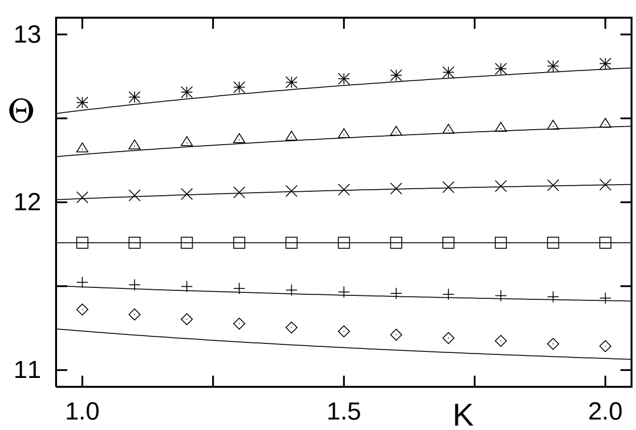

FIG. 2. Dependence of the period Θ on the control amplitude for various delay times τ; from bottom to top, $\tau - T = -1.0$, -0.5, 0.0, 0.5, 1.0, 1.5: numerical simulations (symbols) and analytical expression (2) with $\kappa = -0.9$ (lines). For a few data points outside the stabilized regime the period was estimated from a dominant peak in the power spectrum.

Experiments.—To illustrate the experimental accessibility of our analytical results we have performed measurements on a nonlinear electronic circuit (cf. Fig. 3). The circuit consists of several operational amplifiers (three acting as integrators, two as inverters) with associated feedback components. The nonlinearity is provided by the diodes. The voltages probed at x, y, z can be considered as the degrees of freedom in our experiment. At f_x, f_y, f_z external signals can be fed into the system for control purposes. Typical frequencies of the circuit are about 600 kHz.

Without control the system undergoes a period-doubling cascade to chaos on variation of the resistance R, ending up in a Rössler-type attractor. Topological analysis [9] of this three-dimensional system yielded a frequency of π/T in the Floquet exponent for the unstable period-one orbit of the chaotic attractor. This corresponds to a complete flip of the neighborhood of this orbit. Therefore the orbit is accessible to time-delayed feedback control.

The control device consists of a cascade of electronic delay lines with a limiting frequency of about 3 MHz and several operational amplifiers acting as preamplifier, subtractor, or inverter. The device allows one to apply a control force of the form $F(t) = -K[U(t) - U(t - \tau)]$ with a τ range of 10 ns$-$21 μs. Our feedback scheme consisted of coupling the voltage at z via the control device to f_z.

To check the coincidence with our analytical results we looked for periodic behavior of our nonlinear circuit by sweeping the control amplitude K at fixed τ. By increasing K the system undergoes an inverse period-doubling cascade ending up in a period-one state. This periodic state yields the desired value Θ. A further

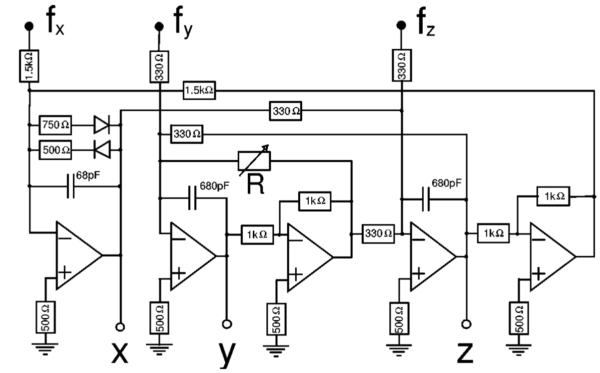

FIG. 3. Experimental setup of the nonlinear electronic circuit without the time-delayed feedback device. Experiments have been performed at $R = 110\ \Omega$.

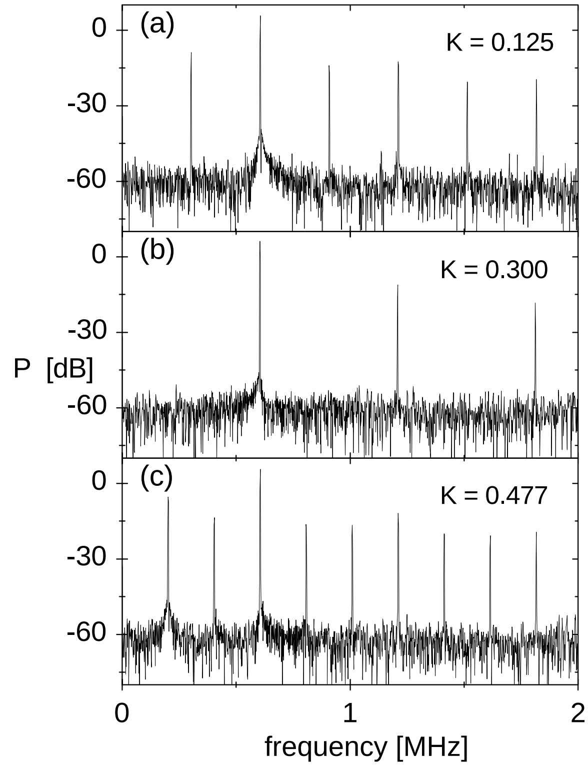

FIG. 4. Power spectrum of $x(t)$ for $\tau - T = 2.5$ ns: (a) below the lower; (b) between both; and (c) above the upper stability threshold.

increase of K results in a Hopf bifurcation destroying the stability of the periodic state (cf. Fig. 4).

Figure 4(a) shows the main frequency at 605.5 kHz and its subharmonic at 302.7 kHz corresponding to the flip bifurcation at the lower stability threshold. At the upper stability threshold a Hopf bifurcation yields an incommensurate frequency component at 201.4 kHz. We checked that frequency locking did not occur. By measuring Θ for various τ and K values, one obtains the grey-shaded surface displayed in Fig. 5. Note that, for correct delay time $\tau = T$, one automatically gets $\Theta = \tau = T$. Within an experimental error of 1 ns the intersection with the surface $\Theta = \tau$ yields a straight line with $\tau = 1.656$ μs.

Since the curvature of the surface in the direction of τ is negligible for τ values close to the real period, i.e., $\pm 10\%$, the coincidence with our analytical expression (2) is quite reasonable in this region. Calculation of the system parameter yields $\kappa = -0.31 \pm 0.01$. For larger delay mismatch Eq. (2) can still be used iteratively in the sense of a Newton method for detecting the exact period T.

In conclusion, we have shown that the period of true periodic unstable orbits can be obtained from the properties of the control signal, at least if a periodic signal can be realized. Our approach is based on the fact that the true periodic orbit of the uncontrolled system is deformed into a fictitious periodic orbit by the control if the delay time differs from the true period. Our analytical expression (2) relates the fictitious period Θ with the true period T, the delay τ, and the control amplitude K. Peculiarities

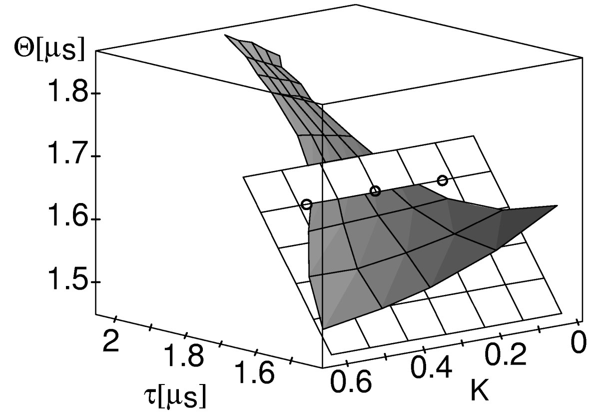

FIG. 5. Measured values of the period Θ depending on τ and K. The white plane corresponds to $\Theta = \tau$. The circles indicate the points where spectra shown in Fig. 4 have been obtained.

of the system enter only through a single parameter κ. Of course, our result does not guarantee that the orbit becomes stable if the delay time is adapted without changing the control amplitude (cf. Fig. 5). However, in order to keep the fictitious orbit stable during such an adaptation process one may, for example, monitor the power spectrum of the control signal (cf. Fig. 4), since an instability is indicated by the occurrence of additional peaks in the spectrum.

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