The Complex Gradient Operator and the $\mathbb{CR}$-Calculus

ECE275A – Lecture Supplement – Fall 2005

Kenneth Kreutz–Delgado
Electrical and Computer Engineering
Jacobs School of Engineering
University of California, San Diego

VERSION ECE275CG-F05v1.3c

1   Introduction

Often signals and system parameters are most conveniently represented as complex-valued vectors. This occurs, for example, in array processing [1], as well as in communication systems [7] when processing narrowband signals using the *equivalent complex baseband* representation [2]. Furthermore, in many important applications one attempts to optimize a scalar *real*-valued measure of performance over the complex parameters defining the signal or system of interest. This is the case, for example, in LMS adaptive filtering where complex filter coefficients are adapted on line. To effect this adaption one attempts to optimize the performance measure by adjustments of the coefficients along its gradient direction [16, 23].

However, an often confusing aspect of complex LMS adaptive filtering, and other similar gradient-based optimization procedures, is that the partial derivative or gradient used in the adaptation of complex parameters is *not* based on the standard complex derivative taught in the standard mathematics and engineering complex variables courses [3]-[6], which exists if and only if a function of a complex variable z is *analytic* in z. This is because a nonconstant *real*-valued function of a complex variable is *not* analytic and therefore is *not* differentiable in the standard textbook complex-variables sense.

Nonetheless, the same real-valued function viewed as a function of the real-valued real and imaginary *components* of the complex variable can have a (real) gradient when partial derivatives are taken with respect to those two (real) components. In this way we can shift from viewing

1

the real-valued function as a non-differentiable mapping between $\mathbb{C}$ and $\mathbb{R}$ to treating it as a differentiable mapping between $\mathbb{R}^2$ and $\mathbb{R}$. Indeed, the modern graduate-level textbook in complex variables theory by Remmert [12] continually and easily shifts back and forth between the real function $\mathbb{R}^2 \to \mathbb{R}$ or $\mathbb{R}^2$ perspective and the complex function $\mathbb{C} \to \mathbb{C}$ perspective of a complex or real scalar-valued function,

$$f(z) = f(r) = f(x, y),$$

of a complex variable $z = x + j\,y$,

$$z \in \mathbb{C} \Leftrightarrow r = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

In particular, when optimizing a real-valued function of a complex variable $z = x + j\,y$ one can work with the equivalent real gradient of the function viewed as a mapping from $\mathbb{R}^2$ to $\mathbb{R}$ in lieu of a nonexistent complex derivative [14]. However, because the real gradient perspective arises within a complex variables framework, a direct reformulation of the problem to the real domain is awkward. Instead, it greatly simplifies derivations if one can represent the real gradient as a redefined, new *complex gradient* operator. As we shall see below, the complex gradient is an extension of the standard complex derivative to nonanalytic functions.

Confusing the issue is the fact that there is no one unique way to consistently define a "complex gradient" which applies to nonanalytic real-valued functions of a complex variable, and authors do not uniformly adhere to the same definition. Thus it is often difficult to resolve questions about the nature or derivation of the complex gradient by comparing authors. Given the additional fact that typographical errors seem to be rampant these days, it is therefore reasonable to be skeptical of the algorithms provided in many textbooks–especially if one is a novice in these matters.

An additional source of confusion arises from the fact that the derivative of a function with respect to a vector can be alternatively represented as a row vector or as a column vector when a space is Cartesian,[1] and both representations can be found in the literature. As done for the development of the real gradient given in [25], in this note we continue to carefully distinguish between the complex *cogradient* operator, which is a *row vector* operator, and the associated *complex gradient* operator which is a *vector* operator which gives the direction of steepest ascent of a real scalar-valued function.

Because of the constant back-and-forth shift between a real function ("$\mathbb{R}$-calculus") perspective and a complex function ("$\mathbb{C}$-calculus") perspective which a careful analysis of nonanalytic complex functions requires [12], we refer to the mathematics framework underlying the derivatives given in this note as a "$\mathbb{CR}$-calculus." In the following, we start by reviewing some of the properties of standard univariate analytic functions, describe the $\mathbb{CR}$-calculus for univariate nonanalytic functions, and then develop a multivariate $\mathbb{CR}$-calculus appropriate for optimization scalar real-valued cost functions of a complex parameter vector. We end the note with some application examples.

---

[1]I.e., is Euclidean with identity metric tensor.

2   The Derivative of a Holomorphic Function

Let $z = x + jy$, for x, y real, denote a complex number and let

$$f(z) = u(x, y) + j\, v(x, y)$$

be a general complex-valued function of the complex number z.[2] In standard complex variables courses it is emphasized that for the complex derivative,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z},$$

to exist in a meaningful way it must be *independent* of the direction with which Δz approaches zero in the complex plane. *This is a very strong condition* to be placed on the function $f(z)$. As noted in an introductory comment from the textbook by Flanigan [6]:

> You will learn to appreciate the difference between a complex analytic function (roughly a complex-valued function $f(z)$ having a complex derivative $f'(z)$) and the real functions $y = f(x)$ which you differentiated in calculus. Don't be deceived by the similarity of the notations $f(z)$, $f(x)$. The complex analytic function $f(z)$ turns out to be much more special, enjoying many beautiful properties not shared by the run-of-the-mill function from ordinary real calculus. The reason [$\cdots$] is that $f(x)$ is merely $f(x)$ whereas the complex analytic function $f(z)$ can be written as
>
> $$f(z) = u(x, y) + iv(x, y),$$
>
> where $z = x + iy$ and $u(x, y)$, $v(x, y)$ are each real-valued *harmonic* functions related to each other in a very strong way: the Cauchy-Riemann equations
>
> $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{1}$$
>
> In summary, the deceptively simple hypothesis that
>
> $$f'(z) \quad \text{exists}$$
>
> forces a great deal of structure on $f(z)$; moreover, this structure mirrors the structure of the harmonic $u(x, y)$ and $v(x, y)$, functions of *two* real variables.[3]

In particular the following conditions are equivalent statements about a complex function $f(z)$ on an open set containing z in the complex plane [6]:

---

[2]Later, in Section 3, we will interchangeably alternate between this notation and the more informative notation $f(z, \bar{z})$. Other useful representations are $f(u, v)$ and $f(x, y)$. *In this section* we look for the (strong) conditions for which $f : z \mapsto f(z) \in \mathbb{C}$ is differentiable as a mapping $\mathbb{C} \to \mathbb{C}$ (in which case we say that f is $\mathbb{C}$-*differentiable*), but in subsequent sections we will admit the *weaker condition* that $f : (x, y) \mapsto (u, v)$ be differentiable as a mapping $\mathbb{R}^2 \to \mathbb{R}^2$ (in which case we say that f is $\mathbb{R}$-*differentiable*); see Remmert [12] for a discussion of these different types of differentiability.

[3]Quoted from page 2 of reference [6]. Note that in the quote $i = \sqrt{-1}$ whereas in this note we take $j = \sqrt{-1}$ following standard electrical engineering practice.

- The derivative $f'(z)$ exists and is continuous.

- The function $f(z)$ is *holomorphic* (i.e, analytic in z).[4]

- The function $f(z)$ satisfies the *Cauchy-Riemann conditions* (1).

- All derivatives of the function $f(z)$ exist and $f(z)$ has a convergent power series.

Furthermore, it is a simple consequence of the Cauchy-Riemann conditions that

$$f(z) = u(x, y) + j\, v(x, y)$$

is holomorphic only if the functions $u(x, y)$ and $v(x, y)$ both satisfy Laplace's equation

$$\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v(x, y)}{\partial x^2} + \frac{\partial^2 v(x, y)}{\partial y^2} = 0.$$

Such functions are known as *harmonic* functions. Thus if either $u(x, y)$ or $v(x, y)$ fail to be harmonic, the function $f(z)$ is not differentiable.[5]

Although many important complex functions are holomorphic, including the functions z^n, e^z, $\ln(z)$, $\sin(z)$, and $\cos(z)$, and hence differentiable in the standard complex variables sense, there are commonly encountered useful functions which are not:

- The function $f(z) = \bar{z}$, where '$\bar{z}$' denotes complex conjugation, fails to satisfy the Cauchy-Riemann conditions.

- The functions $f(z) = \text{Re}(z) = \frac{z+\bar{z}}{2} = x$ and $g(z) = \text{Im}(z) = \frac{z-\bar{z}}{2j} = y$ fail the Cauchy-Riemann conditions.

- The function $f(z) = |z|^2 = \bar{z}z = x^2 + y^2$ is not harmonic.

- Any *nonconstant purely real-valued* function $f(z)$ (for which it must be the case that $v(z, y) \equiv 0$) fails the Cauchy-Riemann condition. In particular the real function $f(z) = |z| = \sqrt{\bar{z}z} = \sqrt{x^2 + y^2}$ is not differentiable.[6]

---

[4]A function is *analytic* on some domain if it can be expanded in a convergent power series on that domain. Although this condition implies that the function has derivatives of all orders, analyticity is a stronger condition than infinite differentiability as there exist functions which have derivatives of all orders but which cannot be expressed as a power series. For a complex-valued function of a complex variable, the term analytic has been replaced in modern mathematics by the entirely synonymous term *holomorphic*. Thus *real-valued* power-series-representable functions of a *real-variable* are analytic (real analytic), while *complex-valued* power-series-representable functions of a *complex-variable* are *holomorphic* (complex analytic).

[5]Because a harmonic function on $\mathbb{R}^2$ satisfies the partial differential equation known as Laplace's equation, by existence and uniqueness of the solution to this partial differential equation its value is completely determined at a point in the interior of *any* simply connected region which contains that point once the values on the boundary (boundary conditions) of that region are specified. This is the reason that contour integration of an analytic complex function works and that we have the freedom to select that contour to make the integration as easy as possible. On the other hand, there is, in general, no equivalent to contour integration for an arbitrary function on $\mathbb{R}^2$. See the excellent discussion in Flanigan [6].

[6]Thus we have the classic result that the only holomorphic real-valued functions are the constant real-valued functions.

Note in particular, the implication of the above for the problem of minimizing the real-valued squared-error loss functional

$$\ell(a) = \mathrm{E}\left\{|\eta_k - \bar{a}\xi_k|^2\right\} = \mathrm{E}\left\{\overline{(\eta_k - \bar{a}\xi_k)}(\eta_k - \bar{a}\xi_k)\right\} \triangleq \mathrm{E}\left\{\bar{e}_k e_k\right\} \tag{2}$$

for finite second-order moments stationary scalar complex random variables ξ_k and η_k, and unknown complex constant $a = a_x + ja_y$. Using the theory of optimization in Hilbert spaces, the minimization can be done by invoking the *projection theorem* (which is equivalent to the *orthogonality principle*) [24]. Alternatively, the minimization can be performed by completing the square. Either procedure will result in the Wiener-Hopf equations, which can then be solved for the optimal complex coefficient variable a.

However, if a gradient procedure for determining the optimum is desired, we are immediately stymied by the fact that the *purely real* nonconstant function $\ell(a)$ is *not* analytic and therefore its derivative with respect to a *does not exist in the conventional sense* of a complex derivative [3]-[6], which applies only to holomorphic functions of a. A way to break this impasse will be discussed in the next section. Meanwhile note that *all* of the real-valued nonholomorphic functions shown above can be viewed as functions of both z *and* its complex conjugate $\bar{z}$, as this fact will be of significance in the following discussion.

3   Extensions of the Complex Derivative – The $\mathbb{CR}$-Calculus

In this section we continue to focus on functions of a *single* complex variable z. The primary references for the material developed here are Nehari [11], Remmert [12], and Brandwood [14].

3.1   A Possible Extension of the Complex Derivative.

As we have seen, in order for the complex derivative of a function of $z = x + j\,y$,

$$f(z) = u(x, y) + j\,v(x, y),$$

to exist in the standard holomorphic sense, the real partial derivatives of $u(x, y)$ and $v(x, y)$ must not only exist, they must also satisfy the Cauchy-Riemann conditions (1). As noted by Flanigan [6]: "This is much stronger than the mere *existence* of the partial derivatives." However, the "mere existence" of the (real) partial derivatives *is* necessary and sufficient for a stationary point of a (necessarily nonholomorphic) non-constant *real-valued* functional $f(z)$ to exist when $f(z)$ is viewed *as a differentiable function of the real and imaginary parts of z,* i.e., as a function over $\mathbb{R}^2$,

$$f(z) = f(x, y) : \mathbb{R}^2 \to \mathbb{R}. \tag{3}$$

Thus the trick is to exploit the real $\mathbb{R}^2$ vector space structure which underlies $\mathbb{C}$ when performing gradient-based optimization. In essence, the remainder of this note is concerned with a thorough discussion of this "trick."

Towards this end, it is convenient to define a generalization or extension of the standard partial derivative to nonholomorphic functions of $z = x + j\,y$ that are nonetheless differentiable with respect to x and y and which incorporates the real gradient information directly within the complex variables framework. After Remmert [12], we will call this the *real-derivative*, or $\mathbb{R}$-*derivative,* of a possibly nonholomorphic function in order to avoid confusion with the standard *complex-derivative,* or $\mathbb{C}$-*derivative*, of a holomorphic function which was presented and discussed in the previous section. Furthermore, *we would like the real-derivative to reduce to the standard complex derivative when applied to holomorphic functions*.

Note that if one rewrites the real-valued loss function (2) in terms of purely real quantities, one obtains (temporarily suppressing the time dependence, k)

$$\ell(a) = \ell(a_x, a_y) = \mathrm{E}\left\{e_x^2 + e_y^2\right\} = \mathrm{E}\left\{(\eta_x - a_x\xi_x - a_y\xi_y)^2 + (\eta_y + a_y\xi_x - a_x\xi_y)^2\right\} . \quad (4)$$

From this we can easily determine that

$$\frac{\partial \ell(a_x, a_y)}{\partial a_x} = -2\,\mathrm{E}\left\{e_x\xi_x + e_y\xi_y\right\} ,$$

and

$$\frac{\partial \ell(a_x, a_y)}{\partial a_y} = -2\,\mathrm{E}\left\{e_x\xi_y - e_y\xi_x\right\} .$$

Together these can be written as

$$\left(\frac{\partial}{\partial a_x} + j\frac{\partial}{\partial a_y}\right)\ell(a) = \frac{\partial \ell(a_x, a_y)}{\partial a_x} + j\frac{\partial \ell(a_x, a_y)}{\partial a_y} = -2\,\mathrm{E}\left\{\xi_k\bar{e}_k\right\} \qquad (5)$$

which looks very similar to the standard result for the real case.

Indeed, equation (5) is the definition of the generalized complex partial derivative often given in engineering textbooks, including references [7]-[9]. However, this is *not* the definition used in this note, which instead follows the formulation presented in [10]-[20]. We do not use the definition (5) because it *does not* reduce to the standard $\mathbb{C}$-derivative for the case when a function $f(a)$ *is* a holomorphic function of the complex variable a. For example, take the simplest case of $f(a) = a$, for which the standard derivative yields $\frac{d}{da}f(a) = 1$. In this case, the definition (5) applied to $f(a)$ unfortunately results in the value 0. Thus we will *not* view the definition (5) as an admissible *generalization* of the standard complex partial derivative, although it does allow the determination of the stationary points of $\ell(a)$.[7]

3.2   The $\mathbb{R}$-Derivative and Conjugate $\mathbb{R}$-Derivative.

There are a variety of ways to develop the formalism discussed below (see [11]-[14]). Here, we roughly follow the development given in Remmert [12] with additional material drawn from Brandwood [14] and Nehari [11].

---

[7]In fact, it is a scaled version of the conjugate $\mathbb{R}$-derivative discussed in the next subsection.

Note that the *nonholomorphic* (nonanalytic in z)[8] functions given as examples in the previous section can all be written in the form $f(z, \bar{z})$, where they *are* holomorphic in $z = x + j\,y$ *for fixed* $\bar{z}$ and holomorphic in $\bar{z} = x - j\,y$ *for fixed* z.[9] It can be shown that *this fact is true in general* for any complex- or real-valued function

$$f(z) = f(z, \bar{z}) = f(x, y) = u(x, y) + j\,v(x, y) \tag{6}$$

of a complex variable for which the real-valued functions u and v are differentiable as functions of the real variables x and y. This fact underlies the development of the so-called *Wirtinger calculus* [12] (or, as we shall refer to it later, the $\mathbb{CR}$-*calculus*.) In essence, the so-called *conjugate coordinates,*

$$\boxed{\text{Conjugate Coordinates:} \quad c \triangleq (z, \bar{z})^T \in \mathbb{C} \times \mathbb{C}, \quad z = x + j\,y \quad \text{and} \quad \bar{z} = x - j\,y} \tag{7}$$

can serve as a formal substitute for the real $r = (x, y)^T$ representation of the point $z = x + j\,y \in \mathbb{C}$ [12].[10] According to Remmert [12], the calculus of complex variables utilizing this perspective was initiated by Henri Poincaré (over 100 years ago!) and further developed by Wilhelm Wirtinger in the 1920's [10]. Although this methodology has been fruitfully exploited by the German-speaking engineering community (see, e.g., references [13] or [33]), it has not generally been appreciated by the English speaking engineering community until relatively recently.[11]

For a general complex- or real-valued function function $f(c) = f(z, \bar{z})$ consider the *pair* of partial derivatives of $f(c)$ *formally*[12] defined by

$$\boxed{\underline{\mathbb{R}\text{-Derivative of } f(c)} \triangleq \left.\frac{\partial f(z, \bar{z})}{\partial z}\right|_{\bar{z} = \text{const.}} \quad \text{and} \quad \underline{\text{Conjugate } \mathbb{R}\text{-Derivative of } f(c)} \triangleq \left.\frac{\partial f(z, \bar{z})}{\partial \bar{z}}\right|_{z = \text{const.}}} \tag{8}$$

---

[8]Perhaps now we can better appreciate the merit of distinguishing between holomorphic and analytic functions. A function can be nonholomorphic (i.e. nonanalytic) in the *complex variable* $z = x + j\,y$ yet still be analytic in the *real variables* x and y.

[9]That is, if we make the substitution $w = \bar{z}$, they are analytic in w for fixed z, and analytic in z for fixed w. This simple insight underlies the development given in Brandwood [14] and Remmert [12].

[10]Warning! The interchangeable use of the various notational forms of f implicit in the statement $f(z) = f(z, \bar{z})$ can lead to confusion. To minimize this possibility we define the term "$f(z)$ (z-only)" to mean that $f(z)$ is independent of $\bar{z}$ (and hence is holomorphic) and the term "$f(\bar{z})$ ($\bar{z}$ only)" to mean that $f(z)$ is a function of $\bar{z}$ only. Otherwise there are no restrictions on $f(z) = f(z, \bar{z})$.

[11]An important exception is Brandwood [14] and the work that it has recently influenced such as [1, 15, 16]. However, these latter references do not seem to fully appreciate the clarity and ease of computation that the Wirtinger calculus ($\mathbb{CR}$-calculus) can provide to the problem of differentiating nonholomorphic function and optimizing real-valued functions of complex variables. Perhaps this is do to the fact that [14] did *not* reference the Wirtinger calculus as such, nor cite the rich body of work which had already existed in the mathematics community ([11, 18, 12]).

[12]These statements are *formal* because one cannot truly vary $z = x + j\,y$ while keeping $\bar{z} = x - j\,y$ constant, and vice versa.

where the formal partial derivatives are taken to be standard complex partial derivatives ($\mathbb{C}$-derivatives) taken with respect to z in the first case and with respect to $\bar{z}$ in the second.[13] For example, with $f(z, \bar{z}) = z^2\bar{z}$ we have

$$\frac{\partial f}{\partial z} = 2z\bar{z} \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}} = z^2\,.$$

As denoted in (8), we call the first expression the $\mathbb{R}$-*derivative* (the *real-derivative*) and the second expression the *conjugate $\mathbb{R}$-derivative* (or $\overline{\mathbb{R}}$-*derivative*).

It is proved in [11, 14, 12] that the $\mathbb{R}$-derivative and $\overline{\mathbb{R}}$-derivative formally defined by (8) can be equivalently written as[14]

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - j\frac{\partial f}{\partial y}\right) \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y}\right) \tag{9}$$

where the partial derivatives with respect to x and y are *true* (i.e., non-formal) partial derivatives of the function $f(z) = f(x, y)$, which is always assumed in this note to be differentiable with respect to x and y (i.e., to be $\mathbb{R}$-differentiable). Thus it is the *right-hand-sides* of the expressions given in (9) which make rigorous the formal definitions of (8).

Note that from equation (9) that we immediately have the properties

$$\frac{\partial z}{\partial z} = \frac{\partial \bar{z}}{\partial \bar{z}} = 1 \quad \text{and} \quad \frac{\partial z}{\partial \bar{z}} = \frac{\partial \bar{z}}{\partial z} = 0\,. \tag{10}$$

Comments:

1. The condition $\frac{\partial f}{\partial \bar{z}} = 0$ is true for an $\mathbb{R}$-differentiable function f if and only the Cauchy-Riemann conditions are satisfied (see [11, 14, 12]). *Thus a function f is holomorphic (analytic in z) if and only if it does not depend on the complex conjugated variable $\bar{z}$, $f(z) = f(z)$ (z only).*[15]

2. The $\mathbb{R}$-derivative, $\frac{\partial f}{\partial z}$, of an $\mathbb{R}$-differentiable function f is equal to the standard $\mathbb{C}$-derivative, $f'(z)$, when $f(z, \bar{z})$ is independent of $\bar{z}$, i.e., when $f(z) = f(z)$ (z only).

3. An $\mathbb{R}$-differentiable function f is holomorphic in $\bar{z}$ (analytic in $\bar{z}$) if and only if it does not depend on the variable z, $f(z, \bar{z}) = f(\bar{z})$ ($\bar{z}$ only), which is true if and only if $\frac{\partial f}{\partial z} = 0$.

---

[13]A careful and rigorous analysis of these formal partial derivatives can be found in Remmert [12]. In [12], a differentiable complex function f is called $\mathbb{C}$-*differentiable* while if f is differentiable as a mapping from $\mathbb{R}^2 \to \mathbb{R}^2$, it is said to be *real-differentiable* ($\mathbb{R}$-differentiable) (See footnote 2). It is shown in [12] that the partial derivatives (8) exist if and only if f is $\mathbb{R}$-differentiable. As discussed further below, throughout this note we assume that all functions are globally *real analytic* ($\mathbb{R}$-analytic), which is a sufficient condition for a function to be globally $\mathbb{R}$-differentiable.

[14]Recall the representation $f = f(x, y) = u(x, y) + j\,v(x, y)$. Note that the relationships (9) make it clear why the partial derivatives (8) exist if and only if f is $\mathbb{R}$-differentiable. (See footnotes 2 and 13).

[15]This obviously provides a simple and powerful characterization of holomorphic and nonholomorphic functions and shows the elegance of the Wirtinger calculus formulation based on the use of conjugate coordinates $(z, \bar{z})$. Note that the two Cauchy-Riemann conditions are replaced by the single condition $\frac{\partial f}{\partial \bar{z}} = 0$. The reader should reexamine the nonholomorphic (nonanalytic in z) functions discussed in the previous section in the light of this condition.

To summarize, an $\mathbb{R}$-differentiable function f is holomorphic (analytic in z) if and only if $f(z) = f(z)$ (z only), which is true if and only if $\frac{\partial f}{\partial \bar{z}} = 0$, in which case the $\mathbb{R}$-derivative coincides with the standard $\mathbb{C}$-derivative, $\frac{\partial f}{\partial z} = f'(z)$. We call the *single condition* $\frac{\partial f}{\partial \bar{z}} = 0$ the *Cauchy-Riemann condition* for f to be holomorphic:

$$\boxed{\text{Cauchy Riemann Condition:} \quad \frac{\partial f}{\partial \bar{z}} = 0} \tag{11}$$

Real Analytic Complex Functions.   Throughout the discussion given above we have been making the assumption that a complex function f is real differentiable ($\mathbb{R}$-differentiable). We henceforth make the stronger assumption that complex functions over $\mathbb{C}$ are globally *real analytic* ($\mathbb{R}$-analytic) over $\mathbb{R}^2$. As discussed above, and rigorously proven in Remmert [12], $\mathbb{R}$-analytic functions are $\mathbb{R}$-differentiable and $\overline{\mathbb{R}}$-differentiable.

A function $f(z)$ has a power series expansion in the complex variable z,

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{1}{2}f''(z_0)(z - z_0)^2 + \cdots + \frac{1}{n!}f^{(n)}(z_0)(z - z_0)^n + \cdots$$

where the complex coefficient $f^{(n)}(z_0)$ denotes an n-times $\mathbb{C}$-derivative of $f(z)$ evaluated at the point z_0, if and only if it is holomorphic. If the function $f(z)$ is not holomorphic over $\mathbb{C}$, so that the above expansion does not exist, but is nonetheless still $\mathbb{R}$-analytic as a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$, then the real and imaginary parts of $f(z) = u(x, y) + j\, v(x, y)$, $z = x + j\, y$, can be expanded in terms of the real variables $r = (x, y)^T$,

$$u(r) = u(r_0) + \frac{\partial u(r_0)}{\partial r}(r - r_0)^T + (r - r_0)^T \frac{\partial}{\partial r}\left(\frac{\partial u(r_0)}{\partial r}\right)^T (r - r_0) + \cdots$$

$$v(r) = v(r_0) + \frac{\partial v(r_0)}{\partial r}(r - r_0)^T + (r - r_0)^T \frac{\partial}{\partial r}\left(\frac{\partial v(r_0)}{\partial r}\right)^T (r - r_0) + \cdots$$

Note that if the $\mathbb{R}$-analytic function is *purely real,* then $f(z) = u(x, y)$ and we have

$$f(r) = f(r_0) + \frac{\partial f(r_0)}{\partial r}(r - r_0)^T + (r - r_0)^T \frac{\partial}{\partial r}\left(\frac{\partial f(r_0)}{\partial r}\right)^T (r - r_0) + \cdots$$

Properties of the $\mathbb{R}$- and $\overline{\mathbb{R}}$-Derivatives.   The $\mathbb{R}$-derivative and $\overline{\mathbb{R}}$-derivative are both *linear operators* which obey the *product rule* of differentiation. The following important and useful properties also hold (see references [11, 12]).[16]

Complex Derivative Identities:

$$\frac{\partial \bar{f}}{\partial \bar{z}} = \overline{\left(\frac{\partial f}{\partial z} \right)} \tag{12}$$

$$\frac{\partial \bar{f}}{\partial z} = \overline{\left(\frac{\partial f}{\partial \bar{z}} \right)} \tag{13}$$

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} \qquad \text{Differential Rule} \tag{14}$$

$$\frac{\partial h(g)}{\partial z} = \frac{\partial h}{\partial g} \frac{\partial g}{\partial z} + \frac{\partial h}{\partial \bar{g}} \frac{\partial \bar{g}}{\partial z} \qquad \text{Chain Rule} \tag{15}$$

$$\frac{\partial h(g)}{\partial \bar{z}} = \frac{\partial h}{\partial g} \frac{\partial g}{\partial \bar{z}} + \frac{\partial h}{\partial \bar{g}} \frac{\partial \bar{g}}{\partial \bar{z}} \qquad \text{Chain Rule} \tag{16}$$

As a simple consequence of the above, note that if $f(z)$ is real-valued then $\bar{f}(z) = f(z)$ so that we have the additional very important identity that

$$f(z) \in \mathbb{R} \;\Rightarrow\; \overline{\left(\frac{\partial f}{\partial z} \right)} = \frac{\partial f}{\partial \bar{z}} \tag{17}$$

As a simple first application of the above, note that the $\overline{\mathbb{R}}$-derivative of $\ell(a)$ can be easily computed from the definition (2) and the above properties to be

$$\frac{\partial \ell(a)}{\partial \bar{a}} = \mathrm{E}\left\{ \bar{e}_k e_k \right\} = \mathrm{E}\left\{ \frac{\partial \bar{e}_k}{\partial \bar{a}} e_k + \bar{e}_k \frac{\partial e_k}{\partial \bar{a}} \right\} = \mathrm{E}\left\{ 0 \cdot e_k - \bar{e}_k \, \xi_k \right\} = -\,\mathrm{E}\left\{ \xi_k \, \bar{e}_k \right\}. \tag{18}$$

which is the same result obtained from the "brute force" method based on deriving expanding the loss function in terms of the real and imaginary parts of a, followed by computing (5) and then using the result (9). Similarly, it can be easily shown that the $\mathbb{R}$-derivative of $\ell(a)$ is given by

$$\frac{\partial \ell(a)}{\partial a} = -\,\mathrm{E}\left\{ \bar{\xi}_k e_k \right\}. \tag{19}$$

Note that the results (18) and (19) are the complex conjugates of each other, which is consistent with the identity (17).

We view the *pair* of formal partial derivatives for a possibly nonholomorphic function defined by (8) as the natural generalization of the single complex derivative ($\mathbb{C}$-derivative) of a holomorphic

---

[16]In the following for $z = x + j\,y$ we define $dz = dx + j\,dy$ and $d\bar{z} = dx - j\,dy$, while $h(g) = h \circ g$ denotes the composition of the two function h and g.

function. The fact that there are *two* derivatives under general consideration does not need to be developed in elementary standard complex analysis courses where it is usually assumed that f is always holomorphic (analytic in z). In the case when f is holomorphic then f is independent of $\bar{z}$ and the conjugate partial derivative is zero, while the extended derivative reduces to the standard complex derivative.

First-Order Optimality Conditions.   As mentioned in the introduction, we are often interested in optimizing a scalar function with respect to the real and imaginary parts $r = (x, y)^T$ of a complex number $z = x + j\, y$. It is a standard result from elementary calculus that a first-order necessary condition for a point $r_0 = (x_0, y_0)^T$ to be an optimum is that this point be a stationary point of the loss function. Assuming differentiability, stationarity is equivalent to the condition that the partial derivatives of the loss function with respect the parameters $r = (x, y)^T$ vanish at the point $r = (x_0, y_0)^T$. The following fact is an easy consequence of the definitions (8) and is discussed in [14]:

- A necessary and sufficient condition for a real-valued function, $f(z) = f(x, y)$, $z = x + j\, y$, to have a stationary point with respect to the real parameters $r = (x, y)^T \in \mathbb{R}^2$ is that its $\overline{\mathbb{R}}$-derivative vanishes.   Equivalently, a necessary and sufficient condition for $f(z) = f(x, y)$ to have a stationary point with respect to $r = (x, y)^T \in \mathbb{R}^2$ is that its $\mathbb{R}$-derivative vanishes.

For example, setting either of the derivatives (18) or (19) to zero results in the so-called Wiener-Hopf equations for the optimal MMSE estimate of a. This result can be readily extended to the multivariate case, as will be discussed later in this note.

The Univariate $\mathbb{CR}$-Calculus.   As noted in [12], the approach we have been describing is known as the Wirtinger calculus in the German speaking countries, after the pioneering work of Wilhelm Wirtinger in the 1920's [10]. Because this approach is based on being able to apply the calculus of *real variables* to make statements about functions of *complex variables*, in this note we use the term "$\mathbb{CR}$-calculus" interchangeable with "Wirtinger calculus."

Despite the important insights and ease of computation that it can provide, it is the case that the use of conjugate coordinates z and $\bar{z}$ (which underlies the $\mathbb{CR}$-calculus) is *not* needed when developing the classical univariate theory of holomorphic (analytic in z) functions.[17] It is only in the multivariate and/or nonholonomic case that the tools of the $\mathbb{CR}$-calculus begin to be indispensible. Therefore it is not developed in the standard courses taught to undergraduate engineering and science students in this country [3]-[6] which have changed little in mode of presentation from the earliest textbooks.[18]

---

[17]"The differential calculus of these operations ... [is] ... largely irrelevant for classical function theory ..." — R. Remmert [12], page 66.

[18]For instance, the widely used textbook by Churchill [3] adheres closely to the format and topics of its first edition which was published in 1948. The latest edition (the 7th at the time of this writing) does appear to have one brief homework problem on the extended and conjugate derivatives.

Ironically, the elementary textbook by Nehari [11] was an attempt made in 1961 (over 40 years ago!) to integrate at least some aspects of the $\mathbb{CR}$-calculus into the elementary treatment of functions of a single complex variable.[19] However, because the vast majority of textbooks treat the univariate case, as long as the mathematics community, and most of the engineering community, was able to avoid dealing with nonholomorphic functions, there was no real need to bring the ideas of the $\mathbb{CR}$-calculus into the mainstream univariate textbooks.

Fortunately, an excellent sophisticated and extensive introduction to univariate complex variables theory and the $\mathbb{CR}$-calculus is available in the textbook by Remmert [12], which is a translation from the 1989 German edition. This book also details the historical development of complex analysis. The highly recommended Remmert and Nehari texts have been used as primary references for this note (in addition to the papers by Brandwood [14] and Van den Bos [27]).

The Multivariate $\mathbb{CR}$-Calculus.   Although one can forgo the tools of the $\mathbb{CR}$-calculus in the case of univariate holomorphic functions, this is not the situation in the multivariate holomorphic case where researchers have long utilized these tools [17]-[20].[20] Unfortunately, multivariate complex analysis is highly specialized and technically abstruse, and therefore virtually all of the standard textbooks are accessible only to the specialist or to the aspiring specialist. It is commonly assumed in these textbooks that the reader has great facility with differential geometry, topology, calculus on manifolds, and differential forms, in addition to a good grasp of advanced univariate complex variables theory. However, because the focus of the theory of multivariate complex functions is on *holomorphic* functions, whereas our concern is the essentially ignored (in this literature) case of nonholomorphic real-valued functionals, it appears to be true that only a very small part of the material presented in these references is useful, primarily for creating a rigorous and self-consistent multivariate $\mathbb{CR}$-calculus framework base on the results given in the papers by Brandwood [14] and Van den Bos [27].

The clear presentation by Brandwood [14] provides a highly accessible aspect of the multivariate $\mathbb{CR}$-calculus as applied to the problem of optimizing real-valued functionals of complex variables.[21] As this is the primary interest of many engineers, this pithy paper is a very useful presentation of just those very few theoretical and practical issues which are needed to get a clear grasp of the problem. Unfortunately, even twenty years after its publication, this paper still is not as widely known as it should be. However, the recent utilization of the Brandwood results in [1, 13, 15, 16] seems to indicate a standardization of the Brandwood presentation of the complex gradient into the mainstream textbooks. The results given in the Brandwood paper [14] are particulary useful when coupled with with the significant extension of Brandwood's results to the problem of computing complex Hessians which has been provided by Van den Bos's paper [27].

---

[19]This is still an excellent textbook that is highly recommended for an accessible introduction to the use of derivatives based on the conjugate coordinates z and $\bar{z}$.

[20]"[The $\mathbb{CR}$-calculus] is quite indispensable in the function theory of several variables." — R. Remmert [12], page 67.

[21]Although, as mentioned in an earlier footnote, Brandwood for some reason did not cite or mention any prior work relating to the use of conjugate coordinates or the Wirtinger calculus.

At this still relatively early stage in the development of a widely accepted framework for dealing with real-valued (nonholomorphic) functions of several complex variables, presumably even the increasingly widely used formalism of Brandwood [14] and Van den Bos [27] potentially has some room for improvement and/or clarification (though this is admittedly a matter of taste). In this spirit, and mindful of the increasing acceptance of the approach in [14] and [27], in the remainder of this note we develop a multivariate $\mathbb{CR}$-calculus framework that is only slightly different than that of [14] and [27], incorporating insights available from the literature on the calculus of multivariate complex functions and complex differential manifolds [17]-[20].[22]

4   Multivariate $\mathbb{CR}$-Calculus

The remaining sections of this note will provide an expanded discussion and generalized presentation of the *multivariate $\mathbb{CR}$-calculus* as presented in Brandwood [14] and Van den Bos [27], and it is assumed that the reader has read these papers, as well as reference [25]. The discussion given below utilizes insights gained from references [17, 18, 19, 20, 21, 22] and adapts notation and concepts presented for the real case in [25].

4.1   The Space $\mathcal{Z} = \mathbb{C}^n$.

We define the n-dimensional column vector $\mathbf{z}$ by

$$\mathbf{z} = \begin{pmatrix} z_1 & \cdots & z_n \end{pmatrix}^T \in \mathcal{Z} = \mathbb{C}^n$$

where $z_i = x_i + j\,y_i$, $i = 1, \cdots, n$, or, equivalently,

$$\mathbf{z} = \mathbf{x} + j\,\mathbf{y}$$

with $\mathbf{x} = (x_1 \cdots x_n)^T$ and $\mathbf{y} = (y_1 \cdots y_n)^T$. The space $\mathcal{Z} = \mathbb{C}^n$ is a vector space over the field of complex numbers with the standard component-wise definitions of vector addition and scalar multiplication. Noting the one-to-one correspondence

$$\mathbf{z} \in \mathbb{C}^n \Leftrightarrow \mathbf{r} = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in \mathcal{R} \triangleq \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$$

it is evident that there exists a natural isomorphism between $\mathcal{Z} = \mathbb{C}^n$ and $\mathcal{R} = \mathbb{R}^{2n}$.

The conjugate coordinates of $\mathbf{z} \in \mathbb{C}^n$ are defined by

$$\bar{\mathbf{z}} = \begin{pmatrix} \bar{z}_1 & \cdots & \bar{z}_n \end{pmatrix}^T \in \mathcal{Z} = \mathbb{C}^n$$

---

[22]Realistically, one must admit that many, and likely most, engineers will be unlikely to make the move from the perspective and tools provided by [14] and [27] (which already enable the engineer to solve most problems of practical interest) to that developed in this note, primarily because of the requirement of some familiarity of (or willingness to learn) concepts of differential geometry at the level presented in [25] (which is at the level of the earlier chapters of [21] and [22]).

We denote the pair of conjugate coordinate vectors $(\mathbf{z}, \bar{\mathbf{z}})$ by

$$\mathbf{c} \triangleq \begin{pmatrix} \mathbf{z} \\ \bar{\mathbf{z}} \end{pmatrix} \in \mathbb{C}^{2n} = \mathbb{C}^n \times \mathbb{C}^n$$

Noting that $\mathbf{c}$, $(\mathbf{z}, \bar{\mathbf{z}})$, $\mathbf{z}$, $(\mathbf{x}, \mathbf{y})$, and $\mathbf{r}$ are alternative ways to denote the *same point* $\mathbf{z} = \mathbf{x} + j\,\mathbf{y}$ in $\mathcal{Z} = \mathbb{C}^n$, for a function

$$\mathbf{f} : \mathbb{C}^n \to \mathbb{C}^m$$

throughout this note we will use the convenient (albeit abusive) notation

$$\mathbf{f}(\mathbf{c}) = \mathbf{f}(\mathbf{z}, \bar{\mathbf{z}}) = \mathbf{f}(\mathbf{z}) = \mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{f}(\mathbf{r}) \in \mathbb{C}^m$$

where $\mathbf{z} = \mathbf{x} + j\,\mathbf{y} \in \mathcal{Z} = \mathbb{C}^n$. We will have more to say about the relationships between these representations later on in Section 6 below.

We further assume that $\mathcal{Z} = \mathbb{C}^n$ is a Riemannian manifold with a hermitian, positive-definite $n \times n$ metric tensor $\Omega_{\mathbf{z}} = \Omega_{\mathbf{z}}^H > 0$. This assumption makes every tangent space[23] $\mathsf{T}_z\mathcal{Z} = \mathbb{C}^n_{\mathbf{z}}$ a Hilbert space with inner product

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \mathbf{v}_1^H \Omega_{\mathbf{z}} \mathbf{v}_2 \qquad \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{C}^n_{\mathbf{z}}.$$

4.2   The Cogradient Operator and the Jacobian Matrix

The Cogradient and Conjugate Cogradient.   Define the *cogradient* and *conjugate cogradient operators* respectively as the *row operators*[24]

$$\boxed{\text{Cogradient Operator:} \quad \frac{\partial}{\partial \mathbf{z}} \triangleq \begin{pmatrix} \frac{\partial}{\partial z_1} & \cdots & \frac{\partial}{\partial z_n} \end{pmatrix}} \tag{20}$$

$$\boxed{\text{Conjugate cogradient Operator:} \quad \frac{\partial}{\partial \bar{\mathbf{z}}} \triangleq \begin{pmatrix} \frac{\partial}{\partial \bar{z}_1} & \cdots & \frac{\partial}{\partial \bar{z}_n} \end{pmatrix}} \tag{21}$$

where $(z_i, \bar{z}_i)$, $i = 1, \cdots, n$ are conjugate coordinates as discussed earlier and the component operators are $\mathbb{R}$-derivatives and $\overline{\mathbb{R}}$-derivatives defined according to equations (8) and (9),

$$\frac{\partial}{\partial z_i} = \frac{1}{2} \left(\frac{\partial}{\partial x_i} - j \frac{\partial}{\partial y_i} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2} \left(\frac{\partial}{\partial x_i} + j \frac{\partial}{\partial y_i} \right), \tag{22}$$

for $i = 1, \cdots, n$.[25] Equivalently, we have

$$\frac{\partial}{\partial \mathbf{z}} = \frac{1}{2} \left(\frac{\partial}{\partial \mathbf{x}} - j \frac{\partial}{\partial \mathbf{y}} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{\mathbf{z}}} = \frac{1}{2} \left(\frac{\partial}{\partial \mathbf{x}} + j \frac{\partial}{\partial \mathbf{y}} \right), \tag{23}$$

---

[23]A tangent space at the point $\mathbf{z}$ is the space of all differential displacements, $d\mathbf{z}$, at the point $\mathbf{z}$ or, alternatively, the space of all velocity vectors $\mathbf{v} = \frac{d\mathbf{z}}{dt}$ at the point $\mathbf{z}$. These are equivalent statements because $d\mathbf{z}$ and $\mathbf{v}$ are scaled version of each other, $d\mathbf{z} = \mathbf{v}dt$. The tangent space $\mathsf{T}_z\mathcal{Z} = \mathbb{C}^n_{\mathbf{z}}$ is a linear variety in the space $\mathcal{Z} = \mathbb{C}^n$. Specifically it is a copy of $\mathbb{C}^n$ affinely translated to the point $\mathbf{z}$, $\mathbb{C}^n_{\mathbf{z}} = \{\mathbf{z}\} + \mathbb{C}^n$.

[24]The rationale behind the terminology "cogradient" is explained in [25].

[25]As before the left-hand-sides of (23) are formal partial derivatives, while the right-hand-sides are actual partial derivatives.

When applying the cogradient operator $\frac{\partial}{\partial \mathbf{z}}$, $\bar{\mathbf{z}}$ is formally treated as a constant, and when applying the conjugate cogradient operator $\frac{\partial}{\partial \bar{\mathbf{z}}}$, $\mathbf{z}$ is formally treated as a constant. For example, consider the scalar-valued function

$$f(\mathbf{c}) = f(\mathbf{z}, \bar{\mathbf{z}}) = z_1 \bar{z}_2 + \bar{z}_1 z_2 \,.$$

For this function we can readily determine by partial differentiation on the z_i and $\bar{z}_i$ components that

$$\frac{\partial f(\mathbf{c})}{\partial \mathbf{z}} = \begin{pmatrix} \bar{z}_2 & \bar{z}_1 \end{pmatrix} \quad \text{and} \quad \frac{\partial f(\mathbf{c})}{\partial \bar{\mathbf{z}}} = \begin{pmatrix} z_2 & z_1 \end{pmatrix} \,.$$

The Jacobian Matrix.   Let $\mathbf{f}(\mathbf{c}) = \mathbf{f}(\mathbf{z}, \bar{\mathbf{z}}) \in \mathbb{C}^m$ be a mapping[26]

$$\mathbf{f} : \mathcal{Z} = \mathbb{C}^n \to \mathbb{C}^m.$$

The generalization of the identity (14) yields the *vector form of the differential rule*,[27]

$$d\mathbf{f}(\mathbf{c}) = \frac{\partial \mathbf{f}(\mathbf{c})}{\partial \mathbf{c}} d\mathbf{c} = \frac{\partial \mathbf{f}(\mathbf{c})}{\partial \mathbf{z}} \, d\mathbf{z} + \frac{\partial \mathbf{f}(\mathbf{c})}{\partial \bar{\mathbf{z}}} \, d\bar{\mathbf{z}} \,, \qquad \text{Differential Rule} \tag{24}$$

where the $m \times n$ matrix $\frac{\partial \mathbf{f}}{\partial \mathbf{z}}$ is called the *Jacobian*, or *Jacobian matrix*, of the mapping $\mathbf{f}$, and the $m \times n$ matrix $\frac{\partial \mathbf{f}}{\partial \bar{\mathbf{z}}}$ the *conjugate Jacobian* of $\mathbf{f}$. only The Jacobian of $\mathbf{f}$ is often denoted by $J_{\mathbf{f}}$ is computed by applying the cogradient operator component-wise to $\mathbf{f}$,

$$J_{\mathbf{f}}(\mathbf{c}) = \frac{\partial \mathbf{f}(\mathbf{c})}{\partial \mathbf{z}} = \begin{pmatrix} \frac{\partial f_1(\mathbf{c})}{\partial \mathbf{z}} \\ \vdots \\ \frac{\partial f_n(\mathbf{c})}{\partial \mathbf{z}} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1(\mathbf{c})}{\partial z_1} & \cdots & \frac{\partial f_1(\mathbf{c})}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(\mathbf{c})}{\partial z_1} & \cdots & \frac{\partial f_n(\mathbf{c})}{\partial z_n} \end{pmatrix} \in \mathbb{C}^{m \times n}, \tag{25}$$

and similarly the conjugate Jacobian, denoted by $J_{\mathbf{f}}^c$ is computing by applying the conjugate cogradient operator component-wise to $\mathbf{f}$,

$$J_{\mathbf{f}}^c(\mathbf{c}) = \frac{\partial \mathbf{f}(\mathbf{c})}{\partial \bar{\mathbf{z}}} = \begin{pmatrix} \frac{\partial f_1(\mathbf{c})}{\partial \bar{\mathbf{z}}} \\ \vdots \\ \frac{\partial f_n(\mathbf{c})}{\partial \bar{\mathbf{z}}} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1(\mathbf{c})}{\partial \bar{z}_1} & \cdots & \frac{\partial f_1(\mathbf{c})}{\partial \bar{z}_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(\mathbf{c})}{\partial \bar{z}_1} & \cdots & \frac{\partial f_n(\mathbf{c})}{\partial \bar{z}_n} \end{pmatrix} \in \mathbb{C}^{m \times n}. \tag{26}$$

With this notation we can write the differential rule as

$$d\mathbf{f}(\mathbf{c}) = J_{\mathbf{f}}(\mathbf{c}) \, d\mathbf{z} + J_{\mathbf{f}}^c(\mathbf{c}) \, d\bar{\mathbf{z}} \,. \qquad \text{Differential Rule} \tag{27}$$

---

[26]It will always be assumed that the components of vector-valued functions are $\mathbb{R}$-differentiable as discussed in footnotes (2) and (13).

[27]At this point in our development, the expression $\frac{\partial \mathbf{f}(\mathbf{c})}{\partial \mathbf{c}} d\mathbf{c}$ only has meaning as a shorthand expression for $\frac{\partial \mathbf{f}(\mathbf{c})}{\partial \mathbf{z}} \, d\mathbf{z} +$ $\frac{\partial \mathbf{f}(\mathbf{c})}{\partial \bar{\mathbf{z}}} \, d\bar{\mathbf{z}}$, each term of which must be interpreted formally as $\mathbf{z}$ and $\bar{\mathbf{z}}$ cannot be varied independently of each other. (Later, we will examine the very special sense in which the a derivative with respect to $\mathbf{c}$ itself can make sense.) Also note that, unlike the real case discussed in [25], the mapping $d\mathbf{z} \mapsto d\mathbf{f}(\mathbf{c})$ is *not* linear in $d\mathbf{z}$. Even when interpreted formally, the mapping is affine in $d\mathbf{z}$, not linear.

Applying properties (12) and (13) component-wise yields the identities

$$\frac{\partial \bar{\mathbf{f}}(\mathbf{c})}{\partial \bar{\mathbf{z}}} = \overline{\left(\frac{\partial \mathbf{f}(\mathbf{c})}{\partial \mathbf{z}}\right)} = \bar{J}_{\mathbf{f}}(\mathbf{c}) \qquad \text{and} \qquad \frac{\partial \bar{\mathbf{f}}(\mathbf{c})}{\partial \mathbf{z}} = \overline{\left(\frac{\partial \mathbf{f}(\mathbf{c})}{\partial \bar{\mathbf{z}}}\right)} = \bar{J}_{\mathbf{f}}^c(\mathbf{c}) \,. \tag{28}$$

Note from (28) that,

$$\bar{J}_{\mathbf{f}}(\mathbf{c}) = \overline{\left(\frac{\partial \mathbf{f}(\mathbf{c})}{\partial \mathbf{z}}\right)} = \frac{\partial \bar{\mathbf{f}}(\mathbf{c})}{\partial \bar{\mathbf{z}}} \neq J_{\mathbf{f}}^c(\mathbf{c}) = \frac{\partial \mathbf{f}(\mathbf{c})}{\partial \bar{\mathbf{z}}}. \tag{29}$$

However, in *the important special case that* $\mathbf{f}(\mathbf{c})$ *is real-valued*, in which case $\bar{\mathbf{f}}(\mathbf{c}) = \mathbf{f}(\mathbf{c})$, we have

$$\mathbf{f}(\mathbf{c}) \in \mathbb{R}^m \;\Rightarrow\; \bar{J}_{\mathbf{f}}(\mathbf{c}) = \frac{\overline{\partial \mathbf{f}(\mathbf{c})}}{\partial \mathbf{z}} = \frac{\partial \mathbf{f}(\mathbf{c})}{\partial \bar{\mathbf{z}}} = J_{\mathbf{f}}^c(\mathbf{c}). \tag{30}$$

With (27) this yields the following important fact which holds for real-valued functions $\mathbf{f}(\mathbf{c})$,[28]

$$\mathbf{f}(\mathbf{c}) \in \mathbb{R}^m \;\Rightarrow\; d\mathbf{f}(\mathbf{c}) = J_{\mathbf{f}}(\mathbf{c})\, d\mathbf{z} + \overline{J_{\mathbf{f}}(\mathbf{c})\, d\mathbf{z}} = 2\,\text{Re}\left\{J_{\mathbf{f}}(\mathbf{c})\, d\mathbf{z}\right\} \,. \tag{31}$$

Consider the composition of two mappings $\mathbf{h} : \mathbb{C}^m \to \mathbb{C}^r$ and $\mathbf{g} : \mathbb{C}^n \to \mathbb{C}^m$,

$$\mathbf{h} \circ \mathbf{g} = \mathbf{h}(\mathbf{g}) : \mathbb{C}^n \to \mathbb{C}^r \,.$$

The vector extensions of the chain rule identities (15) and (16) to $\mathbf{h} \circ \mathbf{g}$ are

$$\frac{\partial \mathbf{h}(\mathbf{g})}{\partial \mathbf{z}} \;=\; \frac{\partial \mathbf{h}}{\partial \mathbf{g}} \frac{\partial \mathbf{g}}{\partial \mathbf{z}} + \frac{\partial \mathbf{h}}{\partial \bar{\mathbf{g}}} \frac{\partial \bar{\mathbf{g}}}{\partial \mathbf{z}} \qquad \text{Chain Rule} \tag{32}$$

$$\frac{\partial \mathbf{h}(\mathbf{g})}{\partial \bar{\mathbf{z}}} \;=\; \frac{\partial \mathbf{h}}{\partial \mathbf{g}} \frac{\partial \mathbf{g}}{\partial \bar{\mathbf{z}}} + \frac{\partial \mathbf{h}}{\partial \bar{\mathbf{g}}} \frac{\partial \bar{\mathbf{g}}}{\partial \bar{\mathbf{z}}} \qquad \text{Chain Rule} \tag{33}$$

which can be written as

$$J_{\mathbf{h} \circ \mathbf{g}} \;=\; J_{\mathbf{h}} J_{\mathbf{g}} + J_{\mathbf{h}}^c \bar{J}_{\mathbf{g}}^c \tag{34}$$

$$J_{\mathbf{h} \circ \mathbf{g}}^c \;=\; J_{\mathbf{h}} J_{\mathbf{g}}^c + J_{\mathbf{h}}^c \bar{J}_{\mathbf{g}} \tag{35}$$

Holomorphic Vector-valued Functions.   By definition the vector-valued function $\mathbf{f}(\mathbf{z})$ is holomorphic (analytic in the complex vector $\mathbf{z}$) if and only if each of its components

$$f_i(\mathbf{c}) = f_i(\mathbf{z}, \bar{\mathbf{z}}) = f_i(z_1, \cdots, z_n, \bar{z}_1, \cdots, \bar{z}_n) \quad i = 1, \cdots, m$$

is holomorphic separately with respect to each of the components $z_j, j = 1, \cdots, n$. In the references [17, 18, 19, 20] it is shown that $\mathbf{f}(\mathbf{z})$ is holomorphic on a domain if and only if it satisfies a matrix Cauchy Riemann condition everywhere on the domain:

$$\boxed{\text{Cauchy Riemann Condition:} \quad J_{\mathbf{f}}^c = \frac{\partial \mathbf{f}}{\partial \bar{\mathbf{z}}} = 0} \tag{36}$$

This shows that *a vector-valued function which is holomorphic on* $\mathbb{C}^n$ *must be a function of* $\mathbf{z}$ *only*, $\mathbf{f}(\mathbf{c}) = \mathbf{f}(\mathbf{z}, \bar{\mathbf{z}}) = \mathbf{f}(\mathbf{z})$ ($\mathbf{z}$ only).

---

[28]The real part of a vector (or matrix) is the vector (or matrix) of the real parts. Note that the mapping $d\mathbf{z} \mapsto d\mathbf{f}(\mathbf{c})$ is not linear.

Stationary Points of Real-Valued Functionals.   Suppose that f is a *scalar* real-valued function from $\mathbb{C}^n$ to $\mathbb{R}$,[29]

$$f : \mathbb{C}^n \to \mathbb{R}\,;\ \mathbf{z} \mapsto f(\mathbf{z})\,.$$

As discussed in [14], the first-order differential condition for a real-valued functional f to be optimized with respect to the real and imaginary parts of $\mathbf{z}$ at the point $\mathbf{z}_0$ is

$$\boxed{\text{Condition I for a Stationary Point:}\quad \frac{\partial f(\mathbf{z}_0, \bar{\mathbf{z}}_0)}{\partial \mathbf{z}} = 0} \qquad (37)$$

That this fact is true is straightforward to ascertain from equations (20) and (23). An equivalent first-order condition for a real-valued functional f to be stationary at the point $\mathbf{z}_0$ is given by

$$\boxed{\text{Condition II for a Stationary Point:}\quad \frac{\partial f(\mathbf{z}_0, \bar{\mathbf{z}}_0)}{\partial \bar{\mathbf{z}}} = 0} \qquad (38)$$

The equivalence of the two conditions (37) and (38) is a direct consequence of (28) and the fact that f is real-valued.

Differentiation of Conjugate Coordinates?   Note that the use of the notation $f(\mathbf{c})$ as shorthand for $f(\mathbf{z}, \bar{\mathbf{z}})$ appears to suggest that it is permissible to take the complex cogradient of $f(\mathbf{c})$ with respect to the conjugate coordinates vector $\mathbf{c}$ by treating the complex vector $\mathbf{c}$ *itself* as the variable of differentiation. *This is not correct.* Only complex differentiation with respect to the complex vectors $\mathbf{z}$ and $\bar{\mathbf{z}}$ is well-defined. Thus, from the definition $\mathbf{c} \triangleq \mathrm{col}(\mathbf{z}, \bar{\mathbf{z}}) \in \mathbb{C}^{2n}$, for $\mathbf{c}$ *viewed as a complex $2n$-dimensional vector,* the correct interpretation of $\frac{\partial}{\partial \mathbf{c}} f(\mathbf{c})$ is given by

$$\frac{\partial}{\partial \mathbf{c}} f(\mathbf{c}) = \left[\frac{\partial}{\partial \mathbf{z}} f(\mathbf{z}, \bar{\mathbf{z}})\,,\ \frac{\partial}{\partial \bar{\mathbf{z}}} f(\mathbf{z}, \bar{\mathbf{z}}) \right]$$

Thus, for example, we have that

$$\frac{\partial}{\partial \mathbf{c}} \mathbf{c}^H \Omega \mathbf{c} \neq \mathbf{c}^H \Omega$$

which would be true **if** it were permissible to take the complex cogradient with respect to the complex vector $\mathbf{c}$ (which it isn't).

Remarkably, however, below we will show that the $2n$-dimensional complex vector $\mathbf{c}$ is an element of an n-dimensional real vector space and that, as a consequence, it is permissible to take the real cogradient with respect to the *real* vector $\mathbf{c}$!

Comments.   *With the machinery developed up to this point, one can solve optimization problems which have closed-form solutions to the the first-order stationarity conditions.* However, to solve general nonlinear problems one must often resort to gradient-based iterative methods. Furthermore, to verify that the solutions are optimal, one needs to check second order conditions which require the construction of the hessian matrix. Therefore, the remainder of this note is primarily concerned with the development of the machinery required to construct the gradient and hessian of a scalar-valued functional of complex parameters.

---

[29]The function f is unbolded to indicate its scalar-value status.

4.3   Biholomorphic Mappings and Change of Coordinates.

Holomorphic and Biholomorphic Mappings.   A vector-valued function $\mathbf{f}$ is holomorphic (analytic) if its components are holomorphic. In this case the function does not depend on the conjugate coordinate $\bar{\mathbf{z}}$, $\mathbf{f}(\mathbf{c}) = \mathbf{f}(\mathbf{z})$, and satisfies the Cauchy-Riemann Condition,

$$J_{\mathbf{f}}^c = \frac{\partial \mathbf{f}}{\partial \bar{\mathbf{z}}} = 0 \,.$$

As a consequence (see (27)),

$$\mathbf{f}(\mathbf{z}) \text{ holomorphic} \;\Rightarrow\; d\mathbf{f}(\mathbf{z}) = J_{\mathbf{f}}(\mathbf{z}) \, d\mathbf{z} = \frac{\partial \mathbf{f}(\mathbf{z})}{\partial \mathbf{z}} \, d\mathbf{z} \,. \tag{39}$$

Note that when $\mathbf{f}$ is holomorphic, the mapping $d\mathbf{z} \mapsto d\mathbf{f}(\mathbf{z})$ is linear, exactly as in the real case.

Consider the composition of two mappings $\mathbf{h} : \mathbb{C}^m \to \mathbb{C}^r$ and $\mathbf{g} : \mathbb{C}^n \to \mathbb{C}^m$,

$$\mathbf{h} \circ \mathbf{g} = \mathbf{h}(\mathbf{g}) : \mathbb{C}^n \to \mathbb{C}^r \,,$$

which are *both holomorphic*. In this case, as a consequence of the Cauchy-Riemann condition (36), the second chain rule condition (35) vanishes, $J_{\mathbf{h} \circ \mathbf{g}}^c = 0$, and the first chain rule condition (34) simplifies to

$$\mathbf{f} \text{ and } \mathbf{g} \text{ holomorphic} \;\Rightarrow\; J_{\mathbf{h} \circ \mathbf{g}} = J_{\mathbf{h}} \, J_{\mathbf{g}} \,. \tag{40}$$

Now consider the holomorphic mapping $\boldsymbol{\xi} = \mathbf{f}(\mathbf{z})$,

$$d\boldsymbol{\xi} = d\mathbf{f}(\mathbf{z}) = J_{\mathbf{f}}(\mathbf{z}) \, d\mathbf{z} \tag{41}$$

and assume that it is invertible,

$$\mathbf{z} = \mathbf{g}(\boldsymbol{\xi}) = \mathbf{f}^{-1}(\boldsymbol{\xi}) \,. \tag{42}$$

If the invertible function $\mathbf{f}$ and its inverse $\mathbf{g} = \mathbf{f}^{-1}$ are *both* holomorphic, then $\mathbf{f}$ (equivalently, $\mathbf{g}$) is said to be *biholomorphic*. In this case, we have that

$$d\mathbf{z} = \frac{\partial \mathbf{g}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}} \, d\boldsymbol{\xi} = J_{\mathbf{g}}(\boldsymbol{\xi}) \, d\boldsymbol{\xi} = J_{\mathbf{f}}^{-1}(\mathbf{z}) \, d\boldsymbol{\xi} \,, \qquad \boldsymbol{\xi} = \mathbf{f}(\mathbf{z}) \,, \tag{43}$$

showing that

$$J_{\mathbf{g}}(\boldsymbol{\xi}) = J_{\mathbf{f}}^{-1}(\mathbf{z}) \,, \qquad \boldsymbol{\xi} = \mathbf{f}(\mathbf{z}) \,. \tag{44}$$

Coordinate Transformations.   Admissible coordinates on a space defined over a space of complex numbers are related via biholomorphic transformations [17, 18, 19, 20]. Thus if $\mathbf{z}$ and $\boldsymbol{\xi}$ are admissible coordinates on $\mathcal{Z} = \mathbb{C}^n$, there *must* exist a biholomorphic mapping relating the two coordinates, $\boldsymbol{\xi} = \mathbf{f}(\mathbf{z})$. This relationship is often denoted in the following (potentially confusing) manner,

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{z}) \,, \quad d\boldsymbol{\xi} = \frac{\partial \boldsymbol{\xi}(\mathbf{z})}{\partial \mathbf{z}} \, d\mathbf{z} = J_{\boldsymbol{\xi}}(\mathbf{z}) \, d\mathbf{z} \,, \quad \frac{\partial \boldsymbol{\xi}(\mathbf{z})}{\partial \mathbf{z}} = J_{\boldsymbol{\xi}}(\mathbf{z}) = J_{\mathbf{z}}^{-1}(\boldsymbol{\xi}) = \left(\frac{\partial \mathbf{z}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}} \right)^{-1} \tag{45}$$

$$\mathbf{z} = \mathbf{z}(\boldsymbol{\xi}) \,, \quad d\mathbf{z} = \frac{\partial \mathbf{z}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}} \, d\boldsymbol{\xi} = J_{\mathbf{z}}(\boldsymbol{\xi}) \, d\boldsymbol{\xi} \,, \quad \frac{\partial \mathbf{z}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}} = J_{\mathbf{z}}(\boldsymbol{\xi}) = J_{\boldsymbol{\xi}}^{-1}(\mathbf{z}) = \left(\frac{\partial \boldsymbol{\xi}(\mathbf{z})}{\partial \mathbf{z}} \right)^{-1} \,, \quad (46)$$

These equations tell us how vectors (elements of any particular tangent space $\mathbb{C}_z^n$) properly transform under a change of coordinates.

In particular under the change of coordinates $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{z})$, a vector $\mathbf{v} \in \mathbb{C}_{\mathbf{z}}^n$ must transform to its new representation $\mathbf{w} \in \mathbb{C}_{\boldsymbol{\xi}(\mathbf{z})}^n$ according to the

$$\boxed{\text{Vector Transformation Law:} \quad \mathbf{w} = \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{z}} \, \mathbf{v} = J_{\boldsymbol{\xi}} \, \mathbf{v}} \qquad (47)$$

For the composite coordinate transformation $\boldsymbol{\eta}(\boldsymbol{\xi}(\mathbf{z}))$, the chain rule yields

$$\boxed{\text{Transformation Chain Rule:} \quad \frac{\partial \boldsymbol{\eta}}{\partial \mathbf{z}} = \frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{\xi}} \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{z}} \quad \text{or} \quad J_{\boldsymbol{\eta} \circ \boldsymbol{\xi}} = J_{\boldsymbol{\eta}} \, J_{\boldsymbol{\xi}}} \qquad (48)$$

Finally, applying the chain rule to the cogradient, $\frac{\partial \mathbf{f}}{\partial z}$, of a an arbitrary holomorphic function $\mathbf{f}$ we obtain

$$\frac{\partial \mathbf{f}}{\partial \boldsymbol{\xi}} = \frac{\partial \mathbf{f}}{\partial \mathbf{z}} \frac{\partial \mathbf{z}}{\partial \boldsymbol{\xi}} \quad \text{for} \quad \boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{z}) \,.$$

This shows that the cogradient, *as an operator on holomorphic functions,* transforms like

$$\boxed{\text{Cogradient Transformation Law:} \quad \frac{\partial (\,\cdot\,)}{\partial \boldsymbol{\xi}} = \frac{\partial (\,\cdot\,)}{\partial \mathbf{z}} \frac{\partial \mathbf{z}}{\partial \boldsymbol{\xi}} = \frac{\partial (\,\cdot\,)}{\partial \mathbf{z}} \, J_{\mathbf{z}} = \frac{\partial (\,\cdot\,)}{\partial \mathbf{z}} \, J_{\boldsymbol{\xi}}^{-1}} \qquad (49)$$

Note that generally the cogradient transforms quite differently than does a vector.

Finally the transformation law for the metric tensor under a change of coordinates can be determined from the requirement that the inner product must be invariant under a change of coordinates. For arbitrary vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{C}_{\mathbf{z}}^n$ transformed as

$$\mathbf{w}_i = J_{\boldsymbol{\xi}} \, \mathbf{v}_i \in \mathbb{C}_{\boldsymbol{\xi}(\mathbf{z})}^n \quad i = 1, 2 \,,$$

we have

$$\langle \mathbf{w}_1, \mathbf{w}_2 \rangle = \mathbf{w}_1^H \, \Omega_{\boldsymbol{\xi}} \, \mathbf{w}_2 = \mathbf{v}_1^H \, J_{\boldsymbol{\xi}}^H \, \Omega_{\boldsymbol{\xi}} \, J_{\boldsymbol{\xi}} \, \mathbf{v}_2 = \mathbf{v}_1^H \, J_{\mathbf{z}}^{-H} \, \Omega_{\boldsymbol{\xi}} \, J_{\mathbf{z}} \, \mathbf{v}_2 = \mathbf{v}_1^H \, \Omega_{\mathbf{z}} \, \mathbf{v}_2 = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle \,.$$

This results in the

$$\boxed{\text{Metric Tensor Transformation Law:} \quad \Omega_{\boldsymbol{\xi}} = J_{\boldsymbol{\xi}}^{-H} \, \Omega_{\mathbf{z}} \, J_{\boldsymbol{\xi}}^{-1} = J_{\mathbf{z}}^H \, \Omega_{\mathbf{z}} \, J_{\mathbf{z}}} \qquad (50)$$

5   The Gradient Operator $\nabla_\mathbf{z}$

1st-Order Approximation of a Real-Valued Function.   Let $f(\mathbf{c})$ be a *real-valued*[30] functional to be optimized with respect to the real and imaginary parts of the vector $\mathbf{z} \in \mathcal{Z} = \mathbb{C}^n$,

$$f : \mathbb{C}^n \to \mathbb{R} \,.$$

As a *real-valued* function, $f(\mathbf{c})$ does not satisfy the Cauchy-Riemann condition (36) and is therefore not holomorphic.

From (31) we have (with $f(\mathbf{z}) = f(\mathbf{z}, \bar{\mathbf{z}}) = f(\mathbf{c})$) that

$$df(\mathbf{z}) = 2\,\text{Re}\left\{ J_f(\mathbf{z})\,d\mathbf{z} \right\} = 2\,\text{Re}\left\{ \frac{\partial f(\mathbf{z})}{\partial \mathbf{z}}\,d\mathbf{z} \right\} \,. \tag{51}$$

This yields the first order relationship

$$f(\mathbf{z} + d\mathbf{z}) = f(\mathbf{z}) + 2\,\text{Re}\left\{ \frac{\partial f(\mathbf{z})}{\partial \mathbf{z}}\,d\mathbf{z} \right\} \tag{52}$$

and the corresponding first-order power series approximation

$$f(\mathbf{z} + \Delta\mathbf{z}) \approx f(\mathbf{z}) + 2\,\text{Re}\left\{ \frac{\partial f(\mathbf{z})}{\partial \mathbf{z}}\,\Delta\mathbf{z} \right\} \tag{53}$$

which will be rederived by other means in Section 6 below.

The Complex Gradient of a Real-Valued Function.   The relationship (51) defines a *nonlinear* functional, $df_\mathbf{c}(\cdot)$, on the tangent space $\mathbb{C}^n_\mathbf{z}$,[31]

$$df_\mathbf{c}(\mathbf{v}) = 2\,\text{Re}\left\{ \frac{\partial f(\mathbf{c})}{\partial \mathbf{z}}\,\mathbf{v} \right\} \,, \qquad \mathbf{v} \in \mathbb{C}^n_\mathbf{z} \,, \ \mathbf{c} = (\mathbf{z}, \bar{\mathbf{z}}) \,. \tag{54}$$

Assuming the existence of a metric tensor $\Omega_\mathbf{z}$ we can write

$$\frac{\partial f}{\partial \mathbf{z}}\,\mathbf{v} = \left[\Omega_\mathbf{z}^{-1}\left(\frac{\partial f}{\partial \mathbf{z}} \right)^H \right]^H \Omega_\mathbf{z}\,\mathbf{v} = (\nabla_\mathbf{z} f)^H\,\Omega_\mathbf{z}\,\mathbf{v} = \langle \nabla_\mathbf{z} f, \mathbf{v} \rangle \,, \tag{55}$$

where $\nabla_\mathbf{z} f$ is the *gradient* of f, defined as

$$\boxed{ \text{Gradient of } f: \quad \nabla_\mathbf{z} f \triangleq \Omega_\mathbf{z}^{-1}\left(\frac{\partial f}{\partial \mathbf{z}} \right)^H } \tag{56}$$

---

[30]And therefore unbolded.

[31]Because this operator is *nonlinear* in $d\mathbf{z}$, unlike the real vector-space case (see the discussion of the real-vector space case given in [25]), we will avoid calling it a "differential operator.".

Consistent with this definition, the gradient operator is defined as

$$\boxed{\text{Gradient Operator:} \quad \nabla_{\mathbf{z}}(\,\cdot\,) \triangleq \Omega_{\mathbf{z}}^{-1} \left(\frac{\partial(\,\cdot\,)}{\partial \mathbf{z}} \right)^{H}} \tag{57}$$

One can show from the coordinate transformation laws for cogradients and metric tensors that the gradient $\nabla_{\mathbf{z}} f$ transforms like a vector and therefore *is* a vector,

$$\nabla_{\mathbf{z}} f \in \mathbb{C}_{\mathbf{z}}^{n}.$$

Equations (54) and (55) yield,

$$df_{\mathbf{c}}(\mathbf{v}) = 2 \operatorname{Re} \left\{ \langle \nabla_{\mathbf{z}} f, \mathbf{v} \rangle \right\}.$$

Keeping $\|\mathbf{v}\| = 1$ we want to find the directions $\mathbf{v}$ of steepest increase in the value of $|df_{\mathbf{c}}(\mathbf{v})|$. We have as a consequence of the Cauchy-Schwarz inequality that for all unit vectors $v \in \mathbb{C}_{\mathbf{z}}^{n}$,

$$|df_{\mathbf{c}}(\mathbf{v})| = 2 \left| \operatorname{Re} \left\{ \langle \nabla_{\mathbf{z}} f, \mathbf{v} \rangle \right\} \right| \leq 2 \left| \langle \nabla_{\mathbf{z}} f, \mathbf{v} \rangle \right| \leq 2 \|\nabla_{\mathbf{z}} f\| \|\mathbf{v}\| = 2 \|\nabla_{\mathbf{z}} f\|.$$

This upper bound is attained if and only if $\mathbf{v} \propto \nabla_{\mathbf{z}} f$, showing that the gradient gives the directions of steepest increase, with $+\nabla_{\mathbf{z}} f$ giving the direction of *steepest ascent* and $-\nabla_{\mathbf{z}} f$ giving the direction of *steepest descent.* The result (57) is derived in [14] for the special case that the metric is Euclidean $\Omega_{\mathbf{z}} = I$.[32]

Note that the first-order necessary conditions for a stationary point to exist is given by $\nabla_{\mathbf{z}} f = 0$, but that it is much easier to apply the simpler condition $\frac{\partial f}{\partial \mathbf{z}} = 0$ which does not require knowledge of the metric tensor. Of course this distinction vanishes when $\Omega_{\mathbf{z}} = I$ as is the case in [14].

Comments on Applying the Multivariate $\mathbb{CR}$-Calculus.   Because the components of the cogradient and conjugate cogradient operators (20) and (21) formally behave like partial derivatives of functions over real vectors, to use them does not require the development of additional vector partial-derivative identities over and above those that already exist for the real vector space case. The real vector space identities and procedures for vector partial-differentiation (as developed, e.g., in [25]) carry over without change, *provided one first carefully distinguishes between those variables which are to be treated like constants and those variables which are to be formally differentiated.*

Thus, although a variety of complex derivative identities are given in various references [14, 15, 16], there is actually no need to memorize or look up additional "complex derivative identities" if one already knows the real derivative identities. In particular, the derivation of the complex

---

[32]Therefore one must be careful to ascertain when a result derived in [14] holds in the general case. Also note the notational difference between this note and [14]. We have $\nabla_{\mathbf{z}}$ denoting the gradient operator while [14] denotes the gradient operator as $\nabla_{\bar{\mathbf{z}}}$ for $\Omega_{\mathbf{z}} = I$. This difference is purely notational.

derivative identities given in references [14, 15, 16] is trivial if one already knows the standard real-vector derivative identities. For example, it is *obviously* the case the

$$\frac{\partial}{\partial \bar{\mathbf{z}}} \left(\mathbf{a}^H \mathbf{z} \right) = \mathbf{a}^H \frac{\partial \mathbf{z}}{\partial \bar{\mathbf{z}}} = 0$$

as $\mathbf{z}$ *is to be treated as a constant* when taking partial derivatives with respect to $\bar{\mathbf{z}}$, so the fact that $\frac{\partial}{\partial \bar{\mathbf{z}}} \mathbf{a}^H \mathbf{z} = 0$ does *not* have to be memorized as a special complex derivative identity. To reiterate, if one already knows the standard gradient identities for real-valued functions of real variables, *there is no need to memorize additional complex derivative identities.*[33] Instead, one can merely use the regular real derivative identities *while keeping track of which complex variables are to be treated as constants.*[34] This is the approach used to easily derive the complex LMS algorithm in the applications section at the end of this note.

To implement a true gradient descent algorithm, one needs to know the metric tensor. The correct gradient, which depends on the metric tensor, is called the "natural gradient" in [26] where it is argued that superior performance of gradient descent algorithms in certain statistical parameter estimation problems occurs when the natural gradient is used in lieu of of the standard "naive" gradient usually used in such algorithms (see also the discussion in [25]). However, the determination of the metric tensor for a specific application can be highly nontrivial and the resulting algorithms significantly more complex, as discussed in [26], although there are cases where the application of the natural gradient methodology is surprisingly straightforward.

To close this section, we mention that interesting and useful applications of the $\mathbb{CR}$-calculus as developed in [14] and [27] can be found in references [13], [28]-[35], and [38], in addition to the plentiful material to be found in the textbooks [1], [15], [16], and [23].

6 $\mathbf{2}^{nd}$-Order Expansions of a Real-Valued Function on $\mathbb{C}^n$

It is common to numerically optimize cost functionals using iterative gradient descent-like techniques [25]. Determination of the gradient of a real-valued loss function via equation (56) allows the use of elementary gradient descent optimization, while the linear approximation of a biholomorphic mapping $\mathbf{g}(\boldsymbol{\xi})$ via (43) enables optimization of the nonlinear least-squares problem using the Gauss-Newton algorithm.[35]

Another commonly used iterative algorithm is the Newton method, which is based on the repeated computation and optimization of the quadratic approximation to the loss function as given

---

[33]This extra emphasis is made because virtually all of the textbooks (even the exemplary text [15]) provide such extended derivative identities and use them to derive results. This sends the message that unless such identities are at hand, one cannot solve problems. Also, it places one at the mercy of typographical errors which may occur when identities are printed in the textbooks.

[34]Thus, in the real case, $\mathbf{x}$ is the variable to be differentiated in $\mathbf{x}^T \mathbf{x}$ and we have $\frac{\partial}{\partial \mathbf{x}} \mathbf{x}^T \mathbf{x} = 2\mathbf{x}^T$, while in the complex case, if we take $\bar{\mathbf{z}}$ *to be treated as constant* and $\mathbf{z}$ to be the differentiated variable, we have $\frac{\partial}{\partial \mathbf{z}} \mathbf{z}^H \mathbf{z} = \mathbf{z}^H \frac{\partial}{\partial \mathbf{z}} \mathbf{z} = \mathbf{z}^H$. Note that in both cases we use the differentiation rules for vector differentiation which are developed initially for the purely real case once we have decided *which variables are to be treated as constant.*

[35]Recall that the Gauss-Newton algorithm is based on iterative re-linearization of a nonlinear model $\mathbf{z} \approx \mathbf{g}(\boldsymbol{\xi})$.

by a power series expansion to second order [25]. Although the first order approximation to the loss function given by (53) was relatively straight-forward to derive, it is is somewhat more work to determine the second order approximation, which is the focus of this section and which will be attacked using the elegant approach of Van den Bos [27].[36] Along the way we will rederive the first order approximation (53) and the Hessian matrix of second partial derivatives of a real scalar-valued function which is needed to verify the optimality of a solution solving the first order necessary conditions.

6.1 Alternative Coordinate Representations of $\mathcal{Z} = \mathbb{C}^n$.

Conjugate Coordinate Vectors $c \in \mathcal{C}$ **Form a Real Vector Space.** The complex space, $\mathbb{C}^n$, of dimension n naturally has the structure of a real space, $\mathbb{R}^{2n}$, of dimension $2n$, $\mathbb{C}^n \approx \mathbb{R}^{2n}$, as a consequence of the equivalence

$$\mathbf{z} = \mathbf{x} + j\,\mathbf{y} \in \mathcal{Z} = \mathbb{C}^n \Leftrightarrow \mathbf{r} = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in \mathcal{R} \triangleq \mathbb{R}^{2n}.$$

Furthermore, as noted earlier, an alternative representation is given by the set of conjugate coordinate vectors

$$\mathbf{c} = \begin{pmatrix} \mathbf{z} \\ \bar{\mathbf{z}} \end{pmatrix} \in \mathcal{C} \subset \mathbb{C}^{2n} \approx \mathbb{R}^{4n},$$

where $\mathcal{C}$ is defined to be the collection of all such vectors $\mathbf{c}$. Note that the set $\mathcal{C}$ is obviously a subset (and *not* a vector subspace)[37] of the complex vector space $\mathbb{C}^{2n}$. Remarkably, *it is also a $2n$ dimensional vector space over the field of real numbers!*

This is straightforward to show. First, in the obvious manner, one can define vector addition of any two elements of $\mathcal{C}$. To show closure under scalar multiplication by a *real* number α is also straight forward,

$$\mathbf{c} = \begin{pmatrix} \mathbf{z} \\ \bar{\mathbf{z}} \end{pmatrix} \in \mathcal{C} \Rightarrow \alpha\,\mathbf{c} = \begin{pmatrix} \alpha\,\mathbf{z} \\ \overline{\alpha\,\mathbf{z}} \end{pmatrix} \in \mathcal{C}.$$

Note that this homogeneity property obviously fails when α is complex.

To demonstrate that $\mathcal{C}$ is $2n$ dimensional, we will construct below the one-to-one transformation, J, which maps $\mathcal{C}$ onto $\mathcal{R}$, and vice versa, thereby showing that $\mathcal{C}$ and $\mathcal{R}$ are isomorphic, $\mathcal{C} \simeq \mathcal{R}$. In this manner $\mathcal{C}$ and $\mathcal{R}$ are shown to be alternative, *but entirely equivalent* (including their dimensions), real coordinate representations for $\mathcal{Z} = \mathbb{C}^n$. The coordinate transformation J is a linear mapping, and therefore also corresponds to the Jacobian of the transformation between the coordinate system $\mathcal{R}$ and the coordinate system $\mathcal{C}$.

---

[36]A detailed exposition of the second order case is given by Abatzoglou, Mendel, & Harada in [38]. See also [34]. The references [38], [27] and [34] all develop the complex Newton algorithm, although with somewhat different notation.

[37]It is, in fact, a $2n$ dimensional submanifold of the space $\mathbb{C}^{2n} \approx \mathbb{R}^{4n}$.

In summary, we have available three *vector space* coordinate representations for representing complex vectors $\mathbf{z} = \mathbf{x} + j\,\mathbf{y}$. The first is the canonical n-dimensional vector space of complex vectors $\mathbf{z} \in \mathcal{Z} = \mathbb{C}^n$ itself. The second is the canonical $2n$-dimensional real vector space of vectors $\mathbf{r} = \mathrm{col}(\mathbf{x}, \mathbf{y}) \in \mathcal{V} = \mathbb{R}^{2n}$, which arises from the natural correspondence $\mathbb{C}^n \approx \mathbb{R}^{2n}$. The third is the $2n$-dimensional real vector space of vectors $\mathbf{c} \in \mathcal{C} \subset \mathbb{C}^{2n}$, $\mathcal{C} \approx \mathbb{R}^{2n}$.

Because $\mathcal{C}$ can be alternatively viewed as a complex subset of $\mathbb{C}^{2n}$ or as a real vector space isomorphic to $\mathbb{R}^{2n}$, we actually have a fourth "representation"; namely the *non-vector space* complex-vector perspective of elements of $\mathcal{C}$ as elements of the space $\mathbb{C}^{2n}$, $\mathbf{c} = \mathrm{col}(\mathbf{z}, \bar{\mathbf{z}})$.[38] This perspective is just the $(\mathbf{z}, \bar{\mathbf{z}})$ perspective used above to analyze general, possibly nonholomorphic, functions $f(\mathbf{z}) = f(\mathbf{z}, \bar{\mathbf{z}})$.

In order to avoid confusion, we will refer to these two alternative interpretations of $\mathbf{c} \in \mathcal{C} \subset \mathbb{C}^{2n}$ as the **c**-real case (respectively, the $\mathcal{C}$-real case) for when we consider the vector $\mathbf{c} \in \mathcal{C} \approx \mathbb{R}^{2n}$ (respectively, the real vector space $\mathcal{C} \approx \mathbb{R}^{2n}$), and the **c**-complex case (respectively, the $\mathcal{C}$-complex case) when we consider a vector $\mathbf{c} \in \mathcal{C} \subset \mathbb{C}^{2n}$ (respectively, the complex subset $\mathcal{C} \subset \mathbb{C}^{2n}$).[39] These two different perspectives of $\mathcal{C}$ are used throughout the remainder of this note.

Coordinate Transformations and Jacobians.   From the fact that

$$\mathbf{z} = \mathbf{x} + j\,\mathbf{y} \quad \text{and} \quad \bar{\mathbf{z}} = \mathbf{x} - j\,\mathbf{y}$$

it is easily shown that

$$\begin{pmatrix} \mathbf{z} \\ \bar{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} I & j\,I \\ I & -j\,I \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$$

where I is the $n \times n$ identity matrix. Defining[40]

$$\mathsf{J} \triangleq \begin{pmatrix} I & j\,I \\ I & -j\,I \end{pmatrix} \tag{58}$$

then results in the mapping

$$\mathbf{c} = \mathbf{c}(\mathbf{r}) = \mathsf{J}\,\mathbf{r}\,. \tag{59}$$

It is easily determined that

$$\mathsf{J}^{-1} = \frac{1}{2}\mathsf{J}^H \tag{60}$$

---

[38]Since when viewed as a subset of $\mathbb{C}^{2n}$ the set $\mathcal{C}$ is *not* a subspace, this view of $\mathcal{C}$ does not result in a true *coordinate* representation.

[39]In the latter case $\mathbf{c} = \mathrm{col}(\mathbf{z}, \bar{\mathbf{z}})$ is understood in terms of the behavior and properties of its components, especially for differentiation purposes, as mentioned earlier, in the *complex case* the derivative $\frac{\partial}{\partial \mathbf{c}}$ is not well-defined in itself, but is defined in terms of the formal derivatives with respect to $\mathbf{z}$ and $\bar{\mathbf{z}}$. As we shall discover below, in the **c**-real case, the derivative $\frac{\partial}{\partial \mathbf{c}}$ is a true real derivative which is well understood in terms of the behavior of the derivative $\frac{\partial}{\partial \mathbf{r}}$.

[40]Except for a trivial reordering of the elements of $\mathbf{r} = (\mathbf{x}^T\,\mathbf{y}^T)^T$, this is the transformation proposed and utilized by Van den Bos [27], who claims in [31] to have been inspired to do so by Remmert. (See, e.g., the discussion on page 87 of [12].)

so that we have the inverse mapping

$$\mathbf{r} = \mathbf{r}(\mathbf{c}) = \mathsf{J}^{-1}\mathbf{c} = \frac{1}{2}\mathsf{J}^H\mathbf{c}. \tag{61}$$

Because the mapping between $\mathcal{R}$ and $\mathcal{C}$ is linear, one-to-one, and onto, both of these spaces are obviously isomorphic real vector spaces of dimension $2n$. The mappings (59) and (61) therefore correspond to an admissible coordinate transformation between the $\mathbf{c}$ and $\mathbf{r}$ representations of $\mathbf{z} \in \mathcal{Z}$. Consistent with this fact, we henceforth assume that the vector calculus (including all of the vector derivative identities) developed in [25] apply to functions over $\mathcal{C}$.

Note that for the coordinate transformation $\mathbf{c} = \mathbf{c}(\mathbf{r}) = \mathsf{J}\mathbf{r}$ we have the Jacobian

$$J_\mathbf{c} \triangleq \frac{\partial}{\partial \mathbf{r}}\,\mathbf{c}(\mathbf{r}) = \frac{\partial}{\partial \mathbf{r}}\,\mathsf{J}\mathbf{r} = \mathsf{J} \tag{62}$$

showing that J is also the Jacobian of the coordinate transformation from $\mathcal{R}$ to $\mathcal{C}$.[41] The Jacobian of the inverse transformation $\mathbf{r} = \mathbf{r}(\mathbf{c})$ is given by

$$J_\mathbf{r} = J_\mathbf{c}^{-1} = \mathsf{J}^{-1} = \frac{1}{2}\mathsf{J}^H. \tag{63}$$

Of course, then, we have the differential relationships

$$d\mathbf{c} = \frac{\partial \mathbf{c}}{\partial \mathbf{r}}\,d\mathbf{r} = J_\mathbf{c}\,d\mathbf{r} = \mathsf{J}d\mathbf{r} \quad \text{and} \quad d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \mathbf{c}}\,d\mathbf{c} = J_\mathbf{r}\,d\mathbf{c} = \frac{1}{2}\mathsf{J}^H d\mathbf{c} \tag{64}$$

which correspond to the first-order relationships[42]

$$\boxed{\text{1st-Order Relationships:} \quad \Delta\mathbf{c} = J_\mathbf{c}\,\Delta\mathbf{r} = \mathsf{J}\Delta\mathbf{r} \quad \text{and} \quad \Delta\mathbf{r} = J_\mathbf{r}\,\Delta\mathbf{c} = \frac{1}{2}\mathsf{J}^H\Delta\mathbf{c}} \tag{65}$$

where the Jacobian J is given by (60) and

$$\Delta\mathbf{c} = \begin{pmatrix} \Delta\mathbf{z} \\ \Delta\bar{\mathbf{z}} \end{pmatrix} \quad \text{and} \quad \Delta\mathbf{r} = \begin{pmatrix} \Delta\mathbf{x} \\ \Delta\mathbf{y} \end{pmatrix} \tag{66}$$

The Cogradient with respect to the Real Conjugate Coordinates Vector $\mathbf{c}$. The reader might well wonder why we didn't just point out that (64) and (65) are merely simple consequences of the linear nature of the coordinate transformations (59) and (61), and thereby skip the intermediate steps given above. The point is, as discussed in [25],[43] that once we identified the Jacobian of a coordinate transformation over a real manifold, we can readily transform between different coordinate representations of *all* vector-like (contravariant) objects, such as the gradient of a functional,

---

[41]We have just proved, of course, the general property of linear operators that they are their own Jacobians.

[42]For a general, *nonlinear,* coordinate transformation this first-order relationships would be *approximate.* However, because the coordinate transformation considered here happens to be *linear,* the relationships are *exact.*

[43]See the discussion surrounding equations (8) and (11) of [25].

and between *all* covector-like (covariant) objects, *such as the cogradient of a functional,* over that manifold. Indeed, as a consequence of this fact we immediately have the important cogradient operator transformations

$$
\text{Cogradient Transf's:} \quad \frac{\partial(\cdot)}{\partial \mathbf{c}} = \frac{\partial(\cdot)}{\partial \mathbf{r}} J_r = \frac{1}{2}\frac{\partial(\cdot)}{\partial \mathbf{r}} \mathsf{J}^H \quad \text{and} \quad \frac{\partial(\cdot)}{\partial \mathbf{r}} = \frac{\partial(\cdot)}{\partial \mathbf{c}} J_c = \frac{\partial(\cdot)}{\partial \mathbf{c}} \mathsf{J} \tag{67}
$$

with the Jacobian J given by (58) and $J_{\mathbf{r}} = J_{\mathbf{c}}^{-1}$.

Equation (67) is *very important as it allows us to easily, yet rigorously, define the cogradient taken with respect to $\mathbf{c}$ as a true (nonformal) differential operator provided that we view $\mathbf{c}$ as an element of the real coordinate representation space $\mathcal{C}$.* The cogradient $\frac{\partial(\cdot)}{\partial \mathbf{c}}$ is well-defined in terms of the cogradient $\frac{\partial(\cdot)}{\partial \mathbf{r}}$ and the "pullback" transformation

$$
\frac{\partial(\cdot)}{\partial \mathbf{c}} = \frac{1}{2}\frac{\partial(\cdot)}{\partial \mathbf{r}} \mathsf{J}^H .
$$

This shows that $\frac{\partial(\cdot)}{\partial \mathbf{c}}$, which was originally defined in terms of the cogradient and conjugate cogradients taken with respect to $\mathbf{z}$ (the $\mathbf{c}$-*complex interpretation* of $\frac{\partial(\cdot)}{\partial \mathbf{c}}$), can be treated as *a real differential operator* with respect to the "real" vector $\mathbf{c}$ (the $\mathbf{c}$-*real interpretation* of $\frac{\partial(\cdot)}{\partial \mathbf{c}}$).[44]

Complex Conjugation. It is easily determined that the operation of complex conjugation, $\mathbf{z} \to \bar{\mathbf{z}}$, is a nonlinear mapping on $\mathcal{Z} = \mathbb{C}^n$. Consider an element $\boldsymbol{\zeta} \in \mathbb{C}^{2n}$ written as

$$
\boldsymbol{\zeta} = \begin{pmatrix} \boldsymbol{\zeta}_{\text{top}} \\ \boldsymbol{\zeta}_{\text{bottom}} \end{pmatrix} \in \mathbb{C}^{2n} = \mathbb{C}^n \times \mathbb{C}^n \quad \text{with} \quad \boldsymbol{\zeta}_{\text{top}} \in \mathbb{C}^n \quad \text{and} \quad \boldsymbol{\zeta}_{\text{bottom}} \in \mathbb{C}^n .
$$

Of course the operation of complex conjugation on $\mathbb{C}^{2n}$, $\boldsymbol{\zeta} \to \bar{\boldsymbol{\zeta}}$, is, in general, a nonlinear mapping.

Now consider the *linear* operation of swapping the top and bottom elements of $\boldsymbol{\zeta}$, $\boldsymbol{\zeta} \to \tilde{\boldsymbol{\zeta}}$, defined as

$$
\boldsymbol{\zeta} = \begin{pmatrix} \boldsymbol{\zeta}_{\text{top}} \\ \boldsymbol{\zeta}_{\text{bottom}} \end{pmatrix} \to \bar{\boldsymbol{\zeta}} = \begin{pmatrix} \boldsymbol{\zeta}_{\text{bottom}} \\ \boldsymbol{\zeta}_{\text{top}} \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\zeta}_{\text{top}} \\ \boldsymbol{\zeta}_{\text{bottom}} \end{pmatrix} = S\boldsymbol{\zeta}
$$

where

$$
S \triangleq \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}
$$

is the *swap operator* on $\mathbb{C}^{2n}$ which obeys the properties

$$
S = S^T = S^{-1} ,
$$

---

[44]Thus we can directly differentiate an expression like $\mathbf{c}^T \Omega \mathbf{c}$ with respect to $\mathbf{c}$ using the standard identities of real vector calculus. (The fact that these identities hold for the $\mathbf{r}$ calculus and be used to prove their validity for the $\mathbf{c}$-real calculus.) More problematic is an expression like $\mathbf{c}^H \Omega \mathbf{c}$. It is not appropriate to take the complex derivative of this expression with respect to the complex vector $\mathbf{c}$ because $\mathbf{c}$, as an element of $\mathbb{C}^n$ is subject to constraints amongst its components. Instead one can use the identity $\bar{\mathbf{c}} = \tilde{\mathbf{c}} = S\mathbf{c}$ to obtain $\mathbf{c}^H \Omega \mathbf{c} = \mathbf{c}^T S \Omega \mathbf{c}$ which can then be differentiated with respect to $\mathbf{c}$. Of course, this latter approach can fail if $S\Omega$ cannot be interpreted in some sense in the field of real numbers. Note that real versus complex differentiation of $\mathbf{c}^H \Omega \mathbf{c}$ with respect to $\mathbf{c}$ would differ by a factor of 2.

showing that S is symmetric and its own inverse, $S^2 = I$. Note that, in general, swapping is *not* equal to complex conjugation, $\tilde{\zeta} \neq \bar{\zeta}$.

The swap operator S will be used extensively throughout the remainder of this note, so it is important to become comfortable with its use and manipulation. The swap operator is a *block permutation matrix* which permutes (swaps)[45] blocks of rows or blocks of columns depending on whether S premultiplies or postmultiplies a matrix. Specifically, let a $2n \times 2n$ matrix A be block partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Then premultiplication by S results in a block swap of the top n rows *en masse* with the bottom n rows,[46]

$$SA = \begin{pmatrix} A_{21} & A_{22} \\ A_{11} & A_{12} \end{pmatrix}.$$

Alternatively, postmultiplication by S results in a block swap of the first n columns with the last n columns,[47]

$$AS = \begin{pmatrix} A_{12} & A_{11} \\ A_{22} & A_{21} \end{pmatrix}.$$

It is also useful to note the result of a "sandwiching" by S,

$$SAS = A = \begin{pmatrix} A_{22} & A_{21} \\ A_{12} & A_{11} \end{pmatrix}.$$

Because S permutes n rows (or columns), it is a product of n elementary permutation matrices, each of which is known to have a determinant which evaluates to -1. As an easy consequence of this, we have

$$\det S = (-1)^n.$$

Other important properties of the swap operator S will be developed as we proceed.

Now note that the subset $\mathcal{C} \in \mathbb{C}^{2n}$ contains precisely those elements of $\mathbb{C}^{2n}$ for which the operations of swapping and complex conjugation coincide,

$$\mathcal{C} = \left\{ \boldsymbol{\zeta} \in \mathbb{C}^{2n} \;\middle|\; \bar{\boldsymbol{\zeta}} = \tilde{\boldsymbol{\zeta}} \right\} \subset \mathbb{C}^{2n},$$

and thus it is true by construction that $\mathbf{c} \in \mathcal{C}$ obeys $\bar{\mathbf{c}} = \tilde{\mathbf{c}}$, *even though swapping and complex conjugation are different operations* on $\mathbb{C}^{2n}$. Now although $\mathcal{C}$ is not a subspace of the complex vector space $\mathbb{C}^{2n}$, it is a *real* vector space in its own right. We see that the *linear* operation of component swapping on the $\mathcal{C}$-space coordinate representation of $\mathcal{Z} = \mathbb{C}^n$ is exactly equivalent

---

[45]"Permutation" is just a fancy term for "swapping."

[46]Matrix premultiplication of A by *any* matrix always yields a row operation.

[47]Matrix postmultiplication of A by any matrix always yields a column operation. The fact that pre- and post-multiplication yield different actions on A is an interesting and illuminating way to interpret the fact that matrix multiplication is noncommutative, $MA \neq AM$.

to the *nonlinear* operation of complex conjugation on $\mathcal{Z}$. It is important to note that complex conjugation and coordinate swapping represent different operations on a vector $\mathbf{c}$ when $\mathbf{c}$ is viewed as an element of $\mathbb{C}^{2n}$.[48]

We can view the linear swap mapping $S : \mathcal{C} \to \mathcal{C}$ as a coordinate transformation (a coordinate "reparameterization"), $\bar{\mathbf{c}} = \tilde{\mathbf{c}} = S\mathbf{c}$, on $\mathcal{C}$. Because S is linear, the Jacobian of this transformation is just S itself. Thus from the cogradient transformation property we obtain the useful identity

$$\frac{\partial(\cdot)}{\partial\bar{\mathbf{c}}} S = \frac{\partial(\cdot)}{\partial\tilde{\mathbf{c}}} S = \frac{\partial(\cdot)}{\partial\mathbf{c}} \tag{68}$$

It is also straightforward to show that

$$I = \frac{1}{2}\mathsf{J}^T S \mathsf{J} \tag{69}$$

for J given by (58)

Let us now turn to the alternative coordinate representation given by vectors $\mathbf{r}$ in the space $\mathcal{R} = \mathbb{R}^{2n}$. Specifically, consider the $\mathcal{R}$ coordinate vector $\mathbf{r}$ corresponding to the change of coordinates $\mathbf{r} = \frac{1}{2}\mathsf{J}^H\mathbf{c}$. Since the vector $\mathbf{r}$ is real, it is its own complex conjugate, $\bar{\mathbf{r}} = \mathbf{r}$.[49] Complex conjugation of $\mathbf{z}$ is the *nonlinear mapping* in $\mathbb{C}^n$

$$\mathbf{z} = \mathbf{x} + j\,\mathbf{y} \to \bar{\mathbf{z}} = \mathbf{x} + j\,(-\mathbf{y}),$$

and corresponds in the representation space $\mathcal{R}$ to the *linear mapping*[50]

$$\mathbf{r} = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \to \check{\mathbf{r}} \triangleq \begin{pmatrix} \mathbf{x} \\ -\mathbf{y} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = C\mathbf{r}$$

where C is the conjugation matrix

$$C \triangleq \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{70}$$

Note that

$$C = C^T = C^{-1},$$

i.e., that C is symmetric, $C = C^T$, and its own inverse, $C^2 = I$. It is straightforward to show that

$$C = \frac{1}{2}\mathsf{J}^H S \mathsf{J} \tag{71}$$

---

[48]As mentioned earlier, $\mathbf{c}$, in a sense, does "double duty" as a representation for $\mathbf{z}$; once as a (true coordinate) representation of $\mathbf{z}$ in the real vector space $\mathcal{C}$, and alternatively as a "representation" of z in the "doubled up" complex space $\mathbb{C}^{2n} = \mathbb{C}^n \times \mathbb{C}^n$. In the development given below, we will switch between these two perspectives of $\mathbf{c}$.

[49]Note that our theoretical developments are consistent with this requirement, as

$$\bar{\mathbf{r}} = \frac{1}{2}\overline{(\mathsf{J}^H\mathbf{c})} = \frac{1}{2}\mathsf{J}^T\bar{\mathbf{c}} = \frac{1}{2}\mathsf{J}^T\tilde{\mathbf{c}} = \frac{1}{2}\mathsf{J}^T S\mathbf{c} = \frac{1}{2}\mathsf{J}^T S\mathsf{J}\mathbf{r} = I\mathbf{r} = \mathbf{r}\,.$$

[50]We refer to $\check{\mathbf{r}}$ as "$\mathbf{r}$-check."

which can be compared to (69). Finally, it is straightforward to show that

$$\mathbf{c} = \mathsf{J}\mathbf{r} \Leftrightarrow \bar{\mathbf{c}} = \tilde{\mathbf{c}} = \mathsf{J}\check{\mathbf{r}} \,. \tag{72}$$

To summarize, we can represent the complex vector $\mathbf{z}$ by either $\mathbf{c}$ or $\mathbf{r}$, where $\mathbf{c}$ has two interpretations (as a complex vector, "$\mathbf{c}$-complex", in $\mathbb{C}^{2n}$, or as an element, "$\mathbf{c}$-real", of the real vector space $\mathcal{C} \approx \mathbb{R}^{2n}$), and we can represent the complex conjugate $\bar{\mathbf{z}}$ by $\bar{\mathbf{c}}$, $\tilde{\mathbf{c}}$, or $\check{\mathbf{r}}$. And complex conjugation, which is a nonlinear operation in $\mathbb{C}^n$, corresponds to linear operators in the $2n$-dimensional isomorphic real vector spaces $\mathcal{C}$ and $\mathcal{R}$.

6.2 Low Order Series Expansions of a Real-Valued Scalar Function.

By noting that a real-valued scalar function of complex variables can be viewed as a function of either $\mathbf{r}$ or $\mathbf{c}$-real or $\mathbf{c}$-complex or $\mathbf{z}$,

$$f(\mathbf{r}) = f(\mathbf{c}) = f(\mathbf{z}) \,,$$

it is evident that one should be able to represent f as a power series in any of these representations. Following the line of attack pursued by [27], by exploiting the relationships (65) and (67) we will readily show the equivalence up to second order in a power series expansion of f.

Up to second order, the multivariate power series expansion of the real-valued function f viewed as an analytic function of vector $\mathbf{r} \in \mathcal{R}$ is given as [25]

$$\boxed{\text{2nd-Order Expansion in } \mathbf{r}: \ f(\mathbf{r} + \Delta\mathbf{r}) = f(\mathbf{r}) + \frac{\partial f(\mathbf{r})}{\partial \mathbf{r}} \Delta\mathbf{r} + \frac{1}{2}\Delta\mathbf{r}^T \mathcal{H}_{\mathbf{rr}}(\mathbf{r})\, \Delta\mathbf{r} + \text{h.o.t.}} \tag{73}$$

where[51]

$$\mathcal{H}_{\mathbf{rr}}(\boldsymbol{\rho}) \triangleq \frac{\partial}{\partial \mathbf{r}}\left(\frac{\partial f(\boldsymbol{\rho})}{\partial \mathbf{r}}\right)^T \quad \text{for} \quad \boldsymbol{\rho}, \mathbf{r} \in \mathcal{R} \tag{74}$$

is the real $\mathbf{r}$-Hessian matrix of second partial derivatives of the real-valued function $f(\mathbf{r})$ with respect to the components of $\mathbf{r}$. It is well known that a real Hessian is symmetric,

$$\mathcal{H}_{\mathbf{rr}} = \mathcal{H}_{\mathbf{rr}}^T \,.$$

However, there is no general guarantee that the Hessian will be a positive definite or positive semidefinite matrix.

It is assumed that the terms $f(\mathbf{r})$ and $f(\mathbf{r} + \Delta\mathbf{r})$ be readily expressed in terms of $\mathbf{c}$ and $\mathbf{c} + \Delta\mathbf{c}$ or $\mathbf{z}$ and $\mathbf{z} + \Delta\mathbf{z}$. Our goal is to determine the proper expression of the linear and quadratic terms of (73) in terms of $\mathbf{c}$ and $\Delta\mathbf{c}$ or $\mathbf{z}$ and $\Delta\mathbf{z}$.

---

[51]When no confusion can arise, one usually drops the subscripts on the Hessian and uses the simpler notation $\mathcal{H}(\boldsymbol{\rho}) = \mathcal{H}_{\mathbf{rr}}(\boldsymbol{\rho})$. (As is done, for example, in [25].) Note that the Hessian is the matrix of second partial derivatives of a *real-valued* scalar function.

Scalar Products and Quadratic Forms on the Real Vector Space $\mathcal{C}$.   Consider two vectors $\mathbf{c} = \mathrm{col}(\mathbf{z}, \bar{\mathbf{z}}) \in \mathcal{C}$ and $\mathbf{s} = \mathrm{col}(\boldsymbol{\xi}, \bar{\boldsymbol{\xi}}) \in \mathcal{C}$. The scalar product for any two such vectors in $\mathcal{C}$-real (i.e., in the *real vector space* $\mathcal{C} \approx \mathbb{R}^{2n}$) is defined by

$$\langle \mathbf{c}, \mathbf{s} \rangle \triangleq \mathbf{c}^T S \mathbf{s} = \bar{\mathbf{c}}^T \mathbf{s} = \mathbf{c}^H \mathbf{s} = \mathbf{z}^H \boldsymbol{\xi} + \bar{\mathbf{z}}^H \bar{\boldsymbol{\xi}} = \mathbf{c}^H \mathbf{s} = \mathbf{z}^H \boldsymbol{\xi} + \overline{\mathbf{z}^H \boldsymbol{\xi}} = 2 \,\mathrm{Re}\, \mathbf{z}^H \boldsymbol{\xi}.$$

The row vector $\mathbf{c}^T S = \mathbf{c}^H$ is a linear functional which maps the elements of $\mathcal{C}$-real into the real numbers. The set of all such linear functionals is a vector space itself and is known as the *dual space, $\mathcal{C}^*$*, of $\mathcal{C}$ [36, 37]. The elements of $\mathcal{C}^*$ are known as *dual vectors* or *covectors*, and the terms "dual vector", "covector", and "linear functional" should all be taken to be synonymous. Given a vector $\mathbf{c} \in \mathcal{C}$, there is a natural one-to-one mapping between $\mathbf{c}$ and a corresponding dual vector, $\mathbf{c}^*$ in $\mathcal{C}^*$ defined by[52]

$$\mathbf{c}^* \triangleq \mathbf{c}^T S = \mathbf{c}^H.$$

Henceforth it is understood that scalar-product expressions like

$$\mathbf{a}^H \mathbf{s} \quad \text{or} \quad \mathbf{c}^H \mathbf{b}$$

where $\mathbf{s} \in \mathcal{C}$ and $\mathbf{c} \in \mathcal{C}$ are known to be elements of $\mathcal{C}$ *are only meaningful if $\mathbf{a}$ and $\mathbf{b}$ are also elements of $\mathcal{C}$.* Thus, *it must be the case* that *both* vectors in a scalar product must belong to $\mathcal{C}$ if it is the case that one of them does, otherwise we will view the resulting scalar as nonsensical.

Thus, for a real-valued function of up to quadratic order in a vector $\mathbf{c} \in \mathcal{C}$,

$$f(\mathbf{c}) = a + \mathbf{b}^H \mathbf{c} + \frac{1}{2} \mathbf{c}^H M \mathbf{c} = a + \mathbf{b}^H \mathbf{c} + \frac{1}{2} \mathbf{c}^H \mathbf{s}, \quad \mathbf{s} = M\mathbf{c}, \tag{75}$$

to be well-posed, it must be the case that $a \in \mathbb{R}$, $\mathbf{b} \in \mathcal{C}$,[53] and $\mathbf{s} = M\mathbf{c} \in \mathcal{C}$.[54] Thus, as we proceed to derive various first and second order functions of the form (75), we will need to check for these conditions. If the conditions are met, we will say that the terms, and the quadratic form, are *admissible* or *meaningful*.

To test whether a vector $\mathbf{b} \in \mathbb{C}^{2n}$ belongs to $\mathcal{C}$ is straightforward:

$$\mathbf{b} \in \mathcal{C} \Leftrightarrow \bar{\mathbf{b}} = S\mathbf{b}. \tag{76}$$

It is rather more work to develop a test to determine if a matrix $M \in \mathbb{C}^{2n \times 2n}$ has the property that it is a linear mapping from $\mathcal{C}$ to $\mathcal{C}$,

$$M \in \mathcal{L}(\mathcal{C}, \mathcal{C}) = \{M \mid M\mathbf{c} \in \mathcal{C}, \ \forall \mathbf{c} \in \mathcal{C} \ \text{and} \ M \text{ is linear}\} \subset \mathcal{L}(\mathbb{C}^{2n}, \mathbb{C}^{2n}) = \mathbb{C}^{2n \times 2n}.$$

---

[52]Warning! Do not confuse the dual vector (linear functional) $\mathbf{c}^*$ with an adjoint operator, which is often also denoted using the "star" notation.

[53]I.e., that $\mathbf{b}^H$ be a bona fide linear function on $\mathcal{C}$, $\mathbf{b}^H = \mathbf{b}^* \in \mathcal{C}^*$.

[54]I.e., because $\mathbf{c}^H = \mathbf{c}^* \in \mathcal{C}^*$, is a linear functional on $\mathcal{C}$, it must have a legitimate object $\mathbf{s}$ to operate on, namely an element $\mathbf{s} = M\mathbf{c} \in \mathcal{C}$.

Note that the fact that $\mathcal{L}(\mathcal{C},\mathcal{C}) \subset \mathcal{L}(\mathbb{C}^{2n},\mathbb{C}^{2n})$ is just the statement that any matrix which maps from $\mathcal{C} \subset \mathbb{C}^{2n}$ to $\mathcal{C} \subset \mathbb{C}^{2n}$ is also obviously a linear mapping from $\mathbb{C}^{2n}$ to $\mathbb{C}^{2n}$. However, *this is just a subset statement*; it is *not* a subspace statement. This is because $\mathcal{L}(\mathcal{C},\mathcal{C})$ is a *real* vector space of linear operators,[55] while $\mathcal{L}(\mathbb{C}^{2n},\mathbb{C}^{2n})$ is a complex vector space of linear operators.[56] Because they are vector spaces over *different fields*, they cannot have a vector-subspace/vector-parent-space relationship to each other.

To determine necessary and sufficient conditions for a matrix $M \in \mathbb{C}^{2n \times 2n}$ to be an element of $\mathcal{L}(\mathcal{C},\mathcal{C})$ suppose that the vector $\mathbf{c} = \mathrm{col}(\mathbf{z}, \bar{\mathbf{z}}) \in \mathcal{C}$ always maps to a vector $\mathbf{s} = \mathrm{col}(\boldsymbol{\xi}, \bar{\boldsymbol{\xi}}) \in \mathcal{C}$ under the action of M, $\mathbf{s} = M\mathbf{c}$. Expressed in block matrix form, this relationship is

$$\begin{pmatrix} \boldsymbol{\xi} \\ \bar{\boldsymbol{\xi}} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \mathbf{z} \\ \bar{\mathbf{z}} \end{pmatrix}.$$

The first block row of this matrix equation yields the conditions

$$\boldsymbol{\xi} = M_{11}\mathbf{z} + M_{12}\bar{\mathbf{z}}$$

while the complex conjugate of the second block row yields

$$\boldsymbol{\xi} = \bar{M}_{22}\mathbf{z} + \bar{M}_{21}\bar{\mathbf{z}}$$

and subtracting these two sets of equations results in the following condition on the block elements of M,

$$(M_{11} - \bar{M}_{22})\mathbf{z} + (M_{12} - \bar{M}_{21})\bar{\mathbf{z}} = 0\,.$$

With $\mathbf{z} = \mathbf{x} + j\,\mathbf{y}$, this splits into the two sets of conditions,

$$[(M_{11} - \bar{M}_{22}) + (M_{12} - \bar{M}_{21})]\mathbf{x} = 0$$

and

$$[(M_{11} - \bar{M}_{22}) - (M_{12} - \bar{M}_{21})]\mathbf{y} = 0.$$

Since these equations must hold for any $\mathbf{x}$ and $\mathbf{y}$, they are equivalent to

$$(M_{11} - \bar{M}_{22}) + (M_{12} - \bar{M}_{21}) = 0$$

and

$$(M_{11} - \bar{M}_{22}) - (M_{12} - \bar{M}_{21}) = 0.$$

Finally, adding and subtracting these two equations yields the necessary and sufficient conditions for M to admissible (i.e., to be a mapping from $\mathcal{C}$ to $\mathcal{C}$),

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \in \mathbb{C}^{2n \times 2n} \text{ is an element of } \mathcal{L}(\mathcal{C},\mathcal{C}) \text{ iff } M_{11} = \bar{M}_{22} \text{ and } M_{12} = \bar{M}_{21}\,. \quad (77)$$

---

[55]I.e., a vector space over the field of real numbers.

[56]I.e., a vector space over the field of complex numbers.

This necessary and sufficient condition is more conveniently expressed in the following equivalent form,

$$M \in \mathcal{L}(\mathcal{C},\mathcal{C}) \Leftrightarrow M = S\bar{M}S \Leftrightarrow \bar{M} = SMS \tag{78}$$

which is straightforward to verify.

Given an arbitrary matrix $M \in \mathbb{C}^{2n \times 2n}$, we can define a natural mapping of M into $\mathcal{L}(\mathcal{C},\mathcal{C}) \subset \mathbb{C}^{2n \times 2n}$ by

$$\mathbf{P}(M) \triangleq \frac{M + S\bar{M}S}{2} \in \mathcal{L}(\mathcal{C},\mathcal{C})\,, \tag{79}$$

in which case the condition (78) has an equivalent restatement as

$$M \in \mathcal{L}(\mathcal{C},\mathcal{C}) \Leftrightarrow \mathbf{P}(M) = M\,. \tag{80}$$

It is straightforward to demonstrate that

$$\mathbf{P}(M) \in \mathcal{C}, \ \ \forall M \in \mathbb{C}^{2n \times 2n} \ \text{ and } \ \mathbf{P}(\mathbf{P}(M)) = \mathbf{P}(M) \tag{81}$$

i.e., that $\mathbf{P}$ is an idempotent mapping of $\mathbb{C}^{2n \times 2n}$ onto $\mathcal{L}(\mathcal{C},\mathcal{C})$, $\mathbf{P}^2 = \mathbf{P}$. However, it is important to note that $\mathbf{P}$ is *not* a linear operator (the action of complex conjugation precludes this) nor a projection operator in the conventional sense of projecting onto a lower dimensional *subspace* as its range space is *not* a subspace of its domain space. However, it is reasonable to interpret $\mathbf{P}$ as a projector of the *manifold* $\mathbb{C}^{2n}$ onto the *submanifold* $\mathcal{C} \subset \mathbb{C}^{2n}$.[57]

A final important fact is that if $M \in \mathbb{C}^{2n \times 2n}$ is invertible, then $M \in \mathcal{L}(\mathcal{C},\mathcal{C})$ if and only if $M^{-1} \in \mathcal{L}(\mathcal{C},\mathcal{C})$, which we state formally as

$$\text{Let } M \text{ be invertible, then } \mathbf{P}(M) = M \text{ iff } \mathbf{P}(M^{-1}) = M^{-1}. \tag{82}$$

I.e., if an invertible matrix M is admissible, then M^{-1} is admissible. The proof is straightforward:

$$\begin{aligned}
M = S\bar{M}S \ \ &\text{and} \ \ M \text{ invertible} \\
\Leftrightarrow M^{-1} \ &= \ \left(S\bar{M}S\right)^{-1} \\
&= \ S(\bar{M})^{-1}S \\
&= \ S\overline{M^{-1}}S\,.
\end{aligned}$$

---

[57]With $\mathbb{C}^{2n \times 2n} \approx \mathbb{R}^{4n \times 4n} \approx \mathbb{R}^{16n^2}$ and $\mathcal{L}(\mathcal{C},\mathcal{C}) \approx \mathcal{L}(\mathbb{R}^{2n}, \mathbb{R}^{2n}) \approx \mathbb{R}^{2n \times 2n} \approx \mathbb{R}^{4n^2}$, it is reasonable to view $\mathbf{P}$ as a linear projection operator from the *vector space* $\mathbb{R}^{16n^2}$ onto the *vector subspace* $\mathbb{R}^{4n^2} \subset \mathbb{R}^{16n^2}$. This allows us to interpret $\mathbf{P}$ as a projection operator from the *manifold* $\mathbb{C}^{2n}$ onto the *submanifold* $\mathcal{C} \subset \mathbb{C}^{2n}$. Once we know that $\mathbf{P}$ is a linear mapping from $\mathbb{C}^{2n}$ into $\mathbb{C}^{2n}$, we can then compute its adjoint operator, $\mathbf{P}^*$, and then test to see if its self-adjoint. If it is, then the projection operator $\mathbf{P}$ is, in fact, an orthogonal projection operator. In the interest of time, this additional computation and test will not be done, as in the remainder of this note we will only exploit the idempotency property of the projector $\mathbf{P}$.

First Order Expansions. Up to first order, the power series expansion of the real-valued function f viewed as a function of $\mathbf{r} \in \mathcal{R}$ is

$$\boxed{\text{First-Order Expansion in } \mathbf{r}: \qquad f(\mathbf{r} + \Delta\mathbf{r}) = f(\mathbf{r}) + \frac{\partial f(\mathbf{r})}{\partial \mathbf{r}} \Delta\mathbf{r} + \text{h.o.t.}} \qquad (83)$$

Focussing our attention first on the linear term $\frac{\partial f(\mathbf{r})}{\partial \mathbf{r}} \Delta\mathbf{r}$, and using the $\mathbf{c}$-real vector space interpretation of $\mathbf{c}$, namely that $\mathbf{c} \in \mathcal{C}$ where, as discussed above, $\mathcal{C}$ is a $2n$-dimensional coordinate space isomorphic to $\mathbb{R}^{2n}$, we have

$$\begin{aligned}
\frac{\partial f}{\partial \mathbf{r}} \Delta\mathbf{r} &= \frac{\partial f}{\partial \mathbf{r}} J_{\mathbf{c}}^{-1} \Delta\mathbf{c} &&\text{(from equation (65))} \\
&= \frac{\partial f}{\partial \mathbf{c}} \Delta\mathbf{c} &&\text{(from equation (67))}
\end{aligned}$$

which yields the first order expansion of f in terms of the parameterization in $\mathbf{c}$,

$$\boxed{\text{First-Order Expansion in } \mathbf{c}: \quad f(\mathbf{c} + \Delta\mathbf{c}) = f(\mathbf{c}) + \frac{\partial f(\mathbf{c})}{\partial \mathbf{c}} \Delta\mathbf{c} + \text{h.o.t.}} \qquad (84)$$

Note that $\frac{\partial f(\mathbf{c})}{\partial \mathbf{c}} \Delta\mathbf{c}$ is real valued. Furthermore, as a consequence of the fact that with $f(\mathbf{c})$ real-valued we have

$$\overline{\left(\frac{\partial f(\mathbf{c})}{\partial \mathbf{c}} \right)^H} = \left(\frac{\partial f(\mathbf{c})}{\partial \overline{\mathbf{c}}} \right)^H = S \left(\frac{\partial f(\mathbf{c})}{\partial \mathbf{c}} \right)^H,$$

which is the necessary and sufficient condition given in (76) that

$$\left(\frac{\partial f(\mathbf{c})}{\partial \mathbf{c}} \right)^H \in \mathcal{C}.$$

Thus $\frac{\partial f(\mathbf{c})}{\partial \mathbf{c}} \in \mathcal{C}^*$ and the term $\frac{\partial f(\mathbf{c})}{\partial \mathbf{c}} \Delta\mathbf{c}$ is admissible in the sense defined earlier. Note that an equivalent condition for the term $\frac{\partial f(\mathbf{c})}{\partial \mathbf{c}} \Delta\mathbf{c}$ to be admissible is that

$$S \left(\frac{\partial f(\mathbf{c})}{\partial \mathbf{c}} \right)^T \in \mathcal{C},$$

which is true if and only if

$$\left(\frac{\partial f(\mathbf{c})}{\partial \mathbf{c}} \right)^T \in \mathcal{C}.$$

This shows a simple inspection of $\frac{\partial f(\mathbf{c})}{\partial \mathbf{c}}$ itself can be performed to test for admissibility of the linear term.[58]

---

[58]In this note, the first order expansion (84) is doing double duty in that it is simultaneously standing for the $\mathbf{c}$-real expansion and the $\mathbf{c}$-complex expansion. A more careful development would make this distinction explicit, in which case one would more carefully explore the distinction between $\left(\frac{\partial f(\mathbf{c})}{\partial \mathbf{c}} \right)^T$ versus $\left(\frac{\partial f(\mathbf{c})}{\partial \mathbf{c}} \right)^H$ in the linear term. Because this note has already become rather notationally tedious, this option for greater precision has been declined. However, greater care must therefore be made when switching between the $\mathcal{C}$-real and $\mathcal{C}$-complex perspectives.

As discussed above, to be meaningful as a true derivative, the derivative with respect to **c** has to be interpreted as a real derivative. This is the **c**-real interpretation of (84). In addition, (84) has a **c**-complex interpretation for which the partial derivative with respect to **c** is not well-defined as a complex derivative as it stands, but rather only makes sense as a shorthand notation for simultaneously taking the complex derivatives with respect to **z** and **z̄**,

$$\frac{\partial}{\partial \mathbf{c}} = \left(\frac{\partial}{\partial \mathbf{z}}, \ \frac{\partial}{\partial \bar{\mathbf{z}}} \right).$$

Thus, to work in the domain of complex derivatives, we must move to the **c**-complex perspective $\mathbf{c} = \mathrm{col}(\mathbf{z}, \bar{\mathbf{z}})$, and then break **c** apart so that we can work with expressions explicitly involving **z** and **z̄**, exploiting the fact that the formal partial derivatives with respect to **z** and **z̄** are well defined. Noting that

$$\frac{\partial}{\partial \mathbf{c}} = \begin{pmatrix} \frac{\partial}{\partial \mathbf{z}} & \frac{\partial}{\partial \bar{\mathbf{z}}} \end{pmatrix} \quad \text{and} \quad \Delta \mathbf{c} = \begin{pmatrix} \Delta \mathbf{z} \\ \Delta \bar{\mathbf{z}} \end{pmatrix}$$

we obtain

$$
\begin{aligned}
\frac{\partial f(\mathbf{c})}{\partial \mathbf{c}} \Delta \mathbf{c} &= \frac{\partial f}{\partial \mathbf{z}} \Delta \mathbf{z} + \frac{\partial f}{\partial \bar{\mathbf{z}}} \Delta \bar{\mathbf{z}} \\
&= \frac{\partial f}{\partial \mathbf{z}} \Delta \mathbf{z} + \overline{\frac{\partial f}{\partial \mathbf{z}} \Delta \mathbf{z}} \qquad (f \text{ is real-valued}) \\
&= 2 \, \mathrm{Re} \left\{ \frac{\partial f}{\partial \mathbf{z}} \Delta \mathbf{z} \right\}
\end{aligned}
$$

which yields the first order expansion of f in terms of the parameterization in **z**,

$$\boxed{\text{First-Order Expansion in } \mathbf{z}: \quad f(\mathbf{z} + \Delta \mathbf{z}) = f(\mathbf{z}) + 2 \, \mathrm{Re} \left\{ \frac{\partial f}{\partial \mathbf{z}} \Delta \mathbf{z} \right\} + \text{h.o.t.}} \qquad (85)$$

This is the rederivation of (53) promised earlier. Note that (85) makes *explicit* the relationship which is *implied* in the **c**-complex interpretation of (84).

We also summarize our intermediate results concerning the linear term in a power series expansion using the **r**, **c** or **z** representations,

$$\boxed{\text{Linear-Term Relationships:} \quad \frac{\partial f}{\partial \mathbf{r}} \Delta \mathbf{r} = \frac{\partial f}{\partial \mathbf{c}} \Delta \mathbf{c} = 2 \, \mathrm{Re} \left\{ \frac{\partial f}{\partial \mathbf{z}} \Delta \mathbf{z} \right\}} \qquad (86)$$

The derivative in the first expression is a real derivative. The derivative in the second expression can be interpreted as a real derivative (the **c**-real interpretation). The derivative in the last expression is a complex derivative; it corresponds to the **c**-complex interpretation of the second term in (86). Note that all of the linear terms are real valued.

We now have determined the first-order expansion of f in terms of **r**, **c**, and **z**. To construct the second-order expansion it remains to examine the second-order term in (73) and some of the properties of the real Hessian matrix (74) which completely specifies that term.

Second Order Expansions.   Note from (73) that knowledge of the real Hessian matrix $\mathcal{H}_{\mathbf{rr}}$ completely specifies the second order term in the real power series expansion of f with respect to $\mathbf{r}$. The goal which naturally presents itself to us at this point is now to reexpress this quadratic-order term in terms of $\mathbf{c}$, which we indeed proceed to do. However, because the canonical coordinates vector $\mathbf{c}$ *has two interpretations,* one as a shorthand for the pair $(\mathbf{z}, \bar{\mathbf{z}}$ (the $\mathbf{c}$-complex perspective) and the other as an element of a real vector space (the $\mathbf{c}$-real perspective), we will rewrite the second order term in two different forms, one (the $\mathbf{c}$-complex form) involving the $\mathbf{c}$-complex Hessian matrix

$$\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}(\boldsymbol{v}) \triangleq \frac{\partial}{\partial \mathbf{c}} \left(\frac{\partial f(\boldsymbol{v})}{\partial \mathbf{c}} \right)^{H} \quad \text{for} \quad \boldsymbol{v}, \mathbf{c} \in \mathcal{C} \subset \mathbb{C}^{2n} \tag{87}$$

and the other (the $\mathbf{c}$-real form) involving the $\mathbf{c}$-real Hessian matrix

$$\mathcal{H}_{\mathbf{cc}}^{\mathbb{R}}(\boldsymbol{v}) \triangleq \frac{\partial}{\partial \mathbf{c}} \left(\frac{\partial f(\boldsymbol{v})}{\partial \mathbf{c}} \right)^{T} \quad \text{for} \quad \boldsymbol{v}, \mathbf{c} \in \mathcal{C} \approx \mathbb{R}^{2n}. \tag{88}$$

In (87), the derivative with respect to $\mathbf{c}$ only has meaning as a short-hand for $\left(\frac{\partial}{\partial \mathbf{z}}, \frac{\partial}{\partial \bar{\mathbf{z}}} \right)$. In (88), the derivative with respect to $\mathbf{c}$ is well-defined via the $\mathbf{c}$-real interpretation.

It is straightforward to show a relationship between the real Hessian $\mathcal{H}_{\mathbf{rr}}$ and the $\mathbf{c}$-complex Hessian $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$,

$$\begin{aligned}
\mathcal{H}_{\mathbf{rr}} &\triangleq \frac{\partial}{\partial \mathbf{r}} \left(\frac{\partial f}{\partial \mathbf{r}} \right)^{T} \\
&= \frac{\partial}{\partial \mathbf{r}} \left(\frac{\partial f}{\partial \mathbf{r}} \right)^{H} \\
&= \frac{\partial}{\partial \mathbf{r}} \left(\frac{\partial f}{\partial \mathbf{c}} \mathsf{J} \right)^{H} && \text{(from equation (67))} \\
&= \frac{\partial}{\partial \mathbf{r}} \left\{ \mathsf{J}^{H} \left(\frac{\partial f}{\partial \mathbf{c}} \right)^{H} \right\} \\
&= \frac{\partial}{\partial \mathbf{c}} \left\{ \mathsf{J}^{H} \left(\frac{\partial f}{\partial \mathbf{c}} \right)^{H} \right\} \mathsf{J} && \text{(from equation (67))} \\
&= \mathsf{J}^{H} \frac{\partial}{\partial \mathbf{c}} \left(\frac{\partial f}{\partial \mathbf{c}} \right)^{H} \mathsf{J} && \text{(From equation (32) of [25])} \\
&= \mathsf{J}^{H} \mathcal{H}_{\mathbf{cc}}^{\mathbb{C}} \mathsf{J}.
\end{aligned}$$

The resulting important relationship

$$\mathcal{H}_{\mathbf{rr}} = \mathsf{J}^{H} \mathcal{H}_{\mathbf{cc}}^{\mathbb{C}} \mathsf{J} \tag{89}$$

between the real and $\mathbf{c}$-complex Hessians was derived in [27] based on the there unjustified (but true) assumption that the second order terms of the powers series expansions of f in terms of $\mathbf{r}$

and **c**-complex must be equal. Here, we reverse this order of reasoning, and will show below the equality of the second order terms in the **c**-complex and **r** expansions as a *consequence* of (89).

Note from (60) that

$$\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}} = \frac{1}{4}\, \mathsf{J}\, \mathcal{H}_{\mathbf{rr}}\, \mathsf{J}^H. \tag{90}$$

Recalling that the Hessian $\mathcal{H}_{\mathbf{rr}}$ is a symmetric matrix,[59] it is evident from (90) that $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$ is *Hermitian*[60]

$$\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}} = \left(\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}\right)^{H}$$

(and hence, like $\mathcal{H}_{\mathbf{rr}}$, has real eigenvalues), and positive definite (semidefinite) if and only $\mathcal{H}_{\mathbf{rr}}$ is positive definite (semidefinite).

As noted by Van den Bos [27], one can now readily relate the values of the eigenvalues of $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$ and $\mathcal{H}_{\mathbf{rr}}$ from the fact, which follows from (60) and (90), that

$$\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}} - \lambda I = \frac{1}{4}\, \mathsf{J}\, \mathcal{H}_{\mathbf{rr}}\, \mathsf{J}^H - \frac{\lambda}{2}\mathsf{J}\mathsf{J}^H = \frac{1}{4}\mathsf{J}\left(\mathcal{H}_{\mathbf{rr}} - 2\lambda I\right)\mathsf{J}^H.$$

This shows that the eigenvalues of the real Hessian matrix are twice the size of the eigenvalues of the complex Hessian matrix (and, as a consequence, must share the same condition number).[61]

Focussing our attention now on the second order term of (73), we have

$$
\begin{aligned}
\frac{1}{2}\Delta\mathbf{r}^T\,\mathcal{H}_{\mathbf{rr}}\,\Delta\mathbf{r} &= \frac{1}{2}\Delta\mathbf{r}^H\,\mathcal{H}_{\mathbf{rr}}\,\Delta\mathbf{r} \\
&= \frac{1}{2}\Delta\mathbf{r}^H\,\mathsf{J}^H\,\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}\,\mathsf{J}\,\Delta\mathbf{r} \qquad \text{(From equation (89))} \\
&= \frac{1}{2}\Delta\mathbf{c}^H\,\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}\,\Delta\mathbf{c}, \qquad \text{(From equation (65))}
\end{aligned}
$$

thereby showing the equality of the second order terms in an expansion of a real-valued function f either in terms of **r** or **c**-complex,[62]

$$\frac{1}{2}\Delta\mathbf{r}^T\,\mathcal{H}_{\mathbf{rr}}\,\Delta\mathbf{r} = \frac{1}{2}\Delta\mathbf{c}^H\,\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}\,\Delta\mathbf{c}. \tag{91}$$

Note that both of these terms are real valued.

With the proof of the equalities 86 and 91, we have (almost) completed a derivation of the

$$\boxed{\text{2nd-Order Expansion in } \mathbf{c}\text{-Complex: } f(\mathbf{c} + \Delta\mathbf{c}) = f(\mathbf{c}) + \frac{\partial f(\mathbf{c})}{\partial \mathbf{c}}\,\Delta\mathbf{c} + \frac{1}{2}\Delta\mathbf{c}^H\,\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}(\mathbf{c})\,\Delta\mathbf{c} + \text{h.o.t.}} \tag{92}$$

---

[59] In the real case, this is a general property of the matrix of second partial derivatives of a scalar function.

[60] As expected, as this is a general property of the matrix of partial derivatives $\frac{\partial}{\partial \mathbf{z}}\left(\frac{\partial f(\mathbf{z})}{\partial \mathbf{z}}\right)^{H}$ of any *real-valued* function $f(\mathbf{z})$.

[61] For a Hermitian matrix, the singular values are the absolute values of the (real) eigenvalues. Therefore the condition number, which is the ratio of the largest to the smallest eigenvalue (assuming a full rank matrix) is given by the ratio of the largest to smallest eigenvalue magnitude.

[62] And thereby providing a proof of this assumed equality in [27].

where the **c**-complex Hessian $\mathcal{H}^{\mathbb{C}}_{\mathbf{cc}}$ is given by equation (87) and is related to the real hessian $\mathcal{H}_{\mathbf{rr}}$ by equations (89) and (90). Note that all of the terms in (92) are real valued. The derivation has not been fully completed because we have not verified that $\Delta\mathbf{c}^H \mathcal{H}^{\mathbb{C}}_{\mathbf{cc}}(\mathbf{c}) \Delta\mathbf{c}$ is admissible in the sense defined above. The derivation will be fully completed once we have verified that $\mathcal{H}^{\mathbb{C}}_{\mathbf{cc}} \in \mathcal{L}(\mathcal{C}, \mathcal{C})$, which we will do below.

The **c**-complex expansion (92) is *not* differentiable with respect to **c**-complex *itself,* which is not well defined, but, if differentiation is required, should be instead interpeted has a short-hand, or implicit, statement involving $\mathbf{z}$ and $\bar{\mathbf{z}}$, for which derivatives are well defined. To explicitly show the the second order expansion of the real-valued function f in terms of the complex vectors $\mathbf{z}$ and $\bar{\mathbf{z}}$, it is convenient to define the quantities

$$\mathcal{H}_{\mathbf{zz}} \triangleq \frac{\partial}{\partial\mathbf{z}}\left(\frac{\partial f}{\partial\mathbf{z}}\right)^H, \quad \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} \triangleq \frac{\partial}{\partial\bar{\mathbf{z}}}\left(\frac{\partial f}{\partial\mathbf{z}}\right)^H, \quad \mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}} \triangleq \frac{\partial}{\partial\mathbf{z}}\left(\frac{\partial f}{\partial\bar{\mathbf{z}}}\right)^H, \quad \text{and} \quad \mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}} \triangleq \frac{\partial}{\partial\bar{\mathbf{z}}}\left(\frac{\partial f}{\partial\bar{\mathbf{z}}}\right)^H. \tag{93}$$

With $\frac{\partial}{\partial\mathbf{c}} = (\frac{\partial}{\partial\mathbf{z}}, \frac{\partial}{\partial\bar{\mathbf{z}}})$, we also have from (87) and the definitions (93) that

$$\mathcal{H}^{\mathbb{C}}_{\mathbf{cc}} = \begin{pmatrix} \mathcal{H}_{\mathbf{zz}} & \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} \\ \mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}} & \mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}} \end{pmatrix}. \tag{94}$$

Thus, using the earlier proven property that $\mathcal{H}^{\mathbb{C}}_{\mathbf{cc}}$ is Hermitian, $\mathcal{H}^{\mathbb{C}}_{\mathbf{cc}} = (\mathcal{H}^{\mathbb{C}}_{\mathbf{cc}})^H$, we immediately have from (94) the *Hermitian conjugate conditions*

$$\mathcal{H}_{\mathbf{zz}} = \mathcal{H}^H_{\mathbf{zz}} \quad \text{and} \quad \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} = \mathcal{H}^H_{\mathbf{z}\bar{\mathbf{z}}} \tag{95}$$

which also hold for $\mathbf{z}$ and $\bar{\mathbf{z}}$ replaced by $\bar{\mathbf{z}}$ and $\mathbf{z}$ respectively.

Some additional useful properties can be shown to be true for the block components of (94) defined in (93). First note that as a consequence of f being a real-valued function, it is straightforward to show the validity of the *conjugation conditions*

$$\overline{\mathcal{H}^{\mathbb{C}}_{\mathbf{cc}}} = \mathcal{H}^{\mathbb{C}}_{\bar{\mathbf{c}}\bar{\mathbf{c}}}$$

or, equivalently,

$$\mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}} = \overline{\mathcal{H}_{\mathbf{zz}}} \quad \text{and} \quad \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} = \overline{\mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}}}, \tag{96}$$

which also hold for $\mathbf{z}$ and $\bar{\mathbf{z}}$ replaced by $\bar{\mathbf{z}}$ and $\mathbf{z}$ respectively. It is also straightforward to show that

$$\mathcal{H}^{\mathbb{C}}_{\mathbf{cc}} = S\mathcal{H}^{\mathbb{C}}_{\bar{\mathbf{c}}\bar{\mathbf{c}}}S = S\overline{\mathcal{H}^{\mathbb{C}}_{\mathbf{cc}}}\,S,$$

for $S = S^T = S^{-1}$ (showing that $\mathcal{H}^{\mathbb{C}}_{\mathbf{cc}}$ and $\mathcal{H}^{\mathbb{C}}_{\bar{\mathbf{c}}\bar{\mathbf{c}}}$ are related by a similarity transformation and therefore share the same eigenvalues[63]), which is precisely the necessary and sufficient condition (78) that the matrix $\mathcal{H}^{\mathbb{C}}_{\mathbf{cc}} \in \mathcal{L}(\mathcal{C}, \mathcal{C})$. This verifies that the term $\Delta\mathbf{c}^H\mathcal{H}^{\mathbb{C}}_{\mathbf{cc}}\Delta\mathbf{c}$ is admissible and

---

[63]Their eigenvectors are complex conjugates of each other, as reflected in the similarity transformation being given by the swap operator S

provides the completion of the proof of the validity of (92) promised earlier. Finally, note that properties (96) and (95) yield the *conjugate symmetry conditions,*

$$\mathcal{H}_{\mathbf{zz}} = \mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^T \quad \text{and} \quad \mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}} = \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}}^T \,, \tag{97}$$

which also hold for $\mathbf{z}$ and $\bar{\mathbf{z}}$ replaced by $\bar{\mathbf{z}}$ and $\mathbf{z}$ respectively.

From equations (66), (91), and (94) we can now expand the second order term in (73) as follows

$$
\begin{aligned}
\frac{1}{2} \Delta \mathbf{r}^T \mathcal{H}_{\mathbf{rr}} \Delta \mathbf{r} &= \frac{1}{2} \Delta \mathbf{c}^H \mathcal{H}_{\mathbf{cc}}^{\mathbb{C}} \Delta \mathbf{c} \\
&= \frac{1}{2} \left(\Delta \mathbf{z}^H \mathcal{H}_{\mathbf{zz}} \Delta \mathbf{z} + \Delta \mathbf{z}^H \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} \Delta \bar{\mathbf{z}} + \Delta \bar{\mathbf{z}}^H \mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}} \Delta \mathbf{z} + \Delta \bar{\mathbf{z}}^H \mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}} \Delta \bar{\mathbf{z}} \right) \\
&= \mathrm{Re} \left\{ \Delta \mathbf{z}^H \mathcal{H}_{\mathbf{zz}} \Delta \mathbf{z} + \Delta \mathbf{z}^H \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} \Delta \bar{\mathbf{z}} \right\}
\end{aligned}
$$

where the last step follows as a consequence of (96).[64] Thus, we have so-far determined that

$$\frac{1}{2} \Delta \mathbf{r}^T \mathcal{H}_{\mathbf{rr}} \Delta \mathbf{r} = \frac{1}{2} \Delta \mathbf{c}^H \mathcal{H}_{\mathbf{cc}}^{\mathbb{C}} \Delta \mathbf{c} = \mathrm{Re} \left\{ \Delta \mathbf{z}^H \mathcal{H}_{\mathbf{zz}} \Delta \mathbf{z} + \Delta \mathbf{z}^H \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} \Delta \bar{\mathbf{z}} \right\} . \tag{98}$$

Combining the results given in (73), (86), and (98) yields the desired expression for the second order expansion of f in terms of $\mathbf{z}$,

$$\boxed{2^{nd}\text{-Order Exp. in } \mathbf{z}: \quad f(\mathbf{z} + \Delta \mathbf{z}) = f(\mathbf{z}) + 2 \, \mathrm{Re} \left\{ \frac{\partial f}{\partial \mathbf{z}} \Delta \mathbf{z} \right\} + \mathrm{Re} \left\{ \Delta \mathbf{z}^H \mathcal{H}_{\mathbf{zz}} \Delta \mathbf{z} + \Delta \mathbf{z}^H \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} \Delta \bar{\mathbf{z}} \right\} + \text{h.o.t.}} \tag{99}$$

We note in passing that Equation (99) is exactly the same expression given as Equation (A.7) of reference [38] and Equation (8) of reference [34], which were both derived via an alternative procedure.

The $\mathbf{c}$-complex expansion shown in Equation (92) is one of two possible alternative second-order representations in $\mathbf{c}$ for $f(\mathbf{c})$ (the other being the $\mathbf{c}$-real expansion), and was used as the starting point of the theoretical developments leading to the $\mathbf{z}$-expansion (99). We now turn to the development of the $\mathbf{c}$-real expansion of $f(\mathbf{c})$, which will be accomplished by writing the second order term of the quadratic expansion in terms of the $\mathbf{c}$-real Hessian $\mathcal{H}_{\mathbf{cc}}^{\mathbb{R}}$.

From the definitions (88), (87), and (93), and using the fact that $\frac{\partial}{\partial \mathbf{c}} = (\frac{\partial}{\partial \mathbf{z}}, \frac{\partial}{\partial \bar{\mathbf{z}}})$, it is straightforward to show that

$$\mathcal{H}_{\mathbf{cc}}^{\mathbb{R}} = \begin{pmatrix} \mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}} & \mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}} \\ \mathcal{H}_{\mathbf{zz}} & \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} \end{pmatrix} = S \begin{pmatrix} \mathcal{H}_{\mathbf{zz}} & \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} \\ \mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}} & \mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}} \end{pmatrix} \tag{100}$$

or[65]

$$\mathcal{H}_{\mathbf{cc}}^{\mathbb{R}} = \mathcal{H}_{\mathbf{c}\bar{\mathbf{c}}}^{\mathbb{C}} = S \mathcal{H}_{\mathbf{cc}}^{\mathbb{C}} = \mathcal{H}_{\bar{\mathbf{c}}\bar{\mathbf{c}}}^{\mathbb{C}} S. \tag{101}$$

---

[64] Alternatively, the last step also follows as a consequence of (95).

[65] Alternative derivations are possible. For example, $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}} = \frac{\partial}{\partial \mathbf{c}} \left(\frac{\partial f}{\partial \mathbf{c}} \right)^H = \frac{\partial}{\partial \mathbf{c}} \left(\frac{\partial f}{\partial \bar{\mathbf{c}}} \right)^T = \frac{\partial}{\partial \mathbf{c}} \left(\frac{\partial f}{\partial \mathbf{c}} S \right)^T = \frac{\partial}{\partial \mathbf{c}} S \left(\frac{\partial f}{\partial \mathbf{c}} \right)^T = S \frac{\partial}{\partial \mathbf{c}} \left(\frac{\partial f}{\partial \mathbf{c}} \right)^T = S \mathcal{H}_{\mathbf{cc}}^{\mathbb{R}} \Rightarrow \mathcal{H}_{\mathbf{cc}}^{\mathbb{R}} = S \mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$, noting that $S = S^T = S^{-1}$.

Note from the first equality in (100) and the conjugate symmetry conditions (97) that the **c**-real Hessian is *symmetric*

$$\mathcal{H}_{\mathbf{cc}}^{\mathbb{R}} = (\mathcal{H}_{\mathbf{cc}}^{\mathbb{R}})^T . \tag{102}$$

Let the SVD of $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$ be

$$\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}} = U \Sigma V^H$$

then from (101) the SVD of $\mathcal{H}_{\mathbf{cc}}^{\mathbb{R}}$ is given by

$$\mathcal{H}_{\mathbf{cc}}^{\mathbb{R}} = U' \Sigma V^H , \quad U' = SU$$

showing that $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$ and $\mathcal{H}_{\mathbf{cc}}^{\mathbb{R}}$ share the same singular values, and hence the same condition number (which is given by the ratio of the largest to smallest singular value). The three Hessian matrices $\mathcal{H}_{\mathbf{rr}}$, $\mathcal{H}_{\mathbf{cc}}^{\mathbb{R}}$, and $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$ are essentially equivalent for investigating numerical issues and for testing whether a proposed minimizer of the second order expansion of $f(\mathbf{r}) = f(\mathbf{c})$ is a local (or even global) minimum. Thus, one can choose to work with the Hessian matrix which is easiest to compute and analyze. This is usually the **c**-complex Hessian $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$, and it is often most convenient to determine numerical stability and optimality using $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$ even when the algorithm is being developed from one of the alternative perspectives (i.e., the real **r** or the **c**-real second order expansion).

Now note that from (101) we immediately and easily have

$$\frac{1}{2} \Delta \mathbf{c}^T \, \mathcal{H}_{\mathbf{cc}}^{\mathbb{R}} \, \Delta \mathbf{c} = \frac{1}{2} \Delta \mathbf{c}^T \, S \, \mathcal{H}_{\mathbf{cc}}^{\mathbb{C}} \, \Delta \mathbf{c} = \frac{1}{2} (S \Delta \mathbf{c})^T \, \mathcal{H}_{\mathbf{cc}}^{\mathbb{C}} \, \Delta \mathbf{c} = \frac{1}{2} \overline{(\Delta \mathbf{c})}^T \, \mathcal{H}_{\mathbf{cc}}^{\mathbb{C}} \, \Delta \mathbf{c} = \frac{1}{2} \Delta \mathbf{c}^H \, \mathcal{H}_{\mathbf{cc}}^{\mathbb{C}} \, \Delta \mathbf{c}$$

showing the equivalence of the **c**-real and **c**-complex second order terms in the expansion of $f(\mathbf{c})$.[66] Combining this result with (98), we have shown the following equivalences between the second order terms in the various expansions of f under consideration in this note:

$$\boxed{\text{2nd-Order Terms: } \frac{1}{2} \Delta \mathbf{r}^T \, \mathcal{H}_{\mathbf{rr}} \, \Delta \mathbf{r} = \frac{1}{2} \Delta \mathbf{c}^T \, \mathcal{H}_{\mathbf{cc}}^{\mathbb{R}} \, \Delta \mathbf{c} = \frac{1}{2} \Delta \mathbf{c}^H \, \mathcal{H}_{\mathbf{cc}}^{\mathbb{C}} \, \Delta \mathbf{c} = \mathrm{Re} \left\{ \Delta \mathbf{z}^H \mathcal{H}_{\mathbf{zz}} \Delta \mathbf{z} + \Delta \mathbf{z}^H \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} \Delta \bar{\mathbf{z}} \right\}}$$

$$\tag{103}$$

where the second order expansion in **r** is given by (73), the **c**-complex expansion by (92), the expansion in terms of **z** by (99), and the **c**-real expansion by

$$\boxed{\text{2nd-Order Expansion in c-Real: } f(\mathbf{c} + \Delta \mathbf{c}) = f(\mathbf{c}) + \frac{\partial f(\mathbf{c})}{\partial \mathbf{c}} \Delta \mathbf{c} + \frac{1}{2} \Delta \mathbf{c}^T \, \mathcal{H}_{\mathbf{cc}}^{\mathbb{R}}(\mathbf{c}) \, \Delta \mathbf{c} + \text{h.o.t.}}$$

$$\tag{104}$$

Note that all of the terms in (103) and (104) are real valued.

The expansion in of $f(\mathbf{c})$ in terms of **c**-*complex* shown in (92) is *not* differentiable with respect to **c** (this is only true for the **c**-*real* expansion). However, (92) is differentiable with respect to **z** and $\bar{\mathbf{z}}$ and can be viewed as a short-hand equivalent to the full $(\mathbf{z}, \bar{\mathbf{z}})$ expansion provided by (99). Therefore, it is Equation (99) which is the natural form for optimization with respect to **c**-complex

---

[66]One can show that the term $\Delta \mathbf{c}^T \, \mathcal{H}_{\mathbf{cc}}^{\mathbb{R}} \, \Delta \mathbf{c}$ is admissible if and only if $\mathcal{H}_{\mathbf{cc}}^{\mathbb{R}} = SM$ for $M \in \mathcal{L}(\mathcal{C}, \mathcal{C})$, which is the case here.

via a derivative-based approach, because only differentiation with respect to the components $(\mathbf{z}, \bar{\mathbf{z}})$ of $\mathbf{c}$-complex is well-posed. On the other hand, differentiation with respect to $\mathbf{c}$-real is well-posed, so that one can optimize (104) by taking derivatives of (104) with respect to $\mathbf{c}$-real itself.

Note that (73), (92), and (104) are the natural forms to use for optimization via "completing the square"(see below). This is because the expansions in terms of $\mathbf{r}$, $\mathbf{c}$-complex, and $\mathbf{c}$-real are less awkward for completing-the-square purposes than the expansion in $\mathbf{z}$ provided by (99).[67] Note that the expansions (73) and (92) are both differentiable with respect to the expansion variable itself and both have a form amenable to optimization by completing the square.

The various second order expansions developed above can be found in references [38], [27] and [34]. In [27], Van den Bos shows the equality of the first, second, and third second-order terms shown in equation (98) but does not mention the fourth (which, anyway, naturally follows from the third term in (98) via a simple further expansion in terms of $\mathbf{z}$ and $\bar{\mathbf{z}}$). The approach used in this note is a more detailed elaboration of the derivations presented in [27]. In reference [34] Yan and Fan show the equality of the first and last terms in (98), but, while they cite the results of Van den Bos [27] regarding the middle terms in (98), do not appear to have appreciated that the fourth term in (98) is an *immediate consequence* of the second or third terms, and derive it from scratch using an alternative, "brute force" approach.

Quadratic Minimization and the Newton Algorithm.   The Newton algorithm for minimizing a scalar function $f(z)$ exploits the fact that it is generally straightforward to minimize the quadratic approximations provided by second order expansions such as (73), (92), (99), and (104). The Newton method starts with an initial estimate of the optimal solution, say $\hat{\mathbf{c}}$, then expands $f(\mathbf{c})$ about the estimate $\hat{\mathbf{c}}$ to second order in $\Delta \mathbf{c} = \mathbf{c} - \hat{\mathbf{c}}$, and then minimizes the resulting second order approximation of $f(\mathbf{c})$ with respect to $\Delta \mathbf{c}$. Having determined an estimated update $\widehat{\Delta \mathbf{c}}$ in this manner, one updates the original estimate $\hat{\mathbf{c}} \leftarrow \hat{\mathbf{c}} + \alpha \widehat{\Delta \mathbf{c}}$, for some small "stepsize" $\alpha > 0$, and then starts the optimization cycle all over again. For appropriate choices of the stepsize α, this iterative approximate quadratic optimization algorithm can result in a sequence of estimates $\hat{\mathbf{c}}_0$, $\hat{\mathbf{c}}_1$, $\hat{\mathbf{c}}_2$, $\cdots$, which converges to the true optimal solution.

Note that the optimal solution to the quadratic approximations provided by (73), (92), and (104) can be *immediately* written down using the "completing-the-square" solution developed in Equations (5)-(7) of [24], assuming that the relevant Hessians are all invertible:

$$\widehat{\Delta \mathbf{r}} \;=\; -\left(\mathcal{H}_{\mathbf{rr}}\right)^{-1}\left(\frac{\partial f(\mathbf{r})}{\partial \mathbf{r}}\right)^{T} \qquad \text{(from the } \mathbf{r} \text{ expansion (73))} \tag{105}$$

$$\widehat{\Delta \mathbf{c}^{\mathbb{C}}} \;=\; -\left(\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}\right)^{-1}\left(\frac{\partial f(\mathbf{c})}{\partial \mathbf{c}}\right)^{H} \qquad \text{(from the } \mathbf{c}\text{-complex expansion (92))} \tag{106}$$

$$\widehat{\Delta \mathbf{c}^{\mathbb{R}}} \;=\; -\left(\mathcal{H}_{\mathbf{cc}}^{\mathbb{R}}\right)^{-1}\left(\frac{\partial f(\mathbf{c})}{\partial \mathbf{c}}\right)^{T} \qquad \text{(from the } \mathbf{c}\text{-real expansion (104))} . \tag{107}$$

---

[67]Although (99) can also be optimized by completing the square.

Solutions (105) and (106) can also be found in Van den Bos [27]. Note that $\widehat{\Delta c^{\mathbb{C}}}$ is an *admissible* solution, i.e., that

$$\widehat{\Delta c^{\mathbb{C}}} \in \mathcal{C}$$

as required for self-consistency of our theory, as a consequence of the fact that $\left(\frac{\partial f(\mathbf{c})}{\partial \mathbf{c}}\right)^H$ and $(\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}})^{-1}$ satisfy

$$\left(\frac{\partial f(\mathbf{c})}{\partial \mathbf{c}}\right)^H \in \mathcal{C} \quad \text{and} \quad (\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}})^{-1} \in \mathcal{L}(\mathcal{C}, \mathcal{C}),$$

with the latter condition a consequence of property (82) and the fact that $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}} \in \mathcal{L}(\mathcal{C}, \mathcal{C})$. If this were not the case, then we generally would have the *meaningless* answer that $\widehat{\Delta c^{\mathbb{C}}} \notin \mathcal{C}$.

The admissibility of the solution (107) follows from the admissibility of (106). This will be evident from the fact, as we shall show, that all of the solutions (105)-(107) must all correspond to the *same update,*

$$\widehat{\Delta c^{\mathbb{C}}} = \widehat{\Delta c^{\mathbb{R}}} = \mathsf{J}\widehat{\Delta r}.$$

Note that

$$
\begin{aligned}
\widehat{\Delta c^{\mathbb{C}}} &= -(\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}})^{-1}\left(\frac{\partial f(\mathbf{c})}{\partial \mathbf{c}}\right)^H \\
&= -\left(\frac{1}{4}\mathsf{J}\mathcal{H}_{\mathbf{rr}}\mathsf{J}^H\right)^{-1}\left(\frac{1}{2}\frac{\partial f(\mathbf{r})}{\partial \mathbf{r}}\mathsf{J}^H\right)^H \qquad \text{(from (67) and (90))} \\
&= \\
&= -\left(\mathsf{J}\mathcal{H}_{\mathbf{rr}}\mathsf{J}^{-1}\right)^{-1}\mathsf{J}\left(\frac{\partial f(\mathbf{r})}{\partial \mathbf{r}}\right)^T \qquad \text{(from (63))} \\
&= -\mathsf{J}(\mathcal{H}_{\mathbf{rr}})^{-1}\left(\frac{\partial f(\mathbf{r})}{\partial \mathbf{r}}\right)^T \\
&= \mathsf{J}\widehat{\Delta r}
\end{aligned}
$$

as required. On the other hand,

$$
\begin{aligned}
\widehat{\Delta c^{\mathbb{R}}} &= -(\mathcal{H}_{\mathbf{cc}}^{\mathbb{R}})^{-1}\left(\frac{\partial f(\mathbf{c})}{\partial \mathbf{c}}\right)^T \\
&= -(S\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}})^{-1}\left(\frac{\partial f(\mathbf{c})}{\partial \mathbf{c}}\right)^T \qquad \text{(from (101))} \\
&= -(\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}})^{-1}\left(\frac{\partial f(\mathbf{c})}{\partial \mathbf{c}}S\right)^T \\
&= -(\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}})^{-1}\left(\frac{\partial f(\mathbf{c})}{\partial \overline{\mathbf{c}}}\right)^T \\
&= -(\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}})^{-1}\left(\frac{\partial f(\mathbf{c})}{\partial \mathbf{c}}\right)^H \\
&= \widehat{\Delta c^{\mathbb{C}}}.
\end{aligned}
$$

Thus, the updates (105)-(107) are indeed equivalent.

The updates (105) and (107), determined via a completing the square argument, can alternatively be obtained by setting the (*real*) derivatives of their respective quadratically-approximated loss functions to zero, and solving the necessary condition for an optimum. Note that if we attempt to (erroneously) take the (*complex*) derivative of (92) with respect to $\mathbf{c}$-complex and then set this expression to zero, the resulting "solution" will be off by a factor of two. In the latter case, we must instead take the derivatives of (99) with respect to $\mathbf{z}$ and $\bar{\mathbf{z}}$ and set the resulting expressions to zero in order to obtain the optimal solution.[68]

At convergence, the Newton algorithm will produce a solution to the necessary first-order condition

$$\frac{\partial f(\hat{\mathbf{c}})}{\partial \mathbf{c}} = 0 \,,$$

and this point will be a local minimum of $f(\cdot)$ if the Hessians are strictly positive definite at this point. Typically, one would verify positive definiteness of the $\mathbf{c}$-complex Hessian at the solution point $\hat{\mathbf{c}}$,

$$\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}(\hat{\mathbf{c}}) = \begin{pmatrix} \mathcal{H}_{\mathbf{zz}}(\hat{\mathbf{c}}) & \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}}(\hat{\mathbf{c}}) \\ \mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}}(\hat{\mathbf{c}}) & \mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}(\hat{\mathbf{c}}) \end{pmatrix} > 0 \,.$$

As done in [38] and [34], the solution to the quadratic minimization problem provided by (105)-(107) can be expressed in a closed form expression which directly produces the solution $\hat{\mathbf{z}} \in \mathbb{C}^n$. To do so, we rewrite the solution (106) for the Newton update $\widehat{\Delta \mathbf{c}}$ as

$$\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}} \, \widehat{\Delta \mathbf{c}} = -\left(\frac{\partial f(\mathbf{c})}{\partial \mathbf{c}}\right)^H$$

which we then write in expanded form in terms of $\mathbf{z}$ and $\bar{\mathbf{z}}$

$$\begin{pmatrix} \mathcal{H}_{\mathbf{zz}} & \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} \\ \mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}} & \mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}} \end{pmatrix} \begin{pmatrix} \widehat{\Delta \mathbf{z}} \\ \widehat{\Delta \bar{\mathbf{z}}} \end{pmatrix} = -\begin{pmatrix} \left(\frac{\partial f}{\partial \mathbf{z}}\right)^H \\ \left(\frac{\partial f}{\partial \bar{\mathbf{z}}}\right)^H \end{pmatrix} \,. \tag{108}$$

Assuming that $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$ is positive definite, then $\mathcal{H}_{\mathbf{zz}}$ is invertible and the second block row in (108) results

$$\widehat{\Delta \bar{\mathbf{z}}} = -\mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{-1} \mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}} \widehat{\Delta \mathbf{z}} - \mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{-1} \left(\frac{\partial f}{\partial \bar{\mathbf{z}}}\right)^H \,.$$

Plugging this into the first block row of (108) then yields the Newton algorithm update equation

$$\widetilde{\mathcal{H}_{\mathbf{zz}}} \, \widehat{\Delta \mathbf{z}} = -\left(\frac{\partial f}{\partial \mathbf{z}}\right)^H + \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} \mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{-1} \left(\frac{\partial f}{\partial \bar{\mathbf{z}}}\right)^H \,, \tag{109}$$

where

$$\widetilde{\mathcal{H}_{\mathbf{zz}}} \triangleq \mathcal{H}_{\mathbf{zz}} - \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} \mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{-1} \mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}}$$

---

[68]This is the procedure used in [38] and [34].

is the *Schur complement* of $\mathcal{H}_{\mathbf{zz}}$ in $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$. Equation (109) is equivalent to the solution given as Equation (A.12) in [38]. Invertibility of the Schur complement $\widetilde{\mathcal{H}_{\mathbf{zz}}}$ follows from our assumption that $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$ is positive definite, and the Newton update is therefore given by

$$\widehat{\Delta \mathbf{z}} = \left(\mathcal{H}_{\mathbf{zz}} - \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}}\mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{-1}\mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}}\right)^{-1}\left\{\mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}}\mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{-1}\left(\frac{\partial f}{\partial \bar{\mathbf{z}}}\right)^{H} - \left(\frac{\partial f}{\partial \mathbf{z}}\right)^{H}\right\}. \tag{110}$$

The matrices $\mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}$ and $\widetilde{\mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}} = \left(\mathcal{H}_{\mathbf{zz}} - \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}}\mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{-1}\mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}}\right)$ in (109) are invertible if and only if $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$ is invertible. Note that invertibility of $\mathcal{H}_{\mathbf{zz}}$ (equivalently, $\mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}} = \overline{\mathcal{H}_{\mathbf{zz}}}$) is not a sufficient condition for the Schur complement to be nonsingular. However, if $\mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} = \overline{\mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}}} = 0$ then invertibility of $\mathcal{H}_{\mathbf{zz}}$ is a necessary and sufficient condition for a solution $\widehat{\Delta \mathbf{z}}$ to exist.

As noted by Yan & Fan [34], the need to guarantee positive definiteness of the Schur complement $\widetilde{\mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}} = \left(\mathcal{H}_{\mathbf{zz}} - \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}}\mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{-1}\mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}}\right)$ is a significant computational burden for an on-line adaptive filtering algorithm to bear. For this reason, to improve the numerical robustness of the Newton algorithm and to provide a substantial simplification, they suggest making the approximation that the block off-diagonal elements of $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$ are zero

$$\mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} = \overline{\mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}}} \approx 0$$

which results in the simpler approximate solution

$$\widehat{\Delta \mathbf{z}} \approx -\mathcal{H}_{\mathbf{zz}}^{-1}\left(\frac{\partial f}{\partial \mathbf{z}}\right)^{H}. \tag{111}$$

The argument given by Yan and Fan supporting the use of the approximation $\mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} \approx 0$ is that as the Newton algorithm converges to the optimal solution $\hat{z} = z_0$, setting $\mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}}$ "to zero implies that we will use a quadratic function to approximate the cost near $\mathbf{z}_0$" [34]. However Yan and Fan do not give a formal definition of a "quadratic function" and this statement is *not generally* true as there is no a priori reason why the off-diagonal block matrix elements of the Newton Hessian should be zero, or approach zero, as we demonstrate in Example 2 of the Applications section below.

However, as we shall discuss later below, setting the block off-diagonal elements to zero *is* justifiable, but not necessarily as an *approximation* to the Newton algorithm. Setting the block off-diagonal elements in the Newton Hessian to zero, results in an *alternative*, "quasi-Newton" algorithm which can be studied in its own right as a competitor algorithm to the Newton algorithm, the Gauss-Newton algorithm, or the gradient descent algorithm.[69]

Nonlinear Least-Squares: Gauss vs. Newton.   In this section we are interested in finding an approximate solution, $\hat{\mathbf{z}}$, to the nonlinear inverse problem

$$\mathbf{g}(\mathbf{z}) \approx y$$

---

[69]That is not to say that there can't be conditions under which the quasi-Newton algorithm does converge to the Newton algorithm. Just as one can give conditions for which the Gauss-Newton algorithm converges to the Newton algorithm, one should be able to do the same for the quasi-Newton algorithm.

for known $y \in \mathbb{C}^m$ and known real-analytic function $\mathbf{g} : \mathbb{C}^n \to \mathbb{C}^m$. We desire a least-squares solution, which is a solution that minimizes the weighted least-squares loss function[70]

$$\ell(\mathbf{z}) = \frac{1}{2} \, \|\mathbf{y} - \mathbf{g}(\mathbf{z})\|_W^2 = \frac{1}{2} \, (\mathbf{y} - \mathbf{g}(\mathbf{z}))^H \, W \, (\mathbf{y} - \mathbf{g}(\mathbf{z}))$$

where W is a Hermitian positive-definite weighting matrix. Although the nonlinear function $\mathbf{g}$ is assumed to be real-analytic, in general it is assumed to be *not* holomorphic (i.e., $\mathbf{g}$ is *not analytic in* $\mathbf{z}$).

In the subsequent development we will analyze the problem using the **c**-real perspective developed in the preceding discussions. Thus, the loss function is assumed to be re-expressible in terms of $\mathbf{c}$,[71]

$$\ell(\mathbf{c}) = \frac{1}{2} \, \|\mathbf{y} - \mathbf{g}(\mathbf{c})\|_W^2 = \frac{1}{2} \, (\mathbf{y} - \mathbf{g}(\mathbf{c}))^H \, W \, (\mathbf{y} - \mathbf{g}(\mathbf{c})) \,. \tag{112}$$

Quantities produced from this perspective[72] may have a different functional form than those produced purely within the $\mathbf{z} \in \mathcal{Z}$ perspective, but the end results will be the same.

We will consider two iterative algorithms for minimizing the loss function (112): The Newton algorithm, discussed above, and the Gauss-Newton algorithm which is usually a somewhat simpler, yet related, method for iteratively finding a solution which minimizes a least-squares function of the form (112).[73]

As discussed earlier, the Newton method is based on an iterative quadratic expansion and minimization of the loss function $\ell(\mathbf{z})$ about a current solution estimation, $\hat{\mathbf{z}}$. Specifically the Newton method minimizes an approximation to $\ell(\mathbf{c}) = \ell(\mathbf{z})$ based on the second order expansion of $\ell(\mathbf{c})$ in $\Delta \mathbf{c}$ about a current solution estimate $\hat{\mathbf{c}} = \mathrm{col}(\hat{\mathbf{z}}, \hat{\bar{\mathbf{z}}})$,

$$\ell(\hat{\mathbf{c}} + \Delta \mathbf{c}) \approx \hat{\ell}(\Delta \mathbf{c})^{\mathrm{Newton}}$$

where

$$\hat{\ell}(\Delta \mathbf{c})^{\mathrm{Newton}} = \ell(\hat{\mathbf{c}}) + \frac{\partial \ell(\hat{\mathbf{c}})}{\partial \mathbf{c}} \, \Delta \mathbf{c} + \frac{1}{2} \Delta \mathbf{c}^H \, \mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}(\hat{\mathbf{c}}) \, \Delta \mathbf{c}. \tag{113}$$

Minimizing the Newton loss function $\hat{\ell}(\Delta \mathbf{c})^{\mathrm{Newton}}$ then results in a correction $\widehat{\Delta \mathbf{c}}^{\mathrm{Newton}}$ which is then used to update the estimate $\hat{\mathbf{c}} \leftarrow \hat{\mathbf{c}} + \alpha \widehat{\Delta \mathbf{c}}^{\mathrm{Newton}}$ for some stepsize $\alpha > 0$. The algorithm then

---

[70]The factor of $\frac{1}{2}$ has been included for notational convenience in the ensuing derivations. If it is removed, some of the intermediate quantities derived subsequently (such as Hessians, etc.) will differ by a factor of 2, although the ultimate answer is independent of any overall constant factor of the loss function. If in your own problem solving ventures, your intermediate quantities appear to be off by a factor of 2 relative to the results given in this note, you should check whether your loss function does or does not have this factor.

[71]Quantities produced from this perspective–such as the Gauss-Newton Hessian to be discussed below–may have a different functional form than those produced purely within the $\mathbf{z} \in \mathcal{Z}$ perspective, but the final answers are the same.

[72]Such as the Gauss-Newton Hessian to be discussed below.

[73]Thus the Newton algorithm is a *general method* that can be used to minimize a variety of different loss functions, while the Gauss-Newton algorithm is a *least-squares estimation method* which is specific to the problem of minimizing the least-squares loss function (112).

starts all over again. As discussed earlier in this note, a "completing-the-square" argument can be invoked to readily show that the correction which minimizes the quadratic Newton loss function is given by

$$\widehat{\Delta \mathbf{c}}^{\text{Newton}} = -\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}(\hat{\mathbf{c}})^{-1} \left(\frac{\partial \ell(\hat{\mathbf{c}})}{\partial \mathbf{c}} \right)^{H} \tag{114}$$

provided that the **c**-complex Hessian $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}(\hat{\mathbf{c}})$ is invertible. Because it defines the second-order term in the Newton loss function and directly enters into the Newton correction, we will often refer to $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}(\hat{\mathbf{c}})$ as the *Newton Hessian.* If we block partition the Newton Hessian and solve for the correction $\widehat{\Delta \mathbf{z}}^{\text{Newton}}$, we obtain the solution (110) which we earlier derived for a more general (possibly non-quadratic).

We now determine the form of the cogradient $\frac{\partial \ell(\hat{\mathbf{c}})}{\partial \mathbf{c}}$ of the least-squares loss function (112). This is done by utilizing the **c**-real perspective which allows us to take (real) cogradients with respect to **c**-real. First, however, it is convenient to define the compound Jacobian of $\mathbf{g}(\hat{\mathbf{c}})$ as

$$G(\hat{\mathbf{c}}) \triangleq \frac{\partial \mathbf{g}(\hat{\mathbf{c}})}{\partial \mathbf{c}} \triangleq \left(\frac{\partial \mathbf{g}(\hat{\mathbf{z}})}{\partial \mathbf{z}} \quad \frac{\partial \mathbf{g}(\hat{\mathbf{z}})}{\partial \bar{\mathbf{z}}} \right) = \left(J_{\mathbf{g}}(\mathbf{c}) \quad J_{\mathbf{g}}^{c}(\mathbf{c}) \right) \in \mathbb{C}^{m \times 2n}. \tag{115}$$

Setting $\mathbf{e} = \mathbf{y} - \mathbf{g}(\mathbf{c})$, we have[74]

$$\begin{aligned}
\frac{\partial \ell}{\partial \mathbf{c}} &= \frac{1}{2} \frac{\partial}{\partial \mathbf{c}} \mathbf{e}^{H} W \mathbf{e} \\
&= \frac{1}{2} \mathbf{e}^{H} W \frac{\partial}{\partial \mathbf{c}} \mathbf{e} + \frac{1}{2} \mathbf{e}^{T} W^{T} \frac{\partial}{\partial \mathbf{c}} \bar{\mathbf{e}} \\
&= -\frac{1}{2} \mathbf{e}^{H} W \frac{\partial \mathbf{g}}{\partial \mathbf{c}} - \frac{1}{2} \mathbf{e}^{T} W^{T} \frac{\partial \bar{\mathbf{g}}}{\partial \mathbf{c}} \\
&= -\frac{1}{2} \mathbf{e}^{H} W G - \frac{1}{2} \mathbf{e}^{T} W^{T} \overline{\left(\frac{\partial \mathbf{g}}{\partial \mathbf{c}} S \right)} \\
&= -\frac{1}{2} \mathbf{e}^{H} W G - \frac{1}{2} \mathbf{e}^{T} W^{T} \overline{G} S
\end{aligned}$$

or

$$\frac{\partial \ell}{\partial \mathbf{c}} = -\frac{1}{2} \mathbf{e}^{H} W G - \frac{1}{2} \overline{\mathbf{e}^{H} W G} S. \tag{116}$$

This expression for $\frac{\partial \ell}{\partial \mathbf{c}}$ is admissible, as required, as it is readily verified that

$$\overline{\left(\frac{\partial \ell}{\partial \mathbf{c}} \right)^{H}} = S \left(\frac{\partial \ell}{\partial \mathbf{c}} \right)^{H}$$

as per the requirement given in (76).

---

[74]Remember that $\frac{\partial}{\partial \mathbf{c}}$ is only well-defined as a derivative within the **c**-real framework.

The linear term in the Newton loss function $\hat{\ell}^{\text{Newton}}$ is therefore given by

$$
\begin{aligned}
\frac{\partial \ell}{\partial \mathbf{c}} \Delta \mathbf{c} &= -\frac{1}{2} \mathbf{e}^H W\, G\, \Delta \mathbf{c} - \frac{1}{2} \overline{\mathbf{e}^H W\, G S\, \Delta \mathbf{c}} \\
&= -\frac{1}{2} \mathbf{e}^H W\, G\, \Delta \mathbf{c} - \frac{1}{2} \overline{\mathbf{e}^H W\, G\, \Delta \mathbf{c}} \\
&= -\operatorname{Re} \left\{ \mathbf{e}^H W\, G\, \Delta \mathbf{c} \right\}.
\end{aligned}
$$

Thus

$$
\frac{\partial \ell}{\partial \mathbf{c}} \Delta \mathbf{c} = -\operatorname{Re} \left\{ \mathbf{e}^H W\, G\, \Delta \mathbf{c} \right\} = -\operatorname{Re} \left\{ (\mathbf{y} - \mathbf{g}(\mathbf{c}))^H W\, G\, \Delta \mathbf{c} \right\}. \tag{117}
$$

If the reader has any doubts as to the validity or correctness of this derivation, she/he is invited to show that the left-hand side of (117) is equal to $2 \operatorname{Re} \left\{ \frac{\partial f}{\partial \mathbf{z}} \Delta \mathbf{z} \right\}$ as expected from equation (86).

Before continuing on to determine the functional form of the **c**-complex Hessian $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}(\hat{\mathbf{c}})$ needed to form the Newton loss function and solution, we turn first to a discussion of the Gauss-Newton algorithm.

Whereas the Newton method is based on an iterative quadratic expansion and minimization of the loss function $\ell(\mathbf{z})$ about a current solution estimation, $\hat{\mathbf{z}}$, The Gauss-Newton method is based on iterative "relinearization" of the system equations $\mathbf{y} \approx \mathbf{g}(\mathbf{z})$ about the current estimate, $\hat{\mathbf{z}}$ and minimization of the resulting approximate least-squares problem. We put "linearization" in quotes because (unless the function $\mathbf{g}$ happens to be holomorphic) generally we are not linearizing $\mathbf{g}$ with respect to z but, rather, we are linearizing with respect to $\mathbf{c} = \operatorname{col}(\mathbf{z}, \bar{\mathbf{z}})$.

Expanding the system equations $\mathbf{y} \approx \mathbf{g}(\mathbf{z})$ about a current estimate $\hat{\mathbf{z}}$, we have

$$
\mathbf{y} - \mathbf{g}(\mathbf{z}) = \mathbf{y} - \mathbf{g}(\hat{\mathbf{z}} + \Delta \mathbf{z}) \approx \mathbf{y} - \left(\mathbf{g}(\hat{\mathbf{z}}) + \frac{\partial \mathbf{g}(\hat{\mathbf{z}})}{\partial \mathbf{z}} \Delta \mathbf{z} + \frac{\partial \mathbf{g}(\hat{\mathbf{z}})}{\partial \bar{\mathbf{z}}} \Delta \bar{\mathbf{z}} \right)
$$

where $\Delta \mathbf{z} = \mathbf{z} - \hat{\mathbf{z}}$ and $\Delta \bar{\mathbf{z}} = \overline{\Delta \mathbf{z}} = \bar{\mathbf{z}} - \bar{\hat{\mathbf{z}}} = \bar{\mathbf{z}} - \hat{\bar{\mathbf{z}}}$. Note that the approximation to $\mathbf{g}$ is not a linear function of $\mathbf{z}$ as complex conjugation is a nonlinear operation on $\mathbf{z}$. However, if $\mathbf{g}$ is holomorphic, then $\frac{\partial \mathbf{g}}{\partial \bar{\mathbf{z}}} \equiv 0$, in which case the approximation is linear in $\mathbf{z}$. Although the approximation of $\mathbf{g}$ generally is not linear in $\mathbf{z}$, it is linear in $\mathbf{c} = \operatorname{col}(\mathbf{z}, \bar{\mathbf{z}})$, and we rewrite the approximation as

$$
\mathbf{y} - \mathbf{g}(\mathbf{c}) = \mathbf{y} - \mathbf{g}(\hat{\mathbf{c}} + \Delta \mathbf{c}) \approx \Delta \mathbf{y} - G(\hat{\mathbf{c}})\, \Delta \mathbf{c} \tag{118}
$$

where $\Delta \mathbf{y} = \mathbf{y} - \mathbf{g}(\hat{\mathbf{z}})$, $\hat{\mathbf{c}} = \operatorname{col}(\hat{\mathbf{z}}, \hat{\bar{\mathbf{z}}})$, $\Delta \mathbf{c} = \mathbf{c} - \hat{\mathbf{c}}$, and $G(\hat{\mathbf{c}})$ is the (compound) Jacobian mapping of $\mathbf{g}$ evaluated at the current estimate $\hat{\mathbf{c}}$ given in Equation (115). With this approximation, the loss function (112) is approximated by the following quadratic loss function (notationally suppressing the dependence on $\hat{\mathbf{c}}$),

$$
\ell(\mathbf{c}) = \ell(\hat{\mathbf{c}} + \Delta \mathbf{c}) \approx \hat{\ell}(\Delta \mathbf{c})^{\text{Gauss}}
$$

where

$$
\begin{aligned}
\hat{\ell}(\Delta\mathbf{c})^{\text{Gauss}} &= \frac{1}{2}\|\Delta\mathbf{y} - G\,\Delta\mathbf{c}\|_W^2 \\
&= \frac{1}{2}(\Delta\mathbf{y} - G\,\Delta\mathbf{c})^H\,W\,(\Delta\mathbf{y} - G\,\Delta\mathbf{c}) \\
&= \frac{1}{2}\|\Delta\mathbf{y}\|^2 - \text{Re}\left\{\Delta\mathbf{y}^H W\,G\,\Delta\mathbf{c}\right\} + \frac{1}{2}\Delta\mathbf{c}^H\,G^H W G\,\Delta\mathbf{c} \\
&= \ell(\hat{\mathbf{c}}) + \frac{\partial\ell(\hat{\mathbf{c}})}{\partial\mathbf{c}}\,\Delta\mathbf{c} + \frac{1}{2}\Delta\mathbf{c}^H\,G^H W G\,\Delta\mathbf{c}. \qquad \text{(from (117)}
\end{aligned}
$$

Unfortunately, the resulting quadratic form

$$
\hat{\ell}(\Delta\mathbf{c})^{\text{Gauss}} = \ell(\hat{\mathbf{c}}) + \frac{\partial\ell(\hat{\mathbf{c}})}{\partial\mathbf{c}}\,\Delta\mathbf{c} + \frac{1}{2}\Delta\mathbf{c}^H\,G^H W G\,\Delta\mathbf{c} \tag{119}
$$

is *not* admissible as it stands.[75] This is because the matrix $G^H W G$ is not admissible,

$$
G^H W G = \left(\frac{\partial\mathbf{g}}{\partial\mathbf{c}}\right)^H W\left(\frac{\partial\mathbf{g}}{\partial\mathbf{c}}\right) \notin \mathcal{L}(\mathcal{C},\mathcal{C}).
$$

This can be seen by showing that the condition (78) is violated:

$$
\begin{aligned}
S\,\overline{G^H W G}\,S &= S\,\overline{\left(\frac{\partial\mathbf{g}}{\partial\mathbf{c}}\right)^H W\left(\frac{\partial\mathbf{g}}{\partial\mathbf{c}}\right)}\,S \\
&= \overline{\left(\frac{\partial\mathbf{g}}{\partial\overline{\mathbf{c}}}\right)^H W\left(\frac{\partial\mathbf{g}}{\partial\overline{\mathbf{c}}}\right)} \\
&= \left(\frac{\partial\overline{\mathbf{g}}}{\partial\mathbf{c}}\right)^H \bar{W}\left(\frac{\partial\overline{\mathbf{g}}}{\partial\mathbf{c}}\right) \\
&\neq \left(\frac{\partial\mathbf{g}}{\partial\mathbf{c}}\right)^H W\left(\frac{\partial\mathbf{g}}{\partial\mathbf{c}}\right).
\end{aligned}
$$

Fortunately, we can rewrite the quadratic form (119) as an equivalent form which is admissible on $\mathcal{C}$. To do this note that $G^H W G$ is Hermitian, so that

$$
\Delta\mathbf{c}^H G^H W G\Delta\mathbf{c} = \overline{\Delta\mathbf{c}^H G^H W G\Delta\mathbf{c}} \in \mathbb{R}\,.
$$

---

[75] And thus the complex Gauss-Newton algorithm is more complicated in form than the real Gauss-Newton algorithm for which the quadratic form (119) is acceptable [25].

Also recall from Equation (79) that $\mathbf{P}(G^H W G) \in \mathcal{L}(\mathcal{C}, \mathcal{C})$ and $\Delta\mathbf{c} \in \mathcal{C} \Rightarrow S\Delta\mathbf{c} = \Delta\bar{\mathbf{c}}$. We have[76]

$$
\begin{aligned}
\Delta\mathbf{c}^H G^H W G \Delta\mathbf{c} &= \Delta\mathbf{c}^H \mathbf{P}(G^H W G)\Delta\mathbf{c} + \Delta\mathbf{c}^H \left(G^H W G - \mathbf{P}(G^H W G)\right)\Delta\mathbf{c} \\
&= \Delta\mathbf{c}^H \mathbf{P}(G^H W G)\Delta\mathbf{c} + \frac{1}{2}\Delta\mathbf{c}^H \left(G^H W G - S\overline{G^H W G}S\right)\Delta\mathbf{c} \\
&= \Delta\mathbf{c}^H \mathbf{P}(G^H W G)\Delta\mathbf{c} + \frac{1}{2}\left(\Delta\mathbf{c}^H G^H W G \Delta\mathbf{c} - \overline{\Delta\mathbf{c}^H G^H W G \Delta\mathbf{c}}\right) \\
&= \Delta\mathbf{c}^H \mathbf{P}(G^H W G)\Delta\mathbf{c} + 0 \\
&= \Delta\mathbf{c}^H \mathbf{P}(G^H W G)\Delta\mathbf{c}.
\end{aligned}
$$

Thus we have shown that on the space of admissible variations, $\Delta\mathbf{c} \in \mathcal{C}$, the inadmissible quadratic form (119) is equivalent to the admissible quadratic form

$$
\hat{\ell}(\Delta\mathbf{c})^{\text{Gauss}} = \ell(\hat{\mathbf{c}}) + \frac{\partial\ell(\hat{\mathbf{c}})}{\partial\mathbf{c}}\,\Delta\mathbf{c} + \frac{1}{2}\Delta\mathbf{c}^H \mathcal{H}_{\mathbf{cc}}^{\text{Gauss}}(\hat{\mathbf{c}})\,\Delta\mathbf{c} \tag{120}
$$

where

$$
\mathcal{H}_{\mathbf{cc}}^{\text{Gauss}}(\hat{\mathbf{c}}) \triangleq \mathbf{P}\left(G^H(\hat{\mathbf{c}})W G(\hat{\mathbf{c}})\right) \tag{121}
$$

denotes the *Gauss-Newton* Hessian.

Note that the Gauss-Newton Hessian $\mathcal{H}_{\mathbf{cc}}^{\text{Gauss}}(\hat{\mathbf{c}})$ is Hermitian and always guaranteed to be at least positive semi-definite, and guaranteed to be positive definite if $\mathbf{g}$ is assumed to be one-to-one (and thereby ensuring that the compound Jacobian matrix G has full column rank). This is in contrast to the Newton (i.e., the **c**-complex) Hessian $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}(\hat{\mathbf{c}})$ which, unfortunately, can be indefinite or rank deficient even though it is Hermitian and even if $\mathbf{g}$ is one-to-one.

Assuming that $\mathcal{H}_{\mathbf{cc}}^{\text{Gauss}}(\hat{\mathbf{c}})$ is invertible, the correction which minimizes the Gauss-Newton loss function (120) is given by

$$
\widehat{\Delta\mathbf{c}}^{\text{Gauss}} = -\mathcal{H}_{\mathbf{cc}}^{\text{Gauss}}(\hat{\mathbf{c}})^{-1}\left(\frac{\partial\ell(\hat{\mathbf{c}})}{\partial\mathbf{c}}\right)^H. \tag{122}
$$

Because of the admissibility of $\mathcal{H}_{\mathbf{cc}}^{\text{Gauss}}$ and $\left(\frac{\partial\ell(\hat{\mathbf{c}})}{\partial\mathbf{c}}\right)^H$, the resulting solution is admissible $\widehat{\Delta\mathbf{c}}^{\text{Gauss}} \in \mathcal{C}$.

Comparing Equations (114) and (122), it is evident that the difference between the two algorithms resides in the difference between the Newton Hessian, $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}(\hat{\mathbf{c}})$, which is the actual **c**-complex Hessian of the least-squares loss function $\ell(\mathbf{c})$, and the Gauss-Newton Hessian $\mathcal{H}_{\mathbf{cc}}^{\text{Gauss}}(\hat{\mathbf{c}})$ which has an unclear relationship to $\ell(\mathbf{c})$.[77] For this reason, we now turn to a discussion of the relationship between $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}(\hat{\mathbf{c}})$ and $\mathcal{H}_{\mathbf{cc}}^{\text{Gauss}}(\hat{\mathbf{c}})$.

---

[76]Note that the following derivation does *not* imply that $G^H W G = \mathbf{P}(G^H W G)$, a fact which would contradict our claim that $G^H W G$ is not admissible. This is because in the derivation we are not allowing arbitrary vectors in $\mathbb{C}^{2n}$ but are only admitting vectors $\Delta\mathbf{c}$ constrained to lie in $\mathcal{C}$, $\Delta\mathbf{c} \in \mathcal{C} \subset \mathbb{C}^{2n}$.

[77]Note that, *by construction,* $\mathcal{H}_{\mathbf{cc}}^{\text{Gauss}}(\hat{\mathbf{c}})$ is the Hessian matrix of the Gauss-Newton loss function. The question is: what is its relationship to the least-squares loss function or the Newton loss function?

We can compute the Newton Hessian $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$ from the relationship (see Equation (101))

$$\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}} = S \, \mathcal{H}_{\mathbf{cc}}^{\mathbb{R}} = S \, \frac{\partial}{\partial \mathbf{c}} \left(\frac{\partial \ell}{\partial \mathbf{c}} \right)^T$$

where $\frac{\partial}{\partial \mathbf{c}}$ is taken to be a $\mathbf{c}$-real cogradient operator. Note from (116) that,

$$\left(\frac{\partial \ell}{\partial \mathbf{c}} \right)^H = -\frac{1}{2} G^H W \mathbf{e} - \frac{1}{2} S \overline{G^H W \mathbf{e}} = \frac{1}{2} \left(B + S \overline{B} \right), \tag{123}$$

where

$$B \triangleq -G^H W \mathbf{e} \tag{124}$$

with $\mathbf{e} = \mathbf{y} - \mathbf{g}(\mathbf{c})$. This results in

$$\left(\frac{\partial \ell}{\partial \mathbf{c}} \right)^T = \overline{\left(\frac{\partial \ell}{\partial \mathbf{c}} \right)^H} = \frac{1}{2} \left(\bar{B} + S B \right),$$

Also note that

$$\frac{\partial \bar{B}}{\partial \mathbf{c}} = \overline{\frac{\partial B}{\partial \bar{\mathbf{c}}}} = \overline{\left(\frac{\partial B}{\partial \mathbf{c}} S \right)} = \overline{\frac{\partial B}{\partial \mathbf{c}}} \, S.$$

We have

$$\mathcal{H}_{\mathbf{cc}}^{\mathbb{R}} = \frac{\partial}{\partial \mathbf{c}} \left(\frac{\partial \ell}{\partial \mathbf{c}} \right)^T = \frac{1}{2} \left(S \frac{\partial B}{\partial \mathbf{c}} + \frac{\partial \bar{B}}{\partial \mathbf{c}} \right)$$

or

$$\mathcal{H}_{\mathbf{cc}}^{\mathbb{R}} = \frac{\partial}{\partial \mathbf{c}} \left(\frac{\partial \ell}{\partial \mathbf{c}} \right)^T = \frac{1}{2} \left(S \frac{\partial B}{\partial \mathbf{c}} + \overline{\frac{\partial B}{\partial \mathbf{c}}} \, S \right). \tag{125}$$

This yields

$$\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}} = S \, \mathcal{H}_{\mathbf{cc}}^{\mathbb{R}} = \frac{1}{2} \left(\frac{\partial B}{\partial \mathbf{c}} + S \, \overline{\frac{\partial B}{\partial \mathbf{c}}} \, S \right) \tag{126}$$

with B given by (124), which we can write as

$$\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}} = S \, \mathcal{H}_{\mathbf{cc}}^{\mathbb{R}} = \mathbf{P} \left(\frac{\partial B}{\partial \mathbf{c}} \right). \tag{127}$$

Recall that $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$ must be admissible. The function $\mathbf{P}(\cdot)$ produces admissible matrices which map from $\mathcal{C}$ to $\mathcal{C}$, and thereby ensures that the right-hand side of equation (127) is indeed an admissible matrix, as required for self-consistency of our development. The presence of the operator $\mathbf{P}$ does not show up in the real case (which is the standard development given in textbooks) as $\frac{\partial B}{\partial \mathbf{c}}$ is automatically symmetric as required for admissibility in the real case [25].

Note that B can be written as

$$B = - \left(\frac{\partial \mathbf{g}}{\partial \mathbf{c}} \right)^H W \left(\mathbf{y} - \mathbf{g} \right) = - \sum_{i=1}^{m} \left(\frac{\partial g_i}{\partial \mathbf{c}} \right)^H \left[W \left(\mathbf{y} - \mathbf{g} \right) \right]_i$$

where g_i and $[W(\mathbf{y} - \mathbf{g})]_i$ denote the i-th components of $\mathbf{g}$ and $W\mathbf{e} = W(\mathbf{y} - \mathbf{g})$ respectively. We can then compute $\frac{\partial B}{\partial \mathbf{c}}$ as

$$
\begin{aligned}
\frac{\partial B}{\partial \mathbf{c}} &= \left(\frac{\partial \mathbf{g}}{\partial \mathbf{c}}\right)^H W \left(\frac{\partial \mathbf{g}}{\partial \mathbf{c}}\right) - \sum_{i=1}^m \frac{\partial}{\partial \mathbf{c}} \left(\frac{\partial g_i}{\partial \mathbf{c}}\right)^H [W(\mathbf{y} - \mathbf{g})]_i \\
&= G^H W G - \sum_{i=1}^m \frac{\partial}{\partial \mathbf{c}} \left(\frac{\partial g_i}{\partial \mathbf{c}}\right)^H [W(\mathbf{y} - \mathbf{g})]_i
\end{aligned}
$$

or

$$
\frac{\partial B}{\partial \mathbf{c}} = G^H W G - \sum_{i=1}^m \frac{\partial}{\partial \mathbf{c}} \left(\frac{\partial g_i}{\partial \mathbf{c}}\right)^H [W\mathbf{e}]_i \, . \tag{128}
$$

Equations (127) and (128) result in

$$
\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}} = \mathcal{H}_{\mathbf{cc}}^{\text{Gauss}} - \sum_{i=1}^m \mathcal{H}_{\mathbf{cc}}^{(i)} \, . \tag{129}
$$

where

$$
\mathcal{H}_{\mathbf{cc}}^{(i)} \triangleq \mathbf{P} \left(\frac{\partial}{\partial \mathbf{c}} \left(\frac{\partial g_i}{\partial \mathbf{c}}\right)^H [W\mathbf{e}]_i\right) \, , \quad i = 1, \cdots, m \, . \tag{130}
$$

Note that Equation (129), which is our final result for the structural form of the Newton Hessian $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$, looks very much like the result for the real case [25]. [78] The first term on the right-hand-side of (129) is the Gauss-Newton Hessian $\mathcal{H}_{\mathbf{cc}}^{\text{Gauss}}$, which is admissible, Hermitian and at least positive semidefinite (under the standard assumption that W is Hermitian positive definite). Below, we will show that the matrices $\mathcal{H}_{\mathbf{cc}}^{(i)}$, $i = 1, \cdots, m$, are also admissible and Hermitian. While the Gauss-Newton Hessian is always positive semidefinite (and always positive definite if $\mathbf{g}$ is one-to-one), the presence of the second term on the right-hand-side of (129) can cause the Newton Hessian to become indefinite, or even negative definite.

We can now understand the relationship between the Gauss-Newton method and the Newton method when applied to the problem of minizing the least-squares loss function. The Gauss-Newton method is an approximation to the Newton method which arises from ignoring the second term on the right-hand-side of (129). This approximation is not only easier to implement, it will generally have superior numerical properties as a consequence of the definiteness of the Gauss-Newton Hessian. Indeed, if the mapping $\mathbf{g}$ is onto, via the Gauss-Newton algorithm one can produce a sequence of estimates $\hat{\mathbf{c}}_k$, $k = 1, 2, 3, \cdots$, which drives $\mathbf{e}(\hat{\mathbf{c}}_k) = \mathbf{y} - \mathbf{g}(\hat{\mathbf{c}}_k)$, and hence the second term on the right-hand-side of (129), to zero as $k \to \infty$. In which case, asymptotically there will be little difference in the convergence properties between the Newton and Gauss-Newton methods. This property is well known in the classical optimization literature, which suggests that by working within the $\mathbf{c}$-real perspective, we may be able to utilize a variety of insights that have

---

[78]The primary difference is due to the presence of the projector $\mathbf{P}$ in the complex Newton algorithm. Despite the similarity, note that it takes much more work to rigorously derive the complex Newton-Algorithm!

been developed for the Newton and Gauss-Newton methods when optimizing over real vector spaces.

To complete the proof of the derivation of (129), it remains to demonstrate that $\mathcal{H}_{\mathbf{cc}}^{(i)}$, $i = 1, \cdots, m$, are admissible and Hermitian. Note that the "raw" matrix

$$[W\mathbf{e}]_i \, \frac{\partial}{\partial \mathbf{c}} \left(\frac{\partial g_i}{\partial \mathbf{c}} \right)^H$$

is neither Hermitian nor admissible because of the presence of the complex scalar factor $[W\mathbf{e}]_i$. Fortunately, the processing of the second matrix of partial derivatives by the operator $\mathbf{P}$ to form the matrix $\mathcal{H}_{\mathbf{cc}}^{(i)}$ via

$$\mathcal{H}_{\mathbf{cc}}^{(i)} = \mathbf{P}(A_{\mathbf{cc}}(g_i))$$

creates a matrix which is both admissible and Hermitian. The fact that $\mathcal{H}_{\mathbf{cc}}^{(i)}$ is admissible is obvious, as the projector $\mathbf{P}$ is idempotent. We will now prove that $\mathcal{H}_{\mathbf{cc}}^{(i)}$ is Hermitian.

Define the matrix

$$A_{\mathbf{cc}}(g_i) \triangleq \frac{\partial}{\partial \mathbf{c}} \left(\frac{\partial g_i}{\partial \mathbf{c}} \right)^H, \tag{131}$$

and note that

$$\left[\frac{\partial}{\partial \mathbf{c}} \left(\frac{\partial g_i}{\partial \mathbf{c}} \right)^H \right]^H = \left[\overline{\frac{\partial}{\partial \mathbf{c}} \left(\frac{\partial \bar{g}_i}{\partial \bar{\mathbf{c}}} \right)^T} \right]^T = \overline{\left[\frac{\partial}{\partial \bar{\mathbf{c}}} \left(\frac{\partial \bar{g}_i}{\partial \mathbf{c}} \right)^T \right]} = \frac{\partial}{\partial \mathbf{c}} \left(\frac{\partial \bar{g}_i}{\partial \mathbf{c}} \right)^H,$$

which shows that $A_{\mathbf{cc}}(g_i)$ has the property that

$$A_{\mathbf{cc}}(g_i)^H = A_{\mathbf{cc}}(\bar{g}_i). \tag{132}$$

Now note that

$$\begin{aligned} S \frac{\partial}{\partial \mathbf{c}} \left(\frac{\partial g_i}{\partial \mathbf{c}} \right)^H S &= S \frac{\partial}{\partial \bar{\mathbf{c}}} \left(\frac{\partial g_i}{\partial \mathbf{c}} \right)^H \\ &= \frac{\partial}{\partial \bar{\mathbf{c}}} \left[S \left(\frac{\partial g_i}{\partial \mathbf{c}} \right)^H \right] \\ &= \frac{\partial}{\partial \bar{\mathbf{c}}} \left(\frac{\partial g_i}{\partial \mathbf{c}} S \right)^H \\ &= \frac{\partial}{\partial \bar{\mathbf{c}}} \left(\frac{\partial g_i}{\partial \bar{\mathbf{c}}} \right)^H, \end{aligned}$$

which establishes the second property that

$$S A_{\mathbf{cc}}(g_i) S = A_{\bar{\mathbf{c}}\bar{\mathbf{c}}}(g_i). \tag{133}$$

Finally note that properties (132) and (133) together yield the property

$$A_{\mathbf{cc}}(g_i)^H = A_{\mathbf{cc}}(\bar{g}_i) = SA_{\bar{\mathbf{cc}}}(\bar{g}_i)S = S\overline{A_{\mathbf{cc}}(g_i)}S .$$

Setting $a_i = [W\mathbf{e}]_i$, we have

$$\mathcal{H}_{\mathbf{cc}}^{(i)} = \mathbf{P}(a_i A_{\mathbf{cc}}(g_i)) = \frac{a_i A_{\mathbf{cc}}(g_i) + S\overline{a_i A_{\mathbf{cc}}(g_i)}S}{2} = \frac{a_i A_{\mathbf{cc}}(g_i) + \bar{a}_i S\overline{A_{\mathbf{cc}}(g_i)}S}{2} = \frac{a_i A_{\mathbf{cc}}(g_i) + \bar{a}_i A_{\mathbf{cc}}(g_i)^H}{2}$$

which is obviously Hermitian. Note that the action of the projector $\mathbf{P}$ on the "raw" matrix $a_i A_{\mathbf{cc}}(g_i)$, is equal to the action of Hermitian symmetrizing the matrix $a_i A_{\mathbf{cc}}(g_i)$.

Below, we will examine the least-squares algorithms at the block-component level, and will show that significant simplifications occur when $\mathbf{g}(\mathbf{z})$ is holomorphic.

Generalized Gradient Descent Algorithms. As in the real case [25], the Newton and Gauss-Newton algorithms can be viewed as special instances of a family of generalized gradient descent algorithms. Given a general real-valued loss function $\ell(\mathbf{c})$ which we wish to minimize[79] and a current estimate, $\hat{\mathbf{c}}$ of optimal solution, we can determine an update of our estimate to a new value $\hat{\mathbf{c}}_{\text{new}}$ which will decrease the loss function as follows.

For the loss function $\ell(\mathbf{c})$, with $\mathbf{c} = \hat{\mathbf{c}} + d\mathbf{c}$, we have

$$d\ell(\hat{\mathbf{c}}) = \ell(\hat{\mathbf{c}} + d\mathbf{c}) - \ell(\hat{\mathbf{c}}) = \frac{\partial\ell(\hat{\mathbf{c}})}{\partial\mathbf{c}}d\mathbf{c}$$

which is just the differential limit of the first order expansion

$$\Delta\ell(\hat{\mathbf{c}};\alpha) = \ell(\hat{\mathbf{c}} + \alpha\Delta\mathbf{c}) - \ell(\hat{\mathbf{c}}) \approx \alpha\frac{\partial\ell(\hat{\mathbf{c}})}{\partial\mathbf{c}}\Delta\mathbf{c} .$$

The stepsize $\alpha > 0$ is a control parameter which regulates the accuracy of the first order approximation assuming that

$$\alpha \to 0 \Rightarrow \alpha\Delta\mathbf{c} \to d\mathbf{c} \quad \text{and} \quad \Delta\ell(\hat{\mathbf{c}};\alpha) \to d\ell(\hat{\mathbf{c}}) .$$

If we assume that $\mathcal{C}$ is a Cartesian space,[80] then the gradient of $\ell(\mathbf{c})$ is given by[81]

$$\nabla_{\mathbf{c}}\ell(\mathbf{c}) = \left(\frac{\partial\ell(\mathbf{c})}{\partial\mathbf{c}}\right)^H .$$

---

[79]The loss function does *not* have to be restricted to the least-squares loss considered above.

[80]I.e., We assume that $\mathcal{C}$ has identity metric tensor. In [25] we call the resulting gradient a Cartesian gradient (if the metric tensor assumption is true for the space of intertest) or a naive gradient (if the metric tensor assumption is false, but made anyway for convenience).

[81]Note for future reference that the gradient has been specifically computed in Equation (123) for the special case when $\ell(\mathbf{c})$ is the least-squares loss function (112).

Take the update to be the generalized gradient descent correction

$$\Delta \mathbf{c} = -Q(\hat{\mathbf{c}}) \left(\frac{\partial \ell(\hat{\mathbf{c}})}{\partial \mathbf{c}} \right)^H = -Q(\hat{\mathbf{c}}) \, \nabla_{\mathbf{c}} \ell(\hat{\mathbf{c}}) \tag{134}$$

where $Q(\hat{\mathbf{c}})$ is a Hermitian matrix function of $\mathbf{c}$ which is assumed to be positive definite when evaluated at the value $\hat{\mathbf{c}}$.[82] This then yields the key stability condition[83]

$$\Delta \ell(\hat{\mathbf{c}}; \alpha) \approx -\alpha \|\nabla_{\mathbf{c}} \ell(\hat{\mathbf{c}})\|_Q^2 \triangleq -\alpha \, \nabla_{\mathbf{c}} \ell(\hat{\mathbf{c}})^H \, Q \, \nabla_{\mathbf{c}} \ell(\hat{\mathbf{c}}) \le 0, \tag{135}$$

where the right-hand-side is equal to zero if and only if

$$\nabla_{\mathbf{c}} \ell(\hat{\mathbf{c}}) = 0 \,.$$

Thus if the stepsize parameter α is chosen small enough, making the update

$$\hat{\mathbf{c}}_{\text{new}} = \hat{\mathbf{c}} + \alpha \Delta \mathbf{c} = \hat{\mathbf{c}} - Q \, \nabla_{\mathbf{c}} \ell(\hat{\mathbf{c}})$$

results in

$$\ell(\hat{\mathbf{c}}_{\text{new}}) = \ell(\hat{\mathbf{c}} + \alpha \Delta \mathbf{c}) = \ell(\hat{\mathbf{c}}) + \Delta \ell(\hat{\mathbf{c}}; \alpha) \approx \ell(\hat{\mathbf{c}}) - \alpha \|\nabla_{\mathbf{c}} \ell(\hat{\mathbf{c}})\|_Q^2 \le \ell(\hat{\mathbf{c}})$$

showing that we *either* have a nontrivial update of the value of $\hat{\mathbf{c}}$ which results in a strict decrease in the value of the loss function, *or* we have no update of $\hat{\mathbf{c}}$ nor decrease of the loss function because $\hat{\mathbf{c}}$ is a stationary point. If the loss function $\ell(\mathbf{c})$ is bounded from below, iterating on this procedure starting from a estimate $\hat{\mathbf{c}}_1$ will produce a sequence of estimates $\hat{\mathbf{c}}_i$, $i = 1, 2, 3, \cdots$, which will converge to a local minimum of the loss function. This simple procedure is the basis for all generalized gradient descent algorithms.

Assuming that we begin with an admissible estimate, $\hat{\mathbf{c}}_1$, for this procedure to be valid, we require that the sequence of estimates $\hat{\mathbf{c}}_i$, $i = 1, 2, 3, \cdots$, be admissible, which is true if the corresponding updates $\Delta \mathbf{c}$ are admissible,

$$\Delta \mathbf{c} = -Q(\hat{\mathbf{c}}_i) \nabla_{\hat{\mathbf{c}}_i} \ell(\hat{\mathbf{c}}_i) = -Q(\hat{\mathbf{c}}_i) \left(\frac{\partial \ell(\hat{\mathbf{c}}_i)}{\partial \hat{\mathbf{c}}_i} \right)^H \in \mathcal{C} \,, \quad i = 1, 2, \cdots \,.$$

We have established the admissibility of $\nabla_{\mathbf{c}} \ell(\mathbf{c}) = \left(\frac{\partial \ell(\mathbf{c})}{\partial \mathbf{c}} \right)^H \in \mathcal{C}$ above. *It is evident that in order for a generalized gradient descent algorithm (GDA) to be admissible it must be the case that Q be admissible,*

Generalized GDA is Admissible $\Leftrightarrow$ Generalized Gradient Q-Matrix is Admissible, $Q \in \mathcal{L}(\mathcal{C}, \mathcal{C})$.

---

[82] The fact that Q is otherwise arbitrary (except for the admissibility criterion discussed below) is what makes the resulting algorithm a *generalized* gradient descent algorithm in the parlance of [25]. When $Q = I$, we obtain the standard gradient descent algorithm.

[83] We interpret the stability condition to mean that for a small enough stepsize $\alpha > 0$, we will have $\Delta \ell(\hat{\mathbf{c}}; \alpha) \le 0$.

Furthermore, *a sufficient condition that the resulting algorithm be stable*[84] *is that Q be Hermitian and positive definite.* Note that given a candidate Hermitian positive definite matrix, Q', which is not admissible,

$$Q' \notin \mathcal{L}(\mathcal{C}, \mathcal{C}) \,,$$

we can transform it into an admissible Hermitian positive definite matrix via the projection

$$Q = \mathbf{P}(Q') \in \mathcal{L}(\mathcal{C}, \mathcal{C}) \,.$$

It can be much trickier to ensure that Q remain positive definite.

If we set

$$Q^{\text{Newton}}(\mathbf{c}) = \mathcal{H}_{\mathbf{cc}}^{\text{Newton}}(\mathbf{c})^{-1}$$

with

$$\mathcal{H}_{\mathbf{cc}}^{\text{Newton}} \triangleq \mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$$

then we obtain the Newton algorithm (114). If we take the loss function to be the least-squares loss function (112) and set

$$Q^{\text{Gauss}}(\mathbf{c}) = \mathcal{H}_{\mathbf{cc}}^{\text{Gauss}}(\mathbf{c})^{-1}$$

we obtain the Gauss-Newton algorithm (122). Whereas the Gauss-Newton algorithm generally has a positive definite Q-matrix (assuming that $g(\mathbf{c})$ is one-to-one), the Newton algorithm can have convergence problems due to the Newton Hessian $\mathcal{H}_{\mathbf{cc}}^{\text{Newton}} = \mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$ becoming indefinite. Note that taking

$$Q^{\text{Simple}} = I \,,$$

which we refer to as the "simple" choice, results in the standard gradient descent algorithm which is always stable (for a small enough stepsize so that the stability condition (135) holds).

The important issue being raised here is the problem of stability versus speed of convergence. It is well-known that the Newton algorithm tends to have a very fast rate of convergence, but at the cost of constructing and inverting the Newton Hessian $\mathcal{H}_{\mathbf{cc}}^{\text{Newton}} = \mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$ and potentially encountering more difficult algorithm instability problems. On the other hand, standard gradient descent ($Q = I$) tends to be very stable and much cheaper to implement, but can have very long convergence times.

The Gauss-Newton algorithm, which is an option available when the loss function $\ell(\mathbf{c})$ is the least-squares loss function (112), is considered an excellent trade-off between the Newton algorithm and standard gradient descent. The Gauss-Newton Hessian $\mathcal{H}_{\mathbf{cc}}^{\text{Gauss}}$ is generally simpler in form and, if $\mathbf{g}(\mathbf{c})$ is one-to-one, is always positive definite. Furthermore, if $\mathbf{g}(\mathbf{c})$ is also onto, assuming the algorithm converges, the Gauss-Newton and Newton algorithms are asymptotically equivalent.

We can also begin to gain some insight into the proposal by Yan and Fan [34] to ignore the block off-diagonal elements of the Newton Hessian,[85]

$$\mathcal{H}_{\mathbf{cc}}^{\text{Newton}} = \mathcal{H}_{\mathbf{cc}}^{\mathbb{C}} = \begin{pmatrix} \mathcal{H}_{\mathbf{zz}} & \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} \\ \mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}} & \mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}} \end{pmatrix} \,.$$

---

[84] Assuming a small enough step size to ensure that the stability condition (135) is satisfied.

[85] The values of the block elements of $\mathcal{H}_{\mathbf{cc}}^{\text{Newton}}$ will be computed for the special case of the least-squares loss function (112) later below.

As mentioned earlier, Yan and Fan make the claim in [34] that the block off-diagonal elements vanish for a quadratic loss function. As noted above, and shown in an example below, this is *not* generally true.[86] However, it *is* reasonable to ask what harm (if any), or what benefit (if any) can accrue by constructing a *new*[87] generalized gradient descent algorithm as a modification to the Newton algorithm created by simply ignoring the block off-diagonal elements in the Newton Hessian and working instead with the simplified *quasi-Newton Hessian,*

$$\mathcal{H}_{\mathbf{cc}}^{\text{quasi-Newton}} \triangleq \widehat{\mathcal{H}}_{\mathbf{cc}}^{\mathbb{C}} \triangleq \begin{pmatrix} \mathcal{H}_{\mathbf{zz}} & 0 \\ 0 & \mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}} \end{pmatrix}.$$

This results in a new generalized gradient descent algorithm, which we call the *quasi-Newton algorithm*, which is somewhere in complexity between the Newton algorithm and standard gradient descent. Note that the hermitian matrix $\mathcal{H}_{\mathbf{zz}}$ is positive definite if and only if $\mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}$ is positive definite. Thus invertibility and positive-definiteness of the quasi-Newton Hessian $\mathcal{H}_{\mathbf{cc}}^{\text{quasi-Newton}} = \widehat{\mathcal{H}}_{\mathbf{cc}}^{\mathbb{C}}$ is equivalent to invertibility and positive definiteness of the block element $\mathcal{H}_{\mathbf{zz}}$.

On the other hand, invertibility and positive definiteness of $\mathcal{H}_{\mathbf{zz}}$ is only a necessary condition for invertibility and positive definiteness of the complete Newton Hessian $\mathcal{H}_{\mathbf{cc}}^{\text{Newton}} = \mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$. Assuming that $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$ is positive definite, we have the well-known factorization

$$\begin{pmatrix} I & 0 \\ -\mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}}\mathcal{H}_{\mathbf{zz}}^{-1} & I \end{pmatrix} \mathcal{H}_{\mathbf{cc}}^{\mathbb{C}} \begin{pmatrix} I & -\mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}}\mathcal{H}_{\mathbf{zz}}^{-1} \\ 0 & I \end{pmatrix} = \begin{pmatrix} \mathcal{H}_{\mathbf{zz}} & 0 \\ 0 & \widetilde{\mathcal{H}}_{\mathbf{zz}} \end{pmatrix} \tag{136}$$

where

$$\widetilde{\mathcal{H}}_{\mathbf{zz}} = \mathcal{H}_{\mathbf{zz}} - \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}}\mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{-1}\mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}}$$

is the Schur complement of $\mathcal{H}_{\mathbf{zz}}$ in $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$. From the factorization (136) we immediately obtain the useful condition

$$\text{rank}\left(\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}\right) = \text{rank}\left(\mathcal{H}_{\mathbf{zz}}\right) + \text{rank}\left(\widetilde{\mathcal{H}}_{\mathbf{zz}}\right). \tag{137}$$

Note from condition (137) that the Newton Hessian $\mathcal{H}_{\mathbf{cc}}^{\text{Newton}} = \mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$ is positive definite if and only if $\mathcal{H}_{\mathbf{zz}}$ *and* its Schur complement $\widetilde{\mathcal{H}_{\mathbf{zz}}}$ are *both* positive definite. Thus it is obviously a more difficult matter to ascertain and ensure the stability of the Newton Hessian than to do the same for the quasi-Newton Hessian.

The quasi-Newton algorithm is constructed by forming the Q matrix from the quasi-Newton Hessian $\mathcal{H}_{\mathbf{cc}}^{\text{quasi-Newton}} = \widehat{\mathcal{H}}_{\mathbf{cc}}^{\mathbb{C}}$,

$$Q^{\text{Pseudo-Newton}} = \left(\mathcal{H}_{\mathbf{cc}}^{\text{quasi-Newton}}\right)^{-1} = \left(\widehat{\mathcal{H}}_{\mathbf{cc}}^{\mathbb{C}}\right)^{-1} = \begin{pmatrix} \mathcal{H}_{\mathbf{zz}}^{-1} & 0 \\ 0 & \mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{-1} \end{pmatrix}$$

which is admissible and hermitian, and positive definite provided $\mathcal{H}_{\mathbf{zz}} = \overline{\mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}}$ is positive definite. Thus, if $\mathcal{H}_{\mathbf{zz}} = \overline{\mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}}$ is positive definite, *the quasi-Newton algorithm is guaranteed to be stable*

---

[86]What is true, as we've noted, is that for a quadratic loss function, the Gauss-Newton and Newton Hessians asymptotically become equal.

[87]I.e., no approximation algorithms are invoked.

(assuming a small enough stepsize $\alpha > 0$ so that the stability condition (135) is satisfied). With this choice of Q in (134), the quasi-Newton update is given by[88]

$$\Delta \mathbf{z}^{\text{quasi-Newton}} = -\mathcal{H}_{\mathbf{zz}}^{-1}\left(\frac{\partial f}{\partial \mathbf{z}}\right)^{H} \tag{138}$$

which is just the simplification shown earlier in Equation (111) and proposed by Yan and Fan in [34]. However, unlike Yan and Fan, we do not present the quasi-Newton algorithm as an approximation to the Newton algorithm, *but rather as one more algorithm in the family of generalized Newton algorithms indexed by the choice of the matrix Q.*

Indeed, recognizing that the Gauss-Newton algorithm potentially has better stability properties than the Newton algorithm, naturally leads us to propose a *quasi-Gauss-Newton* algorithm for minimizing the least-squares lose function (112) as follows. Because the hermitian Gauss-Newton Hessian is admissible, it can be partitioned as

$$\mathcal{H}_{\mathbf{cc}}^{\text{Gauss}} = \begin{pmatrix} U_{\mathbf{zz}} & U_{\bar{\mathbf{z}}\mathbf{z}} \\ U_{\bar{\mathbf{z}}\mathbf{z}} & U_{\mathbf{z}\mathbf{z}} \end{pmatrix}$$

with $U_{\bar{\mathbf{z}}\mathbf{z}} = U_{\bar{\mathbf{z}}\mathbf{z}}^{T}$.[89] The Gauss-Newton Hessian is positive-definite if and only if $U_{\mathbf{zz}}$ (equivalently $\overline{U_{\mathbf{zz}}}$) and its Schur complement $\widetilde{U_{\mathbf{zz}}} = U_{\mathbf{zz}} - U_{\bar{\mathbf{z}}\mathbf{z}}\overline{U_{\mathbf{zz}}}^{-1}\overline{U_{\bar{\mathbf{z}}\mathbf{z}}}$ are invertible.

On the other hand the *quasi-Gauss-Newton Hessian,*

$$\mathcal{H}_{\mathbf{cc}}^{\text{quasi-Gauss}} \triangleq \begin{pmatrix} U_{\mathbf{zz}} & 0 \\ 0 & \overline{U_{\mathbf{zz}}} \end{pmatrix}$$

is positive definite if and only if $U_{\mathbf{zz}}$ is positive definite. Choosing

$$Q^{\text{quasi-Gauss}} = (\mathcal{H}_{\mathbf{cc}}^{\text{quasi-Gauss}})^{-1} = \begin{pmatrix} U_{\mathbf{zz}}^{-1} & 0 \\ 0 & \overline{U_{\mathbf{zz}}}^{-1} \end{pmatrix}$$

results in the *quasi-Gauss-Newton algorithm*

$$\Delta \mathbf{z}^{\text{quasi-Gauss}} = -U_{\mathbf{zz}}^{-1}\left(\frac{\partial f}{\partial \mathbf{z}}\right)^{H} \tag{139}$$

which is guaranteed to be stable (for a small enough stepsize so that the stability condition (135) is satisfied) if $U_{\mathbf{zz}}$ is positive definite.

Note that $\mathcal{H}_{\mathbf{zz}}$ can become indefinite even while $U_{\mathbf{zz}}$ remains positive definite. Thus, the quasi-Gauss-Newton algorithm appears to be generally easier to stabilize than the quasi-Newton algorithm. Furthermore, if $\mathbf{g}$ is onto, we expect that asymptotically the quasi-Gauss-Newton and quasi-Newton algorithm become equivalent. Thus the quasi-Gauss-Newton algorithm is seen to stand in

---

[88]We can ignore the remaining update equation as it is just the complex conjugate of the shown update equation.

[89]The values of these block components will be computed below.

the same relationship to the quasi-Newton algorithm as the Gauss-Newton algorithm does to the Newton algorithm.

Without too much effort, we can construct the block matrix components needed to implement the Newton and Gauss-Newton algorithms developed above in order to minimize the least-squares loss function (112).[90]

Let us first look at the elements needed to implement the Gauss-Newton algorithm. From Equation (121) and the derivations following Equation (119) one obtains

$$U_{\mathbf{zz}} = \frac{1}{2}\left(\left(\frac{\partial \mathbf{g}}{\partial \mathbf{z}}\right)^H W \left(\frac{\partial \mathbf{g}}{\partial \mathbf{z}}\right) + \overline{\left(\frac{\partial \mathbf{g}}{\partial \overline{\mathbf{z}}}\right)^H W \left(\frac{\partial \mathbf{g}}{\partial \overline{\mathbf{z}}}\right)}\right) \tag{140}$$

which is positive definite, assuming that W is positive definite and that $\mathbf{g}$ is one-to-one. Similarly, one finds that

$$U_{\overline{\mathbf{z}}\mathbf{z}} = \frac{1}{2}\left(\left(\frac{\partial \mathbf{g}}{\partial \mathbf{z}}\right)^H W \left(\frac{\partial \mathbf{g}}{\partial \overline{\mathbf{z}}}\right) + \overline{\left(\frac{\partial \mathbf{g}}{\partial \overline{\mathbf{z}}}\right)^H W \left(\frac{\partial \mathbf{g}}{\partial \mathbf{z}}\right)}\right). \tag{141}$$

Also $U_{\overline{\mathbf{z}}\overline{\mathbf{z}}} = \overline{U_{\mathbf{zz}}}$ and $U_{\mathbf{z}\overline{\mathbf{z}}} = \overline{U_{\overline{\mathbf{z}}\mathbf{z}}}$. We have now completely specified the Gauss-Newton Hessian $\mathcal{H}_{\mathbf{cc}}^{\text{Gauss}}$ and the quasi-Gauss-Newton Hessian at the block components level,

$$\mathcal{H}_{\mathbf{cc}}^{\text{Gauss}} = \begin{pmatrix} U_{\mathbf{zz}} & U_{\overline{\mathbf{z}}\mathbf{z}} \\ U_{\mathbf{z}\overline{\mathbf{z}}} & U_{\overline{\mathbf{z}}\overline{\mathbf{z}}} \end{pmatrix} \qquad \mathcal{H}_{\mathbf{cc}}^{\text{quasi-Gauss}} \triangleq \begin{pmatrix} U_{\mathbf{zz}} & 0 \\ 0 & U_{\overline{\mathbf{z}}\overline{\mathbf{z}}} \end{pmatrix}$$

Now *note the important fact that* $U_{\overline{\mathbf{z}}\mathbf{z}} = U_{\mathbf{z}\overline{\mathbf{z}}} = 0$ *when* $\mathbf{g}$ *is holomorphic*! Thus, when $\mathbf{g}$ is holomorphic there is no difference between the Gauss-Newton and pseudo-Gauss-Newton algorithms.[91] Furthermore, when $\mathbf{g}(\mathbf{z})$ is holomorphic, $U_{\mathbf{zz}}$ simplifies to

$$U_{\mathbf{zz}} = \frac{1}{2}\left(\frac{\partial \mathbf{g}}{\partial \mathbf{z}}\right)^H W \left(\frac{\partial \mathbf{g}}{\partial \mathbf{z}}\right) = \frac{1}{2}J_{\mathbf{g}}^H W J_{\mathbf{g}}, \tag{142}$$

where $J_{\mathbf{g}}$ is the Jacobian matrix of $\mathbf{g}$.

Now let us turn to the issue of computing the elements need to implement the Newton Algorithm, recalling that the Newton Hessian is block partitioned as

$$\mathcal{H}_{\mathbf{cc}}^{\text{Newton}} = \mathcal{H}_{\mathbf{cc}}^{\mathbb{C}} = \begin{pmatrix} \mathcal{H}_{\mathbf{zz}} & \mathcal{H}_{\overline{\mathbf{z}}\mathbf{z}} \\ \mathcal{H}_{\mathbf{z}\overline{\mathbf{z}}} & \mathcal{H}_{\overline{\mathbf{z}}\overline{\mathbf{z}}} \end{pmatrix}.$$

One can readily relate the block components $\mathcal{H}_{\mathbf{zz}}$ and $\mathcal{H}_{\overline{\mathbf{z}}\mathbf{z}}$ to the matrices $U_{\mathbf{zz}}$ and $U_{\overline{\mathbf{z}}\mathbf{z}}$ used in the Gauss-Newton and quasi-Gauss-Newton algorithms by use of Equation (129). We find that

$$\mathcal{H}_{\mathbf{zz}} = U_{\mathbf{zz}} - \sum_{i=1}^{m} V_{\mathbf{zz}}^{(i)}$$

---

[90]This, of course, results in only a special case application of the Newton and quasi-Newton algorithms, both of which can be applied to more general loss functions.

[91]Recall that $\mathbf{g}(\mathbf{z})$ is holomorphic (analytic in $\mathbf{z}$) if and only if the Cauchy-Riemann condition $\frac{\partial \mathbf{g}(\mathbf{z})}{\partial \overline{\mathbf{z}}} = 0$ is satisfied.

and

$$V_{\mathbf{zz}}^{(i)} = \frac{1}{2}\left[\left(\frac{\partial}{\partial \mathbf{z}}\left(\frac{\partial g_i(\mathbf{z})}{\partial \mathbf{z}}\right)^H [W\mathbf{e}]_i\right) + \overline{\left(\frac{\partial}{\partial \bar{\mathbf{z}}}\left(\frac{\partial g_i(\mathbf{z})}{\partial \bar{\mathbf{z}}}\right)^H [W\mathbf{e}]_i\right)}\right] \qquad (143)$$

where $\mathbf{e} = \mathbf{y} - \mathbf{g}(\mathbf{z})$. Similarly, we find that

$$\mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} = U_{\bar{\mathbf{z}}\mathbf{z}} - \sum_{i=1}^{m} V_{\bar{\mathbf{z}}\mathbf{z}}^{(i)}$$

and

$$V_{\bar{\mathbf{z}}\mathbf{z}}^{(i)} = \frac{1}{2}\left[\left(\frac{\partial}{\partial \bar{\mathbf{z}}}\left(\frac{\partial g_i(\mathbf{z})}{\partial \mathbf{z}}\right)^H [W\mathbf{e}]_i\right) + \overline{\left(\frac{\partial}{\partial \mathbf{z}}\left(\frac{\partial g_i(\mathbf{z})}{\partial \bar{\mathbf{z}}}\right)^H [W\mathbf{e}]_i\right)}\right] \qquad (144)$$

Furthermore, $V_{\bar{\mathbf{z}}\bar{\mathbf{z}}} = \overline{V_{\mathbf{zz}}}$ and $V_{\mathbf{z}\bar{\mathbf{z}}} = \overline{V_{\bar{\mathbf{z}}\mathbf{z}}}$. Note that neither $V_{\mathbf{zz}}$ nor $V_{\bar{\mathbf{z}}\mathbf{z}}$ vanish when $\mathbf{g}$ is holomorphic, but instead simplify to

$$V_{\mathbf{zz}}^{(i)} = \frac{1}{2}\frac{\partial}{\partial \mathbf{z}}\left(\frac{\partial g_i(\mathbf{z})}{\partial \mathbf{z}}\right)^H [W\mathbf{e}]_i \quad \text{and} \quad V_{\bar{\mathbf{z}}\mathbf{z}}^{(i)} = \frac{1}{2}\frac{\partial}{\partial \bar{\mathbf{z}}}\left(\frac{\partial g_i(\mathbf{z})}{\partial \mathbf{z}}\right)^H [W\mathbf{e}]_i . \qquad (145)$$

We have shown that the relationship between the Newton Hessian and Gauss-Newton Hessian is given by

$$\underbrace{\begin{pmatrix} \mathcal{H}_{\mathbf{zz}} & \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} \\ \mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}} & \mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}} \end{pmatrix}}_{\mathcal{H}_{\mathbf{cc}}^{\text{Newton}}} = \underbrace{\begin{pmatrix} U_{\mathbf{zz}} & U_{\bar{\mathbf{z}}\mathbf{z}} \\ U_{\mathbf{z}\bar{\mathbf{z}}} & U_{\bar{\mathbf{z}}\bar{\mathbf{z}}} \end{pmatrix}}_{\mathcal{H}_{\mathbf{cc}}^{\text{Gauss}}} - \sum_{i=1}^{m}\begin{pmatrix} V_{\mathbf{zz}}^{(i)} & V_{\bar{\mathbf{z}}\mathbf{z}}^{(i)} \\ V_{\mathbf{z}\bar{\mathbf{z}}}^{(i)} & V_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{(i)} \end{pmatrix}$$

In the special case when $\mathbf{g}(\mathbf{z})$ is holomorphic, the relationship becomes

$$\underbrace{\begin{pmatrix} \mathcal{H}_{\mathbf{zz}} & \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} \\ \mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}} & \mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}} \end{pmatrix}}_{\mathcal{H}_{\mathbf{cc}}^{\text{Newton}}} = \underbrace{\begin{pmatrix} U_{\mathbf{zz}} & 0 \\ 0 & U_{\bar{\mathbf{z}}\bar{\mathbf{z}}} \end{pmatrix}}_{\mathcal{H}_{\mathbf{cc}}^{\text{Gauss}}} - \frac{1}{2}\sum_{i=1}^{m}\begin{pmatrix} \frac{\partial}{\partial \mathbf{z}}\left(\frac{\partial g_i(\mathbf{z})}{\partial \mathbf{z}}\right)^H [W\mathbf{e}]_i & \frac{\partial}{\partial \bar{\mathbf{z}}}\left(\frac{\partial g_i(\mathbf{z})}{\partial \mathbf{z}}\right)^H [W\mathbf{e}]_i \\ \frac{\partial}{\partial \bar{\mathbf{z}}}\left(\frac{\partial g_i(\mathbf{z})}{\partial \mathbf{z}}\right)^H [W\mathbf{e}]_i & \frac{\partial}{\partial \mathbf{z}}\left(\frac{\partial g_i(\mathbf{z})}{\partial \mathbf{z}}\right)^H [W\mathbf{e}]_i \end{pmatrix} .$$

This shows that if $\mathbf{g}(\mathbf{z})$ is *holomorphic,* so that the block off-diagonal elements of the Gauss-Newton Hessian vanish, and $\mathbf{g}(\mathbf{z})$ is also *onto,* so that asymptotically we expect that $\mathbf{e} \approx 0$, then the claim of Yan and Fan in [34] that setting the block off-diagonal elements of the Hessian matrix can proved a a good *approximation* to the Hessian matrix is reasonable, at least when optimizing the least-squares loss function. However, when $\mathbf{e} \approx 0$ the Newton least-squares loss function (113) reduces to the Gauss-Newton loss function (120), so that in the least-squares case one may as well make the move immediately to the even simpler Gauss-Newton algorithm (which in this case coincides with the quasi-Gauss-Newton algorithm).

However, the real point to be made is that *any* generalized gradient descent algorithm is worthy of consideration,[92] provided that it is admissible, provably stable, and (at least locally) convergent

---

[92] I.e., we don't have to necessarily invoke an approximation argument.

to the desired optimal solution. After all the standard gradient descent algorithm corresponds to the cheapest "approximation" of all, namely that

$$\mathcal{H}_{\mathbf{cc}}^{\text{Newton}} \approx I$$

and very few will deny the utility of this algorithm, even though as an "approximation" to the Newton algorithm it might be far from correct. The resulting algorithm has intrinsic merit as an algorithm in its own right, namely as the member of the family of gradient descent algorithms corresponding to the simplest choice of the Q-matrix,

$$Q = Q^{\text{Simple}} = I \,.$$

In the end, if the algorithm *works*, it's ok. As it is said, "the proof is in the pudding."[93]

   We see then that we have a variety of algorithms at hand which fit within the framework of generalized gradient descent algorithms. These algorithms are characterized by the specific choice of the Q-matrix in the gradient descent algorithm, and include (roughly in the expected order of decreasing complexity, decreasing ideal performance, and increasing stability when applied to the least-squares loss function): 1) the Newton algorithm, 2) the quasi-Newton algorithm, 3) the Gauss-Newton algorithm, 4) the quasi-Gauss-Newton algorithm, and 5) standard gradient descent. Note that the Newton, quasi-Newton, and standard gradient descent algorithms are general algorithms, while the Gauss-Newton and quasi-Gauss-Newton algorithms are methods for minimizing the least-squares loss function (112).

   For convenience, we will now summarize the generalized gradient descent algorithms that we have developed in this note. In all of the algorithms, the update step is given by

$$\hat{\mathbf{c}} \leftarrow \hat{\mathbf{c}} + \alpha \Delta \mathbf{c}$$

or, equivalently,

$$\hat{\mathbf{z}} \leftarrow \hat{\mathbf{z}} + \alpha \Delta \mathbf{z}$$

for a specific choice of the stepsize $\alpha > 0$. The stability claims made are based on the assumption that α has been chosen small enough to ensure that the stability condition (135) is valid. Furthermore, we use the shorthand notation

$$G(\mathbf{c}) = \frac{\partial \mathbf{g}(\mathbf{c})}{\partial \mathbf{c}}$$

and

$$\mathbf{e}(\mathbf{c}) = \mathbf{y} - \mathbf{g}(\mathbf{c}) \,.$$

---

[93]Of course, we are allowed to ask what the performance of the Q^{Simple} algorithm is relative to the Q^{Newton} algorithm.

1. **Standard (Simple) Gradient Descent.**

   Applies to any smooth loss function which is bounded from below.

   $\mathcal{H}_{\mathbf{cc}}^{\text{simple}}(\hat{\mathbf{c}}) = I$

   $Q^{\text{simple}}(\hat{\mathbf{c}}) = (\mathcal{H}_{\mathbf{cc}}^{\text{simple}}(\hat{\mathbf{c}}))^{-1} = I$

   $\Delta\mathbf{c}^{\text{simple}} = -\nabla_{\mathbf{z}}\ell(\hat{\mathbf{c}}) = -\left(\frac{\partial\ell(\hat{\mathbf{c}})}{\partial\mathbf{c}}\right)^{H}$

   $\Delta\mathbf{z}^{\text{simple}} = -\nabla_{\mathbf{z}}\ell(\hat{\mathbf{z}}) = -\left(\frac{\partial\ell(\hat{\mathbf{z}})}{\partial\mathbf{z}}\right)^{H}$

   Application to Least-Squares Loss Function (112):

   $$\left(\frac{\partial\ell}{\partial\mathbf{c}}\right)^{H} = -\frac{1}{2}G^{H}W\mathbf{e} - \frac{1}{2}S\overline{G^{H}W\mathbf{e}} = \frac{1}{2}\left(B(\hat{\mathbf{c}}) + S\overline{B}(\hat{\mathbf{c}})\right)$$
   $$\text{where } B(\hat{\mathbf{c}}) = -G(\hat{\mathbf{c}})^{H}W e(\hat{\mathbf{c}})$$
   $$\Delta\mathbf{c}^{\text{simple}} = -\frac{1}{2}\left[B(\hat{\mathbf{c}}) + S\overline{B(\hat{\mathbf{c}})}\right]$$
   $$\left(\frac{\partial\ell}{\partial\mathbf{z}}\right)^{H} = -\frac{1}{2}\left[\left(\frac{\mathbf{g}(\hat{\mathbf{z}})}{\partial\mathbf{z}}\right)^{H}W\mathbf{e}(\hat{\mathbf{z}}) + \overline{\left(\frac{\mathbf{g}(\hat{\mathbf{z}})}{\partial\overline{\mathbf{z}}}\right)^{H}W\mathbf{e}(\hat{\mathbf{z}})}\right]$$
   $$\Delta\mathbf{z}^{\text{simple}} = \frac{1}{2}\left[\left(\frac{\mathbf{g}(\hat{\mathbf{z}})}{\partial\mathbf{z}}\right)^{H}W\mathbf{e}(\hat{\mathbf{z}}) + \overline{\left(\frac{\mathbf{g}(\hat{\mathbf{z}})}{\partial\overline{\mathbf{z}}}\right)^{H}W\mathbf{e}(\hat{\mathbf{z}})}\right]$$

   $\mathbf{g}(\mathbf{z})$ holomorphic:

   $$\left(\frac{\partial\ell}{\partial\mathbf{z}}\right)^{H} = -\frac{1}{2}\left(\frac{\mathbf{g}(\hat{\mathbf{z}})}{\partial\mathbf{z}}\right)^{H}W\mathbf{e}(\hat{\mathbf{z}})$$
   $$\Delta\mathbf{z}^{\text{simple}} = \frac{1}{2}\left(\frac{\mathbf{g}(\hat{\mathbf{z}})}{\partial\mathbf{z}}\right)^{H}W\mathbf{e}(\hat{\mathbf{z}})$$

   Generally stable but slow.

2. **Gauss-Newton Algorithm.**

   Applies to the least-squares loss function (112).

   $\mathcal{H}_{\mathbf{cc}}^{\text{Gauss}}(\hat{\mathbf{c}}) = \begin{pmatrix} U_{\mathbf{zz}} & U_{\overline{\mathbf{z}}\mathbf{z}} \\ U_{\mathbf{z}\overline{\mathbf{z}}} & U_{\overline{\mathbf{z}}\overline{\mathbf{z}}} \end{pmatrix}$

   where $U_{\mathbf{zz}}$ is given by (140), $U_{\overline{\mathbf{z}}\overline{\mathbf{z}}} = \overline{U_{\mathbf{zz}}}$, $U_{\overline{\mathbf{z}}\mathbf{z}}$ is given by (141), and $U_{\mathbf{z}\overline{\mathbf{z}}} = \overline{U_{\overline{\mathbf{z}}\mathbf{z}}}$.

   $Q^{\text{Gauss}}(\hat{\mathbf{c}}) = \mathcal{H}_{\mathbf{cc}}^{\text{Gauss}}(\hat{\mathbf{c}})^{-1}$

   $\Delta\mathbf{c}^{\text{Gauss}} = -Q^{\text{Gauss}}(\hat{\mathbf{c}})\left(\frac{\partial\ell(\hat{\mathbf{c}})}{\partial\mathbf{c}}\right)^{H}$ where

   $\left(\frac{\partial\ell}{\partial\mathbf{c}}\right)^{H} = -\frac{1}{2}G^{H}W\mathbf{e} - \frac{1}{2}S\overline{G^{H}W\mathbf{e}} = \frac{1}{2}\left(B(\hat{\mathbf{c}}) + S\overline{B}(\hat{\mathbf{c}})\right)$

   with $B(\hat{\mathbf{c}}) = -G(\hat{\mathbf{c}})^{H}W e(\hat{\mathbf{c}})$

   $\Delta\mathbf{z}^{\text{Gauss}} = \left(U_{\mathbf{zz}} - U_{\overline{\mathbf{z}}\mathbf{z}}U_{\overline{\mathbf{z}}\overline{\mathbf{z}}}^{-1}U_{\mathbf{z}\overline{\mathbf{z}}}\right)^{-1}\left\{U_{\overline{\mathbf{z}}\mathbf{z}}U_{\overline{\mathbf{z}}\overline{\mathbf{z}}}^{-1}\left(\frac{\partial\ell}{\partial\overline{\mathbf{z}}}\right)^{H} - \left(\frac{\partial\ell}{\partial\mathbf{z}}\right)^{H}\right\}$ where

   $\left(\frac{\partial\ell}{\partial\mathbf{z}}\right)^{H} = -\frac{1}{2}\left[\left(\frac{\mathbf{g}(\hat{\mathbf{z}})}{\partial\mathbf{z}}\right)^{H}W\mathbf{e}(\hat{\mathbf{z}}) + \overline{\left(\frac{\mathbf{g}(\hat{\mathbf{z}})}{\partial\overline{\mathbf{z}}}\right)^{H}W\mathbf{e}(\hat{\mathbf{z}})}\right]$; $\left(\frac{\partial\ell}{\partial\overline{\mathbf{z}}}\right)^{H} = \overline{\left(\frac{\partial\ell}{\partial\mathbf{z}}\right)^{H}}$

$\mathbf{g}(\mathbf{z})$ holomorphic:

$U_{\mathbf{z}\mathbf{z}}$ takes the simpler form (142), $U_{\bar{\mathbf{z}}\bar{\mathbf{z}}} = \overline{U_{\mathbf{z}\mathbf{z}}}$, and $U_{\mathbf{z}\bar{\mathbf{z}}} = \overline{U_{\bar{\mathbf{z}}\mathbf{z}}} = 0$.

$$\mathcal{H}_{\mathbf{cc}}^{\text{Gauss}}(\hat{\mathbf{c}}) = \begin{pmatrix} U_{\mathbf{z}\mathbf{z}} & 0 \\ 0 & U_{\bar{\mathbf{z}}\bar{\mathbf{z}}} \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} \left(\frac{\partial \mathbf{g}}{\partial \mathbf{z}}\right)^H W \left(\frac{\partial \mathbf{g}}{\partial \mathbf{z}}\right) & 0 \\ 0 & \overline{\left(\frac{\partial \mathbf{g}}{\partial \mathbf{z}}\right)^H W \left(\frac{\partial \mathbf{g}}{\partial \mathbf{z}}\right)} \end{pmatrix}$$

$$\left(\frac{\partial \ell}{\partial \mathbf{z}}\right)^H = -\tfrac{1}{2} \left(\frac{\mathbf{g}(\hat{\mathbf{z}})}{\partial \mathbf{z}}\right)^H W \mathbf{e}(\hat{\mathbf{z}})$$

$$\Delta \mathbf{z}^{\text{Gauss}} = U_{\mathbf{z}\mathbf{z}}^{-1} \left(\frac{\partial \ell}{\partial \mathbf{z}}\right)^H = \left[\left(\frac{\partial \mathbf{g}(\hat{\mathbf{z}})}{\partial \mathbf{z}}\right)^H W \left(\frac{\partial \mathbf{g}(\hat{\mathbf{z}})}{\partial \mathbf{z}}\right)\right]^{-1} \left(\frac{\mathbf{g}(\hat{\mathbf{z}})}{\partial \mathbf{z}}\right)^H W \mathbf{e}(\hat{\mathbf{z}})$$

Stability generally requires positive definiteness of both $U_{\mathbf{z}\mathbf{z}}$ and its Schur complement: $\widetilde{U}_{\mathbf{z}\mathbf{z}} = U_{\mathbf{z}\mathbf{z}} - U_{\bar{\mathbf{z}}\mathbf{z}} U_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{-1} U_{\mathbf{z}\bar{\mathbf{z}}}$. The need to step for positive-definiteness of the Schur complement can significantly increase the complexity of an on-line adaptive filtering algorithm.

If $\mathbf{g}(\mathbf{z})$ is holomorphic, then stability only requires positive definiteness of the matrix $U_{\mathbf{z}\mathbf{z}} = \left(\frac{\partial \mathbf{g}(\hat{\mathbf{z}})}{\partial \mathbf{z}}\right)^H W \left(\frac{\partial \mathbf{g}(\hat{\mathbf{z}})}{\partial \mathbf{z}}\right)$, which will be the case if $\mathbf{g}(\mathbf{z})$ is one-to-one. Thus, the algorithm may be easier to stabilize when $\mathbf{g}(\mathbf{z})$ is holomorphic.

Convergence tends to be fast.

3. **Pseudo-Gauss-Newton Algorithm.**

Applies to the least-squares loss function (112).

$$\mathcal{H}_{\mathbf{cc}}^{\text{Gauss}}(\hat{\mathbf{c}}) = \begin{pmatrix} U_{\mathbf{z}\mathbf{z}} & 0 \\ 0 & U_{\bar{\mathbf{z}}\bar{\mathbf{z}}} \end{pmatrix}$$

where $U_{\mathbf{z}\mathbf{z}}$ is given by (140) and $U_{\bar{\mathbf{z}}\bar{\mathbf{z}}} = \overline{U_{\mathbf{z}\mathbf{z}}}$.

$$Q^{\text{pseudo-Gauss}}(\hat{\mathbf{c}}) = [\mathcal{H}_{\mathbf{cc}}^{\text{pseudo-Gauss}}(\hat{\mathbf{c}})]^{-1} = \begin{pmatrix} U_{\mathbf{z}\mathbf{z}}^{-1} & 0 \\ 0 & U_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{-1} \end{pmatrix}$$

$$\Delta \mathbf{c}^{\text{pseudo-Gauss}} = -Q^{\text{pseudo-Gauss}}(\hat{\mathbf{c}}) \left(\frac{\partial \ell(\hat{\mathbf{c}})}{\partial \mathbf{c}}\right)^H \quad \text{where}$$

$$\left(\frac{\partial \ell}{\partial \mathbf{c}}\right)^H = -\tfrac{1}{2} G^H W \mathbf{e} - \tfrac{1}{2} S \overline{G^H W \mathbf{e}} = \tfrac{1}{2} \left(B(\hat{\mathbf{c}}) + S \overline{B}(\hat{\mathbf{c}})\right)$$

with $B(\hat{\mathbf{c}}) = -G(\hat{\mathbf{c}})^H W e(\hat{\mathbf{c}})$

$$\Delta \mathbf{z}^{\text{pseudo-Gauss}} = -[U_{\mathbf{z}\mathbf{z}}(\hat{\mathbf{z}})]^{-1} \left(\frac{\partial \ell(\hat{\mathbf{z}})}{\partial \mathbf{z}}\right)^H = \left[\left(\frac{\partial \mathbf{g}}{\partial \mathbf{z}}\right)^H W \left(\frac{\partial \mathbf{g}}{\partial \mathbf{z}}\right) + \overline{\left(\frac{\partial \mathbf{g}}{\partial \bar{\mathbf{z}}}\right)^H W \left(\frac{\partial \mathbf{g}}{\partial \bar{\mathbf{z}}}\right)}\right]^{-1} \left(\frac{\partial \ell(\hat{\mathbf{z}})}{\partial \mathbf{z}}\right)^H \quad \text{where}$$

$$\left(\frac{\partial \ell}{\partial \mathbf{z}}\right)^H = -\tfrac{1}{2} \left[\left(\frac{\mathbf{g}(\hat{\mathbf{z}})}{\partial \mathbf{z}}\right)^H W \mathbf{e}(\hat{\mathbf{z}}) + \overline{\left(\frac{\mathbf{g}(\hat{\mathbf{z}})}{\partial \bar{\mathbf{z}}}\right)^H W \mathbf{e}(\hat{\mathbf{z}})}\right]$$

$\mathbf{g}(\mathbf{z})$ holomorphic:

$U_{\mathbf{z}\mathbf{z}}$ takes the simpler form of (142) , and $U_{\bar{\mathbf{z}}\bar{\mathbf{z}}} = \overline{U_{\mathbf{z}\mathbf{z}}}$.

$$\mathcal{H}_{\mathbf{cc}}^{\text{pseudo-Gauss}}(\hat{\mathbf{c}}) = \begin{pmatrix} U_{\mathbf{z}\mathbf{z}} & 0 \\ 0 & U_{\bar{\mathbf{z}}\bar{\mathbf{z}}} \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} \left(\frac{\partial \mathbf{g}}{\partial \mathbf{z}}\right)^H W \left(\frac{\partial \mathbf{g}}{\partial \mathbf{z}}\right) & 0 \\ 0 & \overline{\left(\frac{\partial \mathbf{g}}{\partial \mathbf{z}}\right)^H W \left(\frac{\partial \mathbf{g}}{\partial \mathbf{z}}\right)} \end{pmatrix}$$

$$\left(\tfrac{\partial \ell}{\partial \mathbf{z}}\right)^H = -\tfrac{1}{2}\left(\tfrac{\mathbf{g}(\hat{\mathbf{z}})}{\partial \mathbf{z}}\right)^H W \mathbf{e}(\hat{\mathbf{z}})$$

$$\Delta \mathbf{z}^{\text{pseudo-Gauss}} = \left[\left(\tfrac{\partial \mathbf{g}(\hat{\mathbf{z}})}{\partial \mathbf{z}}\right)^H W \left(\tfrac{\partial \mathbf{g}(\hat{\mathbf{z}})}{\partial \mathbf{z}}\right)\right]^{-1}\left(\tfrac{\mathbf{g}(\hat{\mathbf{z}})}{\partial \mathbf{z}}\right)^H W \mathbf{e}(\hat{\mathbf{z}})$$

Stability requires positive definiteness of $U_{\mathbf{zz}}(\hat{\mathbf{z}}) = \left(\tfrac{\partial \mathbf{g}(\hat{\mathbf{z}})}{\partial \mathbf{z}}\right)^H W \left(\tfrac{\partial \mathbf{g}(\hat{\mathbf{z}})}{\partial \mathbf{z}}\right)$ which will be the case if $\mathbf{g}(\mathbf{z})$ is one-to-one.

Convergence is expected to be quick but generally slower than for Gauss-Newton due to loss of efficiency due to neglecting the block off-diagonal terms in the Gauss-Newton Hessian (off-set, however, by reduced complexity and possible gains in stability), except for the case when $\mathbf{g}(\mathbf{z})$ is holomorphic, in which case the two algorithms coincide.

4. **Newton-Algorithm.**

   Applies to any smooth loss function which is bounded from below.

   $$\mathcal{H}_{\mathbf{cc}}^{\text{Newton}}(\hat{\mathbf{c}}) = \begin{pmatrix} \mathcal{H}_{\mathbf{zz}}(\hat{\mathbf{c}}) & \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}}(\hat{\mathbf{c}}) \\ \mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}}(\hat{\mathbf{c}}) & \mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}(\hat{\mathbf{c}}) \end{pmatrix}$$

   $$Q^{\text{Newton}}(\hat{\mathbf{c}}) = [\mathcal{H}_{\mathbf{cc}}^{\text{Newton}}(\hat{\mathbf{c}})]^{-1}$$

   $$\Delta \mathbf{c}^{\text{Newton}} = -Q^{\text{Newton}}(\hat{\mathbf{c}})\left(\tfrac{\partial \ell(\hat{\mathbf{c}})}{\partial \mathbf{c}}\right)^H$$

   $$\Delta \mathbf{z}^{\text{Newton}} = \left(\mathcal{H}_{\mathbf{zz}} - \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}}\mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{-1}\mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}}\right)^{-1}\left\{\mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}}\mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{-1}\left(\tfrac{\partial \ell}{\partial \bar{\mathbf{z}}}\right)^H - \left(\tfrac{\partial \ell}{\partial \mathbf{z}}\right)^H\right\}$$

   Application to the Least-Squares Loss Function (112):

   $$\mathcal{H}_{\mathbf{cc}}^{\text{Newton}} = \begin{pmatrix} \mathcal{H}_{\mathbf{zz}} & \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} \\ \mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}} & \mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}} \end{pmatrix} = \begin{pmatrix} U_{\mathbf{zz}} & U_{\bar{\mathbf{z}}\mathbf{z}} \\ U_{\mathbf{z}\bar{\mathbf{z}}} & U_{\bar{\mathbf{z}}\bar{\mathbf{z}}} \end{pmatrix} - \sum_{i=1}^m \begin{pmatrix} V_{\mathbf{zz}}^{(i)} & V_{\bar{\mathbf{z}}\mathbf{z}}^{(i)} \\ V_{\mathbf{z}\bar{\mathbf{z}}}^{(i)} & V_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{(i)} \end{pmatrix}$$
   $$= \mathcal{H}_{\mathbf{cc}}^{\text{Gauss}}(\hat{\mathbf{c}}) - \sum_{i=1}^m \begin{pmatrix} V_{\mathbf{zz}}^{(i)} & V_{\bar{\mathbf{z}}\mathbf{z}}^{(i)} \\ V_{\mathbf{z}\bar{\mathbf{z}}}^{(i)} & V_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{(i)} \end{pmatrix}$$

   $U_{\mathbf{zz}}$ is given by (140), $U_{\bar{\mathbf{z}}\bar{\mathbf{z}}} = \overline{U_{\mathbf{zz}}}$, $U_{\mathbf{z}\bar{\mathbf{z}}}$ is given by (141), $U_{\bar{\mathbf{z}}\mathbf{z}} = \overline{U_{\mathbf{z}\bar{\mathbf{z}}}}$

   $V_{\mathbf{zz}}^{(i)}$ is given by (143), $V_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{(i)} = \overline{V_{\mathbf{zz}}^{(i)}}$, $V_{\mathbf{z}\bar{\mathbf{z}}}^{(i)}$ is given by (144), $V_{\bar{\mathbf{z}}\mathbf{z}}^{(i)} = \overline{V_{\mathbf{z}\bar{\mathbf{z}}}^{(i)}}$.

   $$\Delta \mathbf{c}^{\text{Newton}} = -Q^{\text{Newton}}(\hat{\mathbf{c}})\left(\tfrac{\partial \ell(\hat{\mathbf{c}})}{\partial \mathbf{c}}\right)^H \quad \text{where}$$

   $$\left(\tfrac{\partial \ell}{\partial \mathbf{c}}\right)^H = -\tfrac{1}{2}G^H W \mathbf{e} - \tfrac{1}{2}S\overline{G^H W \mathbf{e}} = \tfrac{1}{2}\left(B(\hat{\mathbf{c}}) + S\overline{B}(\hat{\mathbf{c}})\right)$$
   with $B(\hat{\mathbf{c}}) = -G(\hat{\mathbf{c}})^H W e(\hat{\mathbf{c}})$

   $$\Delta \mathbf{z}^{\text{Newton}} = \left(\mathcal{H}_{\mathbf{zz}} - \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}}\mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{-1}\mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}}\right)^{-1}\left\{\mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}}\mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{-1}\left(\tfrac{\partial \ell}{\partial \bar{\mathbf{z}}}\right)^H - \left(\tfrac{\partial \ell}{\partial \mathbf{z}}\right)^H\right\} \quad \text{where}$$

   $$\left(\tfrac{\partial \ell}{\partial \mathbf{z}}\right)^H = -\tfrac{1}{2}\left[\left(\tfrac{\mathbf{g}(\hat{\mathbf{z}})}{\partial \mathbf{z}}\right)^H W \mathbf{e}(\hat{\mathbf{z}}) + \overline{\left(\tfrac{\mathbf{g}(\hat{\mathbf{z}})}{\partial \bar{\mathbf{z}}}\right)^H W \mathbf{e}(\hat{\mathbf{z}})}\right]; \quad \left(\tfrac{\partial \ell}{\partial \bar{\mathbf{z}}}\right)^H = \overline{\left(\tfrac{\partial \ell}{\partial \mathbf{z}}\right)^H}$$

   $\mathbf{g}(\mathbf{z})$ holomorphic:

   $$\mathcal{H}_{\mathbf{cc}}^{\text{Newton}} = \begin{pmatrix} U_{\mathbf{zz}} & 0 \\ 0 & U_{\bar{\mathbf{z}}\bar{\mathbf{z}}} \end{pmatrix} - \sum_{i=1}^m \begin{pmatrix} V_{\mathbf{zz}}^{(i)} & V_{\bar{\mathbf{z}}\mathbf{z}}^{(i)} \\ V_{\mathbf{z}\bar{\mathbf{z}}}^{(i)} & V_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{(i)} \end{pmatrix} = \mathcal{H}_{\mathbf{cc}}^{\text{pseudo-Gauss}}(\hat{\mathbf{c}}) - \sum_{i=1}^m \begin{pmatrix} V_{\mathbf{zz}}^{(i)} & V_{\bar{\mathbf{z}}\mathbf{z}}^{(i)} \\ V_{\mathbf{z}\bar{\mathbf{z}}}^{(i)} & V_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{(i)} \end{pmatrix}$$

$V_{\mathbf{zz}}^{(i)}$ and $V_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{(i)}$ take the simpler forms of (145), $V_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{(i)} = \overline{V_{\mathbf{zz}}^{(i)}}$, $V_{\mathbf{z}\bar{\mathbf{z}}}^{(i)} = \overline{V_{\bar{\mathbf{z}}\mathbf{z}}^{(i)}}$

$U_{\mathbf{zz}}$ takes the simpler form of (142), $U_{\bar{\mathbf{z}}\bar{\mathbf{z}}} = \overline{U_{\mathbf{zz}}}$

$$\Delta \mathbf{z}^{\text{Newton}} = \left(\mathcal{H}_{\mathbf{zz}} - \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} \mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{-1} \mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}} \right)^{-1} \left\{ \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} \mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{-1} \left(\tfrac{\partial \ell}{\partial \bar{\mathbf{z}}} \right)^H - \left(\tfrac{\partial \ell}{\partial \mathbf{z}} \right)^H \right\} \text{ where}$$

$$\left(\tfrac{\partial \ell}{\partial \mathbf{z}} \right)^H = -\tfrac{1}{2} \left(\tfrac{\mathbf{g}(\hat{\mathbf{z}})}{\partial \mathbf{z}} \right)^H W \mathbf{e}(\hat{\mathbf{z}}); \quad \left(\tfrac{\partial \ell}{\partial \bar{\mathbf{z}}} \right)^H = \overline{\left(\tfrac{\partial \ell}{\partial \mathbf{z}} \right)^H}$$

Stability generally requires positive definiteness of both $\mathcal{H}_{\mathbf{zz}}$ and its Schur complement $\widetilde{\mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}} = \left(\mathcal{H}_{\mathbf{zz}} - \mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} \mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{-1} \mathcal{H}_{\mathbf{z}\bar{\mathbf{z}}} \right)$. The need to step for positive-definiteness of the Schur complement can significantly increase the complexity of an on-line adaptive filtering algorithm.

When minimizing the least-squares loss function, we expect stability to be greater when $\mathbf{g}(\mathbf{c})$ is holomorphic. This is particularly true if $\mathbf{g}(\mathbf{c})$ is also onto and the algorithm is convergent, as we then expect the difference between the Newton and Gauss-Newton Hessians (and hence the difference between the Newton and Gauss-Newton algorithms) to become negligible asymptotically.

The Newton algorithm is known to have very fast convergence properties, provided it can be stabilized.

5. **Pseudo-Newton Algorithm.**

Applies to any smooth loss function which is bounded from below.

$$\mathcal{H}_{\mathbf{cc}}^{\text{pseudo-Newton}}(\hat{\mathbf{c}}) = \begin{pmatrix} \mathcal{H}_{\mathbf{zz}}(\hat{\mathbf{c}}) & 0 \\ 0 & \mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}(\hat{\mathbf{c}}) \end{pmatrix}$$

$$Q^{\text{pseudo-Newton}}(\hat{\mathbf{c}}) = [\mathcal{H}_{\mathbf{cc}}^{\text{pseudo-Newton}}(\hat{\mathbf{c}})]^{-1}$$

$$\Delta \mathbf{c}^{\text{psedudo-Newton}} = -Q^{\text{pseudo-Newton}}(\hat{\mathbf{c}}) \left(\tfrac{\partial \ell(\hat{\mathbf{c}})}{\partial \mathbf{c}} \right)^H$$

$$\Delta \mathbf{z}^{\text{pseudo-Newton}} = - [\mathcal{H}_{\mathbf{zz}}(\hat{\mathbf{z}})]^{-1} \left(\tfrac{\partial \ell(\hat{\mathbf{z}})}{\partial \mathbf{z}} \right)^H$$

Application to the Least-Squares Loss Function (112):

$$\mathcal{H}_{\mathbf{cc}}^{\text{pseudo-Newton}} = \begin{pmatrix} \mathcal{H}_{\mathbf{zz}}(\hat{\mathbf{c}}) & 0 \\ 0 & \mathcal{H}_{\bar{\mathbf{z}}\bar{\mathbf{z}}}(\hat{\mathbf{c}}) \end{pmatrix} = \begin{pmatrix} U_{\mathbf{zz}} - \sum\limits_{i=1}^{m} V_{\mathbf{zz}}^{(i)} & 0 \\ 0 & U_{\bar{\mathbf{z}}\bar{\mathbf{z}}} - \sum\limits_{i=1}^{m} V_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{(i)} \end{pmatrix}$$

$$= \mathcal{H}_{\mathbf{cc}}^{\text{pseudo-Gauss}}(\hat{\mathbf{c}}) - \begin{pmatrix} \sum\limits_{i=1}^{m} V_{\mathbf{zz}}^{(i)} & 0 \\ 0 & \sum\limits_{i=1}^{m} V_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{(i)} \end{pmatrix}$$

$V_{\mathbf{zz}}^{(i)}$ is given by (143) and $V_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{(i)} = \overline{V_{\mathbf{zz}}^{(i)}}$. $U_{\mathbf{zz}}$ is given by (140) and $U_{\bar{\mathbf{z}}\bar{\mathbf{z}}} = \overline{U_{\mathbf{zz}}}$

$$\Delta \mathbf{c}^{\text{pseudo-Newton}} = -Q^{\text{pseudo-Newton}}(\hat{\mathbf{c}}) \left(\tfrac{\partial \ell(\hat{\mathbf{c}})}{\partial \mathbf{c}} \right)^H \text{ where}$$

$$\left(\tfrac{\partial \ell}{\partial \mathbf{c}} \right)^H = -\tfrac{1}{2} G^H W \mathbf{e} - \tfrac{1}{2} S \overline{G^H W \mathbf{e}} = \tfrac{1}{2} \left(B(\hat{\mathbf{c}}) + S \overline{B}(\hat{\mathbf{c}}) \right)$$
with $B(\hat{\mathbf{c}}) = -G(\hat{\mathbf{c}})^H W e(\hat{\mathbf{c}})$

$$\Delta \mathbf{z}^{\text{pseudo-Newton}} = -\left[\mathcal{H}_{\mathbf{zz}}(\hat{\mathbf{z}})\right]^{-1} \left(\frac{\partial \ell(\hat{\mathbf{z}})}{\partial \mathbf{z}}\right)^H = -\left[U_{\mathbf{zz}} - \sum_{i=1}^{m} V_{\mathbf{zz}}^{(i)}\right]^{-1} \left(\frac{\partial \ell(\hat{\mathbf{z}})}{\partial \mathbf{z}}\right)^H \quad \text{where}$$

$$\left(\frac{\partial \ell}{\partial \mathbf{z}}\right)^H = -\frac{1}{2}\left[\left(\frac{\mathbf{g}(\hat{\mathbf{z}})}{\partial \mathbf{z}}\right)^H W \mathbf{e}(\hat{\mathbf{z}}) + \overline{\left(\frac{\mathbf{g}(\hat{\mathbf{z}})}{\partial \bar{\mathbf{z}}}\right)^H W \mathbf{e}(\hat{\mathbf{z}})}\right]$$

$\mathbf{g}(\mathbf{z})$ holomorphic $\Rightarrow$

$U_{\mathbf{zz}}$ takes the simpler form of (142), $U_{\bar{\mathbf{z}}\bar{\mathbf{z}}} = \overline{U_{\mathbf{zz}}}$.

$V_{\mathbf{zz}}^{(i)}$ takes the simpler form (145), $V_{\bar{\mathbf{z}}\bar{\mathbf{z}}}^{(i)} = \overline{V_{\mathbf{zz}}^{(i)}}$

$$\left(\frac{\partial \ell(\hat{\mathbf{z}})}{\partial \mathbf{z}}\right)^H = -\frac{1}{2}\left(\frac{\mathbf{g}(\hat{\mathbf{z}})}{\partial \mathbf{z}}\right)^H W \mathbf{e}(\hat{\mathbf{z}})$$

$$\Delta \mathbf{z}^{\text{pseudo-Newton}} = \frac{1}{2}\left[U_{\mathbf{zz}} - \sum_{i=1}^{m} V_{\mathbf{zz}}^{(i)}\right]^{-1} \left(\frac{\mathbf{g}(\hat{\mathbf{z}})}{\partial \mathbf{z}}\right)^H W \mathbf{e}(\hat{\mathbf{z}})$$

$$= \left[\left(\frac{\partial \mathbf{g}}{\partial \mathbf{z}}\right)^H W \left(\frac{\partial \mathbf{g}}{\partial \mathbf{z}}\right) - \sum_{i=1}^{m} \frac{\partial}{\partial \mathbf{z}}\left(\frac{\partial g_i(\mathbf{z})}{\partial \mathbf{z}}\right)^H [W\mathbf{e}]_i\right]^{-1} \left(\frac{\mathbf{g}(\hat{\mathbf{z}})}{\partial \mathbf{z}}\right)^H W \mathbf{e}(\hat{\mathbf{z}})$$

Stability generally requires positive definiteness of $\mathcal{H}_{\mathbf{zz}}$.

The pseudo-Newton is expected to be fast, but have a loss of efficiency relative to the Newton algorithm. When $\mathbf{g}(\mathbf{z})$ is holomorphic and onto, we expect good performance as asymptotically a stabilized pseudo-Newton algorithm will coincide with the Newton algorithm. If $\mathbf{g}(\mathbf{z})$ is nonholomorphic, the pseudo-Newton and Newton algorithms will not coincide asymptotically, so the speed of the pseudo-Newton algorithm is expected to always lag the Newton algorithm.

The algorithm suggested by Yan and Fan in [34] corresponds in the above taxonomy to the pseudo-Newton algorithm. We see that for obtaining a least-squares solution to the nonlinear inverse problem $\mathbf{y} = \mathbf{g}(\mathbf{z})$, if $\mathbf{g}$ is holomorphic, then the Yan and Fan suggestion can result in a good approximation to the Newton algorithm. However, for nonholomorphic least-squares inverse problems and for other types of optimization problems (including the problem considered by Yan and Fan in [34]), the approximation suggested by Yan and Fan is *not* guaranteed to provide a good *approximation* to the Newton algorithm.[94] However, as we have discussed, it does result in an admissible generalized gradient descent method in its own right, and, as such, one can judge the resulting algorithm on its own merits and in comparison with other competitor algorithms.

Equality Constraints. The classical approach to incorporating equality constraints into the problem of optimizing a scalar cost function is via the method of Lagrange multipliers. The theory of Lagrange multipliers is well-posed when the objective function and constraints are *real-valued* functions of real unknown variables. Note that a vector of p *complex* equality constraint conditions,

$$\mathbf{g}(\mathbf{z}) = 0 \in \mathbb{C}^p$$

---

[94]Such a a claim *might* be true. However, it would have to be justified.

is equivalent to $2p$ *real* equality constraints corresponding to the conditions

$$\operatorname{Re} \mathbf{g}(\mathbf{z}) = 0 \in \mathbb{R}^p \quad \text{and} \quad \operatorname{Im} \mathbf{g}(\mathbf{z}) = 0 \in \mathbb{R}^p \,.$$

Thus, given the problem of optimizing a real scalar-valued loss function $\ell(\mathbf{z})$ subject to a vector of p complex equality constraints constraints $\mathbf{h}(\mathbf{z}) = 0$, one can construct a well-defined lagrangian as

$$\mathfrak{L} = \ell(\mathbf{z}) + \lambda_R^T \operatorname{Re} \mathbf{g}(\mathbf{z}) + \lambda_I^T \operatorname{Im} \mathbf{g}(\mathbf{z}) \,, \tag{146}$$

for *real-valued* p-dimensional lagrange multiplier vectors λ_R and λ_I.

If we define the *complex lagrange multiplier vector* λ by

$$\lambda = \lambda_R + j \, \lambda_I \in \mathbb{C}^p$$

it is straightforward to show that the lagrangian (146) can be equivalently written as

$$\mathfrak{L} = \ell(\mathbf{z}) + \operatorname{Re} \lambda^H \mathbf{g}(\mathbf{z}) \,. \tag{147}$$

One can now apply the multivariate $\mathbb{C}\mathbb{R}$-Calculus developed in this note to find a stationary solution to the Lagrangian (147). Of course, subtle issues involving the application of the $\mathbf{z}$, $\mathbf{c}$-complex, and $\mathbf{c}$-real perspectives to the problem will likely arise on a case-by-case basis.

Final Comments on the 2nd Order Analysis.   It is evident that the analysis of second-order properties of a real-valued function on $\mathbb{C}^n$ is much more complicated than in the purely real case [25], perhaps dauntingly so. Thus, it is perhaps not surprising that very little analysis of these properties can be found in the literature.[95] By far, the most illuminating is the paper by Van den Bos [27], which, unfortunately, is very sparse in its explanation.[96] A careful reading of Van den Bos indicates that he is fully aware that there are two interpretations of $\mathbf{c}$, the real interpretation and the complex interpretation. *This is a key insight.* As we have seen above, it provides a very powerful analysis and algorithm development tool which allows us to switch between the $\mathbf{c}$-real interpretation (which enables us to use the tools and insights of real analysis) and the $\mathbf{c}$-complex perspective (which is shorthand for working at the algorithm implementation level of $\mathbf{z}$ and $\bar{\mathbf{z}}$). The now-classic paper by Brandwood [14] presents a development of the complex vector calculus using the $\mathbf{c}$-complex perspective which, although adequate for the development of first-order algorithms, presents greater difficulties when used as a tool for second order algorithm development. In this note, we've exploited the insight provided by Van den Bos [27] to perform a more careful, yet still preliminary, analysis of second-order Newton and Gauss-Newton algorithms. Much work remains to explore the analytical, structural, numerical, and implementation properties of these, and other second order, algorithms.

---

[95]That I could find. Please alert me to any relevant references that I am ignorant of!

[96]Likely a result of page limitations.

7 Applications

1. A Simple "Nonlinear" Least Squares Problem - I. This is a simple, but interesting, problem which is nonlinear in $z \in \mathbb{C}$ yet linear in $\mathbf{c} \in \mathcal{C} \subset \mathbb{C}^2$.

Let $z \in \mathbb{C}$ be an unknown scalar complex quantity we wish to estimate from multiple iid noisy measurements,

$$y_k = s + n_k,$$

$k = 1, \cdots, n$, of a scalar signal $s \in \mathbb{C}$ which is related to z via

$$s = g(z), \quad g(z) = \alpha z + \beta \bar{z}.$$

where $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$ are known complex numbers. It is assumed that the measurement noise n_k is iid and (complex) Gaussian, $n_k \sim N(0, \sigma^2 I)$, with σ^2 known. Note that the function $g(z)$ is both nonlinear in z (because complex conjugation is a nonlinear operation on z) and nonholomorphic (nonanalytic in z). However, because the problem must be linear in the underlying real space $\mathcal{R} = \mathbf{R}^2$ (a fact which shows up in the obvious fact that the function g is linear in $\mathbf{c}$), we expect that this problem should be exactly solvable, as will be shown to indeed be the case.

Under the above assumptions the maximum likelihood estimate (MLE) is found by minimizing the loss function [15][97]

$$
\begin{aligned}
\ell(z) &= \frac{1}{2n} \sum_{k=1}^{n} \|y_k - g(z)\|^2 \\
&= \frac{1}{n} \sum_{k=1}^{n} \|y_k - \alpha z - \beta \bar{z}\|^2 \\
&= \frac{1}{2n} \sum_{k=1}^{n} \overline{(y_k - \alpha z - \beta \bar{z})}(y_k - \alpha z - \beta \bar{z}) \\
&= \frac{1}{2n} \sum_{k=1}^{n} (\bar{y}_k - \bar{\alpha}\bar{z} - \bar{\beta}z)(y_k - \alpha z - \beta \bar{z}).
\end{aligned}
$$

Note that this is a *nonlinear least-squares problem* as the function $g(z)$ is *nonlinear* in z.[98] Furthermore, $g(z)$ is *nonholomorphic* (nonanalytic in z). Note, however, that although $g(z)$ is *nonlinear in z*, it *is linear* in $\mathbf{c} = (z, \bar{z})^T$, and that as a consequence the loss function $\ell(z) = \ell(\mathbf{c})$ has an *exact* second order expansion in $\mathbf{c}$ of the form (92), which can be verified by a simple expansion of $\ell(z)$ in terms of z and $\bar{z}$ (see below). The corresponding $\mathbf{c}$-complex Hessian matrix (to be computed below) *does not* have zero off-diagonal entries, which shows that a loss function being quadratic does not *alone* ensure that $\mathcal{H}_{\bar{\mathbf{z}}\mathbf{z}} = 0$, a fact which contradicts the claim made in [34].

---

[97]The additional overall factor of $\frac{1}{n}$ has been added for convenience.
[98]Recall that complex conjugation is a nonlinear operation.

Defining the sample average of n samples $\{\xi_1, \cdots, \xi_k\}$ by

$$\langle \xi \rangle \triangleq \frac{1}{n} \sum_{k=1}^{n} \xi_k$$

the loss function $\ell(z)$ can be expanded and rewritten as

$$2\,\ell(z) = \left\langle |y|^2 \right\rangle + \alpha\bar{\beta}z^2 - \left(\alpha \langle \bar{y} \rangle + \bar{\beta} \langle y \rangle\right) z + \left(|\alpha|^2 + |\beta|^2\right) z\bar{z} - \left(\bar{\alpha} \langle y \rangle + \beta \langle \bar{y} \rangle\right) \bar{z} + \bar{\alpha}\beta\bar{z}^2 \quad (148)$$

or

$$\ell(z) = \frac{1}{2} \left\langle |y|^2 \right\rangle - \frac{1}{2} \begin{pmatrix} \alpha \langle \bar{y} \rangle + \bar{\beta} \langle y \rangle & \bar{\alpha} \langle y \rangle + \beta \langle \bar{y} \rangle \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \end{pmatrix} + \frac{1}{4} \begin{pmatrix} z \\ \bar{z} \end{pmatrix}^H \begin{pmatrix} |\alpha|^2 + |\beta|^2 & 2\bar{\alpha}\beta \\ 2\alpha\bar{\beta} & |\alpha|^2 + |\beta|^2 \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \end{pmatrix}.$$

Since this expansion is done using the **z**-perspective, we expect that it corresponds to a second order expansion about the value $\hat{\mathbf{z}} = 0$,

$$\ell(z) = \ell(0) + \frac{\partial \ell(0)}{\partial \mathbf{c}} \mathbf{c} + \frac{1}{2} \mathbf{c}^H \mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}(0) \mathbf{c} \quad (149)$$

with

$$\frac{\partial \ell(0)}{\partial \mathbf{c}} = \begin{pmatrix} \frac{\partial \ell(0)}{\partial \mathbf{z}} & \frac{\partial \ell(0)}{\partial \bar{\mathbf{z}}} \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} \alpha \langle \bar{y} \rangle + \bar{\beta} \langle y \rangle & \bar{\alpha} \langle y \rangle + \beta \langle \bar{y} \rangle \end{pmatrix}$$

and

$$\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}(0) = \frac{1}{2} \begin{pmatrix} |\alpha|^2 + |\beta|^2 & 2\,\bar{\alpha}\beta \\ 2\,\alpha\bar{\beta} & |\alpha|^2 + |\beta|^2 \end{pmatrix}.$$

And indeed this turns out to be the case. Simple differentiation of (148) yields,

$$\frac{\partial \ell(z)}{\partial z} = \alpha\bar{\beta}z + \frac{1}{2}\left(|\alpha|^2 + |\beta|^2\right)\bar{z} - \frac{1}{2}\left(\alpha \langle \bar{y} \rangle + \bar{\beta} \langle y \rangle\right)$$

$$\frac{\partial \ell(z)}{\partial \bar{z}} = \bar{\alpha}\beta\bar{z} + \frac{1}{2}\left(|\alpha|^2 + |\beta|^2\right) z - \frac{1}{2}\left(\bar{\alpha} \langle y \rangle + \beta \langle \bar{y} \rangle\right)$$

which evaluated at zero give the linear term in the quadratic loss function, and further differentiations yield,

$$\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}(z) = \begin{pmatrix} \mathcal{H}_{zz} & \mathcal{H}_{\bar{z}z} \\ \mathcal{H}_{z\bar{z}} & \mathcal{H}_{\bar{z}\bar{z}} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} |\alpha|^2 + |\beta|^2 & 2\,\bar{\alpha}\beta \\ 2\,\alpha\bar{\beta} & |\alpha|^2 + |\beta|^2 \end{pmatrix}$$

which is independent of z. Note that, as expected,

$$\frac{\partial \ell(z)}{\partial \bar{z}} = \overline{\frac{\partial \ell(z)}{\partial z}}.$$

If we set the two partial derivatives to zero, we obtain two stationarity equations for the two stationary quantities z and $\bar{z}$. Solving for z then yields the least-squares estimate of z,[99]

$$\hat{z}_{\text{opt}} = \frac{1}{|\alpha|^2 - |\beta|^2} \left(\bar{\alpha} \langle y \rangle - \beta \langle \bar{y} \rangle\right).$$

---

[99]Note that this answer reduces to the obvious solutions for the two special cases $\alpha = 0$ and $\beta = 0$.

This solution can also be obtained by completing the square on (149) to obtain

$$\hat{\mathbf{c}}_{\mathrm{opt}} = - \left(\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}\right)^{-1} \left(\frac{\partial \ell(0)}{\partial \mathbf{c}}\right)^H$$

An obvious necessary condition for the least-squares solution to exist is that

$$|\alpha|^2 \neq |\beta|^2 .$$

The solution will be a global[100] minimum if the Hessian matrix is positive definite. This will be true if the two leading principal minors are strictly positive, which is true if and only if, again, $|\alpha|^2 \neq |\beta|^2$. Thus, if $|\alpha|^2 \neq |\beta|^2$ the solution given above is a global minimum to the least squares problem.

The condition $|\alpha|^2 = |\beta|^2$ corresponds to *loss of identifiability* of the model

$$g(z) = \alpha z + \beta \bar{z} .$$

To see this, first note that to identify a complex number is equivalent to identifying both the real and imaginary parts of the number. If either of them is unidentifiable, then so is the number.

Now note that the condition $|\alpha|^2 = |\beta|^2$ says that α and β have the same magnitude, but, in general, a different phase. If we call the phase difference ϕ, then the condition $|\alpha|^2 = |\beta|^2$ is equivalent to the condition

$$\alpha = e^{j\phi}\beta ,$$

which yields

$$g(z) = e^{j\phi}\beta z + \beta \bar{z} = e^{j\frac{\phi}{2}}\beta \left(e^{j\frac{\phi}{2}}z + e^{-j\frac{\phi}{2}}\bar{z}\right) = e^{j\frac{\phi}{2}}\beta \left(e^{j\frac{\phi}{2}}z + \overline{e^{j\frac{\phi}{2}}z}\right) = e^{j\frac{\phi}{2}}\beta \operatorname{Re}\left\{e^{j\frac{\phi}{2}}z\right\}.$$

Thus, it is evident that the imaginary part of $e^{j\frac{\phi}{2}}z$ is unidentifiable, and thus the complex number $e^{j\frac{\phi}{2}}z$ itself is unidentifiable. And, since

$$z = e^{-j\frac{\phi}{2}}\left(e^{j\frac{\phi}{2}}z\right) = e^{-j\frac{\phi}{2}}\left(\operatorname{Re}\left\{e^{j\frac{\phi}{2}}z\right\} + j\operatorname{Im}\left\{e^{j\frac{\phi}{2}}z\right\}\right),$$

it is obvious that z is unidentifiable.

Note for the simplest case of $\alpha = \beta$ ($\phi = 0$), we have

$$g(z) = \alpha z + \alpha \bar{z} = \alpha \operatorname{Re}\left\{z\right\}$$

in which case $\operatorname{Im}\left\{z\right\}$, and hence z, is unidentifiable.

---

[100]Because the Hessian is independent of z.

2. A Simple "Nonlinear" Least Squares Problem - II. The "nonlinearity" encountered in the previous example, is in a sense "bogus" and is not a nonlinearity at all, at least when viewed from the **c**-real perspective. Not surprisingly then, we were able to compute an exact solution. Here, we will briefly look at the Newton and Gauss-Newton algorithms applied to the simple problem of Example 1.

In the previous example, we computed the Newton Hessian of the least-squares loss function (148). The difference between the Newton and Gauss-Newton algorithm resides in the difference between the Newton Hessian and the Gauss-Newton Hessian. To compute the Gauss-Newton Hessian, note that

$$y = g(\mathbf{c}) = (\alpha\ \beta)\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = G\mathbf{c}$$

and therefore (since the problem is linear in **c**) we have the not surprising result that

$$G\Delta\mathbf{c} = \frac{\partial g(\mathbf{c})}{\partial \mathbf{c}}\Delta\mathbf{c}$$

with

$$G = (\alpha\ \beta).$$

In this example, the least-squares weighting matrix is $W = I$ and we have

$$G^H W G = G^H G = \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix}(\alpha\ \beta) = \begin{pmatrix} |\alpha|^2 & \bar{\alpha}\beta \\ \bar{\beta}\alpha & |\beta|^2 \end{pmatrix}$$

which is seen to be independent of **c**. From (121), we construct the Gauss-Newton Hessian as

$$\mathcal{H}^{\text{Gauss}}_{\mathbf{cc}} = \mathbf{P}\left(G^H G\right) = \frac{\begin{pmatrix} |\alpha|^2 & \bar{\alpha}\beta \\ \bar{\beta}\alpha & |\beta|^2 \end{pmatrix} + S\overline{\begin{pmatrix} |\alpha|^2 & \bar{\alpha}\beta \\ \bar{\beta}\alpha & |\beta|^2 \end{pmatrix}}S}{2} = \frac{1}{2}\begin{pmatrix} |\alpha|^2 + |\beta|^2 & 2\,\bar{\alpha}\beta \\ 2\,\alpha\bar{\beta} & |\alpha|^2 + |\beta|^2 \end{pmatrix} = \mathcal{H}^{\mathbb{C}}_{\mathbf{cc}}$$

showing that for this simple example the Newton and Gauss-Newton Hessians are the same, and therefore *the Newton and Gauss-Newton algorithms are identical.* As seen from Equations (129) and (131), this is a consequence of the fact that $g(\mathbf{c})$ is linear in **c** as then the matrix of second partial derivatives of g required to compute the difference between the Newton and Gauss-Newton algorithms vanishes

$$A_{\mathbf{cc}}(g) \triangleq \frac{\partial}{\partial \mathbf{c}}\left(\frac{\partial g}{\partial \mathbf{c}}\right)^H = 0.$$

From the derivatives computed in the previous example, we can compute $\left(\frac{\partial \ell(\hat{\mathbf{c}})}{\partial \mathbf{c}}\right)^H$ as

$$\left(\frac{\partial \ell(\hat{\mathbf{c}})}{\partial \mathbf{c}}\right)^H = \begin{pmatrix} \left(\frac{\partial \ell(\hat{\mathbf{c}})}{\partial \mathbf{z}}\right)^H \\ \left(\frac{\partial \ell(\hat{\mathbf{c}})}{\partial \bar{\mathbf{z}}}\right)^H \end{pmatrix} = \begin{pmatrix} \left(\frac{\partial \ell(\mathbf{0})}{\partial \mathbf{z}}\right)^H \\ \left(\frac{\partial \ell(\mathbf{0})}{\partial \bar{\mathbf{z}}}\right)^H \end{pmatrix} + \frac{1}{2}\begin{pmatrix} |\alpha|^2 + |\beta|^2 & 2\,\bar{\alpha}\beta \\ 2\,\alpha\bar{\beta} & |\alpha|^2 + |\beta|^2 \end{pmatrix}\begin{pmatrix} \hat{z} \\ \hat{\bar{z}} \end{pmatrix}$$

or

$$\left(\frac{\partial \ell(\hat{\mathbf{c}})}{\partial \mathbf{c}}\right)^H = \left(\frac{\partial \ell(0)}{\partial \mathbf{c}}\right)^H + \mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}\hat{\mathbf{c}}.$$

The optimal update in the Newton algorithm is therefore given by

$$\widehat{\Delta \mathbf{c}} = -\left(\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}\right)^{-1}\left(\frac{\partial \ell(\hat{\mathbf{c}})}{\partial \mathbf{c}}\right)^H = -\left(\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}\right)^{-1}\left(\frac{\partial \ell(0)}{\partial \mathbf{c}}\right)^H - \hat{\mathbf{c}} = \hat{\mathbf{c}}_{\mathrm{opt}} - \hat{\mathbf{c}}\,.$$

The update step in the Newton algorithm is given by

$$\hat{\mathbf{c}}_{\mathrm{new}} = \hat{\mathbf{c}} + \alpha\widehat{\Delta \mathbf{c}}\,.$$

If we take the "Newton stepsize" $\alpha = 1$, we obtain

$$\hat{\mathbf{c}}_{\mathrm{new}} = \hat{\mathbf{c}} + \widehat{\Delta \mathbf{c}} = \hat{\mathbf{c}} + \hat{\mathbf{c}}_{\mathrm{opt}} - \hat{\mathbf{c}} = \hat{\mathbf{c}}_{\mathrm{opt}}$$

showing that we can attain the optimal solution in only one update step. For the real case, it is well-known that the Newton algorithm attains the optimum in one step for a quadratic loss function. Thus our result is not surprising given that the problem is a linear least-squares problem in $\mathbf{c}$.

Note that the off-diagonal elements of the constant-valued Hessian $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$ *are never zero* and generally are not small relative to the size of the diagonal elements of $\mathcal{H}_{\mathbf{cc}}^{\mathbb{C}}$. This contradicts the statement made in [34] that for a quadratic loss function, the diagonal elements must be zero.[101] However, the pseudo-Newton algorithm proposed in [34] will converge to the correct solution when applied to our problem, but at a slower convergent rate than the full Newton algorithm, which is seen to be capable of providing one-step convergence. We have a trade off between complexity (the less complex pseudo-Newton algorithm versus the more complex Newton algorithm) versus speed of convergence (the slower converging pseudo-Newton algorithm versus the fast Newton algorithm).

3. The Complex LMS Algorithm.   Consider the problem of determining the complex *vector* parameter $a \in \mathbb{C}^n$ which minimizes the following generalization of the loss function (2) to the vector parameter case,

$$\ell(a) = \mathrm{E}\left\{|e_k|^2\right\}, \qquad e_k = \eta_k - a^H\xi_k, \tag{150}$$

for $\eta_k \in \mathbb{C}$ and $\xi_k \in \mathbb{C}^n$. We will assume throughout that the parameter space is Euclidean so that $\Omega_a = I$. The cogradient of $\ell(a)$ with respect to the unknown parameter vector a is given by

$$\frac{\partial}{\partial a}\,\ell(a) = \mathrm{E}\left\{\frac{\partial}{\partial a}\,|e|^2\right\}\,.$$

---

[101] It is true, as we noted above, that for the quadratic loss function associated with a holomorphic nonlinear inverse problem the off-diagonal elements of the Hessian are zero. However, the statement is not true in general.

To determine the cogradient of

$$|e_k|^2 = \bar{e}_k e_k = e_k \bar{e}_k = (\eta_k - a^H \xi_k)\overline{(\eta_k - a^H \xi_k)}$$

note that

$$\bar{e}_k = \overline{(\eta_k - a^H \xi_k)} = (\bar{\eta}_k - \xi_k^H a)$$

and that $e_k = (\eta_k - a^H \xi_k)$ is independent of a. Then we have

$$
\begin{aligned}
\frac{\partial}{\partial a}\, e_k \bar{e}_k &= e_k \frac{\partial}{\partial a}\, (\bar{\eta}_k - \xi_k^H a) \\
&= -e_k \frac{\partial}{\partial a}\, \xi_k^H a \\
&= -e_k \xi_k^H \, .
\end{aligned}
$$

The gradient of $|e_k|^2 = e_k \bar{e}_k$ is given by

$$\nabla_a e_k \bar{e}_k = \left(\frac{\partial}{\partial a}\, e_k \bar{e}_k\right)^H = -\left(e_k \xi_k^H\right)^H = -\xi_k \bar{e}_k \, .$$

Thus, we readily have that the gradient (direction of steepest ascent) of the loss function $\ell(a) = \mathrm{E}\left\{|e_k|^2\right\}$ is

$$\nabla_a \ell(a) = -\mathrm{E}\left\{\xi_k \bar{e}_k\right\} = -\mathrm{E}\left\{\xi_k \left(\bar{\eta}_k - \xi_k^H a\right)\right\} \, .$$

If we set this (or the cogradient) equal to zero to determine a stationary point of the loss function we obtain the standard Wiener-Hopf equations for the MMSE estimate of a.[102]

Alternatively, if we make the *instantaneous stochastic-gradient approximation,*

$$\nabla_a \ell(a) \approx \widehat{\nabla}_a \ell(\widehat{a}_k) \triangleq \nabla_a |e_k|^2 = -\xi_k \bar{e}_k = \xi_k \left(\bar{\eta}_k - \xi_k^H \widehat{a}_k\right) \, ,$$

where $\widehat{a}_k$ is a current estimate of the MMSE value of a and $-\nabla_a \ell(a)$ gives the direction of steepest descent of $\ell(a)$, we obtain the standard LMS on-line stochastic gradient-descent algorithm for learning an estimate of the complex vector a,

$$
\begin{aligned}
\widehat{a}_{k+1} &= \widehat{a}_k - \alpha_k \widehat{\nabla}_a \ell(\widehat{a}_k) \\
&= \widehat{a}_k + \alpha_k \xi_k \bar{e}_k \\
&= \widehat{a}_k + \alpha_k \xi_k \left(\bar{\eta}_k - \xi_k^H \widehat{a}_k\right) \\
&= \left(I - \alpha_k \xi_k \xi_k^H\right) \widehat{a}_k + \alpha_k \xi_k \bar{\eta}_k \, .
\end{aligned}
$$

Thus, we have easily derived the complex LMS algorithm,

$$\text{Complex LMS Algorithm:} \qquad \widehat{a}_{k+1} = \left(I - \alpha_k \xi_k \xi_k^H\right) \widehat{a}_k + \alpha_k \xi_k \bar{\eta}_k \, . \tag{151}$$

---

[102]Which, as mentioned earlier, can also be obtained from the orthogonality principle or completing the square. Thus, if the Wiener-Hopf equations are our only goal there is no need to discuss complex derivatives at all. It is only when a direction of steepest descent is needed in order to implement an on-line adaptive descent-like algorithm that the need for the extended or conjugate derivative arises.

References

[1] *Optimum Array Processing,* H.L. Van Trees, 2002, Wiley Interscience.

[2] *Elements of Signal Detection & Estimation,* C.W. Helstrom, 1995, Prentice Hall.

[3] *Complex Variables and Applications,* 6nd Edition, J. Brown & R. Churchill, 1996 McGraw-Hill, New York.

[4] *Complex Variables,* 2nd Edition, S. Fisher, 1990/1999, Dover Publications, New York.

[5] *Complex Variables and the Laplace Transform for Engineers,* W. LePage, 1980/1961, Dover Publications, New York.

[6] *Complex Variables: Harmonic and Analytic Functions,* F. Flanigan, 1972/1983, Dover Publications, New York.

[7] *Principles of Mobile Communication,* 2nd Edition, G.L. Stuber, 2001, Kluwer, Boston.

[8] *Digital Communication,* E. Lee & D. Messerchmitt, 1988, Kluwer, Boston.

[9] *Introduction to Adaptive Arrays,* R. Monzingo & T. Miller, 1980, Wiley, New York.

[10] "Zur Formalen Theorie der Funktionen von mehr Complexen Veränderlichen," W. Wirtinger, *Math. Ann.,* 97: 357-75, 1927

[11] *Introduction to Complex Analysis,* Z. Nehari, 1961, Allyn & Bacon, Inc.

[12] *Theory of Complex Functions,* R. Remmert, 1991, Springer-Verlag.

[13] *Precoding and Signal Shaping for Digital Transmission*, Robert Fischer, 2002, Wiley-Interscience. Especially Appendix A: "Wirtinger Calculus."

[14] "A Complex Gradient Operator and its Application in Adaptive Array Theory," D.H. Brandwood, *IEE Proceedings H (Microwaves, Optics, and Antennas)* (British), Vol. 130, No. 1, pp. 11-16, Feb. 1983.

[15] *Fundamentals of Statistical Signal Processing: Estimation Theory*, S.M. Kay, 1993, Prentice-Hall, New Jersey. Chapter 15: "Extensions for Complex Data and Parameters."

[16] *Adaptive Filter Theory,* 3rd Edition, S. Haykin, 1991, Prentice-Hall, New Jersey. Especially Appendix B: "Differentiation with Respect to a Vector."

[17] *Theory of Functions on Complex Manifolds,* G.M. Henkin & J. Leiterer, 1984, Birkäuser.

[18] *An Introduction to Complex Analysis in Several Variables,* Revised 3rd Edition, Lars Hörmander, 1990, North-Holland. (First Edition was published in 1966.)

[19] *Several Complex Variables,* Corrected 2nd Edition, 2001, S.G. Krantz, AMS Chelsea Publishing.

[20] *Introduction to Complex Analysis, Part II: Functions of Several Variables,* B.V. Shabat, American Mathematical Society, 1992.

[21] *Geometrical Methods of Mathematical Physics,* Bernard Schutz,1980, Cambridge University Press

[22] *The Geometry of Physics, An Introduction,* (with corrections and additions), Theodore Frankel, 2001, Cambridge University Press.

[23] *Fundamentals of Adaptive Filtering,* A.H. Sayed, 2003, Wiley.

[24] "Finite Dimensional Hilbert Spaces and Linear Inverse Problems," ECE275A Lecture Supplement No. ECE275LS1-F05v1.1, K. Kreutz-Delgado, UCSD ECE Department, October 2005.

[25] "Real Vector Derivatives and Gradient," ECE275A Lecture Supplement No. ECE275LS2-F05v1.0, K. Kreutz-Delgado, UCSD ECE Department, October 2005.

[26] "Natural Gradient Works Efficiently in Learning," Shun-ichi Amari, *Neural Computation,* 10:251-76, 1998

[27] "Complex Gradient and Hessian," A. van den Bos, *IEE Proc.-Vis. Image Signal Processing* (British), 141(6):380-82, December 1994.

[28] "A Cramér-Rao Lower Bound for Complex Parameters," A. van den Bos, *IEEE Transactions on Signal Processing*, 42(10):2859, October 1994.

[29] "Estimation of Complex Fourier Coefficients," A. van den Bos, *IEE Proc.-Control Theory Appl.* (British), 142(3):253-6, May 1995.

[30] "The Multivariate Complex Normal Distribution-A Generalization," A. van den Bos, *IEEE Transactions on Information Theory*, 41(2):537-39, March 1995.

[31] "Price's Theorem for Complex Variates," A. van den Bos, *IEEE Transactions on Information Theory*, 42(1):286-7, January 1996.

[32] "The Real-Complex Normal Distribution," A. van den Bos, *IEEE Transactions on Information Theory*, 44(4):537-39, July 1998.

[33] "Complex Digital Networks: A Sensitivity Analysis Based on the Wirtinger Calculus," D. Franken, *IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications,* 44(9):839-43, November 1997.

[34] "A Newton-Like Algorithm for Complex Variables with Applications in Blind Equalization," G. Yan & H. Fan, *IEEE Proc. Signal Processing,* 48(2):553-6, February 2000.

[35] "Cramér-Rao Lower Bound for Constrained Complex Parameters," A.K. Jagannatham & B.D. Rao, *IEEE Signal Proc. Letters,* 11(11):875-8, November 2004.

[36] *Optimization by Vector Space Methods,* D.G. Luenberger, Wiley, 1969.

[37] *Linear Operator Theory in Engineering and Science,* A.W. Naylor & G.R. Sell, Springer-Verlag, 1982.

[38] "The Constrained Total Least Squares Technique and its Application to Harmonic Super-resolution," T.J. Abatzoglou, J.M. Mendel, & G.A. Harada, *IEEE Transactions on Signal Processing,* 39(5):1070-86, May 1991.