

## UNIT II

### Relations

#### Introduction

The elements of a set may be related to one another. For example, in the set of natural numbers there is the less than' relation between the elements. The elements of one set may also be related to the elements another set.

#### Binary Relation

A binary relation between two sets A and B is a rule R which decides, for any elements, whether a is in relation R to b. If so, then we write  $a R b$ . If a is not in relation R to b, then  $a \not R b$ .

We can also consider  $a R b$  as the ordered pair (a, b) in which case we can define a binary relation from A to B as a subset of  $A \times B$ . This subset is denoted by the relation R.

In general, any set of ordered pairs defines a binary relation.

For example, the relation of father to his child is  $F = \{(a, b) / a \text{ is the father of } b\}$  In this relation F, the first member is the name of the father and the second is the name of the child.

The definition of relation permits any set of ordered pairs to define a relation.

For example, the set S given by

$$S = \{(1, 2), (3, a), (b, a), (b, \text{Joe})\}$$

Definition

The domain D of a binary relation S is the set of all first elements of the ordered pairs in the relation.(i.e)  $D(S) = \{a / \exists b \text{ for which } (a, b) \in S\}$

The range R of a binary relation S is the set of all second elements of the ordered pairs in the relation. (i.e)  $R(S) = \{b / \exists a \text{ for which } (a, b) \in S\}$

For example

For the relation  $S = \{(1, 2), (3, a), (b, a), (b, \text{Joe})\}$

$$D(S) = \{1, 3, b, b\} \text{ and}$$

$$R(S) = \{2, a, a, \text{Joe}\}$$

Let X and Y be any two sets. A subset of the Cartesian product  $X \times Y$  defines a relation, say C. For any such relation C, we have  $D(C) \subseteq X$  and  $R(C) \subseteq Y$ , and the relation C is said to from X to Y. If  $Y = X$ , then C is said to be a relation form X to X. In such case, c is called a relation in X. Thus any relation in X is a subset of  $X \times X$ . The set  $X \times X$  is called a *universal relation* in X, while the empty set which is also a subset of  $X \times X$  is called a *void relation* in X.

For example: Let L denote the relation —less than or equal to $\leq$  and D denote the relation —divides $\mid$  where  $x \mid y$  means — x divides y $\mid$ . Both L and D are defined on the set  $\{1, 2, 3, 4\}$

$L = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$

$D = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$

$L \cap D = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\} = D$

### Properties of Binary Relations:

**Definition:** A binary relation R in a set X is **reflexive** if, for every  $x \in X$ ,  $x R x$ , That is  $(x, x) \in R$ , or R is reflexive in X  $\iff (x) (x \in X \implies x R x)$ .

For example:-

- The relation  $\leq$  is reflexive in the set of real numbers.
- The set inclusion is reflexive in the family of all subsets of a universal set.
- The relation equality of set is also reflexive.
- The relation is parallel in the set lines in a plane.
- The relation of similarity in the set of triangles in a plane is reflexive.

**Definition:** A relation R in a set X is symmetric if for every x and y in X, whenever  $x R y$ , then  $y R x$ . (i.e) R is symmetric in X  $\iff (x) (y) (x \in X \wedge y \in X \wedge x R y \implies y R x)$

For example:-

- The relation equality of set is symmetric.
- The relation of similarity in the set of triangles in a plane is symmetric.
- The relation of being a sister is not symmetric in the set of all people.
- However, in the set females it is symmetric.

**Definition:** A relation R in a set X is **transitive** if, for every x, y, and z are in X, R is transitive in X  $\iff (x) (y) (z) (x \in X \wedge y \in X \wedge z \in X \wedge x R y \wedge y R z \implies x R z)$

For example:-

- The relations  $<$ ,  $\leq$ ,  $>$ ,  $\geq$  and  $=$  are transitive in the set of real numbers
- The relations  $\subset$ ,  $\supset$ ,  $\subseteq$ ,  $\supseteq$  and equality are also transitive in the family of sets.
- The relation of similarity in the set of triangles in a plane is transitive.

**Definition:** A relation  $R$  in a set  $X$  is **irreflexive** if, for every  $x \in X$ ,  $(x, x) \notin R$ .

**For example:-**

- < The relation  $<$  is irreflexive in the set of all real numbers.
- < The relation proper inclusion is irreflexive in the set of all nonempty subsets of a universal set.
- Let  $X = \{1, 2, 3\}$  and  $S = \{(1, 1), (1, 2), (3, 2), (2, 3), (3, 3)\}$  is neither irreflexive nor reflexive.

**Definition:** A relation  $R$  in a set  $x$  is **anti symmetric** if, for every  $x$  and  $y$  in  $X$ ,  
whenever  $x R y$  and  $y R x$ , Then  $x = y$ .

**Symbolically,**  $(x) (y) (x \in X \wedge y \in X \wedge x R y \wedge y R x \Rightarrow x = y)$

**For example**

- The relations  $\leq$ ,  $\geq$  and  $=$  are anti symmetric
- The relation  $\subset$  is anti symmetric in set of subsets.
- The relation  $\text{---divides---}$  is anti symmetric in set of real numbers.
- Consider the relation  $\text{---is a son of---}$  on the male children in a family. Evidently the relation is not symmetric, transitive and reflexive.
- The relation  $\text{---is a divisor of---}$  is reflexive and transitive but not symmetric on the set of natural numbers.
- Consider the set  $H$  of all human beings. Let  $r$  be a relation  $\text{---is married to---}$   
 $R$  is symmetric.
- Let  $I$  be the set of integers.  $R$  on  $I$  is defined as  $a R b$  if  $a - b$  is an even number.  $R$  is an reflexive, symmetric and transitive

### **Equivalence Relation:**

**Definition:** A relation  $R$  in a set  $A$  is called an **equivalence** relation if

- $a R a$  for every  $a$  i.e.  $R$  is reflexive
- $a R b \Rightarrow b R a$  for every  $a, b \in A$  i.e.  $R$  is symmetric
- $a R b$  and  $b R c \Rightarrow a R c$  for every  $a, b, c \in A$ , i.e.  $R$  is transitive.

**For example**

- < The relation equality of numbers on set of real numbers.
- < The relation being parallel on a set of lines in a plane.

**Problem1:** Let

**R in T as  $R = \{(a, b) / (a, b) \in T \text{ and } a \text{ is similar to } b\}$**   
We have to show that relation R is an Equivalence relation

**Solution :**

- < A triangle a is similar to itself.  $a R a$
- < If the triangle a is similar to the triangle b, then triangle b is similar to the triangle a then  $a R b \Rightarrow b R a$
- < If a is similar to b and b is similar to c, then a is similar to c (i.e)  $a R b$  and  $b R c \Rightarrow a R c$ .

**Hence R is an equivalence relation.**

**Problem 2:** Let  $x = \{1, 2, 3, \dots, 7\}$  and  $R = \{(x, y) / x - y \text{ is divisible by } 3\}$  Show that R is an equivalence relation.

**Solution:** For any  $a \in X$ ,  $a - a$  is divisible by 3, Hence  $a R a$ , R is reflexive

For any  $a, b \in X$ , if  $a - b$  is divisible by 3, then  $b - a$  is also divisible by 3, R is symmetric.

For any  $a, b, c \in X$ , if  $a R b$  and  $b R c$ , then  $a - b$  is divisible by 3 and  $b - c$  is divisible by 3. So that  $(a - b) + (b - c)$  is also divisible by 3, hence  $a - c$  is also divisible by 3. Thus R is transitive.

**Hence R is equivalence.**

**Problem3.** Let Z be the set of all integers. Let m be a fixed integer. Two integers a and

b are said to be congruent modulo m if and only if m divides  $a - b$ , in which case we write  $a \equiv b \pmod{m}$ .

This relation is called the relation of congruence modulo m and we can show that is an equivalence relation.

**Solution :**

- <  $a - a = 0$  and m divides  $a - a$  (i.e)  $a R a$ ,  $(a, a) \in R$ , R is reflexive .
- <  $a R b \Rightarrow m \text{ divides } a - b$

m divides  $b - a \Rightarrow b \equiv a \pmod{m} \Rightarrow a R b$  that is R is symmetric.

- $a R b$  and  $b R c \Rightarrow a \equiv b \pmod{m}$  and  $b \equiv c \pmod{m}$   $\Rightarrow m \text{ divides } a - b \text{ and } m \text{ divides } b - c$   
 $\Rightarrow a - b = km$  and  $b - c = lm$  for some  $k, l \in \mathbb{Z}$   
 $\Rightarrow (a - b) + (b - c) = km + lm$   
 $\Rightarrow a - c = (k + l)m$

$$O \quad a \equiv c \pmod{m}$$

$$O \quad a R c$$

**O R is transitive**

**Hence the congruence relation is an equivalence relation.**

### Equivalence Classes:

Let  $R$  be an equivalence relation on a set  $A$ . For any  $a \in A$ , the equivalence class generated by  $a$  is the set of all elements  $b \in A$  such  $a R b$  and is denoted  $[a]$ . It is also called the  $R$  – equivalence class and denoted by  $a \in A$ . i.e.,  $[a] = \{b \in A / b R a\}$

Let  $Z$  be the set of integer and  $R$  be the relation **called —congruence modulo 3** defined by  $R = \{(x, y) / x \hat{I} Z \hat{U} y \hat{I} Z \hat{U} (x-y) \text{ is divisible by } 3\}$

**Then the equivalence classes are**

$$[0] = \{\dots -6, -3, 0, 3, 6, \dots\}$$

$$[1] = \{\dots, -5, -2, 1, 4, 7, \dots\}$$

$$[2] = \{\dots, -4, -1, 2, 5, 8, \dots\}$$

### Composition of binary relations:

**Definition:** Let  $R$  be a relation from  $X$  to  $Y$  and  $S$  be a relation from  $Y$  to  $Z$ . Then the relation  $R \circ S$  is given by  $R \circ S = \{(x, z) / x \hat{I} X \hat{U} z \hat{I} Z \hat{U} y \hat{I} Y \text{ such that } (x, y) \hat{I} R \hat{U} (y, z) \hat{I} S\}$  is called the composite relation of  $R$  and  $S$ .

The operation of obtaining  $R \circ S$  is called the **composition of relations**.

**Example:** Let  $R = \{(1, 2), (3, 4), (2, 2)\}$  and

$$S = \{(4, 2), (2, 5), (3, 1), (1, 3)\}$$

$$\text{Then } R \circ S = \{(1, 5), (3, 2), (2, 5)\} \text{ and } S \circ R = \{(4, 2), (3, 2), (1, 4)\}$$

It is to be noted that  $R \circ S \neq S \circ R$ .

$$\text{Also } R \circ (S \circ T) = (R \circ S) \circ T = R \circ S \circ T$$

**Note:** We write  $R \circ R$  as  $R^2$ ;  $R \circ R \circ R$  as  $R^3$  and so on.

### Definition

Let  $R$  be a relation from  $X$  to  $Y$ , a relation  $\check{R}$  from  $Y$  to  $X$  is called the converse of  $R$ , where the ordered pairs of  $\check{R}$  are obtained by interchanging the numbers in each of the ordered pairs of  $R$ . This means for  $x \hat{I} X$  and  $y \hat{I} Y$ , that  $x R y \text{ ó } y \check{R} x$ .

Then the relation  $\check{R}$  is given by  $\check{R} = \{(x, y) / (y, x) \hat{I} R\}$  is called the converse of  $R$  Example:

$$\text{Let } R = \{(1, 2), (3, 4), (2, 2)\}$$

$$\text{Then } \check{R} = \{(2, 1), (4, 3), (2, 2)\}$$

**Note:** If  $R$  is an equivalence relation, then  $\check{R}$  is also an equivalence relation.

**Definition** Let  $X$  be any finite set and  $R$  be a relation in  $X$ . The relation  $R^+ = R \cup R^2 \cup R^3 \dots$  in  $X$  is called the *transitive closure* of  $R$  in  $X$

Example: Let  $R = \{(a, b), (b, c), (c, a)\}$ .

Now  $R^2 = R \circ R = \{(a, c), (b, a), (c, b)\}$

$R^3 = R^2 \circ R = \{(a, a), (b, b), (c, c)\}$

$R^4 = R^3 \circ R = \{(a, b), (b, c), (c, a)\} = R$

$R^5 = R^3 \circ R^2 = R^2$  and so on.

Thus,  $R^+ = R \cup R^2 \cup R^3 \cup R^4 \cup \dots$

$= R \cup R^2 \cup R^3.$

$= \{(a, b), (b, c), (c, a), (a, c), (b, a), (c, b), (a, a), (b, b), (c, c)\}$

**We see that  $R^+$  is a transitive relation containing  $R$ . In fact, it is the smallest transitive relation containing  $R$ .**

## Partial Ordering Relations:

### Definition

A binary relation  $R$  in a set  $P$  is called *partial order relation* or *partial ordering* in  $P$  iff  $R$  is reflexive, anti symmetric, and transitive.

A partial order relation is denoted by the symbol  $\leq$ . If  $\leq$  is a partial ordering on  $P$ , then the ordered pair  $(P, \leq)$  is called a *partially ordered set* or a *poset*.

< Let  $R$  be the set of real numbers. The relation —less than or equal to  $<$  or  $\leq$ , is a partial ordering on  $R$ .

< Let  $X$  be a set and  $r(X)$  be its power set. The relation subset,  $\subseteq$  on  $X$  is partial ordering.

< Let  $S_n$  be the set of divisors of  $n$ . The relation  $D$  means —divides $<$  on  $S_n$ , is partial ordering on  $S_n$ .

In a partially ordered set  $(P, \leq)$ , an element  $y \in P$  is said to cover an element  $x \in P$  if  $x < y$  and if there does not exist any element  $z \in P$  such that  $x \leq z$  and  $z \leq y$ ; that is,  $y$  covers  $x \iff (x < y \implies (x \leq z \leq y \implies x = z \vee z = y))$

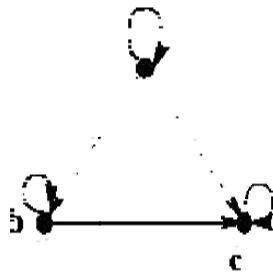
A partial order relation  $\leq$  on a set  $P$  can be represented by means of a diagram known as a Hasse diagram or partial order set diagram of  $(P, \leq)$ . In such a diagram, each element is represented by a small circle or a dot. The circle for  $x \in P$  is drawn below the circle for  $y \in P$  if  $x < y$ , and a line is drawn between  $x$  and  $y$  if  $y$  covers  $x$ .

If  $x < y$  but  $y$  does not cover  $x$ , then  $x$  and  $y$  are not connected directly by a single line. However, they are connected through one or more elements of  $P$ .

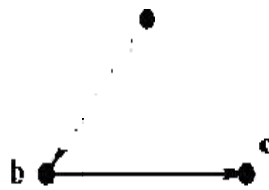
## Hasse Diagram:

A Hasse diagram is a digraph for a poset which does not have loops and arcs implied by the transitivity.

Example 10: For the relation  $\{< a, a >, < a, b >, < a, c >, < b, b >, < b, c >, < c, c >\}$  on set  $\{a, b, c\}$ , the Hasse diagram has the arcs  $\{< a, b >, < b, c >\}$  as shown below

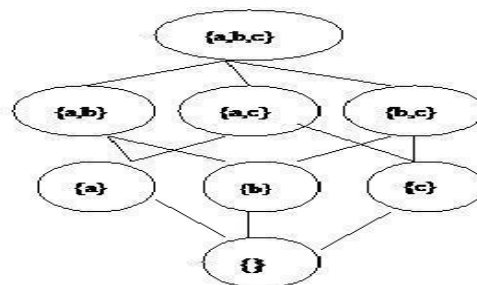


Digraph for Partial Order



Hasse Diagram

Ex: Let  $A$  be a given finite set and  $r(A)$  its power set. Let  $\hat{I}$  be the subset relation on the elements of  $r(A)$ . Draw Hasse diagram of  $(r(A), \hat{I})$  for  $A = \{a, b, c\}$



## Lattice and its Properties:

### Introduction:

A lattice is partially ordered set  $(L, \leq)$  in which every pair of elements  $a, b \in L$  has a greatest lower bound and a least upper bound.

The glb of a subset,  $\{a, b\} \subseteq L$  will be denoted by  $a * b$  and the lub by  $a \vee b$ .

.

Usually, for any pair  $a, b \in L$ ,  $\text{GLB } \{a, b\} = a * b$ , is called the meet or product and  $\text{LUB } \{a, b\} = a \vee b$ , is called the join or sum of  $a$  and  $b$ .





For any  $a \in L$ ,  $a \leq a$ ,  $a \leq \text{LUB} \{a, b\} \Rightarrow a \leq a * (a \wedge b)$ . On the other hand,  $\text{GLB} \{a, a \wedge b\} \leq a$  i.e.,  $(a \wedge b) \wedge a$ , hence  $a * (a \wedge b) = a$

### Theorem 1

Let  $(L, \leq)$  be a lattice with the binary operations  $*$  and  $\wedge$  denote the operations of meet and join respectively For any  $a, b \in L$ ,

$$a \leq b \iff a * b = a \iff a \wedge b = b$$

Proof

Suppose that  $a \leq b$ . we know that  $a \leq a$ ,  $a \leq \text{GLB} \{a, b\}$ , i.e.,  $a \leq a * b$ .

But from the definition of  $a * b$ , we get  $a * b \leq a$ .

$$\text{Hence } a \leq b \Rightarrow a * b = a \quad \dots \quad (1)$$

Now we assume that  $a * b = a$ ; but is possible only if  $a \leq b$ ,

$$\text{that is } a * b = a \Rightarrow a \leq b \quad \dots \quad (2)$$

From (1) and (2), we get  $a \leq b \iff a * b = a$ .

Suppose  $a * b = a$ .

$$\text{then } b \wedge (a * b) = b \wedge a = a \wedge b \quad \dots \quad (3)$$

$$\text{but } b \wedge (a * b) = b \quad (\text{by iv}). \dots \quad (4)$$

Hence  $a \wedge b = b$ , from (3)  $\Rightarrow$  (4)

Suppose  $a \wedge b = b$ , i.e.,  $\text{LUB} \{a, b\} = b$ , this is possible only if  $a \leq b$ , thus (3)  $\Rightarrow$  (1)

(1)  $\Rightarrow$  (2)  $\Rightarrow$  (3)  $\Rightarrow$  (1). Hence these are equivalent.

Let us assume  $a * b = a$ .

Now  $(a * b) \wedge b = a \wedge b$

We know that by absorption law,  $(a * b) \wedge b = b$

$$\text{so that } a \wedge b = b, \text{ therefore } a * b = a \text{ and } a \wedge b = b \quad (5)$$

$$\text{similarly, we can prove } a \wedge b = b \text{ and } a * b = a \quad (6)$$

From (5) and (6), we get

$$a * b = a \text{ and } a \wedge b = b$$

Hence the theorem.

**Theorem 2** For any  $a, b, c \in L$ , where  $(L, \leq)$  is a lattice.  $b$

$$\leq c \iff \{ a * b \leq a * c \text{ and } \\ \{ a \wedge b \leq a \wedge c \}$$

**Proof** Suppose  $b \leq c$ . we have proved that  $b \leq a \iff b * c = b \dots \dots \dots (1)$

$$\begin{aligned} \text{Now consider } (a * b) * (a * c) &= (a * a) * (b * c) \\ &= a * (b * c) \quad (\text{by Idempotent}) \\ &= a * b \quad (\text{by (1)}) \end{aligned}$$

Thus  $(a * b) * (a * c) = a * b$  which  $\Rightarrow (a * b) \leq (a * c)$

$$\begin{aligned} \text{c) Similarly } (a \wedge b) \wedge (a \wedge c) &= (a \wedge a) \wedge (b \wedge c) \\ &= a \wedge (b \wedge c) \\ &= a \wedge c \end{aligned}$$

which  $\Rightarrow (a \wedge b) \leq (a \wedge c)$

note: These properties are known as **isotonicity**.

## Functions

### Introduction

A function is a special type of relation. It may be considered as a relation in which each element of the domain belongs to only one ordered pair in the relation. Thus a function from A to B is a subset of  $A \times B$  having the property that for each  $a \in A$ , there is one and only one  $b \in B$  such that  $(a, b) \in G$ .

### Definition

Let A and B be any two sets. A relation f from A to B is called a function if for every  $a \in A$  there is a unique  $b \in B$  such that  $(a, b) \in f$ .

**Note that the definition of function requires that a relation must satisfy two additional conditions in order to qualify as a function.**

The first condition is that every  $a \in A$  must be related to some  $b \in B$ , (i.e) the domain of f must be A and not merely subset of A. The second requirement of uniqueness can be expressed as  $(a, b) \in f \wedge (b, c) \in f \Rightarrow b = c$

Intuitively, a function from a set A to a set B is a rule which assigns to every element of A, a unique element of B. If  $a \in A$ , then the unique element of B assigned to a under f is denoted by f

(a). The usual notation for a function f from A to B is  $f: A \rightarrow B$  defined by  $a \mapsto f(a)$  where  $a \in A$ ,  $f(a)$  is called the image of a under f and a is called pre image of f(a).

- < Let  $X = Y = \mathbf{R}$  and  $f(x) = x^2 + 2$ .  $D_f = \mathbf{R}$  and  $R_f \subseteq \mathbf{R}$ .
- < Let X be the set of all statements in logic and let  $Y = \{\text{True}, \text{False}\}$ . A mapping  $f: X \rightarrow Y$  is a function.
- < A program written in high level language is mapped into a machine language by a compiler. Similarly, the output from a compiler is a function of its input.
- < Let  $X = Y = \mathbf{R}$  and  $f(x) = x^2$  is a function from  $X \rightarrow Y$ , and  $g(x^2) = x$  is not a function from  $X \rightarrow Y$ .

**A mapping  $f: A \rightarrow B$  is called one-to-one (injective or 1-1) if distinct elements of A are mapped into distinct elements of B. (i.e) f is one-to-one if**

$$a_1 = a_2 \Rightarrow f(a_1) = f(a_2) \text{ or equivalently } f(a_1) \neq f(a_2) \Rightarrow a_1 \neq a_2$$

*For example,  $f: \mathbf{N} \rightarrow \mathbf{N}$  given by  $f(x) = x$  is 1-1 where N is the set of a natural numbers.*

A mapping  $f: A \rightarrow B$  is called **onto (surjective)** if for every  $b \in B$  there is an  $a \in A$  such that  $f(a) = b$ . i.e. if every element of B has a pre-image in A. Otherwise it is called **into**.

*For example,  $f: \mathbf{Z} \rightarrow \mathbf{Z}$  given by  $f(x) = x + 1$  is an onto mapping. A mapping is both 1-1 and onto is called bijective*

*For example*  $f: \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x + 1$  is bijective.

**Definition:** A mapping  $f: \mathbb{R} \rightarrow b$  is called a **constant mapping** if, for all  $a \in A$ ,  $f(a) = b$ , a fixed element.

*For example*  $f: \mathbb{Z} \rightarrow \mathbb{Z}$  given by  $f(x) = 0$ , for all  $x \in \mathbb{Z}$  is a constant mapping.

### Definition

A mapping  $f: A \rightarrow A$  is called the identity mapping of  $A$  if  $f(a) = a$ , for all  $a \in A$ . Usually it is denoted by  $I_A$  or simply  $I$ .

### Composition of functions:

If  $f: A \rightarrow B$  and  $g: B \rightarrow C$  are two functions, then the composition of functions  $f$  and  $g$ , denoted by  $g \circ f$ , is the function is given by  $g \circ f: A \rightarrow C$  and is given by

$$g \circ f = \{(a, c) / a \in A \wedge c \in C \wedge \exists b \in B : f(a) = b \wedge g(b) = c\} \text{ and } (g \circ f)(a) = (g(f(a)))$$

Example 1: Consider the sets  $A = \{1, 2, 3\}$ ,  $B = \{a, b\}$  and  $C = \{x, y\}$ . Let  $f: A \rightarrow B$  be defined by  $f(1) = a$ ;  $f(2) = b$  and  $f(3) = b$  and Let  $g: B \rightarrow C$  be defined by  $g(a) = x$  and  $g(b) = y$

(i.e)  $f = \{(1, a), (2, b), (3, b)\}$  and  $g = \{(a, x), (b, y)\}$ . Then  $g \circ f: A \rightarrow C$  is defined by

$$(g \circ f)(1) = g(f(1)) = g(a) = x$$

$$(g \circ f)(2) = g(f(2)) = g(b) = y$$

$$(g \circ f)(3) = g(f(3)) = g(b) = y$$

$$\text{i.e., } g \circ f = \{(1, x), (2, y), (3, y)\}$$

If  $f: A \rightarrow A$  and  $g: A \rightarrow A$ , where  $A = \{1, 2, 3\}$ , are given by

$$f = \{(1, 2), (2, 3), (3, 1)\} \text{ and } g = \{(1, 3), (2, 2), (3, 1)\}$$

Then  $g \circ f = \{(1, 2), (2, 1), (3, 3)\}$ ,  $f \circ g = \{(1, 1), (2, 3), (3, 2)\}$

$$f \circ f = \{(1, 3), (2, 1), (3, 2)\} \text{ and } g \circ g = \{(1, 1), (2, 2), (3, 3)\}$$

Example 2: Let  $f(x) = x+2$ ,  $g(x) = x-2$  and  $h(x) = 3x$  for  $x \in \mathbb{R}$ , where  $\mathbb{R}$  is the set of real numbers.

$$\text{Then } f \circ f = \{(x, x+4) / x \in \mathbb{R}\}$$

$$f \circ g = \{(x, x) / x \in \mathbb{R}\}$$

$$g \circ f = \{(x, x) / x \in \mathbb{R}\}$$

$$g \circ g = \{(x, x-4) / x \in \mathbb{R}\}$$

$$h \circ g = \{(x, 3x-6) / x \in \mathbb{R}\}$$

$$h \circ f = \{(x, 3x+6) / x \in \mathbb{R}\}$$

### Inverse functions:

Let  $f: A \rightarrow B$  be a one-to-one and onto mapping. Then, its inverse, denoted by  $f^{-1}$  is given by  $f^{-1} = \{(b, a) / (a, b) \in f\}$ . Clearly  $f^{-1}: B \rightarrow A$  is one-to-one and onto.

Also we observe that  $f \circ f^{-1} = IB$  and  $f^{-1} \circ f = IA$ .  
If  $f^{-1}$  exists then  $f$  is called invertible.

For example: Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = x + 2$

Then  $f^{-1}: \mathbb{R} \rightarrow \mathbb{R}$  is defined by  $f^{-1}(x) = x - 2$

Theorem: Let  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  be two one to one and onto functions. Then  $g \circ f$  is also one to one and onto function.

#### Proof

Let  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  be two one to one and onto functions. Let  $x_1, x_2 \in X$

$$\langle g \circ f(x_1) = g \circ f(x_2),$$

$$\langle g(f(x_1)) = g(f(x_2)),$$

$$\langle g(x_1) = g(x_2) \text{ since } [f \text{ is } 1-1]$$

$$x_1 = x_2 \text{ since } [g \text{ is } 1-1]$$

so that  $g \circ f$  is 1-1.

By the definition of composition,  $g \circ f: X \rightarrow Z$  is a function.

We have to prove that every element of  $z \in Z$  is an image element for some  $x \in X$  under  $g \circ f$ .

Since  $g$  is onto  $\exists y \in Y$  :  $g(y) = z$  and  $f$  is onto from  $X$  to  $Y$ ,

$$\exists x \in X : f(x) = y.$$

$$\text{Now, } (g \circ f)(x) = g(f(x))$$

$$= g(y) \quad [\text{since } f(x) = y]$$

$$= z \quad [\text{since } g(y) = z] \text{ which shows that } g \circ f \text{ is onto.}$$

Theorem  $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$  (i.e) the inverse of a composite function can be expressed in terms of the composition of the inverses in the reverse order.

**Proof.**  $f: A \rightarrow B$  is one to one and onto.  $g: B \rightarrow C$  is one to one and onto.

$g \circ f: A \rightarrow C$  is also one to one and onto.  $P(g \circ f)^{-1}$ :

$C \rightarrow A$  is one to one and onto.

Let  $a \in A$ , then there exists an element  $b \in B$  such that  $f(a) = b$   $P a = f^{-1}(b)$

(c). Now  $b \in B$   $P$  there exists an element  $c \in C$  such that  $g(b) = c$   $P b = g^{-1}(c)$ .

Then  $(g \circ f)(a) = g[f(a)] = g(b) = c$   $P a = (g \circ f)^{-1}(c)$ ..... (1)

$$(f^{-1} \circ g^{-1})(c) = f^{-1}(g^{-1}(c)) = f^{-1}(b) = a$$

$$P a = (f^{-1} \circ g^{-1})(c)$$

) ....(2) Combining (1) and (2), we have  $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Theorem: If  $f: A \rightarrow B$  is an invertible mapping ,  
then  $f \circ f^{-1} = I_B$  and  $f^{-1} \circ f = I_A$

Proof:  $f$  is invertible, then  $f^{-1}$  is defined by  $f(a) = b \iff f^{-1}(b) = a$  where  $a \in A$  and  $b \in B$  .

Now we have to prove that  $f \circ f^{-1} = I_B$

. Let  $b \in B$  and  $f^{-1}(b) = a, a \in A$

then  $f \circ f^{-1}(b) = f(f^{-1}(b))$

$= f(a) = b$

therefore  $f \circ f^{-1}(b) = b \forall b \in B \Rightarrow f \circ f^{-1} =$

$I_B$  Now  $f^{-1} \circ f(a) = f^{-1}(f(a)) = f^{-1}(b) = a$

therefore  $f^{-1} \circ f(a) = a \forall a \in A \Rightarrow f^{-1} \circ f = I_A$ .

Hence the theorem.

## Recursive Functions:

The term "recursive function" is often used informally to describe any function that is defined with recursion. There are several formal counterparts to this informal definition, many of which only differ in trivial respects.

Kleene (1952) defines a "partial recursive function" of nonnegative integers to be any function  $f$  that is defined by a noncontradictory system of equations whose left and right sides are composed from

(1) function symbols (for example,  $f, g, h$ , etc.), (2) variables for nonnegative integers (for example,  $x, y, z$ , etc.), (3) the constant 0, and (4) the successor function  $S(x) = x + 1$ .

For example,

$$f(x, 0) = 0 \quad (1)$$

$$f(x, S(y)) = g(f(x, y), x) \quad (2)$$

$$g(x, 0) = x \quad (3)$$

$$g(x, S(y)) = S(g(x, y)) \quad (4)$$

defines  $f(x, y)$  to be the function  $x, y$  that computes the product of  $x$  and  $y$ .

Note that the equations might not uniquely determine the value of  $f$  for every possible input, and in that sense the definition is "partial." If the system of equations determines the value of  $f$  for every input, then the definition is said to be "total." When the term "recursive function" is used alone, it is usually implicit that "total recursive function" is intended. Note that some authors use the term "general recursive function" to mean partial recursive function, although others use it to mean "total recursive function."

The set of functions that can be defined recursively in this manner is known to be equivalent to the set of functions computed by Turing machines and by the lambda calculus.

### Algebraic systems:

An algebraic system, loosely speaking, is a set, together with some operations on the set. Before formally defining what an algebraic system is, let us recall that a  $n$ -ary operation (or operator) on a set  $A$  is a function whose domain is  $A^n$  and whose range is a subset of  $A$ . Here,  $n$  is a non-negative integer. When  $n=0$ , the operation is usually called a nullary operation, or a constant, since one element of  $A$  is singled out to be the (sole) value of this operation. A finitary operation on  $A$  is just an  $n$ -ary operation for some non-negative integer  $n$ .

Definition. An *algebraic system* is an ordered pair  $(A, O)$ , where  $A$  is a set, called the underlying set of the algebraic system, and  $O$  is a set, called the operator set, of finitary operations on  $A$ .

We usually write  $A$ , instead of  $(A, O)$ , for brevity.

A prototypical example of an algebraic system is a group, which consists of the underlying set  $G$ , and a set  $O$  consisting of three operators: a constant  $e$  called the multiplicative identity, a unary operator called the multiplicative inverse, and a binary operator called the multiplication.

For a more comprehensive listing of examples, please see this entry.

#### Remarks.

- < An algebraic system is also called algebra for short. Some authors require that  $A$  be non-empty. Note that  $A$  is automatically non-empty if  $O$  contains constants. A *finite algebra* is an algebra whose underlying set is finite.
- < By definition, all operators in an algebraic system are finitary. If we allow  $O$  to contain infinitary operations, we have an *infinitary algebraic system*. Other generalizations are possible. For example, if the operations are allowed to be multivalued, the algebra is said to be a *multialgebra*. If the operations are not everywhere defined, we get a *partial algebra*. Finally, if more than one underlying set is involved, then the algebra is said to be *many-sorted*.

The study of algebraic systems is called the theory of universal algebra. The first important thing in studying algebraic system is to compare systems that are of the same "type". Two algebras are said to have the same *type* if there is a one-to-one correspondence between their operator sets such that an  $n$ -ary operator in one algebra is mapped to an  $n$ -ary operator in the other algebra.

## Examples:

Some recurring universes:  $\mathbb{N}$ =natural numbers;  $\mathbb{Z}$ =integers;  $\mathbb{Q}$ =rational numbers;  $\mathbb{R}$ =real numbers;  $\mathbb{C}$ =complex numbers.

$\mathbb{N}$  is a pointed unary system, and under addition and multiplication, is both the standard interpretation of Peano arithmetic and a commutative semiring.

Boolean algebras are at once semigroups, lattices, and rings. They would even be abelian groups if the identity and inverse elements were identical instead of complements.

## Group-like structures

- < Nonzero  $\mathbb{N}$  under addition (+) is a magma.
- <  $\mathbb{N}$  under addition is a magma with an identity.
- <  **$\mathbb{Z}$  under subtraction (−) is a quasigroup.**
- < Nonzero  $\mathbb{Q}$  under division (÷) is a quasigroup.  $a^{-1} * b$ , and  $y * a = b$  if  $y = b * a^{-1}$ .
- < Every group is a loop, because  $a * x = b$  if and only if  $x = a^{-1} * b$  and only if  $y = b * a^{-1}$ .
- < 2x2 matrices(of non-zero determinant) with matrix multiplication form a group.
- <  $\mathbb{Z}$  under addition (+) is an abelian group.
- < Nonzero  $\mathbb{Q}$  under multiplication (×) is an abelian group.
- < Every cyclic group  $G$  is abelian, because if  $x, y$  are in  $G$ , then  $xy = yx$ . In particular,  $\mathbb{Z}$  is an abelian group under addition, as is the integers modulo  $n$   $\mathbb{Z}/n\mathbb{Z}$ .
- < A monoid is a category with a single object, in which case the composition of morphisms and the identity morphism interpret monoid multiplication and identity element, respectively.
- < The Boolean algebra **2** is a boundary algebra.

## General Properties:

### Property of Closure

If we take two *real numbers* and multiply them together, we get another real number. (The real numbers are all the rational numbers and all the irrational numbers.) Because this is always true, we say that the real numbers are "closed under the operation of multiplication": there is no way to escape the set. When you combine any two elements of the set, the result is also included in the set.

Real numbers are also closed under addition and subtraction. They are not closed under the square root operation, because the square root of -1 is not a real number.

## Inverse

The inverse of something is that thing turned inside out or upside down. The inverse of an operation undoes the operation: division undoes multiplication.

A number's *additive inverse* is another number that you can add to the original number to get the additive identity. For example, the additive inverse of 67 is -67, because  $67 + -67 = 0$ , the additive identity.

Similarly, if the product of two numbers is the *multiplicative identity*, the numbers are *multiplicative inverses*. Since  $6 * 1/6 = 1$  (the multiplicative identity), the multiplicative inverse of 6 is  $1/6$ .

Zero does not have a multiplicative inverse, since no matter what you multiply it by, the answer is always 0, not 1.

## Equality

The equals sign in an equation is like a scale: both sides, left and right, must be the same in order for the scale to stay in balance and the equation to be true.

The *addition property of equality* says that if  $a = b$ , then  $a + c = b + c$ : if you add the same number to (or subtract the same number from) both sides of an equation, the equation continues to be true.

The *multiplication property of equality* says that if  $a = b$ , then  $a * c = b * c$ : if you multiply (or divide) by the same number on both sides of an equation, the equation continues to be true.

The *reflexive property of equality* just says that  $a = a$ : anything is congruent to itself: the equals sign is like a mirror, and the image it "reflects" is the same as the original.

The *symmetric property of equality* says that if  $a = b$ , then  $b = a$ .

The *transitive property of equality* says that if  $a = b$  and  $b = c$ , then  $a = c$ .

## Semi groups and monoids:

In the previous section, we have seen several algebraic system with binary operations. Here we consider an algebraic system consisting of a set and an associative binary operation on the set and then the algebraic system which possess an associative property with an identity element. These algebraic systems are called semigroups and monoids.



## Semi group

Let  $S$  be a nonempty set and let  $*$  be a binary operation on  $S$ . The algebraic system  $(S, *)$  is called a semi-group if  $*$  is associative

$$\text{if } a * (b * c) = (a * b) * c \text{ for all } a, b, c \in S.$$

Example The  $N$  of natural numbers is a semi-group under the operation of usual addition of numbers.

## Monoids

Let  $M$  be a nonempty set with a binary operation  $*$  defined on it. Then  $(M, *)$  is called a monoid if

- $*$  is associative

(i.e)  $a * (b * c) = (a * b) * c$  for all  $a, b, c \in M$  and  
there exists an element  $e$  in  $M$  such that

$$a * e = e * a = a \text{ for all } a \in M$$

$e$  is called the identity element in  $(M, *)$ .

It is easy to prove that the identity element is unique. From the definition it follows that  $(M, *)$  is a semigroup with identity.

Example1 Let  $S$  be a nonempty set and  $r(S)$  be its power set. The algebras  $(r(S), \cup)$  and  $(r(S), \cap)$  are monoids with the identities  $f$  and  $S$  respectively.

Example2 Let  $N$  be the set of natural numbers, then  $(N, +)$ ,  $(N, \times)$  are monoids with the identities  $0$  and  $1$  respectively.

## Groups Sub Groups:

Recalling that an algebraic system  $(S, *)$  is a semigroup if the binary operation  $*$  is associative. If there exists an identity element  $e \in S$ , then  $(S, *)$  is monoid. A further condition is imposed on the elements of the monoid, i.e., the existence of an inverse for each element of  $S$  then the algebraic system is called a group.

### Definition

Let  $G$  be a nonempty set, with a binary operation  $*$  defined on it. Then the algebraic system  $(G, *)$  is called a group if

- $*$  is associative i.e.  $a * (b * c) = (a * b) * c$  for all  $a, b, c \in G$ .
- there exists an element  $e$  in  $G$  such that  $a * e = e * a = a$  for all  $a \in G$
- for each  $a \in G$  there is an element denoted by  $a^{-1}$  in  $G$  such that  $a * a^{-1} = a^{-1} * a = e$ ,  $a^{-1}$  is called the inverse of  $a$ .

From the definition it follows that  $(G, *)$  is a monoid in which each element has an inverse w.r.t.  $*$  in  $G$ .

A group  $(G, *)$  in which  $*$  is commutative is called an abelian group or a commutative group. If  $*$  is not commutative then  $(G, *)$  is called a non-abelian group or non-commutative group.

The order of a group  $(G, *)$  is the number of elements of  $G$ , when  $G$  is finite and is denoted by  $o(G)$  or  $|G|$

Examples 1.  $(\mathbb{Z}_5, +)$  is an abelian group of order 5.

2.  $G = \{1, -1, i, -i\}$  is an abelian group with the binary operation  $x$  is defined as  $1 \times 1 = 1, -1 \times -1 = 1, i \times i = -1, -i \times -i = 1, \dots$

## Homomorphism of semigroups and monoids

### Semigroup homomorphism.

Let  $(S, *)$  and  $(T, D)$  be any two semigroups. A mapping  $g: S \rightarrow T$  such that any two elements  $a, b \in S$ ,  $g(a * b) = g(a) D g(b)$  is called a semigroup homomorphism.

### Monoid homomorphism

Let  $(M, *, e_M)$  and  $(T, D, e_T)$  be any two monoids. A mapping  $g: M \rightarrow T$  such that any two elements  $a, b \in M$ ,

$$g(a * b) = g(a) D g(b)$$

$$\text{and } g(e_M) = e_T$$

is called a monoid homomorphism.

**Theorem 1** Let  $(S, *)$ ,  $(T, D)$  and  $(V, \Delta)$  be semigroups. A mapping  $g: S \rightarrow T$  and  $h: T \rightarrow V$  be semigroup homomorphisms. Then  $(h \circ g): S \rightarrow V$  is a semigroup homomorphism from  $(S, *)$  to  $(V, \Delta)$ .

**Proof.** Let  $a, b \in S$ . Then

$$\begin{aligned} (h \circ g)(a * b) &= h(g(a * b)) \\ &= h(g(a) D g(b)) \\ &= h(g(a)) \Delta h(g(b)) \\ &= (h \circ g)(a) \Delta (h \circ g)(b) \end{aligned}$$

**Theorem 2** Let  $(S, *)$  be a given semigroup. There exists a homomorphism  $g: S \rightarrow SS$ , where  $(SS, \circ)$  is a semigroup of function from  $S$  to  $S$  under the operation of composition.

**Proof** For any element  $a \in S$ , let  $g(a) = f_a$  where  $f_a \in SS$  and  $f_a$  is defined by

$$f_a(b) = a * b \quad \text{for any } a, b \in S$$

$$g(a * b) = f_{a * b}$$

$$\text{Now } f_{a * b}(c) = (a * b) * c = a * (b * c)$$

$$\text{where } = f_a(f_b(c)) = (f_a \circ f_b)(c).$$

Therefore,  $g(a * b) = f a * b = f a \circ f b = g(a) \circ g(b)$ , this shows that  $g: S \rightarrow SS$  is a homomorphism.

**Theorem 3** For any commutative monoid  $(M, *)$ , the set of idempotent elements of  $M$  forms a submonoid.

**Proof.** Let  $S$  be the set of idempotent elements of  $M$ .

Since the identity element  $e \in M$  is idempotent,  $e \in S$ .

Let  $a, b \in S$ , so that  $a * a = a$  and  $b * b = b$

Now  $(a * b) * (a * b) = (a * b) * (b * a) \quad [(M, *) \text{ is a commutative monoid}]$

$$= a * (b * b) * a$$

$$= a * b * a$$

$$= a * a * b$$

$$= a * b$$

Hence  $a * b \in S$  and  $(S, *)$  is a submonoid.

### Isomorphism:

–1

In abstract algebra, an isomorphism is a bijective map  $f$  such that both  $f$  and its inverse  $f^{-1}$  are homomorphisms, i.e., *structure-preserving* mappings. In the more general setting of category theory, an **isomorphism** is a morphism  $f: X \rightarrow Y$  in a category for which there exists an "inverse"  $f^{-1}: Y \rightarrow X$ , with the property that both  $f$

$$f \circ f^{-1} = \text{id}_Y \text{ and } f^{-1} \circ f = \text{id}_X.$$