Lecture 18

Recall:-

Given $\underline{x} \neq \underline{0}$ in $\mathbb{R}^n$, Then there exists a householder transformation P such that

$$P\underline{x} = \alpha \, \underline{e}_1 \quad \text{for some } \alpha \in \mathbb{R}.$$

where

$$P = I - \frac{\underline{u}\,\underline{u}^t}{2q^2}$$

where

$$2q^2 = \|\underline{x}\|^2 + \|\underline{x}\| \, x_1 \cdot \text{sing}(x_1)$$

$$= \|\underline{x}\| \left(\|\underline{x}\| + \text{sign}(x_1)\, x_1 \right)$$

$$\alpha = -\text{sign}(x_1)\, \|\underline{x}\|$$

&

$$\underline{u} = \begin{pmatrix} x_1 \mp m \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \begin{pmatrix} x_1 + \text{sign}(x_1)\, \|\underline{x}\| \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\text{sign}(x_1) = \begin{cases} 1 & \text{if } x_1 > 0 \\ -1 & \text{if } x_1 < 0. \end{cases}$$

Orthogonal reduction of a matrix into a triangular form & QR-decomposition by householder transformations :

Let $A_{m\times n}$ be a real matrix.

$$A_{m\times n} = [a_{ij}]_{m\times n}.$$

Let $\underline{x} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}_{m\times 1}$ = the first Column of A.

Assume $\underline{x} \neq \underline{0}$

$\therefore$ By applying the above theorem, there exists a householder transformation P_m such that $P_m \underline{x} = \alpha_{11} \underline{e_1}$, $e_1 \in \mathbb{R}^m$.

for some $\alpha_{11} \in \mathbb{R}$

where $\alpha_{11} = -\text{sign}(x_1) \|\underline{x}\|$
$= -\text{sign}(a_{11}) \|\underline{a}\|$.

$$\& \quad P_m A = \left[P_m \underline{x} \quad P_m \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \quad \cdots\cdots \quad P_m \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} \right]$$

$$= \begin{bmatrix} \alpha_{11} & \alpha_{12} \cdots & \alpha_{1n} \\ 0 & \begin{pmatrix} b_{22} & \cdots & b_{2n} \\ \vdots & & \\ b_{m2} & \cdots & b_{mn} \end{pmatrix} & \\ \vdots & & \\ 0 & & \end{bmatrix} \quad \text{(say)}$$

Let $U_1 = P_m$.

Now Consider the matrix $\begin{bmatrix} b_{22} & \cdots & b_{2n} \\ \vdots & & \\ b_{m2} & \cdots & b_{mn} \end{bmatrix}_{(m-1)\times(n-1)}$

Set $\underline{x}$ = the first Colum of the above matrix.

$$= \begin{pmatrix} b_{22} \\ \vdots \\ b_{m2} \end{pmatrix}$$

Assume $\underline{x} \neq \underline{0}$.

Now by applying the above thm, there exists a householder transformation P_{m-1} such that $P_{m-1}\underline{x} = \alpha_{22}\underline{e_1}$,

$$\alpha_{22} = -\operatorname{sign}(b_{22})\,\|\underline{x}\|.$$

&

$$\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P_{m-1} \end{array}\right]_{m\times m} P_m A = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P_{m-1} \end{array}\right] \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \\ 0 & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ 0 & \alpha_{22} & \cdots & \alpha_{2n} \\ 0 & 0 & \begin{pmatrix} c_{33} & \cdots & c_{3n} \\ \vdots & & \vdots \\ c_{m3} & \cdots & c_{mn} \end{pmatrix} \\ \vdots & \vdots & \\ 0 & 0 & \end{bmatrix}$$

Let $U_2 = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P_{m-1} \end{array}\right]_{m\times m}$.

Repeating the above process we get at k^{th} step, is

$$U_{k+1} = \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & P_{m-k} \end{array}\right], \quad k = 1, 2, \ldots, r-1$$

$$U_1 = P_m$$

Let $Q = U_1 U_2 \cdots U_r$, $r = \min(m-1, n)$

$$Q^t A = U_r^t \cdots U_1^t A$$

$$= U_r \cdots U_1 A \quad (\because U_1^t = U_1, \ldots, U_r^t = U_r)$$

$$= \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ 0 & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots \; 0 & \alpha_{nn} \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \text{ if } m > n$$

$$= \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & & \alpha_{1n} \\ 0 & \alpha_{22} & & \cdots & \alpha_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \alpha_{mm} & \cdots & \alpha_{mn} \end{bmatrix} \quad \text{if } m \leq n$$

Note that Q is an orthogonal matrix.

Thus $Q^t A = R$, where R is of the above form

$$\Rightarrow \quad A = QR,$$

where Q is orthogonal & R is upperΔler as above.

This decomposition is known as the <u>QR-decomposition of the matrix</u> A.

<u>LU-decomposition:</u> $A_{m\times n}$

Let U be an orthogonal matrix such that $\underset{n\times n}{U} A^t$ = upperΔler matrix $= L^t$

where L is a lower Δler matrix.

$$\Rightarrow \quad A^t = U^t L^t = (LU)^t$$

$$\Rightarrow \quad A = LU$$

where L is lowerΔler & U is an orthogonal matrix.

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1) Determine the QR-decomposition of the matrix A by using householder transformation.

where $A = \begin{bmatrix} 0 & 3 & 50 \\ 3 & 5 & 25 \\ 4 & 0 & 25 \end{bmatrix}$

Sol:- let $\underline{x} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} \neq \underline{0}$.

$$\|\underline{x}\| = \sqrt{0^2+3^2+4^2} = 5.$$

To find a householder transformation P_3 such that

$$P_3 \underline{x} = \alpha \underline{e}_1, \quad \underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

$$P_3 = I - \frac{\underline{u}\underline{u}^t}{2q^2},$$

where $\underline{u} = \begin{pmatrix} x_1 + \text{sing}[x_1] \|\underline{x}\| \\ x_2 \\ \vdots \\ x_3 \end{pmatrix}$

$$2q^2 = \|\underline{x}\| \left(\|\underline{x}\| + \text{sign}(x_1)\, x_1 \right)$$

$\text{sign}(x_1) = 1$

$$2q^2 = 5(5+0) = 25.$$

$$\underline{u} = \begin{pmatrix} 0+1(5) \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$$

$$\therefore P_3 = I - \frac{1}{2q^2} \underline{u}\,\underline{u}^t.$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{25} \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix} \begin{pmatrix} 5 & 3 & 4 \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{25} \begin{bmatrix} 25 & 15 & 20 \\ 15 & 9 & 12 \\ 20 & 12 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -15/25 & -20/25 \\ -15/25 & 16/25 & -12/25 \\ -20/25 & -12/25 & 9/25 \end{bmatrix} = U_1 \text{ (say)}.$$

Now $$P_3 A = U_1 A = U_1 \begin{bmatrix} 0 & 3 & 50 \\ 3 & 5 & 25 \\ 4 & 0 & 25 \end{bmatrix}$$

$$= \frac{1}{25} \begin{bmatrix} -125 & -75 & (-35)(25) \\ 0 & 35 & -650 \\ 0 & -120 & -1075 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -3 & -35 \\ 0 & 7/5 & -26 \\ 0 & -24/5 & -43 \end{bmatrix}$$

For the matrix $\begin{bmatrix} 7/5 & -26 \\ -24/5 & -43 \end{bmatrix}$

determine P_2 such that $P_2 \underline{x} = \alpha_{22} \underline{e}_1$.

$$\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Let $\underline{x} = \begin{pmatrix} 7/5 \\ -24/5 \end{pmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ (say)

$$\|\underline{x}\| = \sqrt{\frac{49}{25} + \frac{576}{25}} = 5.$$

$$\text{sign}(x_1) = 1$$

$$2q^2 = \|\underline{x}\| \left(\|\underline{x}\| + \text{sign}(x_1)\, x_1 \right)$$

$$= 5\left(5 + 7/5\right)$$

$$= 32.$$

$$P_2 = I_2 - \frac{1}{2q^2} \underline{u}\,\underline{u}^t,$$

where $\underline{u} = \begin{pmatrix} x_1 + \text{sign}(x_1) \|x_1\| \\ x_2 \end{pmatrix}$

$$= \begin{pmatrix} 7/5 + 5 \\ -24/5 \end{pmatrix} = \begin{pmatrix} 32/5 \\ -24/5 \end{pmatrix}$$

$$\therefore P_2 = I_2 - \frac{1}{32}\begin{pmatrix} 32/5 \\ -24/5 \end{pmatrix}\begin{pmatrix} \frac{32}{5} & \frac{-24}{5} \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{32}\begin{bmatrix} \frac{(32)^2}{25} & \frac{(-32)(24)}{25} \\ \frac{(-32)(24)}{25} & \frac{(24)^2}{25} \end{bmatrix}$$

$$= \begin{bmatrix} -7/25 & \frac{24}{25} \\ \frac{24}{25} & 7/25 \end{bmatrix}$$

Let $U_2 = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P_2 \end{array}\right]_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -7/25 & 24/25 \\ 0 & 24/25 & 7/25 \end{bmatrix}$

Now $U_2(U_1 A) = \frac{1}{25}\begin{bmatrix} 25 & 0 & 0 \\ 0 & -7 & 24 \\ 0 & 24 & 7 \end{bmatrix}\begin{bmatrix} -5 & -3 & -35 \\ 0 & 7/5 & -26 \\ 0 & -\frac{24}{5} & -43 \end{bmatrix}$

$$= \begin{bmatrix} -5 & -3 & -35 \\ 0 & -5 & -34 \\ 0 & 0 & -37 \end{bmatrix} = R \text{ (say)}.$$

Let $Q = U_1 U_2$

Then $A = QR$.

$$Q = Y\,U_2 = \frac{1}{(25)^2}\begin{bmatrix} 0 & -15 & -20 \\ -15 & 16 & -12 \\ -20 & -12 & 9 \end{bmatrix}\begin{bmatrix} 25 & 0 & 0 \\ 0 & -7 & 24 \\ 0 & 24 & 7 \end{bmatrix}$$

$$= \frac{1}{25}\begin{bmatrix} 0 & -15 & -20 \\ -15 & -16 & 12 \\ -20 & 12 & -9 \end{bmatrix}$$

Thus we determined the QR-decomposition of A.

<u>Cholesky decomposition:</u>

Let $A_{n\times n}$ be a symmetrix matrix.

Assume A is +ve definite or +ve semidefinite.

Then all the eigenvalues of A are +ve or non-ve.

Let Q be an orthogonal matrix such that $Q^t A Q = \text{diagonal}(d_1, \ldots, d_n)$

Let $D = \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{bmatrix}$

Let $P = DQ^t$.

Then $A = Q D^2 Q^t$

$$= (QD)(DQ^t)$$
$$= (QD^t)(DQ^t)$$
$$= (DQ^t)^t(DQ^t)$$
$$= P^tP.$$

By using the QR-decomposition applying to P there exists a matrix $H_{n\times n}$ which is a product of householder transformations such that

$HP = L^t$, where L is a lower Δ^{lar} matrix

$$\Rightarrow \boxed{P = H^tL^t.}$$

Now
$$A = P^tP = (H^tL^t)^t(H^tL^t)$$
$$= (LH)(H^tL^t)$$
$$= L(HH^t)L^t$$
$$= LL^t \quad (\because H \text{ is orthogonal})$$

Thus $\boxed{A = LL^t}$ where L is a Lower Δ^{lar} matrix provided A is +ve def or +ve semi def.

This decomposition is known as Cholesky

decomposition of A.

(11) Find the cholesky decomposition of the matrix $A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

(check that A is +ve def)

Sol: The eigenvalues of A are 2, 4, 4.

To find an orthogonal matrix Q such that

$$Q^t A Q = \begin{bmatrix} 2 & & 0 \\ & 4 & \\ 0 & & 4 \end{bmatrix}.$$

Eigenvectors corresponding to $\lambda = 2$:

$$A \underline{x} = 2\underline{x}$$

$$\Rightarrow (A - 2I)\underline{x} = \underline{0}.$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underline{0}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}. \quad \forall x \in \mathbb{R}.$$

Eigenvectors Corr. to $\lambda = 4$:

$$(A-4I)\underline{x} = \underline{0}.$$

$$\Rightarrow \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underline{0}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Consider the eigenvectors.

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Let $$Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q is an orthogonal matrix

& $$Q^{t}AQ = Q^{-1}AQ = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Let $$D = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Let $P = DQ^{t}$.

$$= \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ -\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

& $A = P^t P$.

TO find $H_{3\times 3}$ such that $HP = L^t$:

Let $\underline{x} = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 0 \end{pmatrix}$.

$$\|\underline{x}\| = \sqrt{2+1} = \sqrt{3}.$$

$$\text{sign}(x_1) = 1.$$

$$2q^2 = \|\underline{x}\| \left(\|\underline{x}\| + \text{sign}(x_1)\, x_1\right)$$

$$= 3 + \sqrt{3}.$$

$$\underline{u} = \begin{pmatrix} x_1 + \text{sign}(x_1)\|\underline{x}\| \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1+\sqrt{3} \\ -\sqrt{2} \\ 0 \end{pmatrix}.$$

$$P_3 = I_3 - \frac{1}{2q^2}\underline{u}\,\underline{u}^t$$

$$= \begin{bmatrix} -1/\sqrt{3} & \frac{\sqrt{2}}{\sqrt{3}} & 0 \\ \sqrt{2}/\sqrt{3} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = U_1 \text{ (say)}.$$

$$U_1 P = \begin{bmatrix} -\sqrt{3} & 1/\sqrt{3} & 0 \\ 0 & 2\sqrt{2}/\sqrt{3} & 0 \\ 0 & 0 & 2 \end{bmatrix} = R \quad \text{upper } \Delta^{\text{lar}} \text{ matrix}$$

$$= L^t \qquad \boxed{H = U_1}.$$

now $\quad P = (L U_1)^t$

$$A = P^t P$$
$$= (LU_1)(U_1^t L^t)$$
$$= LL^t.$$

where $L = \begin{bmatrix} -\sqrt{3} & 0 & 0 \\ 1/\sqrt{3} & \frac{2\sqrt{2}}{\sqrt{3}} & 0 \\ 0 & 0 & 2 \end{bmatrix}$

verify that $\quad LL^t = A. \qquad$ cholesky decomp.

Alternative method:

$A = LL^t$.

Let $\quad L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ (say).

$A = LL^t$.

$$\Rightarrow \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$\Rightarrow \quad \text{"} \quad = \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2+l_{22}^2 & l_{21}l_{31}+l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21}+l_{32}l_{22} & l_{31}^2+l_{32}^2+l_{33}^2 \end{bmatrix}$$

$\Rightarrow \quad 3 = l_{11}^2, \quad -1 = l_{11}l_{21}, \quad 0 = l_{11}l_{31}.$

$l_{21}^2 + l_{22}^2 = 3, \quad l_{21}l_{31} + l_{22}l_{32} = 0$

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 4$

$\Rightarrow \quad \boxed{l_{11} = \pm\sqrt{3}.} \qquad \boxed{l_{21} = \frac{-1}{\pm\sqrt{3}} = \frac{1}{\mp\sqrt{3}}}$

$\boxed{l_{31} = 0} \qquad \frac{1}{3} + l_{22}^2 = 3$

$\Rightarrow \quad l_{22}^2 = 3 - \frac{1}{3} = 8/3$

$\boxed{l_{22} = \pm\sqrt{8/3} = \pm\frac{2\sqrt{2}}{\sqrt{3}}.}$

$\left(\pm\frac{1}{\ldots}\right)(\ldots) + \left(\pm 2\ldots\right)l \ldots = 0$

$$\left(\mp\sqrt{3}\right)(0) + \left(\frac{\alpha}{\sqrt{3}}\right) l_{32} = 0.$$

$$\Rightarrow \boxed{l_{32} = 0}.$$

$$l_{33}^2 = 4 \Rightarrow \boxed{l_{33} = \pm 2}$$

$$\therefore \quad L = \begin{bmatrix} \pm\sqrt{3} & 0 & 0 \\ \frac{1}{\mp\sqrt{3}} & \pm\frac{2\sqrt{2}}{\sqrt{3}} & 0 \\ 0 & 0 & \pm 2 \end{bmatrix}$$

Check that $LL^t = A$.

Choleky decomposition of A.

(Singular value decomposition) Theorem:-

Let A be an $m\times n$ matrix over $\mathbb{C}$.
Then there exists unitary matrices $U_{m\times m}, V_{n\times n}$ such that $A = UDV^*$, where

$$
D = \begin{cases}
\text{diagonal}(\lambda_1, \cdots, \lambda_r, 0, \cdots, 0), & \text{if } m = n \\[2ex]
\left[\begin{array}{ccccc|ccc}
\lambda_1 & & & & 0 & 0 & \cdots & 0 \\
 & \ddots & & & & \vdots & & \vdots \\
 & & \lambda_r & & & & & \\
 & & & 0 & & & & \\
0 & & & & 0 & 0 & \cdots & 0
\end{array}\right] & \text{if } m < n \\[6ex]
\left[\begin{array}{ccccc}
\lambda_1 & & & & 0 \\
 & \ddots & & & \\
 & & \lambda_r & & \\
 & & & 0 & \\
0 & & & & 0 \\ \hline
0 & \cdots & & \cdots & 0 \\
\vdots & & & & \vdots \\
0 & \cdots & & \cdots & 0
\end{array}\right], & \text{if } m > n
\end{cases}
$$

where $\lambda_1, \cdots, \lambda_r$ are +ve real numbers.

& $r \leq \min\{m, n\}$.

($A = UDV^*$ is known as singular value decomposition of A (SVD) of A.)