Ex. Determine the characteristic roots and the characteristic vectors of the matrix:

$$A=\begin{bmatrix} 2 & 0 & 0 \\ 4 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

eigenvalues $\lambda = 2, 2, 3.$

Note: eigenvalues of a triangular matrix are its diagonal elements.

Eigenspace of $\lambda = 2$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_3 = 0, \quad x_1 = 0, \quad x_2 = \alpha.$

eigenvector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix}.$

Geometric multiplicity of $\lambda = 2$: $\underline{1}$

Algebraic " " $\lambda = 2$: 2

Eigenspace of $\lambda = 3$: $\begin{bmatrix} -1 & 0 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3: \text{free param.} \end{array}$

eigenspace: $\alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$

Algebraic multiplicity of $\lambda = 3$: 1

Geometric multiplicity of $\lambda = 3$: 1.

Ex. Find the dimension of the eigenspace of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Char. polynomial $(\lambda-1)^3 = 0$

$\Rightarrow \lambda = 1, 1, 1.$

Algebraic multiplicity of $\lambda = 1$: 3

eigenspace: $(A-\lambda I)x = 0 \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_2 = \alpha_1 \quad x_3 = \alpha_2 : \quad x_1 = -\alpha_2$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} \cancel{0}\, 1 \\ 0 \\ 1 \end{bmatrix}$. Dimension of eigenspace: 2.

Ex. Find the dimension of eigenspace of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\lambda = 1, 1, 1.$

$(A-\lambda I)x = 0 \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_3 = \alpha_3, \quad x_2 = \alpha_2, \quad x_1 = \alpha_1$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Dim. of eigenspace = 3.

Similarity of Matrices:

An $n\times n$ matrix B is called similar to an $n\times n$ matrix A if

$$B = P^{-1}AP$$

for some non-singular matrix P.

Th. If B is similar to A then B has the same eigenvalues as A. If x is an eigenvector of A then $y = P^{-1}x$ is an eigenvector of B corresponding to the same eigenvalue.

Proof:

$$Ax = \lambda x \;\Rightarrow\; \lambda P^{-1}x = P^{-1}Ax$$

$$\Rightarrow \lambda P^{-1}x = P^{-1}A(PP^{-1})x$$

$$= P^{-1}AP(P^{-1}x)$$

$$\lambda(P^{-1}x) = B(P^{-1}x)$$

$\Rightarrow \lambda$ is an eigenvalue of B and $P^{-1}x$ is an eigenvector corresponding to the eigenvalue λ.

Th. If A and B are square matrixes similar to each other then they have the same characteristic polynomial.

Proof: $B = P^{-1}AP$

$$\det(B-\lambda I) = \det\left(P^{-1}AP - P^{-1}(\lambda I)P\right)$$

$$= \det\left(P^{-1}(A-\lambda I)P\right)$$

$$= \det(P^{-1})\det(A-\lambda I)\det(P)$$

$$= \det(A-\lambda I).$$

Diagonalization of a matrix

Def: A square matrix A is said to be diagonalizable if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix (i.e., A is similar to a diagonal matrix)

Theorem: Let A be an $n\times n$ matrix. Then A is diagonalizable iff A has n linearly independent eigenvectors.

Theorem: If an $n\times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

Remark: The matrix P which diagonalizes A is called Model matrix of A whose columns are the eigenvectors corresponding to different eigenvalues.

Example 1:

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \quad \text{eigenvalues} \quad 1 \ \& \ 6$$

$$\text{eigenvectors} \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} -1/5 & 4/5 \\ 1/5 & 1/5 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

Example: 2: $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

eigenvalues $2, \ 2, \ 8$

eigenvectors $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix} \qquad P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Example: 3: $A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ eigenvalues $2 \quad 2 \quad 3$

eigenvectors $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow$ The given matrix is not diagonalizable.

Application of Diagonalization:

a) Power of Matrices

$$P^{-1}AP = D$$

$$\Rightarrow A = PDP^{-1}$$

$$A^2 = (PDP^{-1})(PDP^{-1})$$

$$= PD(P^{-1}P)DP^{-1}$$

$$= PD^2P^{-1}$$

Similarly. $A^3 = (PD^2P^{-1})(PDP^{-1}) = PD^3P^{-1}$

and $A^n = PD^nP^{-1}$

Example: Find A^5 for $A = \begin{bmatrix} 1 & 4 \\ 1/2 & 0 \end{bmatrix}$

eigenvalues -1 2

eigenvectors $\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix}$ $\quad P^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix}$

$$A^5 = \frac{1}{6}\begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} (-1)^5 & 0 \\ 0 & 2^5 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 32 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 44 \\ 5.5 & 10 \end{bmatrix}$$

b) Solution of system of linear differential equation

Consider the system of linear equation

$$\dot{X}(t) = A\,X(t)$$

Let us assume that A is diagonalizable, then

$$D = P^{-1}AP \Rightarrow A = PDP^{-1}$$

$$\Rightarrow \dot{X}(t) = P D P^{-1} X(t)$$

$$\Rightarrow P^{-1}\dot{X}(t) = D P^{-1} X(t)$$

OR $$\left[P^{-1}X(t)\right]' = D\left[P^{-1}X(t)\right]$$

Substitute: $P^{-1}X(t) =: Y(t)$

then $$\dot{Y}(t) = DY(t)$$

$$\Rightarrow \begin{bmatrix} \dot{y}_1(t) \\ \vdots \\ \dot{y}_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & \ddots & 0 \\ 0 & \ddots & \lambda_n \end{bmatrix} \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix}$$

$$\Rightarrow \dot{y}_i(t) = \lambda_i y_i(t) \Rightarrow y_i(t) = c_i e^{\lambda_i t} \quad \forall i$$

$$\Rightarrow X(t) = PY(t).$$