Simple linear regression.

$D = \{(x_i, y_i) \mid$ x_i's are non-stochastic, y_i's are stochastic $\}$.

$\epsilon_i \overset{iid}{\sim} N(0, \sigma^2)$.

<u>Model</u> : $y_i = \beta_0 + \beta_1 x_i + \epsilon_i \quad \forall\, i = 1, 2 \dots n.$

Gauss-Markov Model.

Unknown parameters are $\underline{\beta_0, \beta_1, \sigma^2}$.

$E(y_i) = \beta_0 + \beta_1 x_i$ as $E(\epsilon_i) = 0$ and x_i's are non stochastic.

Hence.

$$\underset{\sim}{Y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad \underset{\sim}{\epsilon} = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{pmatrix} \quad \underset{\sim}{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \quad X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = (\underset{\sim}{1} \;\; \underset{\sim}{x})$$

$$\underset{\sim}{Y} = X \underset{\sim}{\beta} + \underset{\sim}{\epsilon}. \text{ where } \underset{\sim}{\epsilon} \sim N(\underset{\sim}{0}, I_n \sigma^2).$$

$$\Rightarrow \quad \underset{\sim}{Y} = \underset{\sim}{X}\beta + \underset{\sim}{\epsilon} \sim N(X\underset{\sim}{\beta}, I_n \sigma^2)$$

- Y_i's are independent but not identically distributed.

→ Methods of estimation for $\beta_0, \beta_1, \sigma^2$

- Least square method.
- Maximum likelihood method.

LS condition : To minimize.

$$S(\beta_0, \beta_1) = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$$

$$= \| \underset{\sim}{Y} - X\underset{\sim}{\beta} \|_2^2 = (\underset{\sim}{Y} - X\underset{\sim}{\beta})^T (\underset{\sim}{Y} - X\underset{\sim}{\beta})$$

$$\hat{\beta} = (\hat{\beta}_0, \hat{\beta}_1) = \underset{(\beta_0, \beta_1)}{\arg\min}\; S(\beta_0, \beta_1)$$

Solve $\left.\frac{\partial S}{\partial \beta_0}\right|_{\hat{\beta}_0} = 0$ and $\left.\frac{\partial S}{\partial \beta_1}\right|_{\hat{\beta}_1} = 0$.

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} y_i x_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

$$\Rightarrow \boxed{\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} \quad \text{and} \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}}$$

HW1:

① $\frac{\partial \underset{\sim}{b}^T A}{\partial \underset{\sim}{b}} = A^T$

② $\frac{\partial \underset{\sim}{b}^T A \underset{\sim}{b}}{\partial \underset{\sim}{b}} = b^T(A + A^T)$

$\Big\} = X^T Y = X^T X \underset{\sim}{\hat{\beta}}$ Normal eqration.

Show that $\underset{\sim}{\hat{\beta}} = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = \underline{(X^T X)^{-1} X^T Y}$

When all x_i are same $(X^T X)$ is not invertable.

⊛ Show that regression line is passing through $(\bar{x}, \bar{y})$

Regression line is

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

$$= \bar{y} - \hat{\beta}_1 \bar{x} + \hat{\beta}_1 x$$

$$= \bar{y} + \hat{\beta}_1 (x - \bar{x})$$

If we replace $x = \bar{x}$ then $\hat{y} = \bar{y}$

$\Rightarrow$ Regression line is passing through $(\bar{x}, \bar{y})$.

③

- We know that $\hat{\beta}_1 = \dfrac{S_{xy}}{S_{xx}}$

$$= \sum_{i=1}^{n} \frac{(x_i - \bar{x})}{S_{xx}} y_i$$

$$= \underset{\sim}{\alpha}^T \underset{\sim}{y} \quad \text{where } \alpha_i = \frac{(x_i - \bar{x})}{S_{xx}}, \; i = 1, 2 \ldots n.$$

S_{xy}

$= \sum (x_i - \bar{x})(y_i - \bar{y})$

$= \sum (x_i - \bar{x}) y_i$

$= \sum (y_i - \bar{y}) x_i$

$\Rightarrow \hat{\beta}_1$ is a linear estimator of β_1.

$$\hat{\beta}_1 \sim N\left(\underset{\sim}{\alpha}^T X \underset{\sim}{\beta}, \underset{\sim}{\alpha}^T I_n \sigma^2 \underset{\sim}{\alpha}\right) \quad \Big| \quad \underset{\sim}{Y} \sim N\left(X \underset{\sim}{\beta}, I_n \sigma^2\right)$$

$$\sigma^2 \underset{\sim}{\alpha}^T I_n \underset{\sim}{\alpha} = \sigma^2 \underset{\sim}{\alpha}^T \underset{\sim}{\alpha} = \sigma^2 \sum_{i=1}^{n} \alpha_i^2 = \sigma^2 \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{S_{xx}^2}$$

$$= \sigma^2 \frac{S_{xx}}{S_{xx}^2} = \sigma^2 / S_{xx} = Var(\hat{\beta}_1)$$

$$E(\hat{\beta}_1) = \underset{\sim}{\alpha}^T X \underset{\sim}{\beta} = \underset{\sim}{\alpha}^T \left(\underset{\sim}{1} \; \underset{\sim}{x}\right) \underset{\sim}{\beta}$$

$$= \left(\underset{\sim}{\alpha}^T \underset{\sim}{1} \;\; \underset{\sim}{\alpha}^T \underset{\sim}{x}\right) \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

$$= \beta_1.$$

$$0 = \underset{\sim}{\alpha}^T \underset{\sim}{1} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})}{S_{xx}}$$

$$\underset{\sim}{\alpha}^T \underset{\sim}{x} = \sum_{i=1}^{n} \frac{(x_i - \bar{x}) \cdot x_i}{S_{xx}} = \frac{S_{xx}}{S_{xx}} = \underline{1}.$$

$$\hat{\beta}_1 \sim N\left(\beta_1, \sigma^2 / S_{xx}\right)$$

$\Rightarrow \hat{\beta}_1$ is an unbiased estimator of β_1.

HW. 2. Express $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$ as a linear estimator of β_0. and show that.

$$\hat{\beta}_0 \sim N\left(\beta_0, \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)\right).$$

$$\begin{aligned}
\hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x} \\
&= \frac{1}{n}\underset{\sim}{1}^T \underset{\sim}{y} - \bar{x} \cdot \underset{\sim}{\alpha}^T \underset{\sim}{y}. \\
&= \left(\frac{1}{n}\underset{\sim}{1} - \bar{x}\underset{\sim}{\alpha}\right)^T \underset{\sim}{y}. \\
&\equiv \underset{\sim}{\theta}^T \underset{\sim}{y}.
\end{aligned}
\qquad \theta_i = \left(\frac{1}{n} - \frac{\bar{x}(x_i - \bar{x})}{S_{xx}}\right)$$

$$\hat{\beta}_0 \sim N\left(\underset{\sim}{\theta}^T X \underset{\sim}{\beta},\ \underset{\sim}{\theta}^T I_n \sigma^2 \underset{\sim}{\theta}\right)$$

$$\left.\begin{aligned}
E(\hat{\beta}_0) &= \beta_0 \\
Var(\hat{\beta}_0) &= \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right).
\end{aligned}\right\} \text{HW.}$$

- Estimation of σ^2.

$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$. is the regression line.

$\hat{\underset{\sim}{y}} = \hat{\beta}_0 + \hat{\beta}_1 \underset{\sim}{x}$ $\quad$ $\underset{\sim}{x}$ is known to us.

$\Rightarrow \hat{\underset{\sim}{y}} = X \hat{\underset{\sim}{\beta}}$. $\qquad X = [\underset{\sim}{1} \; \underset{\sim}{x}]$

$\Rightarrow \hat{y} = X(X^T X)^{-1} X^T \underset{\sim}{y}$. $\qquad \hat{\underset{\sim}{\beta}} = (X^T X)^{-1} X^T \underset{\sim}{y}$.

Let us denote $P_X = X(X^T X)^{-1} X^T$

- P_X is an orthogonal projection matrix of $\mathcal{C}(X)$.
- $P_X^T = P_X$. symmetric
- $P_X^2 = P_X$. idempotent.

$\hat{\underset{\sim}{Y}}$ is the orthogonal projection of $\underset{\sim}{Y}$ in $\mathcal{C}(X)$. and $\underset{\sim}{e}$ is orthogonal to $\hat{\underset{\sim}{Y}}$.

As $\hat{\underset{\sim}{y}} = P_X \underset{\sim}{Y}$. $\longrightarrow$ prediction vector.

define. $\underset{\sim}{e} = \underset{\sim}{Y} - \hat{\underset{\sim}{Y}} = (I - P_X)\underset{\sim}{Y}$ $\rightarrow$ estimated error vector.

- ~~$\mathcal{C}(\ldots)$~~
- $(I_n - P_X)^T = (I_n - P_X)$ symmetric
- $(I - P_X)^2 = (I - P_X)$. idempotent.

$\underset{\sim}{e} = (I - P_X)\underset{\sim}{Y} \sim N(0, (I - P_X)\sigma^2)$.

$\hat{\underset{\sim}{Y}} = P_X \underset{\sim}{Y} \sim N(X\beta, P_X \sigma^2)$

$\underset{\sim}{Y} \sim N(X\underset{\sim}{\beta}, \sigma^2 I_n)$

- $\text{cov}(\hat{\underset{\sim}{Y}}, \underset{\sim}{e})$.

$= \text{cov}(P_X \underset{\sim}{Y}, (I - P_X)\underset{\sim}{Y})$.

$$\boxed{\text{cov}(A\underset{\sim}{Y}, B\underset{\sim}{Y}) = A \Sigma . B^T}$$

$= \sigma^2 P_X I_n (I_n - P_X) = \sigma^2 0 = 0$ matrix.

Also. $\hat{y}^T e = 0$ scalar..

- $\hat{y}$ is an orthogonal projection of y in $\mathcal{C}(\underset{\sim}{X})$.
- $\hat{\underset{\sim}{y}}^T \underset{\sim}{e} = 0$.
- $\mathrm{cov}(\hat{\underset{\sim}{y}}, \underset{\sim}{e}) = 0$
- $\hat{\underset{\sim}{y}}, \underset{\sim}{e}$ are independently distributed. when $\underset{\sim}{y} \sim N(X\beta, \sigma^2 I_n)$.

$\cancel{\sum(\hat{y}_i - \bar{y})^2}$

$$SSR = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \underset{\sim}{e}^T \underset{\sim}{e}.$$

$$= \left[(I - P_x)\underset{\sim}{y}\right]^T \left[(I - P_x)\underset{\sim}{y}\right]$$

$$= \underset{\sim}{y}^T (I - P_x)^T (I - P_x) \underset{\sim}{y}.$$

$$= \underset{\sim}{y}^T (I - P_x) \underset{\sim}{y}.$$

$$\boxed{\underset{\sim}{y}^T (I - P_x) \underset{\sim}{y} \sim \sigma^2 \chi^2_{n-2,\, \underline{0}}}$$

$$df = \mathrm{rank}(I - P_x) = (n-2)$$

$$ncp = \underset{\sim}{\mu}^T \cancel{\sigma^2} (I_n - P_x) \underset{\sim}{\mu}.$$

$$= \cancel{\sigma^2} (X\underset{\sim}{\beta})^T (I - P_x) X \underset{\sim}{\beta}.$$

$$= \cancel{\sigma^2} \underset{\sim}{\beta}^T \left[X^T (I - P_x) X\right] \underset{\sim}{\beta}$$

$$= \underset{\sim}{\beta}^T \, 0 \, \underset{\sim}{\beta} = 0$$

matrix   scalar

$\underset{\sim}{y}^T (I - P_x) \underset{\sim}{y}$

$= \sigma^2 \left(\frac{y}{\sigma}\right)^T (I - P) \left(\frac{y}{\sigma}\right)$

~~ncp~~ $= \sigma^{-2} \left(\frac{X\beta}{\sigma}\right)^T (I - P) \left(\frac{X\beta}{\sigma}\right)$

0

as $(I - P_x)$ is symmetric and idempotent.

$\underset{\sim}{y} \sim N(X\underset{\sim}{\beta}, \sigma^2 I_n)$

$\mathrm{rank}(P_x) = 2$.

$\frac{y}{\sigma} \sim N\left(\frac{X\beta}{\sigma}, I_n\right)$

$\boxed{AGA = A}$

$\underline{X^T X - X^T X (X^T X)^- X^T X = 0}$

SSR = Residual Sum of square = $\sum_{i=1}^{n}(y_i - \hat{y})^2$

$$E(SSR) = E(e^T e) = E\left(\sum_{i=1}^{n}(y_i - \hat{y})^2\right)$$

$$= E\left(\underset{\sim}{y}^T (I - P_X) \underset{\sim}{y}\right) = (n-2)\sigma^2.$$

$$\Rightarrow E\left[\frac{\sum_{i=1}^{n}(y_i - \hat{y})^2}{n-2}\right] = \sigma^2.$$

$$\hat{\sigma}^2 = \frac{SSR}{n-2}.$$ Unbiased estimator.

HW: 3 (a) Find the maximum liklihood estimator of $\hat{\beta}_0, \hat{\beta}_1, \sigma^2$.

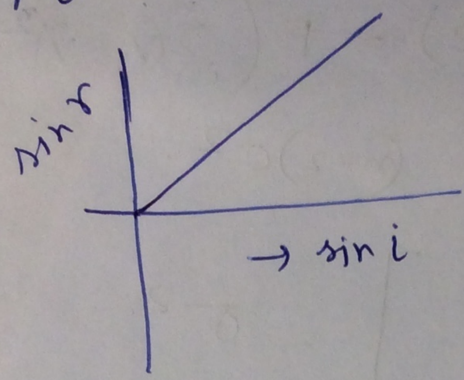
(b) Are they unbiased estimators of their corresponding parameter?

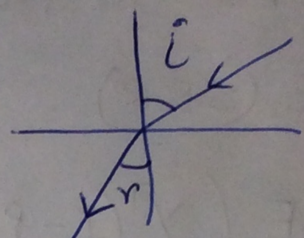
$$\hat{\beta}_{0\,mle} = \hat{\beta}_{0\,LS}$$

$$\hat{\beta}_{1\,mle} = \hat{\beta}_{1\,LS}.$$

$$\hat{\sigma}^2_{mle} = \frac{SSR}{n} \neq \frac{SSR}{n-2} = \hat{\sigma}^2_{LS}.$$

① $H_0: \beta_0 = 0$ vs $H_1 = \beta_0 \neq 0$.





$$\frac{\sin i}{\sin r} = C.$$

$\hat{\beta}_0 \sim$.

σ^2 has to be estimated.

- Find test statistic and the distribution of test statis under Ho.
- Conclude with the test rule.

Hypothesis testing for SLR.

① $H_0: \beta_0 = b_0$ vs $H_1: \beta_0 \neq b_0$.

Lelvel of test $\alpha = 0.05$ or 0.01 or 0.1.

we know: $\hat{\beta}_0 \sim N\left(\beta_0, \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)\right)$

$$\Rightarrow \frac{\hat{\beta}_0 - \beta_0}{\sqrt{\sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)}} \sim N(0,1)$$

Under Ho $$\frac{\hat{\beta}_0 - b_0}{\sqrt{\sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)}} \sim N(0,1).$$

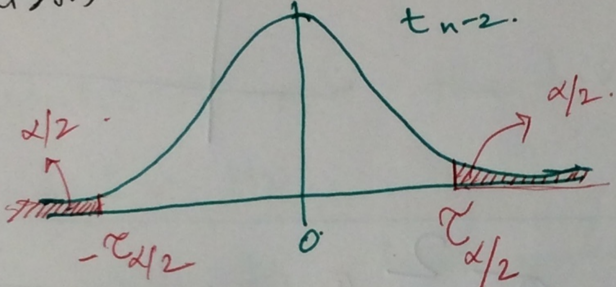
As we don't know σ^2 we use its estimated value $\hat{\sigma}^2 = \frac{1}{(n-2)}\left(S_{yy} - \frac{S_{xy}^2}{S_{xx}}\right)$

Under H_0

$$T_0 = \frac{\hat{\beta}_0 - b_0}{\sqrt{\hat{\sigma}^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)}} \sim t_{n-2}$$

This the distribution of test statistic under H_0.

level α.



Test rule

If T_0(obsered) $> \tau_{\alpha/2}$
or. T_0(obsued) $< -\tau_{\alpha/2}$.

$\Leftrightarrow |T_0(\text{obs})| > \tau_{\alpha/2}$.

Conclusion: then reject H_0 in favour of H_1 at level α.

What is $100(1-\alpha)\%$ CI of β_0?

$$\frac{\hat{\beta}_0 - \beta_0}{\sqrt{\hat{\sigma}^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)}} \sim t_{n-2}.$$

$$\Rightarrow P\left(-\tau_{\alpha/2} < \frac{\hat{\beta}_0 - \beta_0}{\sqrt{\hat{\sigma}^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)}} < \tau_{\alpha/2}\right) = 1-\alpha.$$

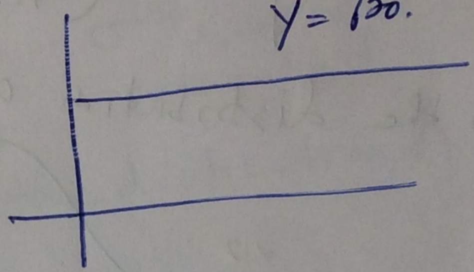
$$P\left(\hat{\beta}_0 - \tau_{\alpha/2}\sqrt{\hat{\sigma}^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)} < \beta_0 < \hat{\beta}_0 + \tau_{\alpha/2}\sqrt{\hat{\sigma}^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)}\right) = 1-\alpha.$$

HW (a) $H_0: \beta_1 = b_1$ vs $\beta_1 > b_1$

Perform a test at level $\alpha = 0.05$

(b) Find $100(1-\alpha)\%$ CI of β_1.

If: $b_1 = 0$



$H_0: \sigma^2 = \sigma_0^2 = f(S_{xy})$ vs $\sigma^2 > \sigma_0^2$.

$$\hat{\sigma}^2 = \frac{SSR}{n-2}$$

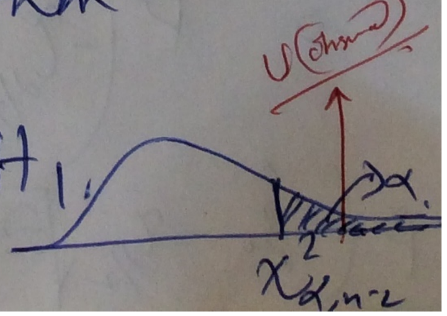
$$\Rightarrow SSR = \cancel{\hat{\sigma}^2(n-2)} \sim \sigma^2 \chi^2_{n-2}$$

$$\Rightarrow \frac{SSR}{\sigma^2} \sim \chi^2_{n-2}.$$

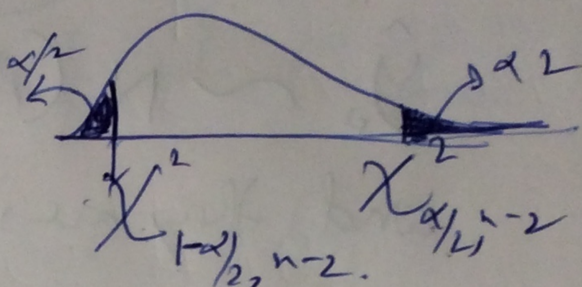
Under Ho $U = \frac{SSR}{\sigma_0^2} \sim \chi^2_{n-2}$.

For large value of U we will reject Ho.

If $\frac{SSR}{\sigma_0^2} > \chi^2_{\alpha, n-2}$ then we reject Ho in favor of H_1.



$$\frac{SSR}{\sigma^2} \sim \chi^2_{n-2}.$$



$$\Rightarrow P\left(\chi^2_{1-\alpha/2,\, n-2} < \frac{SSR}{\sigma^2} < \chi^2_{\alpha/2,\, n-2}\right) = 1-\alpha.$$

$$= P\left(\frac{SSR}{\chi^2_{\alpha/2,\, n-2}} < \sigma^2 < \frac{SSR}{\chi^2_{1-\alpha/2,\, n-2}}\right) = 1-\alpha.$$

Prediction.

Let (y_0, x_0) also satisfies the regression model.

$$y_0 = \beta_0 + \beta_1 x_0 + \epsilon_0 \qquad \epsilon_0 \sim N(0, \sigma^2)$$

we know x_0 we want to predict y_0.

predicted value $\boxed{\hat{y}_0 = \hat{\beta}_0 + \hat{\beta}_1 x_0}$ then the distribution of

$$= \bar{y} + \hat{\beta}_1 (x_0 - \bar{x})$$

$$\hat{y}_0 \sim N\left(\beta_0 + \beta_1 x_0,\ \sigma^2\left(\frac{1}{n} + \frac{(x_0-\bar{x})^2}{S_{xx}}\right)\right)$$

as. $COV(\bar{y}, \hat{\beta}) = 0.$ Home work

$y_0 \sim N(\beta_0 + \beta_1 x_0, \sigma^2)$ by assumption.

$\hat{y}_0 \sim N\left(\beta_0 + \beta_1 x_0, \sigma^2\left(\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{xx}}\right)\right)$

and they are independent.

$$y_0 - \hat{y}_0 \sim N\Big(0, \underbrace{\sigma^2\left(1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{xx}}\right)}_{= \sigma_{y_0}^2}\Big)$$

$$\frac{y_0 - \hat{y}_0}{\sqrt{\sigma_{y_0}^2}} \sim N(0,1)$$

If we use $\hat{\sigma}^2$ inplace σ^2.

$$\frac{y_0 - \hat{y}_0}{\sqrt{\hat{\sigma}_{y_0}^2}} \sim t_{n-2}.$$

$$P\left(\hat{y}_0 - \tau_{\alpha/2, n-2}\sqrt{\hat{\sigma}_{y_0}^2} < y_0 < \hat{y}_0 + \tau_{\alpha/2}\sqrt{\hat{\sigma}_{y_0}^2}\right) = 1-\alpha.$$

$100(1-\alpha)\%$ Prediction interval.

Consider paired random variables (X, Y)

then the regression of Y on $X = x$ is

$$E(Y \mid X = x) = \int_{-\infty}^{\infty} y\, f(y \mid x)\, dy$$

which is a function of x.

In particular if $(X, Y) \sim N(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$

then we are interested to obtain the regression of Y on $X = x$ or X on $Y = y$.

$$f(y \mid X = x) = \frac{f(x, y)}{f_X(x)} \Rightarrow f(x, y) = \underline{f(y \mid x) f_X(x)}.$$

If $(X, Y) \sim N(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$

$X \sim N(\mu_x, \sigma_x^2)$

$$f(x, y) = \frac{e^{-\frac{1}{2}\frac{1}{(1-\rho^2)}\left\{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right\}}}{2\pi\, \sigma_x \sigma_y \sqrt{1-\rho^2}}$$

$$= \frac{e^{-\frac{1}{2}\left(\frac{1}{1-\rho^2}\right)\left\{\left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{y-\mu_y}{\sigma_y}\right)\left(\frac{x-\mu_x}{\sigma_x}\right) + \rho^2\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right\}}\, e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}{2\pi\, \sigma_x \sigma_y \sqrt{1-\rho^2}}$$

$$= \frac{e^{-\frac{1}{2}\left(\frac{1}{1-\rho^2}\right)\left\{\left(\frac{y-\mu_y}{\sigma_y} - \rho\frac{x-\mu_x}{\sigma_x}\right)^2\right\}}\, e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}{\sqrt{2\pi}\, \sigma_y \sqrt{1-\rho^2}\, \sqrt{2\pi}\, \sigma_x}$$

$$= \left[\frac{e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}{\sqrt{2\pi}\, \sigma_x}\right]\left[\frac{e^{-\frac{1}{2}\left\{\frac{1}{\sigma_y^2(1-\rho^2)}\left(y - \left[\mu_y + \rho\frac{\sigma_y}{\sigma_x}(x-\mu_x)\right]\right)^2\right\}}}{\sqrt{2\pi}\, \sqrt{\sigma_y^2(1-\rho^2)}}\right].$$

As $X \sim N(\mu_x, \sigma_x^2)$

$$Y|X \sim N\left(\mu_y + \rho\frac{\sigma_y}{\sigma_x}(x-\mu_x),\ \sigma_y^2(1-\rho^2)\right)$$

by symmetry

$$X|Y \sim N\left(\mu_x + \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y),\ \sigma_x^2(1-\rho^2)\right).$$

Note: $V(Y) \geq V(Y|X)$

i.e. $\sigma_y^2 \geq \sigma_y^2(1-\rho^2)$. $\quad |\rho| \leq 1$

Note: $\beta_0 = \mu_y \cancel{\#} - \rho\frac{\sigma_y}{\sigma_x}\mu_x$.

$\beta_1 = \rho\frac{\sigma_y}{\sigma_x}$.

If we want to test. $H_0: \beta_1 = 0$ vs $H_1: \beta_1 \neq 0$.

$\Leftrightarrow H_0: \rho = 0$ vs $H_1: \rho \neq 0$.

$$\hat{\rho} = r = \frac{\sum(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum(x_i-\bar{x})^2\sum(y_i-\bar{y})^2}} = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$$

Under H_0: ~~$\beta_1 = 0$~~ i.e. $\rho = 0$.

$$\boxed{\frac{r\sqrt{n-2}}{\sqrt{1-r^2}} \sim t_{n-2}.}$$

n is the sample size.

$H_0: \rho = \rho_0 \neq 0.$ Vs $H_1: \rho \neq \rho_0.$

We can use large sample test:

Define. $Z = \frac{1}{2} \log \frac{1+r}{1-r} = \tanh^{-1}(r).$

$$\mu_Z = \frac{1}{2} \log \frac{1+\rho_0}{1-\rho_0} = \tanh^{-1}(\rho_0).$$

$$\sigma_Z^2 = (n-3)^{-1}$$ where n is the sample size.

$$\left(\frac{Z - \mu_Z}{\sigma_Z}\right) \sim N(0,1)$$ when

<u>n is large.</u>