Lecture 2

Theorem:- (1) Every vector space has a basis.

(2) Any two bases of a vector space have the same cardinality. This number is called the dimension of the vector space.

Def:- Let V be a vector space over F. Let $B \subset V$ be a basis of V. Then $\dim(V) := |B|$.

Example:- (i) $\{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis of $\mathbb{R}^3$.

(ii) $\{1, x, x^2, \ldots\}$ is a basis of $\mathbb{R}[x]$.

Characterization of basis

Thm:- Let V be a vector sp. over a field F. Let $B \subset V$ be a subset of V. Then the following are equivalent:

(i) B is a basis of V

(ii) B is a maximal l.i subset of V

(iii) B is a minimal spanning subset of V.

Def:- Let A be an $m \times n$ matrix.
A matrix obtained from A by deleting some rows & some columns of A is

Some ...
called a submatrix of A.

Let $A = [a_{ij}]_{m\times n}$.

A submatrix denoted $A(i_1,\dots,i_r \mid j_1,\dots,j_s)$ of A is the matrix of size $r\times s$ whose entries are as

$$A(i_1,\dots,i_r \mid j_1,\dots,j_s) = \begin{bmatrix} a_{i_1j_1} & a_{i_1j_2} & \cdots & a_{i_1j_s} \\ a_{i_2j_1} & a_{i_2j_2} & \cdots & a_{i_2j_s} \\ \vdots & & & \\ a_{i_rj_1} & a_{i_rj_2} & \cdots & a_{i_rj_s} \end{bmatrix}_{r\times s}$$

where

$$1 \le i_1 < \cdots < i_r \le m$$
$$1 \le j_1 < \cdots < j_s \le n.$$

Def:- The determinants of square submatrices of A are called the minors of A.

Examples:- (1) $A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 3 & 5 \\ -1 & 0 & 1 & -1 \end{bmatrix}$

2x2 minors: $\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$, $\begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix}$, $\begin{vmatrix} 3 & 5 \\ 1 & -1 \end{vmatrix}$, $\begin{vmatrix} 2 & 4 \\ 0 & -1 \end{vmatrix}$

....

3x3 minors: $|A(1,2,3 \mid 1,2,3)| = \left|\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ -1 & 0 & 1 \end{bmatrix}\right|$

$|A(1,2,3 \mid 1,3,4)| = \left|\begin{bmatrix} 1 & -1 & 4 \\ 0 & 3 & 5 \end{bmatrix}\right|$

$\begin{bmatrix} -1 & 1 & 4 \end{bmatrix}$,

- $\begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}$ is not a submatrix of A.

Def:- Let $A_{m\times n}$ be an $m\times n$ matrix, say $A \neq \underline{0}$. Then the <u>rank</u> of A is defined as the integer $r \geq 0$ such that there exists a $r\times r$ non-zero minor of A and all minors of A of size $> r$ are all zeros.

i.e.) Rank of A is the largest integer r such that there exists an $r\times r$ minor which is non-zero.

Def:- The determinants of the submatrices of the form $A(i_1,\ldots,i_r \mid i_1,\ldots,i_r)$ are called the <u>principal minors</u> of $A_{m\times n}$, where $1 \leq i_1 < \cdots < i_r \leq \min\{m,n\}$

Def:- The <u>leading principal minors</u> of $A_{n\times n}$ are determinants of $A(1|1)$, $A(1,2|1,2)$, $\ldots$, $A(1,2,\ldots,n \mid 1,2,\ldots,n)$.

Recall:- ① Let $B_{m\times n}$ be a matrix which row equivalent to $A_{m\times n}$. Then

$\text{rank}[A] = \text{rank}[B]$.

Defn Let V, W be vector spaces over a field F. A map $T: V \to W$ is said to be a <u>linear transformation</u> (l.t), if

(i) $T(\underline{v_1} + \underline{v_2}) = T(\underline{v_1}) + T(\underline{v_2}) \;\; \forall \underline{v_1}, \underline{v_2} \in V$.

(ii) $T(\lambda \underline{v}) = \lambda T(\underline{v}) \;\; \forall \underline{v} \in V, \; \forall \lambda \in F$.

<u>Examples</u>:– ① Let $A_{m\times n}$ be an $m\times n$ real matrix. Then define a ~~map~~ map $T: \mathbb{R}^n \to \mathbb{R}^m$ as

$$T(\underline{v}) = A\underline{v} \quad \forall \underline{v} \in \mathbb{R}^n.$$

check that T is a L.T.

② $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T(x, y, z) = (x+y, \, y+z)$

L.T.

<u>Convention</u>:– All vectors in $\mathbb{R}^n$ we write as column vectors (column matrix)

i.e., $$\mathbb{R}^n = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \middle| \; a_1, \dots, a_n \in \mathbb{R} \right\}$$
$$= \left\{ (a_1, \dots, a_n) \; \middle| \; a_1, \dots, a_n \in \mathbb{R} \right\}$$

③ $T: V \to W$, $T(\underline{v}) = \underline{0} \;\; \forall \underline{v} \in V$. is a L.T called <u>zero transformation</u>.

Defn Let $T: V \to W$ be a L.T. Then the null space of T or kernel of T is defined as $N(T) := \{ \underline{v} \in V \mid T(\underline{v}) = \underline{0} \}$.

check that $N(T) \subseteq V$ is a subspace of V & its dimension is called the nullity of T.

i.e., $\text{nullity}(T) := \dim(N(T))$.

Defn Let $T: V \to W$ be a L.T. Then the range of T or image of T is defined as

$$R(T) := \{ T(\underline{v}) \mid \underline{v} \in V \}$$
$$= \{ \underline{w} \in W \mid \underline{w} = T(\underline{v}) \text{ for some } \underline{v} \in V \}$$

Check that $R(T)$ is a subsp. of W.

& its dimension is called the rank of T.

i.e., $\text{rank}(T) = \dim(R(T))$.

Theorem (Rank-nullity thm)

Let V, W be vector space over a field F. Let $T: V \to W$ be a L.T. Then

$$\text{rank}(T) + \text{nullity}(T) = \dim(V).$$

i.e. $\dim(R(T)) + \dim(N(T)) = \dim(V)$.

Proposition:— Let $A_{m\times n}$, $B_{n\times p}$ be matrices.

Then $\mathrm{rank}(AB) \leq \min\{\mathrm{rank}(A), \mathrm{rank}(B)\}$.

proof:— We will show: $\mathrm{rank}(AB) \leq \mathrm{rank}(A)$
& $\mathrm{rank}(AB) \leq \mathrm{rank}(B)$.

claim 1: $N(B) \subseteq N(AB)$.

Pf of claim 1:— Let $\underline{v} \in N(B)$.

$B: \mathbb{R}^p \to \mathbb{R}^n$
$B(\underline{v}) = B\underline{v}$ $\forall \underline{v} \in \mathbb{R}^p$.

$\Rightarrow B\underline{v} = \underline{0}$.

$\Rightarrow A(B\underline{v}) = A\underline{0} = \underline{0}$

$\Rightarrow (AB)\underline{v} = \underline{0}$

$\Rightarrow \underline{v} \in N(AB)$

$\therefore N(B) \subseteq N(AB) \subseteq \mathbb{R}^p$. subspaces.

$\Rightarrow \dim(N(B)) \leq \dim(N(AB))$

($\because$ if $W \subseteq V$ subsp. then $\dim(W) \leq \dim(V)$)

$$\Rightarrow \underbrace{p - \dim(N(B))}_{\substack{\| \\ \mathrm{rank}(B)}} \geq \underbrace{p - \dim(N(AB))}_{\substack{\| \\ \mathrm{rank}(AB)}}$$

by rank-nullity Thm.

$\therefore$ $\boxed{\mathrm{rank}(AB) \leq \mathrm{rank}(B)}$.

claim 2: $R(AB) \subseteq R(A)$

proof of claim 2: Let $\underline{w} \in R(AB)$

$\Rightarrow \underline{w} = (AB)(\underline{v})$ for some $\underline{v}$.

$\Rightarrow$ $\underline{w} = \underbrace{A(B\underline{v})}_{\in R(A)} = A(\underline{u})$ for some $\underline{u}$, $\underline{u} = B\underline{v}$

$\therefore$ $\underline{w} \in R(A)$

$\therefore$ $R(AB) \subseteq R(A)$

$\Rightarrow$ $\underbrace{\dim[R(AB)]}_{= \text{rank}(AB)} \leq \underbrace{\dim(R(A))}_{= \text{rank}(A)}$.

$\therefore$ $\boxed{\text{rank}(AB) \leq \text{rank}(A)}$

Eg: Let A be any $m \times n$ real matrix.

(1) Define a L.T $T_A : \mathbb{R}^n \to \mathbb{R}^m$

$T_A(\underline{v}) = A\underline{v}$ $\forall \underline{v} \in \mathbb{R}^n$.

$A \mapsto T_A$

Corollary (consequence):–

Let $A_{n\times n}$ be a matrix & Suppose $P_{n\times n}$ is an invertible matrix.

Then $\text{rank}(A) = \text{rank}(PA) = \text{rank}(AP)$

proof:– $\text{rank}(A) = \text{rank}(P^{-1}(PA))$

$\Rightarrow$

$\leq \text{rank}(PA)$

$\leq \text{rank}(A).$

$\Rightarrow \text{rank}(A) = \text{rank}(PA).$

Let $A_{m\times n}$ be an $m\times n$ matrix.

$A = [\underline{c_1}\ \underline{c_2} \cdots \underline{c_n}]$

where $\underline{c_1}, \underline{c_2}, \ldots, \underline{c_n}$ are column vectors of A which lies in $\mathbb{R}^m$.

llly $A = \begin{bmatrix} \underline{r_1} \\ \underline{r_2} \\ \vdots \\ \underline{r_m} \end{bmatrix}$ where $\underline{r_1}, \ldots, \underline{r_m}$ are the row vectors of A which lies in $\mathbb{R}^n$.

Blocks of matrices.

<u>Definition:</u> Let A be an $m\times n$ matrix. Then the <u>rowspace</u> of A is defined as the subspace spanned by the row vectors of A in $\mathbb{R}^n$.

i.e. $\text{rowspace}(A) \doteq \text{Span}(\{\underline{r_1}, \ldots, \underline{r_m}\}) \subseteq \mathbb{R}^n$.

where $\underline{r_1}, \ldots, \underline{r_m}$ are the rows of A.

llly the column space of A is defined as

Illy the column space of A is defined as the subspace of $\mathbb{R}^m$ spanned by the column vectors of A.

i.e., $\text{Columnspace}(A) = \text{Span}(\{\underline{c}_1, \dots, \underline{c}_n\}) \subseteq \mathbb{R}^m$.
where $\underline{c}_1, \dots, \underline{c}_n$ are the columns of A.

Defs The dimension of rowspace of $A_{m \times n}$ is called the <u>rowrank</u> of A.

Illy the dimension of column space of $A_{m \times n}$ is called the <u>column rank</u> of A.

<u>Observations</u>:

① $$\text{rowspace}(A_{m\times n}) = \left\{ \lambda_1 \underline{r}_1 + \dots + \lambda_m \underline{r}_m \;\middle|\; \lambda_1, \dots, \lambda_m \in \mathbb{R} \right\}$$
$$= \left\{ \begin{bmatrix} \underline{r}_1 & \cdots & \underline{r}_m \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} \;\middle|\; \lambda_i \in \mathbb{R} \;\forall i \right\}$$
$$= \left\{ A^t \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix} \;\middle|\; \lambda_i \in \mathbb{R} \;\forall i \right\}$$
$$= R(A^t).$$

② $$\text{Columnspace}(A_{m\times n}) = \left\{ \lambda_1 \underline{c}_1 + \dots + \lambda_n \underline{c}_n \;\middle|\; \lambda_i \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} \underline{c_1} & \underline{c_2} & \cdots & \underline{c_n} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \;\middle|\; \lambda_i \in \mathbb{R} \; \forall i \right\}$$

$$= \left\{ A \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \;\middle|\; \lambda_i \in \mathbb{R} \; \forall i \right\}$$

$$= R(A).$$

<u>Proposition:-</u> Let $A_{m \times n}$ be an $m \times n$ matrix. Then $\operatorname{rank}(A) = \operatorname{rank}(A^t)$.

<u>proof:-</u>

<u>claim:-</u> $A\underline{x} = \underline{0} \iff A^t A \underline{x} = \underline{0} \quad \forall \underline{x} \in \mathbb{R}^n.$

<u>Pf. of claim:-</u>

<u>$\Rightarrow$:</u> Assume $A\underline{x} = \underline{0}$.

$$\Rightarrow A^t(A\underline{x}) = A^t \underline{0} = \underline{0}$$

$$\Rightarrow (A^t A)\underline{x} = \underline{0}.$$

<u>$\Leftarrow$:</u> Assume $A^t A \underline{x} = \underline{0}$.

To show: $A\underline{x} = \underline{0}$.

We have $A^t A \underline{x} = \underline{0}$

$$\Rightarrow \underset{1\times n}{\underline{x}^t}\,\underset{n\times m}{A^t}\,\underset{m\times n}{A}\,\underset{n\times 1}{\underline{x}} = \underline{x}^t\underline{0} = 0$$

$$\Rightarrow (A\underline{x})^t(A\underline{x}) = 0 \qquad \left(\because (AB)^t = B^tA^t\right)$$

$$\Rightarrow |A\underline{x}|^2 = 0$$

$$\Rightarrow |A\underline{x}| = 0$$

$$\Rightarrow A\underline{x} = 0.$$

was required

$$\underline{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

$$|\underline{y}| = \sqrt{y_1^2 + \cdots + y_m^2}$$

$$\underline{y}^t\underline{y} = y_1^2 + \cdots + y_m^2 = |\underline{y}|^2$$

Thus the claim is true.

claim $\Rightarrow$

$$N(A) = \{\underline{x} \in \mathbb{R}^n \,/\, A\underline{x} = \underline{0}\}$$

$$= \{\underline{x} \in \mathbb{R}^n \,/\, A^tA\underline{x} = 0\}$$

$$= N(A^tA).$$

$$\therefore \boxed{N(A) = N(A^tA)}$$

$$\Rightarrow \text{nullity}(A) = \text{nullity}(A^tA)$$

$$\Rightarrow n - \text{nullity}(A) = n - \text{nullity}(A^tA)$$

$$\Rightarrow \text{rank}(A) = \text{rank}(A^tA)$$

(By rank-nullity thm).

$$\leq \text{rank}(A^t).$$

$$(\because rk(AB) \leq rk(A))$$

$\therefore$ $\operatorname{rank}(A) \leq \operatorname{rank}(A^t)$.

Ily we can prove $\operatorname{rank}(A^t) \leq \operatorname{rank}(A)$ by considering A^t instead of A in the above proof. (EXERCISE)

Thus $\boxed{\operatorname{rank}(A) = \operatorname{rank}(A^t)}$.