# Lecture 6

Definition:- Let $A$ be an $n \times n$ matrix over a field $F$. Then $A$ is said to be diagonalizable over $F$, if there exists a basis for $F^n$ consisting of eigenvectors of $A$.

Equivalently, there exists an invertible matrix $P_{n\times n}$ such that $P^{-1}AP = D$, where $D$ is a diagonal matrix.

Examples:- ① $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ over $F = \mathbb{R}$.

Eigenvalues of $A$ are $0, 0$.

Eigenspace corresponding to $\lambda = 0$:

$$E_0(A) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; A\begin{pmatrix} x \\ y \end{pmatrix} = 0\begin{pmatrix} x \\ y \end{pmatrix} = \underline{0} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; y = 0 \right\}$$

$$= \underline{\left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}}$$

The matrix $A$ to be diagonalizable over $\mathbb{R}$ means we need to find a basis of $\mathbb{R}^2$ of eigenvectors of $A$

$\Rightarrow$ we need to find two l.i eigenvectors of $A$.

This is not possible because any two eigenvectors of $A$ are l.d.

$\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ 0 \end{pmatrix}$ l.d.

Check! $x \neq 0, y \neq 0$

$\begin{pmatrix} x \\ 0 \end{pmatrix} = \left(\frac{x}{y}\right)\begin{pmatrix} y \\ 0 \end{pmatrix}$

$\therefore$ The given matrix $A$ is NOT diagonalizable.

<u>Theorem</u>:- Let $A$ be an $n \times n$ matrix over a field $F$. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $A$. Then $A$ is diagonalizable over $F$

$\Updownarrow$

Alg. multiplicity $(\lambda_j)$ = Geom. multiplicity $(\lambda_j)$ $\forall j = 1, 2, \dots, n$.

(2) $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -7 \\ 0 & 0 & -1 \end{bmatrix}$ over $F = \mathbb{R}$.

Eigenvalues $= 2, 2, -1$.

$$E_2(A) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle/ A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle/ (A - 2I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underline{0} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & -7 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; \begin{matrix} 3y + 4z = 0 \\ -7z = 0 \\ -3z = 0 \end{matrix} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x \in \mathbb{R} \right\}$$

$\therefore$ geo. mult$(2) = \dim(E_2(A))$
$= 1$

Alg. mult $(2) = 2$

$\therefore$ Alg. mult $(2) \neq$ geo. mult $(2)$

$\therefore$ $A$ is not diagonizable.

(3) $A = \begin{bmatrix} +1 & 2 \\ 4 & -1 \end{bmatrix}$ over $\mathbb{R}$.

Char eq. $\det(\lambda I - A) = 0$

$\det\left(\begin{bmatrix} \lambda-1 & -2 \\ -4 & \lambda+1 \end{bmatrix}\right) = 0$

$\Rightarrow$

$\lambda^2 - 1 - 8 = 0$

$\Rightarrow \lambda^2 = 9 \Rightarrow \lambda = \pm 3.$

$$E_{-3}(A) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; (A + 3I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; \begin{bmatrix} 4 & 2 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \right\}$$

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; \begin{bmatrix} \; \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; \begin{array}{l} 4x + 2y = 0 \\ \text{i.e. } -2x = y \end{array} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ -2x \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}$$

$$= \left\{ x \begin{pmatrix} 1 \\ -2 \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}$$ geo. mult $(-3) = 1$

$= \text{alg. mult}(-3)$.

$$E_3(A) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; (A - 3I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

i.e. $x - y = 0$

$$= \left\{ \begin{pmatrix} x \\ x \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}$$

$$= \left\{ x \begin{pmatrix} 1 \\ 1 \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}$$

$\therefore$ geo. mult $(3) = 1 = \text{alg. mult}(3)$.

$\therefore$ A is diagonalizable.

Let $\underline{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ eigenvector corr to $\lambda = -3$

$\underline{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ " " $\lambda = 3$.

Let $P = [\underline{v}_1 \;\; \underline{v}_2]$ $\{\underline{v}_1, \underline{v}_2\}$ l.i

$$= \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$ $\therefore$ Basis for $\mathbb{R}^2$

$$P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -3 & 3 \\ 6 & 3 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -9 & 0 \\ 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix} = D$$

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Remark: Suppose $\underset{n\times n}{A}$ is diagonalizable. $P^{-1}AP = D$

Then $A^{100} = PD^{100}P^{-1}$!

$(A = PDP^{-1})$

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Let $A$ be a $2\times 2$ matrix with real entries.

Let $\lambda = a+ib$ be an eigenvalue of $A$

& $\underline{v} = \underline{x} + i\underline{y}$ be an eigenvector corresponding to the eigenvalue $\lambda = a+ib$

$a, b, \in \mathbb{R}$

$\underline{x}, \underline{y} \in \mathbb{R}^2$.

Now $A\underline{v} = \lambda \underline{v}$

$\Rightarrow \quad A(\underline{x} + i\underline{y}) = (a+ib)(\underline{x} + i\underline{y})$

$\Rightarrow \quad A\underline{x} + iA\underline{y} = (a\underline{x} - b\underline{y}) + i(a\underline{y} + b\underline{x})$

$$\Rightarrow \quad \left.\begin{aligned} A\underline{x} &= a\underline{x} - b\underline{y} \\ A\underline{y} &= a\underline{y} + b\underline{x}. \end{aligned}\right\} \text{ --- } (*)$$

Now $\begin{bmatrix} A\underline{x} & A\underline{y} \end{bmatrix} = A\begin{bmatrix} \underline{x} & \underline{y} \end{bmatrix}$

$= \begin{bmatrix} a\underline{x} - b\underline{y} & a\underline{y} + b\underline{x} \end{bmatrix}$ (by $(*)$)

$= \begin{bmatrix} \underline{x} & \underline{y} \end{bmatrix} \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

Let $P = \begin{bmatrix} \underline{x} & \underline{y} \end{bmatrix}$

Then $AP = P\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

if $P$ is invertible, then $\boxed{P^{-1}AP = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}}$

$a, b \in \mathbb{R}$.

EXERCISE 1:

Let $A_{3\times 3}$ be a matrix over $\mathbb{R}$.

Let $A$ has real eigenvalue $\lambda$ & complex eigenvalue $a \pm ib$ ($a, b \in \mathbb{R}$).

Let $\underline{x}$, $\underline{y} \pm i\underline{z}$ be their corresponding eigenvectors of $A$. respectively.

Then there exists a matrix $P_{3\times 3}$ such that
$$AP = P \begin{bmatrix} \lambda & 0 & 0 \\ 0 & a & b \\ 0 & -b & a \end{bmatrix}$$
where $P = [\underline{x} \;\; \underline{y} \;\; \underline{z}]$

<u>EXERCISE 2</u>: Let $A$ be a $2n\times 2n$ matrix over $\mathbb{R}$

Let $a_j \pm i b_j$ $j = 1, 2, \ldots, n$ be the eigenvalues of $A$, where $a_j, b_j \in \mathbb{R}$ $\forall j$.

Then there exists a $2n\times 2n$ matrix $P$ such that
$$AP = P \begin{bmatrix} A_1 & & & O \\ & A_2 & & \\ & & \ddots & \\ O & & & A_n \end{bmatrix}_{2n\times 2n}$$

where each $A_k = \begin{bmatrix} a_k & b_k \\ -b_k & a_k \end{bmatrix}$ $\forall k = 1, 2, \ldots, n$

$$\begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_n \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \cdots & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix} & \cdots & \begin{matrix} 0 \\ 0 \end{matrix} \\ \vdots & & \ddots & \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \cdots & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{pmatrix} a_n & b_n \\ -b_n & a_n \end{pmatrix} \end{bmatrix}_{2n\times 2n}$$

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**Def:-** Let $A$ be an $n\times n$ matrix over some field $F$ ($\mathbb{R}$ or $\mathbb{C}$). Then the minimal polynomial of $A$ is defined as the least degree non-zero polynomial $p(x)$ such that $p(A)=0$

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**Cayley-Hamilton Theorem:-** Let $A$ be an $n\times n$ matrix and $p(x)$ be its characteristic polynomial. Then $p(A) = \underline{0}$.

**Remark:-** Char. poly. need not be equal to the minimal poly.

Note:- Suppose $p(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$

$\alpha_i \in F$.

$$p(A) = \alpha_0 I + \alpha_1 A + \cdots + \alpha_n A^n$$ is an $n\times n$ matrix.

<u>Example</u> ① $A = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}$

Eigenvalues $= \pm 3$.

Char. poly $= p(\lambda) = \lambda^2 - 9$.   $p(A) = 0$.

$(A - 3I) \neq 0$, $(A + 3I) \neq 0$

But $(A - 3I)(A + 3I) = p(A) = 0$.

minimal poly $= \lambda^2 - 9$.

$\lambda - 3$

$\lambda + 3$

$\lambda^2 - 9$

<u>Division Algorithm for polynomials.</u>

Let $F = \mathbb{R}$ or $\mathbb{C}$.

Let $f(x), g(x)$ be polynomials in the variable $x$ i.e. $f(x), g(x) \in F[x]$.

Then there exists polynomials $q(x)$, & $r(x)$ such that $f(x) = g(x)q(x) + r(x)$,

where $r(x) = 0$ or $\deg(r(x)) < \deg(g(x))$

<u>Theorem</u> minimal polynomial divides the char. poly.

i.e. Let $A$ be an $n\times n$ matrix & $p(x)$ be its char. poly. & $q(x)$ be its minimal polynomial. Then $q(x) \mid p(x)$.

i.e. $p(x) = q(x) \cdot h(x)$ for some polynomial $h(x)$.

<u>proof:-</u> Use the division algorithm,

$$p(x) = q(x) \cdot h(x) + r(x)$$

for some polynomials $h(x)$, $r(x)$ such that $r(x) = 0$ or $\deg(r(x)) < \deg(q(x))$.

$$p(A) = \overset{= \underline{0}}{q(A)}\, h(A) + r(A)$$

$p(A) = \underline{0}$ by Cayley - Hamilton thm.

$\Rightarrow r(A) = \underline{0}$. where $r(x) = 0$ or $\deg(r(x)) < \deg(q(x))$

If $\deg(r(x)) < \deg(q(x))$, then this is a contradiction to the minimality of $q(x)$.

$\Rightarrow r(x) = 0$

$\therefore$ $\boxed{p(x) = q(x) h(x)}$ $\Rightarrow q(x) \mid p(x)$.

(1) Find $A^{593}$, where $A = \begin{bmatrix} -3 & -4 \\ 2 & 3 \end{bmatrix}$.

Sol: The eigenvalues of $A$ are $1, -1$.

Let $f(x) = x^{593}$.

char. poly $= p(x) = x^2 - 1$.

Division Alg: $\underline{f(x) = p(x)\, q(x) + r(x)}$

$r(x) = 0$ or

$\deg[r(x)] < \deg(p(x)) = 2$

$f(A) = p(A)\, q(A) + r(A)$

$= \underline{0} + r(A)$ (by Cayley-Hamilton Thm)

$= r(A)$, where $\deg(r(x)) \le 1$.

Let $r(x) = \alpha_0 + \alpha_1 x$ for some scalars $\alpha_0, \alpha_1$.

$\therefore$ $f(A) = r(A) = \alpha_0 I + \alpha_1 A$. ✓

Suppose $\lambda$ is an eigenvalue of $A$, then

$f(\lambda) = p(\lambda)\, q(\lambda) + r(\lambda)$

$= 0 + r(\lambda)$

$\therefore$ $f(\lambda) = r(\lambda)$ ✓

$\therefore \underline{f(\lambda) = r(\lambda)}$ $\forall$ $\lambda$ eigenvalues of $A$.

$\therefore \lambda^{593} = f(\lambda) = \alpha_0 + \alpha_1 \lambda$

$\lambda = -1: \quad (-1)^{593} = \alpha_0 - \alpha_1$

$\lambda = 1: \quad 1^{593} = \alpha_0 + \alpha_1$

$$\Rightarrow \begin{aligned} -1 &= \alpha_0 - \alpha_1 \\ 1 &= \alpha_0 + \alpha_1 \end{aligned}$$

$+$

$0 = 2\alpha_0 \Rightarrow \boxed{\alpha_0 = 0}$

$\Rightarrow \boxed{\alpha_1 = 1}$

$\therefore \quad f(x) = \alpha_0 + \alpha_1 x$

$= 0 + x.$

$\Rightarrow f(A) = \alpha_0 + \alpha_1 A$

$= A$

$\Rightarrow \boxed{A^{593} = A} = \begin{bmatrix} -3 & -4 \\ 2 & 3 \end{bmatrix}.$

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<u>Theorem</u>:— Let $A$ be an $n \times n$ matrix

& $p(x)$ be its characteristic poly.

Let $f(x) = p(x)\, q(x) + r(x)$, where $r(x) = 0$

or $\deg(r(\lambda)) < \deg(p(\lambda))$.

If $\lambda$ is an eigenvalue of $A$ of multiplicity $k$, then

$$f(\lambda) = r(\lambda)$$

$$\frac{d f(\lambda)}{d\lambda} = \frac{d r(\lambda)}{d\lambda}$$

$$\frac{d^2 f(\lambda)}{d\lambda^2} = \frac{d^2 r(\lambda)}{d\lambda^2}$$

$$\vdots$$

$$\frac{d^{k-1} f(\lambda)}{d\lambda^{k-1}} = \frac{d^{k-1} r(\lambda)}{d\lambda^{k-1}}.$$