Lecture 16

Let $\underline{x} \in \mathbb{C}^n$.

$$\|\underline{x}\|_p = \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p} \geq 0$$

$$\|\alpha \underline{x}\|_p = \left(\sum_{j=1}^{n} |\alpha x_j|^p\right)^{1/p}$$

$$= \left(\sum_{j=1}^{n} |\alpha|^p |x_j|^p\right)^{1/p}$$

$$= |\alpha| \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}$$

$$= |\alpha| \, \|\underline{x}\|_p. \qquad \forall \alpha \in \mathbb{C}.$$

Theorem (Hölder's inequality) :-

Let $p>1$, $q>1$ be real numbers such that $\frac{1}{p}+\frac{1}{q}=1$ (such a pair is called a Conjugate pair)

Then for any $\underline{x}, \underline{y} \in \mathbb{C}^n$,

$$\sum_{j=1}^{n} |x_j y_j| \leq \|\underline{x}\|_p \, \|\underline{y}\|_q.$$

proof:- step 1:

Let $f(t) = 1-\lambda + \lambda t - t^{\lambda}$ $\quad 0<\lambda<1$

To show that $\alpha^{\lambda}\beta^{1-\lambda} \leq \lambda\alpha + (1-\lambda)\beta \longrightarrow (*)$

$\forall\, \alpha, \beta \geq 0.$

Notice that $f(t) \geq 0 \quad \forall\ 0 \leq t \leq 1.$

Suppose either $\alpha = 0$ or $\beta = 0$, then $(*)$ is trivially true.

Assume $\alpha \neq 0,\ \beta \neq 0$ & say $\alpha < \beta$.

Then $f\left(\frac{\alpha}{\beta}\right) \geq 0$

$$\Rightarrow (1-\lambda) + \lambda\frac{\alpha}{\beta} - \frac{\alpha^{\lambda}}{\beta^{\lambda}} \geq 0.$$

$$\Rightarrow (1-\lambda)\beta + \lambda\alpha - \alpha^{\lambda}\beta^{1-\lambda} \geq 0.$$

$$\Rightarrow \alpha^{\lambda}\beta^{1-\lambda} \leq \lambda\alpha + (1-\lambda)\beta$$

Thus $(*)$ is true.

<u>step 2</u>:

Apply $(*)$ for $\lambda = \frac{1}{p} < 1$

$$\alpha = |\hat{x}_j|^p := \frac{|x_j|^p}{\|x\|^p}$$

$$\beta = |\hat{y}_j|^q := \frac{|y_j|^q}{\|\underline{y}\|_q^q}.$$

Then we get

$$\left(|\hat{x}_j|^p\right)^{1/p}\left(|\hat{y}_j|^q\right)^{1/q} \le \frac{1}{p}|\hat{x}_j|^p + \frac{1}{q}|\hat{y}_j|^q$$

$$\Rightarrow \quad |\hat{x}_j \hat{y}_j| \le \frac{1}{p}|\hat{x}_j|^p + \frac{1}{q}|\hat{y}_j|^q$$

$$\forall j = 1, \ldots, n.$$

$$\Rightarrow \quad \sum_{j=1}^{n} |\hat{x}_j \hat{y}_j| \le \frac{1}{p}\sum_{j=1}^{n}|\hat{x}_j|^p + \frac{1}{q}\sum_{j=1}^{n}|\hat{y}_j|^q$$

$$= \frac{1}{p}\sum_{j=1}^{n}\frac{|x_j|^p}{\|\underline{x}\|_p^p} + \frac{1}{q}\sum_{j=1}^{n}\frac{|y_j|^q}{\|\underline{y}\|_q^q}.$$

$$= \frac{1}{p} \cdot \frac{1}{\|\underline{x}\|_p^p}\left(\sum_{j=1}^{n}|x_j|^p\right)$$

$$+\frac{1}{q}\frac{1}{\|\underline{y}\|_q^q}\left(\sum_{j=1}^{n}|y_j|^q\right)$$

$$=\frac{1}{p}\cdot\frac{\|\underline{x}\|_p^p}{\|\underline{x}\|_p^p}+\frac{1}{q}\frac{\|\underline{y}\|_q^q}{\|\underline{y}\|_q^q}$$

$$=\frac{1}{p}+\frac{1}{q}$$

$$=1$$

Thus $\displaystyle\sum_{j=1}^{n}|\hat{x}_j\hat{y}_j| \leq 1.$

$$\Rightarrow \sum_{j=1}^{n}\frac{|x_j y_j|}{\|\underline{x}\|_p\|\underline{y}\|_q}\leq 1$$

$$\Rightarrow \sum_{j=1}^{n}|x_j y_j| \leq \|\underline{x}\|_p\|\underline{y}\|_q.$$

Thus we proved the Hölder's inequality

<u>Theorem (Minkowski's inequality)</u> :-

$\forall\ \underline{x},\underline{y}\in\mathbb{C}^n$, $\quad \|\underline{x}+\underline{y}\|_p \leq \|\underline{x}\|_p+\|\underline{y}\|_p.$

For $p>1$.

proof:-

Let $\frac{1}{p}+\frac{1}{q}=1$.

Now for any scalars $\alpha, \beta \in \mathbb{C}$ we have

$$|\alpha+\beta|^{p} = |\alpha+\beta|\,|\alpha+\beta|^{p-1}$$

$$= |\alpha+\beta|\,|\alpha+\beta|^{p/q}.$$

$$\leq |\alpha|\,|\alpha+\beta|^{p/q} + |\beta|\,|\alpha+\beta|^{p/q} \longrightarrow (*)$$

Let $\underline{x}=(x_1, \dots, x_n)$, $\underline{y}=(y_1, \dots, y_n)$

Apply $(*)$ to the scalars x_j, y_j.

$$|x_j+y_j|^{p} \leq |x_j|\,|x_j+y_j|^{p/q} + |y_j|\,|x_j+y_j|^{p/q}$$

$$\Rightarrow \sum_{j=1}^{n} |x_j+y_j|^{p} \leq \sum_{j=1}^{n} \left|x_j\,(x_j+y_j)^{p/q}\right| + \sum_{j=1}^{n} \left|y_j\,(x_j+y_j)^{p/q}\right|$$

$$\leq \left(\sum_{j=1}^{n} |x_j|^{p}\right)^{1/p} \left(\sum_{j=1}^{n} \left(|x_j+y_j|^{p/q}\right)^{q}\right)^{1/q}$$

$$+ \left(\sum_{j=1}^{n} |y_j|^{p}\right)^{1/p} \left(\sum_{j=1}^{n} \left(|x_j+y_j|^{p/q}\right)^{q}\right)^{1/q}$$

(by apply Hölder inequality to both the terms on RHS)

$$\Rightarrow \sum_{j=1}^{n} |x_j+y_j|^p \le \left(\|x\|_p + \|y\|_p\right)\left(\sum_{j=1}^{n} |x_j+y_j|^p\right)^{1/q}$$

$$\Rightarrow \left(\sum_{j=1}^{n} |x_j+y_j|^p\right)^{1-\frac{1}{q} = \frac{1}{p}} \le \|x\|_p + \|y\|_p.$$

$$\Rightarrow \|x+y\|_p \le \|x\|_p + \|y\|_p.$$

Thus $\|-\|_p$ p-norm is a vector norm on $\mathbb{C}^n$.

<u>Definition</u>:- Let $\|-\|_a$, $\|-\|_b$ be two norms on $\mathbb{C}^n$. Then we say that these two norms are <u>equivalent</u>, if there exists $M, N > 0$ such that

$$M \|\underline{x}\|_a \leq \|\underline{x}\|_b \leq N \|\underline{x}\|_a$$

$$\forall \underline{x} \in \mathbb{C}^n.$$

<u>Theorem</u>:- All norms on $\mathbb{C}^n$ are equivalent.

<u>proof</u>:- We show that all norms on $\mathbb{C}^n$ are equivalent to the Euclian norm on $\mathbb{C}^n$.

Let $S = \{ \underline{x} \in \mathbb{C}^n \mid \|\underline{x}\|_2 = 1 \}$.

S is closed & bounded.

Let $\underline{y} = \dfrac{x}{\|\underline{x}\|_2} \in \mathbb{C}^n$. for $\underline{x} \neq \underline{0}$ in $\mathbb{C}^n$,

We have $\underline{y} \in S$. $\left(\because \|\underline{y}\|_2 = \left\| \dfrac{\underline{x}}{\|\underline{x}\|_2} \right\|_2 = \dfrac{1}{\|\underline{x}\|_2} \|\underline{x}\|_2 = 1 \right)$

Let $\|-\|$ be any arbitrary norm on $\mathbb{C}^n$.

Note that $\|-\|$ is always a continuous function

$$\|-\| : \mathbb{C}^n \to \mathbb{R}$$

$$\underline{x} \mapsto \|\underline{x}\|$$

By

By Weierstrass Theorem, $\|-\|$ attains its maximum $M>0$ & minimum $m>0$ on S.

i.e., $m \le \|\underline{y}\| \le M \quad \forall \underline{y} \in S$.

$\Rightarrow \quad m \le \left\| \frac{\underline{x}}{\|\underline{x}\|_2} \right\| \le M$ for any $\underline{x} \ne \underline{0}$ in $\mathbb{C}^n$.

$\Rightarrow \quad m \|\underline{x}\|_2 \le \|\underline{x}\| \le M \|\underline{x}\|_2$.

$\forall \underline{x}$ in $\mathbb{C}^n$.

$\|-\|_2$ & $\|-\|$ are equivalent.

Proposition:- For any $\underline{x} \in \mathbb{C}^n$,

① $\|\underline{x}\|_\infty \le \|\underline{x}\|_1 \le n \|\underline{x}\|_\infty$

② $\|\underline{x}\|_\infty \le \|\underline{x}\|_2 \le \sqrt{n} \|\underline{x}\|_\infty$

③ $\|\underline{x}\|_2 \le \|\underline{x}\|_1 \le \sqrt{n} \|\underline{x}\|_2$.

④ $\frac{1}{\sqrt{n}} \|\underline{x}\|_1 \le \|\underline{x}\|_2 \le \|\underline{x}\|_1$

⑤ $\frac{1}{n} \|\underline{x}\|_1 \le \|\underline{x}\|_\infty \le \|\underline{x}\|_1$

(b) $\frac{1}{\sqrt{n}} \|\underline{x}\|_2 \leq \|\underline{x}\|_\infty \leq \|\underline{x}\|_2.$

proof:- (1) we have $|x_j| \leq |\underline{x}_1| + \cdots + |x_n|$

$\forall j = 1, \cdots, n.$

$$\Rightarrow \max\{|x_j| \,/\, j=1,\cdots,n\} \leq \sum_{j=1}^{n} |x_j|$$

$$\Rightarrow \|\underline{x}\|_\infty \leq \|\underline{x}\|_1.$$

Now
$$\|\underline{x}\|_1 = |x_1| + \cdots + |x_n|$$
$$\leq \max_j |x_j| + \max_j |x_j| + \cdots$$
$$\cdots + \max_j |x_j|$$
$$= n \max_j |x_j|$$
$$= n \|\underline{x}\|_\infty.$$

$$\therefore \|\underline{x}\|_1 \leq n \|\underline{x}\|_\infty.$$

(2) consider
$$\|\underline{x}\|_\infty^2 = \left(\max_j |x_j|\right)^2$$
$$= \max_j |x_j|^2$$
$$\leq |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2$$

$$= \|\underline{x}\|_2^2$$

$$\Rightarrow \quad \|\underline{x}\|_\infty \le \|\underline{x}\|_2 .$$

Now $\quad \|\underline{x}\|_2^2 = |x_1|^2 + \cdots + |x_n|^2$

$$\le \max_j |x_j|^2 + \cdots + \max_j |x_j|^2$$

$$= n \max_j |x_j|^2$$

$$= n \, \|\underline{x}\|_\infty^2$$

Thus $\quad \|\underline{x}\|_2^2 \le n \, \|\underline{x}\|_\infty^2$

$$\Rightarrow \quad \|\underline{x}\|_2 \le \sqrt{n} \, \|\underline{x}\|_\infty .$$

③ $\quad \|\underline{x}\|_2^2 = |x_1|^2 + \cdots + |x_n|^2$

$$\le \left(|x_1| + \cdots + |x_n|\right)^2 = \|\underline{x}\|_1^2$$

$$\Rightarrow \quad \|\underline{x}\|_2 \le \|\underline{x}\|_1 .$$

Cauchy-Schwartz inequality:

$$\sum_{j=1}^{n} |x_j y_j| \le \|\underline{x}\|_2 \, \|\underline{y}\|_2$$

Let $\quad \underline{y} = (1, 1, \cdots, 1) \in \mathbb{C}^n .$

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Then $\|x\|_1 = \sum_{j=1}^{n} |x_j| \le \|x\|_2 \cdot \sqrt{n}$.

$\therefore \|x\|_1 \le \sqrt{n}\, \|x\|_2$.

(4), (5), (6): EXERCISE.

Matrix norm

A matrix norm is a function $\|-\|$ defined on the set of all $m\times n$ matrices over $\mathbb{C}$, satisfying the following conditions:

$$\|-\| : M_{m\times n}(\mathbb{C}) \longrightarrow \mathbb{R},$$

(i) $\|A\| \geq 0 \quad \forall A \in M_{m\times n}(\mathbb{C})$

& $\|A\| = 0 \iff A = \underline{0}$

(ii) $\|\alpha A\| = |\alpha|\, \|A\|, \quad \forall \alpha \in \mathbb{C}$
$\forall A \in M_{m\times n}(\mathbb{C})$.

(iii) $\|A+B\| \le \|A\| + \|B\|, \quad \forall A, B \in M_{m\times n}(\mathbb{C})$.

(iv) if $m=n$, i.e., for any square matrices
A, B of size n×n we have

A, B of $M_{n \times n}$, we have

$$\|AB\| \leq \|A\| \, \|B\|.$$

Examples ① $V = M_{m \times n}(\mathbb{C})$.

For $A \in V$, $\|A\|_F := \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$

Known as Frobenius norm or Euclidean norm on V.

We are interested in studying the matrix norms which are compatible with underline vector norms.

i.e., $$\|\underset{n\times n}{A}\ \underset{n\times 1}{\underline{x}}\| \leq \|A\| \, \|\underline{x}\| \quad \forall\, \underline{x} \in \mathbb{C}^n.$$

Induced matrix norm.

A vector norm defined on $\mathbb{C}^p$ $(p = m \text{ or } n)$ induces a matrix norm on $M_{m\times n}(\mathbb{C})$ defined as

$$\|A\| := \max_{\substack{\underline{x} \in \mathbb{C}^n \\ \& \ \|\underline{x}\| = 1}} \left\{ \|A\underline{x}\| \right\}$$

$$= \max_{\substack{\underline{x} \neq \underline{0} \\ \text{in } \mathbb{C}^n}} \left(\frac{\|A\underline{x}\|}{\|\underline{x}\|} \right).$$

check that this is infact a matrix norm.

Also $\|A\underline{x}\| \leq \|A\| \, \|\underline{x}\| \quad \forall \, \underline{x} \in \mathbb{C}^n$.

Example:- ① $V = M_{2\times 3}(\mathbb{C})$

on $\mathbb{C}^2$ take $\|-\|_1$

& on $\mathbb{C}^3$, take $\|-\|_2$

Then the induced norm on V from the $\|-\|_1$ norm on $\mathbb{C}^2$ & $\|-\|_2$ on $\mathbb{C}^3$

is

$$\|A\| = \max_{\substack{\underline{x} \neq \underline{0} \\ \text{in } \mathbb{C}^3}} \left(\frac{\|\underset{2\times 3}{A}\,\underset{3\times 1}{\underline{x}}\|_1}{\|\underline{x}\|_2} \right)$$

② Let $V = M_{3\times 3}(\mathbb{C})$.

The induced norm on V induced from 2-norm on $\mathbb{C}^3$ is defined as

$$\|A\| = \max_{\substack{x \neq 0 \\ \text{in } \mathbb{C}^3}} \left(\frac{\|Ax\|_2}{\|x\|_2} \right)$$

$$= \max_{\substack{x \in \mathbb{C}^3 \\ \& \|x\|_2 = 1}} \left(\|Ax\|_2 \right)$$

is known as induced 2-norm on $V = M_{3\times 3}(\mathbb{C})$ & we denote by $\|A\|_2$.

More generally, the <u>induced p-norm</u> on $V = M_{n\times n}(\mathbb{C})$, is defined as denoted as

$$\|A\|_p := \max_{\substack{x \neq 0 \\ \text{in } \mathbb{C}^n}} \left(\frac{\|Ax\|_p}{\|x\|_p} \right)$$

$$= \max_{\substack{x \in \mathbb{C}^n \\ \|x\|_p = 1}} \left(\|Ax\|_p \right)$$

<u>Qn:</u> How can we find the induced p-norms ?

<u>Theorem:</u> Let $A \in M_{n\times n}(\mathbb{C})$. $A = [a_{ij}]_{n\times n}$.

Then

(a) $\|A\|_1 = \max_{j=1,\dots,n} \left(\sum_{i=1}^{n} |a_{ij}| \right)$

= The maximum of the absolute column sums of A.

(b) $\|A\|_\infty = \max_{i=1,\dots,n} \left(\sum_{j=1}^{n} |a_{ij}| \right)$

= The maximum of the absolute row sums of A.

Example 1:

(1) $A = \begin{bmatrix} 1 & i \\ -i+1 & -2 \end{bmatrix}$

$\|A\|_1 = \max\{ |1| + |-i+1|,\ |i| + |-2| \}$

$= \max\{ 1+\sqrt{2},\ 1+2 \}$

$= \max\{ 1+\sqrt{2},\ 3 \}$

$= 3$

$\|A\|_\infty = \max\{ |1| + |i|,\ |-i+1| + |-2| \}$

$= \max\{ 2,\ 2+\sqrt{2} \}$

$= 2+\sqrt{2}.$

Th

Theorem:- Let $A \in M_{n\times n}(\mathbb{C})$. Then the induced 2-norm of A is given by

$$\|A\|_2 = \max_{\substack{x \neq 0 \\ \text{in } \mathbb{C}^n}} \left(\frac{\|Ax\|_2}{\|x\|_2} \right)$$

$$= \sqrt{\lambda_{max}}.$$

where λ_{max} is the largest eigenvalue of A^*A.

Example:- $A = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$

$$A^*A = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

The eigenvalues of A^*A

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = 0 \implies 2 - 2\lambda - \lambda + \lambda^2 - 1 = 0.$$

$$\implies \lambda^2 - 3\lambda + 1 = 0.$$

$$\lambda = \frac{3 \pm \sqrt{9-4}}{2}$$

$$= \frac{3 \pm \sqrt{5}}{2}.$$

$$\lambda_{max} = \frac{3+\sqrt{5}}{2}.$$

$$\therefore \|A\|_2 = \sqrt{\frac{3+\sqrt{5}}{2}}.$$

Theorem :— Suppose A is an invertible $n \times n$ matrix. Then

$$\|A^{-1}\|_2 = \frac{1}{\sqrt{\lambda_{min}}}.$$

where λ_{min} is the smallest eigenvalue of $A^* A$.

For the above example,

$$\|A^{-1}\|_2 = \frac{1}{\sqrt{\frac{3-\sqrt{5}}{2}}} = \sqrt{\frac{2}{3-\sqrt{5}}}.$$

Theorem (Rayleigh quotient) :—

Let H be a Hermitian matrix of

size $n \times n$ & $\underline{x} \neq \underline{0}$ in $\mathbb{C}^n$. Then the quotient $R(\underline{x}) := \dfrac{\langle H\underline{x}, \underline{x} \rangle}{\langle \underline{x}, \underline{x} \rangle}$ $\forall\, \underline{x} \neq \underline{0}$ in $\mathbb{C}^n$ — Standard inner product.

(known as Rayleigh quotient)

Then
$$\max_{\underline{x} \neq \underline{0}} R(\underline{x}) = \max_j \lambda_j(H)$$

$$\& \quad \min_{\underline{x} \neq \underline{0}} R(\underline{x}) = \min_j \lambda_j(H)$$

where $\lambda_j(H)$ is the eigenvalue of H.