Multivariate Analysis

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Regression Analysis
Multivariate Analysis

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Expectation

Definition

Let $\mathbf{X} = (X_1, X_2, \cdots, X_n)^T$ be a random vector with finite expectation for each of the component the we define expectation of a random vector as

$$E(\mathbf{X}) = (E(X_1), E(X_2), \cdots, E(X_n))^T.$$

Similarly if $\mathbf{Y} = ((Y_{ij}))_{m\times n}$ is a random matrix with finite expectation for each of the component the we define expectation of a random matrix as $E(\mathbf{Y}) = ((E(Y_{ij})))_{m\times n}$.

Dispersion

Definition

The dispersion matrix or the variance-covariance matrix is

$$D(\mathbf{X}) = ((Cov(X_i, X_j)))_{n\times n} = E[(\mathbf{X} - E(\mathbf{X}))(\mathbf{X} - E(\mathbf{X}))^T] = Cov(\mathbf{X}, \mathbf{X})$$

NOTE:
(1) $Cov(\mathbf{U}_p, \mathbf{V}_q) = ((Cov(U_i, V_j)))_{p\times q}$
(2) $E(\mathbf{X} + \mathbf{b}) = E(\mathbf{X}) + \mathbf{b}$
(3) $D(\mathbf{X} + \mathbf{b}) = D(\mathbf{X})$
(4)$Cov(\mathbf{X} + \mathbf{b}, \mathbf{Y} + \mathbf{c}) = Cov(\mathbf{X}, \mathbf{Y})$

Results

Let $\mathbf{X}$ be a random vector with n-components such that $E(\mathbf{X}) = \mu$ and $D(\mathbf{X}) = \Sigma$ then

1. $E(l^T\mathbf{X}) = l^T\mu$, where $l \in \mathbb{R}^n$ is a constant vector
2. $D(l^T\mathbf{X}) = l^T\Sigma l$
3. $E(\mathbf{AX}) = \mathbf{A}\mu$, where $\mathbf{A} \in \mathbb{R}^{p\times n}$ is a constant matrix
4. $D(\mathbf{AX}) = \mathbf{A}\Sigma\mathbf{A}^T$ and $Cov(\mathbf{AX}, \mathbf{BX}) = \mathbf{A}\Sigma\mathbf{B}^T$
5. If $Cov(\mathbf{U}_p, \mathbf{V}_q) = \Gamma$ then $Cov(\mathbf{AU}, \mathbf{BV}) = \mathbf{A}\Gamma\mathbf{B}^T$

Exercise

Let $\mathbf{X}$ be a random vector with n-components such that $E(\mathbf{X}) = \mu$ and $D(\mathbf{X}) = \Sigma$ then

Note1 [3]. It will imply [1].

Note2 [5]. It will imply [2] and [4].

Note3 $D(\mathbf{X})$ is a p.s.d. matrix.

Theorems

Theorem 1

Let $\mathbf{X}$ be a random vector with n-components such that $E(\mathbf{X}) = \mu$ and $D(\mathbf{X}) = \Sigma$. Show that $E(\mathbf{X}^T A\mathbf{X}) = trace(\Sigma A) + \mu^T A\mu$

Theorems

Theorem 2

Let $\mathbf{X}$ be a random vector with n-components such that $E(\mathbf{X}) = \mu$ and $D(\mathbf{X}) = \Sigma$ then $P((\mathbf{X} - \mu) \in \mathcal{C}(\Sigma)) = 1$.

Theorems

Theorem 3

Let $\mathbf{X}$ be a random vector with n-components such that $E(\mathbf{X}) = \mu$ and $D(\mathbf{X}) = \Sigma$ with $Rank(\Sigma) = r \leq n$. Also assume that $\Sigma = BB^T$ where B is a $(n \times r)$ matrix and C is a left inverse of B i.e. $CB = \mathbf{I}_r$. Define $\mathbf{Y} = C(\mathbf{X} - \mu)$. Show that

(i) $E(\mathbf{Y}) = \mathbf{0}$

(ii) $D(\mathbf{Y}) = \mathbf{I}_r$

(iii) $\mathbf{X} = \mu + B\mathbf{Y}$ with probability 1.

Definition

Multivariate Normal

A random vector $\mathbf{X}$ is said to follow multivariate normal $N(\boldsymbol{\mu}, \Sigma)$ if it has a density

$$f(\mathbf{x}) = \frac{\exp\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\}}{(\sqrt{2\pi})^n \sqrt{|\Sigma|}}$$

for some $\boldsymbol{\mu} \in \mathbf{R}^n$ and p.s.d. Σ

1. If $\mathbf{X} \sim N(\boldsymbol{\mu}, \Sigma)$ then $A\mathbf{X} \sim N(A\boldsymbol{\mu}, A\Sigma A^T)$
2. If $\mathbf{X} \sim N(\boldsymbol{\mu}, \Sigma)$ then there exists B and its left inverse C such that $\mathbf{Y} = C(\mathbf{X}-\boldsymbol{\mu}) \sim N(\mathbf{0}, \mathbf{I}_r)$ and $\mathbf{X} = \boldsymbol{\mu} + B\mathbf{Y}$ with probability one.

Defination

χ^2-distribution

If $\mathbf{X} \sim N(\boldsymbol{\mu}, \mathbf{I}_n)$ then $\mathbf{X}^T\mathbf{X}$ is said to follow Chi-squared distribution with degrees of freedom (d.f.) n and non-centrality parameter (n.c.p) $\boldsymbol{\mu}^T\boldsymbol{\mu}$.

Note: If $\mathbf{X} \sim N(\mu, \mathbf{I}_n)$, show that $E(\mathbf{X}^T\mathbf{X}) = n + \boldsymbol{\mu}^T\boldsymbol{\mu}$

Theorem

χ^2-distribution

If $\mathbf{X} \sim N(\boldsymbol{\mu}, \mathbf{I}_n)$ then $\mathbf{X}^T A \mathbf{X}$ has Chi-squared distribution iff A is idempotent. Moreover $\mathbf{X}^T A \mathbf{X} \sim \chi^2_{df=Rank(A), ncp=\boldsymbol{\mu}^T A \boldsymbol{\mu}}$

Corollary : If A_1 and A_2 are symmetric and idempotent matrices such that $Q = A_1 - A_2$ be a p.s.d. matrix then $\mathbf{X}^T Q \mathbf{X}$ and $\mathbf{X}^T A_2 \mathbf{X}$ are independently distributed.

Independence

Theorem

Let $\mathbf{X} \sim N(\boldsymbol{\mu}, \mathbf{I}_n)$ and A is symmetric and $CA = \mathbf{0}$ then $\mathbf{X}^T A\mathbf{X}$ and $C\mathbf{X}$ are independently distributed.

Theorem

Cochran's Theorem

Let $\mathbf{X} \sim N(\boldsymbol{\mu}, \mathbf{I}_n)$ and $\mathbf{X}^T A \mathbf{X} \equiv \sum_{i=1}^{k} \mathbf{X}^T A_i \mathbf{X}$ where A_is are symmetric and A is an idempotent matrix. Then $\mathbf{X}^T A_i \mathbf{X} \sim \chi^2_{Rank(A_i), \boldsymbol{\mu}^T A_i \boldsymbol{\mu}}$ and they are independent.

T-statistic

Let $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$. Define $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2$

1. Find the distribution of $\bar{X}$ and S^2.
2. Show that they are independently distributed
3. Construct t-statistic from it .
4. Construct F-statistic from it.

References

1 (APPENDIX C1) Introduction to Linear Regression Analysis, 5th Edition By Douglas C. Montgomery, Elizabeth A. Peck, G. Geoffrey Vining

2 . [CHAPTER 3] Linear Models : An Integrated Approach By Debasis Sengupta, Sreenivasa Rao Jammalamadaka