Simple linear regression.

$$D = \left\{ (x_i, y_i) \;\middle|\; \begin{matrix} x_i\text{'s are non-stochastic} \\ y_i\text{'s are stochastic} \end{matrix} \right\}$$

$$\epsilon_i \overset{iid}{\sim} N(0, \sigma^2).$$

Model : $y_i = \beta_0 + \beta_1 x_i + \epsilon_i \quad \forall i = 1, 2 \cdots n.$

Gauss-Markov Model.

Unknown parameters are $\beta_0, \beta_1, \sigma^2$.

$$E(y_i) = \beta_0 + \beta_1 x_i$$ as $E(\epsilon_i) = 0$ and x_i's are non stochastic.

Hence.

$$\underset{\sim}{Y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad \underset{\sim}{\epsilon} = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{pmatrix} \quad \underset{\sim}{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \quad X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = (\underset{\sim}{1} \; \underset{\sim}{x})$$

$$\underset{\sim}{Y} = X\underset{\sim}{\beta} + \underset{\sim}{\epsilon}. \text{ where } \underset{\sim}{\epsilon} \sim N(\underset{\sim}{0}, I_n\sigma^2).$$

$$\Rightarrow \underset{\sim}{Y} = \underset{\sim}{X}\beta + \underset{\sim}{\epsilon} \sim N(X\underset{\sim}{\beta}, I_n\sigma^2)$$

- Y_i's are independent but not identically distributed.

→ Methods of estimation for $\beta_0, \beta_1, \sigma^2$

- Least square method.
- Maximum likelihood method.

LS condition : To minimize.

$$S(\beta_0, \beta_1) = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$$

$$= \| \underset{\sim}{Y} - X\underset{\sim}{\beta} \|_2^2 = (\underset{\sim}{Y} - X\underset{\sim}{\beta})^T (\underset{\sim}{Y} - X\underset{\sim}{\beta})$$

$$\hat{\beta} = (\hat{\beta}_0, \hat{\beta}_1) = \underset{(\beta_0, \beta_1)}{\arg\min}\; S(\beta_0, \beta_1)$$

Solve $\left.\frac{\partial S}{\partial \beta_0}\right|_{\hat\beta_0} = 0$ and $\left.\frac{\partial S}{\partial \beta_1}\right|_{\hat\beta_1} = 0$.

$$\hat\beta_1 = \frac{\sum_{i=1}^{n} y_i x_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n}(x_i-\bar{x})^2}.$$

$$\Rightarrow \boxed{\hat\beta_1 = \frac{S_{xy}}{S_{xx}} \text{ and } \hat\beta_0 = \bar{y} - \hat\beta_1 \bar{x}}$$

HW1:

① $\frac{\partial \underset{\sim}{b}^T A}{\partial \underset{\sim}{b}} = A^T$.

② $\frac{\partial \underset{\sim}{b}^T A \underset{\sim}{b}}{\partial \underset{\sim}{b}} = b^T(A+A^T)$.

$= X^TY = X^TX\hat{\underset{\sim}{\beta}}$

Normal eqration.

Show that $\hat{\underset{\sim}{\beta}} = \begin{pmatrix}\hat\beta_0 \\ \hat\beta_1\end{pmatrix} = (X^TX)^{-1}X^TY$

When all x_i's are same (X^TX) is not invertable.

Show that regression line is passing through $(\bar{x},\bar{y})$

⊛ Show that regression line is passing through $(\bar{x},\bar{y})$

Regression line is

$$\hat{y} = \hat\beta_0 + \hat\beta_1 x.$$
$$= \bar{y} - \hat\beta_1\bar{x} + \hat\beta_1 x.$$
$$= \bar{y} + \hat\beta_1(x-\bar{x})$$

If we replace $x=\bar{x}$ then $\hat{y}=\bar{y}$

$\Rightarrow$ Regression line is passing through $(\bar{x},\bar{y})$.

- We know that $\hat{\beta}_1 = \dfrac{S_{xy}}{S_{xx}}$

$$= \sum_{i=1}^{n} \frac{(x_i - \bar{x})}{S_{xx}} y_i$$

$$= \underset{\sim}{\alpha}^T \underset{\sim}{y} \qquad \text{where } \alpha_i = \frac{(x_i - \bar{x})}{S_{xx}}, \; i = 1, 2 \dots n.$$

S_{xy}

$= \sum (x_i - \bar{x})(y_i - \bar{y})$

$= \sum (x_i - \bar{x}) y_i$

$= \sum (y_i - \bar{y}) x_i$

$\Rightarrow \hat{\beta}_1$ is a linear estimator of β_1.

$\hat{\beta}_1 \sim N(\underset{\sim}{\alpha}^T X \underset{\sim}{\beta}, \underset{\sim}{\alpha}^T I_n \sigma^2 \underset{\sim}{\alpha})$ $\quad | \quad \underset{\sim}{Y} \sim N(X\underset{\sim}{\beta}, I_n \sigma^2)$

$$\sigma^2 \underset{\sim}{\alpha}^T I_n \underset{\sim}{\alpha} = \sigma^2 \underset{\sim}{\alpha}^T \underset{\sim}{\alpha} = \sigma^2 \sum_{i=1}^{n} \alpha_i^2 = \sigma^2 \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{S_{xx}^2}$$

$$= \sigma^2 \frac{S_{xx}}{S_{xx}^2} = \sigma^2 / S_{xx} = Var(\hat{\beta}_1)$$

$$E(\hat{\beta}_1) = \underset{\sim}{\alpha}^T X \underset{\sim}{\beta} = \underset{\sim}{\alpha}^T (\underset{\sim}{1} \; \underset{\sim}{x}) \underset{\sim}{\beta}$$

$$= (\underset{\sim}{\alpha}^T \underset{\sim}{1} \;\; \underset{\sim}{\alpha}^T \underset{\sim}{x}) \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

$$= (0 \quad 1) \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

$$= \beta_1.$$

$$0 = \underset{\sim}{\alpha}^T \underset{\sim}{1} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})}{S_{xx}}$$

$$\underset{\sim}{\alpha}^T \underset{\sim}{x} = \sum_{i=1}^{n} \frac{(x_i - \bar{x}) \cdot x_i}{S_{xx}} = \frac{S_{xx}}{S_{xx}} = 1.$$

$$\hat{\beta}_1 \sim N\left(\beta_1, \sigma^2 / S_{xx}\right)$$

$\Rightarrow \hat{\beta}_1$ is an unbiased estimator of β_1.

HW. 2. Express $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$ as a linear estimator of β_0. and show that.

$$\hat{\beta}_0 \sim N\left(\beta_0, \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)\right).$$

$$\begin{aligned}
\hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x} \\
&= \frac{1}{n} \underset{\sim}{1}^T \underset{\sim}{y} - \bar{x} \cdot \underset{\sim}{\alpha}^T \underset{\sim}{y} \\
&= \left(\frac{1}{n}\underset{\sim}{1} - \bar{x}\underset{\sim}{\alpha}\right)^T \underset{\sim}{y} \\
&\equiv \underset{\sim}{\theta}^T \underset{\sim}{y}
\end{aligned}$$

$$\theta_i = \left(\frac{1}{n} - \frac{\bar{x}(x_i - \bar{x})}{S_{xx}}\right)$$

$$\hat{\beta}_0 \sim N\left(\underset{\sim}{\theta}^T X \underset{\sim}{\beta},\ \underset{\sim}{\theta}^T I_n \sigma^2 \underset{\sim}{\theta}\right)$$

$$\left.\begin{aligned}
E(\hat{\beta}_0) &= \beta_0 \\
Var(\hat{\beta}_0) &= \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)
\end{aligned}\right\} \text{HW}$$

- Estimation of σ^2.

$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$. is the regression line.

$\hat{\underset{\sim}{y}} = \hat{\beta}_0 + \hat{\beta}_1 \underset{\sim}{x}$ $\quad \underset{\sim}{x}$ is known to us.

$\Rightarrow \hat{\underset{\sim}{y}} = X \hat{\underset{\sim}{\beta}}$. $\quad X = [\underset{\sim}{1} \; \underset{\sim}{x}]$

$\Rightarrow \hat{y} = X(X^T X)^{-1} X^T \underset{\sim}{y}$. $\quad \hat{\underset{\sim}{\beta}} = (X^T X)^{-1} X^T \underset{\sim}{y}$.

Let us denote $P_X = X(X^T X)^{-1} X^T$

- P_X is an orthogonal projection matrix of $\mathcal{C}(X)$.
- $P_X^T = P_X$. symmetric
- $P_X^2 = P_X$. idempotent.

$\hat{\underset{\sim}{y}}$ is the orthogonal projection of $\underset{\sim}{y}$ in $\mathcal{C}(X)$. and $\underset{\sim}{e}$ is orthogonal to $\hat{\underset{\sim}{y}}$.

As $\hat{\underset{\sim}{y}} = P_X \underset{\sim}{y}$. $\longrightarrow$ prediction vector.

define. $\underset{\sim}{e} = \underset{\sim}{y} - \hat{\underset{\sim}{y}} = (I - P_X)\underset{\sim}{y}$ $\rightarrow$ estimated error vector.

- ~~$\mathcal{C}(\ldots)$~~
- $(I_n - P_X)^T = (I_n - P_X)$ symmetric
- $(I - P_X)^2 = (I - P_X)$ idempotent.

$\underset{\sim}{e} = (I - P_X)\underset{\sim}{y} \sim N(0, (I - P_X)\sigma^2)$.

$\hat{\underset{\sim}{y}} = P_X \underset{\sim}{y} \sim N(X\beta, P_X \sigma^2)$

$\underset{\sim}{y} \sim N(X\underset{\sim}{\beta}, \sigma^2 I_n)$

- $\text{cov}(\hat{\underset{\sim}{y}}, \underset{\sim}{e})$.

$= \text{cov}(P_X \underset{\sim}{y}, (I - P_X)\underset{\sim}{y})$.

$$\boxed{\text{cov}(A\underset{\sim}{y}, B\underset{\sim}{y}) = A\Sigma . B^T}$$

$= \sigma^2 P_X I_n (I_n - P_X) = \sigma^2 \mathbf{0} = \mathbf{0}$ matrix.

Also. $\hat{y}^T e = 0$ scalar..

$SSR =$ Residual Sum of square $= \sum_{i=1}^{n} (y_i - \hat{y})^2$

$$E(SSR) = E(e^T e) = E\left(\sum_{i=1}^{n}(y_i - \hat{y})^2\right)$$

$$= E\left(\underset{\sim}{y}^T (I - P_x) \underset{\sim}{y}\right) = (n-2)\sigma^2.$$

$$\Rightarrow E\left[\frac{\sum_{i=1}^{n}(y_i - \hat{y})^2}{n-2}\right] = \sigma^2 .$$

$$\hat{\sigma}^2 = \frac{SSR}{n-2} \qquad \underline{\text{Unbiased estimator.}}$$

HW 3 (a) Find the maximum liklihood estimator of $\hat{\beta}_0, \hat{\beta}_1, \sigma^2$.

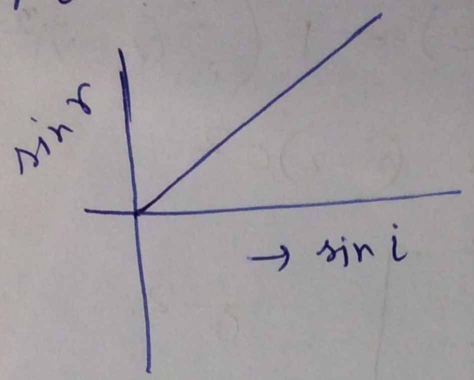
(b) Are they unbiased estimator of their corresponding parameter?

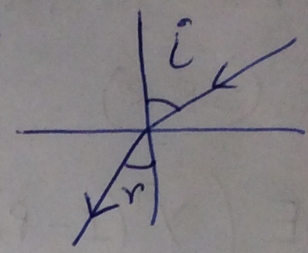
$$\hat{\beta}_{0\,mle} = \hat{\beta}_{0\,LS}$$

$$\hat{\beta}_{1\,mle} = \hat{\beta}_{1\,LS}.$$

$$\hat{\sigma}^2_{mle} = \frac{SSR}{n} \neq \frac{SSR}{n-2} = \hat{\sigma}^2_{LS}.$$

1. $H_0: \beta_0 = 0$ vs $H_1 = \beta_0 \neq 0.$





$$\frac{\sin i}{\sin r} = C.$$

$\hat{\beta}_0 \sim$

σ^2 has to be estimated.

- Find test statistic and the distribution of test statis under Ho.
- Conclude with the test rule.

Hypothesis testing for SLR.

1. $H_0 : \beta_0 = b_0$ vs $H_1: \beta_0 \neq b_0.$

Level of test $\alpha = 0{\cdot}05$. or $0{\cdot}01$. or $0{\cdot}1$.

we know: $$\hat{\beta}_0 \sim N\left(\beta_0, \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)\right)$$

$$\Rightarrow \frac{\hat{\beta}_0 - \beta_0}{\sqrt{\sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)}} \sim N(0,1)$$

Under Ho $$\frac{\hat{\beta}_0 - b_0}{\sqrt{\sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)}} \sim N(0,1).$$

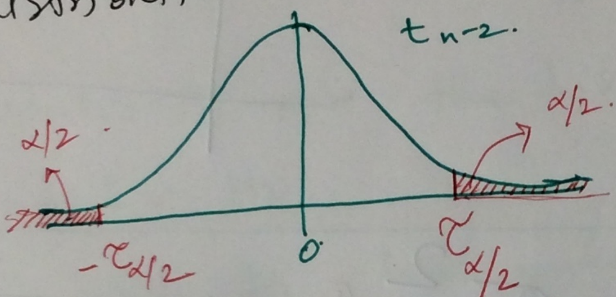
As we don't know σ^2 we use its estimated value $\hat{\sigma}^2 = \frac{1}{(n-2)}\left(S_{yy} - \frac{S_{xy}^2}{S_{xx}}\right)$

Under Ho

$$T_0 = \frac{\hat{\beta}_0 - b_0}{\sqrt{\hat{\sigma}^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)}} \sim t_{n-2}$$

This the distribution of test statistic under Ho.

level α.



Test rule

If $T_0(\text{obsered}) > \tau_{\alpha/2}$

or. $T_0(\text{obsued}) < -\tau_{\alpha/2}$.

$\Leftrightarrow |T_0(\text{obs})| > \tau_{\alpha/2}$.

condition

then reject Ho in favour of H_1 at level α.

What is $100(1-\alpha)\%$ CI of β_0?

$$\frac{\hat{\beta}_0 - \beta_0}{\sqrt{\hat{\sigma}^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)}} \sim t_{n-2}.$$

$$\Rightarrow P\left(-\tau_{\alpha/2} < \frac{\hat{\beta}_0 - \beta_0}{\sqrt{\hat{\sigma}^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)}} < \tau_{\alpha/2}\right) = 1-\alpha.$$

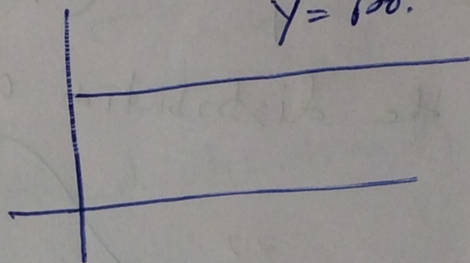
$$P\left(\hat{\beta}_0 - \tau_{\alpha/2}\sqrt{\hat{\sigma}^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)} < \beta_0 < \hat{\beta}_0 + \tau_{\alpha/2}\sqrt{\hat{\sigma}^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)}\right) = 1-\alpha.$$

Hw (a) $H_0: \beta_1 = b_1$ vs $\beta_1 > b_1$

Perform a test at level $\alpha = 0.05$.

(b) Find $100(1-\alpha)\%$ CI of β_1.

If $b_1 = 0$



$H_0: \sigma^2 = \sigma_0^2 = f(x,y)$ vs $\sigma^2 > \sigma_0^2$.

$$\hat{\sigma}^2 = \frac{SSR}{n-2}$$

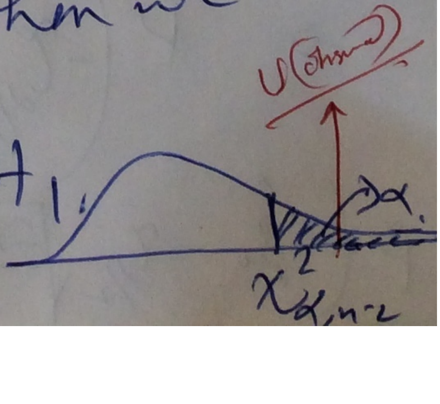
$$\Rightarrow SSR = \cancel{\hat{\sigma}^2(n-2)} \sim \sigma^2 \chi^2_{n-2}$$

$$\Rightarrow \frac{SSR}{\sigma^2} \sim \chi^2_{n-2}.$$

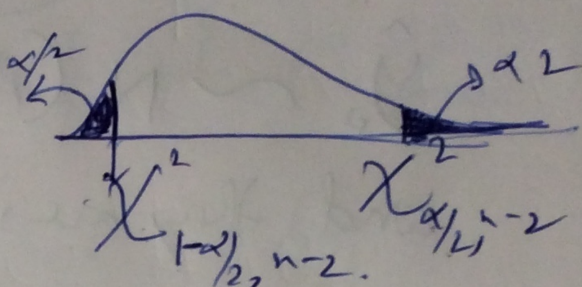
Under Ho $U = \dfrac{SSR}{\sigma_0^2} \sim \chi^2_{n-2}$.

For large value of U we will reject Ho.

If $\dfrac{SSR}{\sigma_0^2} > \chi^2_{\alpha, n-2}$ then we reject Ho in favor of H_1.



$$\frac{SSR}{\sigma^2} \sim \chi^2_{n-2}.$$



$$\Rightarrow P\left(\chi^2_{1-\alpha/2,\, n-2} < \frac{SSR}{\sigma^2} < \chi^2_{\alpha/2,\, n-2}\right) = 1-\alpha.$$

$$= P\left(\frac{SSR}{\chi^2_{\alpha/2,\, n-2}} < \sigma^2 < \frac{SSR}{\chi^2_{1-\alpha/2,\, n-2}}\right) = 1-\alpha.$$

Prediction.

Let (y_0, x_0) also satisfies the regression model.

$$y_0 = \beta_0 + \beta_1 x_0 + \epsilon_0 \qquad \epsilon_0 \sim N(0, \sigma^2)$$

We know x_0 we want to predict y_0.

predicted value $\boxed{\hat{y}_0 = \hat{\beta}_0 + \hat{\beta}_1 x_0}$ then the distribution of

$$= \bar{y} + \hat{\beta}_1 (x_0 - \bar{x})$$

$$\hat{y}_0 \sim N\left(\beta_0 + \beta_1 x_0,\ \sigma^2\left(\frac{1}{n} + \frac{(x_0-\bar{x})^2}{S_{xx}}\right)\right)$$

as. $\mathrm{COV}(\bar{y}, \hat{\beta}) = 0$. Home work.

$y_0 \sim N\left(\beta_0 + \beta_1 x_0, \sigma^2\right)$ by assumption.

$\hat{y}_0 \sim N\left(\beta_0 + \beta_1 x_0, \sigma^2\left(\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{xx}}\right)\right)$

and they are independent.

$$y_0 - \hat{y}_0 \sim N\left(0, \underbrace{\sigma^2\left(1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{xx}}\right)}_{= \sigma_{y_0}^2}\right)$$

$$\frac{y_0 - \hat{y}_0}{\sqrt{\sigma_{y_0}^2}} \sim N(0,1)$$

If we use $\hat{\sigma}^2$ inplace σ^2.

$$\frac{y_0 - \hat{y}_0}{\sqrt{\hat{\sigma}_{y_0}^2}} \sim t_{n-2}.$$

$$P\left(\hat{y}_0 - \tau_{\alpha/2, n-2}\sqrt{\hat{\sigma}_{y_0}^2} < y_0 < \hat{y}_0 + \tau_{\alpha/2}\sqrt{\hat{\sigma}_{y_0}^2}\right) = 1-\alpha.$$

$100(1-\alpha)\%$ Prediction interval.