MATRIX REPRESENTATION OF A LINEAR MAP.

Let $T: V \to W$ be a linear transformation from an n-dimensional vector space to an m-dimensional vector space W.

Let $\alpha = (v_1, v_2, \dots, v_n)$ ordered basis of V

$\beta = (w_1, w_2, \dots, w_m)$ ordered basis of W.

Consider,

$$\underbrace{T(v_1)}_{\in W} = a_{11}w_1 + a_{21}w_2 + \dots + a_{m1}w_m$$

$$T(v_2) = a_{12}w_1 + a_{22}w_2 + \dots + a_{m2}w_m$$

$$\vdots$$

$$T(v_n) = a_{1n}w_1 + a_{2n}w_2 + \dots + a_{mn}w_m$$

In short,

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i \quad , \quad j = 1, 2, \dots n.$$

for some scalars, a_{ij} $(i = 1, 2 \dots m,\ j = 1, 2, \dots, n)$.

For any vector $x \in V$, let

$$x = \sum_{j=1}^{n} x_j v_j$$

Consider

$$T(x) = \sum_{j=1}^{n} x_j \, T(v_j)$$

$$= \sum_{j=1}^{n} x_j \sum_{i=1}^{m} a_{ij} \, w_i$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) w_i$$

$$= \left(\sum_{j=1}^{n} a_{1j} x_j \right) w_1 + \left(\sum_{j=1}^{n} a_{2j} x_j \right) w_2 + \cdots + \left(\sum_{j=1}^{n} a_{mj} x_j \right) w_m$$

Note that

$$[T(x)]_\beta = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \sum_{j=1}^{n} a_{2j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} x_j \end{bmatrix}$$

coordinate vector of $T(x)$ w.r.t. β.

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= A \, [x]_\alpha$$

Hence,

$$[T(x)]_\beta = A[x]_\alpha$$

That is, for any $x \in V$, the coordinate vector $[T(x)]_\beta$ of $T(x)$ in W is just the product of a fixed matrix A and the coordinate vector $[x]_\alpha$ of x.

Note that

$$A = \begin{bmatrix} a_{11}, a_{12}, \cdots, a_{1n} \\ a_{21}, a_{22} \cdots a_{2n} \\ \vdots \\ a_{m1}\ a_{m2} \cdots a_{mn} \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ [T(v_1)]_\beta & [T(v_2)]_\beta & \cdots & [T(v_n)]_\beta \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

Columns are coordinate vectors of $T(v_j)$.

Def: The matrix A is called the associated matrix for T (or the matrix representation of T) with respect to the ordered bases α and β and is denoted by $A = [T]_\alpha^\beta$.

When $V = W$ and $\alpha = \beta$ we simply write $[T]_\alpha$.

$$\boxed{[T(x)]_\beta = [T]_\alpha^\beta [x]_\alpha}$$

Theorem: Let $T: V \to W$ be linear transformation from n-dimensional vector space V to an m-dimensional vector space W. For fixed ordered bases $\alpha = \{v_1, v_2, \ldots, v_n\}$ for V and $\beta = \{w_1, w_2, \ldots w_m\}$ for W, there corresponds to a unique associated $m \times n$ matrix $[T]_\alpha^\beta$ for T such that for any vector $x \in V$ the coordinate vector $[T(x)]_\beta$ of $T(x)$ w.r.t. β is given as a matrix product of the associated matrix $[T]_\alpha^\beta$ for T and the coordinate vector $[x]_\alpha$ i.e., $[T(x)]_\beta = [T]_\alpha^\beta [x]_\alpha$.

Example: Let V be the vector space of functions with basis $S = \{\sin t, \cos t, e^{3t}\}$ and let $D: V \to V$ be the differential operator defined by

$$D(f(t)) = \frac{d}{dt}(f(t)).$$

Find $[D]_S$.

<u>Solution</u>:

$$D(\sin t) = \cos t = 0 \cdot (\sin t) + 1 \cdot (\cos t) + 0 \cdot (e^{3t})$$

$$D(\cos t) = -\sin t = -1 (\sin t) + 0 \cdot (\cos t) + 0 \cdot (e^{3t})$$

$$D(e^{3t}) = 3e^{3t} = 0 \cdot (\sin t) + 0 (\cos t) + 3 (e^{3t})$$

Hence
$$[D]_S = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Check: Find $D(\sin t + 2\cos t + e^{3t})$ with the help of $[D]_S$.

Coordinates of $(\sin t + 2\cos t + e^{3t})$ are $[1, 2, 1]^T$

hence,
$$[D]_S \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = [D(\sin t + 2\cos t + e^{3t})]_S$$

$$\Rightarrow D(\sin t + 2\cos t + e^{3t}) = -2 \cdot \sin t + 1 \cdot \cos t + 3e^{3t}$$
$$= -2\sin t + \cos t + 3e^{3t}.$$

<u>Verified!</u>

Ex 1: Let $F\colon \mathbf{R}^2 \to \mathbf{R}^2$ be the linear operator defined by $F(x, y) = (2x + 3y,\ 4x - 5y)$.

(*a*) Find the matrix representation of F relative to the basis $S = \{u_1, u_2\} = \{(1, 2),\ (2, 5)\}$.

(1) First find $F(u_1)$, and then write it as a linear combination of the basis vectors u_1 and u_2. (For notational convenience, we use column vectors.) We have

$$F(u_1) = F\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}8\\-6\end{bmatrix} = x\begin{bmatrix}1\\2\end{bmatrix} + y\begin{bmatrix}2\\5\end{bmatrix} \quad \text{and} \quad \begin{aligned} x + 2y &= 8\\ 2x + 5y &= -6\end{aligned}$$

Solve the system to obtain $x = 52$, $y = -22$. Hence $F(u_1) = 52u_1 - 22u_2$.

(2) Next find $F(u_2)$, and then write it as a linear combination of u_1 and u_2:

$$F(u_2) = F\left(\begin{bmatrix}2\\5\end{bmatrix}\right) = \begin{bmatrix}19\\-17\end{bmatrix} = x\begin{bmatrix}1\\2\end{bmatrix} + y\begin{bmatrix}2\\5\end{bmatrix} \quad \text{and} \quad \begin{aligned} x + 2y &= 19\\ 2x + 5y &= -17\end{aligned}$$

Solve the system to get $x = 129$, $y = -55$. Thus $F(u_2) = 129u_1 - 55u_2$.

Now write the coordinates of $F(u_1)$ and $F(u_2)$ as columns to obtain the matrix

$$[F]_S = \begin{bmatrix}52 & 129\\-22 & -55\end{bmatrix}$$

(*b*) Find the matrix representation of F relative to the (usual) basis $E = \{e_1, e_2\} = \{(1, 0),\ (0, 1)\}$.

Find $F(e_1)$ and write it as a linear combination of the usual basis vectors e_1 and e_2, and then find $F(e_2)$ and write it as a linear combination of e_1 and e_2. We have

$$\begin{aligned} F(e_1) &= F(1, 0) = (2, 2) &&= 2e_1 + 4e_2\\ F(e_2) &= F(0, 1) = (3, -5) &&= 3e_1 - 5e_2\end{aligned} \quad \text{and so} \quad [F]_E = \begin{bmatrix}2 & 3\\4 & -5\end{bmatrix}$$

Note that the coordinates of $F(e_1)$ and $F(e_2)$ form the columns, not the rows, of $[F]_E$. Also, note that the arithmetic is much simpler using the usual basis of $\mathbf{R}^2$.

Ex2

Consider the following matrix A, which may be viewed as a linear operator on $\mathbf{R}^2$, and basis S of $\mathbf{R}^2$:

$$A = \begin{bmatrix}3 & -2\\4 & -5\end{bmatrix} \quad \text{and} \quad S = \{u_1, u_2\} = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\5\end{bmatrix}\right\}$$

(1) First we write $A(u_1)$ as a linear combination of u_1 and u_2. We have

$$A(u_1) = \begin{bmatrix}3 & -2\\4 & -5\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}-1\\-6\end{bmatrix} = x\begin{bmatrix}1\\2\end{bmatrix} + y\begin{bmatrix}2\\5\end{bmatrix} \quad \text{and so} \quad \begin{aligned} x + 2y &= -1\\ 2x + 5y &= -6\end{aligned}$$

Solving the system yields $x = 7$, $y = -4$. Thus $A(u_1) = 7u_1 - 4u_2$.

(2) Next we write $A(u_2)$ as a linear combination of u_1 and u_2. We have

$$A(u_2) = \begin{bmatrix}3 & -2\\4 & -5\end{bmatrix}\begin{bmatrix}2\\5\end{bmatrix} = \begin{bmatrix}-4\\-7\end{bmatrix} = x\begin{bmatrix}1\\2\end{bmatrix} + y\begin{bmatrix}2\\5\end{bmatrix} \quad \text{and so} \quad \begin{aligned} x + 2y &= -4\\ 2x + 5y &= -7\end{aligned}$$

Solving the system yields $x = -6$, $y = 1$. Thus $A(u_2) = -6u_1 + u_2$. Writing the coordinates of $A(u_1)$ and $A(u_2)$ as columns gives us the following matrix representation of A:

$$[A]_S = \begin{bmatrix}7 & -6\\-4 & 1\end{bmatrix}$$

Remark: Suppose we want to find the matrix representation of A relative to the usual basis $E = \{e_1, e_2\} = \{[1, 0]^T, [0, 1]^T\}$ of $\mathbf{R}^2$. We have

$$A(e_1) = \begin{bmatrix} 3 & -2 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3e_1 + 4e_2$$

$$A(e_2) = \begin{bmatrix} 3 & -2 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \end{bmatrix} = -2e_1 - 5e_2$$

and so $\quad [A]_E = \begin{bmatrix} 3 & -2 \\ 4 & -5 \end{bmatrix}$

Note that $[A]_E$ is the original matrix A. This result is true in general:

> The matrix representation of any $n \times n$ square matrix A over a field K relative to the usual basis E of K^n is the matrix A itself; that is,
> $$[A]_E = A$$

Ex 3: Let $T: \mathbb{R}^3 - \mathbb{R}^2$ be a linear transformation defined by $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y+z \\ y-z \end{bmatrix}$. Determine the matrix of the linear transformation T with respect to the ordered bases:

$$X = \left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\} \text{ in } \mathbb{R}^3 \quad \text{and} \quad Y = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} \text{ in } \mathbb{R}^2$$

Consider

$$T\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 1 \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \times \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$$T\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0 \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \times \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$T\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \times \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

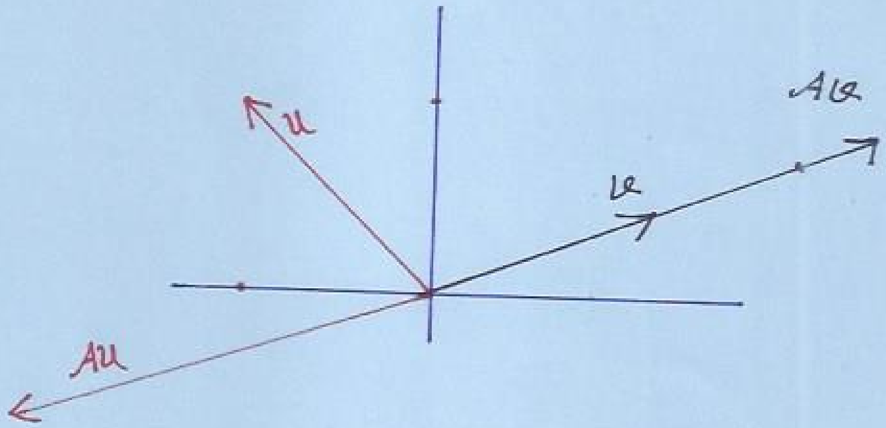
Therefore, the matrix of linear transformation with respect to the given basis is

$$[T]_X^Y = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

EIGENVALUES AND EIGENVECTORS

Consider $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $u = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$Au = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$$



$$Av = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \boxed{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

<u>Def.</u> Let A be any square matrix (real or complex). A scalar λ is called an eigenvalue of A if there exists a non-zero vector x such that

$$Ax = \lambda x$$

The vector x is an eigenvector associated with the eigenvalue λ.

Geometrically, an eigenvector of a matrix A is a nonzero vector x in the $\mathbb{R}^n$ such that the vectors x and Ax are parallel.

Algebraically, an eigenvector x is a nontrivial solution of the homogeneous system $(A - \lambda I)x = 0$ of linear equations, i.e., an eigenvector x is a nonzero vector in the nullspace of $(A - \lambda I)$.

How to find eigen values and eigenvectors

Consider $(A-\lambda I)x = 0$

Two unknowns λ and x.

Note that $(A-\lambda I)x=0$ has a nontrivial solution x iff λ satisfies the equation

$$\det(A-\lambda I) = 0 \qquad \left(c_0\lambda^n + c_1\lambda^{n-1} + \cdots + c_n = 0\right)$$

The above equation is called the characteristic equation of A.

Eigenvalues (or characteristic roots) are the roots of the characteristic equation $\det(A-\lambda I)=0$.

Eigenvectors (or ~~eigen~~ characteristic vectors) of A can be determined by solving the homogeneous system of equations

$$(A-\lambda I)x = 0 \quad \text{for each eigenvalue } \lambda.$$

The nullspace $\mathrm{Null}(A-\lambda I)$ is called the eigenspace of A corresponding to eigenvalue λ.

CAYLEY - HAMILTON THEOREM:

Every square matrix satisfies its own characteristic equation, i.e.,

$$c_0A^n + c_1A^{n-1} + \cdots + c_nI_n = 0.$$

Example: Use Cayley - Hamilton theorem to find

A^{-1} when $A = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$

Characteristic equation of A is

$$\begin{vmatrix} 2-\lambda & 4 \\ 3 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad 10 - 7\lambda + \lambda^2 - 12 = 0$$

$$\Rightarrow \quad \lambda^2 - 7\lambda - 2 = 0$$

By Cayley - Hamilton theorem

$$A^2 - 7A - 2I_n = 0$$

$$\Rightarrow A(A-7I) = 2I_n$$

$$\Rightarrow A^{-1} = \frac{1}{2}(A-7I)$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} -5 & 4 \\ 3 & -2 \end{bmatrix}$$

Theorem: If x is characteristic vector of A corresponding to the characteristic value λ then kx is also a characteristic vector corresponding to the same characteristic value λ. Here k is any nonzero scalar.

$$Ax = \lambda x \Rightarrow k(Ax) = k(\lambda x)$$

$$\Rightarrow A(kx) = \lambda(kx)$$

$\Rightarrow$ kx is a characteristic vector corresponding to the characteristic value λ.

Theorem: If x is a characteristic vector of a matrix A then x cannot correspond to more than one characteristic value of A.

Proof: Let us assume

$$Ax = \lambda_1 x \quad \& \quad Ax = \lambda_2 x$$

$$\Rightarrow \lambda_1 x = \lambda_2 x$$

$$\Rightarrow (\lambda_1 - \lambda_2)x = 0$$

$$\Rightarrow \lambda_1 - \lambda_2 = 0 \text{ since } x \neq 0.$$

$$\Rightarrow \lambda_1 = \lambda_2$$

Theorem: Two eigenvectors of a square matrix A corresponding to two distinct eigenvalues of A are linearly independent.

Solution: Let x_1, x_2 be the ~~eigenvalues~~ of A corresponding to two distinct eigenvalues λ_1, λ_2, respectively.

Then, $Ax_1 = \lambda_1 x_1 \quad \& \quad Ax_2 = \lambda_2 x_2$

Consider $c_1 x_1 + c_2 x_2 = 0 \quad — ①, \quad c_1, c_2 \in \mathbb{R}$.

then, $c_1 A x_1 + c_2 A x_2 = 0$

$$\Rightarrow c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 = 0 \quad — ②$$

$$① \,\&\, ② \Rightarrow \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} c_1 x_1 \\ c_2 x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Clearly $\begin{vmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{vmatrix} = \lambda_2 - \lambda_1 \neq 0$

$$\Rightarrow c_1 x_1 = 0 \quad \& \quad c_2 x_2 = 0$$

Since $x_1 \neq 0 \;\&\; x_2 \neq 0 \Rightarrow c_1 = 0 = c_2$

Hence x_1 and x_2 are linearly independent.

Th. If $x_1, x_2, \ldots, x_r$ be r ~~eigenvalues~~ eigenvectors of an $n \times n$ matrix A corresponding to r distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$, respectively. Then $x_1, x_2, \ldots, x_r$ are linearly independent.

PROPERTIES OF EIGENVALUES AND EIGENVECTORS:

Let λ be an eigenvalue of A and x be its corresponding eigenvector. Then,

1) αA has eigenvalue $\alpha\lambda$ and corresponding eigenvector x.

$$Ax = \lambda x \Rightarrow (\alpha A)x = (\alpha\lambda)x.$$

2) A^m has eigenvalues λ^m and corresponding eigenvectors x for any positive integer m.

$$Ax = \lambda x \Rightarrow A \cdot Ax = A(\lambda x) \Rightarrow A^2 x = \lambda Ax = \lambda\lambda x = \lambda^2 x$$

$$\Rightarrow \lambda^2 \text{ is the eigenvalue of } A^2$$

3) $A - kI$ has eigenvalue $\lambda - k$ and corresponding vector x.

$$Ax = \lambda x \Rightarrow Ax - kIx = \lambda x - kx \Rightarrow$$

$$(A - kI)x = (\lambda - k)x$$

4) A^{-1} (if it exists) has eigenvalue $1/\lambda$ and corresponding eigenvector x.

$$Ax = \lambda x \Rightarrow A^{-1}Ax = A^{-1}\lambda x \Rightarrow A^{-1}x = \frac{1}{\lambda}x.$$

5) A and A^T has same eigenvalues.

6) For a real matrix A if $\alpha + i\beta$ is an eigenvalue then its conjugate $\alpha - i\beta$ is also an eigenvalue.

Theorem: The characteristic roots of a Hermitian matrix are real.

Proof: Recall: A is Hermitian $\Leftrightarrow A^* = A$

Let λ be a characteristic root of A and x its eigenvector.

Then,
$$Ax = \lambda x$$

Premultiplying both sides by x^*.

$$x^* A x = x^* \lambda x = \lambda x^* x \quad \text{--- (1)}$$

Taking conjugate transform both sides.

$$(x^* A x)^* = (\lambda x^* x)^*$$

$$\Rightarrow x^* A^* x = \bar{\lambda} x^* x \qquad (\text{as } (x^*)^* = x)$$

$$\Rightarrow x^* A x = \bar{\lambda} x^* x \quad \text{--- (2)}$$

From (1) & (2) $\Rightarrow \lambda x^* x = \bar{\lambda} x^* x$

$\Rightarrow (\lambda - \bar{\lambda}) x^* x = 0$

$\Rightarrow \lambda - \bar{\lambda} = 0$ since $x^* x \neq 0$.

$\Rightarrow \lambda = \bar{\lambda}$

$\Rightarrow \lambda$ is real.

In a similar way, we can prove the following:

1) The characteristic roots of a real symmetric matrix are all real.

2) The characteristic roots of a skew-Hermitian matrix are either purely imaginary or zero

3) The characteristic roots of a real skew-symmetric matrix are either purely imaginary or zero.

Theorem: The characteristic roots of a unitary matrix are of unit modulus.

Proof: Recall: unitary matrix $A^*A = I$

Consider

$$Ax = \lambda x \quad \text{—①}$$

$$\Rightarrow (Ax)^* = (\lambda x)^* \Rightarrow x^*A^* = \bar{\lambda} x^* \quad \text{—②}$$

$$① \& ② \Rightarrow (x^*A^*)(Ax) = (\bar{\lambda}x^*)(\lambda x)$$

$$\Rightarrow x^*(A^*A)x = \bar{\lambda}\lambda x^*x$$

$$\Rightarrow x^* I x = \bar{\lambda}\lambda x^*x$$

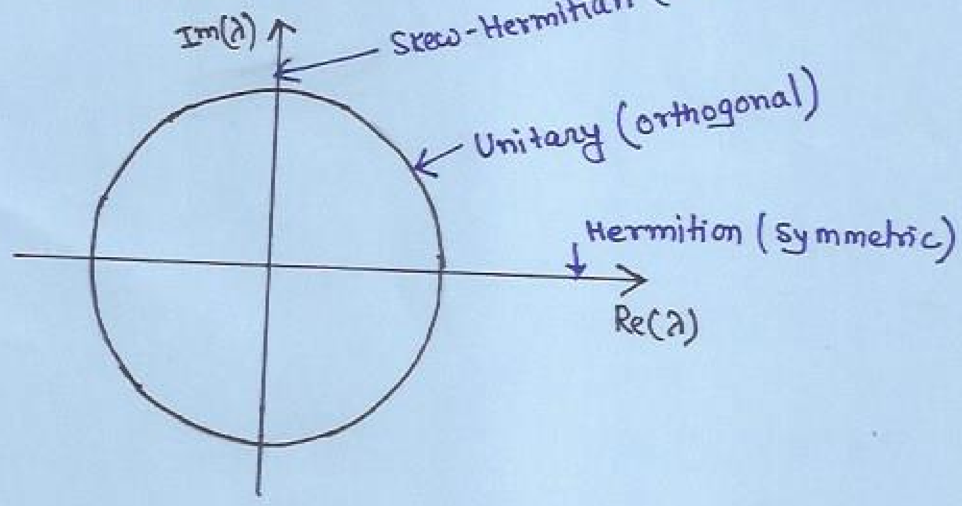
$$\Rightarrow x^*x(1-\bar{\lambda}\lambda) = 0$$

$$\Rightarrow 1-\bar{\lambda}\lambda = 0 \quad \text{as } x^*x \neq 0$$

$$\Rightarrow \bar{\lambda}\lambda = 1 \Rightarrow |\lambda|^2 = 1.$$

Corollary: The characteristic roots of an orthogonal matrix are of unit modulus.

LOCATION OF EIGENVALUES:



Examble

Find Eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$$

Sol: Characteristic polynomial is

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & \sqrt{2} \\ \sqrt{2} & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(1-\lambda) - 2 = 0$$

$$\Rightarrow 2 - 3\lambda + \lambda^2 - 2 = 0$$

$$\Rightarrow \lambda(\lambda - 3) = 0$$

Thus the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 3$.

$\lambda_1 = 0$: $(A - \lambda I)x = 0$

$$\Rightarrow \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & \sqrt{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Choose $x_2 = \alpha$

$$x_1 = -\frac{\sqrt{2}\alpha}{2} = -\frac{\alpha}{\sqrt{2}}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 1 \end{bmatrix}$$

$\lambda_2 = 3$: $$\begin{bmatrix} -1 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & \sqrt{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Choose $x_2 = \alpha$, $x_1 = \sqrt{2}\alpha$.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}.$$

Note that eigenvectors are linearly independent.

Ex. Find eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Characteristic polynomial: $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -2 & 0 \\ -2 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(3-\lambda)(5-\lambda) + 2\left(-2(5-\lambda)\right) = 0$$

$$\Rightarrow (\lambda - 1)(\lambda - 5)^2 = 0$$

Eigenvalues $\lambda_1 = 1, \quad \lambda_{2,3} = 5$

Eigenvectors: $\lambda_1 = 1$: $(A - \lambda I)x = 0$

$$\Rightarrow \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} \boxed{2} & -2 & 0 \\ 0 & 0 & \boxed{4} \\ 0 & 0 & 0 \end{bmatrix}$$

$x_2 = \alpha$
$x_3 = 0$
$x_1 = 2\alpha/2 = \alpha$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda = 5$: $\begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$x_2 = \alpha \quad x_3 = \beta \quad x_1 = -\alpha$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Two linearly indep. eigenvector correspond. to $\lambda = 5$: $[-1, 1, 0]^T$ & $[0, 0, 1]^T$.

Ex. Find a basis for the eigenspace of

$$A=\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Ch. equation:

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 1 & 0 & 0 & 0 \\ 0 & 2-\lambda & 1 & 0 & 0 \\ 0 & 0 & 2-\lambda & 1 & 0 \\ 0 & 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = 2, 2, 2, 2, 2.$$

eigenvector corresponding to $\lambda = 2$:

$$(A-\lambda I)x = 0$$

$$\begin{array}{c} x_1 \\ \begin{bmatrix} 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = \alpha, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 0, \quad x_5 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \text{Basis} = \{(1,0,0,0,0)^T\}.$$

Algebraic Multiplicity: multiplicity of λ as a root of the characteristic polynomial.

Geometric Multiplicity: dimension of the eigenspace of λ.
(number of linearly independent eigenvectors corresponding to an eigenvalue λ).

Theorem: The geometric multiplicity of an eigenvalue λ of a matrix A does not exceed its algebraic multiplicity.

$$\boxed{\text{Geometric multiplicity} \leq \text{Algebraic multiplicity}}$$

Ex. Find the characteristic roots and the corresponding characteristic vectors of the matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Characteristic equation:

$$|A - \lambda I| = 0$$

$$\Rightarrow (2-\lambda)(\lambda-2)(\lambda-8) = 0 \Rightarrow \lambda = 2, 2, 8.$$

Algebraic multiplicity of $\lambda = 2$ is 2.
" " of $\lambda = 8$ is 1.

Eigenvector corresponding to $\lambda = 8$:

$$(A-\lambda I)x = 0$$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} \boxed{-2} & -2 & 2 \\ 0 & \boxed{-3} & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

These equations possess $3-2=1$ linearly independent solution.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \alpha \neq 0, \ \alpha \in \mathbb{R}.$$

Geometric multiplicity of $\lambda = 8$ is 1.

Eigenvector corresponding to $\lambda = 2$:

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -2 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

These equations possess $3-1=2$ linearly indep. solutions.

$x_3 = \alpha_2$

$x_2 = \alpha_1$ $\quad x_1 = \frac{1}{2}(\alpha_1 - \alpha_2)$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$$

Geometric multiplicity of $\lambda = 2$ is 2.