ACHELON FORM: $A_{m\times n}\, x_{n\times 1} = b_{m\times 1}$

$$[A'|b'] = \left[\begin{array}{cccccccc|c} \blacksquare & * & * & * & \cdots & * & * & & * \\ 0 & 0 & \blacksquare & * & \cdots & * & * & & * \\ 0 & \cdots & 0 & \blacksquare & * & \cdots & * & & * \\ \vdots & & \vdots & 0 & \vdots & \vdots & \vdots & & \vdots \\ 0 & & 0 & 0 & \cdots & \blacksquare & * & * & * \\ 0 & \cdots & & & & \cdots & & 0 & \circledast \\ \vdots & & & & & & & & \\ 0 & \cdots & & & & \cdots & & 0 & \circledast \end{array}\right] \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \\ \end{array}\right\} \text{Pivot Rows } (r) \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{Zero Rows } (m-r) \end{array}$$

$\blacksquare$: Pivot Element $\neq 0$; $\circledast$ $*$ other elements (can be 0)

Example: Solve:

$$x_1 + x_2 + x_3 = 6$$
$$3x_1 + 3x_2 + 4x_3 = 20$$
$$2x_1 + x_2 + 3x_3 = 13$$

Augmented Matrix:

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & 3 & 4 & 20 \\ 2 & 1 & 3 & 13 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$; $R_3 \to R_3 - 2R_1$

$$[A|b] \sim \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 6 \\ 0 & \boxed{-1} & 1 & 1 \\ 0 & 0 & \boxed{1} & 2 \end{array}\right]$$

Back subst.

$$x_3 = 2$$
$$x_2 = 1$$
$$x_1 = 3$$

Note that $\left.\text{no. of pivots} = \text{no. of unknowns}\right\} \Rightarrow$ Unique Solution

CASE 1:

If $\circledast \neq 0$

NO SOLUTION

CASE 2:

If $\circledast = 0$ & $r = n$

UNIQUE SOLUTION

CASE 3:

If $\circledast = 0$ & $r < n$

IFINITELY MANY SOLUTIONS.

FREE VARIABLES $(n-r)$

Note that r cannot be greater than n.

Example:

$$4y + 3z = 8$$
$$2x - z = 2$$
$$3x + 2y = 5$$

$$\Leftrightarrow \begin{bmatrix} 0 & 4 & 3 \\ 2 & 0 & -1 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 5 \end{bmatrix}$$

$$[A|b] = \left[\begin{array}{ccc:c} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & 2 \\ 3 & 2 & 0 & 5 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\sim \left[\begin{array}{ccc:c} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 3 & 2 & 0 & 5 \end{array}\right]$$

$R_3 \to R_3 - \frac{3}{2} R_1$

$$\sim \left[\begin{array}{ccc:c} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 0 & 2 & \frac{3}{2} & 2 \end{array}\right]$$

$R_3 \to R_3 - \frac{R_2}{2}$

$$\sim \left[\begin{array}{ccc:c} \boxed{2} & 0 & -1 & 2 \\ 0 & \boxed{4} & 3 & 8 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

$0 \cdot x_3 = -2 \Rightarrow \quad 0 = -2$ (inconsistent)

This shows the system has NO solution.

Example:

Consider

$$\begin{bmatrix} 0 & 4 & 3 \\ 2 & 0 & -1 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 7 \end{bmatrix}$$

$$[A|b] = \left[\begin{array}{ccc:c} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & 2 \\ 3 & 2 & 0 & 7 \end{array}\right]$$

$$\sim \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & & \end{array} \\ \left[\begin{array}{ccc:c} \boxed{2} & 0 & -1 & 2 \\ 0 & \boxed{4} & 3 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_3 \to$ free variable

$\Rightarrow$ x_3 can be taken arbitrarily

let us take $x_3 = \alpha$ then: $x_2 = \frac{1}{4}(8 - 3\alpha)$

& $x_1 = \frac{1}{2}(2 + \alpha)$

One can also write in vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}}_{\substack{\text{satisfies} \\ Ax=b}} + \alpha \underbrace{\begin{bmatrix} \frac{1}{2} \\ -3/4 \\ 1 \end{bmatrix}}_{\substack{\text{satisfies} \\ Ax=0}}$$

$$x = x_p + x_h$$

x_p: one particular solution of $Ax = b$

x_h: Solution of homog. system $Ax = 0$

Example: solve the system of equations $Ax = b$ with

$$[A|b] = \left[\begin{array}{ccccc:c} 1 & 2 & -2 & -1 & 1 & 1 \\ 2 & 4 & -4 & 0 & 3 & 2 \\ -1 & -2 & 3 & 3 & 4 & 3 \\ 3 & 6 & -7 & 1 & 1 & \beta \end{array}\right] \quad ; \qquad \beta \in \mathbb{R}$$

$$\begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \\ R_4 \to R_4 - 3R_1 \end{array} \quad \sim \left[\begin{array}{ccccc:c} 1 & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 & 4 \\ 0 & 0 & -1 & 4 & -2 & \beta-3 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \quad \sim \left[\begin{array}{ccccc:c} 1 & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 5 & 4 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & 4 & -2 & \beta-3 \end{array}\right]$$

$$R_4 \to R_4 + R_2 \quad \sim \left[\begin{array}{cccccc} 1 & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 5 & 4 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 6 & 3 & \beta+1 \end{array}\right]$$

$$R_4 \to R_4 - 3R_3 \quad \sim \left[\begin{array}{ccccc:c} 1 & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 5 & 4 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta+1 \end{array}\right]$$

Case I: $\beta \neq -1 \rightarrow$ NO SOLUTION

Case II: $\beta = -1$:

$$\sim \begin{array}{c} \begin{array}{cccccc} x_1 & & x_3 & x_4 & & \end{array} \\ \left[\begin{array}{ccccc:c} \boxed{1} & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 2 & 5 & 4 \\ 0 & 0 & 0 & \boxed{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccccc} & x_2 & & & x_5 & \end{array} \end{array} \leftarrow \text{free variables}$$

Take $x_2 = \alpha_1$

$x_5 = \alpha_2$

$$x_4 = -\tfrac{1}{2}\alpha_2$$
$$x_3 = 4 - 4\alpha_2$$
$$x_1 = 9 - 2\alpha_1 - 9.5\alpha_2$$

Writting in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \underbrace{\begin{bmatrix} 9 \\ 0 \\ 4 \\ 0 \\ 0 \end{bmatrix}}_{\substack{\text{satisfies} \\ Ax=b}} + \underbrace{\alpha_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -9.5 \\ 0 \\ -4 \\ -0.5 \\ 1 \end{bmatrix}}_{\text{satisfies } Ax=0}$$

$$x = x_p + x_h$$

Note:

- Solution of $Ax=0$ (homog. system) : x_h
- vectors that generate solutions of $Ax=0$ are

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \ \& \ \begin{bmatrix} -9.5 \\ 0 \\ -4 \\ -0.5 \\ 1 \end{bmatrix}$$

- These generators are called basis of vector space (solution space, or null space of $Ax=0$).
- Free variable(s) is (are) responsible for infinitely many solutions.
- An invertible matrix has no free variable.

[$Ax=b \Rightarrow x = A^{-1}b$ (unique solution)]