Lecture 8

Theorem :– For every Hermitian matrix $H_{n\times n}$, there exists a unitary matrix $S_{n\times n}$ such that $S^*HS = \text{diagonal}(\lambda_1, \ldots, \lambda_n)$

$$= \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}_{n\times n}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of H.

i.e) Every Hermitian matrix is diagonizable.

proof:– We prove by Induction on n.

$n=1$: the statement is trivially true.

Assume $n=2$:

Let $\underline{x_1}$ be an eigenvector of A corresponding to the eigenvalue λ_1.

& $\|\underline{x_1}\| = 1$.

Choose a vector $\underline{x_2}$ such that $\underline{x_2}$ is orthogonal to $\underline{x_1}$ & $\|\underline{x_2}\| = 1$.

i.e $\langle \underline{x_1}, \underline{x_2} \rangle = 0$ & $\|\underline{x_2}\| = 1$.

(We can always find such vector by using Gram-Schmidt orthonormalization.)

Let $U = \begin{bmatrix} \underline{x}_1 & \underline{x}_2 \end{bmatrix}_{2\times 2}$. Then U is a unitary matrix.

Now
$$U^* H U = \begin{bmatrix} \underline{x}_1^* \\ \underline{x}_2^* \end{bmatrix} H \begin{bmatrix} \underline{x}_1 & \underline{x}_2 \end{bmatrix}$$

$$= \begin{bmatrix} \underline{x}_1^* \\ \underline{x}_2^* \end{bmatrix} \begin{bmatrix} H\underline{x}_1 & H\underline{x}_2 \end{bmatrix}$$

$H\underline{x}_1 = \lambda_1 \underline{x}_1$ & $\underline{x}_1^* \underline{x}_1 = 1$

$$= \begin{bmatrix} \underline{x}_1^* H \underline{x}_1 & \underline{x}_1^* H \underline{x}_2 \\ \underline{x}_2^* H \underline{x}_1 & \underline{x}_2^* H \underline{x}_2 \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} \lambda_1 & \underline{x}_1^* H \underline{x}_2 \\ \lambda_1 \underline{x}_2^* \underline{x}_1 & \underline{x}_2^* H \underline{x}_2 \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} \lambda_1 & \underline{x}_1^* H \underline{x}_2 \\ 0 & \underline{x}_2^* H \underline{x}_2 \end{bmatrix}$$

Now
$$(U^* H U)^* = U^* H^* (U^*)^*$$
$$= U^* H U \quad (\because H^* = H)$$

$$\Rightarrow \begin{bmatrix} \bar{\lambda}_1 & 0 \\ (\underline{x}_1^* H \underline{x}_2)^* & (\underline{x}_2^* H \underline{x}_2)^* \end{bmatrix} = \begin{bmatrix} \lambda_1 & \underline{x}_1^* H \underline{x}_2 \\ 0 & \underline{x}_2^* H \underline{x}_2 \end{bmatrix}$$

$$\Rightarrow \quad \bar{\lambda}_1 = \lambda_1 \quad \& \quad \boxed{\underline{x}_1^* H \underline{x}_2 = 0}.$$

$$\therefore \quad U^* H U = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \underline{x}_2^* H \underline{x}_2 \end{bmatrix}$$

Let $\lambda_2 = \underline{x}_2^* H \underline{x}_2$.

Then $U^* H U = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$\therefore$ U is our required unitary matrix.

Thus the statement is true for $n=2$.

Assume $n \geq 3$ & the induction hypothesis.

Let $\underline{x}_1$ be an eigenvector of H corresponding to the eigenvalue λ_1. & $\|\underline{x}_1\| = 1$.

$$\therefore \quad H\underline{x}_1 = \lambda_1 \underline{x}_1.$$

choose $\underline{x}_2, \ldots, \underline{x}_n$ vectors in $\mathbb{C}^n$ such that $\{\underline{x}_1, \ldots, \underline{x}_n\}$ is an orthonormal set.

This is always possible by Gram–Schmidt.

i.e. $\langle \underline{x}_i, \underline{x}_j \rangle = 0 \ \forall i \neq j$ & $\|\underline{x}_j\| = 1 \ \forall j$

Let

Let $U = [\underline{x_1}\ \underline{x_2} \cdots \underline{x_n}]_{n\times n}$

Now
$$U^* H U = \begin{bmatrix} \underline{x_1}^* \\ \underline{x_2}^* \\ \vdots \\ \underline{x_n}^* \end{bmatrix} H [\underline{x_1}\ \underline{x_2} \cdots \underline{x_n}]$$

$$= \begin{bmatrix} \underline{x_1}^* \\ \vdots \\ \underline{x_n}^* \end{bmatrix} [H\underline{x_1}\ \ H\underline{x_2} \cdots H\underline{x_n}]$$

$$= \begin{bmatrix} \underline{x_1}^* H \underline{x_1} & \underline{x_1}^* H \underline{x_2} & \cdots & \underline{x_1}^* H \underline{x_n} \\ \underline{x_2}^* H \underline{x_1} & \underline{x_2}^* H \underline{x_2} & \cdots & \underline{x_2}^* H \underline{x_n} \\ \vdots & \vdots & & \vdots \\ \underline{x_n}^* H \underline{x_1} & \underline{x_n}^* H \underline{x_2} & \cdots & \underline{x_n}^* H \underline{x_n} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 \underline{x_1}^* \underline{x_1} & \underline{x_1}^* H \underline{x_2} & \cdots & \underline{x_1}^* H \underline{x_n} \\ \lambda_1 \underline{x_2}^* \underline{x_1} & \underline{x_2}^* H \underline{x_2} & \cdots & \underline{x_2}^* H \underline{x_n} \\ \vdots & \vdots & & \vdots \\ \lambda_1 \underline{x_n}^* \underline{x_1} & \underline{x_n}^* H \underline{x_2} & \cdots & \underline{x_n}^* H \underline{x_n} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & \underline{x_1}^* H \underline{x_2} & \cdots & \underline{x_1}^* H \underline{x_n} \\ 0 & \underline{x_2}^* H \underline{x_2} & \cdots & \underline{x_2}^* H \underline{x_n} \\ \vdots & \vdots & & \vdots \\ \vdots & \underline{x_n}^* H \underline{x_2} & & \underline{x_n}^* H \underline{x_n} \end{bmatrix}$$

$$\begin{bmatrix} \vdots \\ 0 & x_n^* H x_2 & \cdots & x_n^* H x_n \end{bmatrix}$$

Since H is a Hermitian matrix, we have $U^* H U$ is also Hermitian.

$$\Rightarrow (U^* H U)^* = U^* H U.$$

$$\Rightarrow \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ (x_1^* H x_2)^* & & \cdots & (x_n^* H x_2)^* \\ \vdots & & & \\ (x_1^* H x_n)^* & & \cdots & (x_n^* H x_n)^* \end{bmatrix} = \begin{bmatrix} \lambda_1 & x_1^* H x_2 & \cdots & x_1^* H x_n \\ 0 & \vdots & & \vdots \\ \vdots & & & \\ 0 & x_n^* H x_2 & \cdots & x_n^* H x_n \end{bmatrix}$$

$$\Rightarrow x_1^* H x_2 = x_1^* H x_3 = \cdots = x_1^* H x_n = 0.$$

$$\therefore U^* H U = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & x_2^* H x_2 & \cdots & x_2^* H x_n \\ \vdots & \vdots & & \vdots \\ 0 & x_n^* H x_2 & \cdots & x_n^* H x_n \end{bmatrix}$$

where $\begin{pmatrix} x_2^* H x_2 & \cdots & x_2^* H x_n \\ \vdots & & \vdots \\ x_n^* H x_2 & \cdots & x_n^* H x_n \end{pmatrix} = G$ (Say)

$$= \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & G \end{array}\right]$$

$G_{(n-1)\times(n-1)}$.

check that G is a Hermitian matrix.

($G^* = G$)

$\therefore$ G is a Hermitian matrix of size $(n-1) \times (n-1)$.

$\therefore$ By Induction hypothesis, there exists a unitary matrix $T_{(n-1)\times(n-1)}$ such that

$$T^* G T = \text{diagonal}(\lambda_2, \dots, \lambda_n)$$

Let $P = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & T \end{array}\right]_{n \times n}$.

Let $S = UP$

Consider

$$\begin{aligned}
S^* H S &= (UP)^* H (UP) \\
&= (P^* U^*) H (UP) \\
&= P^* (U^* H U) P \\
&= P^* \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & G \end{array}\right] P \\
&= \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & T \end{array}\right]^* \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & G \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & T \end{array}\right]
\end{aligned}$$

$$= \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & T^* \end{array}\right] \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & GT \end{array}\right]$$

$$= \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & T^*GT \end{array}\right]$$

$$= \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$= \text{diagonal}(\lambda_1, \ldots, \lambda_n).$$

Check that S is a Hermitian matrix.

Corollary: For every symmetric matrix $A_{n\times n}$ there exists an orthogonal matrix $S_{n\times n}$ such that $S^tAS = \text{diagonal}(\lambda_1, \ldots, \lambda_n)$

In particular A is diagonalizable.

Corollary: For every square matrix $A_{n\times n}$ over $\mathbb{C}$, there exists a unitary matrix $S_{n\times n}$ such that $S^*AS =$ an upper triangular matrix.

proof: Note that the property of Hermitian matrix was used in the above proof to reduce the elements above the main diagonal to zero of a tranfered matrix.

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① Find an orthogonal matrix $S_{3\times 3}$ such that S^tAS is an upper Δlar matrix, where $A = \begin{bmatrix} 0 & 1 & 1 \\ -2 & 3 & 2 \\ -3 & 3 & 4 \end{bmatrix}$.

Sol:- The eigenvalues of A are

$$\det(A-\lambda I) = 0.$$

$$\Rightarrow (\lambda-1)^2(\lambda-5) = 0.$$

$$\Rightarrow \lambda = 1, 1, 5.$$

The eigenvectors corresponding to $\lambda = 1$:

$$(A-I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underline{0}$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 1 \\ -2 & 2 & 2 \\ -3 & 3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$

$\Rightarrow \quad -x+y+z=0.$

$\Rightarrow \quad \boxed{x = y+z}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y+z \\ y \\ z \end{pmatrix} = y\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Let $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. $\underline{v}_1$ is an eigenvector of A.

Let $\underline{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\underline{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ such that

$\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ l.i set.

Now use Gram-Schmidt orthonormalization:

$$\underline{y}_1 = \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

$$\underline{y}_2 = \underline{v}_2 - \frac{\langle \underline{v}_2, \underline{v}_1 \rangle}{\langle \underline{v}_1, \underline{v}_1 \rangle}\underline{y}_1$$

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix}$$

$$\underline{y}_3 = \underline{v}_3 - \frac{\langle \underline{v}_3, \underline{y}_1 \rangle}{\langle \underline{y}_1, \underline{y}_1 \rangle}\underline{y}_1 - \frac{\langle \underline{v}_3, \underline{y}_2 \rangle}{\langle \underline{y}_2, \underline{y}_2 \rangle}\underline{y}_2.$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - 0 - 0$$

$\therefore$ Let $\underline{x}_1 = \dfrac{\underline{y}_1}{\|\underline{y}_1\|} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ — an eigenvector of A

$\lambda = 1.$

$$\underline{x}_2 = \frac{\underline{y}_2}{\|\underline{y}_2\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\underline{x_3} = \frac{y_3}{\|y_3\|} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$\therefore \{\underline{x_1}, \underline{x_2}, \underline{x_3}\}$ is an orthonormal set.

Let $U = \begin{bmatrix} \underline{x_1} & \underline{x_2} & \underline{x_3} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$U^*AU = U^tAU = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ -2 & 3 & 2 \\ -3 & 3 & 4 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -2 & 4 & 3 \\ -2 & 2 & 1 \\ -3\sqrt{2} & 3\sqrt{2} & 4\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 6 & 3\sqrt{2} \\ 0 & 4 & \sqrt{2} \\ 0 & 6\sqrt{2} & 8 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3/\sqrt{2} \\ 0 & 2 & 1/\sqrt{2} \\ 0 & 3\sqrt{2} & 4 \end{bmatrix}$$

$$\begin{pmatrix} 2 & 1/\sqrt{2} \\ 3\sqrt{2} & 4 \end{pmatrix} = G.$$

Let $G = \begin{bmatrix} 2 & 1/\sqrt{2} \\ 3\sqrt{2} & 4 \end{bmatrix}$.

The eigenvalues of G are $1, 5$.

Eigenvectors corr. to $\lambda = 1$.

$(A - I)(x)$

$$(G - I)\begin{pmatrix} x \\ y \end{pmatrix} = \underline{0}$$

$$\Rightarrow \begin{bmatrix} 1 & 1/\sqrt{2} \\ 3\sqrt{2} & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x = -y/\sqrt{2} \quad \Rightarrow \quad y = -\sqrt{2}\,x$$

$\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$ is an eigenvector of G. corr to $\lambda = 1$.

Let $\underline{u}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$ eigenvect of G & $\|\underline{u}_1\| = 1$.

Choose $\underline{u}_2$ orthogonal to $\underline{u}_1$.

say $\underline{u}_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$.

$\{\underline{u}_1, \underline{u}_2\}$ is an orthonormal set.

Let $T = [\underline{u}_1 \ \underline{u}_2] = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix}$

Let $P = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & T \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{3} & \sqrt{2}/\sqrt{3} \\ 0 & -\sqrt{2}/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$

Let $S = UP$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ \cdots \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{3} & \sqrt{2}/\sqrt{3} \\ \cdots \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -\sqrt{2}/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -\sqrt{2}/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$S^tAS = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{6}} & 1/\sqrt{6} & \frac{-\sqrt{2}}{\sqrt{3}} = \frac{-2}{\sqrt{6}} \\ -1/\sqrt{3} & \frac{1}{\sqrt{3}} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ -2 & 3 & 2 \\ -3 & 3 & 4 \end{bmatrix} S.$$

$$= \begin{bmatrix} -\sqrt{2} & \frac{4}{\sqrt{2}} & 3/\sqrt{2} \\ 4/\sqrt{6} & -4/\sqrt{6} & -7/\sqrt{6} \\ -5/\sqrt{3} & 5/\sqrt{3} & 5/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & \frac{-1}{\sqrt{3}} \\ 1/\sqrt{2} & \frac{1}{\sqrt{6}} & 1/\sqrt{3} \\ 0 & -\sqrt{2}/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 9/\sqrt{6} \\ 0 & 1 & -5/\sqrt{2} \\ 0 & 0 & 5 \end{bmatrix}$$ = upper $\triangle^{lr}$ matrix.

Theorem: Let H be a Hermitian matrix. Let λ be an eigenvalue of H of multiplicity m. Then there exists a set of m

orthogonal eigenvectors associated with λ. (l.i)

<u>proof</u> – First we show that there exists m l.i eigenvectors corr to eigenvalue λ.

By above theorem, there exists a unitary matrix $U_{n\times n}$ such that

$$U^*HU = \text{diagonal matrix}(\lambda_1, \ldots, \lambda_m, \lambda_{m+1}, \ldots, \lambda_n)$$

where $\lambda_1 = \cdots = \lambda_m = \lambda$.

& $\lambda_j \neq \lambda \quad \forall j = m+1, \ldots, n$.

The columns of U say $\underline{x_1}, \ldots, \underline{x_m}, \ldots, \underline{x_n}$ is an orthonormal set. ($\because U$ is unitary).

Now $U^*(\lambda I - H)U = \lambda I - U^*HU$

$$= \lambda I - \text{diagonal matrix}(\lambda_1, \ldots, \lambda_n)$$

$$= \begin{pmatrix} 0 & & & & \\ & \ddots & & & 0 \\ & & 0 & & \\ & & & \lambda - \lambda_m & \\ 0 & & & & \ddots \\ & & & & & \lambda - \lambda_n \end{pmatrix}$$

$\therefore$ rank $(U^*(\lambda I - H)U) = n - m$

$\Rightarrow$ rank $(\lambda I - H) = n - m$

$\Rightarrow$ nullity of $\lambda I - A$ is equal to m. (by rank-nullity thm)

$$\Rightarrow \dim(\underbrace{N(\lambda I - A)}_{= E_\lambda(A)}) = m.$$

$\Rightarrow$ there exists m l.i eigenvectors corr to λ.

Now use the Gram-Schmidt orthonormalization we get m orthonormal set of eigenvectors corr. to the eigenvalue λ of A.
