Lecture 15

Vector norms & matrix norms

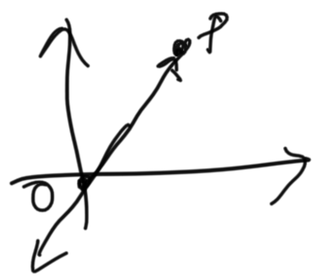
$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\langle \underline{x}, \underline{x} \rangle = \|\underline{x}\|^2 = \underline{x}^t \underline{x} = \begin{cases} |x_1|^2 + \cdots + |x_n|^2, & \text{if } \underline{x} \in \mathbb{C}^n \\ x_1^2 + \cdots + x_n^2 & \text{if } \underline{x} \in \mathbb{R}^n. \end{cases}$$

std innerproduct

$$\|\underline{x}\| = \sqrt{|x_1|^2 + \cdots + |x_n|^2}.$$

"Euclidean norm".



Definition:-

A real valued function $\| - \|$ defined on each element of a vector space V is defined as a vector norm or norm on V if the map $\| - \| : V \to \mathbb{R}$ satisfying the following conditions:

(i) $\|\underline{v}\| \geq 0$ & $\|\underline{v}\| = 0 \Leftrightarrow \underline{v} = \underline{0}$. $\forall \underline{v} \in V$.

(ii) $\|\alpha \underline{v}\| = |\alpha| \, \|\underline{v}\| \quad \forall \underline{v} \in V, \forall \alpha \in F$

(iii) $\| \underline{u} + \underline{v} \| \le \| \underline{u} \| + \| \underline{v} \|$, $\forall\, \underline{u}, \underline{v} \in V$.

(triangular inequality).

Examples:-

① Let $V = \mathbb{C}^n$ over $F = \mathbb{C}$.

Define $\| \underline{x} \|_2 := \sqrt{|x_1|^2 + \cdots + |x_n|^2}$

$\forall\, \underline{x} \in \mathbb{C}^n$.

$\| - \|_2$ is a norm on $\mathbb{C}^n$.

- $\| \underline{x} \|_2 \ge 0$ & $\| \underline{x} \|_2 = 0 \iff x_1 = \cdots = x_n = 0$.
  i.e. $\underline{x} = \underline{0}$.

- $$\| \alpha \underline{x} \|_2 = \sqrt{|\alpha x_1|^2 + \cdots + |\alpha x_n|^2}$$
  $$= \sqrt{|\alpha|^2 (|x_1|^2 + \cdots + |x_n|^2)}$$
  $$= |\alpha| \sqrt{|x_1|^2 + \cdots + |x_n|^2}$$
  $$= |\alpha| \, \| \underline{x} \|_2$$

- $\| \underline{x} + \underline{y} \|_2 \le \| \underline{x} \|_2 + \| \underline{y} \|_2$

  We will prove later by using Cauchy–Schwartz inequality.

$V = \mathbb{C}^n$

$V = \mathbb{C}$

(2) $\|\underline{x}\|_1 = |x_1| + |x_2| + \dots + |x_n| = \sum_{j=1}^{n} |x_j|$

norm on $\mathbb{C}^n$ called "1-norm".

$$\|\underline{x} + \underline{y}\|_1 = |x_1 + y_1| + \dots + |x_n + y_n|$$
$$\leq |x_1| + |y_1| + \dots + |x_n| + |y_n|$$
$$\leq \|\underline{x}\|_1 + \|\underline{y}\|_1. \qquad \forall\, \underline{x}, \underline{y} \in \mathbb{C}^n.$$

(3) For any $1 \leq p < \infty$,

$$\|\underline{x}\|_p := \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p}$$

We will prove that this defines a norm on $\mathbb{C}^n$, called "p-norm".

(4) $V = \mathbb{C}^n$, $\quad \|\underline{x}\|_\infty = \max\{|x_1|, \dots, |x_n|\}$ $\quad \forall\, \underline{x} \in \mathbb{C}^n$

is a norm on V called "∞-norm" or "sup-norm". (EXERCISE)

Theorem (Cauchy - Schwartz inequality):—

$$|\underline{x}^* \underline{y}| \leq \|\underline{x}\|_2 \|\underline{y}\|_2 \qquad \forall\ \underline{x}, \underline{y} \in \mathbb{C}^n.$$

& the equality holds $\Leftrightarrow$ $\underline{y} = \alpha \underline{x}$

for $\alpha = \dfrac{\underline{x}^* \underline{y}}{\underline{x}^* \underline{x}}$, $\underline{x} \neq 0$

$\in \mathbb{C}$.

proof:— Assume $\underline{x} \neq \underline{0}$ & $\underline{y} \neq \underline{0}$ in $\mathbb{C}^n$.

Set $\alpha = \dfrac{\underline{x}^* \underline{y}}{\underline{x}^* \underline{x}} = \dfrac{\underline{x}^* \underline{y}}{\|\underline{x}\|_2^2}$

$$\Rightarrow \quad \alpha\, \underline{x}^* \underline{x} = \underline{x}^* \underline{y}$$

$$\Rightarrow \quad \boxed{\underline{x}^* (\alpha \underline{x} - \underline{y}) = 0.}$$

Now $0 \leq \|\alpha \underline{x} - \underline{y}\|_2^2 = (\alpha \underline{x} - \underline{y})^* (\alpha \underline{x} - \underline{y})$

$$= (\bar{\alpha}\, \underline{x}^* - \underline{y}^*)(\alpha \underline{x} - \underline{y})$$

$$= \bar{\alpha}\, \underline{x}^* (\alpha \underline{x} - \underline{y}) - \underline{y}^* (\alpha \underline{x} - \underline{y})$$

$$= 0 - \underline{y}^* (\alpha \underline{x} - \underline{y})$$

$$= -\alpha\, \underline{y}^* \underline{x} + \underline{y}^* \underline{y}.$$

$$= \frac{(-\underline{y}^*\underline{x})(\underline{x}^*\underline{y})}{\underline{x}^*\underline{x}} + \underline{y}^*\underline{y}$$

$$= -\frac{\overline{(\underline{x}^*\underline{y})}\,(\underline{x}^*\underline{y})}{\|\underline{x}\|_2^2} + \|\underline{y}\|_2^2$$

$$= \frac{-|\underline{x}^*\underline{y}|^2 + \|\underline{x}\|_2^2\,\|\underline{y}\|_2^2}{\|\underline{x}\|_2^2}$$

Thus $\quad 0 \le -|\underline{x}^*\underline{y}|^2 + \|\underline{x}\|_2^2\,\|\underline{y}\|_2^2$

$$\Rightarrow \quad |\underline{x}^*\underline{y}| \le \|\underline{x}\|_2\,\|\underline{y}\|_2.$$

Suppose equality holds.

i.e., $\quad |\underline{x}^*\underline{y}| = \|\underline{x}\|_2\,\|\underline{y}\|_2$

$$\Rightarrow \quad \|\alpha\underline{x} - \underline{y}\|_2^2 = 0.$$

$$\Rightarrow \quad \|\alpha\underline{x} - \underline{y}\|_2 = 0$$

$$\Rightarrow \quad \alpha\underline{x} - \underline{y} = \underline{0}$$

$$\Rightarrow \quad \underline{y} = \alpha\,\underline{x}.$$

Conversely, let $\underline{y} = \alpha \underline{x}$. Then

$$|\underline{x}^* \underline{y}| = |\underline{x}^*(\alpha \underline{x})| = |\alpha|\ |\underline{x}^* \underline{x}|$$

$$= |\alpha| . \|\underline{x}\|_2^2$$

$$= (|\alpha|\ \|\underline{x}\|_2)\ \|\underline{x}\|_2$$

$$= \|\alpha \underline{x}\|_2 . \|\underline{x}\|_2$$

$$= \|\underline{y}\|_2 . \|\underline{x}\|_2 .$$

$\therefore$ equality holds.

Triangular inequality for the 2-norm

To show: $\|\underline{x} + \underline{y}\|_2 \leq \|\underline{x}\|_2 + \|\underline{y}\|_2 \quad \forall \underline{x}, \underline{y} \in \mathbb{C}^n$.

Consider $\|\underline{x}+\underline{y}\|_2^2 = (\underline{x}+\underline{y})^*(\underline{x}+\underline{y})$

$$= (\underline{x}^* + \underline{y}^*)(\underline{x}+\underline{y})$$

$$= \underline{x}^*\underline{x} + \underline{x}^*\underline{y} + \underline{y}^*\underline{x} + \underline{y}^*\underline{y}$$

$$= \underline{x}^*\underline{x} + (\underline{x}^*\underline{y} + \overline{\underline{x}^*\underline{y}}) + \underline{y}^*\underline{y}$$

$$= \underline{x}^*\underline{x} + 2\,\mathrm{Re}(\underline{x}^*\underline{y}) + \underline{y}^*\underline{y}$$

$$\leq \underline{x}^*\underline{x} + 2\,|\underline{x}^*\underline{y}| + \underline{y}^*\underline{y}$$

$$\leq \underline{x}^*\underline{x} + 2\,\|\underline{x}\|_2\,\|\underline{y}\|_2 + \underline{y}^*\underline{y}$$

(by using Cauchy–Schwartz ineq)

$$\leq \|\underline{x}\|_2^2 + 2\|\underline{x}\|_2\,\|\underline{y}\|_2 + \|\underline{y}\|_2^2$$

$$= \left(\|\underline{x}\|_2 + \|\underline{y}\|_2\right)^2$$

Thus $\|\underline{x}+\underline{y}\|_2^2 \leq \left(\|\underline{x}\|_2 + \|\underline{y}\|_2\right)^2$

$$\Rightarrow \|\underline{x}+\underline{y}\|_2 \leq \|\underline{x}\|_2 + \|\underline{y}\|_2.$$

Result:- $\big|\, \|\underline{x}\| - \|\underline{y}\| \,\big| \leq \|\underline{x}-\underline{y}\| \quad \forall \underline{x}, \underline{y} \in \mathbb{C}^n.$

proof:-

$$\|\underline{x}\| = \|(\underline{x}-\underline{y}) + \underline{y}\|$$

$$\leq \|\underline{x}-\underline{y}\| + \|\underline{y}\| \quad \text{(by triangular ineq.)}$$

$$\Rightarrow \|\underline{x}\| - \|\underline{y}\| \leq \|\underline{x}-\underline{y}\|$$

& $\|\underline{y}\| = \|(\underline{x}-\underline{y}) - \underline{x}\| \leq \|\underline{x}-\underline{y}\| + \|\underline{x}\|.$

$$\Rightarrow \|\underline{y}\| - \|\underline{x}\| \leq \|\underline{x}-\underline{y}\|.$$

Thus $\big|\, \|\underline{x}\| - \|\underline{y}\| \,\big| \leq \|\underline{x}-\underline{y}\|.$

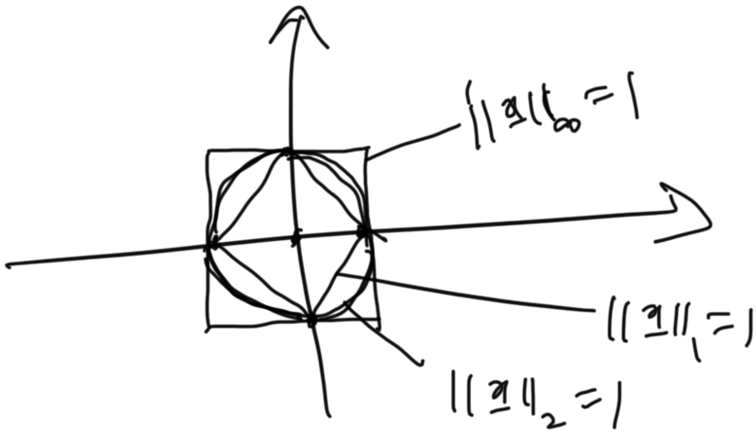
$\underline{x} \in \mathbb{R}^n$

$$\|\underline{x}\|_1 = \sum_{j=1}^{n} |x_j|$$

$\|\underline{x}\|_1 = 1.$

$\|\underline{x}\|_2 = 1$

$\|\underline{x}\|_\infty = 1$



In fact,

$$\|\underline{x}\|_\infty = \lim_{p\to\infty} \|\underline{x}\|_p = \lim_{p\to\infty} \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}$$

$$= \max\{|x_1|, \dots, |x_n|\}.$$

(EXERCISE).