Lecture 19

Proof of the theorem:-

Consider A^*A, which is Hermitian.

Also A^*A is +ve def or +ve semi-def.

$$\left(\because \underset{Q(\underline{z})}{\underset{\shortparallel}{\underline{z}^* A^* A \underline{z}}} = (A\underline{z})^*(A\underline{z}) = \underset{\text{std inner product}}{\langle A\underline{z}, A\underline{z} \rangle} \geq 0 \quad \forall \underline{z}. \right)$$

Let $\lambda_1^2, \dots, \lambda_r^2, 0, \dots, 0$ be the eigenvalues of A^*A. & $\lambda_j \neq 0 \ \forall j = 1, \dots, r$, $r \leq \min(m, n)$

Also there exists a unitary matrix $V_{n\times n}$ such that

$$V^*(A^*A)V = \begin{pmatrix} \lambda_1^2 & & & & & \\ & \ddots & & & O & \\ & & \lambda_r^2 & & & \\ & & & 0 & & \\ & O & & & \ddots & \\ & & & & & 0 \end{pmatrix}_{n\times n}.$$

Set $\underset{m\times n}{A} V_{n\times n} = [\underline{x}_1 \dots \underline{x}_n]_{m\times n}$ (say).

where $\underline{x}_j \in \mathbb{C}^m \quad \forall j$.

Then

$$(AV)^*(AV) = \begin{pmatrix} \lambda_1^2 & & & & & \\ & \ddots & & & O & \\ & & \lambda_r^2 & & & \\ & & & 0 & & \\ & O & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} \underline{x}_1^* \\ \vdots \\ \underline{x}_n^* \end{bmatrix} [\underline{x}_1 \dots \underline{x}_n] = \quad ''$$

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$$\Rightarrow \begin{bmatrix} \underline{x}_1^* \underline{x}_1 & \underline{x}_1^* \underline{x}_2 & \cdots & \underline{x}_1^* \underline{x}_n \\ \vdots & & & \vdots \\ \underline{x}_n^* \underline{x}_1 & \underline{x}_n^* \underline{x}_2 & \cdots & \underline{x}_n^* \underline{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & & & & 0 \\ & \ddots & & & \\ & & \lambda_r^2 & & \\ & & & 0 & \\ 0 & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

$$\Rightarrow \quad \underline{x}_l^* \underline{x}_j = \begin{cases} \lambda_l^2 & \text{if } l = j = 1, \cdots, r \\ 0 & \text{otherwise.} \end{cases}$$

&

$$\underline{x}_l^* \underline{x}_l = 0 \quad \text{for } l = r+1, \cdots, n.$$

$$\left(\Rightarrow \boxed{\underline{x}_l = \underline{0}}, \ l = r+1, \cdots, n \right).$$

$\Rightarrow \{\underline{x}_1, \cdots, \underline{x}_r\}$ is an orthogonal set.

Let $\underline{u}_j = \dfrac{\underline{x}_j}{\lambda_j}$ for $j = 1, \cdots, r$.

$\Rightarrow \{\underline{u}_1, \cdots, \underline{u}_r\}$ is an orthonormal set in $\mathbb{C}^m$.

Extend this set to an orthonormal set which is a basis of $\mathbb{C}^m$, say $\{\underline{u}_1, \cdots, \underline{u}_r, \cdots, \underline{u}_m\}$

Let $U = [\underline{u}_1 \cdots \underline{u}_m]_{m \times m}$.

Then U is unitary.

Now $AV = [\underline{x}_1 \cdots \underline{x}_r \ \underline{0} \cdots \underline{0}]_{m \times n}$

$$= \left[\lambda_1 \underline{u}_1 \cdots \lambda_r \underline{u}_r \;\; \underline{0} \cdots \underline{0} \right]$$

$$= UD$$

where D is as in the statement of the theorem.

$$\Rightarrow \boxed{A = UDV^*}$$

<u>Remark:-</u> ① The columns of U, V are called the <u>singular bases</u> of A.

② The numbers $\lambda_1, \ldots, \lambda_r$ together with $\lambda_{r+1} = 0, \ldots, \lambda_n = 0$ which are the +ve square roots of the eigenvalues of A^*A. known as the <u>singular values</u> of A

& $A = UDV^*$ is known as the <u>singular value decomposition</u> (SVD) of A.

③ $$\boxed{A = UDV^* = \sum_{k=1}^{r} \lambda_k \underline{u}_k \underline{v}_k^*.}$$

where $\underline{u_k}$ is the k^{th} Column of U
& $\underline{v_k}$ is the k^{th} Column of V.

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① Find the SVD of $A = \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \\ 1 & 1-i \end{bmatrix}$.

Sol:- $A^*A = \begin{bmatrix} 1 & 1-i & 1 \\ 1+i & 2 & 1+i \end{bmatrix} \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \\ 1 & 1-i \end{bmatrix}$

$= \begin{bmatrix} 1+1+1+1 & 1-i+2-2i+1-i \\ 1+i+2+2i+1+i & 1+1+4+1+1 \end{bmatrix}$

$= \begin{bmatrix} 4 & 4-4i \\ 4+4i & 8 \end{bmatrix} = 4\begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}$.

The eigenvalues of A^*A are $12, 0$.

Let $\lambda_1^2 = 12$, $\lambda_2 = 0$.

To find $V_{2\times 2}$ such V is unitary &

$V^*(A^*A)V = \begin{bmatrix} 12 & 0 \\ 0 & 0 \end{bmatrix}$.

Eigenvectors corr. to 12 of A^*A:

$$A^*A\begin{pmatrix} x \\ y \end{pmatrix} = 12\begin{pmatrix} x \\ y \end{pmatrix}.$$

$$\Rightarrow \quad 4\begin{bmatrix} x+(1-i)y \\ (1+i)x+2y \end{bmatrix} = 12\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} x+(1-i)y &= 3x \\ (1+i)x+2y &= 3y. \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} -2x+(1-i)y &= 0 \\ (1+i)x-y &= 0. \end{aligned}$$

$$\Rightarrow \quad y=(1+i)x.$$

$$\therefore \quad \begin{pmatrix} x \\ y \end{pmatrix} = y\begin{pmatrix} 1 \\ 1+i \end{pmatrix}.$$

Let $\quad v_1 = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}.$

Eigenvectors corresponding to 0 of A^*A:

$$A^*A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} x+(1-i)y &= 0 \\ (1+i)x+2y &= 0. \end{aligned}$$

$$\Rightarrow \quad \boxed{x = (i-1)y} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} i-1 \\ 1 \end{pmatrix} y.$$

Let $\underline{v}_2 = -\begin{pmatrix} i-1 \\ 1 \end{pmatrix}.$

$$\underline{v}_1^* \underline{v}_2 = -(1 \quad 1-i)\begin{pmatrix} i-1 \\ 1 \end{pmatrix} = i-1+1-i = 0.$$

Let $V = \left[\dfrac{\underline{v}_1}{\|\underline{v}_1\|} \quad \dfrac{+\underline{v}_2}{\|\underline{v}_2\|} \right] = \begin{bmatrix} 1/\sqrt{3} & (1-i)/\sqrt{3} \\ (i+1)/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}$

Now $(AV)^*(AV) = \begin{bmatrix} 12 & 0 \\ 0 & 0 \end{bmatrix}$

$$AV = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \\ 1 & 1-i \end{bmatrix} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{3}} \begin{bmatrix} 3 & 0 \\ 3(1+i) & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 0 \\ \sqrt{3}(1+i) & 0 \\ \sqrt{3} & 0 \end{bmatrix}$$

$= [\underline{x}_1 \quad \underline{x}_2]$ (say)

$= [\lambda_1 \underline{u}_1 \quad \underline{0}]$

where $\lambda_1 = \sqrt{12} = 2\sqrt{3}.$

& $\underline{u}_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1+i \\ 1 \end{pmatrix} \in \mathbb{C}^3.$

Consider the basis $\{\underline{u}_1, \underline{e}_2, \underline{e}_3\}$ of $\mathbb{C}^3$.

l.i.

Let $\underline{y}_1 = \underline{u}_1 = \begin{pmatrix}1/2\\(1+i)/2\\1/2\end{pmatrix}$.

$$\underline{y}_2 = \underline{e}_2 - \frac{\langle \underline{e}_2, \underline{y}_1\rangle}{\langle \underline{y}_1, \underline{y}_1\rangle}\underline{y}_1.$$

$$= \begin{pmatrix}0\\1\\0\end{pmatrix} - \frac{(1-i)/2}{1}\begin{pmatrix}1/2\\(1+i)/2\\1/2\end{pmatrix}.$$

$$= \frac{1}{4}\begin{pmatrix}i-1\\2\\i-1\end{pmatrix}.$$

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$$y_3 = \frac{1}{2}\begin{pmatrix}-1\\0\\1\end{pmatrix}.$$

Let $\underline{u}_1 = \frac{1}{2}\begin{pmatrix}1\\1+i\\1\end{pmatrix}$, $\underline{u}_2 = \frac{\underline{y}_2}{\|\underline{y}_2\|} = \frac{\sqrt{2}}{4}\begin{pmatrix}i-1\\2\\i-1\end{pmatrix}$

$$\underline{u}_3 = \frac{\underline{y}_3}{\|\underline{y}_3\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}.$$

Let $U = \begin{bmatrix}\underline{u}_1 & \underline{u}_2 & \underline{u}_3\end{bmatrix}$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1/\sqrt{2} & (i-1)/2 & -1\\ (i+1)/\sqrt{2} & 1 & 0\\ 1/\sqrt{2} & \underline{i-1} & 1\end{bmatrix}$$

Check that

$$U^* A V = \begin{bmatrix} 2\sqrt{3} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = D.$$

$$\Rightarrow \quad \underline{A = UDV^*}.$$