Lecture 12

A quadratic form on $\mathbb{R}^n$ is

$Q(\underline{x}) = \underline{x}^t A \underline{x}$ homogeneous polynomial of degree 2 in the variables $x_1, \dots, x_n$.

We can always assume A is symmetric.

<u>Eg:-</u> $Q\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2x_1^2 - x_1 x_2 + 3x_2^2$

$$= [x_1 \; x_2] \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [x_1 \; x_2] \underbrace{\begin{bmatrix} 2 & -1/2 \\ -1/2 & 3 \end{bmatrix}}_{\text{Symmetric.}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

<u>Real Quadratic forms.</u>

$$Q(\underline{x}) = \underline{x}^t A \underline{x} = \sum_{i,j=1}^{n} x_i x_j a_{ij}, \quad A \text{ is symmetric matrix.}$$

Q is a real quadratic form on $\mathbb{R}^n$.

We know there exists an orthogonal matrix S such that $S^t A S = \text{diagonal}(\lambda_1, \dots, \lambda_n)$

Now $Q(\underline{x}) = \underline{x}^t \, S \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} S^t \underline{x}$.

Let $\underline{y} = S^t \underline{x}$

$$Q(\underline{x}) = \underline{y}^t \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \underline{y}$$

$$= \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 \quad \text{— diagonal form}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A.

$\therefore\ Q(\underline{x}) = Q'(\underline{y})$, Q' is in diagonal form.

i.e., Every quadratic form is equivalent to a diagonal form.

<u>Theorem (Principal axis thm):</u>

Let $Q(\underline{x}) = \underline{x}^t A \underline{x}$ be a real quadratic form on $\mathbb{R}^n$, where A is symmetric. Then Q is equivalent to a diagonal form

$\lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$. (Canonical form).

Qn: how to find a canonical form of a given quadratic form?

<u>Lagrange reduction method.</u>

(i) Let $Q(x_1, x_2, x_3) = 4x_1^2 + 10x_2^2 + 11x_3^2 - 4x_1x_2 + 12x_1x_3 - 12x_2x_3$.

$$= 4\left(x_1^2 - x_1x_2 + 3x_1x_3\right) + 10x_2^2 + 11x_3^2 - 12x_2x_3.$$

$$= 4\left(x_1^2 - \underline{x_1\left(x_2 - 3x_3\right)}\right) + 10x_2^2 + 11x_3^2 - 12x_2x_3.$$

$$= 4\left(x_1 - \frac{1}{2}\left(x_2 - 3x_3\right)\right)^2 - \left(x_2 - 3x_3\right)^2 + 10x_2^2 + 11x_3^2 - 12x_2x_3$$

$$= 4\left(x_1 - \frac{x_2}{2} + \frac{3}{2}x_3\right)^2 + 9x_2^2 - 6x_2x_3 + 2x_3^2$$

$$= 4\left(x_1 - \frac{x_2}{2} + \frac{3}{2}x_3\right)^2 + 9\left(x_2^2 - \frac{2}{3}x_2x_3\right) + 2x_3^2$$

$$= \quad '' \quad + 9\left(x_2 - \frac{x_3}{3}\right)^2 - x_3^2 + 2x_3^2$$

$$= 4\left(x_1 - \frac{x_2}{2} + \frac{3}{2}x_3\right)^2 + 9\left(x_2 - \frac{x_3}{3}\right)^2 + x_3^2$$

$$= 4y_1^2 + 9y_2^2 + y_3^2$$ — diagonal form (need not be unique).

where

$$y_1 = x_1 - \frac{x_2}{2} + \frac{3}{2}x_3$$

$$y_2 = x_2 - \frac{x_3}{2}$$

$$y_3 = x_3.$$

$$\underline{y} = \begin{bmatrix} 1 & -1/2 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix} \underline{x}.$$

Let $Q(\underline{x}) = \underline{x}^t A \underline{x}$, A symmetric on $\mathbb{R}^n$.

<u>Def:-</u> The <u>discriminant</u> of Q is defined as $\det(A)$.

- Q is said to be <u>non-singular</u> if $\det(A) \neq 0$.
- Q is said to be <u>singular</u> if $\det(A) = 0$.

We know

$$Q(\underline{x}) = c_1 y_1^2 + \cdots + c_r y_r^2$$

where all $c_i \neq 0$ $\forall i = 1, 2, \ldots, r$.

$(r \leq n)$.

Rearrange the terms if necessary & write the quadratic form as

$$Q(\underline{x}) = \alpha_1 y_1^2 + \cdots + \alpha_k y_k^2 - \alpha_{k+1} y_{k+1}^2 - \cdots - \alpha_r y_r^2.$$

where all $\alpha_i > 0$ $\forall i = 1, 2, \ldots, r$

Definition:- The number r in the above diagonal form is called the <u>rank</u> of the quadratic form Q.

Def:- The number of +ve terms k in the above diagonal form is called the <u>index</u> of the quadratic form Q.

Def:- The excess number of +ve terms over −ve terms in the above diagonal form is called the <u>signature</u> of Q.

ie signature $(Q) = k - (r-k)$
$= 2k - r.$

where $r =$ rank (Q)
$k =$ index (Q).

Examples:- ① $Q(x) = -y_1^2 - y_2^2 + y_3^2$ in $\mathbb{R}^3$.

rank $(Q) = 3$
index $(Q) = 1$

$$\text{signature}(Q) = 2(1) - 3 = -1.$$

Theorem (Sylvester Law of inertia):

Under all real, non-singular transformation the rank r, the index k (& hence the signature) of a real quadratic form Q are invariants.

Theorem (classification of quadratic forms):

Two real quadratic forms are equivalent

$\Updownarrow$

they have the same rank & index.

Example:- $Q_1(x) = x_1^2 - x_2^2$ in $\mathbb{R}^3$

$Q_2(x) = -x_1^2 + 2x_2^2 - x_3^2$ in $\mathbb{R}^3$.

$\text{rank}(Q_1) = 2$

$\text{rank}(Q_2) = 3$
$\neq \text{rank}(Q_1)$

$\therefore$ Q_1, Q_2 are not equivalent.

Definition:

Let $Q(\underline{x}) = \underline{x}^t A \underline{x}$, '$A$' symmetric matrix on $\mathbb{R}^n$.

(i) Q is said to be <u>positive definite</u> (PD) if $Q(\underline{x}) \geq 0 \ \forall \underline{x} \in \mathbb{R}^n$ & $Q(\underline{x}) > 0 \ \forall \underline{x} \neq \underline{0}$ in $\mathbb{R}^n$.

(ii) Q is said to be <u>negative definite</u> (ND) if $Q(\underline{x}) < 0 \ \forall \underline{x} \neq \underline{0}$ in $\mathbb{R}^n$.

(iii) Q is said to be <u>positive semi-definite</u> (PSD) if $Q(\underline{x}) \geq 0 \ \forall \underline{x} \in \mathbb{R}^n$ & $Q(\underline{x}) = 0$ for some $\underline{x} \neq \underline{0}$ in $\mathbb{R}^n$.

(iv) Q is said to be <u>negative semi-definite</u> (NSD) if $Q(\underline{x}) \leq 0 \ \forall \underline{x} \in \mathbb{R}^n$ & $Q(\underline{x}) = 0$ for some $\underline{x} \neq \underline{0}$ in $\mathbb{R}^n$.

(v) Q is said to be <u>indefinite</u> (ID) if $Q(\underline{x}) \gneq 0$ for some $\underline{x} \neq \underline{0}$ in $\mathbb{R}^n$ & $Q(\underline{x}) \lneq 0$ for some $\underline{x} \neq \underline{0}$ in $\mathbb{R}^n$.

Defn: A real symmetric matrix $A_{n\times n}$ is said to be <u>+ve definite matrix</u> (or −ve def etc) if the corresponding quadratic form $Q(\underline{x}) = \underline{x}^t A \underline{x}$ is +ve definite (or −ve def. etc).

Examples:-

① $Q(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^2 \geq 0$

& $Q(\underline{x}) > 0 \quad \forall\, \underline{x} \neq \underline{0}$.

Q is +ve definite.

② $Q(x_1, x_2, x_3) = x_1^2 + 2x_2^2 \geq 0 \quad \forall\, \underline{x} \in \mathbb{R}^3$

& $Q(\underbrace{0, 0, 1}_{\neq \underline{0}}) = 0$

Q is +ve semi-definite.

③ $Q(x_1, x_2, x_3) = -x_1^2 - 2x_2^2 - 5x_3^2 \leq 0 \quad \forall\, \underline{x}$

& $Q(\underline{x}) < 0 \quad \forall\, \underline{x} \neq \underline{0}$

$\therefore Q$ is −ve definite.

④

(4) $Q(x_1, x_2, x_3) = -x_1^2 - 3x_3^2 \leq 0 \quad \forall \underline{x}$

& $Q(\underbrace{0, 1, 0}_{\neq \underline{0}}) = 0$

Q is $-$ve semi-definite.

(5) $Q(x_1, x_2, x_3) = -x_1^2 + x_2^2 + 3x_3^2$.

$Q(\underbrace{1, 0, 0}_{\neq \underline{0}}) = -1 < 0$

& $Q(\underbrace{0, 1, 1}_{\neq \underline{0}}) = 4 > 0$.

Q is indefinite.

(6) $Q(x_1, x_2, x_3) = -x_1^2 + 2x_2^2$ indefinite.

Theorem: Let $Q(\underline{x}) = \underline{x}^t A \underline{x}$, $A = A^t$ on $\mathbb{R}^n$.
Let Q has rank r, index k & signature s.

Then

(i) Q is $+$ve def. $\Longleftrightarrow$ $r = s = n$.

(ii) Q is $+$ve semi-def. $\Longleftrightarrow$ $r = s < n$.

(iii) Q is $-$ve def. $\iff$ $r = -s = n$.

(iv) Q is $-$ve semi-def $\iff$ $r = -s < n$.

(v) Q is indef. $\iff$ $|s| < r$.

Example: (i) $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + 3x_3^2$.

$n = 3$

$$\text{rank}(Q) = 3 = r$$
$$\text{index}(Q) = 3 = k$$
$$\text{signature}(Q) = 3 = s$$

(2) $Q(x_1, x_2, x_3) = -x_1^2 \cancel{+} x_2^2$

$n = 3$
$r = 2$
$k = 0$
$s = 2k - r$
$= -2$.

$\therefore$ $r = -s < n$.

$\therefore$ Q is $-$ve semi-def.

Theorem: Let $Q(\underline{x}) = \underline{x}^t A \underline{x}$ on $\mathbb{R}^n$, $A = A^t$.

Suppose Q is $+$ve definite. Then

(a) $\det(A) > 0$

(b) every principal minors of A is $+$ve

(c) $a_{ii} > 0$ $\forall i = 1, \ldots, n$, when $A = [a_{ij}]_{n \times n}$

Theorem: $Q(\underline{x}) = \underline{x}^t A \underline{x}$ on $\mathbb{R}^n$.

If Q is PSD, then

(a) $\det(A) = 0$.

(b) every principal minor of A is non-ve.

(c) $a_{ii} > 0$ if x_i^2 appears in $Q(\underline{x})$.

Thm (Sylvester Criterion for +ve definiteness of real quadratic forms).

Let $Q(\underline{x}) = \underline{x}^t A \underline{x}$ on $\mathbb{R}^n$, $A = A^t$. Then

Q is +ve def. $\iff$ all the leading principal minors of A are +ve.

i.e. $a_{11} > 0$, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$, $\cdots$ $\det(A) > 0$.

i.e. $\underbrace{|A(1|1)|}_{1}$, $\underbrace{|A(1,2|1,2)|}_{2}$, $\cdots$, $\underbrace{|A(1,\dots,n-1|1,\dots,n-1)|}_{n-1}$, $\underbrace{|A|}_{n}$

Example:

(1) $Q(x_1, x_2, x_3) = \underline{x}^t \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 5 \\ -1 & 5 & 2 \end{bmatrix} \underline{x}$.

$\underbrace{|A(i_1,\dots i_r|i_1 \dots i_r)|}_{\text{principals}}$

1, $\begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} = 3$, $\det(A) = 1(-4) - 1(8)$
$= -1 < 0$.

Q is not +ve def.

Theorem:- A real quadratic form $Q(\underline{x}) = \underline{x}^t A \underline{x}$ on $\mathbb{R}^n$, $A = A^t$, is $-$ve definite

$$\Updownarrow$$

All the principal minors of A of even order (size) are $+$ve & those of odd order (size) are $-$ve.

$\boxed{A_{n\times n} \text{ order } n}$

Theorem:- $Q(\underline{x}) = \underline{x}^t A \underline{x}$, is $+$ve semi-definite.

$$\Updownarrow$$

A is singular & all its principal minors are non-$-$ve.

Theorem!- $Q(\underline{x}) = \underline{x}^t A \underline{x}$ is $-$ve semi-definite

$$\Updownarrow$$

A is singular & all its principal minors are ≤ 0.

Theorem: $Q(\underline{x}) = \underline{x}^t A \underline{x}$ is indefinite

$\Updownarrow$

the following hold

(a) A has a −ve principal minor of even order

&

(b) A has a +ve principal minor of odd order & a −ve principal minor of odd order.

(equivalent to none of above conditions for PD, PSD, ND, NSD ?)

Example:

$$Q(\underline{x}) = \underline{x}^t \begin{bmatrix} 2 & 1/2 & -3/2 \\ 1/2 & 5 & 3/2 \\ -3/2 & 3/2 & 1 \end{bmatrix} \underline{x}.$$

principal minors of order 1: 2, 5, 1.

principal minors of order 2: $|A(1,2|1,2)|$, $|A(1,3|1,3)|$, $|A(2,3|2,3)|$

: $\frac{39}{4}$, $\frac{-1}{4}$, $\frac{1}{4}$.

principal minors of order 3: $\det(A) = -33/4$.

$\therefore$ Q is indefinite.

Theorem $Q(\underline{x}) = \underline{x}^t A \underline{x}$, $A = A^t$.

(a) Q is +ve definite $\iff$ all the eigenvalues of A are +ve

(b) Q is −ve def. $\iff$ all the eigenvalues of A are −ve.

(c) Q is +ve semi-definite $\iff$ all the eigenvalues of A are non-ve & at least one zero eigenvalue.

(d) Q is −ve semi-definite $\iff$ all the eigenvalues of A are ≤ 0 & at least one zero eigenvalue.

(e) Q is indefinite $\iff$ A has atleast one +ve eigenvalue & at least one −ve eigenvalue.