Linear Mapping (Linear Transformation)

Let X and Y be any two vector space. A mapping $F: X \to Y$ is called a *linear mapping* or *linear transformation* if it satisfies the following two conditions:

i) for any two vectors $u, v \in X$, $F(u+v) = F(u) + F(v)$

ii) for any scalar K and vector $u \in X$, $F(Ku) = K F(u)$

Remark: i) Note that for $K=0$: $F(0) = 0$.

Thus every linear mapping takes the zero vector into the zero vector.

ii) The two conditions above can be combined into one:

$$F(k_1 u + k_2 v) = k_1 F(u) + k_2 F(v)$$

Example: Let $F: \mathbb{R}^3 \to \mathbb{R}^3$ with

$$F(x, y, z) = (x, y, 0)$$

Let $u = (a, b, c)$ $v = (a', b', c')$

then
$$\begin{aligned} F(u+v) &= F(a+a', b+b', c+c') \\ &= (a+a', b+b', 0) \\ &= (a, b, 0) + (a', b', 0) \\ &= F(u) + F(v) \end{aligned}$$

For any constant K.

$$F(Ku) = F(Ka, Kb, Kc) = (Ka, Kb, 0) = K(a, b, 0)$$
$$= K F(u)$$

$\Rightarrow$ F is linear.

Exomple: Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ with $F(x,y) = (x+1, y+2)$

Check wheter F is linear or not?

It is not linear since $F(0,0) = (1,2) \neq (0,0)$

Matrices as linear Mapping (transformation)

Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$.

Then any real $m \times n$ matrices A gives a transformation of $\mathbb{R}^n$ into $\mathbb{R}^m$,

$$y = Ax \quad , \quad x \in \mathbb{R}^n, \, y \in \mathbb{R}^m$$

Since $A(u+v) = Au + Av$

& $A(\lambda u) = \lambda A u$

$\Rightarrow$ This is a linear transformation.

Kernel and Image of a linear mapping.

Let $F: X \to Y$ be linear mapping

$$\operatorname{Ker} F = \{x \in X : F(x) = 0\}$$

$$\operatorname{Im} F = \{y \in Y : \text{there exists } x \in X \text{ for which } F(x) = y\}$$

Theorem: Let $F: X \to Y$ be a linear mapping. Then the kernel of F is a subspace of X and image of F is a subspace of Y.

Theorem: (Imp) Suppose $x_1, x_2, \cdots, x_m$ span a vector space X and suppose $F: X \to Y$ is linear. Then $F(x_1), F(x_2) \cdots F(x_m)$ span $\operatorname{Im} F$.

Idea: Let $y \in \operatorname{Im} F$. Then $\exists\, x \in X : F(x) = y$.

Also $x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m$

Therefore, $y = F(x) = \alpha_1 F(x_1) + \alpha_2 F(x_2) + \cdots + \alpha_m F(x_m)$

$\Rightarrow$ The vectors $F(x_1), F(x_2), \cdots, F(x_m)$ Span $\operatorname{Im} F$.

Example: $F(x,y,z) = (x,y,0)$

$$\operatorname{Im} F = \{(a,b,c) : c = 0\} = xy \text{ plane}$$

$$\operatorname{Ker} F = \{(a,b,c) : a = 0, b = 0\} = z\text{-axis}$$

KERNEL & IMAGE OF MATRIX MAPPING

Consider $A : \mathbb{R}^4 \to \mathbb{R}^3$ with

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix}$$

Take usual basis $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ of $\mathbb{R}^4$.

Then Ae_1, Ae_2, Ae_3, Ae_4 spans the image of A.

$\Rightarrow Ae_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}, Ae_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}, Ae_3 = \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix}, Ae_4 = \begin{bmatrix} a_4 \\ b_4 \\ c_4 \end{bmatrix}.$

Thus the image of A is precisely the column space of A.

The kernel of A consists all vectors x for which $Ax = 0$

$\Rightarrow$ The kernel of A is precisely the NULL SPACE of A.

Rank and Nullity of a linear Mapping:

Let $F: X \to Y$ be a linear mapping, then

$$\text{rank}(F) = \dim(\text{Im}\, F)$$

$$\text{nullity}(F) = \dim(\ker F)$$

Theorem: Let X be a vector space of finite dimension and let $F: X \to Y$ be a linear map. Then

$$\text{rank}(F) + \text{nullity}(F) = \dim X$$

Example: Let $F: \mathbb{R}^4 \to \mathbb{R}^3$ be a linear mapping defined by

$$F(x,y,z,t) = (x-y+z+t,\ 2x-2y+3z+4t,\ 3x-3y+4z+5t)$$

Find a basis and dimension of

a) the image of F

b) Kernel of F.

Sol: We know that the vectors

$F(1,0,0,0) = (1,2,3)$, $F(0,1,0,0) = (-1,-2,-3)$

$F(0,0,1,0) = (1,3,4)$, $F(0,0,0,1) = (1,4,5)$

Span Im F.

To find basis:

consider
$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $(1,2,3)$ & $(0,1,1)$ form a basis of $\operatorname{Im} F$. and $\dim(\operatorname{Im} F) = 2$

b) Kernel of F:

Set $F(x,y,z,t) = 0$

$$\left.\begin{aligned} \Rightarrow\ & x - y + z + t = 0 \\ & 2x - 2y + 3z + 4t = 0 \\ & 3x - 3y + 4z + 5t = 0 \end{aligned}\right\} \text{Null space is the kernel of } F.$$

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -2 & 3 & 4 \\ 3 & -3 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{array}{c} \begin{array}{cccc} x & & z & \end{array} \\ \begin{bmatrix} \boxed{1} & -1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} & y & & t \end{array} \end{array}$$

Free variables y & t

$\dim(\ker F) = \text{nullity}(F) = 2$

let $t = \alpha_1$ and $y = \alpha_2$, $z = -2\alpha_1$, $x = -\alpha_1 + 2\alpha_1 + \alpha_2$
$= \alpha_1 + \alpha_2$

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus $(1,0,-2,1)$ & $(1,1,0,0)$ form a basis of $\ker F$.

Example: Let $u = (1,1,3)$, $v = (3,2,-2)$

$L(u) = (4,1,1,1)$ and $L(v) = (-5,1,-3,3)$.

Assume further that L is a linear transformation from $\mathbb{R}^3 \to \mathbb{R}^4$. If $w = (5,4,4)$ and $y = (2,1,7)$, find $L(w)$ and $L(y)$, if possible.

Sol: i) express w as L.C. of u & v, i.e.,

$$\lambda_1 u + \lambda_2 v = w$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 1 & 2 & 4 \\ 3 & -2 & 4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & -11 & -11 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \lambda_2 = 1, \quad \lambda_1 = 2$$

$$\Rightarrow 2u + v = w$$

$$\Rightarrow L(w) = 2L(u) + L(v) = 2\cdot(4,1,1,1) + (-5,1,-3,3)$$

$$= (3,3,-1,5)$$

ii) Express y as L.C. of u & v, i.e.,

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 1 & 2 & 1 \\ 3 & -2 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & -11 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 12 \end{array}\right]$$

INCONSISTENT.

$\Rightarrow$ y cannot be expressed as L.C. of u & v.

Thus, we cannot compute $L(y)$ from the information given.

Th. Let T be a linear map from $\mathbb{R}^n \to \mathbb{R}^m$. Then, there is an $m \times n$ matrix A such that $T(x) = Ax$, for all x in $\mathbb{R}^n$.

Proof.: Define standard basis in $\mathbb{R}^n$:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Then, any x in $\mathbb{R}^n$ can be written as

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$
$$= \sum_{j=1}^{n} x_j e_j$$

Linearity of T implies

$$T(x) = T\left(\sum_{j=1}^{n} x_j e_j\right) = \sum_{j=1}^{n} x_j T(e_j)$$

Let A be $m \times n$ matrix whose columns are $T(e_j)$, $j = 1, 2, \cdots, n$.

Then, $T(x) = Ax$ gives the mapping.

Example: Consider $T: \mathbb{R}^2 \to \mathbb{R}^3$ given as

$$T(x, y) = (2x + 3y, -x + 5y, 4x - 3y)$$

$$T(e_1) = (2, -1, 4) \qquad T(e_2) = (3, 5, -3)$$

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 5 \\ 4 & -3 \end{bmatrix}.$$ Note that $T(x, y) = A\begin{bmatrix} x \\ y \end{bmatrix}$.

Coordinates: Let V be an n-dimensional vector space (over $\mathbb{R}$) with ordered basis $S = \{u_1, u_2, \cdots u_n\}$. Then any vector $v \in V$ can be expressed uniquely as a linear comb. of the basis vectors in S, say,

$$v = \lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n.$$

These n scalars $\lambda_1, \lambda_2, \cdots \lambda_n$ are called the coordinates of v relative to the basis S.

Notation: $[v]_S = [\lambda_1, \lambda_2, \cdots \lambda_n]^T$.

Example: Consider a basis $u_1 = (1,-1,0)$, $u_2 = (1,1,0)$, $u_3 = (0,1,1)$ of $\mathbb{R}^3$. Find the coordinates of $v = (5,3,4)$ relative to the basis $\{u_1, u_2, u_3\} = B$.

$$v = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3$$

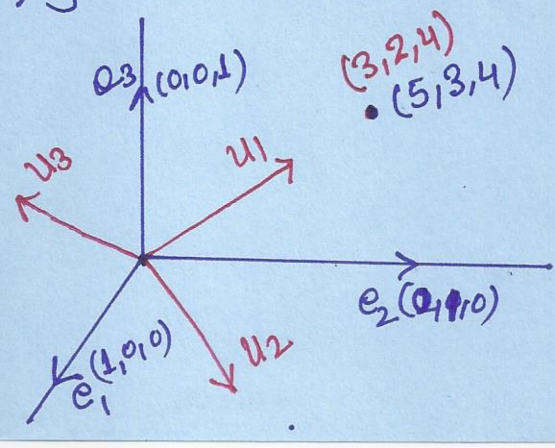
$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ -1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 1 & 0 & 5 \\ 0 & \boxed{2} & 1 & 8 \\ 0 & 0 & \boxed{1} & 4 \end{array}\right]$$

$\Rightarrow \lambda_3 = 4, \ \lambda_2 = 2, \ \lambda_1 = 3$

$[5,3,4]_B = [3,2,4]$.

Let $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ standard basis.

Then $[5,3,4]_S = [5,3,4]$



MATRIX REPRESENTATION OF A LINEAR MAP.

Let $T: V \to W$ be a linear transformation from an n-dimensional vector space to an m-dimensional vector space W.

Let $\alpha = (v_1, v_2, \cdots, v_n)$ ordered basis of V.

$\beta = (w_1, w_2, \cdots, w_m)$ ordered basis of W.

Consider,

$$\underbrace{T(v_1)}_{\in W} = a_{11} w_1 + a_{21} w_2 + \cdots + a_{m1} w_m$$

$$T(v_2) = a_{12} w_1 + a_{22} w_2 + \cdots + a_{m2} w_m$$

$$\vdots$$

$$T(v_n) = a_{1n} w_1 + a_{2n} w_2 + \cdots + a_{mn} w_m$$

In short,

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i \quad , \quad j = 1, 2, \cdots n.$$

for some scalars, a_{ij} $(i = 1, 2 \cdots m,\ j = 1, 2, \cdots, n)$.

For any vector $x \in V$, let

$$x = \sum_{j=1}^{n} x_j v_j$$

Consider

$$T(x) = \sum_{j=1}^{n} x_j \, T(v_j)$$

$$= \sum_{j=1}^{n} x_j \sum_{i=1}^{m} a_{ij} \, w_i$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) w_i$$

$$= \left(\sum_{j=1}^{n} a_{1j} x_j \right) w_1 + \left(\sum_{j=1}^{n} a_{2j} x_j \right) w_2 + \cdots + \left(\sum_{j=1}^{n} a_{mj} x_j \right) w_m$$

Note that

$$[T(x)]_\beta = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \sum_{j=1}^{n} a_{2j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} x_j \end{bmatrix}$$

coordinate vector of $T(x)$ w.r.t. β.

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= A \, [x]_\alpha$$

Hence,

$$[T(x)]_\beta = A[x]_\alpha$$

That is, for <u>any</u> $x \in V$, the coordinate vector $[T(x)]_\beta$ of $T(x)$ in W is just the product of a fixed matrix A and the coordinate vector $[x]_\alpha$ of x.

Note that

$$A = \begin{bmatrix} a_{11}, & a_{12}, & \cdots, & a_{1n} \\ a_{21}, & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ [T(v_1)]_\beta & [T(v_2)]_\beta & \cdots & [T(v_n)]_\beta \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

Columns are coordinate vectors of $T(v_j)$.

<u>Def</u>: The matrix A is called the associated matrix for T (or the matrix representation of T) with respect to the ordered bases α and β and is denoted by $A = [T]_\alpha^\beta$.

When $V = W$ and $\alpha = \beta$ we simply write $[T]_\alpha$.

$$\boxed{[T(x)]_\beta = [T]_\alpha^\beta \, [x]_\alpha}$$