Beta & Gamma Function

Definition:

Beta function:

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad , \quad m>0, n>0$$

Gamma function:

$$\overline{|n} = \int_0^\infty e^{-x} x^{n-1}\,dx \quad , \quad n>0.$$

Convergence of Beta function:

Case-I: $m, n \geq 1$, the integral is proper. Hence it is convergent.

Case-II: $m, n < 1$.

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \underbrace{\int_0^c x^{m-1}(1-x)^{n-1}\,dx}_{I_1} + \underbrace{\int_c^1 x^{m-1}(1-x)^{n-1}\,dx}_{I_2}$$

where $0<c<1$.

Consider

$$I_1 = \int_0^c x^{m-1}(1-x)^{n-1}\,dx$$

Then.

$$\lim_{x\to 0+} x^{\mu}\, x^{m-1}(1-x)^{n-1} = \lim_{x\to 0} x^{\mu+m-1}(1-x)^{n-1}$$

$$= 1 \quad \text{if} \quad \mu+m-1 = 0$$

$$\Rightarrow \mu = -m+1.$$

If $0<m<1$, then $0<\mu<1$ and hence the integral converges.

If $m \leq 0$ then $\mu \geq 1$ and hence the integral diverges.

Similarly: consider $I_2 = \int_c^1 x^{m-1} (1-x)^{n-1} dx$

$$\lim_{x \to 1} (1-x)^{\mu} \cdot x^{m-1} (1-x)^{n-1} = \lim_{x \to 1} x^{m-1} (1-x)^{\mu+n-1}$$

If $0<n<1$, the integral converges

If $n \leq 0$, the integral diverges.

Therefore

$\int_0^1 x^{m-1} (1-x)^{n-1} dx$ converges if both m & $n > 0$.

otherwise it is divergent.

Convergence of Gamma function:

$$\overline{|n} = \int_0^{\infty} x^{n-1} e^{-x} dx$$

Case I: $n \geq 1$

The integrand is bounded in $0 < x \leq a$, where a is arbitrary.

We check convergence of $\int_a^{\infty} x^{n-1} e^{-x} dx$

Consider $\lim\limits_{x \to \infty} x^{\mu} f(x) = \lim\limits_{x \to \infty} \frac{x^{\mu} \cdot x^{n-1}}{e^{x}}$

$= 0$ for all values of μ and n.

Using μ test $(\mu > 1)$, the integral $\int_a^{\infty} x^{n-1} e^{-x} dx$ is convergent for all values of n.

Case II: Let $0<n<1$: Then

$$\int_0^\infty e^{-x} x^{n-1}\, dx = \int_0^a e^{-x}\cdot x^{n-1}\, dx + \underbrace{\int_a^\infty e^{-x} x^{n-1}\, dx}_{\text{converges (see above)}}$$

Note that $\lim_{x\to 0} x^{\mu} x^{n-1} e^{-x} = 1$ if $\mu+n-1=0$, i.e.,

if $\mu = 1-n$

Since n lies between 0 & 1, μ also lies between 0 & 1.

Hence $\int_0^a e^{-x} x^{n-1}\, dx$ is convergent.

Therefore the integral converges for $0<n<1$.

Case III Let $n \leq 0$.

$$\lim_{x\to 0} x^{\mu} x^{n-1} e^{-x} \qquad \text{Take } \mu = 1:$$

$$\lim_{x\to 0} x^{n} e^{-x} = \begin{cases} 1, & n=0 \\ \infty & n<0 \end{cases}$$

$\Rightarrow \int_0^a e^{-x} x^{n-1}\, dx$ diverges.

PROPERTIES OF BETA and GAMMA function:

a) $B(m,n) = B(n,m)$

Subst. $1-x=y$ ..

b) Evaluation of $B(m,n)$

Suppose n is a positive integer.

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Integrating by parts keeping $(1-x)^{n-1}$ as first function.

$$B(m,n) = \left[\frac{x^m}{m}(1-x)^{n-1}\right]_0^1 + \int_0^1 \frac{x^m}{m}(n-1)(1-x)^{n-2}\,dx$$

$$= \frac{(n-1)}{m}\int_0^1 x^m (1-x)^{n-2}\,dx$$

$$\vdots$$

$$= \frac{(n-1)(n-2)\cdots 1^{=(n-(n-1))}}{m(m+1)\cdots(m+n-2)}\int_0^1 x^{m+n-2}\,dx$$

$$= \frac{\lfloor(n-1)}{m(m+1)\cdots(m+n-2)(m+n-1)}$$

If m is a positive integer

$$B(m,n) = \frac{\lfloor(m-1)}{n(n+1)\cdots(n+m-1)}$$

If both m and n are integer

$$B(m,n) = \frac{\lfloor(n-1)\,\lfloor(m-1)}{\lfloor m+n-1}$$

c) Evaluation of Gamma function:

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\, dx$$

integrating by parts:

$$= -x^n e^{-x}\Big|_0^\infty + \int_0^\infty n x^{n-1} e^{-x}\, dx$$

$$\boxed{\Gamma(n+1) = n\,\Gamma(n)}$$

Note that if n is an integer

$$\Gamma(n) = (n-1)(n-2)\cdots 3\cdot 2\cdot \Gamma(1)$$

where $\Gamma(1) = \int_0^\infty e^{-x}\, dx = -e^{-x}\Big|_0^\infty = 1.$

$$\Rightarrow \boxed{\Gamma(n) = \lfloor(n-1)\text{, if } n \text{ is a positive int.}}$$

d) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx$$

Subst. $x = y^2 \Rightarrow dx = 2y\, dy$

$$\Rightarrow \Gamma(n) = \int_0^\infty y^{2n-1} e^{-y^2}\, 2\, dy$$

Set $n = \frac{1}{2} \Rightarrow \Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty y^0 e^{-y^2}\, dy$

$$= 2\int_0^\infty e^{-y^2}\, dy = 2\cdot\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

$$\Rightarrow \boxed{\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}}$$

Different forms of Γn :

a) $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$

Subst. $x = \lambda y \Rightarrow dx = \lambda dy$

$$\Rightarrow \Gamma n = \int_0^\infty e^{-\lambda y} \cdot \lambda^{n-1} y^{n-1} \lambda\, dy$$

$$\Rightarrow \boxed{\int_0^\infty e^{-\lambda y}\, y^{n-1}\, dy = \frac{\Gamma n}{\lambda^n}}$$

b) Subst. $x^n = z \Rightarrow n x^{n-1} dx = dz$

$$\Rightarrow \Gamma n = \int_0^\infty e^{-z^{1/n}} \frac{1}{n} dz \Rightarrow \boxed{\int_0^\infty e^{-z^{1/n}}\, dz = n\Gamma n = \Gamma n+1}$$

c) Subst $e^{-x} = t \Rightarrow -e^{-x} dx = dt$

$$\Rightarrow \Gamma n = -\int_1^0 \left[\ln\left(\frac{1}{t}\right)\right]^{n-1} dt$$

$$\Rightarrow \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{n-1} dt = \Gamma n$$

Different forms of Beta function:

a) $B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$

Subst. $x = \frac{1}{1+y} \Rightarrow dx = -\frac{1}{(1+y)^2} dy$

$$\boxed{B(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy}$$

b) $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, \cos^{2n-1}\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2n-1}\theta \, \cos^{2m-1}\theta \, d\theta$$

Relation between Gamma & Beta function:

We know for m and n being integers

$$B(m,n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$$

$$\Rightarrow \boxed{B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}}$$

This result also holds for $m, n > 0$ (not necessary only for integers)

Some other deductions:

1. $\boxed{\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}}$ $0 < n < 1$.

We know $B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$, putting $m = 1-n$

$$\Rightarrow B(1-n, n) = \Gamma(1-n)\,\Gamma(n) = \int_0^\infty \frac{y^{n-1}}{(1+y)}\, dy = \frac{\pi}{\sin n\pi} \text{ if } 0<n<1$$

(Complicated, Residue th. comp. ana.)

2. $\boxed{\Gamma(n+1)\,\Gamma(1-n) = \frac{n\pi}{\sin n\pi}}$

3. Put $n = 1/2$ in 1. $\Rightarrow \Gamma\left(\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}\right) = \pi$

$$\Rightarrow \boxed{\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}}$$

4. We know

$$B(m,n) = 2\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta \, d\theta = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

Let $2m-1 = p$ & $2n-1 = q$

$$\Rightarrow \boxed{\int_0^{\pi/2} \cos^p\theta \sin^q\theta \, d\theta = \frac{\Gamma\frac{(p+1)}{2} \, \Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}}$$

Let $p=0$ then

$$\boxed{\int_0^{\pi/2} \sin^q\theta \, d\theta = \frac{\Gamma\frac{q+1}{2}}{\Gamma\frac{q+2}{2}} \cdot \frac{\sqrt{\pi}}{2}}$$

Let $q=0$ then

$$\boxed{\int_0^{\pi/2} \cos^p\theta \, d\theta = \frac{\Gamma\frac{p+1}{2}}{\Gamma\frac{p+2}{2}} \cdot \frac{\sqrt{\pi}}{2}}$$

Let $p=0$, $q=0$ then

$$\frac{\pi}{2} = \frac{1}{2}\left(\Gamma\frac{1}{2}\right)^2 \Rightarrow \boxed{\Gamma\frac{1}{2} = \sqrt{\pi}}$$

5. $B(m,n) = B(m+1,n) + B(m,n+1)$

$$\text{R.H.S} = \frac{\Gamma m+1 \, \Gamma n}{\Gamma m+n+1} + \frac{\Gamma m \, \Gamma n+1}{\Gamma m+n+1}$$

$$= \frac{\Gamma m \, \Gamma n}{\Gamma m+n+1}(m+n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n} = B(m,n).$$

Example - 1: Evaluate

$$\int_0^1 x^4 (1-\sqrt{x})^5 \, dx$$

let $\sqrt{x} = t$ or $x = t^2 \Rightarrow dx = 2t\,dt$

$$\int_0^1 t^8 (1-t)^5 \, 2t \, dt$$

$$= 2\int_0^1 t^9 (1-t)^5 \, dt$$

$$= 2 \cdot B(10,6) = 2 \cdot \frac{\Gamma 10 \, \Gamma 6}{\Gamma 16} = 2 \cdot \frac{\underline{|9} \, \underline{|5}}{\underline{|15}} = \frac{1}{15015}$$

Example 2. Show that $\int_0^{\pi/2} (\cot\theta)^{1/2} d\theta = \frac{\pi}{\sqrt{2}}$

$$I = \int_0^{\pi/2} (\cot\theta)^{1/2} d\theta = \int_0^{\pi/2} \cos^{1/2}\theta \, \sin^{-1/2}\theta \, d\theta$$

$$= \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right) \Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{2\,\Gamma\left(\frac{-\frac{1}{2}+\frac{1}{2}+2}{2}\right)} = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{2}$$

$$= \frac{1}{2} \Gamma\left(\frac{1}{4}\right) \Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{1}{2} \cdot \frac{\pi}{\sin(\pi/4)}$$

$$= \frac{\pi}{\sqrt{2}}.$$

Differentiation under integral sign (Leibnitz Rule)

Let $\Phi(\alpha) = \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha)\,dx$

$$\Delta\Phi = \Phi(\alpha+\Delta\alpha) - \Phi(\alpha)$$

$$= \int_{u_1(\alpha+\Delta\alpha)}^{u_2(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\,dx - \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha)\,dx$$

$$= \int_{u_1(\alpha+\Delta\alpha)}^{u_1(\alpha)} f(x,\alpha+\Delta\alpha)\,dx + \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha+\Delta\alpha)\,dx$$

$$+ \int_{u_2(\alpha)}^{u_2(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\,dx - \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha)\,dx$$

$$= \int_{u_1(\alpha)}^{u_2(\alpha)} \left[f(x,\alpha+\Delta\alpha) - f(x,\alpha)\right] dx + \int_{u_2(\alpha)}^{u_2(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\,dx$$

$$- \int_{u_1(\alpha)}^{u_1(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\,dx$$

Using mean value theorem:

$$\int_{u_1(\alpha)}^{u_2(\alpha)} \left[f(x,\alpha+\Delta\alpha) - f(x,\alpha)\right] dx = \Delta\alpha \int_{u_1(\alpha)}^{u_2(\alpha)} f_\alpha(x,\xi_1)\,dx$$

$$\int_{u_2(\alpha)}^{u_2(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\,dx = f(\xi_2,\alpha+\Delta\alpha)\left[u_2(\alpha+\Delta\alpha) - u_2(\alpha)\right]$$

$$\int_{u_1(\alpha)}^{u_1(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\,dx = f(\xi_3,\alpha+\Delta\alpha)\left[u_1(\alpha+\Delta\alpha) - u_1(\alpha)\right]$$

Where $\xi_1 \in (\alpha, \alpha+\Delta\alpha)$, $\xi_2 \in (u_2(\alpha), u_2(\alpha+\Delta\alpha))$, $\xi_3 \in (u_1(\alpha), u_1(\alpha+\Delta\alpha))$

Dividing by $\Delta\alpha$:

$$\frac{\Delta\phi}{\Delta\alpha} = \int_{u_1(\alpha)}^{u_2(\alpha)} f_\alpha(x,\xi_1)\,dx + f(\xi_2,\alpha+\Delta\alpha)\frac{\Delta u_2}{\Delta\alpha} - f(\xi_3,\alpha+\Delta\alpha)\frac{\Delta u_1}{\Delta\alpha}$$

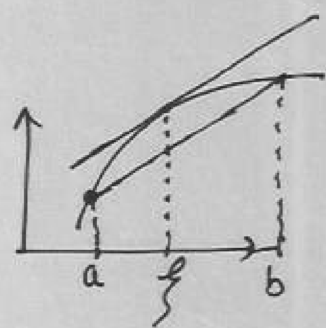
Taking the limit as $\Delta\alpha \to 0$,

$$\frac{d\phi}{d\alpha} = \int_{u_1(\alpha)}^{u_2(\alpha)} f_\alpha(x,\alpha)\,dx + f(u_2(\alpha),\alpha)\frac{du_2}{d\alpha} - f(u_1(\alpha),\alpha)\frac{du_1}{d\alpha}$$

Note: We have used the following mean value theorems in the above proof.

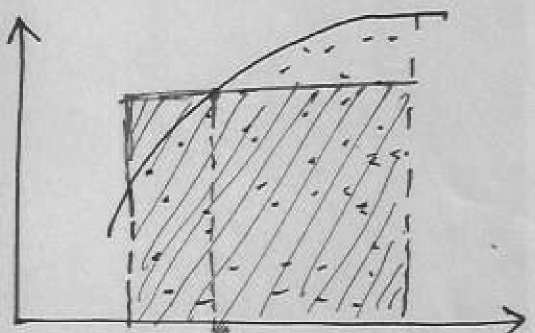
I. Lagrange mean value theorem:

$$\frac{f(b)-f(a)}{b-a} = f'(\xi) \quad ; \quad \xi \in (a,b)$$



II. Mean value theorem of the integral calculus:

$$\int_a^b f(x)\,dx = (b-a)\,f(\xi_1) \quad ; \quad \xi_1 \in (a,b)$$



A Particular Case: Assume that $u_1(\alpha)$ and $u_2(\alpha)$ are some constants. Then,

$$\frac{d\Phi(\alpha)}{d\alpha} = \int_a^b \frac{\partial f}{\partial \alpha}(x,\alpha)\, dx$$

OR

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)\, dx = \int_a^b \frac{\partial f}{\partial \alpha}(x,\alpha)\, dx.$$

Note: Leibnitz rule is not applicable, in general, in the case of improper integrals. In all examples given in this lecture we assume that differentiation under integral sign is valid.

Example: Show that

$$\int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}\, dx = \frac{\pi}{2}\ln(1+a) \text{ if } a \geq 0.$$

Let

$$\varphi(a) = \int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}\, dx$$

$$\Rightarrow \varphi'(a) = \int_0^\infty \frac{1}{(1+x^2)(1+a^2x^2)}\, dx$$

$$= \int_0^\infty \frac{1}{(1-a^2)}\left[\frac{1}{1+x^2} - \frac{a^2}{1+a^2x^2}\right] dx$$

$$= \frac{1}{(1-a^2)}\left[\tan^{-1} x - a\tan^{-1} ax\right]_0^\infty = \frac{1}{(1-a^2)}\frac{\pi}{2}(1-a)$$

$$\Rightarrow \varphi'(a) = \frac{\pi}{2(1+a)}$$

Integrading

$$\varphi(a) = \frac{\pi}{2} \ln(1+a) + c$$

Note that $\varphi(0) = 0$

$$\Rightarrow 0 = \frac{\pi}{2} \ln(1) + c \Rightarrow c = 0$$

$$\Rightarrow \varphi(a) = \frac{\pi}{2} \ln(1+a)$$

Example: Prove $\int_0^\infty e^{-x^2} \cos \alpha x \, dx = \frac{\sqrt{\pi}}{2} e^{-\frac{\alpha^2}{4}}$

$$\varphi(\alpha) = \int_0^\infty e^{-x^2} \cos \alpha x \, dx$$

$$\varphi'(\alpha) = -\int_0^\infty e^{-x^2} \sin \alpha x \cdot x \, dx$$

Integrating right hand side by parts

$$\varphi'(\alpha) = \frac{e^{-x^2}}{2} \sin \alpha x \Big|_0^\infty + \int_0^\infty \left(-\frac{e^{-x^2}}{2}\right) \cos \alpha x \cdot \alpha \cdot dx$$

$$= -\frac{\alpha}{2} \varphi(\alpha)$$

$$\Rightarrow \frac{\varphi'(\alpha)}{\varphi(\alpha)} = -\frac{\alpha}{2} \Rightarrow \ln \varphi(\alpha) = -\frac{\alpha^2}{4} + c$$

$$\Rightarrow \varphi(\alpha) = c_1 e^{-\alpha^2/4}$$

Note that $\varphi(0) = \int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$

$$\Rightarrow \sqrt{\pi}/2 = c_1$$

$$\Rightarrow \int_0^\infty e^{-x^2} \cos \alpha x \, dx = \frac{\sqrt{\pi}}{2} e^{-\alpha^2/4}.$$

Example: Starting with a suitable integral, show that

$$\int_0^x \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right) + \frac{x}{2a^2(x^2+a^2)}$$

Solution: Consider $\varphi(a,x) = \int_0^x \frac{dx}{(x^2+a^2)} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\Big|_0^x$

$$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

Diff. w.r.t a:

$$\frac{\partial\varphi}{\partial a} = \int_0^x -\frac{1}{(x^2+a^2)^2}\cdot 2a\,dx = \frac{1}{a}\frac{1}{\left(1+\frac{x^2}{a^2}\right)}\left(-\frac{x}{a^2}\right) - \frac{1}{a^2}\tan^{-1}\left(\frac{x}{a}\right)$$

$$\Rightarrow \int_0^x \frac{1}{(x^2+a^2)^2}\,dx = \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right) + \frac{x}{2a^2}\frac{1}{(x^2+a^2)}$$

Example: Let $\varphi(\alpha) = \int_\alpha^{\alpha^2} \frac{\sin\alpha x}{x}\,dx$. Find $\varphi'(\alpha)$ where $\alpha \neq 0$.

$$\varphi'(\alpha) = \int_\alpha^{\alpha^2} \frac{\cos\alpha x}{x}\cdot x\,dx + 2\alpha\cdot\frac{\sin\alpha^3}{\alpha^2} - \frac{\sin\alpha^2}{\alpha}$$

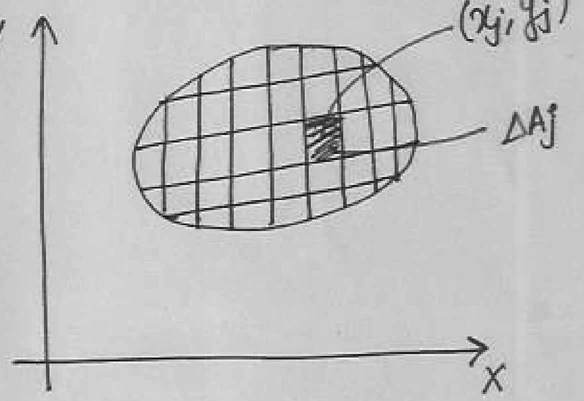
$$= \frac{\sin\alpha x}{\alpha}\Big|_\alpha^{\alpha^2} + \frac{2\sin\alpha^3}{\alpha} - \frac{\sin\alpha^2}{\alpha}$$

$$= \frac{3\sin\alpha^3 - 2\sin\alpha^2}{\alpha}.$$

MULTIPLE INTEGRALS

Double Integrals: Let $f(x,y)$ be defined in a closed region D of the xy plane. Divide D into n subregions of area ΔA_j, $j = 1, 2, \ldots, n$. Let (x_j, y_j) be some point of ΔA_j. Then Consider

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(x_j, y_j)\,\Delta A_j$$

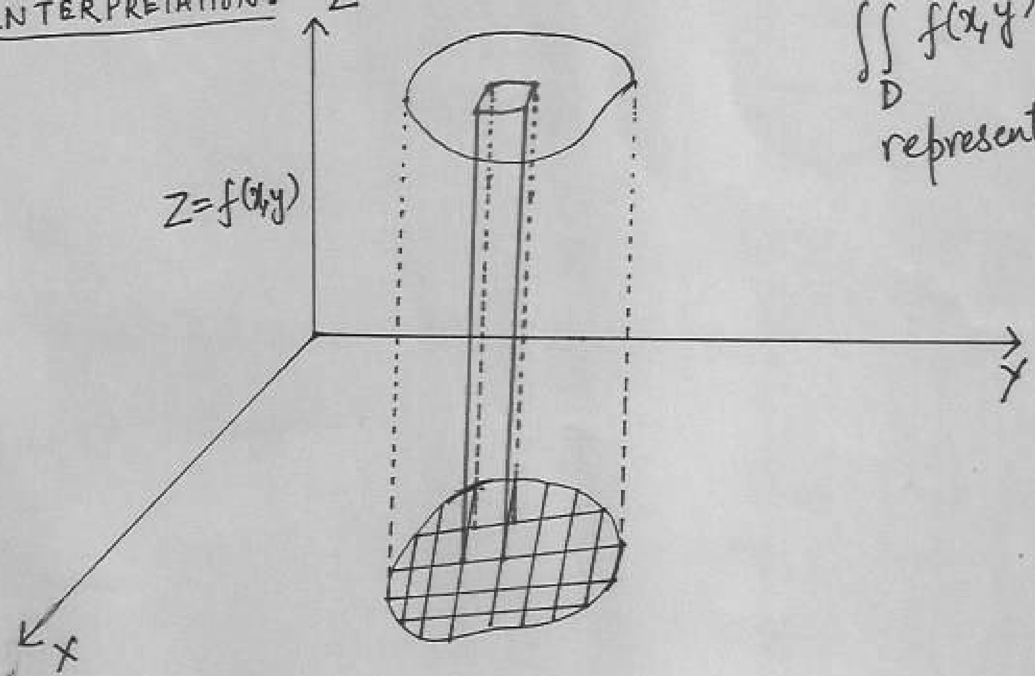


If this limit exists, then it is denoted by

$$\iint_D f(x,y)\,dA \quad \text{or} \quad \iint_D f(x,y)\,dx\,dy$$

Note: It can be proved that the above limit exists if $f(x,y)$ is continuous or piecewise continuous in D.

PHYSICAL INTERPRETATION:



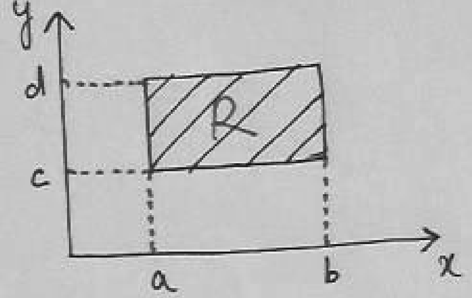
$\iint_D f(x,y)\,dA$ represents Volume

Evaluation of Double Integrals

a) If $f(x,y)$ is continuous* on rectangular region

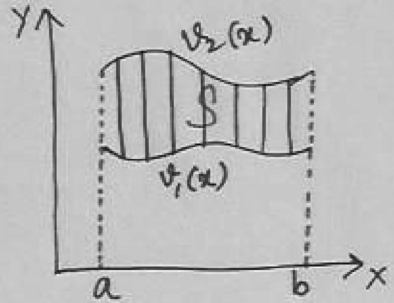
$R: a \le x \le b,\ c \le y \le d$, then

$$\iint_R f(x,y)\,dA = \int_c^d \underbrace{\left\{\int_a^b f(x,y)\,dx\right\}}_{\psi(y)} dy = \int_a^b \underbrace{\left\{\int_c^d f(x,y)\,dy\right\}}_{\psi(x)} dx$$



* or $f(x,y)$ is defined and bounded on R.

b)



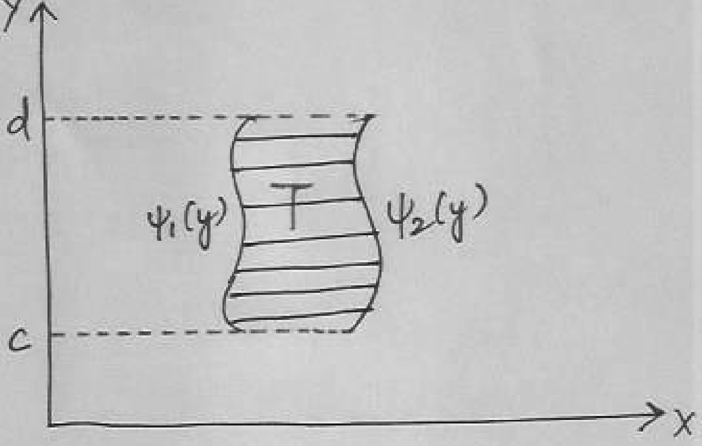
- $v_1(x)$ and $v_2(x)$ are continuous between 'a' and 'b'.
- $f(x,y)$ be defined and bounded on S.

Then,

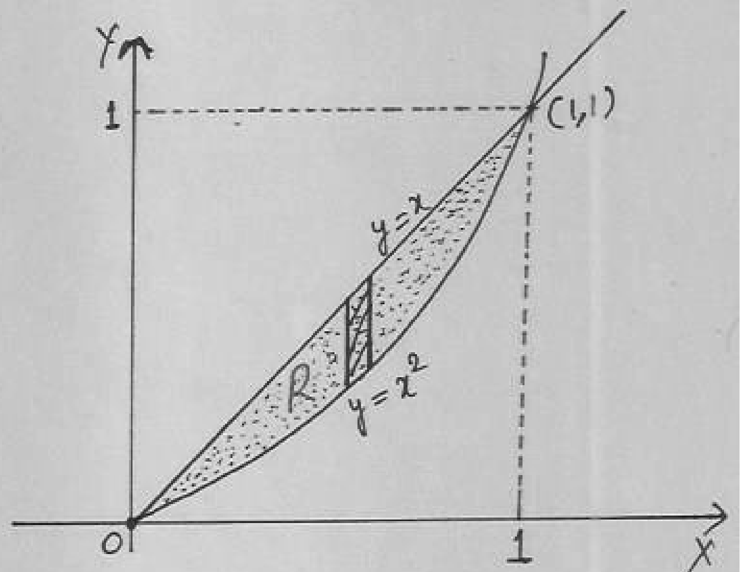
$$\iint_S f(x,y)\,dA = \int_a^b \int_{v_1(x)}^{v_2(x)} f(x,y)\,dy\,dx$$

c)

$$\iint_T f(x,y)\,dA = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\,dy$$



Example-1: Evaluate $\iint_R xy(x+y)\,dA$ where R is the region bounded by the line $y=x$ and the curve $y=x^2$.



Solution: $I = \int_{x=0}^{1} \int_{x^2}^{x} xy(x+y)\,dy\,dx$

$$= \int_0^1 \left[\frac{y^2}{2}\cdot x^2 + x\cdot\frac{y^3}{3}\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left[\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3}\right] dx$$

$$= \int_0^1 \left[\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3}\right] dx$$

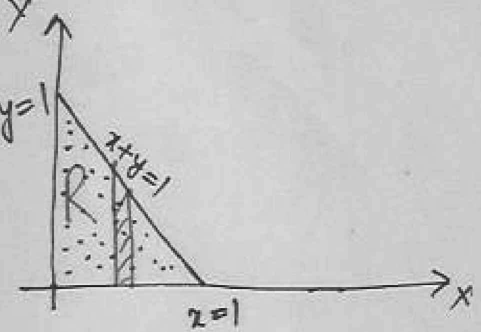
$$= \frac{5}{6}\cdot\frac{1}{5} - \frac{1}{2}\cdot\frac{1}{7} - \frac{1}{3}\cdot\frac{1}{8} = \frac{3}{56}$$

or

$$I = \int_{y=0}^{1} \int_{x=y}^{\sqrt{y}} xy(x+y)\,dx\,dy = \dots = \frac{3}{56}$$

<u>Example-2</u>: Evaluate $\iint_R e^{2x+3y}\,dx\,dy$, R is a triangle bounded by $x=0$, $y=0$ and $x+y=1$.

$$I = \int_{x=0}^{1}\int_{y=0}^{1-x} e^{2x+3y}\,dy\,dx$$



$$= \int_0^1 e^{2x}\left[\frac{e^{3y}}{3}\right]_0^{1-x} dx$$

$$= \int_0^1 e^{2x}\,\frac{1}{3}\cdot\left\{e^{3-3x}-1\right\} dx$$

$$= \frac{1}{3}\int_0^1 \left(e^{3-x}-e^{2x}\right) dx = \frac{1}{3}\left[-e^{3-x}-\frac{e^{2x}}{2}\right]_0^1$$

$$= -\frac{1}{3}\left[e^2+\frac{e^2}{2}-e^3-\frac{1}{2}\right] = -\frac{1}{3}\left[\frac{3e^2}{2}-e^3-\frac{1}{2}\right]$$

Ans...

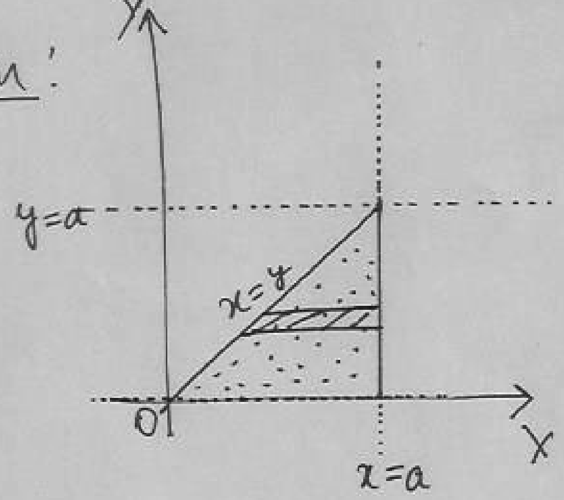
<u>Change of Order of Integration</u>:

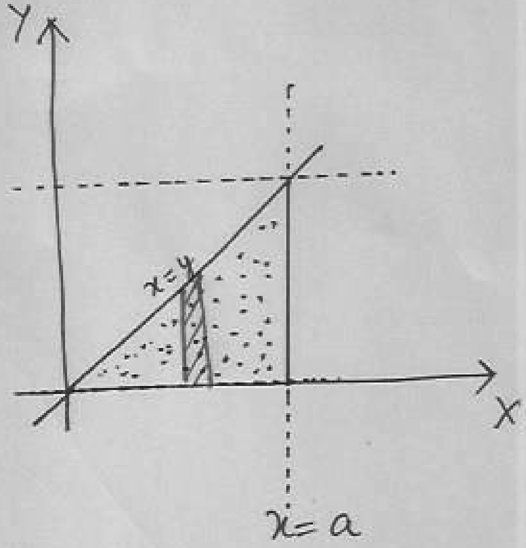
Why? To make the integration easier.

<u>Example</u>: Change the order of integration

$$\int_{y=0}^{a}\int_{x=y}^{a} \frac{x}{x^2+y^2}\,dx\,dy$$ and evaluate.

Solution:





$$\int_{y=0}^{a}\int_{x=y}^{a}\frac{x}{x^2+y^2}\,dx\,dy = \int_{x=0}^{a}\int_{y=0}^{x}\frac{x}{x^2+y^2}\,dy\,dx$$

$$= \int_{x=0}^{a}\left[x\cdot\frac{1}{x}\cdot\tan^{-1}\left(\frac{y}{x}\right)\right]_0^x dx$$

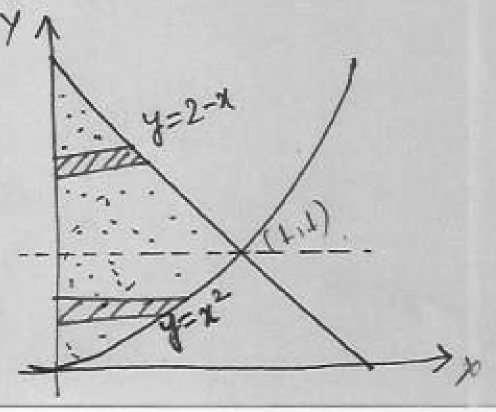
$$= \int_0^a\left[\tan^{-1}(1)-\tan^{-1}(0)\right]dx$$

$$= \frac{\pi}{4}(x)_0^a = \frac{\pi a}{4}$$

Example: Change the order of integration

$$\int_0^1\int_{y=x^2}^{2-x} xy\,dy\,dx$$ and evaluate.

Solution: $\int_0^1\int_{y=x^2}^{2-x} xy\,dy\,dx$



$$= \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy \, dx \, dy + \int_{y=1}^{2} \int_{x=0}^{2-y} xy \, dx \, dy$$

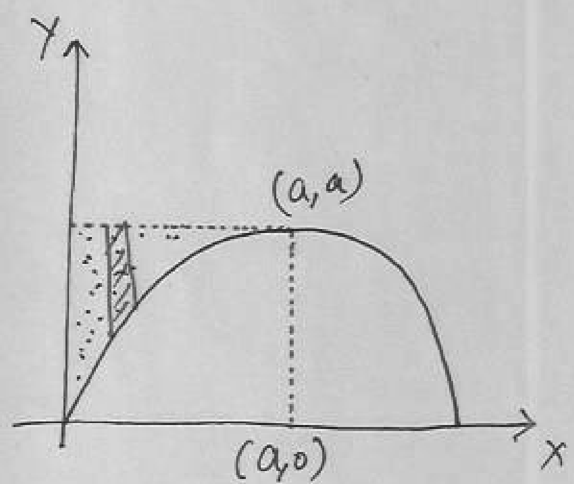
$$= \cdots$$

$$= \frac{3}{8}. \quad \underline{\text{Ans}} \ldots$$

<u>Example</u>: change the order of integration

$$\int_{y=0}^{a} \int_{x=0}^{a-\sqrt{a^2-y^2}} \frac{xy \cdot \log(x+a)}{(x-a)^2} \, dx \, dy \text{ and evaluate.}$$

<u>Solution</u>:



$$\int_{y=0}^{a} \int_{x=0}^{a-\sqrt{a^2-y^2}} \cdots \, dx dy = \int_{x=0}^{a} \int_{y=\sqrt{a^2-(x-a)^2}}^{a} \frac{xy \log(x+a)}{(x-a)^2} \, dy dx$$

$$= \int_0^a \frac{x \log(x+a)}{(x-a)^2} \cdot \frac{1}{2} \cdot \left[a^2 - \{ a^2 - (x-a)^2 \} \right] dx$$

$$= \frac{1}{2} \int_0^a x \log(x+a) \, dx$$

$$= \frac{1}{2} \left[\left\{ \frac{x^2}{2} \log(x+a) \right\}_0^a - \int_0^a \frac{x^2}{2} \cdot \frac{1}{(x+a)} \, dx \right]$$

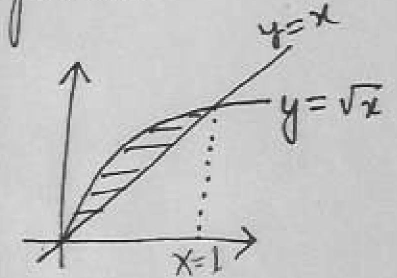
$$= \frac{1}{2} \left[\left\{ \frac{a^2}{2} \log(2a) \right\} - \frac{1}{2} \int_0^a \left[(x-a) + \frac{a^2}{x+a} \right] dx \right]$$

$$= \frac{1}{2}\left[\frac{a^2\log(2a)}{2} - \frac{1}{2}\left\{\frac{a^2}{2} - a^2 + a^2\log(2a) - a^2\log a\right\}\right]$$

$$= \frac{a^2}{8}\left[1 + 2\log a\right]$$

Example → Change the order of integration

$$\int_0^1 \int_x^{\sqrt{x}} f(x,y)\,dy\,dx$$



Ans: $\displaystyle\int_0^1 \int_{y^2}^{y} f(x,y)\,dx\,dy$

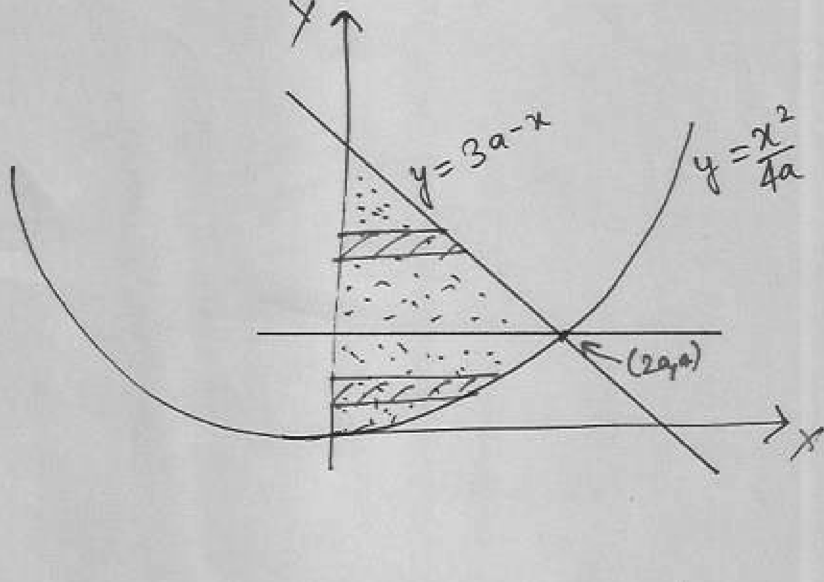
Q: Change the order of integration in

$$I = \int_0^{2a} \int_{\frac{x^2}{4a}}^{3a-\cancel{x}} F(x,y)\,dy\,dx$$

Solution:

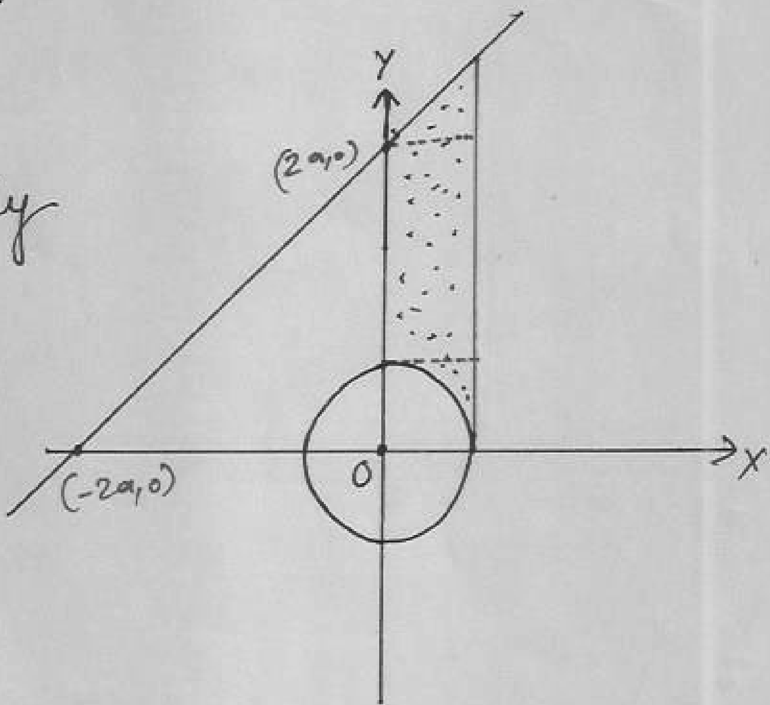
$$I = \int_0^a \int_0^{2\sqrt{ay}} F(x,y)\,dx\,dy$$

$$+ \int_a^{3a} \int_0^{3a-y} F(x,y)\,dx\,dy$$



Example: Change the order of integration in

$$I=\int_0^a \int_{\sqrt{a^2-x^2}}^{x+2a} F(x,y)\,dy\,dx$$

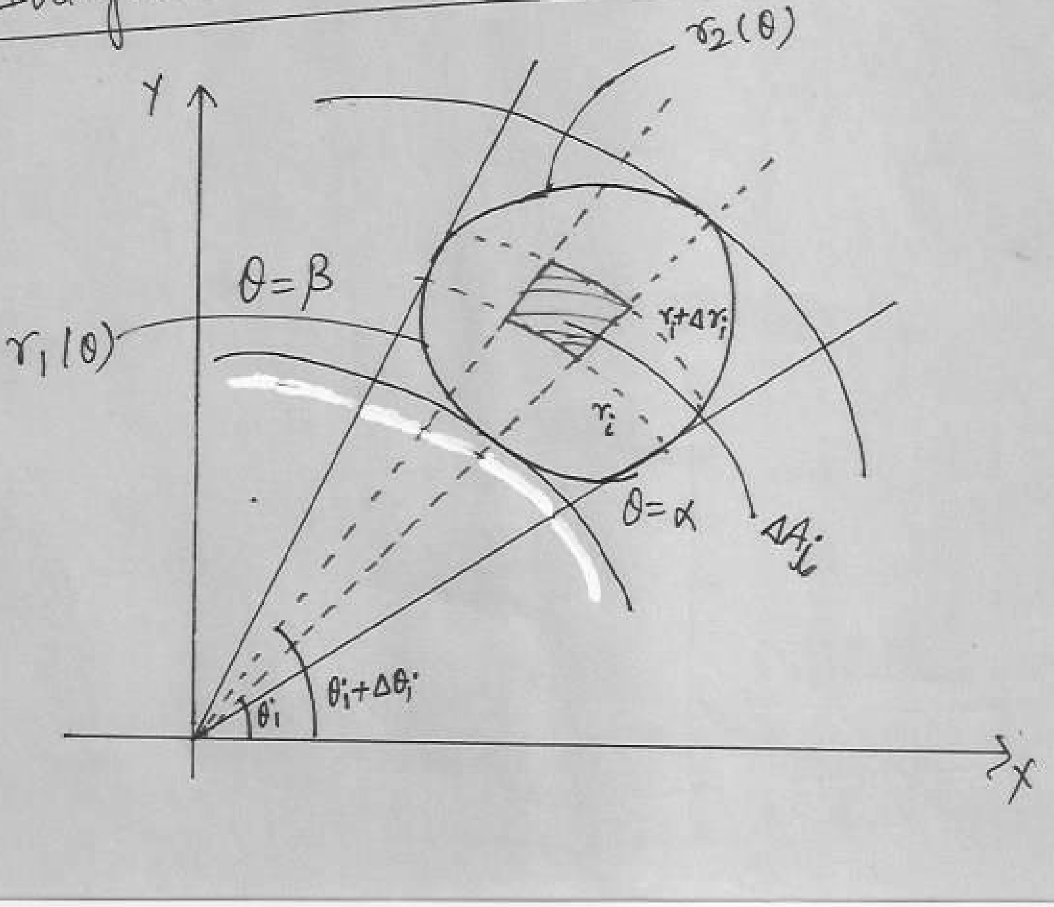


Solution: $I=\int_0^a \int_{\sqrt{a^2-y^2}}^{a} f(x,y)\,dx\,dy$

$$+\int_a^{2a}\int_0^a F(x,y)\,dx\,dy$$

$$+\int_{2a}^{3a}\int_{y-2a}^{a} F(x,y)\,dx\,dy$$

Double Integral in Polar Co-ordinate



$$\Delta A_i = (r_i + \Delta r_i)^2 \frac{\Delta \theta_i}{2} - r_i^2 \frac{\Delta \theta_i}{2}$$

$$= \cancel{2r_i\Delta\Delta}\cdot\left(2r_i \Delta r_i + \Delta r_i^2\right) \frac{\Delta \theta_i}{2}$$

$$= \left(\frac{2r_i + \Delta r_i}{2}\right) \cdot \Delta r_i \, \Delta \theta_i$$

$$= \left(r_i + \frac{\Delta r_i}{2}\right) \Delta r_i \, \Delta \theta_i$$

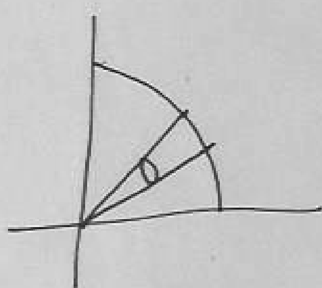
$$= r_i^* \, \Delta r_i \, \Delta \theta_i$$

$$I = \lim_{n \to \infty} \sum_{j=1}^{n} f(r_j, \theta_j) \, \Delta A_j$$

$$I = \int_{\theta=\alpha}^{\beta} \int_{r=r_1(\theta)}^{r=r_2(\theta)} f(r,\theta) \, r \, dr \, d\theta$$

<u>Example</u> → Compute area of first quadrant of a circle.

$$A = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r \, dr \, d\theta = \frac{a^2}{2} \cdot \frac{\pi}{2} = \frac{\pi a^2}{4}$$



Change of Variables:

Analogous to the method of substitution in Single Variable.

$$x = g(t)$$

$$\int_a^b f(x)\,dx = \int_c^d f(g(t))\,\underbrace{g'(t)}_{=\frac{dx}{dt}}\,dt$$

where $a = g(c)$ and $b = g(d)$

We can change variables in two dimensional case.

Let the variables x, y in the double integral.

$$\iint_R f(x,y)\,dx\,dy$$

be changed to new-variables u, v by means of relations.

$$x = \phi(u,v) \quad , \quad y = \psi(u,v)$$

then double integral is transformed into

$$\iint_{R'} f\{\phi(u,v), \psi(u,v)\}\, \underbrace{|J|\,du\,dv}$$

where $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$ Jacobian

R' is the region in uv plane which corresponds to the region R in the xy-plane.

Special Case: Cartesian to Polar Co-ordinate

$x = r\cos\theta \,, \quad y = r\sin\theta$

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

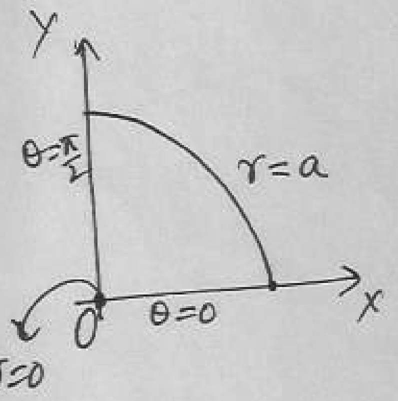
$$\Rightarrow \iint_S f(x,y)\,dx\,dy = \iint_T f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Example 1: Volume of one Octant of a sphere of radius a,

$$\iint_S \sqrt{a^2-x^2-y^2}\,dx\,dy$$

where S is the first quadrant of the circular disk

$$x^2+y^2 \le a^2$$



Change of variables

$x = r\cos\theta \,, \quad y = r\sin\theta$

$|J| = r$

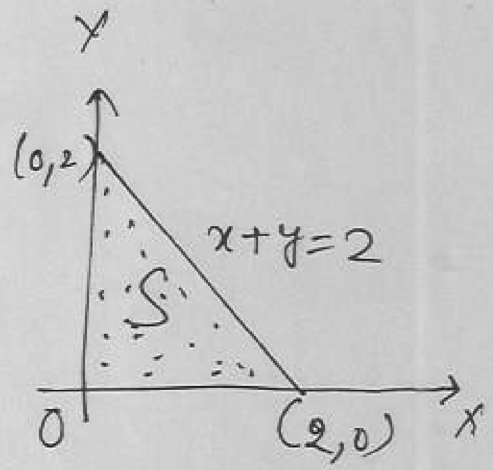
$$\iint_S \sqrt{a^2-x^2-y^2}\,dx\,dy = \iint_R \sqrt{a^2-r^2}\; r\,dr\,d\theta$$

$$\iint_R \sqrt{a^2-r^2}\; r\,dr\,d\theta = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} \sqrt{a^2-r^2}\; r\,dr\,d\theta$$

$$= \frac{\pi}{2}\cdot\left(-\frac{1}{2}\right)\frac{(a^2-r^2)^{3/2}}{3/2}\Bigg|_0^a$$

$$= \frac{\pi}{2}\cdot\left(-\frac{1}{2}\right)(-a^3)\cdot\frac{2}{3}$$

$$= \frac{\pi}{6}a^3 .$$



Example-2: $\displaystyle\iint_S e^{(y-x)/(y+x)}\,dx\,dy$

Change of variables

$$\begin{aligned} y-x&=u \\ y+x&=v \end{aligned} \quad\Rightarrow\quad x=\frac{v-u}{2}, \qquad y=\frac{v+u}{2}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{4}-\frac{1}{4} = -\frac{1}{2}$$

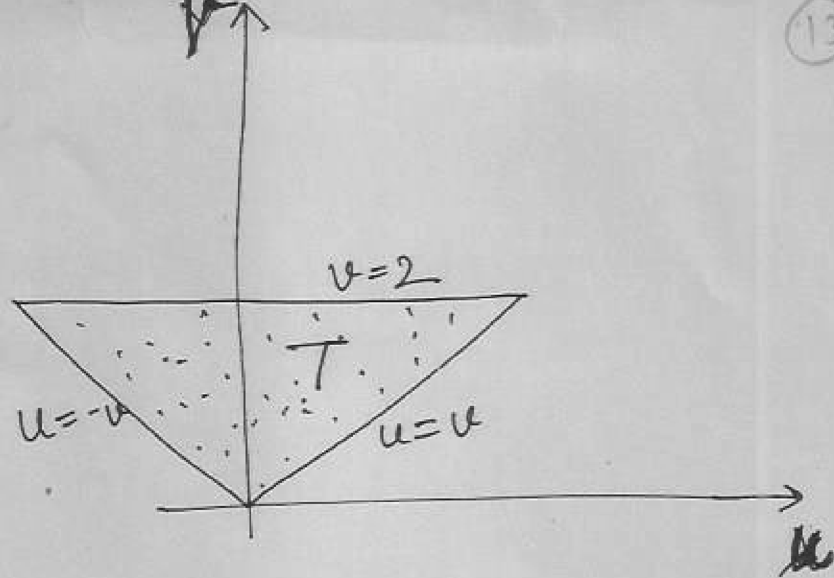
Domain in the uv-plane

line $x=0$ maps to $v=u$

line $y=0$ maps to $v=-u$

line $x+y=2$ maps to $v=2$

$$\iint_S e^{(y-x)/(y+x)}\,dx\,dy$$



$$= \iint_T e^{u/v}\,\frac{1}{2}\,du\,dv$$

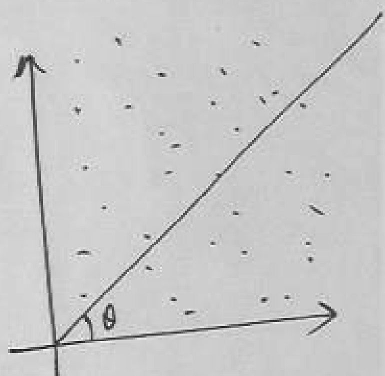
$$= \frac{1}{2}\int_{v=0}^{2}\int_{u=-v}^{v} e^{u/v}\,du\,dv$$

$$= \frac{1}{2}\int_0^2 v\left(e-\frac{1}{e}\right)dv$$

$$= e-\frac{1}{e}.$$

Example: Change into polar coordinates and evaluates

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dy\,dx$$



$$x=r\cos\theta,\quad y=r\sin\theta$$

$$\Rightarrow \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dy\,dx = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^\infty d\theta = \int_0^{\pi/2}\frac{1}{2}\,d\theta = \frac{\pi}{4}.$$

Note: Let $I = \int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-y^2} dy$

$$\Rightarrow I^2 = \int_0^\infty e^{-x^2} dx \cdot \int_0^\infty e^{-y^2} dy$$

$$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy = \frac{\pi}{4}$$

$$\boxed{I = \frac{\sqrt{\pi}}{2}}$$

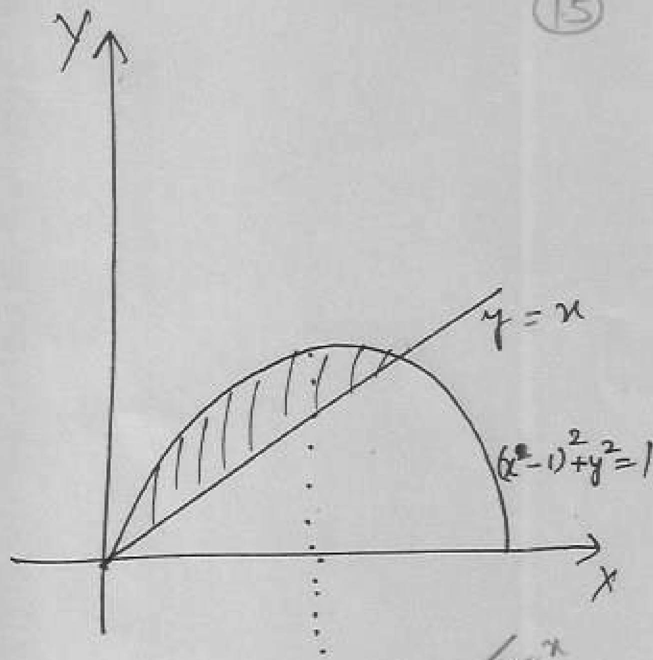
$$\Rightarrow \boxed{\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}}$$

Example: Evaluate $\int_0^1 \int_x^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx$ by changing to polar coordinates.

Solution: The region of integration is bounded by

$y = x$, $y = \sqrt{2x-x^2}$, $x=0$ & $x=1$

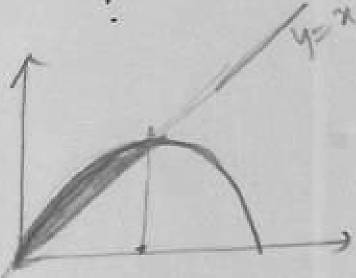
Polar Equation of the circle



$$(r\cos\theta-1)^2+r^2\sin^2\theta=1$$

$$\Rightarrow r^2-2r\cos\theta=0$$

$$\Rightarrow r=2\cos\theta$$



$$\int_0^1\int_x^{\sqrt{2x-x^2}}(x^2+y^2)\,dy\,dx$$

$$=\int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{2\cos\theta} r^2\cdot r\,dr\,d\theta$$

$$=\int_{\pi/4}^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2\cos\theta}d\theta=\int_{\pi/4}^{\pi/2}4\cos^4\theta\cdot d\theta$$

$$=\int_{\pi/4}^{\pi/2}(2\cos^2\theta)^2\,d\theta=\int_{\pi/4}^{\pi/2}(1+\cos2\theta)^2\,d\theta$$

$$=\int_{\pi/4}^{\pi/2}\left[1+\cos^2 2\theta+2\cos2\theta\right]d\theta$$

$$=\int_{\pi/4}^{\pi/2}\left[1+\frac{1}{2}(1+\cos4\theta)+2\cos2\theta\right]d\theta$$

$$= \cdots = \frac{1}{8}(3\pi - 8)$$

Example: Evaluate the integral

$\iint_R \sqrt{x^2+y^2}\, dx\,dy$ by changing to polar co-ordinates, where R is the region in the x-y plane bounded by the circles $x^2+y^2=4$ and $x^2+y^2=9$.

Solution: $x = r\cos\theta$, $y = r\sin\theta$

$$|J| = r$$

$$I = \int_0^{2\pi}\int_2^3 r \cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^3}{3}\right]_2^3 d\theta$$

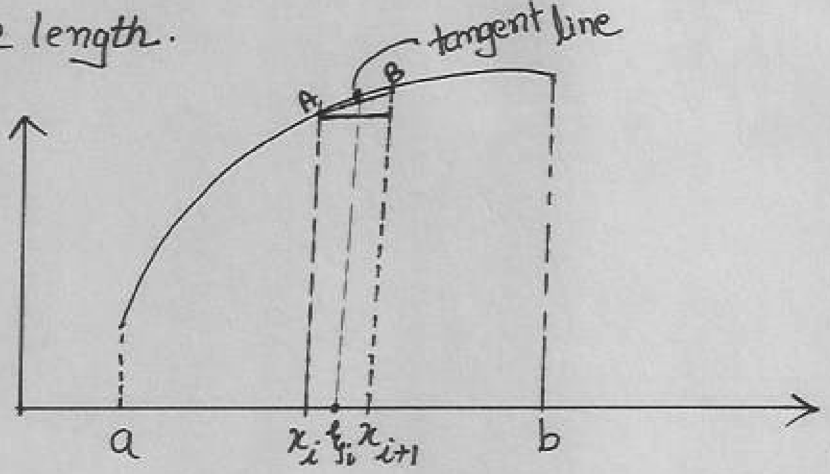
$$= \left(\frac{27-8}{3}\right) 2\pi$$

$$= \frac{19}{3} \cdot 2\pi$$

$$= \frac{38}{3} \cdot \pi$$

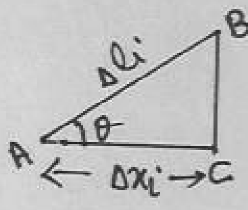
COMPUTING THE AREA OF A SURFACE:

Let us consider the case of 1-dimension, i.e. computation of the curve length.



length of the curve

$$L = \lim_{n\to\infty} \sum_{i=1}^{n-1} \Delta l_i$$



$$\frac{\Delta x_i}{\Delta l_i} = \cos\theta$$

$$\Rightarrow \Delta l_i = \Delta x_i \frac{1}{\cos\theta}.$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n-1} \sqrt{1+f'(\xi_i)^2} \cdot \Delta x_i$$

$$= \int_a^b \sqrt{1+(f'(x))^2} \cdot dx$$

Also $\cos\theta = \dfrac{1}{\sqrt{1+\tan^2\theta}}$

$$\frac{1}{\cos\theta} = \sqrt{1+\tan^2\theta}$$

$$= \sqrt{1+f'(\xi_i)^2}$$

$$\Rightarrow \Delta l_i = \Delta x_i \sqrt{1+f'(\xi_i)^2}$$

In two dimension case we consider tangent plane instead of tangent line and similar to one dimensional case we get surface area.

$$\boxed{S = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy}$$

wher D is the projection of the surface in xy-plane.

Similarly if the equation is given in the form:

$$x = \mu(y,z) \text{ or in the form } y = \psi(x,z)$$

then

$$S = \iint_{\tilde{D}} \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\, dy\, dz$$

$$S = \iint_{\tilde{\tilde{D}}} \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\, dx\, dz.$$

where $\tilde{D}$ and $\tilde{\tilde{D}}$ are the domains in the yz and xz planes in which the given surface is projected.

<u>Example</u>: Compute the surface area of the sphere

$$x^2+y^2+z^2 = R^2$$

<u>Solution</u>: Equation of the surface

$$z = \sqrt{R^2 - x^2 - y^2} \quad \text{(upper half)}$$

In this case:
$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{R^2-x^2-y^2}}$$

$$\frac{\partial z}{\partial y} = -\frac{y}{\sqrt{R^2-x^2-y^2}}$$

Domain of integration: $x^2+y^2 \le R^2$

$$S = 2\int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{+\sqrt{R^2-x^2}} \underbrace{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}}_{\frac{R}{\sqrt{R^2-x^2-y^2}}}\, dy\, dx$$

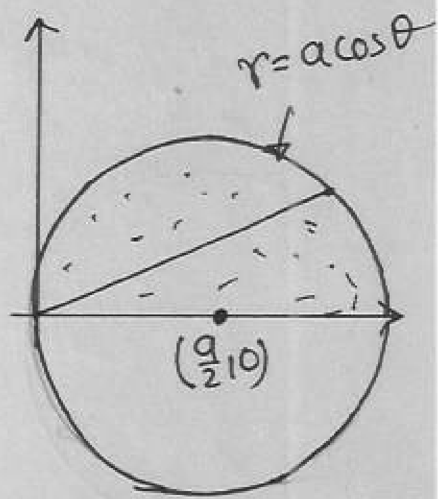
transformation to polar coordinate gives:

$$S = 2\int_0^{2\pi}\int_0^{R} \frac{R}{\sqrt{R^2-r^2}}\, r\cdot dr\, d\theta$$

$$= 2\pi \cdot 2R\left(-\sqrt{R^2-r^2}\right)_0^{R}$$

$$= 4\pi R^2$$

Question: Find the area of that part of the sphere $x^2+y^2+z^2=a^2$ which is cut off by the cylinder $x^2+y^2=ax$.



$$x^2+y^2-ax=0 \Rightarrow \left(x-\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}$$

$$S = 2\cdot 2\int_0^{\pi/2}\int_{r=0}^{a\cos\theta} \frac{a}{\sqrt{a^2-r^2}}\, r\cdot dr\, d\theta$$

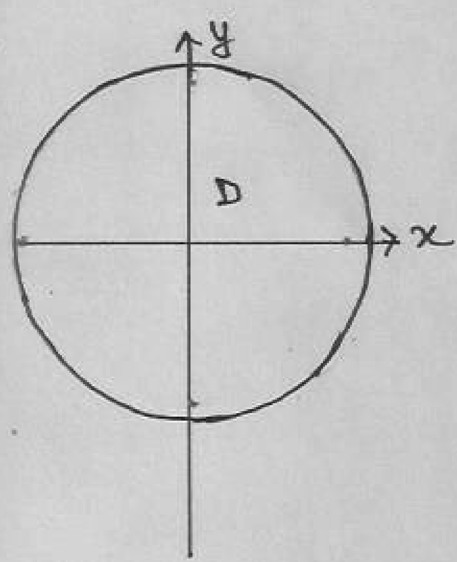
$$= 4\cdot a\int_0^{\pi/2}\left(-\sqrt{a^2-r^2}\right)_0^{a\cos\theta} d\theta$$

$$= 4a\cdot\int_0^{\pi/2}\left[-a\sin\theta + a\right] d\theta$$

$$= 4a\cdot\left[\left[a\cos\theta\right]_0^{\pi/2} + a\{\theta\}_0^{\pi/2}\right] = 4a\cdot\left[-a + a\cdot\frac{\pi}{2}\right]$$

$$= 2a^2(\pi-2).$$

Ex. Determine the surface area of the part of $z = xy$ that lies in the cyl. $x^2+y^2=1$.



Solution:

$$z = f(x,y) = xy$$

$$z_x = y \quad \& \quad z_y = x$$

$$S = \iint_D \sqrt{1+x^2+y^2}\, dA$$

In polar coordinate

$$S = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r\sqrt{1+r^2}\, dr\, d\theta$$

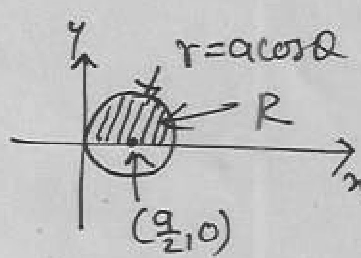
$$= \int_0^{2\pi} \frac{1}{2}\,\frac{2}{3}\left[(1+r^2)^{3/2}\right]_0^1 d\theta = \frac{2\pi}{3}\left(2^{3/2}-1\right) \quad \text{Ans.}$$

Evaluation of volume:

$$V = \iint_D \underset{f(x,y)}{\underset{\parallel}{z}}\, dx\, dy \quad \text{OR} \quad \iint_{\bar{D}} \mu(y,z)\, dy\, dz$$

$$\text{OR} \quad \iint_{\bar{\bar{D}}} \psi(x,z)\, dx\, dz.$$

Example: Find the volume common to sphere $x^2+y^2+z^2=a^2$ and a circular cylinder $x^2+y^2=ax$.



Required volume:

$$V = 4\iint_R z \, dx\, dy$$

$$= 4\iint_R \sqrt{a^2-x^2-y^2}\, dx\, dy$$

Subst. $x = r\cos\theta$ $y = r\sin\theta$

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{a\cos\theta} \sqrt{a^2-r^2}\; r\, dr\, d\theta$$

$$= \frac{4}{-2}\int_0^{\pi/2} \frac{2}{3}\left(a^2-r^2\right)^{3/2}\Big|_0^{a\cos\theta} d\theta$$

$$= -2\cdot\frac{2}{3}\cdot\int_0^{\pi/2}\left(a^3\sin^3\theta - a^3\right) d\theta$$

$$= \frac{-4}{3}a^3\cdot\left[\frac{2}{3}-\frac{\pi}{2}\right] = \frac{2}{9}a^3\left(3\pi-4\right)$$

Ans.

TRIPLE INTEGRALS

Divide the region V into n sub-regions of respective volumes $\delta V_1, \delta V_2, \cdots \delta V_n$. Let (x_r, y_r, z_r) be an arbitrary point in the rth sub-region.

Consider the sum

$$\sum_{j=1}^{n} f(x_j, y_j, z_j)\, \delta V_j$$

If the limit of this sum exists as $n \to \infty$ and $\delta V_j \to 0$, then

$$\iiint_V f(x,y,z)\, dV = \lim_{n \to \infty} \sum_{j=1}^{n} f(x_j, y_j, z_j)\, \delta V_j$$

Evaluation:

$$\iiint_V f(x,y,z)\, dV = \int_{z=a}^{b} \left\{ \int_{y=\varphi_1(z)}^{\varphi_2(z)} \left\{ \int_{x=f_1(y,z)}^{f_2(y,z)} f(x,y,z)\, dx \right\} dy \right\} dz$$

Note: Similar to double integrals, the order of integration is immaterial if the limits of integration are constants.

$$\int_a^b \int_c^d \int_e^f F(x,y,z)\, dx\, dy\, dz = \int_e^f \int_c^d \int_a^b F(x,y,z)\, dz\, dy\, dx$$

$$= \int_c^d \int_e^f \int_a^b F(x,y,z)\, dz\, dx\, dy$$

Example: Evaluate $I = \int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$

$$I = \int_0^a \int_0^x e^{x+y+z} \Big|_0^{x+y} dy\, dx$$

$$= \int_0^a \int_0^x \left(e^{2(x+y)} - e^{x+y}\right) dy\, dx$$

$$= \int_0^a \frac{e^{2(x+y)}}{2} \Big|_0^x dx - \int_0^a e^{x+y} \Big|_0^x dx$$

$$= \frac{1}{2}\left(\int_0^a \left\{\left(e^{4x} - e^{2x}\right) - 2\left(e^{2x} - e^{x}\right)\right\} dx\right.$$

$$= \frac{1}{2} \int_0^a \left(e^{4x} - 3e^{2x} + 2e^{x}\right) dx$$

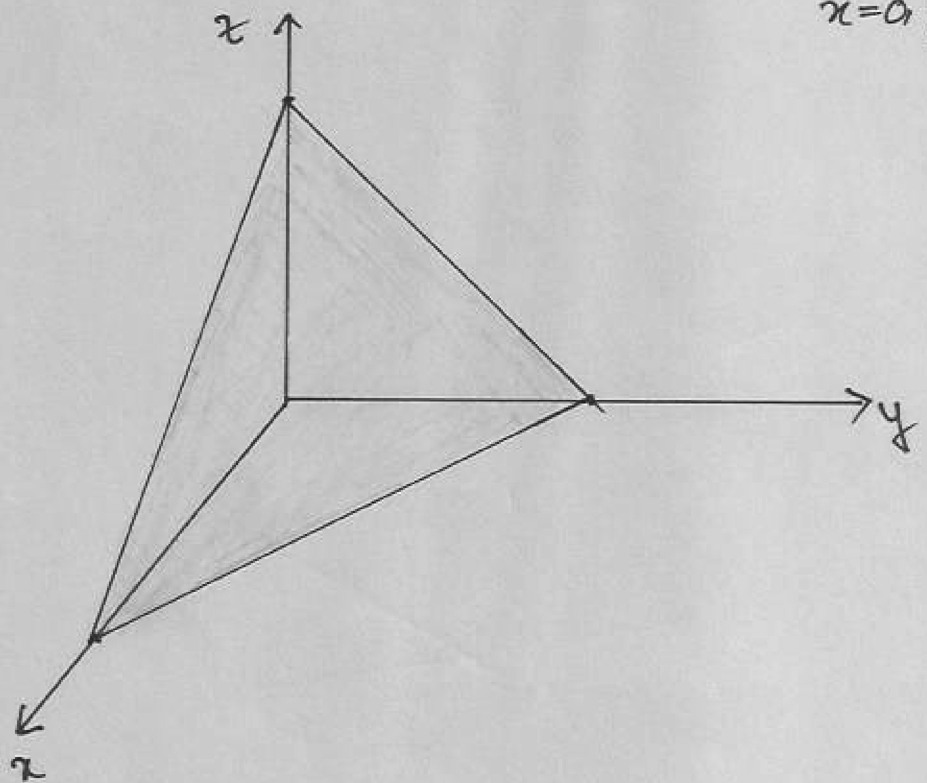
$$= \frac{1}{2}\left[\frac{e^{4x}}{4}\Big|_0^a - \frac{3}{2} e^{2x}\Big|_0^a + 2e^{x}\Big|_0^a\right]$$

$$= \frac{1}{2}\left[\frac{e^{4a}}{4} - \frac{3}{2} e^{2a} + 2e^{a} - \frac{1}{4} + \frac{3}{2} - 2\right]$$

$$= \frac{1}{2}\left[\frac{e^{4a}}{4} - \frac{3}{2} e^{2a} + 2e^{a} - \frac{3}{4}\right]$$

$$= \frac{e^{4a}}{8} - \frac{3}{4} e^{2a} + e^{a} - \frac{3}{8}$$ An

Example: Evaluate $\iiint_R \frac{dx\,dy\,dz}{(x+y+z+1)^3}$; R is the region bounded by $x=0,\ y=0,\ z=0$ & $x+y+z=1$



$$I = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} \frac{1}{(x+y+z+1)^3} \cdot dz\,dy\,dx$$

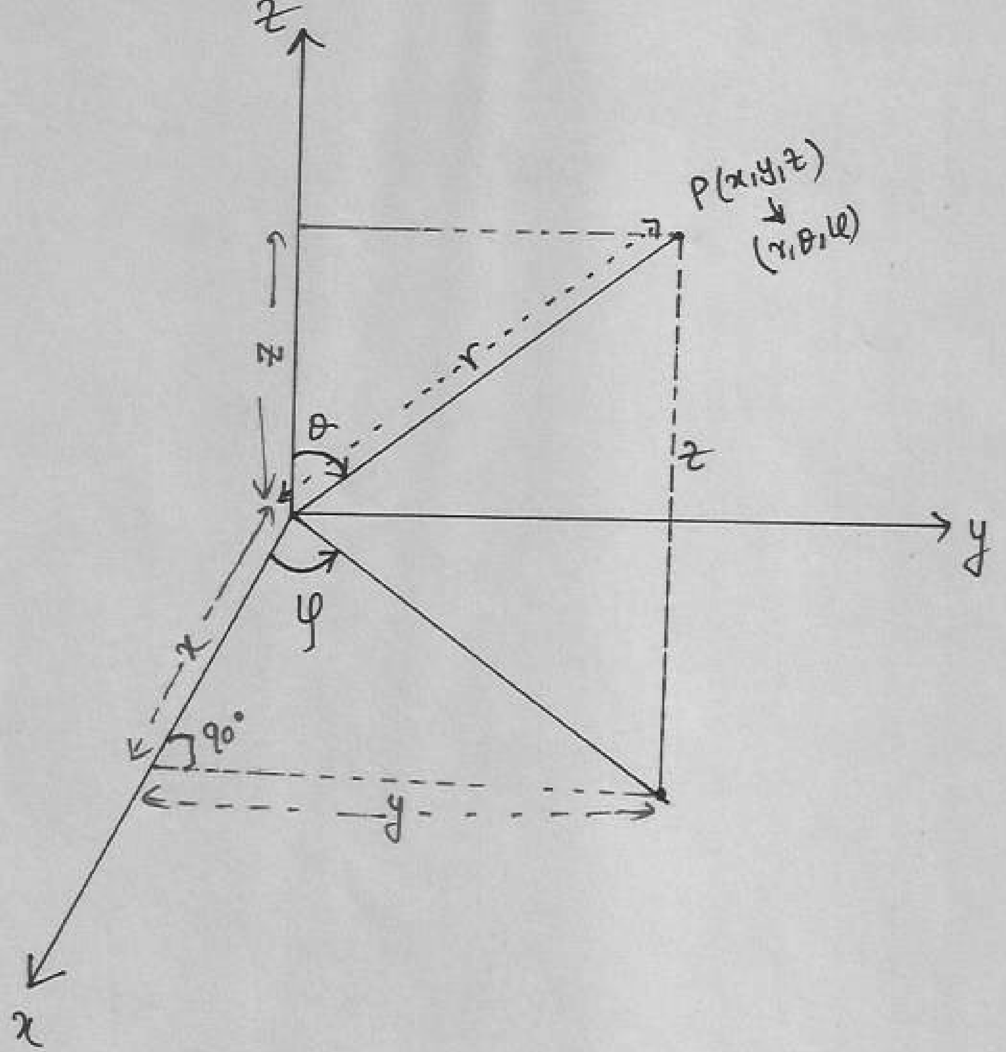
$$= \int_0^1 \int_0^{1-x} \left[-\frac{1}{2}(x+y+z+1)^{-2}\right]_0^{1-x-y} dy\,dx$$

$$\vdots$$

$$= \frac{1}{2}\left[\ln 2 - \frac{5}{8}\right]$$ Ans

Change of Variables in TRIPLE integrals:

- Cartesian co-ordinate (x, y, z) to spherical polar coordinates



$$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$$

Note that $x^2+y^2+z^2=r^2$

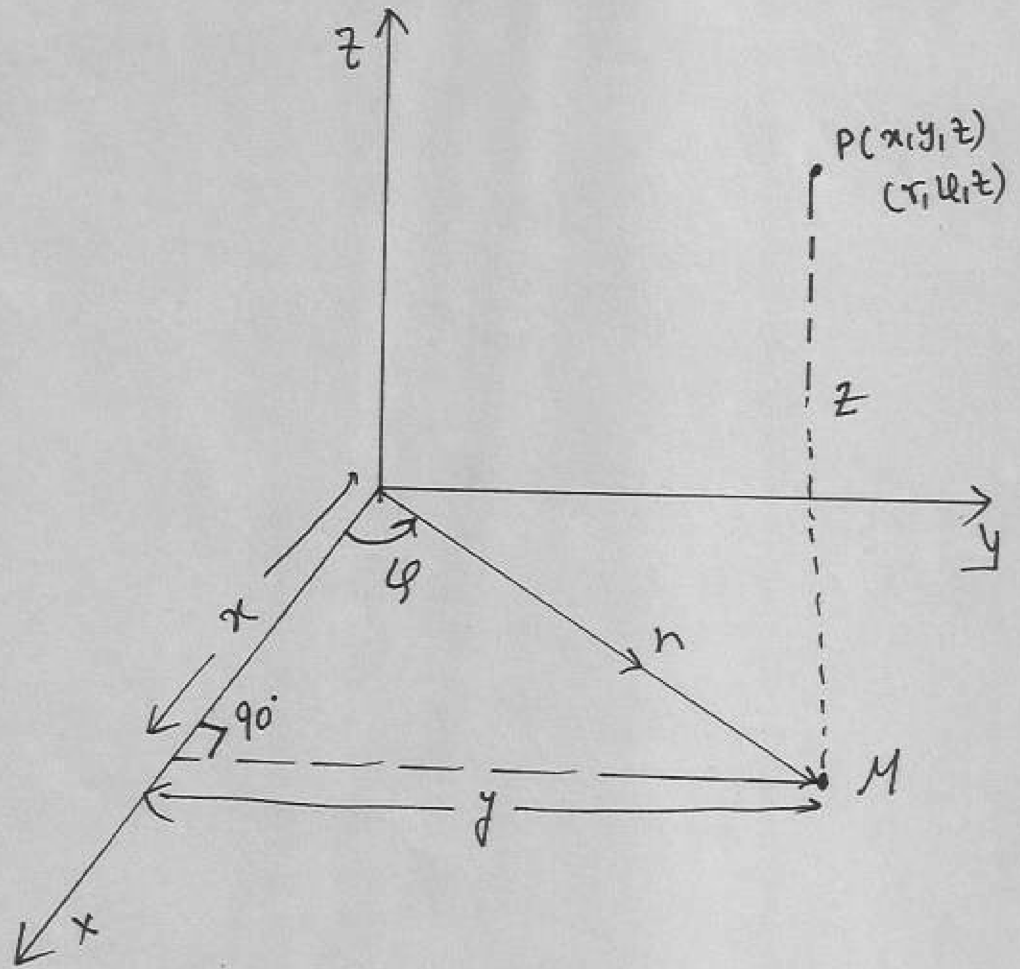
$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta$$

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_{\tilde{D}} f(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)\, r^2\sin\theta\, dr\,d\theta\,d\varphi.$$

ii) Cartesian Coordinates (x, y, z) to Cylindrical coordinates (r, φ, z)

$$(x, y, z) \rightarrow (r, \varphi, z)$$



$$x = r\cos\varphi$$

$$y = r\sin\varphi$$

$$z = z$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial z} \end{vmatrix} = r$$

$$\iiint_D f(x,y,z)\, dx\, dy\, dz = \iiint_{\tilde{D}} f(r\cos\varphi, r\sin\varphi, z)\, r\, dr\, d\varphi\, dz$$

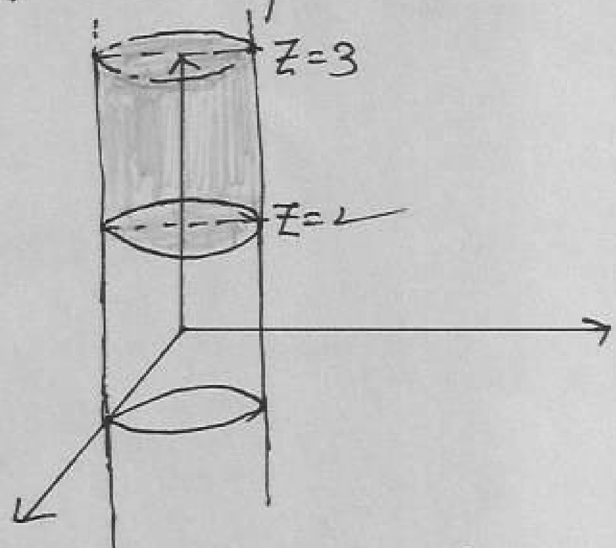
Ex: Changing to cylindrical coordinate, evaluate

$$\iiint z(x^2+y^2)\,dx\,dy\,dz \,; \qquad x^2+y^2 \le 1, \quad 2 \le z \le 3$$

Solution: $x = r\cos\theta \qquad y = r\sin\varphi \qquad z = z$

Note that $x^2+y^2 = r^2$

& $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \vdots & & \\ \vdots & & \end{vmatrix} = r$



$$\iiint z(x^2+y^2)\,dx\,dy\,dz = \int_{z=2}^{3}\int_{\varphi=0}^{2\pi}\int_{r=0}^{1} z\cdot r^2\, r\, dr\,d\varphi\,dz$$

$$= \int_2^3 \int_0^{2\pi} \frac{1}{4} z \, d\varphi\, dz$$

$$= \frac{1}{4}\, 2\pi\, \frac{1}{2}(9-4) = \frac{5\pi}{4}.$$

Example: Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} \frac{1}{\sqrt{x^2+y^2+z^2}}\, dz\, dy\, dx$

by changing into spherical polar coordinate.

Solution:

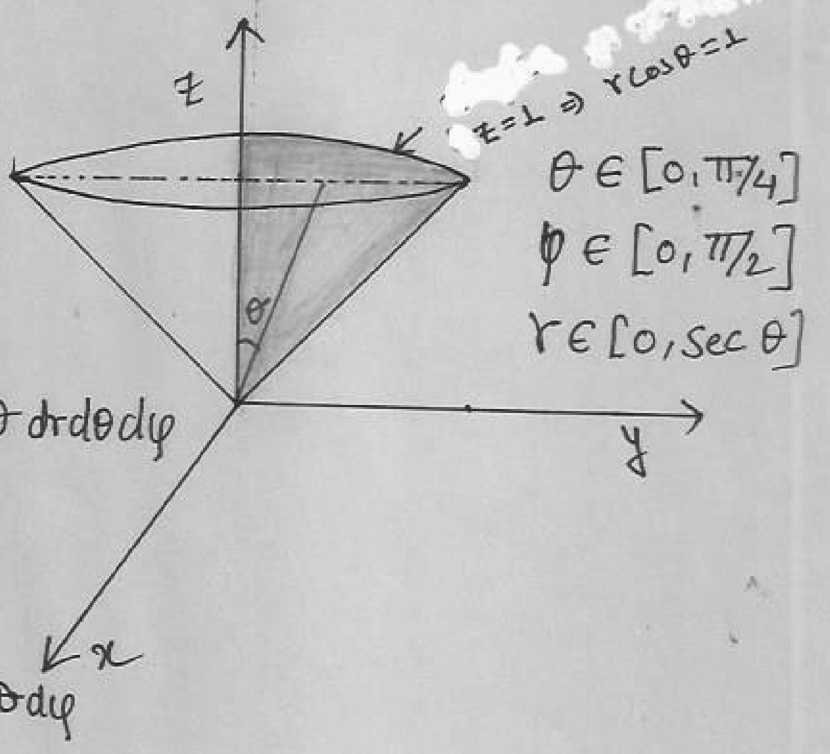
$$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$$

$$J = r^2\sin\theta \qquad x^2+y^2+z^2 = r^2$$

x varies from 0 to 1

y varies from 0 to $y=\sqrt{1-x^2}$ i.e., $y^2+x^2=1$

z varies from $\sqrt{x^2+y^2}$, i.e. $z^2 = x^2+y^2$ (cone) to 1



$\theta \in [0, \pi/4]$

$\varphi \in [0, \pi/2]$

$r \in [0, \sec\theta]$

$$I = \int_0^{\pi/2} \int_{\theta}^{\pi/4} \int_{r=0}^{\sec\theta} \frac{1}{r}\cdot r^2\sin\theta\, dr\, d\theta\, d\varphi$$

$$= \int_0^{\pi/2}\int_0^{\pi/4} \frac{1}{2}\sec^2\theta\sin\theta\, d\theta\, d\varphi$$

$$= \frac{\pi}{4}\int_0^{\pi/4} \sec\theta\tan\theta\, d\theta$$

$$= \frac{\pi}{4}\sec\theta\Big|_0^{\pi/4} = \frac{(\sqrt{2}-1)\pi}{4}$$

Ex. Evaluate $$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz\, dy\, dx$$

by changing to spherical polar coordinates.

Sol: $x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$

$J = r^2\sin\theta.$

$$I = \int_{\theta=0}^{\pi/2} \int_{\varphi=0}^{\pi/2} \int_{r=0}^{1} \frac{r^2\sin\theta}{\sqrt{1-r^2}}\, dr\, d\varphi\, d\theta$$

First evaluate: $\int_0^1 \frac{r^2}{\sqrt{1-r^2}}\, dr$ subst $r = \sin t$, $dr = \cos t\, dt$

$$= \int_0^{\pi/2} \frac{\sin^2 t}{\cos t} \cos t\, dt = \frac{\pi}{4}$$

$$I = \frac{\pi}{4} \int_0^{\pi/2} \int_0^{\pi/2} \sin\theta\, d\varphi\, d\theta$$

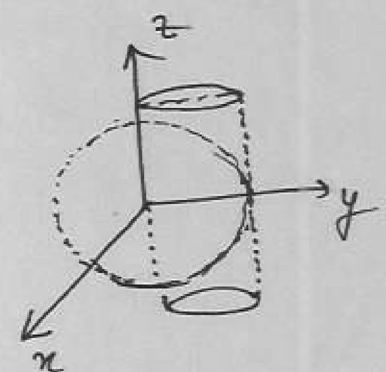
$$= \frac{\pi}{4} \cdot \frac{\pi}{2} \cdot \left[-\cos\theta\right]_0^{\pi/2}$$

$$= \frac{\pi^2}{8} \cdot 1.$$

$$= \frac{\pi^2}{8}$$ Ans

Q. Using triple integral find the volume common to a sphere $x^2+y^2+z^2=a^2$ and a circular cylinder $x^2+y^2=ax$.

$$V=\iiint_V dx\,dy\,dz = \iiint_V dz\,dy\,dx$$



$$= 4\int_0^a \int_{y=0}^{\sqrt{ax-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$

$$= 4\int_0^a \int_0^{\sqrt{ax-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$$

⋮

proceed as in double integral

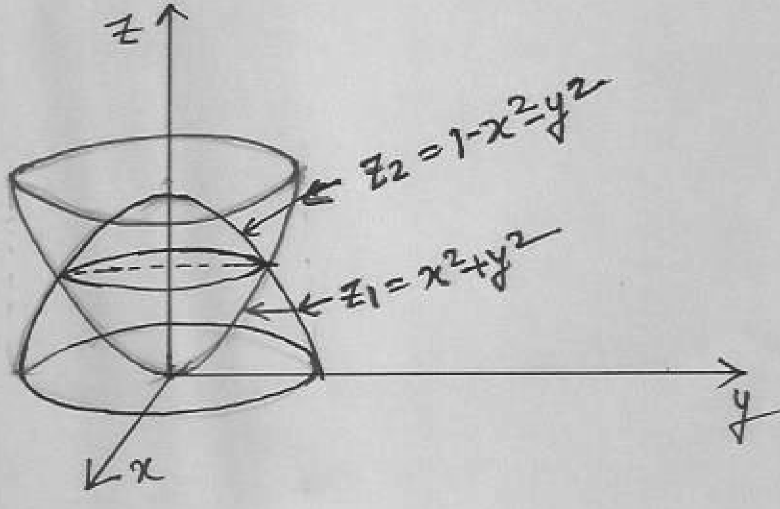
$$= \frac{2}{9}a^3\left(\pi-\frac{4}{3}\right).$$

Q. Find the volume of the solid formed by two paraboloids: $z_1=x^2+y^2$ & $z_2=1-x^2-y^2$

Intersecting curve:

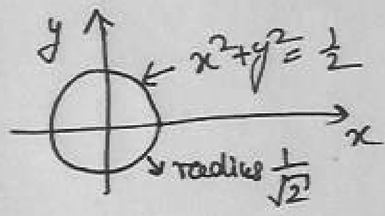
$x^2+y^2=1-x^2-y^2$

$\Rightarrow x^2+y^2=\frac{1}{2}$



$$V = \iiint_V dx\,dy\,dz = \int_{x=-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{y=-\sqrt{\frac{1}{2}-x^2}}^{+\sqrt{\frac{1}{2}-x^2}} \int_{x^2+y^2}^{1-x^2-y^2} dz\,dy\,dx$$

Projection on xy plane:



Changing to cylindrical coordinates

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$V = 4\int_0^{\pi/2} \int_{r=0}^{\frac{1}{\sqrt{2}}} \int_{z=r^2}^{1-r^2} r\,dz\,dr\,d\theta$$

$$= 4\int_0^{\pi/2} \int_{r=0}^{\frac{1}{\sqrt{2}}} r\cdot(1-r^2-r^2)\,dr\,d\theta$$

$$= 2\pi \int_0^{\frac{1}{\sqrt{2}}} r(1-2r^2)\,dr$$

$$= 2\pi\left[\frac{1}{2}\left(\frac{1}{2}-0\right) - \frac{2}{4}\cdot\left(\frac{1}{4}-0\right)\right]$$

$$= 2\pi\cdot\left[\frac{1}{4}-\frac{1}{8}\right]$$

$$= 2\pi\cdot\frac{1}{8}$$

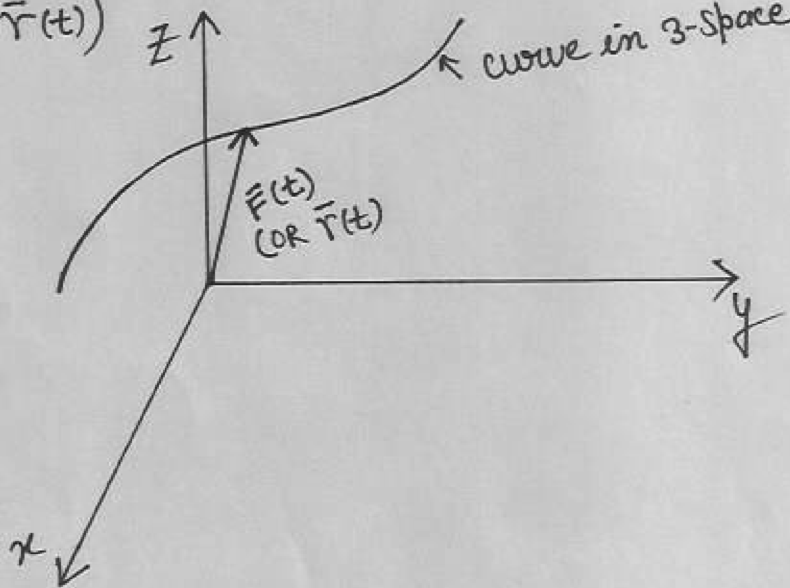
$$= \frac{\pi}{4}$$ Ans.

SCALAR and VECTOR Fields

Vector function of one variable: (Parametric representation of curves ~~surfaces~~)

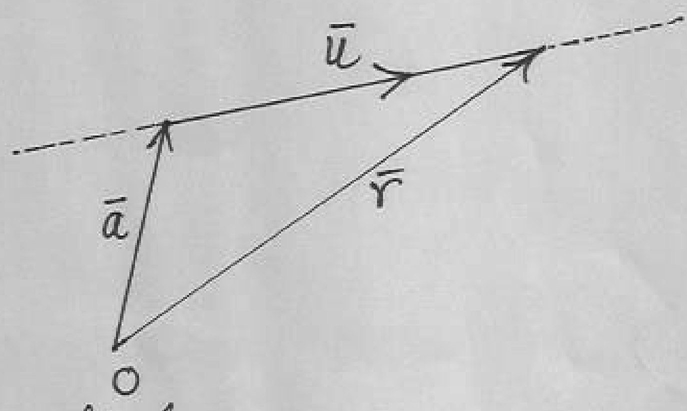
$$\vec{F}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}, \quad a \le t \le b$$

(OR $\bar{r}(t)$)



OR in 2d space $\bar{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}, \quad a \le t \le b$

Examples! Let $\bar{a}$ be the position vector of a particular fixed point on the line and $\bar{u}$ be the vector pointing along the line.



Equation of the straight line: $\bar{r} = \bar{a} + \lambda \bar{u}$

Limit and Continuity of vector functions:

<u>limit:</u> $\lim_{t\to a} |\bar{r}(t) - \bar{l}| = 0$

<u>Continuity:</u> The function $\bar{r}(t)$ is said to be continuous at $t=a$ if

(i) $\bar{r}(t)$ is defined in some neighbourhood of a

(ii) $\lim_{t\to a} \bar{r}(t)$ exists and

(iii) $\lim_{t\to a} \bar{r}(t) = \bar{r}(a)$.

<u>Differentiability</u>: $\bar{r}(t)$ is said to be differentiable if

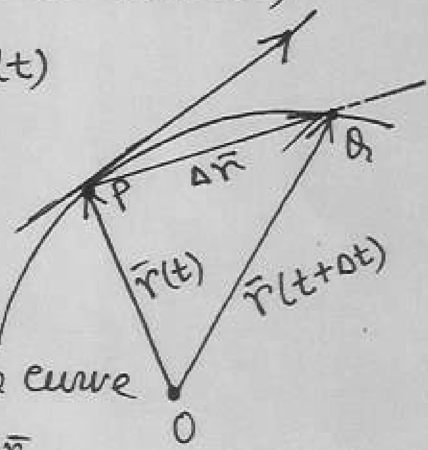
$$\lim_{\Delta t\to 0} \frac{\bar{r}(t+\Delta t) - \bar{r}(t)}{\Delta t} \text{ exists.}$$

Let $\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ be the parametric representation of a curve C, then

$$\frac{d\bar{r}}{dt} = r'(t) = \frac{dx(t)}{dt}\hat{i} + \frac{dy(t)}{dt}\hat{j} + \frac{dz(t)}{dt}\hat{k}$$

Geometric representation of $r'(t)$: (tangent to a curve)

Note that the direction of $\Delta\bar{r} = \bar{r}(t+\Delta t) - \bar{r}(t)$ and $\frac{\Delta \bar{r}}{\Delta t}$ is the same.



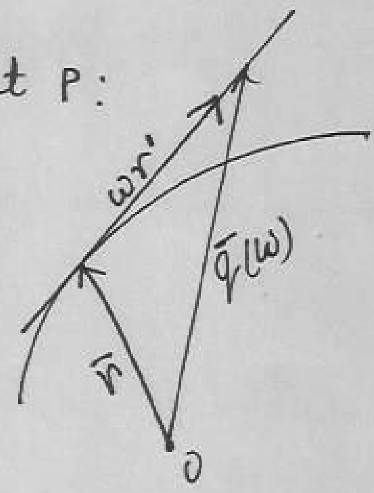
Then the limiting position of the vector $\frac{\Delta\bar{r}}{\Delta t}$, i.e., $\lim_{\Delta t\to 0}\frac{\Delta\bar{r}}{\Delta t}$ is the tangent to the curve at P.

tangent vector $= \bar{r}'(t) = \lim_{\Delta t\to 0} \frac{\Delta\bar{r}}{\Delta t}$.

Unit tangent vector $\bar{u} = \dfrac{\bar{r}'(t)}{|\bar{r}'(t)|}$

Equation of the tangent to c at P:

$$\bar{q}(\omega) = \bar{r} + \omega \bar{r}'$$



Partial derivatives of a vector function:

Let $\bar{r} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ & v_1, v_2, v_3 are differentiable functions of n variables $t_1, t_2, \cdots t_n$.

Then the partial derivative of v with respect to t_j is given by

$$\frac{\partial \bar{v}}{\partial t_j} = \frac{\partial v_1}{\partial t_j}\hat{i} + \frac{\partial v_2}{\partial t_j}\hat{j} + \frac{\partial v_3}{\partial t_j}\hat{k}$$

Example: (i) Find $v'(t)$ for $v(t) = (\cos t + t^2)(t\hat{i} + \hat{j} + 2\hat{k})$

$$v'(t) = (3t^2 - t\sin t + \cos t)\hat{i} + (2t - \sin t)(\hat{j} + 2\hat{k})$$

ii) Partial derivatives:

$$\bar{r}(t_1, t_2) = a\cos t_1\, i + a\sin t_1\, j + t_2 K$$

$$\frac{\partial \bar{r}}{\partial t_1} = -a\sin t_1\, i + a\cos t_1\, j$$

$$\frac{\partial \bar{r}}{\partial t_2} = \bar{K}$$

□

Vector Field:

A vector field in 3d space is a 3 components vector and the components are function of 3 variables.

A vector field in the plane is a 2 component vector whose components are functions of two variables.

A vector field in 3d-space:

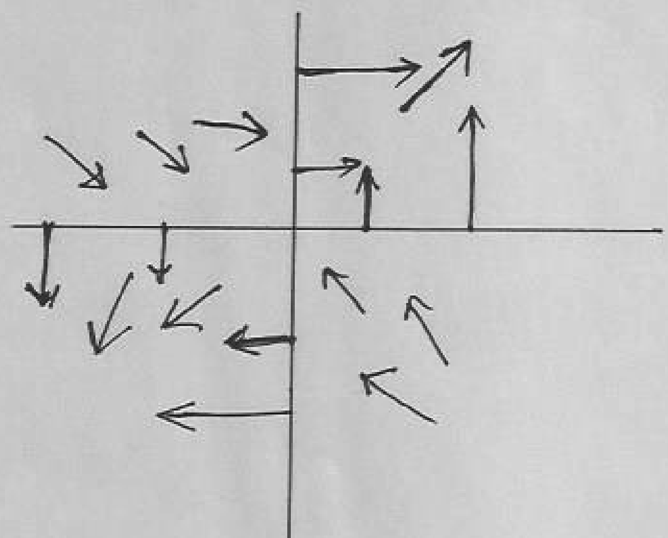
$$G(x,y,z) = f(x,y,z)\,\hat{i} + g(x,y,z)\,\hat{j} + h(x,y,z)\,\hat{k}$$

A vector field in 2d-space:

$$K(x,y) = f(x,y)\,\hat{i} + g(x,y)\,\hat{j}$$

Example: Velocity of the air within a room.

Example: $U(x,y) = y\,i + x\,j$



Similarly, a scalar field is defined (temperature inside a room)

Example: $T(x,y) = x^2+y^2$; visualization through level curves $x^2+y^2 = \text{const}$.

level curves of scalar field $T(x,y) = x^2+y^2$.

(level surface in 3D case)

- Gradient of a scalar function $f(x,y,z)$ is a vector given by

$$\text{grad } f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

- Nebla or Del operator:

$$\nabla \equiv \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}\hat{k} \quad \text{or} \quad \nabla \equiv \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

So, $\text{grad } f = \nabla f$.

- If a surface is given by $f(x,y,z) = c$, then $\nabla f(P)$ is the vector normal to the surface $f(x,y,z) = C$ at the point P.

Consider a smoth curve C on the surface passing through the point P on the surface. Let $\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ be the position vector of P.

Since the curve lies on the surface, we have

$$f(x(t), y(t), z(t)) = C$$

Then $\frac{d}{dt} f(x(t), y(t), z(t)) = 0 \Rightarrow \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt} = 0$

$$\Rightarrow \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)\cdot\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right) = 0$$

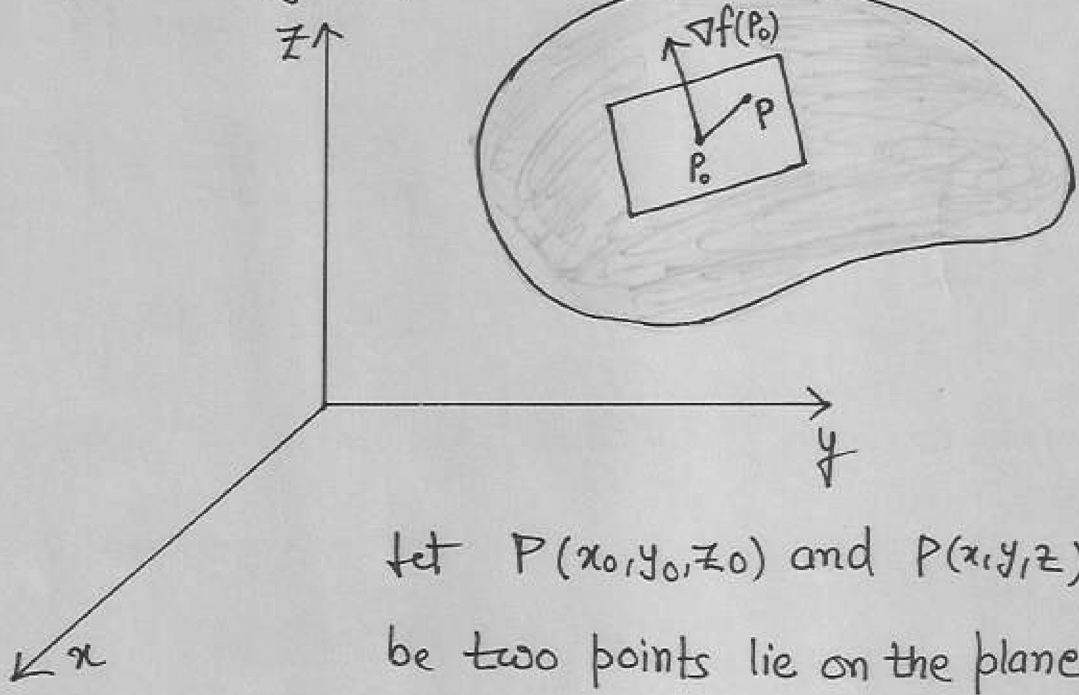
$$\Rightarrow \nabla f \cdot \bar{r}'(t) = 0$$

Note that $\bar{r}'(t)$ is a tangent vector at P and lies in the tangent plane at P. $\Rightarrow \nabla f(P)$ is a vector normal to the surface $f(x,y,z) = C$ at P.

- Unit normal vector to a surface $f(x,y,z) = C$

$$\hat{n} = \frac{\nabla f}{|\nabla f|}$$

- Equation of the tangent plane:



Let $P(x_0, y_0, z_0)$ and $P(x, y, z)$ be two points lie on the plane.

$\Rightarrow$ $\vec{P_0P}$ lies on the tangent plane

$$\Rightarrow \vec{P_0P} \cdot \nabla f(P_0) = 0 \quad \text{(Perpendicular lines)}$$

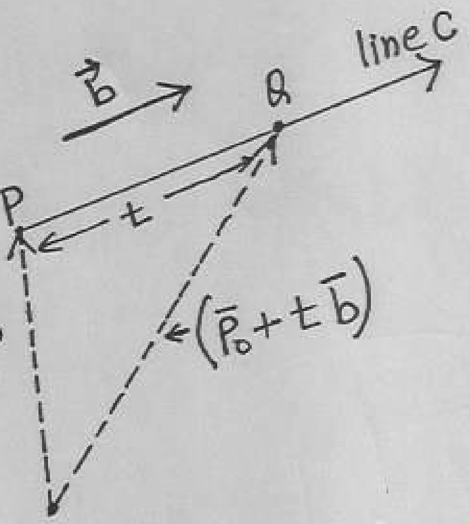
$$\Rightarrow \left((x-x_0)\hat{i} + (y-y_0)\hat{j} + (z-z_0)\hat{k}\right) \cdot \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial k}\hat{k}\right) = 0$$

$$\Rightarrow \boxed{(x-x_0)\frac{\partial f}{\partial x}(P_0) + (y-y_0)\frac{\partial f}{\partial y}(P_0) + (z-z_0)\frac{\partial f}{\partial z}(P_0) = 0}$$

- DIRECTIONAL DERIVATIVE OF $f(x,y,z)$ ALONG $\vec{b}$

  - Generalization of the notion of partial derivatives

In partial derivative: Direction is parallel to one of the coordinate axes.



let $|\bar{b}| = 1$.

Position vector of the line c is: $r(t) = \bar{P}_0 + t\bar{b} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

Using chain rule:

$$\underset{t \to 0}{\lim} \frac{f(Q) - f(P)}{t} = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$\uparrow$

rate of change of f in the direction of $\vec{b}$

$$= \left(\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z}\hat{k}\right) \cdot \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt} j + \frac{dz}{dt} k\right)$$

$$= \nabla f \cdot \frac{d\bar{r}}{dt}$$

$$= \nabla f \cdot \vec{b} \qquad \left(\nabla f|_P \cdot \vec{b}\right)$$

At any point P, the directional derivative of f represents the rate of change in f along $\bar{b}$ at the point P, it is denoted by

$$D_b f = \nabla f \cdot \bar{b}$$

Remark: Directional derivative of f in the i direction

$$= \nabla f \cdot \hat{i}$$

$$= \frac{\partial f}{\partial x}.$$

Maximum rate of change of a scalar field

Note that

Rate of change of f in the direction of $\vec{b}$ is

$$D_b f = \bar{\nabla} f \cdot \bar{b} = |\bar{\nabla} f| \, |\bar{b}| \cos\theta = |\bar{\nabla} f| \cos\theta$$

$$\Rightarrow \quad -|\nabla f| \le D_b f \le |\nabla f| \quad \text{since } -1 \le \cos\theta \le 1$$

$\Rightarrow$ Rate of change is maximum when θ is 0, that is, in the direction of $\bar{\nabla} f$.

$\Rightarrow$ Rate of change is minimum when θ is π, that is, in the opposite direction of $\bar{\nabla} f$.

$\Rightarrow$ Gradient vector $\bar{\nabla} f$ points in the direction in which f increases most rapidly and $-\nabla f$ points in the direction in which f decreases most rapidly.

Example: Find the unit normal to the surface $x^2+y^2-z=0$ at the point $(1,1,2)$.

Solution: Define $f = x^2+y^2-z \Rightarrow \nabla f = (2x, 2y, -1)^T$

$\nabla f(1,1,2) = (2,2,-1)^T$

Unit normal vector $\hat{n} = \dfrac{1}{\sqrt{4+4+1}}(2,2,-1)^T = \left(\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{1}{3}\right)^T$

The other unit normal vector is $-\hat{n} = \left(-\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{1}{3}\right)^T$.

Example: Find the directional derivative of the scalar field $f = 2x + y + z^2$ in the direction of the vector $(1,1,1)$ and evalute this at the origin.

Sol: $\nabla f = (2, 1, 2z)$

$$D_{(1,1,1)} f = \nabla f \cdot \frac{(1,1,1)}{\sqrt{3}}$$

$$= (2, 1, 2z) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$= \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} z$$

Value at the origin : $\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \sqrt{3}$.

Conservative vector field:

A vector field $\bar{v}$ is said to be conservative if the vector function can be written as the gradient of a scalar function f, that is, $\bar{v} = \nabla f$.

The function f is called a potential function or a potential of $\bar{v}$.

Example: Show that the vector field $\bar{F} = (2x+y, x, 2z)$ is conservative.

Sol: F is conservative if it can be written as $\bar{F} = \nabla \varphi$.

$$\Rightarrow \underbrace{\frac{\partial \varphi}{\partial x} = 2x + y}, \quad \frac{\partial \varphi}{\partial y} = x, \quad \frac{\partial \varphi}{\partial z} = 2z$$

$$\varphi = x^2 + xy + h(y,z) \Rightarrow x = x + \frac{\partial h}{\partial y} \Rightarrow \frac{\partial h}{\partial y} = 0 \Rightarrow h \text{ is indep. of } y.$$

Using the last eq. $2z = 0 + \frac{dh}{dz} \Rightarrow h = z^2 + c$

$$\Rightarrow \varphi = x^2 + xy + z^2 + c$$

Ans.

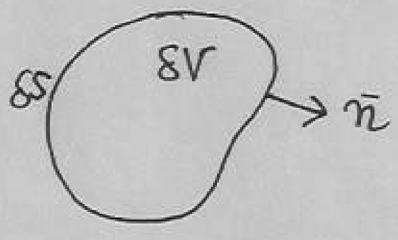
Divergence of a vector field

The divergence of a vector field $\bar{V}$ is defined as

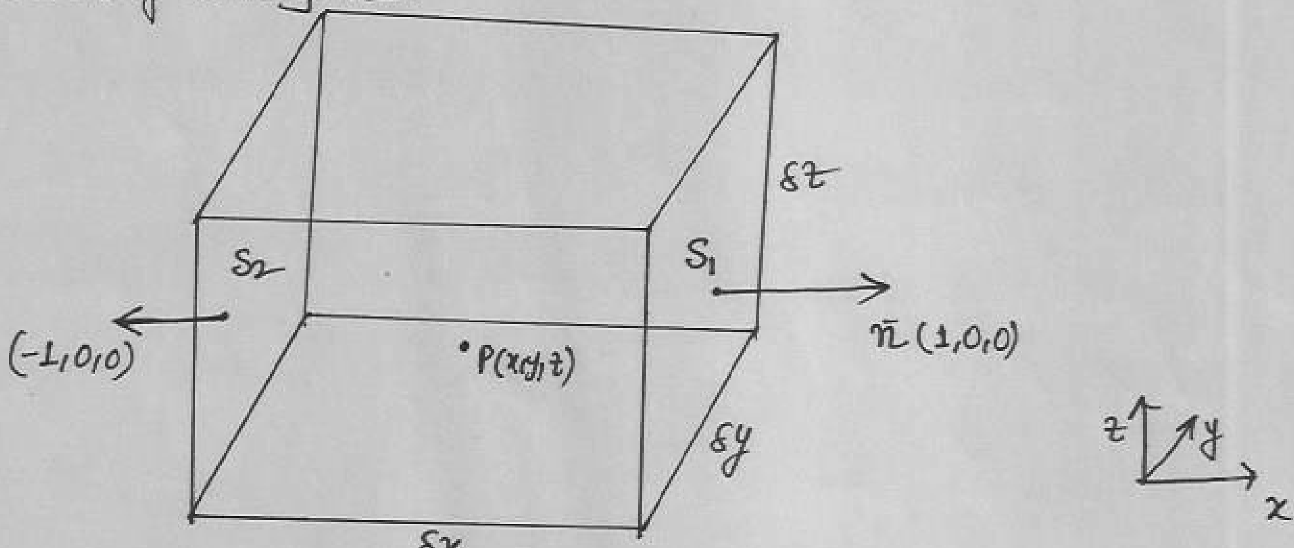
$$\operatorname{div} \bar{V} = \lim_{\delta V \to 0} \frac{1}{\delta V} \iint_{\delta S} \bar{V} \cdot \bar{n} \, ds$$

flux of the vector field $\bar{V}$ out of a small closed surface.

where δV is a small volume enclosing P with surface δS and $\bar{n}$ is the outward pointing normal to δS.



Computation of Divergence:



$$\iint_{S_1} \bar{u} \cdot \bar{n} \, ds \simeq u_1\left(x + \frac{\delta x}{2}, y, z\right) \delta y \, \delta z$$

$$\iint_{S_2} \bar{u} \cdot \bar{n} \, ds \simeq -u_1\left(x - \frac{\delta x}{2}, y, z\right) \delta y \, \delta z$$

$$\Rightarrow \iint_{S_1 + S_2} \bar{u} \cdot \bar{n} \, ds \simeq \left(u_1\left(x + \frac{\delta x}{2}, y, z\right) - u_1\left(x - \frac{\delta x}{2}, y, z\right) \right) \delta y \, \delta z$$

$$= \frac{\partial u_1}{\partial x} \delta x \, \delta y \, \delta z \simeq \frac{\partial u_1}{\partial x} \delta V$$

Similarly from other sides

$$\iint_{S_3+S_4} \bar{u}\cdot\bar{n}\, ds \simeq \frac{\partial u_2}{\partial y}\,\delta V$$

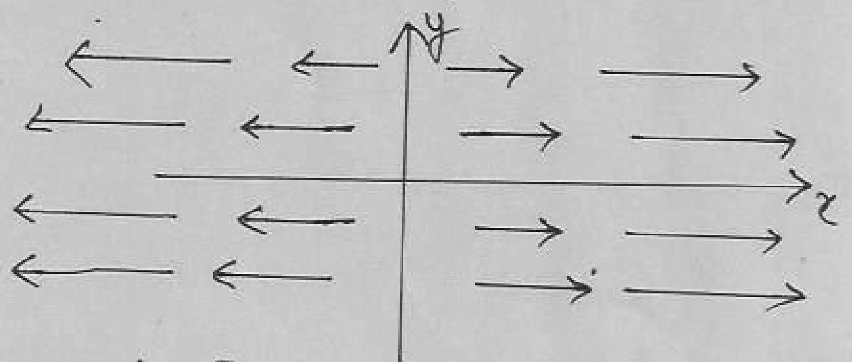
& $$\iint_{S_5+S_6} \bar{u}\cdot\bar{n}\, ds \simeq \frac{\partial u_3}{\partial z}\,\delta V$$

Therefore, $\operatorname{div} \bar{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}$

OR $$\boxed{\operatorname{div} \bar{u} = \bar{\nabla}\cdot\bar{u}}$$

Physical Interpretation: Divergence can be interpreted as the rate of expansion or compression of the vector field.

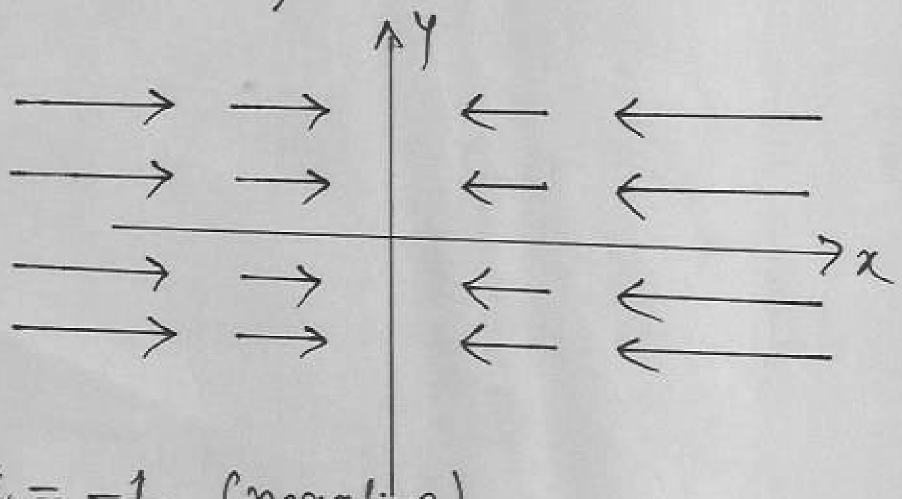
Example-1: $\bar{u} = (x, 0, 0)$



Tendency of fluid is EXPANSION.

$\operatorname{div} \bar{u} = 1$ (positive)

Example 2: $\bar{u} = (-x, 0, 0)$

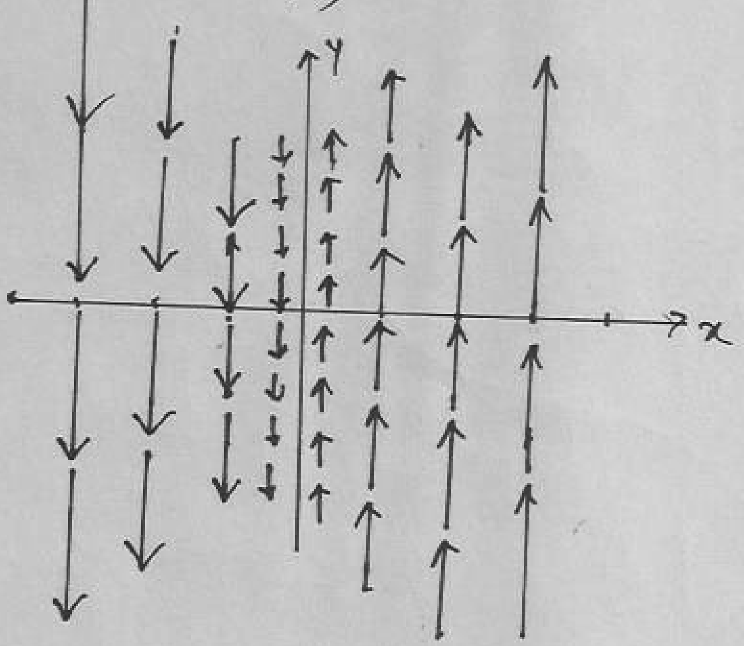


Tendency of the fluid is compression

$\operatorname{div} \bar{u} = -1$ (negative)

Example: (3):

$\bar{u} = (0, x, 0)$



Neither expanding nor contracting

$$\operatorname{div}(\bar{u}) = 0$$

Note: A vector field $\bar{v}$ for which $\bar{\nabla}\cdot\bar{v} = 0$ everywhere is said to be solenoidal. The relation $\operatorname{div}\bar{v} = 0$ is also known as the condition of incompressibility.

CURL OF A VECTOR FIELD:

Curl of a vector field is given by

$$\operatorname{Curl}\bar{F} = \bar{\nabla}\times\bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

where $\bar{v} = (v_1\hat{i} + v_2\hat{j} + v_3\hat{k})$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}.$$

Physical Interpretation: Its signifies the tendency of ROTATION. The vector $\operatorname{Curl}\bar{v}$ is directed along the axis of rotation with magnitude twice the angular speed.

Example: (Same as discussed in divergence section)

i) $\bar{u} = (x, 0, 0)$

$$\nabla \times u = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 0 & 0 \end{vmatrix} = i \cdot 0 - j \cdot 0 + k \cdot 0 = 0$$

No sense of rotation

ii) $\bar{u} = (-x, 0, 0)$

Again $\nabla \times u = 0 \Rightarrow$ No sense of rotation

iii) $\bar{u} = (0, x, 0)$

$$\nabla \times u = \hat{k}$$

$\rightarrow$ Rotation is about an axis in the z-direction.

NOTE: A vector field $\bar{u}$ for which $\nabla \times \bar{u} = 0$ everywhere is said to be irrotational.

Curl and Conservative vector field: Suppose $\bar{u}$ is conservative, i.e.,

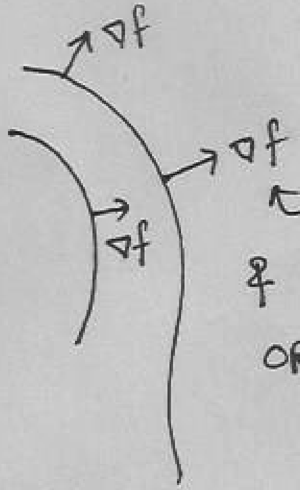
$$\bar{u} = \nabla \varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z}\right)$$

$$\nabla \times \bar{u} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{vmatrix} = i\left(\frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \varphi}{\partial y}\right)\right) + j\left(\frac{\partial}{\partial z}\left(\frac{\partial \varphi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial z}\right)\right)$$
$$+ k\left(\frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial x}\right)\right) = 0$$

Any vector field that can be written as the gradient of a scalar field is IRROTATIONAL.

Summary

a)



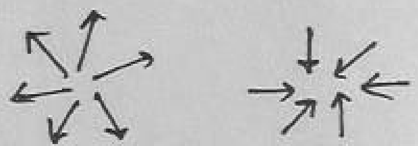
GRADIENT

← Normal vector

& max rate of change of a scalar field

OR greatest rate of change (increase) of a function

b] DIVERGENCE:



Tendency of compression or expansion

c) CURL:

Tendency to rotate

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$\nabla \cdot \bar{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}$$

If $\nabla \cdot \bar{u} = 0$, $\bar{u}$ is said to be SOLENOIDEL

If $\nabla \times \bar{u} = 0$, $\bar{u}$ is said to be IRROTATIONAL

If $\bar{u} = \nabla f$, $\bar{u}$ is said to be CONSERVATIVE.

Line Integrals:

Definitions:

Smooth curves: Let $\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ denote the position vector of a point $P(x,y,z)$ in three dimensional space.

If $\bar{r}(t)$ posses a continuous first order derivative for all values of t under consideration then the curve is known as smooth.

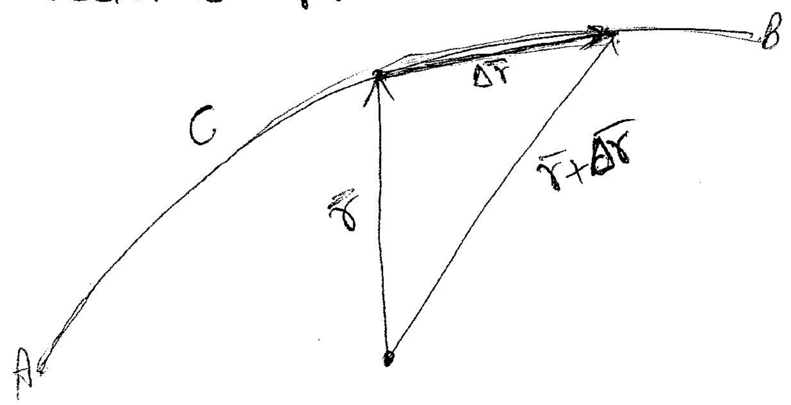
piecewise smooth if it is madeup of a finite number of smooth curves.

Simple Closed curve: A closed smooth curve which does not intersect itself anywhere is known as simple closed curve.

Smooth surfaces: A surface $\bar{r} = \bar{f}(u,v)$ is said to be smooth if $\bar{f}(u,v)$ possesses continuous first order partial derivatives.

Line integrals: (work done by a force).

Let a force $\bar{F}$ act upon a particle which is displaced along a given curve C in space from the point P whose position vector is $\bar{r}$.



first divide the curve C into a large number of small pieces.

Consider the work done when the particle moves from the position $\bar{r}$ to $\bar{r}+\Delta\bar{r}$.

On this small section of the curve C the work done is $\bar{F}\cdot\Delta\bar{r}$

Total work done $W \simeq \sum_{i=1}^{N} \bar{F}_i \cdot \Delta\bar{r}_i$

The line integral is defined as:

$$\boxed{\int_C \bar{F}\cdot d\bar{r} = \lim_{N\to\infty} \sum_{i=1}^{N} \bar{F}_i \cdot \Delta\bar{r}_i}$$

Evaluation of the line integral:

$$\boxed{\int_C \bar{F}\cdot d\bar{r} = \int_a^b \bar{F}(\bar{r}(t))\cdot \frac{d\bar{r}}{dt}\,dt}$$

In component form: $\bar{F}(\bar{r}) = \hat{i}\,F_1(x,y,z) + \hat{j}\,F_2(x,y,z) + \hat{k}\,F_3(x,y,z)$

$d\bar{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$. Then

$$\boxed{\int_C \bar{F}\cdot d\bar{r} = \int_C F_1\,dx + F_2\,dy + F_3\,dz}$$

Example: Find the work done by $\bar{F} = (y - x^2)\bar{i} + (z - y^2)\bar{j} + (x - z^2)\bar{k}$ over the curve $\bar{r}(t) = t\hat{i} + t^2\hat{j} + t^3\bar{k}$, $0 \le t \le 1$ from $(0,0,0)$ to $(1,1,1)$.

Solution: $$\frac{d\bar{r}}{dt} = \hat{i} + 2t\hat{j} + 3t^2\bar{k}$$

$$\bar{F}(\bar{r}t) = (t^2 - t^2)\hat{i} + (t^3 - t^4)\hat{j} + (t - t^6)\bar{k}$$

$$= (t^3 - t^4)\hat{j} + (t - t^6)\bar{k}$$

$$\bar{F} \cdot \frac{d\bar{r}}{dt} = 2t(t^3 - t^4) + 3t^2(t - t^6)$$

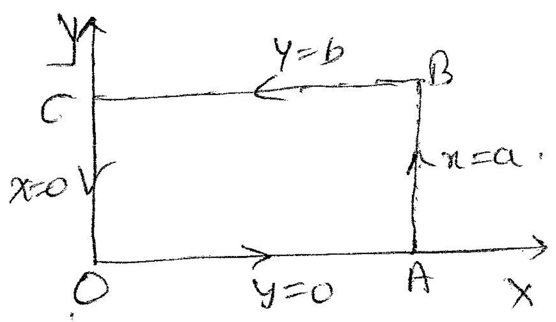
$$= 2t^4 - 2t^5 + 3t^3 - 3t^8$$

$$\int_C \bar{F} \cdot d\bar{r} = \int_{t=0}^{1} (2t^4 - 2t^5 + 3t^3 - 3t^8)\, dt$$

$$= \frac{29}{60}.$$ Ans.

Example: 2: Evaluate $\int_C \bar{F} \cdot d\bar{r}$ $\quad \bar{F} = (x^2 + y^2)\hat{i} - 2xy\hat{j}$

C: rectangle in xy plane bounded by ~~y=0, y=b, x=0, x=a~~
$y = 0, x = a, y = b, x = 0$



$$\int_C \vec{F} \cdot d\bar{r} = \int_C \left((x^2 + y^2)\hat{i} - 2xy\hat{j} \right) \cdot \left(dx\,\hat{i} + dy\,\hat{j} \right).$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C \left[(x^2+y^2)\,dx - 2xy\,dy\right].$$

Along OA: $y=0$, $dy=0$ & x varies from 0 to a.

$$\int_{OA} \bar{F}\cdot d\bar{r} = \int_0^a x^2\,dx = \frac{a^3}{3}$$

Along AB: $$\int_{AB} \bar{F}\cdot d\bar{r} = \int_0^b -2a\cdot y\,dy = -ab^2.$$

Along BC: $$\int_{BC} \bar{F}\cdot d\bar{r} = \int_a^0 (x^2+b^2)\,dx$$

$$= -\left[\frac{a^3}{3} + ab^2\right]$$

Along DO: $$\int_{CO} \bar{F}\cdot d\bar{r} = \int_b^0 0\cdot dy = 0$$

$$\Rightarrow \int_C \bar{F}\cdot d\bar{r} = -2ab^2$$

Ans.

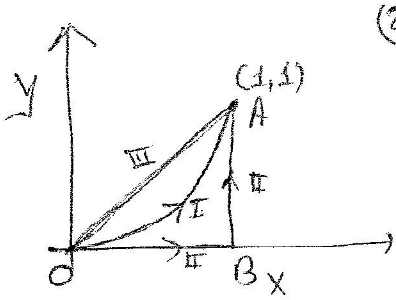
<u>Example:</u> If $\bar{F} = y\hat{i} - x\hat{j}$ Evaluate $\int_C \bar{F}\cdot d\bar{r}$ from $(0,0)$ to $(1,1)$ along the following ~~that~~ path:

i) the parabola $y = x^2$

ii) the straight line $(0,0)$ to $(1,0)$ and then to $(1,1)$

iii) the straight line joining $(0,0)$ to $(1,1)$.

Solution:



$$\bar{r} = x\hat{i} + y\hat{j}$$

$$d\bar{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C (y\hat{i} - x\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_C y\,dx - x\,dy.$$

i) parabola $y = x^2 \Rightarrow dy = 2x\,dx$. x varies from 0 to 1.

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 x^2\,dx - x\cdot 2x\,dx$$

$$= \int_0^1 -x^2\,dx = -\frac{1}{3}.$$

ii) along OB & then BA:

$$\int_C \bar{F}\cdot d\bar{r} = \int_{OB} (y\,dx - x\,dy) + \int_{BA} (y\,dx - x\,dy)$$

along OB, $y = 0$ $dy = 0$

along BA $x = 1$ $dx = 0$,

$$\int_C \bar{F}\cdot d\bar{r} = \int_{y=0}^{1} -dy = -1.$$

iii) Along straight line $\bar{O}A$. along $\bar{O}A$ $y = x$.
$dy = dx$.

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 (x\,dx - x\,dx) = 0.$$

Ans.

Example: If $\bar{F} = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\bar{k}$ evaluate $\int_c \bar{F}\cdot d\bar{r}$ where c is the straight line joining $(0,0,0)$ to $(1,1,1)$.

Solution: Equation of the line:

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} = t \text{ (parameter)}$$

$$x = t \qquad y = t \qquad z = t$$

$$\bar{r} = t\hat{i} + t\hat{j} + t\bar{k}$$

$$\frac{d\bar{r}}{dt} = (\hat{i} + \hat{j} + \bar{k})$$

$$F = (3t^2 + 6t)\hat{i} - 14t^2\hat{j} + 20t^3\bar{k}$$

$$\int_c \bar{F}\cdot d\bar{r} = \int_0^1 [(3t^2+6t)\hat{i} - 14t^2 j + 20t^3\bar{k}]\cdot[\hat{i}+\hat{j}+\bar{k}]\,dt$$

$$= \int_0^1 [3t^2 + 6t - 14t^2 + 20t^3]\,dt$$

$$= \int_0^1 [-11t^2 + 6t + 20t^3]\,dt$$

$$= \frac{-11}{3} + \frac{6}{2} + \frac{20}{4}$$

$$= \frac{13}{3} \quad \text{Ans}.$$

Example: Find the total work done in moving a particle in a force field $\bar{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = t^2+1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to 2.

Solution:

$$\int_c \bar{F}\cdot d\bar{r} = \int_c 3(t^2+1)(2t^2)(2t) - 5t^3(4t) + 10(t^2+1)3t^2\,dt$$

$$= 303 \quad \text{Ans}.$$

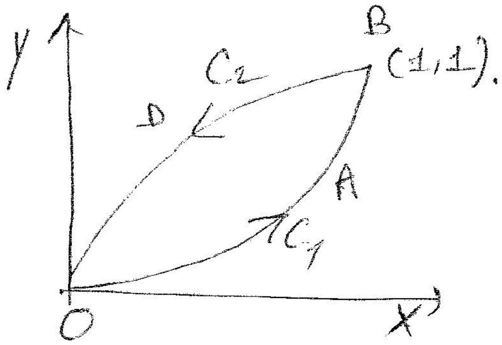
Note: The integral around a closed curve, $\oint$, is called Circulation integral.

Example: Find the circulation of F around the curve C where
$\bar{F} = (2x+y^2)\hat{i} + (3y-4x)\hat{j}$ and C in the curve
$y=x^2$ for $(0,0)$ to $(1,1)$ and the curve $y^2=x$ from $(1,1)$ to $(0,0)$.

Solution: $\bar{r} = x\hat{i} + y\hat{j}$

$d\bar{r} = \hat{i}dx + \hat{j}dy$

$F \cdot d\bar{r} = (2x+y^2)dx + (3y-4x)dy$



$$\int_C \bar{F}\cdot d\bar{r} = \int_{C_1} \bar{F}\cdot d\bar{r} + \int_{C_2} \bar{F}\cdot d\bar{r}$$

Along OAB: $y=x^2$ $dy = 2x\,dx$.

$$\int_{C_1} \bar{F}\cdot d\bar{r} = \int_{C_1} (2x+x^4)dx + (3\cdot x^2 - 4x)\cdot 2x\,dx.$$

$$= \int_0^1 (x^4+6x^3-8x^2+2x)\,dx$$

$$= \left[\frac{x^5}{5} + 6\frac{x^4}{4} - 8\cdot\frac{x^3}{3} + 2\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{30}.$$

Along C_2: $x=y^2$ $dx = 2y\,dy$. y varies from 0 to 1.

$$\int_{C_2} \bar{F}\cdot d\bar{r} = \int_{y=1}^{0} (2y^2+y^2)\,2y\,dy + (3y-4\cdot y^2)\cdot \cancel{2ydx}\,dy.$$

$$= -\int_0^1 6y^3 - 4y^2 + 3y$$

$$= -5/3.$$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \frac{1}{30} - \frac{5}{3} = -\frac{49}{30} \quad \text{Ans.}$$

Problem set 6

Spring 2018

MATHEMATICS-II (MA10002)(Integral Calculus)

1. Discuss the convergence of the following integrals using definition:

   i. $\int_0^1 \frac{1}{1-x} dx$

   ii. $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$

   iii. $\int_0^2 \frac{1}{\sqrt{x(2-x)}} dx$

   iv. $\int_1^\infty \frac{1}{x \log x} dx$

   v. $\int_1^\infty \frac{1}{(1+x)\sqrt{x}} dx$

   vi. $\int_1^3 \frac{10x}{(x^2-9)^{\frac{1}{3}}} dx$

2. Discuss the convergence of the following integrals:

   i. $\int_0^1 \frac{1}{(x+1)(x+2)\sqrt{x(1-x)}} dx$

   ii. $\int_0^{\frac{\pi}{2}} \frac{1}{e^x - \cos x} dx$

   iii. $\int_0^1 \frac{x^{p-1}+x^{-p}}{1+x} dx$

   iv. $\int_0^{\frac{\pi}{2}} \log(\sin x) dx$

   v. $\int_0^\infty \frac{1-\cos x}{x^2} dx$

   vi. $\int_0^\infty \frac{\cos x}{\sqrt{x^3+x}} dx$

   vii. $\int_0^\infty (\frac{1}{x^2} - \frac{1}{x \sinh x}) dx$

   viii. $\int_0^1 x^{n-1} \log x \, dx$

   ix. $\int_1^\infty \frac{e^x}{\sqrt{x^2 - \frac{1}{2}}} dx$

   x. $\int_0^1 \frac{1}{1-x^4} dx$

3. Show that the improper integral $\int_0^\infty \mid \frac{\sin x}{x} \mid dx$ is not convergent.

4. Examine the convergence of $\int_0^\infty \frac{1}{e^x - x} dx$.

5. Show that the improper integral $\int_0^\infty \frac{\sin x(1-\cos x)}{x^n} dx$ is convergent if $0 < n < 4$.

6. Evaluate $\int_0^\infty \frac{5\sin(4x) - 4\sin(5x)}{x^2} dx$.

7. Show that $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is convergent if and only if $m, n > 0$. Find the value of the integral for $m = \frac{5}{2}, n = \frac{7}{2}$.

8. Show that $\int_0^\infty x^{m-1}e^{-x}dx$ is convergent if and only if $m > 0$. Find the value of the integral for $m = 2019$.

9. Evaluate $\int_0^1 x^4(1-\sqrt{x})^5 dx$.

10. Express the following integral in terms of Gamma function: $\int_0^\infty \frac{x^a}{a^x}dx, \ (a > 1)$.

11. Prove that $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}, 0 < n < 1$(Using $\int_0^\infty \frac{x^{n-1}}{1+x}dx = \frac{\pi}{\sin(n\pi)}$).

12. Prove that

    i. $\int_0^1 (\log \frac{1}{y})^{n-1}dy = \Gamma(n)$

    ii. $\int_0^{\frac{\pi}{2}} \tan^p \theta d\theta = \frac{\pi}{2} \sec \frac{p\pi}{2}$ and indicate the restriction on the values of p.

    iii. $\int_a^b (x-a)^{m-1}(b-x)^{n-1}dx = (b-a)^{m+n-1}B(m,n), m > 0, n > 0$.

    iv. $\int_0^1 \frac{1}{(1-x^6)^{\frac{1}{6}}}dx = \pi/3$.

    v. $\int_0^1 \frac{x^2dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}$

13. Using Beta and Gamma functions, evaluate the integral

$$I = \int_{-1}^{1} (1-x^2)^n dx, \text{ where n is a positive integer.}$$

14. Show that

$$\Gamma(2n) = \frac{2^{2n-1}}{\sqrt{\pi}}\Gamma(n+\frac{1}{2})\Gamma(n)$$

and

$$\Gamma(\frac{1}{4})\Gamma(\frac{3}{4}) = \pi\sqrt{2}.$$

Hints and answers

1. Use definition.
   i. divergent
   ii. convergent
   iii. convergent
   iv. divergent
   v. convergent
   vi. convergent
2. i. convergent. Calculate $\lim_{x\to 0+} \sqrt{x} f(x)$ and $\lim_{x\to 1-} \sqrt{1-x} f(x)$ and apply μ test.
   ii. divergent. Calculate $\lim_{x\to 0+} \frac{x}{e^x - \cos x}$ and apply μ test.
   iii. convergent if $0 < p < 1$. Use μ test on $\int_0^1 \frac{x^{p-1}}{1+x} dx$ and $\int_0^1 \frac{x^{-p}}{1+x} dx$.
   iv. convergent. Use comparison test using the function $\frac{1}{\sqrt{x}}$.
   v. convergent. Use comparison test.
   vi. convergent. Use comparison test on the integral $\int_0^\infty \frac{1}{\sqrt{x^3+x}} dx$.
   vii. convergent. Use comparison test.
   viii. convergent if and only if $n > 0$. Use the fact $\lim_{x\to 0+} x^r \log x = 0$ for all $r > 0$ and use comparison test.
   ix. divergent. Use comparison test.
   x. divergent. Use comparison test.
3. divergent. Use $\int_0^{n\pi} |\frac{\sin x}{x}| dx = \sum_{r=1}^n \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x} dx$ and show $\int_0^{n\pi} |\frac{\sin x}{x}| dx \geq \frac{2}{\pi}(1 + \frac{1}{2} + \cdots + \frac{1}{n})$ and take $n \to \infty$.
4. convergent. Use the function e^{-x} and use comparison test.
5. Use comparison test.
6. $20 \log(\frac{5}{4})$.
7. Use comparison test. The value is $\frac{3\pi}{2^8}$.
8. Use comparison test. The value is 2018!.
9. Value of the integral $\frac{1}{15015}$.

10. Value of the integral $\frac{\Gamma(a+1)}{(\log a)^{a+1}}$.

11. First prove $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$, then change the value of m, n suitably.

12. Prove that

    i. Put $y = e^x$.

    ii. $p \in (-1,\ 1)$(Use the solution of Problem 11).

    iii. Put $y = \frac{x-a}{b-a}$.

    iv. Put $t = x^6$ and use Problem 11.

    v. Put $x^2 = \sin\theta$ for the first integral and $x^2 = \tan\theta$ for the second integral.

13. $\frac{2^{2n+1}(n!)^2}{(2n+1)!}$, put $x = 2t - 1$.

14. Use the relation $\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$ and put $m = n$ in the equation. For the next part , put $n = \frac{1}{4}$.

Problem Set - 7

SPRING 2018

MATHEMATICS-II (MA10002) (Integral Calculus)

1. Evaluate the integral $\int_0^1 \frac{x^\alpha - 1}{\log x}\, dx$, $(\alpha > -1)$ by applying differentiating under the integral sign.

2. Using differentiation under integral sign prove the following:

   (i) $\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}\, dx = \frac{\pi}{2}\log(a+1)$, where $a \geq 0$ and $a \neq 1$,

   (ii)Using the result $\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$, show that $\int_0^\infty e^{-x^2}\cos(2\alpha x)\, dx = \frac{\sqrt{\pi}}{2}e^{-\alpha^2}$,

   (iii) $\int_0^t \frac{\log(1+tx)}{1+x^2}\, dx = \frac{\tan^{-1}(t)}{2}\log(1+t^2)$

3. Let $f(x,t) = (x+t^3)^2$ then

   (i) find $\int_0^1 f(x,t)\, dx$.

   (ii) Prove that $\frac{d}{dt}\int_0^1 f(x,t)\, dx = \int_0^1 \frac{\partial}{\partial t} f(x,t)\, dx$.

4. For any real numbers x and t, let

$$f(x,t) = \begin{cases} \frac{xt^3}{(x^2+t^2)^2} & \text{if } x \neq 0, t \neq 0 \\ 0 & \text{if } x = 0, t = 0 \end{cases}$$

   and $F(t) = \int_0^1 f(x,t)\, dx$. Is $\frac{d}{dt}\int_0^1 f(x,t)\, dx = \int_0^1 \frac{\partial}{\partial t} f(x,t)\, dx$? Give the justification.

5. Find the value of the integral $\int_0^\infty \frac{e^{-\alpha x}\sin x}{x}\, dx$, where $\alpha > 0$ and deduce that

   (i) $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$

   (ii) $\int_0^\infty \frac{\sin ax}{x}\, dx = \frac{\pi}{2}$

6. Find the value of the following integrals

   (i) $\int_0^\infty (e^{-x} - e^{-tx})\frac{dx}{x}$

(ii) $\displaystyle\int_0^\infty \frac{e^{-x}}{x}(a - \frac{1}{x} + \frac{1}{x}e^{-ax})\, dx$

(iii) $\displaystyle\int_0^1 \frac{x^a - x^b}{\log x}\, dx$

7. Find the value of $\displaystyle\int_0^\pi \frac{dx}{a + b\cos x}$ when $(a > 0, |b| < a)$

and deduce that $\displaystyle\int_0^\pi \frac{dx}{(a + b\cos x)^2} = \frac{\pi a}{(a^2 - b^2)^{\frac{3}{2}}}$.

8. Evaluate the following integrals over the region D

(i) $\displaystyle\int\int_D xy\, dxdy$, where D is the region bounded by the x-axis, the line $y = 2x$ and the parabola $y = x^2/4a$.

(ii) $\displaystyle\int\int_D e^{x^2}\, dxdy$ where the region D is given by $R : 2y \le x \le 2$ and $0 \le y \le 1$.

(iii) $\displaystyle\int\int_D x^2\, dxdy$, where D is the region in the first quadrant bounded by the hyperbola $xy = 16$ and the lines $y = x, y = 0$ and $x = 8$.

(iv) $\displaystyle\int\int_D \sqrt{xy - y^2}\, dydx$, where D is a traiangle with vertices $(0, 0), (10, 1)$ and $(1, 1)$.

9. Evaluate the following integrals by changing the order of integration

(i) $\displaystyle\int_0^{\pi/2}\int_x^{\pi/2} \frac{\sin y}{y}\, dydx,$

(ii) $\displaystyle\int_0^1\int_x^1 e^{y^2}\, dxdy,$

(iii) $\displaystyle\int_0^2\int_0^{y^2/2} \frac{y}{\sqrt{x^2 + y^2 + 1}}\, dxdy,$

(iv) $\displaystyle\int_0^1\int_{x^2}^{2-x} xy\, dydx,$

(v) $\displaystyle\int_0^\infty\int_0^x e^{-xy} y\, dydx,$

Problem Set 7 - Hints and answers

1. $\log(\alpha+1)$. Consider $\phi(\alpha) = \displaystyle\int_0^1 \frac{x^\alpha - 1}{\log x}\, dx$, first differentiate wrt α, then integrate.

2. Prove that

   i. Proceed as 1.

   ii. Proceed as 1.

   iii. Proceed as 1.

3. i. $\frac{1}{3} + t^3 + t^6$.

   ii. Prove that L.H.S.=R.H.S.= $3t^2 + 6t^5$

4. First compute $F(t)$, then compute $\frac{\partial}{\partial t} f(x,t)$ and show that $\frac{\partial}{\partial t} f(x,t)$ is not a continuous function of (x,t) at (0,0).

5. $\frac{\pi}{2} - \tan^{-1}(\alpha)$. Process as 1.

   i. Put $\alpha = 0$.

   ii. Put $x = ay$ in (i).

6. i. First differentiate wrt t, then integrate.

   ii. First differentiate wrt a, then integrate.

   iii. First differentiate wrt a, then integrate.

7. Use $\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} = 1$ and $\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \cos x$. Answer: $\dfrac{\pi}{\sqrt{a^2 - b^2}}$, then differentiate wrt a.

8. i. $\frac{2048}{3} a^4$.

   ii. $\frac{1}{4}(e^4 - 1)$. Use change of order of integration.

   iii. 448.

   iv. 6.

9. i. 1.

   ii. $\frac{1}{2}(e-1)$.

   iii. $\frac{5}{4}\ln 5 - 1$.

   iv. $\frac{3}{8}$.

   v. $\dfrac{\sqrt{\pi}}{2}$. Use $\displaystyle\int_0^\infty e^{-y^2}\, dy = \frac{\sqrt{\pi}}{2}$.

Problem Set - 8

Spring 2018

MATHEMATICS-II (MA10002)

1. (a) Find the jacobian of the following transformations T.

   (i) $T : x + y = u,\ y = uv$. Find $J = \dfrac{\partial(x, y)}{\partial(u, v)}$.

   (ii) $T : x = 2u + 3v,\ y = 2u - 3v$. Find $J = \dfrac{\partial(x, y)}{\partial(u, v)}$.

   (iii) $T : x + y + z = u,\ x + y = uv,\ x = uvw$. Find $J = \dfrac{\partial(x, y, z)}{\partial(u, v, w)}$.

   (iv) $T : x = r \cos\phi \sin\theta,\ y = r \sin\phi \sin\theta,\ z = r \cos\theta$. Find $J = \dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$.

   (b) Evaluate the double integrations using change of variable.

   (i) Evaluate $\displaystyle\iint_R \sqrt{x^2 + y^2}\, dxdy$, the field of integration being R, the region in xy plane bounded by the circle $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. [Hint: $x = r\cos\theta$, $y = r\sin\theta$.]

   (ii) Using the transformation $x+y = u$, $y = uv$, show that $\displaystyle\iint_E e^{\frac{y}{x+y}}\, dxdy = \frac{1}{2}(e-1)$ where E is the triangle bounded by $x = 0$, $y = 0$, $x + y = 1$.

   (iii) Evaluate $\displaystyle\iint_R (x + y)\, dA$, where R is the trapezoidal region with vertices given by $(0, 0)$, $(5, 0)$, $(\frac{5}{2}, \frac{5}{2})$ and $(\frac{5}{2}, -\frac{5}{2})$ using the transformation $x = 2u + 3v$ and $y = 2u - 3v$.

   (c) Evaluate

$$\iint_R \frac{\sqrt{a^2b^2 - b^2x^2 - a^2y^2}}{\sqrt{a^2b^2 + b^2x^2 + a^2y^2}}\, dxdy,$$

   the field of integration being R, the positive quadrant of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. [Hint: change ellipse to a circle using $x = au, y = bv$.]

2. Show that $\displaystyle\iint_E y\, dxdy = \frac{1}{3}a^3 - \frac{a^2}{2}k + \frac{b}{4}k^2 + \frac{1}{6}k^3$ where $k = -\frac{b}{2} + \sqrt{a^2 + \frac{b^2}{4}}$ and E is the region in the first quadrant bounded by x-axis, the curves $x^2 + y^2 = a^2$, $y^2 = bx$.

3. Find the value of the following triple integrals.

   a) $\displaystyle\iiint_R (x + y + z)\, dxdydz$ where $R : 0 \le x \le 1,\ 1 \le y \le 2,\ 2 \le z \le 3$.

   b) $\displaystyle\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\, dzdydx.$

4. Compute $\iiint \frac{dxdydz}{(1+x+y+z)^3}$ if the region of integration is bounded by the co-ordinate planes and the plane $x+y+z=1$.

5. Evaluate $\iiint x^2yz\,dxdydz$ throughout the volume bounded by the planes $x=0$, $y=0$, $z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, by using $x=av$, $y=bv$, $z=cw$.

6. Using spherical co-ordinate evaluate $\iiint (x^2+y^2+z^2)\,dxdydz$ enclosed by the sphere $x^2+y^2+z^2=1$.

7. Evaluate $\iiint\limits_R y\,dV$ where R is the region lies below the plane $z=x+1$ above the xy-plane and between the cylinders $x^2+y^2=1$ and $x^2+y^2=4$.

8. Find the surface area of the cylinder $x^2+z^2=4$ inside the cylinder $x^2+y^2=4$.

9. Find the surface area of the sphere $x^2+y^2+z^2=9$ lying inside the cylinder $x^2+y^2=3y$.

10. Find the surface area of the section of the cylinder $x^2+y^2=a^2$ made by the plane $x+y+z=a$.

11. Find the volume of the solid bounded by the parabolic $y^2+z^2=4x$ and the plane $x=5$.

12. Calculate the volume of the solid bounded by the following surfaces

$$z=0,\ x^2+y^2=1,\ x+y+z=3.$$

13. Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$.

******************** END ********************

Problem Set - 8 Hints and Answers **Spring 2018**

MATHEMATICS-II (MA10002)

1. (a) (i) Ans: u.
   (ii) Ans: -12.
   (iii) Ans: $-u^2v$.
   (iv) Ans: $r^2 \sin\theta$.

   (b) (i) Ans: $\frac{14}{3}\pi$. [Hint: Take $x = r\cos\theta$, $y = r\sin\theta$.]
   (ii) Ans: $\frac{1}{2}(e-1)$.
   (iii) Ans: $\frac{125}{4}$.

   (c) Ans: $\frac{\pi}{4}ab\left(\frac{\pi}{2}-1\right)$. [Hint: Change ellipse to a circle using $x = au, y = bv$.]

2. [Hint: Divide the region E into two subregions E_1, E_2 both of which are quadratic with respect to y-axis.]

3. a) Ans: $\frac{9}{2}$.
   b) Ans: $\frac{8}{3}\log 2 - \frac{19}{9}$.

4. Ans: $\frac{1}{2}\log 2 - \frac{5}{16}$.

5. Ans: $\frac{a^3b^2c^2}{2520}$.

6. Ans: $\frac{4\pi}{5}$.

7. Ans: 0. [Hint: Use cylindrical coordinates.]

8. Ans: 32.

9. Ans: $18(\pi - 2)$.

10. Ans: $\sqrt{3}\pi a^2$.

11. Ans: 50π.

12. Ans: 3π.

13. Ans: 16π.

******************* END *******************

1. (a) Find whether the vectors $2x^3+x^2+x+1$, x^3+3x^2+x-2 and x^3+2x^2-x+3 of $P(x)$, the vector space of all polynomials over the real field, are linearly independent or not.

   (b) Let $V = \mathbb{R}^2$ be the set of all ordered pairs (x, y) of real numbers. Examine whether V is a vector space over $\mathbb{R}$ with the following two operations:

$$(x, y) + (x_1, y_1) = (x + x_1, y + y_1),$$

$$c(x, y) = (|c|x, |c|y),$$

   where $(x, y), (x_1, y_1) \in V$ and $c \in \mathbb{R}$. [2 + 2 = 4 marks]

2. (a) Find the dimension of the subspace V of $\mathbb{R}^4$, spanned by $S = \{(1, -1, 2, 3), (11, 24, 29, 61), (2, 3, 5, 10)\}$. Also find a subset of S, which is a basis of V.(NO MARKS will be awarded if the basis is not a subset of S)

   (b) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by $T(x, y, z) = (2x - 3y + 4z, -x - y - z)$. Then

      (i) find a basis of $N(T)$, the null space of T,

      (ii) find the dimension of the range space $R(T)$ of T. [2 + 2 = 4 marks]

3. (a) Using rank concept of matrices, find the real constants a, b, c, d, if exist, such that the graph of the function

$$f(x) = ax^3 + bx^2 + cx + d,$$

   passes through the points $(-1, 5), (-2, 7), (2, 11)$ and $(3, 37)$.

   (b) Find a system of linear equations whose set of solutions is given by

$$S = \{(1 - a, 1 + a, a) : a \in \mathbb{R}\}$$

   [3 + 1 = 4 marks]

4. (a) Check the diagonalizability of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   If it is diagonalizable, then find a diagonal matrix D which is similar to A.

(b) Find all real values a and b such that the matrix

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & ia \\ \frac{-i}{\sqrt{2}} & b \end{bmatrix}$$

is unitary. [3 + 2 = 5 marks]

5. (a) Rearrange the augmented matrix of the following system by only row-interchanges to place maximum-magnitude(absolute value) element of each row of the coefficient matrix in diagonal position, and then make these diagonal elements unity by appropriate elementary row-operations to augmented matrix(with entries 4 decimal places rounding):

$$\begin{aligned} 0.45x_1 + 0.30x_2 - 15.00x_3 &= 14.28 \\ 4.50x_1 + 0.15x_2 + 0.30x_3 &= 1.57 \\ 0.15x_1 - 10.50x_2 + 0.45x_3 &= -3.86. \end{aligned}$$

(b) Write down the three algebraic equations(not in matrix form) of Jacobi iteration scheme from the modified augmented matrix, obtained in (a), determining $x_i^{(n+1)}, i = 1, 2, 3; n = 0, 1, 2, \ldots$.

(c) Compute all iteration values up to 4 decimal places (round off) in tabular form under columns $n, x_1^{(n)}, x_2^{(n)}, x_3^{(n)}$, for $n = 0, 1, 2, 3$ only, using the equations obtained in (b), with starting solution $x_i^{(0)} = 0$, for $i = 1, 2, 3$.

(d) From the modified augmented matrix obtained in (a), write down only the three algebraic equations(not in matrix form) of Gauss-Seidel method determining $x_i^{(n+1)}, i = 1, 2, 3; n = 0, 1, 2, \ldots$.. [1 + 0.5 + 2 + 0.5 = 4 marks]

6. (a) Find a root of the equation $e^{2x} = x + 6$ correct up to 4 decimal places by using the Newton-Raphson method starting with initial guess $x_0 = 0.97$.

(b) The algebraic equation $x^4 + x - 1 = 0$ has a root α lying in the interval $[0.5, 1.0]$. Starting with this interval, use bisection method to find an interval of width 0.125 which contains α. [2 + 2 = 4 marks]

7. (a) Determine the interpolation polynomial by means of Lagrange interpolation formula for the points $(-2, -10), (0, 2), (1, 8)$.

(b) Suppose the function $f(x) = \ln(1 + x)$ is approximated using polynomial interpolation with $x = 0$ and $x = 1$. Let $p(x)$ denote the interpolating polynomial. Find an upper bound (as small as possible) for the error magnitude $\max\limits_{0 \leq x \leq 1} |f(x) - p(x)|$ without determining $p(x)$.

[2 + 3 = 5 marks]

Solution Model I

DB
26.02.2018

Q1) a) Find whether the vectors $2x^3+x^2+x+1$, x^3+3x^2+x-2 & x^3+2x^2-x+3 of $P(x)$, the vector space of all polynomials over the real field, are linearly independent or not.

Solⁿ:- Let a, b, c be scalars (reals) such that

$$a(2x^3+x^2+x+1)+b(x^3+3x^2+x-2)+c(x^3+2x^2-x+3)=\theta$$

$$\Rightarrow (2a+b+c)x^3+(a+3b+2c)x^2+(a+b-c)x+(a-2b+3c)=\theta \rightarrow (1)$$

Equating the co-efficients of like powers of x on both sides of (1), we get —

$$\left.\begin{aligned} 2a+b+c&=0\\ a+3b+2c&=0\\ a+b-c&=0\\ a-2b+3c&=0 \end{aligned}\right\} \rightarrow (2) \longrightarrow (1M)$$

The co-efficient matrix A of the system of eqⁿ (2) is —

$$A=\begin{pmatrix}2&1&1\\1&3&2\\1&1&-1\\1&-2&3\end{pmatrix} \sim \begin{pmatrix}1&3&2\\2&1&1\\1&1&-1\\1&-2&3\end{pmatrix} R_1 \sim R_2$$

$$\sim \begin{pmatrix}1&3&2\\0&-5&-3\\0&-2&-3\\0&-5&1\end{pmatrix} \begin{matrix}R_2\to R_2-2R_1\\R_3\to R_3-R_1\\R_4\to R_4-R_1\end{matrix} \sim \begin{pmatrix}1&3&2\\0&1&3/5\\0&-2&-3\\0&-5&1\end{pmatrix} R_2\to -\tfrac{1}{5}R_2$$

$$\sim \begin{pmatrix}1&3&2\\0&1&3/5\\0&0&-9/5\\0&0&4\end{pmatrix} \begin{matrix}R_3\to R_3+2R_2\\R_4\to R_4+5R_2\end{matrix}$$

$$\sim \begin{pmatrix}1&3&2\\0&1&3/5\\0&0&1\\0&0&4\end{pmatrix} R_3\to \left(-\tfrac{5}{9}\right)R_3 \sim \begin{pmatrix}1&3&2\\0&1&3/5\\0&0&1\\0&0&0\end{pmatrix} R_4\to R_4-4R_3$$

which is in echelon form.

$\therefore$ rank $(A)=3=$ no. of non-zero rows in its echelon form.

$=$ no. of unknowns a, b, c in the system of eqⁿ(2)

$\therefore$ the system (2) has the only sol^n $a=0, b=0, c=0$

$\therefore$ the given set of vectors is linearly independent. $\longrightarrow$ (1M)

N.B. : Full marks have not been given if eq^n (2) is solved without showing the steps clearly.

Q1) b) Let $V = \mathbb{R}^2$ be the set of all ordered pairs (x, y) of real numbers. Examine whether V is a vector space over $\mathbb{R}$ with the following two operations:

$$(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$$

$$c(x, y) = (|c|x, |c|y),$$

where $(x, y), (x_1, y_1) \in V$ and $c \in \mathbb{R}$.

Sol^n:- Given $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$

$$c(x, y) = (|c|x, |c|y).$$

We shall show that in this case, the postulate

$$(a+b)\alpha = a\alpha + b\alpha, \ \forall a, b \in F \ \& \ \alpha \in V \text{ fails.}$$

Let $\alpha = (x, y)$ & $a, b \in \mathbb{R}$. We have,

$$(a+b)\alpha = (a+b)(x, y) = (|a+b|x, |a+b|y) \longrightarrow (1) \text{ (by def}^n\text{)}$$

Also, $a\alpha + b\alpha = a(x, y) + b(x, y)$

$= (|a|x, |a|y) + (|b|x, |b|y)$ (by def^n of scalar multiplication)

$= (|a|x + |b|x, |a|y + |b|y)$ (by def^n of addition of vectors)

$= (\{|a| + |b|\}x, \{|a| + |b|\}y) \longrightarrow (2)$

$\longrightarrow$ (1M)

Since, $|a+b| \le |a| + |b|$, therefore from (1) & (2), we conclude that in general

$$(a+b)\alpha \ne a\alpha + b\alpha. \longrightarrow (1M)$$

Hence, $V(\mathbb{R})$ is not a vector space.

Q 2 (a)

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 11 & 24 & 29 & 61 \\ 2 & 3 & 5 & 10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 5 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\dim(V) = 2$ ——— (1 M)

Any two vectors in S form a basis of V. — (1 M)

For eg. $\{(1,-1,2,3), (2,3,5,10)\}$.

(b) (i) $N(T) = \text{span}\left(\left\{\begin{pmatrix} -7/5 \\ 2/5 \\ 1 \end{pmatrix}\right\}\right)$

Basis of $N(T) = \left\{\begin{pmatrix} -7/5 \\ 2/5 \\ 1 \end{pmatrix}\right\}$ — (1 M)

or any non-zero vector in $N(T)$ is a basis of $N(T)$.

(ii) By Rank-nullity theorem,

$$\dim(R(T)) = 3 - \dim(N(T))$$
$$= 3 - 1$$
$$= 2$$

or any valid method.

$\dim(R(T)) = 2$. ——→ (1 M).

Solution

3. (a) Find the real constants a, b, c, d, if exist, such that the graph of the function

$$f(x) = ax^3 + bx^2 + cx + d,$$

passes through the points $(-1, 5), (-2, 7), (2, 11)$ and $(3, 37)$.

(b) Find a system of linear equations whose set of solutions is given by

$$S = \{(1 - a, 1 + a, a) : a \in \mathbb{R}\}$$

[3 + 1 = 4 marks]

(a)

$$f(-1) = -a + b - c + d = 5 \quad -(1)$$

$$f(-2) = -8a + 4b - 2c + d = 7 \quad -(2)$$

$$f(2) = 8a + 4b + 2c + d = 11 \quad -(3)$$

$$f(3) = 27a + 9b + 3c + d = 37 \quad -(4)$$

$$(1) \times 8 \rightarrow \quad -\cancel{8a} + 8b - 8c + 8d = 40$$

$$(2) \times 1 \rightarrow \quad -\cancel{8a} + 4b - 2c + d = 7$$

$$(+) \quad - \quad + \quad - \quad -$$

$$4b - 6c + 7d = 33 \quad -(5)$$

$$(1) \times 8 \rightarrow \quad -\cancel{8a} + 8b - 8c + 8d = 40$$

$$(3) \times 1 \rightarrow \quad \cancel{8a} + 4b + 2c + d = 11$$

$$12b \cancel{6} - 6c + 9d = 51 \quad -(6)$$

$$(1) \times 27 \rightarrow \quad -\cancel{27a} + 27b - 27c + 27d = 135$$

$$(4) \times 1 \rightarrow \quad \cancel{27a} + 9b + 3c + d \; \cancel{8} = 37$$

$$36b - 24c + 28d = 172 \quad -(7)$$

$$(5) \times 3 \rightarrow \quad 1\cancel{2b} - 18c + 21d = 99$$

$$(6) \times 1 \rightarrow \quad 1\cancel{2b} - 6c + 9d = 51$$

$$- \quad + \quad - \quad -$$

$$-12c + 12d = 48 \quad -(8)$$

(2)

2
33
9
———
297

(5) $\times 9$ $\quad 36b - 54c + 63d = \cancel{200}\ 297$

(7) $\times 1$ $\quad 36b - 24c + 28d = 172$

(−) $\quad$ (+) $\quad$ − $\quad$ −

$-30c + 35d = 125$ — (9)

2
24
5
———
120

(8) $\times \frac{5}{2}$ $\quad -30c + \frac{\cancel{65}\ 30}{2} d = 120$

(9) → $\quad -30c + 35d = 125$

$\quad$ − $\quad$ −

+

$-\frac{5}{2} d = -5$

$\frac{65}{2} - 35$

$-\frac{5}{2} d$

$\boxed{d = 1}$

From (9), $-30c = 90$

$\boxed{c = -3}$

From (5), $4b + 18 + 7 = 33$

$4b = 33 - 25 = 8$

$\boxed{b = 2}$

From (1), $-a + 2 + 3 + 1 = 5$

$\boxed{a = 1}$

$$\boxed{\therefore f(x) = x^3 + 2x^2 - 3x + 1}$$

x

(b) $\quad x + y = 2, \quad x + z = 1$

Marks distribution

③ ⓐ

Formulating the equations & intermediate

Steps $\longrightarrow$ 1 mark

$a = 1 \quad b = 2 \longrightarrow$ 1 mark

$c = -3 \quad d = 1 \longrightarrow$ 1 mark.

Remark:-

① If only one coefficient in the equation is correct, then one mark (out of 3) is awarded.

③ ⓑ Any linear equation(s) satisfying the given solution ~~is~~ awarded one mark.

If the equation contains a, then no marks shall be awarded.

4. (a) Check the diagonalizability of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If it is diagonalizable, then find a diagonal matrix D which is similar to A.

(b) Find all real values a and b such that the matrix

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & ia \\ \frac{-i}{\sqrt{2}} & b \end{bmatrix}$$

is unitary. [3 + 2 = 5 marks]

Solution & Marking scheme:

4(a). Determining eigenvalues: $\lambda = 1, 1, -1$ [$\frac{1}{2}$ mark]

Checking diagonalization: [2 marks]

(This may be done by one of the following:

(i) establishing algebraic multiplicity = geom. multiplicity for BOTH the distinct eigenvalues.

(ii) finding 3 linearly independent eigenvectors of A.

(iii) finding an invertible matrix P and verifying $P^{-1}AP = D$, where D is given below.)

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ [$\frac{1}{2}$ mark]

4(b). $(a, b) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ [1 mark]

and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ [1 mark]

Note: If a student writes $a = \pm\frac{1}{\sqrt{2}}$, $b = \pm\frac{1}{\sqrt{2}}$ then ONLY one mark will be awarded.

AG
31.1.18

Q5. Solution (4M)

a)
$$\begin{pmatrix} .45 & .3 & -15 & 14.28 \\ 4.5 & .15 & .3 & 1.57 \\ .15 & -10.5 & .45 & -3.86 \end{pmatrix} \xrightarrow{R_{12}} \begin{pmatrix} 4.5 & .15 & .3 & 1.57 \\ .45 & .3 & -15 & 14.28 \\ .15 & -10.5 & .45 & -3.86 \end{pmatrix}$$

$$\xrightarrow{R_{23}} \begin{pmatrix} 4.5 & .15 & .3 & 1.57 \\ .15 & -10.5 & .45 & -3.86 \\ .45 & .3 & -15 & 14.28 \end{pmatrix} \xrightarrow[-\frac{1}{10.5}R_2,\ -\frac{1}{15}R_3]{\frac{1}{4.5}R_1} \begin{pmatrix} 1.0000 & 0.0333 & 0.0667 & 0.3489 \\ -0.0143 & 1.0000 & -0.0429 & 0.3676 \\ -0.0300 & -0.0200 & 1.0000 & -0.9520 \end{pmatrix}$$

0.5M 0.5M

b)
$$x_1^{(n+1)} = 0.3489 - 0.0333\, x_2^{(n)} - 0.0667\, x_3^{(n)}$$
$$x_2^{(n+1)} = 0.3676 + 0.0143\, x_1^{(n)} + 0.0429\, x_3^{(n)}$$
$$x_3^{(n+1)} = -0.9520 + 0.0300\, x_1^{(n)} + 0.0200\, x_2^{(n)}$$

0.5M

c)

n	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$
0	0.0000	0.0000	0.0000
1	0.3489	0.3676	-0.9520
2	0.4002	0.3317	-0.9342
3	0.4002	0.3332	-0.9334

4X0.5=2M

d)
$$x_1^{(n+1)} = 0.3489 - 0.0333\, x_2^{(n)} - 0.0667\, x_3^{(n)}$$
$$x_2^{(n+1)} = 0.3676 + 0.0143\, x_1^{(n+1)} + 0.0429\, x_3^{(n)}$$
$$x_3^{(n+1)} = -0.9520 + 0.0300\, x_1^{(n+1)} + 0.0200\, x_2^{(n+1)}$$

0.5M

— End —

K.G.
19.2.2018

6. (a) Find a root of the equation $e^{2x} = x+6$ correct upto 4 decimal places by using Newton-Raphson method starting with initial guess $x_0 = 0.97$.

Sol^n: $$x_{n+1} = x_n - \frac{e^{2x_n} - x_n - 6}{2e^{2x_n} - 1}$$

$$= \frac{2e^{2x_n} x_n - x_n - e^{2x_n} + x_n + 6}{2e^{2x_n} - 1}$$

$$= \frac{(2x_n - 1)e^{2x_n} + 6}{2e^{2x_n} - 1} \quad \text{1M}$$

$x_0 = 0.97$.

$$\left.\begin{aligned} x_1 &= 0.97087 \\ x_2 &= 0.97087 \end{aligned}\right\} \text{1M}$$

$\therefore$ Root = 0.9709

6. (b) The algebraic eqn.: $x^4 + x - 1 = 0$ has a root α lying in the interval $[0.5, 1.0]$. Starting with this interval, use bisection method to find an interval of width 0.125 which contains α.

Sol^n: $f(0.5) = -0.4375$; $f(1.0) = 1$

$\frac{0.5+1}{2} = 0.75$; $f(0.75) = 0.06640$

$\therefore$ root lies between $(0.5, 0.75)$ 1M

$\frac{0.5+0.75}{2} = 0.625$; $f(0.625) = -0.2224$

$\therefore$ root lies between $(0.625, 0.75)$ 1M

which is of width 0.125.

N.B. Full marks have <u>not</u> been given if the root has been derived after several iterations, but the wanted interval is not written clearly.

7(a) Determine the interpolation polynomial by means of Lagrange Interpolation Formula for the points $(-2,-10)$, $(0,2)$, and $(1,8)$.

[2 marks]

7(b) Suppose the function $f(x) = \ln(1+x)$ is approximated using polynomial interpolation with $x = 0$ and $x = 1$. Let $p(x)$ denote the interpolating polynomial. Find an upper bound (sharp) for the error magnitude $\max\limits_{0 \le x \le 1} |f(x) - p(x)|$ without determining $p(x)$ explicitely.

[3 marks]

Solution:

7(a)

Lagrange Interpolation Formula:

$$p_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)\,(x_0-x_2)}\, y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)\,(x_1-x_2)}\, y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)\,(x_2-x_1)}\, y_2$$ **[1 marks]**

Application to given points:

$$p_2(x) = \frac{(x-0)(x-1)}{(-2-0)\,(-2-1)}\,(-10) + \frac{(x+2)(x-1)}{(0+2)(0-1)}\,(2) + \frac{(x+2)(x-0)}{(1+2)(1-0)}\,(8) = \mathbf{6x+2}$$ **[1 marks]**

7(b)

$$\max_{0 \le x \le 1} |f(x) - p(x)| \le \max_{0 \le t \le 1} \frac{|f^{(2)}(t)|}{2!} \max_{0 \le x \le 1} |(x-x_0)(x-x_1)|$$
$$= \frac{1}{2} \max_{0 \le t \le 1} \frac{1}{(1+t)^2} \max_{0 \le x \le 1} |(x-0)(x-1)|$$

Note that

$$\max_{0 \le t \le 1} \frac{1}{(1+t)^2} = 1$$ **[1 marks]**

$$\max_{0 \le x \le 1} |x(x-1)| = \frac{1}{4}$$ **[1 marks]**

Hence,

$$\max_{0 \le x \le 1} |f(x) - p(x)| \le \frac{1}{8}$$ **[1 marks]**