Lecture 7

Functions of matrices.

$A_{n\times n}$ matrix.

e^{A}, $\sin(A)$, --- ??

Def:- A sequence of $n\times n$ matrices, say $\{B_k\}_{k=0}^{\infty}$ where $B_k = \left[b_{ij}^{(k)}\right]_{n\times n}$ is said to converge to a matrix $B = [b_{ij}]_{n\times n}$, if

$b_{ij}^{(k)} \to b_{ij}$, as $k\to\infty$, $\forall\, i, j$.

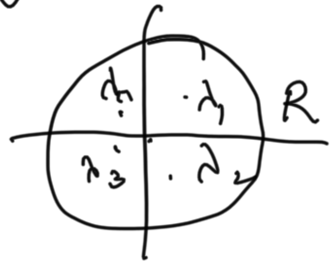
Def:- The infinite series $\sum_{k=0}^{\infty} B_k$ converges to the matrix B, if the sequence of partial sums $\{S_k\}_{k=0}^{\infty}$ converges to B,

where $S_k = \sum_{j=0}^{k} B_j$, $\forall\, k \geq 0$.

Theorem:- Let $z = x+iy$, $x, y \in \mathbb{R}$. If $f(z)$ has the Taylor series $\sum_{k=0}^{\infty} a_k z^k$ which converges for $|z| < R$ $(R > 0)$

& if the eigenvalues $\lambda_1, \dots, \lambda_n$ of an $n \times n$ matrix 'A' have the property that

$$|\lambda_j| < R \quad \forall j = 1, 2, \dots, n,$$



then $\sum_{k=0}^{\infty} a_k A^k$ will Converge to an $n \times n$ matrix which is defind to be $f(A)$. If such a case, $f(A)$ is said to be welldefined.

Examples:

① $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots \qquad \forall x \in \mathbb{R}$

$\| \qquad\qquad |x| < \infty$

$f(x) \qquad\qquad R = +\infty$

$A_{n \times n}$ $\lambda_1, \dots, \lambda_n$ its eigenvalues.

Then e^A is welldefined ??

By above thm,

$|\lambda_1| < \infty$

$\vdots$

$|\lambda_n| < \infty$

$$e^A = I + \frac{A}{1!} + \frac{1}{2!} A^2 + \cdots$$

(2) Sin(A) !

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \quad \forall z$$

For any $n \times n$ matrix, A, $\sin(A)$ is welldefined

& $\sin(A) = A - \frac{A^3}{3!} + \frac{1}{5!}A^5 - \cdots$

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(1) Find e^A, where $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

Sol:- The eigenvalues of A are $\lambda_1 = 5$, $\lambda_2 = -1$.

Let $f(A) = e^A$, $f(\lambda) = e^\lambda$

$r(A) = \alpha_0 I + \alpha_1 A$

$r(\lambda) = \alpha_0 + \alpha_1 \lambda$.

Now $f(A) = r(A)$.

$f(\lambda) = r(\lambda)$ for λ eigenvalues of A.

$f(\lambda_1) = r(\lambda_1)$

$\Rightarrow f(5) = r(5)$

$\Rightarrow \boxed{e^5 = \alpha_0 + 5\alpha_1}$

$f(\lambda_2) = r(\lambda_2)$

$\Rightarrow f(-1) = r(-1)$

$\boxed{e^{-1} = \alpha_0 - \alpha_1}$

Subtracting we get, $e^5 - e^{-1} = 6\alpha_1$

$$\Rightarrow \boxed{\alpha_1 = \frac{e^5 - e^{-1}}{6}}$$

$$\therefore \boxed{\alpha_0 = e^{-1} + \alpha_1 = \frac{5e^{-1}+e^5}{6}}$$

$$\therefore f(A) = r(A) = \alpha_0 I + \alpha_1 A$$

$$\Rightarrow e^A = \left(\frac{5e^{-1}+e^5}{6}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \left(\frac{e^5-e^{-1}}{6}\right)\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}$$

$$= \begin{bmatrix}\frac{4e^{-1}+2e^5}{6} & \frac{e^5-e^{-1}}{3}\\ \frac{2(e^5-e^{-1})}{3} & \frac{2e^{-1}+4e^5}{6}\end{bmatrix}$$

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(2) Find $\sin(A)$, $A = \begin{bmatrix}\pi & 1 & 0\\ 0 & \pi & 0\\ 4 & 1 & \pi/2\end{bmatrix}$

Sol: The eignvalues of A are

$\lambda_1 = \pi/2$, $\lambda_2 = \lambda_3 = \pi$.

Let $f(A) = \sin(A)$ | $r(A) = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$

$f(\lambda) = \sin(\lambda)$ | $r(\lambda) = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2$.

$f'(\lambda) = \cos \lambda.$ | $r'(\lambda) = \alpha_1 + 2\alpha_2 \lambda$

$f(\lambda_1) = r(\lambda_1) \Rightarrow f(\pi/2) = r(\pi/2)$

$f(\lambda_2) = r(\lambda_2) \Rightarrow f(\pi) = r(\pi)$

$f'(\lambda_2) = r'(\lambda_2) \Rightarrow f'(\pi) = r'(\pi)$

$\Rightarrow$

$$\sin(\pi/2) = 1 = \alpha_0 + \alpha_1(\pi/2) + \alpha_2 \cdot \frac{\pi^2}{4}$$
$$\sin(\pi) = 0 = \alpha_0 + \alpha_1 \pi + \alpha_2 \pi^2$$
$$\cos(\pi) = -1 = \alpha_1 + 2\alpha_2 \pi$$

$$\Rightarrow \quad \alpha_0 = \frac{1}{\pi^2}(4\pi^2 - \pi^3)$$

$$\alpha_1 = \frac{1}{\pi^2}(-8\pi + 3\pi^2)$$

$$\alpha_2 = \frac{1}{\pi^2}(4 - 2\pi)$$

$$\therefore \sin(A) = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$$

$$A^2 = \begin{bmatrix} \pi & 1 & 0 \\ 0 & \pi & 0 \\ 4 & 1 & \pi/2 \end{bmatrix} \begin{bmatrix} \pi & 1 & 0 \\ 0 & \pi & 0 \\ 4 & 1 & \pi/2 \end{bmatrix}$$

$$= \begin{bmatrix} \pi^2 & 2\pi & 0 \\ 0 & \pi^2 & 0 \\ 6\pi & 4 + \frac{3\pi}{2} & \frac{\pi^2}{4} \end{bmatrix}$$

$$\therefore \sin A = \frac{1}{\pi^2}(4\pi^2 - \pi^3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$+ \frac{1}{\pi^2}(-8\pi + 3\pi^2) \begin{bmatrix} \pi & 1 & 0 \\ 0 & \pi & 0 \\ 4 & 1 & \pi/2 \end{bmatrix}$$

$$+\frac{1}{\pi^2}(4-2\pi)\begin{bmatrix} \pi^2 & 2\pi & 0 \\ 0 & \pi^2 & 0 \\ 6\pi & 4+\frac{3\pi}{2} & \frac{\pi^2}{4} \end{bmatrix}$$

$$=\frac{1}{\pi^2}\begin{bmatrix} 0 & -\pi^2 & 0 \\ 0 & 0 & 0 \\ -8\pi & 16-10\pi & \pi^2 \end{bmatrix}$$ (Check it!)

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③ Find e^{At}, where $A=\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

('t' parameter)

Soln Let $B = At = \begin{bmatrix} t & 2t \\ 4t & 3t \end{bmatrix}$

Find e^B..

The eigenvalues of B are $5t, -t$.

Let $f(B) = e^B = e^{At}$

$\gamma(B) = \alpha_0 I + \alpha_1 B$
$= \alpha_0 I + \alpha_1 At$

$f(\lambda) = e^\lambda$

$\gamma(\lambda) = \alpha_0 + \alpha_1 \lambda.$

$f(\lambda) = \gamma(\lambda)$

Now $f(5t) = \gamma(5t)$

$$f(-t) = r(-t)$$

$$\Rightarrow \quad e^{5t} = \alpha_0 + \alpha_1 5t$$

$$e^{-t} = \alpha_0 + \alpha_1(-t).$$

$$\Rightarrow \quad \alpha_0 = \frac{1}{6}\left(e^{5t} + 5e^{-t}\right)$$

$$\alpha_1 = \frac{1}{6t}\left(e^{5t} - e^{-t}\right)$$

$$\therefore \quad e^{B} = r(B) = \alpha_0 I + \alpha_1 B.$$

$$= \frac{1}{6}\left(e^{5t} + 5e^{-t}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{6t}\left(e^{5t} - e^{-t}\right)\begin{bmatrix} t & 2t \\ 4t & 3t \end{bmatrix}$$

$$\therefore \quad \boxed{e^{At} = \frac{1}{6}\begin{bmatrix} 2e^{5t} + 4e^{-t} & 2e^{5t} - 2e^{-t} \\ 4e^{5t} - 4e^{-t} & 4e^{5t} + 2e^{-t} \end{bmatrix}}$$