Lecture 10

Theorem :- Every $n \times n$ matrix A possesses n l.i generalized eigenvectors (liges).
Generalized eigenvectors corresponding to distinct eigenvalues are l.i.
If λ is an eigenvalue of A of multiplicity m, then there exists m l.i gen. eigenvectors corr. to λ.

- There are infinitely many ways to choose liges.

Def :- A set of n liges is called a Canonical basis of A if it is composed entirely of chains. (union)

How to determine a Canonical basis of $A_{n\times n}$??

Let λ_j be an eigenvalue of $A_{n\times n}$ of multiplicity ν.
First find the ranks of the matrices

$$(A-\lambda_j I),\ (A-\lambda_j I)^2, \ldots, (A-\lambda_j I)^m,$$

where m is the least non-ve integer such that $\operatorname{rank}\left((A-\lambda_j I)^m\right) = n-\nu$.

Now define $\rho_k := \operatorname{rank}\left((A-\lambda_j I)^{k-1}\right) - \operatorname{rank}\left((A-\lambda_j I)^k\right)$

$\forall k = 1, 2, \dots, m.$

Note: $\text{rank}\left((A - \lambda_j I)^0\right) = \text{rank}(I) = n.$

Then the number of l.i gen. eigenvectors of type k, corresponding to the eigenvalue λ_j that will apper in a canonical basis of A, is equal to ρ_k, $(k = 1, \dots, m)$.

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(1) Find a canonical basis of $A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Sol:- The eigenvalues of A are $1, 1, 1, 1.$

$\lambda = 1$ is an eigenvalue of multiplicty $\nu = 4.$

$n = 4,\ \nu = 4.$

To find least m such that $(A - I)^m$ has rank $n - \nu = 0$.

i.e.

$$A - I = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq \underset{\sim}{0}$$

$$(A - I)^2 = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore\ m = 2$

$\therefore$ $m_1 = 2$. $\qquad$ $\text{rank}\left[(A-I)^2\right] = 0$

$$\rho_1 = \text{rank}\left[(A-I)^0\right] - \text{rank}(A-I)$$
$$= 4 - 2 = 2$$
$$\rho_2 = \text{rank}(A-I) - \text{rank}\left((A-I)^2\right)$$
$$= 2 - 0 = 2$$

$\therefore$ Number of l.i.ges of type 1 that appears in a canonical basis of A is $\rho_1 = 2$

$\therefore$ Number of l.i.ges of type 2 that appear in a canonical basis of A is $\rho_2 = 2$.

Let $\underline{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ be a gen. eigenvector of type 2 corr to $\lambda = 1$.

$$\Rightarrow (A-I)\underline{x} \neq \underline{0} \quad \& \quad (A-I)^2\underline{x} = \underline{0}$$

$$\Rightarrow \begin{pmatrix} y-w \\ 0 \\ w \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $w \neq 0$ or $y - w \neq 0$.

$\Rightarrow$ either y or w must be non-zero.

& x, z are arbitrary.

Choose two l.i. gen. eigenvectors of type

say $\underline{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\underline{y}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Now

Now $$\underline{x}_1 = (A - I)\,\underline{x}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\underline{y}_1 = (A - I)\,\underline{y}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Then $\{\underbrace{\underline{x}_2}_{\text{type 2}}, \underbrace{\underline{x}_1}_{\text{type 1}}, \underbrace{\underline{y}_2}_{\text{type 2}}, \underbrace{\underline{y}_1}_{\text{type 1}}\}$ is a <u>Canonical</u> <u>basis</u> of A.

(<u>In fact it</u> is a basis of the v.s. F^4).

<u>Def:-</u> $A_{n\times n}$ matrix having eigenvalues $\lambda_1, \dots, \lambda_n$. Then a matrix of the form

$$S_k = \begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_k \end{bmatrix}_{n_k \times n_k}.$$

known as the <u>Jordan block</u> corresponding to the eigenvalue λ_k of A.

<u>Def:-</u> A square $A_{n\times n}$ having eigenvalues $\lambda_1, \dots, \lambda_n$; is said to be in <u>Jordan Canonical form</u> if it is a diagonal matrix or can be expressed in either of the

following block diagonal matrices:

$$\begin{bmatrix} D & & & \\ & S_1 & & O \\ & & \ddots & \\ O & & & S_r \end{bmatrix}_{n\times n} \quad \text{or} \quad \begin{bmatrix} S_1 & & O \\ & \ddots & \\ O & & S_r \end{bmatrix}_{n\times n}$$

Where D is a diagonal matrix & each S_j is a Jordan block.

<u>Eg:-</u>

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$ in Jordan canonical form.

or $$A = \begin{bmatrix} \begin{pmatrix} 1 & & \\ & -1 & \\ & & 2 \end{pmatrix} & O \\ O & \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \end{bmatrix}$$ ".

<u>Theorem:-</u> Every square matrix $A_{n\times n}$ having eigenvalues $\lambda_1, \ldots, \lambda_n$, is similar to a Jordan Canonical matrix.

i.e. there exists an invertible matrix M such that $M^{-1}AM = J$

Jordan canonical form.

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① Find a matrix that is in Jordan canonical form that is similar to $A = \begin{bmatrix} 0 & 4 & 2 \\ -3 & 8 & 3 \\ 4 & -8 & -2 \end{bmatrix}$

Sol:

The eigenvalues of A are $2, 2, 2$.

First find a canonical basis of A.

To find least m such that $(A-2I)^m$ has rank $3-3=0$.

$$(A-2I) = \begin{bmatrix} -2 & 4 & 2 \\ -3 & 6 & 3 \\ 4 & -8 & -4 \end{bmatrix}$$ & its rank $= 1$

$$(A-2I)^2 = \begin{bmatrix} -2 & 4 & 2 \\ -3 & 6 & 3 \\ 4 & -8 & -4 \end{bmatrix} \begin{bmatrix} -2 & 4 & 2 \\ -3 & 6 & 3 \\ 4 & -8 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $m = 2$.

$\rho_1 = \text{rank}(I) - \text{rank}(A-2I) = 3-1 = 2$

$\rho_2 = \text{rank}(A-2I) - \text{rank}((A-2I)^2) = 1$

Let $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be a gen. eigenvector of type 2

Corr to $\lambda = 2$.

$(A-2I)\underline{x} \neq \underline{0}$ & $(A-2I)^2 \underline{x} = \underline{0}$

$\Rightarrow \quad -2x + 4y + 2z \neq 0$

$\Rightarrow \quad -x + 2y + z \neq 0.$

Choose $\underline{x}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ gen. eigenvector of type 2

Corr $\lambda = 2$.

$$\underline{x}_1 = (A-2I)\underline{x}_2 = \begin{bmatrix} -2 & 4 & 2 \\ -3 & 6 & 3 \\ 4 & -8 & -4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$$

Choose $\underline{y}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. check it is a gen. eigenvector of type 1.

So that $\{\underline{x}_1, \underline{y}_1\}$ l. i.

$\therefore$ Canonical basis $= \{\underline{x}_2, \underline{x}_1, \underline{y}_1\}$.

Now consider the matrix

$$M = \begin{bmatrix} \underline{y}_1 & \underline{x}_1 & \underline{x}_2 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 3 & 0 \\ 0 & -4 & 1 \end{bmatrix}$$

1 1 2

invertible.

Check that $M^{-1}AM = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = J$ Jordan Canonical form.

Normal matrices.

Defn: An $n \times n$ matrix A is said to be a normal matrix if $A^*A = AA^*$. $\left(A^* = \bar{A}^t\right)$

Examples: orthogonal matrices, Symmetric matrices, Unitary matrices, Hermitian matrices, are all normal.

Theorem: Over $\mathbb{C}$, $A_{n\times n}$ is normal

$\Updownarrow$

A has an orthogonal set of n eigenvectors.

proof: $\Uparrow$: Suppose A has an orthogonal set of n eigenvectors of A.

Say $\underline{x_1}, \ldots, \underline{x_n}$ associated to $\lambda_1, \ldots, \lambda_n$ respectively. (all need not be distinct)

distinct)

Let $U = \begin{bmatrix} \frac{\underline{x}_1}{\|\underline{x}_1\|} & \frac{\underline{x}_2}{\|\underline{x}_2\|} & \cdots & \frac{\underline{x}_n}{\|\underline{x}_n\|} \end{bmatrix}_{n\times n}$.

U is a unitary matrix.

& $U^*AU = \begin{bmatrix} \underline{x}_1^*/\|\underline{x}_1\| \\ \vdots \\ \frac{\underline{x}_n^*}{\|\underline{x}_n\|} \end{bmatrix} A \begin{bmatrix} \frac{\underline{x}_1}{\|\underline{x}_1\|} & \cdots & \frac{\underline{x}_n}{\|\underline{x}_n\|} \end{bmatrix}$

$= U^* \begin{bmatrix} \frac{A\underline{x}_1}{\|\underline{x}_1\|} & \cdots & \frac{1}{\|\underline{x}_n\|} A\underline{x}_n \end{bmatrix}$

$= \begin{bmatrix} \frac{\underline{x}_1^*}{\|\underline{x}_1\|} \\ \vdots \\ \frac{\underline{x}_n^*}{\|\underline{x}_n\|} \end{bmatrix} \begin{bmatrix} \frac{\lambda_1 \underline{x}_1}{\|\underline{x}_1\|} & \cdots & \frac{\lambda_n}{\|\underline{x}_n\|} \underline{x}_n \end{bmatrix}$

$= \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$.

$\Rightarrow \boxed{A = U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^*}$

$A^* = (U^*)^* \begin{bmatrix} \bar{\lambda}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{\lambda}_n \end{bmatrix} U^*$

$= U \begin{bmatrix} \bar{\lambda}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{\lambda}_n \end{bmatrix} U^*$

Now $A^*A = U \begin{bmatrix} \bar{\lambda}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{\lambda}_n \end{bmatrix} U^* U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^*$

$= U \begin{bmatrix} \bar{\lambda}_1\lambda_1 & & 0 \\ & \ddots & \\ \ldots \end{bmatrix} U^*$

$$\begin{bmatrix} & & \\ 0 & & \lambda_n\lambda_n \end{bmatrix}$$

& $AA^* = U\begin{bmatrix} \lambda_1\bar{\lambda}_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n\bar{\lambda}_n \end{bmatrix} U^*$

i.e. $AA^* = A^*A$.

$\therefore$ A is normal.

II: Assume A is normal. i.e. $AA^* = A^*A$

To show: A has an orthogonal set of n eigenvectors.

We know that there exists a unitary matrix S such that

$S^*AS = V$, an upper Δler matrix.

Let $V = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ 0 & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & v_{nn} \end{bmatrix}$ $\quad A = SVS^*$

Now $AA^* = A^*A \Rightarrow (SVS^*)(SVS^*)^* = (SVS^*)^*(SVS^*)$

$\Rightarrow (SVS^*)(SV^*S^*) = (SV^*S^*)(SVS^*)$

$\Rightarrow SVV^*S^* = SV^*VS^*$

$\Rightarrow VV^* = V^*V$.

$\therefore$ V is normal.

$$\Rightarrow \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ 0 & v_{22} & \cdots & v_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & v_{nn} \end{bmatrix} \begin{bmatrix} \overline{v_{11}} & 0 & \cdots & 0 \\ \overline{v_{12}} & \overline{v_{22}} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \overline{v_{1n}} & \overline{v_{2n}} & \cdots & \overline{v_{nn}} \end{bmatrix}$$

$$= \begin{bmatrix} \overline{v_{11}} & 0 & \cdots & 0 \\ \overline{v_{12}} & \overline{v_{22}} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \overline{v_{1n}} & \overline{v_{2n}} & \cdots & \overline{v_{nn}} \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ 0 & v_{22} & \cdots & v_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & v_{nn} \end{bmatrix}$$

equate the $(1,1)^{th}$ entry both sides:

$$|v_{11}|^2 + \cdots + |v_{1n}|^2 = |v_{11}|^2$$

$$\Rightarrow \quad v_{12} = \cdots = v_{1n} = 0.$$

equate the $(2,2)^{th}$ entry both sides:

$$|v_{22}|^2 + \cdots + |v_{2n}|^2 = |v_{22}|^2$$

$$\Rightarrow \quad v_{23} = \cdots = v_{2n} = 0$$

$$\vdots$$

$$v_{n-1,n} = 0$$

Thus $V = \begin{bmatrix} v_{11} & 0 & \cdots & 0 \\ 0 & v_{22} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & v_{nn} \end{bmatrix}$

" V is a diagonal matrix.

$$S^*AS = \text{diagonal}(v_{11}, --, v_{nn})$$

Then the columns of S are n orthogonal eigenvectors of A.

<u>Corollary</u>:- Every normal matrix over $\mathbb{C}$ is diagonizable.

<u>Qn</u>:- Give an example of a diagonizable matrix which is not normal.
