$\mathcal{C}(AA^T) = \mathcal{C}(A)$

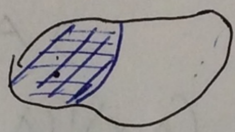
<u>Part I</u> $\mathcal{C}(AA^T) \subseteq \mathcal{C}(A)$ (from property 2)

<u>Part II</u> To show $\mathcal{C}(A) \subseteq \mathcal{C}(AA^T)$.

Let $\boldsymbol{l} \in [\mathcal{C}(AA^T)]^{\perp}$

$\Leftrightarrow \boldsymbol{l}^T(AA^T) = \boldsymbol{0}^T$

$\Rightarrow \boldsymbol{l}^T(AA^T)\boldsymbol{l} = \boldsymbol{0}^T\boldsymbol{l}$



$\Rightarrow (A^T\boldsymbol{l})^T(A^T\boldsymbol{l}) = 0$

$\Rightarrow A^T\boldsymbol{l} = \boldsymbol{0}$

$\Rightarrow \boldsymbol{l}^T A = \boldsymbol{0}^T$

$\Leftrightarrow \boldsymbol{l} \in (\mathcal{C}(A))^{\perp}$

$\Rightarrow (\mathcal{C}(AA^T))^{\perp} \subseteq (\mathcal{C}(A))^{\perp}$.

$\Leftrightarrow \mathcal{C}(A) \subseteq \mathcal{C}(AA^T)$. (proved).

$$\frac{1}{n}\left[\sum_{i=1}^{n}(x_i - \bar{x})^2\right]$$

$$\sum x_i = n\bar{X}$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n}(x_i^2 - 2x_i\bar{x} + \bar{x}^2)\right]$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n}x_i^2 - 2\bar{x}\left(\sum_{i=1}^{n}x_i\right) + n\bar{X}^2\right]$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n}x_i^2 - 2\bar{x}(n\bar{x}) + n\bar{x}^2\right]$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n}x_i^2 - n\bar{x}^2\right]$$

$$= \frac{1}{n}\sum x_i^2 - \bar{X}^2.$$

1. Express sample mean $\frac{1}{n}\sum_{i=1}^{n} x_i$ in vector Notation:

$$\overline{X} = \left(\tfrac{1}{n}\ \tfrac{1}{n} \cdots \tfrac{1}{n}\right)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \frac{1}{n}\underset{\sim}{1}^T \underset{\sim}{X}$$

$$\underset{\sim}{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Vector dot product

2. Express sample variance $\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2$ in vector notation:

$$\frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^2 = \frac{1}{n}\left(\sum x_i^2\right) - \overline{x}^2 = \frac{1}{n}\underset{\sim}{X}^T\underset{\sim}{X} - \frac{1}{n^2}\left(\underset{\sim}{1}^T\underset{\sim}{X}\right)^2$$

$$= \frac{1}{n}\left(\underset{\sim}{X}^T I_n \underset{\sim}{X}\right) - \frac{1}{n^2}\left[X^T\underset{\sim}{1}\,\underset{\sim}{1}^T X\right]$$

$$= \frac{1}{n}\left[\underset{\sim}{X}^T I_n \underset{\sim}{X} - \underset{\sim}{X}^T\left(\frac{1}{n}\underset{\sim}{1}\,\underset{\sim}{1}^T\right)\underset{\sim}{X}\right]$$

$$= \frac{1}{n}\underset{\sim}{X}^T\left(I_n - \frac{1}{n}\underset{\sim}{1}\,\underset{\sim}{1}^T\right)\underset{\sim}{X}$$

Quadratic form.

$\underset{\sim}{1}^T\underset{\sim}{1} = n$.

Is $\left(I_n - \frac{1}{n}\underset{\sim}{1}\,\underset{\sim}{1}^T\right)$ an idempotent matrix? YES

3. Consider $f_1(x) = 1$, $f_2(x) = x-2$, $f_3(x) = (x-2)^2$ construct a basis

~~Does~~ Does $\{f_1(x), f_2(x), f_3(x)\}$ of $\mathbb{P}_2$?

$$f_1(x) = 1 = (1\,0\,0)\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix}$$

$$f_2(x) = x-2 = (-2\,1\,0)\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix}$$

$$f_3(x) = (x-2)^2 = x^2 - 4x + 4$$

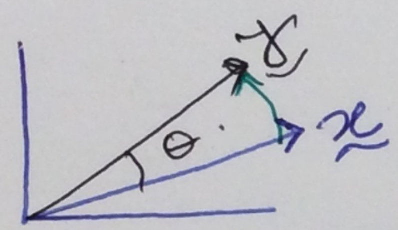
$$= (4 - 4 \;\; 1)\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix}$$

$$\mathcal{C}\begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{R}^3 = \mathrm{sp}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix}\right\}$$

$$\equiv \mathbb{P}_2$$

(4) Let $\underset{\sim}{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\underset{\sim}{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

what is the relation between $(x_1^2 + x_2^2)$ and $(y_1^2 + y_2^2)$?



$Y = A\underset{\sim}{X}$

A is orthogonal matrix.

$\Rightarrow A^T A = I.$

(5). $\underline{\underline{\sum y_i^2}} = \underset{\sim}{Y}^T \underset{\sim}{Y} = (Ax)^T (Ax) = x^T (A^T A) x = \underset{\sim}{X}^T I \underset{\sim}{X}$

$= x^T x = \underline{\underline{\sum x_i^2}}.$