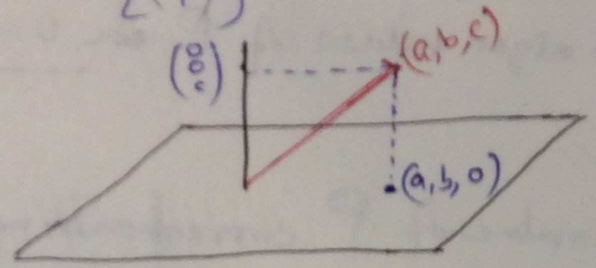
$z\text{-axis} = sp\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ $\quad xy\text{-plane} = sp\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$



$(a,b,0)\begin{pmatrix}0\\0\\c\end{pmatrix} = 0$

$P_{xy} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$ $\quad \underset{\sim}{v} = \begin{pmatrix}a\\b\\c\end{pmatrix}$ $\quad P_{xy}\underset{\sim}{v} = \begin{pmatrix}a\\b\\0\end{pmatrix}$

$(I_3 - P_{xy}) = \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$ $\quad (I - P_{xy})\underset{\sim}{v} = \begin{pmatrix}0\\0\\c\end{pmatrix}$

$$sp\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\} \cap sp\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\} = \begin{pmatrix}0\\0\\0\end{pmatrix} \neq \phi$$

* Projection matrix is an idempotent matrix.

Let S be a sub-space of V.
and P_S be a projection matrix of $S \subseteq V$. } To show $P_S^2 = P_S$.

Let $\underset{\sim}{v} \in V \Rightarrow P_S\underset{\sim}{v} \in S. \quad \forall \underset{\sim}{v} \in V.$

Hence $P_S(P_S\underset{\sim}{v}) = P_S\underset{\sim}{v}.$ as $P_S\underset{\sim}{v} \in S, \forall \underset{\sim}{v} \in V$

$\Rightarrow P_S^2\underset{\sim}{v} = P_S\underset{\sim}{v} \quad \forall \underset{\sim}{v} \in V.$

$\Rightarrow \boxed{P_S^2 = P_S}$. (proved).

(*) Idempotent matrix has eigen values 0 and 1.

If $P^2 = P$ then the eigen values of P are 0 and 1

Let λ be an eigen value of P corresponding to the eigen vector $\underset{\sim}{x} \neq \underset{\sim}{0} \in V$. Then

$$P\underset{\sim}{x} = \lambda \underset{\sim}{x} \quad \text{when } \underset{\sim}{x} \neq \underset{\sim}{0}$$

$$\Rightarrow P(P\underset{\sim}{x}) = P\lambda\underset{\sim}{x}, \quad \underset{\sim}{x} \neq \underset{\sim}{0}$$

$$\Rightarrow P^2\underset{\sim}{x} = \lambda P\underset{\sim}{x} \quad \underset{\sim}{x} \neq \underset{\sim}{0}$$

$$\Rightarrow Px = \lambda.(\lambda\underset{\sim}{x}) \quad \underset{\sim}{x} \neq \underset{\sim}{0}, \; P^2x = Px, \; Px = \lambda x$$

$$\Rightarrow \lambda\underset{\sim}{x} = \lambda^2\underset{\sim}{x} \quad \underset{\sim}{x} \neq 0.$$

$$\Rightarrow \underset{\sim}{x}^T\lambda\underset{\sim}{x} = \underset{\sim}{x}^T\lambda^2\underset{\sim}{x} \quad x \neq 0.$$

$$\Rightarrow (\underset{\sim}{x}^T\underset{\sim}{x})\lambda = (\underset{\sim}{x}^T\underset{\sim}{x})\lambda^2 \quad \underset{\sim}{x}^T\underset{\sim}{x} > 0 \text{ when } \underset{\sim}{x} \neq 0$$

$$\Rightarrow \lambda = \lambda^2 \Rightarrow \boxed{\lambda = 0, 1}$$

proved.

$$\text{Trace}(P) = \text{Sum of eigen values} = \text{Rank of } P.$$