LINEAR ALGEBRA

SYSTEM OF LINEAR EQUATIONS:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

m equations

n unknowns

Consider the four arrays:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

All of these arrays are examples of MATRICES.

The above system of equations can be expressed in matrix form as:

$$Ax = b$$

A : Coefficient matrix

x : vector of unknowns

b : right hand side vector

It is convenient to define Augmented matrix

$$[A|b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Def: A system of equations is consistent if it has at least one solution, and inconsistent if it has no solution.

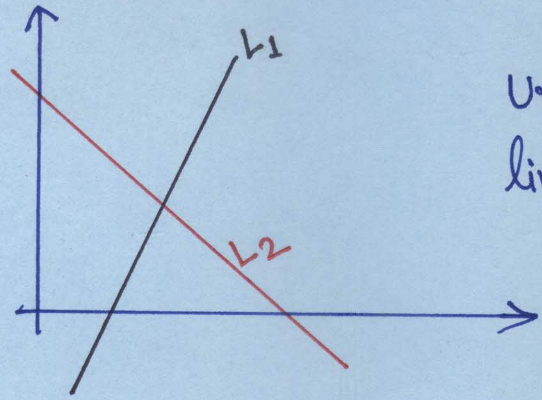
SOLUTION OF SYSTEM OF LINEAR EQUATIONS:

Consider the system of two unknowns:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}\right\} \text{represent straight lines}$$

Possible cases of solution:

a)



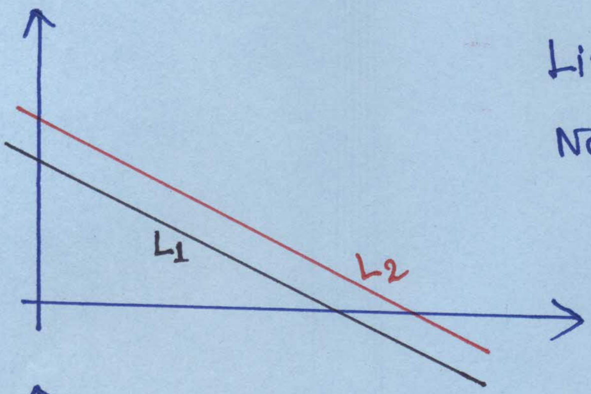
Unique solution
lines intersect

Example:

$-x+y=1$

$2x+y=4$

b)



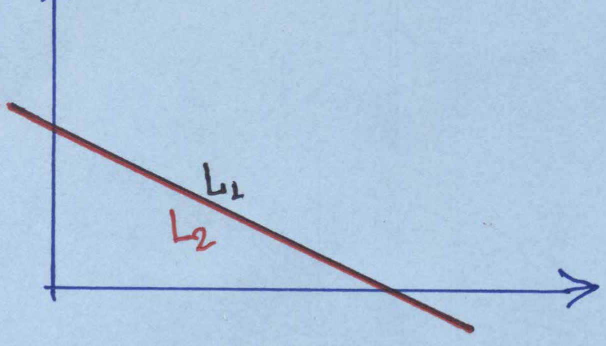
Lines are parallel
No Solution

Example:

$-x+y=1$

$x-y=2$

c)



Lines coinside
Infinitely many solutions

Example:

$-x+y=1$

$2x-2y=-2$

Similarly, case of 3 unknowns can be interpreted with the help of planes (hyperplanes)

SOLUTION METHODS:

A) Method of determinants - Cramer's rule
B) Matrix inversion method, $Ax=b \Rightarrow x=A^{-1}b$
C) Gauss Elimination Method
} Direct Method (Exact solution)

D) Iterative Method - Jacobi & Gauss Seidel Method } Approximate solution

GAUSS-ELIMINATION METHOD:

Consider
$$6x+4y=2 \quad -(1)$$
$$3x-5y=-34 \quad -(2)$$

STEP 1: Multiply eq. (1) by $\frac{1}{2}$ and substract it from (2)

$$6x+4y=2 \quad -(3)$$
$$-7y=-35 \quad -(4)$$

STEP 2: Solution

$$y=5$$
$$x=\frac{1}{6}(2-4\times5)=-3$$

Using augmented matrix

$$\left[\begin{array}{cc:c} 6 & 4 & 2 \\ 3 & -5 & -34 \end{array}\right]$$ corresponding to (1) & (2)

$R_2 \to R_2-\frac{1}{2}R_1$: $$\left[\begin{array}{cc:c} 6 & 4 & 2 \\ 0 & -7 & -35 \end{array}\right]$$ corresponding to (3) & (4)

← This is called Echelon form

In short: $[A \vdots b] \xrightarrow[\text{Elimination}]{\text{Gauss}} [A' | b']$ ← Echelon form

Back substitution : $y=5$, $x=-3$.

ELEMENTARY ROW OPERATIONS OR TRANSFORMATIONS FOR MATRICES

1. Interchange of ith and jth rows $R_i \leftrightarrow R_j$

2. Multiplication of the ith row by a non-zero number λ,

$$R_i \to \lambda R_i$$

3. Addition of λ times the jth row to the ith row

$$R_i \to R_i + \lambda R_j$$

EQUIVALANCE OF MATRICES

If B be $m \times n$ matrix obtained from $m \times n$ matrix A by finite number of elementary transformations of A, then A is called equivalent to B, denoted by $A \sim B$ (A is equivalent to B).

PROPERTIES OF AN EQUIVALANCE RELATION $\sim$

(i) Reflexivity $A \sim A$

(ii) Symmetry : if $A \sim B$ then $B \sim A$

(iii) Transitivity : If $A \sim B$, $B \sim C$ then $A \sim C$