SPANNING SET: Let V be a vector space over $\mathbb{R}$. Vectors $v_1, v_2, \dots, v_n$ in V are said to span V or to form a spanning set of V if every v in V is a linear combination of the vectors $v_1, v_2, \dots, v_n$, that is, if there exist scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ in $\mathbb{R}$ such that

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

Examples: Let us take $V = \mathbb{R}^3$.

CLAIM: The vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ form a spanning set of $\mathbb{R}^3$.

Take any vector $v = \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} \in \mathbb{R}^3$, then

$$\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = v_1\begin{pmatrix}1\\0\\0\end{pmatrix} + v_2\begin{pmatrix}0\\1\\0\end{pmatrix} + v_3\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

CLAIM: The vectors $w_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, w_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}, w_3 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ form a spanning set of $\mathbb{R}^3$.

$$\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \lambda_1\begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda_2\begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda_3\begin{pmatrix}1\\0\\0\end{pmatrix} \Rightarrow \begin{aligned}\lambda_1 &= v_3\\ \lambda_2 &= v_2 - v_3\\ \lambda_3 &= v_1 - v_3 - (v_2 - v_3)\\ &= v_1 - v_2\end{aligned}$$

CLAIM: The vectors $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}$ form a spanning set of $\mathbb{R}^3$.

$$\left[\begin{array}{cccc|c}1&1&1&1&v_1\\1&1&0&0&v_2\\1&0&0&1&v_3\end{array}\right] \sim \left[\begin{array}{cccc|c}1&1&1&1&v_1\\0&0&-1&-1&v_2-v_1\\0&-1&-1&0&v_3-v_1\end{array}\right] \sim \begin{array}{c}\begin{array}{cccc}\lambda_1&\lambda_2&\lambda_3&\end{array}\\ \left[\begin{array}{cccc|c}\boxed{1}&1&1&1&v_1\\0&\boxed{-1}&-1&0&v_3-v_1\\0&0&\boxed{-1}&-1&v_2-v_1\end{array}\right]\\ \begin{array}{cccc}&&&\lambda_4\end{array}\end{array}$$

Solvable but nonunique set of λ's

CLAIM: The vectors $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\3\\5\end{pmatrix}, \begin{pmatrix}1\\5\\9\end{pmatrix}$ do NOT span $\mathbb{R}^3$.

$$\left[\begin{array}{ccc|c}1&1&1&v_1\\2&3&5&v_2\\3&5&9&v_3\end{array}\right] \sim \left[\begin{array}{ccc|c}1&1&1&v_1\\0&1&3&v_2-2v_1\\0&2&6&v_3-3v_1\end{array}\right] \sim \left[\begin{array}{ccc|c}1&1&1&v_1\\0&1&3&v_2-2v_1\\0&0&0&v_3-2v_2+v_1\end{array}\right]$$

INCONSISTENT

Do not span $\mathbb{R}^3$

LINEAR INDEPENDENCE OF VECTORS:

Let V be a vector space. A finite set of vectors $\{v_1, v_2, \ldots, v_n\}$ of V is said to be linearly independent if

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0 \Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$$

(λ_i's are scalars)

Example: Investigate linear independence of

$$v_1 = (1,-1,0),\ v_2 = (0,1,-1),\ v_3 = (0,0,1)$$

Consider $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0$

$$\Rightarrow \lambda_1(1,-1,0) + \lambda_2(0,1,-1) + \lambda_3(0,0,1) = (0,0,0)$$

$$\Rightarrow (\lambda_1, -\lambda_1+\lambda_2, -\lambda_2+\lambda_3) = (0,0,0)$$

$$\Rightarrow \lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 0$$

$\Rightarrow$ The given set of vectors is linearly independent.

Example: Check if $v_1 = (0,0)$, $v_2 (1,2)$ are linearly independent.

$$\lambda_1 (0,0) + \lambda_2 (1,2) = (0,0)$$

We know that $\lambda_1 (0,0) + 0 \cdot (1,2) = (0,0) \quad \forall \lambda_1 \in \mathbb{R}$

$\Rightarrow$ Given vectors are linearly dependent.

NOTE: If 0 is one of the vectors in the set, then the set must be linearly dependent, as

$$\lambda \cdot 0 + 0 \cdot v_1 + 0 \cdot v_2 + \cdots 0 \cdot v_n = 0, \text{ for any } \lambda \in \mathbb{R}$$

Example : Examine if the set

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

is linearly independent.

Consider, $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \quad \begin{aligned} &\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0 \\ &\lambda_1 + \lambda_2 = 0 \\ &\lambda_1 + \lambda_4 = 0 \end{aligned}$$

Augmented matrix $[A|b] = \left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array}\right]$

$$\sim \left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{ccccc} \lambda_1 & \lambda_2 & \lambda_3 & & \end{array} \\ \left[\begin{array}{cccc:c} \boxed{1} & 1 & 1 & 1 & 0 \\ 0 & \boxed{-1} & -1 & 0 & 0 \\ 0 & 0 & \boxed{-1} & -1 & 0 \end{array}\right] \end{array}$$

$\lambda_4 \leftarrow$ free variable

Take $\lambda_4 = 1$. $\quad \lambda_3 = -1, \ \lambda_2 = 1, \ \lambda_1 = -1$.

$\Rightarrow \boxed{-v_1 + v_2 - v_3 + v_4 = 0}$ The set is linearly dependent.

Example: Examine if the set $\{(2,1,1), (1,2,2), (1,1,1)\}$ is linearly dependent in $\mathbb{R}^3$.

$$[A|b] = \left[\begin{array}{ccc:c} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc:c} 2 & 1 & 1 & 0 \\ 0 & 3/2 & 1/2 & 0 \\ 0 & 3/2 & 1/2 & 0 \end{array}\right] \sim \left[\begin{array}{cccc} \boxed{2} & 1 & 1 & 0 \\ 0 & \boxed{3/2} & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\lambda_3 \leftarrow$ free variab

$\Rightarrow$ The set is linearly dependent. Observe: $\boxed{v_1 + v_2 - 3v_3 = 0}$

BASIS & DIMENSION OF A VECTOR SPACE

DIMENSION: maximum number of linearly independent vectors in a vector space V.

BASIS: Set of these linearly independent vectors.

OR

Basis is a set of linearly independent set that span a vector space V.
The number of elements in a basis is called the DIMENSION of the vector space V.

- Note that every vector in V can be written (uniquely) as a linear combination of the basis vectors.
- The vector space $\{0\}$ is defined to have dimension ZERO.

EXAMPLES:

1) Vector space $\mathbb{R}^n$:

Consider $e_1 = (1,0,0 \cdots 0)^T$, $e_2 = (0,1,0,\cdots 0)^T$
$e_n = (0,0 \cdots 1)^T$.

Note that these vectors are linearly independent and we have, for any $v = (v_1, v_2 \cdots v_n)^T$:

$v = v_1 e + v_2 e_2 + \cdots + v_n e_n$.

Dimension: n.

2) Vector space of all $r \times s$ matrices

Consider, for example, 2×3 matrices

The vectors $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdots \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

These vectors are linearly independent and span the vector space of all 2×3 matrices.

The dimension : $2\times 3 = 6$.

Basis in example 1) and 2) are called usual or standard basis.

3) Vector space $P_n(t)$ of all polynomials of degree $\leq n$.

The set $S = \{1, t, t^2, \ldots, t^n\}$ is a basis for $P_n(t)$.

Dimension: $(n+1)$.