Multivariate Analysis.

$$\underset{\sim}{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \underset{\sim}{\mu} = E(\underset{\sim}{X}) = \begin{bmatrix} E(x_1) \\ E(x_2) \\ \vdots \\ E(x_n) \end{bmatrix} \qquad E|x_i| < \infty \ \forall\, i = 1, 2 \cdots n.$$

$$Y = ((Y_{ij}))_{m\times n}. \qquad E(Y) = ((E\,Y_{ij}))_{m\times n}.$$

$$D(\underset{\sim}{X}) = ((\mathrm{cov}(X_i, X_j)))_{n\times n}.$$

$$= ((E[(X_i - E(X_i))(X_j - E(X_j))]))_{n\times n}.$$

$$= ((E(X_i X_j) - E(X_i)E(X_j)))_{n\times n}.$$

$$= ((E(X_i X_j) - \mu_i \mu_j))_{n\times n}.$$

$$= E\left[(\underset{\sim}{X} - \underset{\sim}{\mu})(\underset{\sim}{X} - \underset{\sim}{\mu})^T\right] = \Sigma_X$$

(1) $E(\underset{\sim}{X} + \underset{\sim}{b}) = E(\underset{\sim}{X}) + \underset{\sim}{b} = \underset{\sim}{\mu} + \underset{\sim}{b}.$

(2) $D(\underset{\sim}{X} + \underset{\sim}{b}) = D(\underset{\sim}{X}).$

(c) $\mathrm{cov}(\underset{\sim}{X} + \underset{\sim}{b}, \underset{\sim}{Y} + \underset{\sim}{c}) = \mathrm{cov}(\underset{\sim}{X}, \underset{\sim}{Y}) = \Gamma_{xy}.$

$\underset{\sim}{b}, \underset{\sim}{c}$ are constant vectors.

$$\Gamma_{xy} = ((\mathrm{cov}(X_i, Y_j)))_{p\times q}.$$

(1) $E(A\underset{\sim}{X}) = A E(\underset{\sim}{X}) = A\underset{\sim}{\mu}.$

If $A = l^T$ then $\cancel{E(l^T \mu)}$ $\underset{\sim}{l}^T \underset{\sim}{\mu} = E(\underset{\sim}{l}^T \underset{\sim}{X}) = E\left(\sum_{i=1}^n l_i X_i\right) = \sum_{i=1}^n l_i E(X_i) = \sum_{i=1}^n l_i \mu_i$

(2) $\mathrm{cov}(\underset{\sim}{U}, \underset{\sim}{V}) = \Gamma \Rightarrow \mathrm{cov}(AU, BV) = A\Gamma B^T.$

$\Rightarrow$ If $\underset{\sim}{U} = \underset{\sim}{V} = \underset{\sim}{X} \Rightarrow \mathrm{cov}(A\underset{\sim}{X}, B\underset{\sim}{X}) = A\Sigma B^T.$

$\Rightarrow \underset{\sim}{l}^T = A = B \Rightarrow \mathrm{cov}(\underset{\sim}{l}^T\underset{\sim}{X}, \underset{\sim}{l}^T\underset{\sim}{X}) = \cancel{D(X)} = \underset{\sim}{l}^T \Sigma \underset{\sim}{l}$

$\mathrm{Var}(\underset{\sim}{l}^T\underset{\sim}{X}) = D(\underset{\sim}{l}^T\underset{\sim}{X}).$

$\underset{\sim}{X}$ is a random vector.

$D(\underset{\sim}{X}) = \Sigma$. Then $D(\underset{\sim}{X}) = \Sigma$ is a p.s.d matrix.

Consider $\underset{\sim}{l}^T \neq \underset{\sim}{0}^T$ then.

$$D(\underset{\sim}{l}^T \underset{\sim}{X}) = V(\underset{\sim}{l}^T \underset{\sim}{X}) \geq 0 \quad \forall \ \underset{\sim}{l}^T \neq \underset{\sim}{0}^T.$$

$$\Rightarrow \quad \underset{\sim}{l}^T \Sigma l \geq 0 \quad \forall \ \underset{\sim}{l}^T \neq \underset{\sim}{0}^T.$$

$\Rightarrow$ Σ is a p.s.d or n.n.d matrix.

Also Σ is a symmetric matrix. $\Sigma = \Sigma^T$.

Th. Let $E(\underset{\sim}{X}) = \underset{\sim}{\mu}$, $D(\underset{\sim}{X}) = \Sigma$.

then. $P((\underset{\sim}{X} - \underset{\sim}{\mu}) \in \mathcal{C}(\Sigma)) = 1$

To prove this it is enough to show that

if $\underset{\sim}{l} \in (\mathcal{C}(\Sigma))^{\perp}$ then $\underset{\sim}{l}^T(\underset{\sim}{X} - \underset{\sim}{\mu}) = 0$.

If $l \in (\mathcal{C}(\Sigma))^{\perp}$.

$\Leftrightarrow \ \underset{\sim}{l}^T \Sigma = \underset{\sim}{0}^T.$

$\Rightarrow \ \underset{\sim}{l}^T \Sigma l = \underset{\sim}{0}^T \underset{\sim}{l}.$

$\Rightarrow \ l^T \Sigma l = 0$

$\Rightarrow \ D(\underset{\sim}{l}^T \underset{\sim}{X}) = 0.$

$\Rightarrow \ D(\underset{\sim}{l}^T(\underset{\sim}{X} - \underset{\sim}{\mu})) = 0.$

① $D(\underset{\sim}{l}^T(\underset{\sim}{X} - \underset{\sim}{\mu})) = 0.$

② $E(\underset{\sim}{l}^T(\underset{\sim}{X} - \underset{\sim}{\mu})) = 0$

$\Rightarrow P(\underset{\sim}{l}^T(\underset{\sim}{X} - \underset{\sim}{\mu}) = 0) = 1.$

$\therefore \underset{\sim}{l}$ is orthogonal to $(\underset{\sim}{X} - \underset{\sim}{\mu})$.

$$\Rightarrow \quad P((\underset{\sim}{X} - \underset{\sim}{\mu}) \in \mathcal{C}(\Sigma)) = 1.$$

$$E(x^T A x) = tr(A\Sigma) + \mu^T A \mu. \qquad E(\underset{\sim}{x}) = \underset{\sim}{\mu}. \quad D(\underset{\sim}{x}) = \Sigma.$$

Special case.

- If $X \sim N(\underset{\sim}{\mu}, \Sigma)$

$$E(\underset{\sim}{x}^T \underset{\sim}{x}) = E(\underset{\sim}{x}^T I_n \underset{\sim}{x}) = tr(I\Sigma) + \underset{\sim}{\mu}^T\underset{\sim}{\mu}$$

$$= tr(\Sigma) + \underset{\sim}{\mu}^T\underset{\sim}{\mu}.$$

Non-central χ^2.

- If $\underset{\sim}{X} \sim N(\underset{\sim}{\mu}, I_n)$ $\quad \underset{\sim}{x}^T\underset{\sim}{x} \sim \chi^2_{n,\, ncp = \underset{\sim}{\mu}^T\underset{\sim}{\mu}}$.

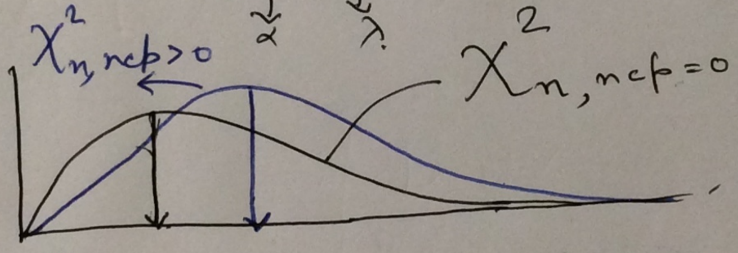
$$E(\underset{\sim}{x}^T\underset{\sim}{x}) = n + \underset{\sim}{\mu}^T\underset{\sim}{\mu} = n + \sum_{i=1}^{n} \mu_i^2 = E\left(\sum x_i^2\right) = \sum_{i=1}^{n}(1 + \mu_i^2)$$

- If $\underset{\sim}{x} \sim N(\underset{\sim}{0}, I_n)$ $\Rightarrow$ $x_1\, x_2 \cdots x_n$ iid $N(0,1)$.

$$E(\underset{\sim}{x}^T\underset{\sim}{x}) = n + 0 = n. = E(\chi^2_n)$$

If $\underset{\sim}{X} \sim N(\underset{\sim}{0}, I_n)$ then $\underset{\sim}{x}^T\underset{\sim}{x} \sim \chi^2_n$. (central χ^2)

$$\chi^2_n \equiv G\Big(\underbrace{\tfrac{n}{2}}_{\alpha}, \underbrace{\tfrac{1}{2}}_{\lambda}\Big). \qquad E(G(\alpha,\lambda)) = \frac{\alpha}{\lambda}.$$



$\sum x_i^2 = \underset{\sim}{x}^T\underset{\sim}{x}$.

$$E\left(\chi^2_{n,\, ncp = \mu^T\mu}\right) = n + \underset{\sim}{\mu}^T\underset{\sim}{\mu}$$

$$E\left(\chi^2_{n,\, ncp = 0}\right) = n + 0.$$