Problem Sheet 2 (Matrix Algebras–MA20107)

(1) Reduce the following quadratic forms into canonical form and find the rank, index, signature of the quadratic forms:

(a) $x_1x_2 + x_2x_3 + x_3x_1$

(b) $(x_1 + x_2 + x_3)x_2$

(c) $4x_1^2 + x_2^2 + 8x_1x_2 + 2x_1x_3 + 2x_2x_3$

(d) $\begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, where $A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

(e) $\begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, where $A = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

(f) $\begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, where $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 9 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

(g) $x_1x_2 - x_3^2$.

(2) Check the definiteness of all the quadratic forms in probem (3).

(3) Determine the value of a for which the matirx $A = \begin{bmatrix} a & 1 & 2 \\ 1 & a & 3 \\ 2 & 3 & a \end{bmatrix}$ is negative definite.

(4) Show that a real symmetric matrix A is positive definite if and only if A^p is positive definite, for $p > 0$ interger.

(5) Show that a real symmetric matrix A of rank r is positive semi-definite if and only if there exists a matrix P of rank r such that $A = P^TP$.

(6) Test whether the quadratic forms $P = x_1^2 - 2x_1x_2 + 3x_2^2$, $Q = x_1x_2 - x_2^2$ are equivalent or not over $\mathbb{R}$.

(7) Using Lagrange's reduction transform the following quadratic forms into diagonal form and also find the transformation.

(a) $4x_1^2 + x_2^2 + 9x_3^2 - 4x_1x_2 + 12x_1x_3$

(b) $x_1x_2 + x_2x_3 + x_3x_1$

(c) $x_1x_2 - x_3x_2$.

(8) Let V be the vector space of all $n \times n$ matrices over $\mathbb{C}$ and $A \in V$. Show that the map $f : V \times V \to \mathbb{C}$, defined by $f(X, Y) = trace(X^TAY)$, for $X, Y \in V$, is bilinear.

(9) Find the matrix of the following bilinear forms $b(\mathbf{x}, \mathbf{y})$.

(a) $-2x_1y_1 - x_1y_2 + 2x_2y_1 - x_3y_1 + 3x_3y_2$

(b) $3x_1y_1 + x_1y_2 + x_2y_1 - 2x_2y_2 - 4x_2y_3 - 4x_3y_2 + 3x_3y_3$

(c) $8x_1y_1 + 12x_1y_3 - 2x_2y_2 + 12x_3y_1 - 2x_3y_3$.

(10) Find the matrix representation of the bilinear forms in the Problem (9) with respect to the ordered bases
$B_1 = \{(1,0,1),(0,1,1),(1,1,0)\}, B_2 = \{(-1,2,1),(0,2,1),(0,0,-1)\}$.