Exercises 2

① Let A be an $m \times n$ matrix. Then show that $\text{rank}(A) = \dim[R(A)]$ where $A : \mathbb{R}^n \to \mathbb{R}^m$

$$\underline{v} \mapsto A\underline{v}.$$

② Let $V = \mathbb{R}^2$, Define

$$\langle \underline{x}, \underline{y} \rangle = 2x_1y_1 - x_1y_2 - x_2y_1 + 5x_2y_2 = [x_1\ x_2]\begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

check whether $\langle , \rangle$ is an inner product or not.

③ Show that the following:

(i) The eigenvalues of a symmetric matrix over $\mathbb{R}$ (or a Hermitian matrix over $\mathbb{C}$) are real.

(ii) The eigenvalues of a skew-symm or skew-Hermitian matrix are zero or purely imaginary.

(iii) The eigenvalues of an orthogonal matrix or unitary matrix have absolute value 1.

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① Proof: Let $A_{m\times n}$ matrix.

Let row-rank $(A) = k$

WLOG assume the first k row vectors of A forms a basis of rowspace(A).
$(\mathbb{R}^n)$

say $\underline{r_1}, \dots, \underline{r_k}$

Let $B = \begin{bmatrix} \underline{r_1} \\ \vdots \\ \underline{r_k} \end{bmatrix}_{k\times n}$ submatrix of A.

rowspace(B) = rowspace(A)

$\therefore$ rowrk(B) = rowrk$(A) = k$.

$\quad\ \ ||$

Columnrk(B)

WLOG $\underline{c_1}, \dots, \underline{c_k}$ be the first k columns of B which forms a basis for the Columnsp(B).

Let $M_{k\times k}$ be the submatrix of B whose rows are $\underline{r}_1, \ldots, r_k$ & columns are $\underline{c}_1, \ldots, \underline{c}_k$.

Of course M is also a submatrix of A.

Now $\det(M) \neq 0$.

Thus we find a $k\times k$ submatrix whose det. is non-zero.

Also if we take any submatrix B' of A of $k+1$ rows of A, then they are l.d.

This implies that all submatrix of B' of size $(k+1)\times(k+1)$ have $\det = 0$.

$\Rightarrow$ all $(k+1)\times(k+1)$ submatrices of A have $\det = 0$. $\boxed{B'_{(k+1)\times n}}$

Hly for any $s > k$.

$\therefore$ $k = \text{rank}(A)$.

$\text{rowrk}(A) = \dim(R(A^t))$

$= \dim(R(A))$.

Observation

$R(A) = \text{Column sp}(A)$

$R(A^t) = \text{Row sp}(A)$

$\dim(R(A)) = \dim(R(A^t))$

$\underline{x}, \underline{y} \in \mathbb{R}^n$ $\quad \underline{x}\cdot\underline{y} = x_1y_1 + \cdots + x_ny_n$

$$\underline{x}\cdot\underline{y} = |\underline{x}|\cdot|\underline{y}| \cos\theta.$$

$$= \underline{y}^t \underline{x}$$

$$\underline{x} \perp \underline{y} \iff \underline{x}\cdot\underline{y} = 0$$

$$= \langle \underline{x}, \underline{y} \rangle$$

std. i.p.

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② sol:-

$$\langle \underline{x}, \underline{x} \rangle = 2x_1^2 - 2x_1x_2 + 5x_2^2$$

$$= (x_1^2 - 2x_1x_2 + x_2^2) + x_1^2 + 4x_2^2$$

$$= (x_1 - x_2)^2 + x_1^2 + 4x_2^2$$

$$\geq 0$$

$$\langle \underline{x}, \underline{x} \rangle = 0 \Rightarrow x_1 - x_2 = 0,\ x_1 = 0,\ x_2 = 0$$

$$\Rightarrow \underline{x} = \underline{0}.$$

(i) Let $A_{n\times n}$ be a Hermitian matrix.

$$\text{i.e.}\quad A = A^* = \bar{A}^t.$$

- Let λ be an eigenvalue of A & $\underline{v}$ be its corresponding eigenvector.

<u>To show:</u> λ is a real number.

$$A\underline{v} = \lambda \underline{v}$$

$$\left(\overline{A\underline{v}}\right)^t = \left(\overline{\lambda \underline{v}}\right)^t$$

$$\Rightarrow \overline{\underline{v}}^t \bar{A}^t = \bar{\lambda}\, \overline{\underline{v}}^t$$

$$\Rightarrow \left(\overline{\underline{v}}^t A\right) \underline{v} = \left(\bar{\lambda}\, \overline{\underline{v}}^t\right) \underline{v}.$$

$$\Rightarrow \overline{\underline{v}}^t \left(A\underline{v}\right) = \bar{\lambda}\, \overline{\underline{v}}^t \underline{v}$$

$$\Rightarrow \underline{v}^* \left(\lambda \underline{v}\right) = \bar{\lambda}\, \underline{v}^* \underline{v}$$

$$\Rightarrow \lambda \left(\underline{v}^* \underline{v}\right) = \bar{\lambda} \left(\underline{v}^* \underline{v}\right) \qquad \left(\underline{v}^* \underline{v} = |\underline{v}|^2 \neq 0\right)$$

$$\Rightarrow \lambda = \bar{\lambda}$$

$\therefore$ λ is real.

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④ **Theorem:** Let $A_{n\times n}$ be a matrix & $\lambda_1, \dots, \lambda_r$ be the distinct eigenvalues of A. Suppose $\underline{v}_1, \dots, \underline{v}_r$ be the corresponding eigenvectors of $\lambda_1, \dots, \lambda_r$ respectively. Then

$\{\underline{v}_1, \dots, \underline{v}_r\}$ is l.i.

proof: We have $A\underline{v}_1 = \lambda_1 \underline{v}_1$

$$\vdots$$

$$A\underline{v}_r = \lambda_r \underline{v}_r.$$

Suppose $c_1 \underline{v_1} + \cdots + c_r \underline{v_r} = \underline{0}$ for some $c_1, \ldots, c_r \in F$. (*)

To show: $c_1 = \cdots = c_r = 0$.

we prove by induction on r.

$\underline{r=1}$: $\{\underline{v_1}\}$ l.i
$\underline{v_1} \neq \underline{0}$

$r=2$: $\{\underline{v_1}, \underline{v_2}\}$ $c_1 \underline{v_1} + c_2 \underline{v_2} = \underline{0}$ $\to$ (1)

$\Rightarrow$ $A(c_1 \underline{v_1} + c_2 \underline{v_2}) = \underline{0}$

$\Rightarrow$ $c_1 A\underline{v_1} + c_2 A\underline{v_2} = \underline{0}$

$\Rightarrow$ $c_1 \lambda_1 \underline{v_1} + c_2 \lambda_2 \underline{v_2} = \underline{0}$ — (2)

(1) $\times \lambda_1$ $\Rightarrow$

$$
\begin{array}{rl}
 & c_1 \lambda_1 \underline{v_1} + c_2 \lambda_1 \underline{v_2} = \underline{0} \\
- & c_1 \lambda_1 \underline{v_1} + c_2 \lambda_2 \underline{v_2} = \underline{0} \\
\hline
+ & c_2 (\lambda_1 - \lambda_2) \underline{v_2} = \underline{0}
\end{array}
$$

But $\lambda_1 \neq \lambda_2$, $\underline{v_2} \neq \underline{0}$.

$\Rightarrow$ $\boxed{c_2 = 0}$ $\Rightarrow$ $\boxed{c_1 = 0}$

i.e. True for $r = 2$

true for $r=2$.

Assume $r \geq 3$.

$A \cdot (*) \implies A(c_1 \underline{v}_1 + \dots + c_r \underline{v}_r) = \underline{0}$

$\implies c_1 A\underline{v}_1 + \dots + c_r A \underline{v}_r = \underline{0}$

$\implies \boxed{c_1 \lambda_1 \underline{v}_1 + \dots + c_r \lambda_r \underline{v}_r = \underline{0}}$ $A*$

$\lambda_r (*) \implies c_1 \lambda_r \underline{v}_1 + \dots + c_{r-1} \lambda_r \underline{v}_{r-1} + \cancel{c_r \lambda_r \underline{v}_r} = \underline{0}$

$** \implies c_1 \lambda_1 \underline{v}_1 + \dots + c_{r-1} \lambda_{r-1} \underline{v}_{r-1} + \cancel{c_r \lambda_r \underline{v}_r} = \underline{0}$

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$\implies c_1(\lambda_r - \lambda_1)\underline{v}_1 + \dots + c_{r-1}(\lambda_r - \lambda_{r-1})\underline{v}_{r-1} = \underline{0}$

By induction hypothesis we get

$c_1(\lambda_r - \lambda_1) = 0, \dots, c_{r-1}(\lambda_r - \lambda_{r-1}) = 0$

$\implies c_1 = \dots = c_{r-1} = 0$

$\implies c_r = 0.$

$\therefore \{\underline{v}_1, \dots, \underline{v}_r\}$ l.i.