Lecture 13

Hermitian forms.

Definition:- Let $H_{n\times n}$ be any Hermitian matrix.

Then any polynomial of the form

$$Q(x_1,\dots,x_n) = x^* H x$$
$$= \sum_{j=1}^{n}\sum_{k=1}^{n} h_{jk}\,\bar{x}_j x_k .$$

$x_1,\dots,x_n$ are Complex variables,

is called a <u>Hermitian form</u> of order n on $\mathbb{C}^n$.

- H is called the matrix of the Hermitian form Q.

<u>Examples</u>- ① $Q(x_1,x_2) = [\bar{x}_1\ \bar{x}_2]\underbrace{\begin{bmatrix} 1 & i+1 \\ -i+1 & 0 \end{bmatrix}}_{=H}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\boxed{H^* = H}$

$$= [\bar{x}_1\ \bar{x}_2]\begin{bmatrix} x_1 + (i+1)x_2 \\ x_1(-i+1) \end{bmatrix}$$

$$= \bar{x}_1 x_1 + \bar{x}_1 x_2 (i+1) + x_1\bar{x}_2(-i+1)$$

$(\quad) + \cdots x_2(i+1))$

$$= \bar{x}_1 x_1 + \bar{x}_1 x_2 (i+1) + \left(\overline{\bar{x}_1 x_2 (i+1)}\right)$$

$$= \bar{x}_1 x_1 + 2\,\mathrm{Re}\left(\bar{x}_1 x_2 (i+1)\right) \in \mathbb{R}$$

Q is a Hermitian form.

(2) $$Q(x_1, x_2, x_3) = \underline{x}^* \underbrace{\begin{bmatrix} 1 & i & -i \\ -i & -1 & 2 \\ i & 2 & 0 \end{bmatrix}}_{=H} \underline{x}$$

Hermitian matrix.

$$= \underline{x}^* \begin{bmatrix} x_1 + i x_2 - i x_3 \\ -i x_1 - x_2 + 2 x_3 \\ i x_1 + 2 x_2 \end{bmatrix}$$

$$= \bar{x}_1 x_1 + i \bar{x}_1 x_2 - \underline{i \bar{x}_1 x_3} + (-i) x_1 \bar{x}_2$$
$$+ (x_2)(\bar{x}_2) + \boxed{2 \bar{x}_2 x_3} + \underline{i x_1 \bar{x}_3} + \boxed{2 x_2 \bar{x}_3}$$

$$= \bar{x}_1 x_1 + 2\,\mathrm{Re}\left(i \bar{x}_1 x_2\right) + 2\,\mathrm{Re}\left(i x_1 \bar{x}_3\right)$$
$$+ \bar{x}_2 x_3 + 4\,\mathrm{Re}\left(\bar{x}_2 x_3\right) \in \mathbb{R}$$

$Q(\underline{x}) = \underline{x}^* H \underline{x}$ Hermitian form.

- det(H) is called the <u>discriminant</u> of Q
- Q is called singular (non-singular) if H is singular (non-singular)

<u>Def:</u> A hermitian form $Q(\underline{x})$ is said to be in <u>diagonal form</u> or <u>canonical form</u>

if $Q(\underline{y}) = c_1 \bar{y}_1 y_1 + c_2 \bar{y}_2 y_2 + \cdots + c_n \bar{y}_n y_n$.

where $c_1, \ldots, c_n \in \mathbb{R}$

$= c_1 |y_1|^2 + \cdots + c_n |y_n|^2$

<u>Theorem:</u> Every Hermitian form is equivalent to a diagonal form.

Let $Q(\underline{x}) = c_1 \bar{y}_1 y_1 + \cdots + c_r \bar{y}_r y_r$

where c_i's are non-zero real nos.

$\Rightarrow Q(x) = \alpha_1 \bar{y}_1 y_1 + \cdots + \alpha_k \bar{y}_k y_k - \alpha_{k+1} \bar{y}_{k+1} y_{k+1}$

$- \cdots - \alpha_r \bar{y}_r y_r.$

where $\alpha_1, \cdots, \alpha_r$ are +ve real nos.

<u>Def</u>:-
- k is called the <u>index</u> of Q
- r is called the <u>rank</u> of Q.
- The signature of Q is defined as

$$2k - r.$$

The definitions for PD, ND, PSD, NSD, ID of Hermitian forms are same as for the real quadratic forms definitions.

<u>Theorem</u>:- (Sylvester criterion for definiteness of Hermitian forms).

A Hermitian form $Q(\underline{x}) = \underline{x}^* H \underline{x}$ is +ve definite

$$\Updownarrow$$

All the leading principal minors of H are +ve.

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Theorem:-

A Hermitian form $Q(\underline{x}) = \underline{x}^* H \underline{x}$ is negative definite

$\Updownarrow$

All the principal minors of H of odd order are -ve & all the principal minors of H of even order are +ve.

Theorem:- ① A Hermitian form $Q(\underline{x}) = \underline{x}^* H \underline{x}$ is positive semi-definite

$\Updownarrow$

H is singular & the principal minors of H are ≥ 0.

② Q is negative semi-definite

$\Updownarrow$

H is singular & the principal minors of even order of H are ≥ 0 & the principal minors of odd order of H are ≤ 0.

Theorem: $Q(\underline{x}) = \underline{x}^* H \underline{x}$ is indefinite

$$\Updownarrow$$

At least one the following conditions is satisfied.

(a) H has a $-$ve principal minor of even order

(b) H has a $+$ve principal minor of odd order and a $-$ve principal minor of odd order.