Lecture 9

Generalized eigenvectors of a square matrix.

Recall: Let $A_{n\times n}$ be a matrix.

$\underline{x} \neq 0$ in $\mathbb{C}^n$ is an eigenvector of A corr. to eigenvalue λ.

$\Rightarrow \quad A\underline{x} = \lambda \underline{x}$

$\Rightarrow \quad (A-\lambda I)\underline{x} = \underline{0}$

Suppose λ is an eigenvalue of A of multiplicity m.

Then there may not exists m l.i eigenvectors of A corresponding to λ.

For eg: For $A=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, The eigenvalue $\lambda = 0$ has multiplicity 2 but 'A' does not have 2 l.i eigenvectors of A corr. to $\lambda = 0$.

Question:– Can we have some kind of m vectors which are l.i corr to an eigenvalue λ of A of multiplicity m?

Def: Let A be an $n \times n$ matrix over $\mathbb{C}$. A vector $\underline{x} \in \mathbb{C}^n$ is called a generalized eigenvector of type r of A corresponding to the eigenvalue λ, if

$$(A - \lambda I)^r \underline{x} = \underline{0} \quad \& \quad (A - \lambda I)^{r-1} \underline{x} \neq \underline{0}.$$

Remark: generalized eigenvectors of type 1 are precisely the eigenvectors of A corr to λ.

what about their existence?

Let $V_1 = \{ \underline{x} \in \mathbb{C}^n \mid (A - \lambda I)\underline{x} = \underline{0} \} \subseteq \mathbb{C}^n$

$V_2 = \{ \underline{x} \in \mathbb{C}^n \mid (A - \lambda I)^2 \underline{x} = \underline{0} \} \subseteq \mathbb{C}^n$

$\vdots$

$V_k = \{ \underline{x} \in \mathbb{C}^n \mid (A - \lambda I)^k \underline{x} = \underline{0} \} \subseteq \mathbb{C}^n$

$\vdots$

$\Rightarrow \; V_1 \subseteq V_2 \subseteq \dots \subseteq V_k \subseteq \dots \subseteq \mathbb{C}^n$.

$\Rightarrow$ there exists k_0 s.t

$$V_{k_0-1} \underset{\neq}{\subset} V_{k_0} = V_{k_0+1} = \cdots$$

$\Rightarrow$ we can find a vector $\underline{x} \in V_{k_0}$ & $\underline{x} \notin V_{k_0-1}$

$\Rightarrow$ $(A-\lambda I)^{k_0}\underline{x} = \underline{0}$ & $(A-\lambda I)^{k_0-1}\underline{x} \neq \underline{0}$.

$\Rightarrow$ $\underline{x}$ is a gen. eigenvector of type k_0 corr. to λ of A.

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① Let $A = \begin{bmatrix} 5 & 0 & 2 \\ 2 & 1 & 1 \\ -5 & 1 & -1 \end{bmatrix}$. It is known that A has a gen. eigenvector of type 2 corresponding to the eigenvalue $\lambda = 2$. Find those.

Sol: Set $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be a gen. eigenvector of A of type 2 corr. to $\lambda = 2$.

$$(A-2I)^2\underline{x} = \underline{0} \quad \& \quad (A-2I)\underline{x} \neq \underline{0}$$

$$(A-2I)^2 = \begin{bmatrix} 3 & 0 & 2 \\ 2 & -1 & 1 \\ -5 & 1 & -3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 2 \\ 2 & -1 & 1 \\ -5 & 1 & -3 \end{bmatrix}$$

$=$

$$= \begin{bmatrix} -1 & 2 & 0 \\ -1 & 2 & 0 \\ 2 & -4 & 0 \end{bmatrix}$$

$$(A-2I)^2 \underline{x} = \underline{0} \Rightarrow \begin{bmatrix} -1 & 2 & 0 \\ -1 & 2 & 0 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x + 2y = 0$$

$$\Rightarrow \boxed{x = 2y}$$

$$\& \quad (A-2I)\underline{x} \neq \underline{0} \Rightarrow \begin{bmatrix} 3 & 0 & 2 \\ 2 & -1 & 1 \\ -5 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3x + 2z \\ 2x - y + z \\ -5x + y - 3z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6y + 2z \\ 3y + z \\ -9y - 3z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3y + z \neq 0$$

$$\Rightarrow \boxed{z \neq -3y}$$

All those vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $x = 2y$
& $z \neq -3y$

are the gen. eigenvectors of type 2, Corr to $\lambda = 2$.

For eg $\underline{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is a gen. eigenvector.

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② Find gen. eigenvectors of type 2 Corr. to the eigenvalue $\lambda = 4$ of $A = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$

Sol:- Let $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ be a gen. eigenvector of type 2 corr. to $\lambda = 4$.

$$(A - 4I)^2 \underline{x} = \underline{0} \quad \& \quad (A - 4I)\underline{x} \neq \underline{0}.$$

$$(A - 4I)^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Now $(A - 4I)\underline{x} \neq \underline{0} \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} y \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \boxed{y \neq 0}$$

$\therefore$ All vectors $\begin{pmatrix} x \\ y \end{pmatrix}$, $y \neq 0$ are gen. eigenvectors of A corr to $\lambda = 4$

of A … 4.

Def: Let $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{C}^n$ be a gen. eigenvector of type m corresponding to the eigenvalue λ of $A_{n\times n}$. Then the <u>chain generated by $\underline{x}$</u> is the set of vectors $\{\underline{x} = \underline{x_m}, \underline{x_{m-1}}, \dots, \underline{x_1}\}$,

where
$$\underline{x_m} = \underline{x}$$
$$\underline{x_{m-1}} := (A-\lambda I)\,\underline{x_m}$$
$$\underline{x_{m-2}} := (A-\lambda I)^2\,\underline{x_m} = (A-\lambda I)\,\underline{x_{m-1}}.$$
$$\vdots$$
$$\underline{x_1} = (A-\lambda I)^{m-1}\,\underline{x_m} = (A-\lambda I)\,\underline{x_2}.$$

Thus: $\{\underline{x}, (A-\lambda I)\underline{x}, (A-\lambda I)^2\underline{x}, \dots, (A-\lambda I)^{m-1}\underline{x}\}$ is the chain gen. by $\underline{x}$.

In general notation,
$$\underline{x_j} := (A-\lambda I)^{m-j}\,\underline{x_m} \quad \forall\, j = 1, 2, \dots, m-1.$$

Theorem: From above notation, $\underline{x}_j$ is a generalized eigenvector of type j corresponding to the eigenvalue λ of A.

proof: To show: $(A-\lambda I)^j \underline{x}_j = \underline{0}$

& $(A-\lambda I)^{j-1} \underline{x}_j \neq \underline{0}$.

Consider
$$(A-\lambda I)^j \underline{x}_j = (A-\lambda I)^j \left((A-\lambda I)^{m-j} \underline{x}_m\right)$$
$$= (A-\lambda I)^m \underline{x}_m$$
$$= \underline{0}$$

&
$$(A-\lambda I)^{j-1} \underline{x}_j = (A-\lambda I)^{m-1} \underline{x}_m$$
$$\neq \underline{0}$$

Theorem: A chain is a l.i set of vectors.

i.e., If $\underline{x} \in \mathbb{C}^n$ is a gen. eigenvector of type m corresponding to the eigenvalue λ of A, then $\left\{\underline{x}, (A-\lambda I)\underline{x}, \ldots, (A-\lambda I)^{m-1}\underline{x}\right\}$ is a l.i set of m vectors.

proof:— Let $\underline{x_j} = (A-\lambda I)^{m-j} \underline{x_m}$, $\forall j = 1, 2, \ldots, m-1$.

$\underline{x_m} = \underline{x}$

To show: $\{\underline{x_m}, \ldots, x_1\}$ is l.i.

Suppose $c_m \underline{x_m} + \cdots + c_1 \underline{x_1} = \underline{0} \rightarrow (\$)$

for some $c_1, \ldots, c_m \in \mathbb{C}$.

Now pre-multiply by $(A-\lambda I)^{m-1}$, then we get

$$c_m (A-\lambda I)^{m-1} \underline{x_m} + c_{m-1} (A-\lambda I)^{m-1} x_{m-1} + \cdots$$

$$\cdots + c_1 (A-\lambda I)^{m-1} \underline{x_1} = \underline{0} \rightarrow (*)$$

Now for any $j = 1, 2, \ldots, m-1$

$$(A-\lambda I)^{m-1} \underline{x_j} = (A-\lambda I)^{m-1-j} \left((A-\lambda I)^j \underline{x_j} \right)$$

$$= (A-\lambda I)^{m-1-j} \underline{0} \quad (\because x_j \text{ is gen. eigenvect of type } j \text{ by above thm})$$

$$= \underline{0}$$

$\therefore$ From $(*)$ we get,

$$\underbrace{c_m (A-\lambda I)^{m-1} \underline{x}_m}_{\neq \underline{0}} = \underline{0}$$

$$\Rightarrow \boxed{c_m = 0}$$

Now pre-multiply (8) with $(A-\lambda I)^{m-2}$, we get $c_{m-1} = 0$

$\vdots$

$c_1 = 0.$

$\therefore$ the chain gen. by $\underline{x}$ is l.i