Problem Sheet 4 (Matrix Algebra–MA20107)

(1) Determine $\| \mathbf{x} \|_p$, for $p = 1, 2, \infty$, where $\mathbf{x}$ is

(a) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} i \\ i \\ i \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix}$ in $\mathbb{C}^n$.

(2) Prove that for any matrix and vector norms

(a) for any $m \times n$ matrices, $| \| A \| - \| B \| | \leq \| A - B \|$

(b) for any n-dimensional column vectors, $\mathbf{x}, \mathbf{y}$, $| \| \mathbf{x} \| - \| \mathbf{y} \| | \leq \| \mathbf{x} - \mathbf{y} \|$.

(c) For any induced matrix norm $\| - \|$, show that $\| AB \| \leq \| A \| \| B \|$.

(3) For any norm $\| - \|$ induced from an inner product, show that

$$\| x + y \|^2 + \| x - y \|^2 = 2(\| x \|^2 + \| y \|^2).$$

(4) Determine the induced matrix norm $\| A \|_p$, for $p = 1, 2, \infty$, where A is

(i) $\begin{bmatrix} 1 & i & -i \\ 0 & 2i & 1+i \\ -7 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1+i & 1-i & 1 \\ 2 & -6 & 0 \\ 4 & i & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & -3 & -1 \\ 0 & 3 & 0 \\ 3 & 0 & 1 \end{bmatrix}$.

(5) Show that $\| A \| = n \max_{i,j} |a_{ij}|$ is a matrix norm on the space of all $n \times n$ matrices, where $A = [a_{ij}]$.

(6) Show that if P is a unitary matrix, then $\| P \|_2 = 1$, where $\| - \|_2$ denote the induced 2-norm.

(7) Prove that for any $n \times n$ matrix A over $\mathbb{C}$, the eigenvalues of A^*A are non-negative, and if 0 is an eigenvalue of A^*A, then it is also an eigenvalue of AA^*.

(8) Prove that $\| UA \|_2 = \| A \|_2 = \| AV \|_2$, for any U, V unitary matrices.

(9) Find the QR-decomposition of the following matrices by using the Householder transformations

(i) $\begin{bmatrix} -2 & -2 & -4 \\ 2 & 3 & 2 \\ 3 & 2 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & -3 & -1 \\ 0 & 3 & 0 \\ 4 & 5 & 4 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$.

(10) Find the Cholesky decomposition of the following matices

$$\text{(i)} \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix} \qquad \text{(ii)} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \text{(iii)} \begin{bmatrix} 6 & 4 & -2 \\ 4 & 12 & -4 \\ -2 & -4 & 13 \end{bmatrix}$$