Lecture 4

Proof of claim 2:-

Let $\underline{y} \in R(B) \cap N(A)$.

$\Rightarrow \quad \underline{y} = B\underline{z}$ for some $\underline{z} \in \mathbb{R}^p$.

& $\quad A\underline{y} = \underline{0}$

$\Rightarrow \quad A(B\underline{z}) = \underline{0}$

$\Rightarrow \quad AB\underline{z} = \underline{0}$

$\Rightarrow \quad \underline{z} \in N(AB)$

$\Rightarrow \quad \underline{z} = \lambda_1 \underline{x}_1 + \cdots + \lambda_q \underline{x}_q + \cdots + \lambda_k \underline{x}_k$

for some $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$

$\Rightarrow \quad \underline{y} = B\underline{z} = \lambda_1 B\underline{x}_1 + \cdots + \lambda_q B\underline{x}_q + \cdots$

$\cdots + \lambda_k B\underline{x}_k$

$= \underline{0} + \cdots + \underline{0} + \lambda_{q+1} B\underline{x}_{q+1} + \cdots$

$\cdots + \lambda_k B\underline{x}_k$

$\in \text{Span}(\{B\underline{x}_{q+1}, \ldots, B\underline{x}_k\})$

Clearly $\text{span}(\{B\underline{x}_{q+1}, \ldots, B\underline{x}_k\}) \subseteq R(B) \cap N(A)$.

Thus $R(B) \cap N(A) = \text{span}(\{B\underline{x}_{q+1}, \ldots, B\underline{x}_k\})$

This proves claim 2.

$\therefore \dim(R(B) \cap N(A)) = k - q$

$= (p - \text{rank}(AB)) - (p - \text{rank}(B))$

$= \text{rank}(B) - \text{rank}(AB)$

$= \text{rank}(B) - \text{rank}(AB)$

This proves the theorem.

Theorem (Sylvester Inequality)

Let $A_{m\times n}$, $B_{n\times p}$ be matrices. Then

$$\text{rank}[A] + \text{rank}[B] - n \le \text{rank}(AB) \le \min\{\text{rank}(A), \text{rank}[B]\}.$$

proof:- The second inequality already proved.

Let us prove the first inequality.

We have $\dim(N(A)) = n - \text{rank}(A)$ (by rank-nullity thm)

& $N(A) \cap R(B) \subseteq N(A)$ subsp.

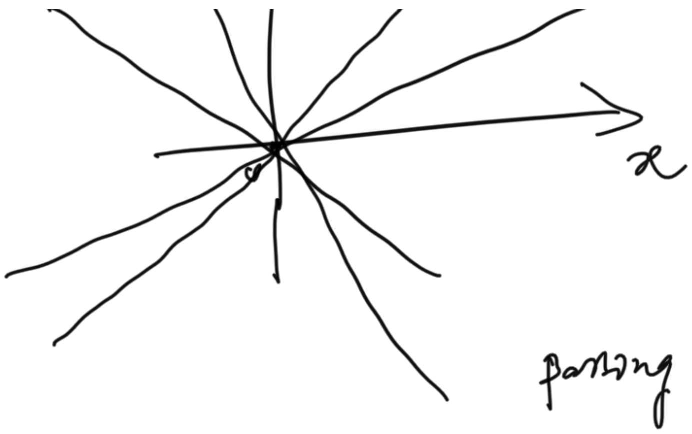
$$\Rightarrow \dim\big(N(A) \cap R(B)\big) \le \dim(N(A)).$$

$$\Rightarrow \underbrace{\text{rank}(B) - \text{rank}(AB)}_{\text{(by above theorem)}} \le n - \text{rank}(A)$$

$$\Rightarrow \boxed{\text{rank}(B) + \text{rank}(A) - n \le \text{rank}(AB)}.$$

Geometry Interpretation.

Let $U = \text{span}\left(\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}\right) = \left\{\begin{pmatrix}a\\a\end{pmatrix} \middle/ a \in \mathbb{R}\right\}$



$x = y$

$\dim(V) = 1$

Any 1-dim. subspace is a line passing through $\underline{0}$.

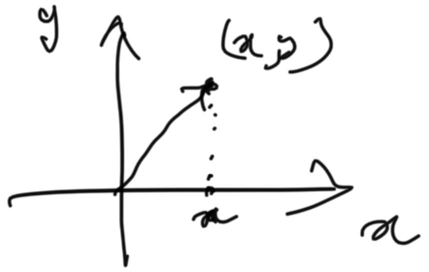
Let V, W be vector spaces over F. Let $T : V \to W$ be a L.T.

i.e. $T(\underline{v} + \underline{w}) = T(\underline{v}) + T(\underline{w}) \quad \forall \underline{v}, \underline{w} \in V$

$T(\lambda \underline{v}) = \lambda T(\underline{v}) \quad \forall \lambda \in F$
$\forall \underline{v} \in V$

Examples

(1)

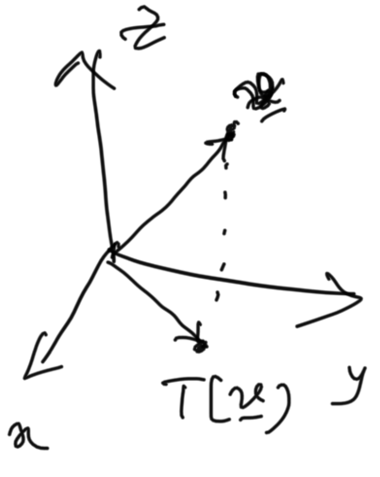


$\mathbb{R}^2$

$T : \mathbb{R}^2 \to \mathbb{R}$ ~~reflexion w.r.t. x-axis~~ (projection)

$T(x, y) = x$

(2)



T = the projection on to xy-plane

$T : \mathbb{R}^3 \to \mathbb{R}^2, \quad T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

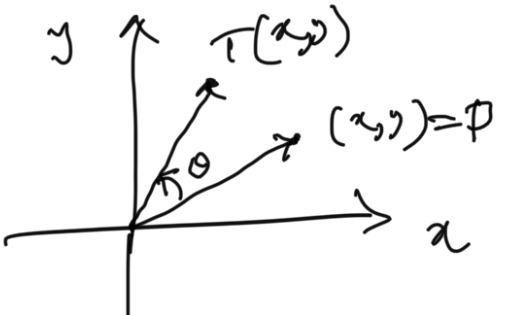
L.T

(3) $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{A \text{ (say)}} \begin{bmatrix} x \\ y \end{bmatrix}$$ where θ is fixed.

$\forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$

L.T



$T(x,y) = ?$

$T(x,y)$ = The point obtained by rotating $\overrightarrow{OP}$ in the anticlockwise direction with an angle θ.

Matrix representation of a L.T.

Let V, W be v.sps / F. Let $T: V \to W$ be a L.T.

Let $B_1 = \{\underline{x_1}, \dots, \underline{x_n}\}$ be a basis for V &

$B_2 = \{\underline{y_1}, \dots, \underline{y_m}\}$ " " W.

Then we have

$$T(\underline{x_1}) = a_{11}\underline{y_1} + a_{12}\underline{y_2} + \dots + a_{1m}\underline{y_m}$$

for some $a_{11}, \ldots, a_{1m} \in F$.

$T(\underset{\sim}{x_2}) = a_{21}\underline{y_1} + a_{22}\underline{y_2} + \cdots + a_{2m}\underset{\sim}{y_m}$, for some $a_{21}, \ldots, a_{2m} \in F$.

$\vdots$

$T(\underset{\sim}{x_n}) = a_{n1}\underline{y_1} + a_{n2}\underline{y_2} + \cdots + a_{nm}\underset{\sim}{y_m}$

for some $a_{n1}, \ldots, a_{nm} \in F$.

Then the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}^t$$

is called the matrix representation of T w.r.t the bases $\mathcal{B}_1, \mathcal{B}_2$ & we denote by $[T]_{\mathcal{B}_1, \mathcal{B}_2}$.

Note:- Let V be vector sp. & $U, W \subseteq V$ subsp. Then the sum $U+W$ is said to be a direct sum if every vector in $U+W$ can be uniquely written as a sum of a vector in U & a vector in W, & written as $U \oplus W$.

For $V = \mathbb{R}^2$, $U = \{(a, \ldots) \mid a \in \mathbb{R}\}$

Eg: $V = \mathbb{R}^2$, $U = \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} \,\middle|\, a \in \mathbb{R} \right\}$

$W = \left\{ \begin{pmatrix} b \\ 0 \end{pmatrix} \,\middle|\, b \in \mathbb{R} \right\}$

Then check that $U + W$ is not a direct sum.

Inner product space (i.p.s)

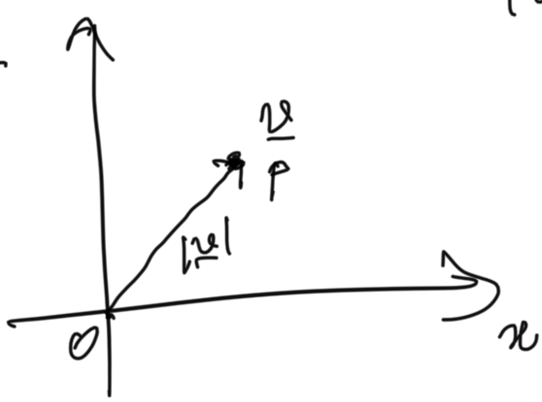
Recall:– $\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$.

$|\underline{v}| = \text{length of } \underline{v} = \sqrt{v_1^2 + \dots + v_n^2}$.

if $\underline{v} \in \mathbb{C}^n$, $|\underline{v}| = \sqrt{|v_1|^2 + \dots + |v_n|^2}$

when $v_1, \dots, v_n \in \mathbb{C}$

$|v_1|$ = modulus of the complex no v_1.



$|\overrightarrow{OP}|$.

Let $F = \mathbb{R}$ or $\mathbb{C}$.

Def:– Let V be a vector space over F.

An inner product on V is a function

An inner product on V is a function

$\langle\ ,\ \rangle : V \times V \to F$ satisfying the following conditions:

(i) $\langle \underline{v}, \underline{v} \rangle \geq 0 \quad \forall \underline{v} \in V$ &
(real no)

$\langle \underline{v}, \underline{v} \rangle = 0 \iff \underline{v} = \underline{0}$.

(ii) $\langle \underline{v}+\underline{w}, \underline{u} \rangle = \langle \underline{v}, \underline{u} \rangle + \langle \underline{w}, \underline{u} \rangle$
$\forall\ \underline{u}, \underline{v}, \underline{w} \in V$.

(iii) $\langle c\underline{v}, \underline{w} \rangle = c \langle \underline{v}, \underline{w} \rangle \quad \forall\ \underline{v}, \underline{w} \in V$
$\forall\ c \in F$

(iv) $\langle \underline{v}, \underline{w} \rangle = \overline{\langle \underline{w}, \underline{v} \rangle}$ (Complex Conjugate)
$\forall\ \underline{v}, \underline{w} \in V$.

Then $\langle , \rangle$ is called an innerproduct on V & a vectorsp V together with an inner product $\langle , \rangle$ is called an <u>inner product space</u> (i.p.s) & we denote as $(V, \langle , \rangle)$.

Examples:- (i) $V = \mathbb{C}^n$ ...

① $V = \mathbb{C}^n$. Define

$$\langle\, ,\, \rangle : V \times V \to \mathbb{C}$$

$$\langle \underline{v}, \underline{w} \rangle := \underset{1\times n}{\underline{w}^*} \underset{n\times 1}{\underline{v}} \quad \forall \underline{v}, \underline{w} \in V.$$

where $\underline{w}^* = \overline{\underline{w}}^t$

$$\underline{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \qquad \overline{\underline{w}} = \begin{pmatrix} \overline{w_1} \\ \vdots \\ \overline{w_n} \end{pmatrix}$$

To show:

$\langle\, ,\, \rangle$ is an innerproduct.

(i)

$$\langle \underline{v}, \underline{v} \rangle = \underline{v}^* \underline{v}$$

$$= \begin{pmatrix} \overline{v_1} & \overline{v_2} & \cdots & \overline{v_n} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$= \overline{v_1} v_1 + \cdots + \overline{v_n} v_n$$

$$= |v_1|^2 + \cdots + |v_n|^2$$

$$\geq 0$$

& $\langle \underline{v}, \underline{v} \rangle = 0 \iff |v_1|^2 + \cdots + |v_n|^2 = 0$

$$\iff |v_1| = \cdots = |v_n| = 0$$

$$\iff v_1 = \cdots = v_n = 0$$

$$\iff \underline{v} = \underline{0}.$$

(ii) $\langle \underline{v} + \underline{w}, \underline{u} \rangle = \underline{u}^* (\underline{v} + \underline{w})$

$$= \underline{u}^* \underline{v} + \underline{u}^* \underline{w} \qquad \forall \underline{u}, \underline{v}, \underline{w} \in \mathbb{C}^n.$$

$$= \langle v, u \rangle + \langle$$

$\langle -, \underline{u} \rangle + \langle \underline{w}, \underline{u} \rangle.$

(iii) $\langle c\underline{v}, \underline{w} \rangle = \underline{w}^*(c\underline{v}) = c\underline{w}^*\underline{v} = c\langle \underline{v}, \underline{w} \rangle.$

(iv) $\overline{\langle \underline{v}, \underline{w} \rangle} = \overline{(\underline{w}^*\underline{v})} = \overline{\underline{w}^*}\,\overline{\underline{v}}$

$= \underline{w}^t\overline{\underline{v}}$

$= (\underline{w}^t\overline{\underline{v}})^t$

$= \overline{\underline{v}}^t(\underline{w}^t)^t$

$= \underline{v}^*\underline{w}$

$= \langle \underline{w}, \underline{v} \rangle \quad \forall\, \underline{v}, \underline{w} \in \mathbb{C}^n.$

$\therefore \langle , \rangle$ is an innerproduct.
Known as standard innerproduct or
usual innerproduct on $\mathbb{C}^n$.

(2) on $V = \mathbb{R}^n$, $\langle \underline{v}, \underline{w} \rangle := \underline{w}^t\underline{v} \in \mathbb{R}$
check that this is an innerproduct
Known as standard innerproduct or
usual inner product on $\mathbb{R}^n$.

(3) $V = \mathbb{R}^2$, define $\langle \underline{x}, \underline{y} \rangle := 2x_1y_1 - x_1y_2 - x_2y_1 + 5x_2y_2$

$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

check that $\langle , \rangle$ is an innerproduct on $\mathbb{R}^n$. (EXERCISE).

Let $(V, \langle , \rangle)$ be an i.p.s over F. Define

$\|\underline{x}\| := \sqrt{\langle \underline{x}, \underline{x} \rangle}$ known as the norm of x or length of $\underline{x}$.

$\underline{x} \in V$,

$\|\underline{x} - \underline{y}\|$ = the distance between $\underline{x}$ & $\underline{y}$.

where $\underline{x}, \underline{y} \in V$

$$\|\underline{x} - \underline{y}\| = \sqrt{\langle \underline{x}-\underline{y}, \underline{x}-\underline{y} \rangle}.$$

Examples:-

① $V = \cancel{\mathbb{R}^n}. \mathbb{C}^n$ with usual inner product $\langle , \rangle$.

$$\langle \underline{u}, \underline{v} \rangle = \underline{v}^* \underline{u}.$$

$$\|\underline{v}\| = \sqrt{\langle \underline{v}, \underline{v} \rangle}$$

$$= \sqrt{\underline{v}^* \underline{v}}$$

$$= \sqrt{|\underline{v}|^2} = |\underline{v}| = \text{length of } \underline{v}.$$

$$\|\underline{v} - \underline{w}\| = \sqrt{\langle \underline{v}-\underline{w}, \underline{v}-\underline{w} \rangle}$$

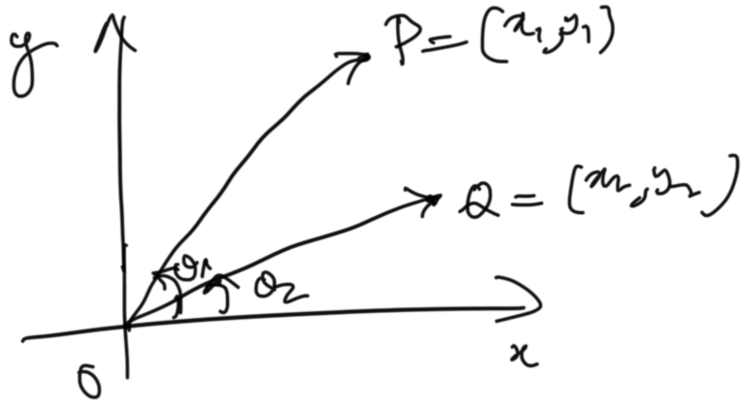
$$= \sqrt{(\underline{v}-\underline{w})^*(\underline{v}-\underline{w})}$$

$$= \sqrt{|\underline{v}-\underline{w}|^2}$$

$$= |\underline{v}-\underline{w}|$$

$$= \sqrt{(|v_1-w_1|)^2 + \cdots + (|v_n-w_n|)^2}$$

= distance between $\underline{v}$ & $\underline{w}$.



Then $\cos(\theta_1-\theta_2)$ = cosine of the angle between $\overline{OP}$ & $\overline{OQ}$

$$= \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$$

$$= \frac{x_1}{\sqrt{x_1^2+y_1^2}} \cdot \frac{x_2}{\sqrt{x_2^2+y_2^2}} + \frac{y_1}{\sqrt{x_1^2+y_1^2}} \cdot \frac{y_2}{\sqrt{x_2^2+y_2^2}}$$

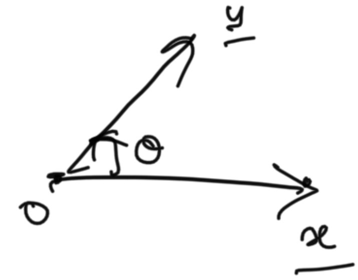
$$= \frac{x_1 x_2 + y_1 y_2}{\|\underline{x}\| \, \|\underline{y}\|}, \quad \underline{x} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad \underline{y} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

$$= \frac{\langle \underline{x}, \underline{y} \rangle}{}$$ — std. innerproduct.

$$\|\underline{x}\| \; \|\underline{y}\|$$

Thus if θ is the angle between $\underline{x}, \underline{y} \in \mathbb{R}^n$ or $\mathbb{C}^n$

Then
$$\boxed{\cos\theta = \frac{\langle \underline{x}, \underline{y} \rangle}{\|\underline{x}\| \; \|\underline{y}\|}}.$$



- $\underline{x}$ & $\underline{y}$ are <u>perpendicular</u> or <u>orthogonal</u> if $\cos\theta = 0$ i.e. $\langle \underline{x}, \underline{y} \rangle = 0$.
  i.e. $\underline{y}^* \underline{x} = 0$

<u>Def:</u> We say $\underline{x}, \underline{y} \in V$ are <u>orthogonal</u> in an i.p.s $(V, \langle , \rangle)$ over F, if
$$\langle \underline{x}, \underline{y} \rangle = 0.$$

<u>Def:</u> A set of vectors $S = \{\underline{v}_1, \dots, \underline{v}_n\}$ in an inner product space $(V, \langle , \rangle)$ is called an <u>orthogonal set</u> if each vector of S is orthogonal to every other vector in S

every other vector in S.

i.e. $\langle \underline{v_i}, \underline{v_j} \rangle = 0 \quad \forall\, i \neq j, \quad i, j = 1, 2, \dots, n.$

Example:- ① $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is an orthogonal set w.r.t std. inner product.

② $\left\{ \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}$ is an orthogonal set in $\mathbb{R}^3$.

$$\left\langle \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\rangle = 1(2) + (-2)(1) + 3(0)$$
$$= 0.$$

Def:- A vector $\underline{v}$ in an i.p.s $(V, \langle , \rangle)$ is said to be a unit vector if $\|\underline{v}\| = 1$.

i.e. $\langle \underline{v}, \underline{v} \rangle^{1/2} = 1$

Note:- Normalization of any non-zero vector $\underline{v}$ in V is $\dfrac{\underline{v}}{\|\underline{v}\|} = \underline{w}$.

Then $\underline{w}$ is a unit vector.

$$\|\underline{w}\| = \left\| \frac{\underline{v}}{\|\underline{v}\|} \right\| = \frac{1}{\|\underline{v}\|} \|\underline{v}\| = 1.$$

Defn A set of vectors $S = \{\underline{v}_1, \ldots, \underline{v}_n\}$ in an i.p.s $(V, \langle , \rangle)$ is said to be an <u>orthonormal</u> set, if

(i) S is an orthogonal set

i.e. $\langle \underline{v}_i, \underline{v}_j \rangle = 0 \quad \forall j \neq i$

(ii) $\|\underline{v}_j\| = 1 \quad \forall j = 1, 2, \ldots, n.$

i.e.

$$\langle \underline{v}_i, \underline{v}_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

$$\forall i, j = 1, 2, \ldots, n.$$

Example: $\{\underline{e}_1, \ldots, \underline{e}_n\}$ is an orthonormal set in the std. i.p.s $(\mathbb{R}^n, \langle , \rangle)$.

$$\underline{e}_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j^{th}.$$

Theorem Every orthogonal set of non zero vectors is l.i & hence every orthonormal set is l.i.

proof: Let $S = S$

Let $S = \{\underline{v}_1, \ldots, \underline{v}_n\}$ be an orthogonal set in $(V, \langle , \rangle)$.

To show: S is l.i.

Suppose $c_1 \underline{v}_1 + \cdots + c_n \underline{v}_n = \underline{0}$ for some $c_1, \ldots, c_n \in F$.

Now for any j,

$$\langle c_1 \underline{v}_1 + \cdots + c_n \underline{v}_n , \underline{v}_j \rangle = \langle 0, \underline{v}_j \rangle = 0$$

$$\Rightarrow \langle c_1 \underline{v}_1, \underline{v}_j \rangle + \langle c_2 \underline{v}_2, \underline{v}_j \rangle + \cdots + \langle c_n \underline{v}_n, \underline{v}_j \rangle = 0$$

(by (ii) in innerproduct definition)

$$\Rightarrow c_1 \langle \underline{v}_1, \underline{v}_j \rangle + c_2 \langle \underline{v}_2, \underline{v}_j \rangle + \cdots + c_n \langle \underline{v}_n, \underline{v}_j \rangle = 0.$$

But S is an orthogonal set

$$\Rightarrow c_j \langle \underline{v}_j, \underline{v}_j \rangle = 0.$$

But $\langle \underline{v}_j, \underline{v}_j \rangle \neq 0$ because $\underline{v}_j \neq \underline{0}$ $\forall j$.

$$\Rightarrow c_j = 0. \quad \forall j$$

$$\Rightarrow c_1 = \cdots = c_n = 0.$$

Thus S is l.i.