Problems

① Suppose $U, W \subseteq V$ are subspaces of a vector space V over F. Then show that

(i) $U+W \subseteq V$ is a subspace of V

(ii) $U \cap W \subseteq V$ is a subspace of V.

② Let $U, W \subseteq V$ be subspaces. Then show that

(i) $\dim(U+W) \leq \dim(U) + \dim(W)$.

__Definition__ Let $U, W \subseteq V$ be subspaces of V. Then we say that V is a direct sum of U & W i.e., $V = U \oplus W$, if every vector of $\underline{v} \in V$ can be uniquely written as $\underline{v} = \underline{u} + \underline{w}$ for some unique $\underline{u} \in U$, $\underline{w} \in W$.

③ Let $V = \mathbb{R}^2$, $U = \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} \middle| a \in \mathbb{R} \right\}$, $W = \left\{ \begin{pmatrix} 0 \\ b \end{pmatrix} \middle| b \in \mathbb{R} \right\}$. Then show that

$V = U \oplus W$. & $\dim(V) = \dim(U) + \dim(W)$.

④ More generally, if $V = U \oplus W$, then show that $\dim(V) = \dim(U) + \dim(W)$.

Def: A sum $U + W$ is called a <u>direct sum</u> if every vector of $U+W$ can be write uniquely as a sum of a vector in U & a vector in W. & we write as $U \oplus W$.

⑤ Show that $U+W$ is a direct sum

i.e., $U+W = U \oplus W \iff U \cap W = \{\underline{0}\}$.

Solution

① Given $U, W \subseteq V$ subspaces.

<u>To Show:</u> $U+W$ is a subspace of V.

Let $\underline{x} \in U+W$, $\underline{y} \in U+W$

Then $\underline{x} = \underline{u_1} + \underline{w_1}$, $\underline{y} = \underline{u_2} + \underline{w_2}$

for some $\underline{u_1}, \underline{u_2} \in U$, $\underline{w_1}, \underline{w_2} \in W$.

(a) $\underline{x} - \underline{y} = (\underline{u_1} + \underline{w_1}) - (\underline{u_2} + \underline{w_2})$

$= (\underline{u_1} - \underline{u_2}) + (\underline{w_1} - \underline{w_2})$

$\in U + W$

(b) For

$\lambda \in F$, $\underline{x} = \underline{u} + \underline{w} \in U+W$.

$$\lambda \underline{x} = \lambda(\underline{u} + \underline{w}) = \lambda \underline{u} + \lambda \underline{w} \in U+W$$

$\therefore$ $U+W$ is a subsp. of V.

To show: $U \cap W \subseteq V$ is a subspace of V.

i.e, To show: (i) For $\underline{v}_1, \underline{v}_2 \in U \cap W$, then $\underline{v}_1 - \underline{v}_2 \in U \cap W$

(ii) for $\lambda \in F$, $\underline{v} \in U \cap W$, $\lambda \underline{v} \in U \cap W$.

Solution of (2):

Given $U, W \subseteq V$ are subspaces.

Let $\{\underline{x}_1, \dots, \underline{x}_n\}$ be a basis of U

& let $\{\underline{y}_1, \dots, \underline{y}_m\}$ be a basis of W.

claim:- $\boxed{U+W = \text{span}(\{\underline{x}_1, \dots, \underline{x}_n, \underline{y}_1, \dots, \underline{y}_m\})}$.

Pf. of claim:- clearly $\underline{x}_i = \underline{x}_i + \underline{0} \in U+W \quad \forall i$

& $\underline{y}_j = \underline{0} + \underline{y}_j \in U+W \quad \forall j$

$\therefore$ $\{\underline{x}_1, \dots, \underline{x}_n, \underline{y}_1, \dots, \underline{y}_m\} \subseteq U+W$.

$\Rightarrow$ $\boxed{\text{span}(\{\underline{x}_1, \dots, \underline{x}_n, \underline{y}_1, \dots, \underline{y}_m\}) \subseteq U+W}$.

Let $\underline{v} \in U+W$.

$\underline{v} = \underline{u} + \underline{w}$ for some $\underline{u} \in U$, $\underline{w} \in W$.

$\Rightarrow$ $\underline{u} = a_1 \underline{x_1} + \cdots + a_n \underline{x_n}$ for some $a_i \in F$

& $\underline{w} = b_1 \underline{y_1} + \cdots + b_m \underline{y_m}$ for some $b_j \in F$.

Now

$\underline{v} = \underline{u} + \underline{w} = a_1 \underline{x_1} + \cdots + a_n \underline{x_n} + b_1 \underline{y_1} + \cdots + b_m \underline{y_m}$

$= $ a l.c of $\{\underline{x_1}, \ldots, \underline{x_n}, \underline{y_1}, \ldots, \underline{y_m}\}$

$\in \text{Span}(\{\underline{x_1}, \ldots, \underline{x_n}, \underline{y_1}, \ldots, \underline{y_m}\})$.

$\therefore$ $\boxed{U + W \subseteq \text{Span}(\{\underline{x_1}, \ldots, \underline{x_n}, \underline{y_1}, \ldots, \underline{y_m}\})}$.

Thus claim is true.

From the claim

$$\dim(U+W) \leq n+m.$$
$$= \dim(U) + \dim(W).$$

Theorem:- $U, W \subseteq V$ subspaces. Then

$$\boxed{\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W).}$$

Example:- (1) $V = \mathbb{R}^2$

$$U = \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} \middle| a \in \mathbb{R} \right\} \subseteq V$$

$$W = \left\{ \begin{pmatrix} b \\ 0 \end{pmatrix} \middle| b \in \mathbb{R} \right\} \subseteq V.$$

Note that $U + W$ ~~sum~~ is <u>not a direct sum</u>.

because there exists vectors in $U + W$ that

We can not write uniquely as a sum of a vector in U & a vector in W.

$\cancel{U \oplus W} = \{0\}$ (In fact every non zero vector $\underset{\in U}{\begin{pmatrix} a \\ 0 \end{pmatrix}} \cancel{+} \underset{\in W}{\begin{pmatrix} b \\ 0 \end{pmatrix}}$ $(a \neq b)$

$= \underset{\in U}{\begin{pmatrix} b \\ 0 \end{pmatrix}} + \underset{\in W}{\begin{pmatrix} a \\ 0 \end{pmatrix}}$

(2) Let $V = \mathbb{R}^2$, $U = \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} \middle/ a \in \mathbb{R} \right\}$

$W = \left\{ \begin{pmatrix} 0 \\ b \end{pmatrix} \middle/ b \in \mathbb{R} \right\}$

Then $U+W$ is a direct sum.

ie, $U+W = \underline{U \oplus W}$.

Let $\underline{v} = \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix} \in U+W$

$= \begin{pmatrix} a \\ b \end{pmatrix}$ Uniquely.

Problem (5): $U+W = U \oplus W$

$\Updownarrow$

$U \cap W = \{0\}$

proof:- $\Uparrow$ ~~Let~~ Assume $U \cap W = \{0\}$.

To show: $U+W = U \oplus W$.

Let $\underline{v} \in U+W$.

Suppose $v = u_1 + w_1$

$$\underline{u}_1 + \underline{w}_1 = \underline{u}_2 + \underline{w}_2$$

for some $\underline{u}_1, \underline{u}_2 \in U$

$\underline{w}_1, \underline{w}_2 \in W$.

$$\Rightarrow \quad \underline{u}_1 - \underline{u}_2 = \underline{w}_2 - \underline{w}_1 \in W \cap U = \{\underline{0}\}$$

$$\Rightarrow \quad \underline{u}_1 - \underline{u}_2 = \underline{w}_2 - \underline{w}_1 = \underline{0}$$

$$\Rightarrow \quad \underline{u}_1 = \underline{u}_2 \ \& \ \underline{w}_1 = \underline{w}_2 .$$

$\Downarrow$ II: EXERCISE.

Theorem: $\dim(U \oplus W) = \dim(U) + \dim(W)$.

Theorem: Let $A_{m \times n}$ be a matrix. Then

$$\dim(R(A)) = \operatorname{rank}(A).$$

$A : \mathbb{R}^n \to \mathbb{R}^m$

$\underline{x} \mapsto A\underline{x}$.

L.T.
