Vector Space: A vector space (over $\mathbb{R}$) is a set V of elements, called vectors, together with two binary algebraic operations, called vector addition (+) and scalar multiplication (·), if the following axioms hold

1) $u+v \in V$, for all $u, v \in V$
2) $\lambda u \in V$ for all $\lambda \in \mathbb{R}$, all $u \in V$

$\Big\}$ (closure properties)

3) $u+v = v+u$, $\forall\, u, v \in V$ commutativity
4) $u+(v+w) = (u+v)+w$, $\forall\, u, v, w \in V$ associativity
5) $\exists\, 0$ (zero vector) in V such that $u+0 = u$, $\forall\, u \in V$
6) For each $u \in V$, $\exists$ a vector in V, denoted by $-u$ (negative of u) such that
$$u+(-u) = 0 \text{ (zero vector)}$$
7) $\lambda(\mu u) = (\lambda\mu)u$, $\forall\, \lambda, \mu \in \mathbb{R}$, and $\forall\, u \in V$, Associativity
8) $\lambda(u+v) = \lambda u + \lambda v$, $\forall\, \lambda \in \mathbb{R}$, $\forall\, u, v \in V$, distributivity
9) $(\lambda+\mu)u = \lambda u + \mu u$, $\forall\, \lambda, \mu \in \mathbb{R}$, $u \in V$, distributivity
10) For each $u \in V$, $1\cdot u = u$, 1 being the identity element in $\mathbb{R}$.

Elements of V are called vectors while the elements of $\mathbb{R}$ (or F, $\mathbb{C}$) are called scalars.

Examples: 1) Vector space $\mathbb{R}^n$ over $\mathbb{R}$

Let V be the set of all ordered n-tubles
$$\{(a_1, a_2, \cdots, a_n);\ a_i \in \mathbb{R}\}$$
Let $+$ and $\bullet$ are defined as
$$(a_1, a_2, \cdots, a_n) + (b_1, b_2, \cdots, b_n) = (a_1+b_1, a_2+b_2, \cdots, a_n+b_n)$$

2) Polynomial space $P_n(t)$ over $\mathbb{R}$:

Let $P_n(t)$ denotes the set of all polynomials of degree less than or equal to n, that is, the set of all polynomial

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_s t^s$$

where $s \leq n$ and $a_i \in \mathbb{R}$

3) Matrix space $M_{m\times n}$:

Set of all $m\times n$ matrices with elements from $\mathbb{R}$

4). Solution of $Ax = 0$ form a vector space

$$V = \{x \in \mathbb{R}^n : A_{m\times n} x_{n\times 1} = O_{m\times 1}\},$$ called NULL SPACE

It is easy to verify that $V \subseteq \mathbb{R}^n$ and forms a vector space, therefore it is called a Vector subspace.

(Let x_1 & x_2 be two solutions of $Ax = 0$, ie,

$Ax_1 = 0$ & $Ax_2 = 0$

$\Rightarrow A(x_1 + x_2) = Ax_1 + Ax_2 = 0$

& $A(\lambda x_1) = \lambda(Ax_1) = 0$.)

SUBSPACES:

Let V be a vector space (over $\mathbb{R}$) and let W be a subset of V. Then W is a subspace of V if W is itself a vector space (over $\mathbb{R}$) with respect to the same operations as in V.

CRITERIA FOR IDENTIFYING SUBSPACES:

For every $u, v \in W$ & $\lambda \in \mathbb{R}$, the following closure properties should hold

$$u + v \in W$$

$$\& \quad \lambda u \in W$$

TRIVIAL SUBSPACES OF V:

- The set $\{0\}$
- The whole set V itself

Example:

1. Let U consists of all vectors from $\mathbb{R}^3$ whose entries are equal; that is,

$$U = \{(a, b, c) : a = b = c\}$$

2. Let $V = \{\underset{(x_1, x_2)}{x} \in \mathbb{R}^2 ; \ x_1 \geq 0 ; \ x_2 \geq 0\}$

Does V form a vector space of $\mathbb{R}^2$?

ANS: NO, because $-1 \cdot x \notin V$.

LINEAR COMBINATION OF VECTORS; $v_1, v_2, \dots, v_n$.

An expression of the type

$$\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n, \quad \text{where } \lambda_i \in \mathbb{R},$$

is called L.C. of vectors $v_1, v_2, \dots v_n$.

REMARK: Let V be a vector space (over $\mathbb{R}$). A vector v in V is a linear comb. of vectors $v_1, v_2, \dots, v_n$ in V if there exist scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ in $\mathbb{R}$ such that

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n.$$

Example: Let $\alpha = (4,3,5)$, $\beta = (0,1,3)$, $\gamma = (2,1,1)$, $\delta = (4,2,2)$

Examine if
i) α is a linear combination of β and γ.
ii) β is a linear combination of γ and δ.
iii) γ is a linear combination of α and β.

i) $\begin{pmatrix}4\\3\\5\end{pmatrix} = \lambda_1 \begin{pmatrix}0\\1\\3\end{pmatrix} + \lambda_2 \begin{pmatrix}2\\1\\1\end{pmatrix} \Rightarrow$

$4 = 2\lambda_2 \Rightarrow \lambda_2 = 2$
$3 = \lambda_1 + \lambda_2 \Rightarrow \lambda_1 = 1$
$5 = 3\lambda_1 + \lambda_2 \leftarrow$ satisfied for $\lambda_1 = 1$ & $\lambda_2 = 2$.

Hence $\alpha = \beta + 2\gamma$.

ii) $\begin{pmatrix}0\\1\\3\end{pmatrix} = \lambda_1 \begin{pmatrix}2\\1\\1\end{pmatrix} + \lambda_2 \begin{pmatrix}4\\2\\2\end{pmatrix} \Rightarrow$

$0 = 2\lambda_1 + 4\lambda_2$
$\left.\begin{matrix} 1 = \lambda_1 + 2\lambda_2 \\ 3 = \lambda_1 + 2\lambda_2 \end{matrix}\right\}$ INCONSISTENT!

$\Rightarrow \beta$ is not a linear combination of γ & δ.

iii) Clearly from (i), γ is a linear combination of α and β.

SPAN OF $v_1, v_2, \ldots v_n$ OR LINEAR SPAN of $v_1, v_2, \ldots, v_n$.

The collection of all linear comb. of $v_1, v_2, \ldots v_n$ is called the linear span of $v_1, v_2, \ldots, v_n$.

It is denoted by $\mathrm{SPAN}(v_1, v_2, \ldots, v_n)$

i.e.,

$$\mathrm{SPAN}(v_1, v_2, \ldots, v_n) = \left\{ \sum_{i=1}^{n} \lambda_i v_i \;\middle|\; \lambda_i \in \mathbb{R} \right\}$$

THEOREM: Let S be a subset of a vector space V. Then $\mathrm{SPAN}(S)$ is a subspace of V and this is the smallest subspace containing the set S.

Idea of proof: Let $S = \{u_1, u_2, \ldots u_n\}$

Suppose u & $v \in \mathrm{SPAN}(S)$ then

$$u = \sum_{i=1}^{n} \lambda_i u_i \quad \text{and} \quad v = \sum_{i=0}^{n} \mu_i u_i$$

then $u + v = \sum_{i=1}^{n} \underbrace{(\lambda_i + \mu_i)}_{\in \mathbb{R}} u_i \Rightarrow u + v \in \mathrm{SPAN}(S)$

Also, $cu = \sum_{i=1}^{n} \underbrace{(c\lambda_i)}_{\in \mathbb{R}} u_i \Rightarrow cu \in \mathrm{SPAN}(S)$