Lecture 14

Theorem (Jacobi method to find a diagonal form of a real quadratic form):

Let $Q(\underline{x}) = \underline{x}^t A \underline{x}$, be a real quadratic form on $\mathbb{R}^n$ of rank r, where A is symmetric. Then there exists a non-singular transformation $\boxed{\underline{x} = P\underline{y}}$ with $\det(P) = 1$ & P is an upper Δ^{ler} matrix, such that

$$Q(\underline{x}) = c_1 y_1^2 + \cdots + c_r y_r^2$$

$$\Updownarrow$$

Δ_j's are non-~~zeros~~ $\forall j = 1, 2, \ldots, r$

i.e. $\Delta_j \neq 0 \quad \forall j = 1, 2, \ldots, r$

where Δ_j's are the leading principal minors of A of order j. $A = [a_{ij}]_{n \times n}$

i.e. $\underline{\Delta_0 = 1}$, $\Delta_1 = a_{11}$, $\Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$,

$$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \ldots, \Delta_n = |A|.$$

& $\underline{c_j = \Delta_j / \Delta_{j-1}}$, $\forall j = 1, 2, \dots, r$.

Examples ① $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 4 \\ -1 & 4 & -1 \end{bmatrix}$

$Q(\underline{x}) = \underline{x}^t A \underline{x}$.

Find a diagonal form of Q by using Jacobi method.

Sol^n

$r = \text{rank}(A) = 3$

$\Delta_0 = 1$

$\Delta_1 = 1 \neq 0$

$\Delta_2 = \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = -4 \neq 0$

$\Delta_3 = |A| = 1(-16) - 2(-2+4) - (8)$

$= -16 - 4 - 8$

$= -28 \neq 0$.

$\therefore \quad Q(\underline{x}) = c_1 y_1^2 + c_2 y_2^2 + c_3 y_3^2$

where $c_1 = \dfrac{\Delta_1}{\Delta_0} = \dfrac{1}{1} = 1$

$c_2 = \dfrac{\Delta_2}{\Delta_1} = \dfrac{-4}{1} = -4$

$c_3 = \dfrac{\Delta_3}{\Delta_2} = \dfrac{-28}{-4} = 7$

$\therefore \quad Q(x) = y_1^2 - 4y_2^2 + 7y_3^2.$

Indefinite