Lecture 3

Recall:- Maximally l.i set S means if we add a vector $v \in V$ & $v \notin S$ to S i.e. $S \cup \{v\}$ is not l.i.

Theorem:- Let A be an $m \times n$ matrix. Then rowrank(A) = Columnrank(A) = rank(A).

proof:- We have

$$\text{rowspace}(A) = R(A^t)$$

$$\Rightarrow \text{rowrank}(A) = \dim(R(A^t)) = \text{rank}(A^t).$$

$$\text{Columnspace}(A) = R(A)$$

$$\Rightarrow \text{Columnrank}(A) = \dim(R(A)) = \text{rank}(A) \quad \text{(EXERCISE)}$$

But we proved that rank(A) = rank(A^t).

$$\therefore \boxed{\text{rowrk}(A) = \text{Columnrk}(A) = \text{rk}(A).}$$

Example:-

(1) Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -1 \end{bmatrix}_{2\times 3}$.

$$\text{rowspace}(A) = \text{Span}\Big(\{\underbrace{(1,-1,2), (0,3,-1)}_{l.i}\}\Big) \subseteq \mathbb{R}^3.$$

$$\Rightarrow \text{rowrk}(A) = 2.$$

$$\text{Columnspace}(A) = \text{span}\Big(\{\underbrace{(1,0), (-1,3), (2,-1)}\}\Big) \subseteq \mathbb{R}^2$$

$$(2,-1) = \frac{5}{3}(1,0) \cancel{+} -\frac{1}{3}(-1,3)$$

$\therefore$ columnsp$(A) = $ span$\left(\{ \underbrace{(1,0), (-1,3)}_{l.i} \} \right)$

$\therefore$ columnrk$(A) = 2$.

rank$(A) = 2$.

<u>Theorem</u>:- Let $A_{m \times n}$, $B_{m \times n}$ be matrices.

Then rank$(A+B) \leq$ rank(A) + rank(B).

<u>proof</u>:-

Let rank$(A) = r$, rank$(B) = p$.

Let $\{\underline{x}_1, \ldots, \underline{x}_r\}$ be the maximal l.i columns of A. in $\mathbb{R}^m$

& let $\{\underline{y}_1, \ldots, \underline{y}_p\}$ be the maximal l.i columns of B. in $\mathbb{R}^m$.

Infact they are bases of columnspaces of A & B respectively.

Let $\underline{z}_1, \ldots, \underline{z}_n$ be the columns of A+B. in $\mathbb{R}^m$.

$\Rightarrow$ Columnsp$(A+B) = $ span$\left(\{ \underline{z}_1, \ldots, \underline{z}_n \} \right)$.

But each $\underline{z}_i$ = i^{th} Column of A+B.

= (i^{th} column of A) + (i^{th} column of B)

= (a l.c of $\underline{x}_1, \ldots, \underline{x}_r$)

$$+\left(\text{a l.c of } \underline{y}_1, \dots, \underline{y}_p\right)$$

$$\in \operatorname{Span}\left(\{\underline{x}_1, \dots \underline{x}_r\}\right) + \operatorname{span}\left(\{\underline{y}_1, \dots \underline{y}_p\}\right)$$

$\forall\ i = 1, 2, \dots, n$

$$\Rightarrow \quad \operatorname{Span}\left(\{\underline{z}_1, \dots, \underline{z}_n\}\right) \subseteq \operatorname{Span}\left(\{\underline{x}_1, \dots \underline{x}_r\}\right) + \operatorname{span}\left(\{\underline{y}_1, \dots, \underline{y}_p\}\right)$$

$$\Rightarrow \quad \operatorname{columnsp}(A+B) \subseteq \operatorname{columnsp}(A) + \operatorname{Columnsp}(B)$$

$$\Rightarrow \quad \dim\left(\operatorname{columnsp}(A+B)\right) \leq \dim\left(\operatorname{Columnsp}(A) + \operatorname{Columnsp}(B)\right)$$

$$\leq \dim\left(\operatorname{columnsp}(A)\right) + \dim\left(\operatorname{columnsp}(B)\right).$$

(EXERICSE)

$$\Rightarrow \quad \operatorname{columnrk}(A+B) \leq \operatorname{columnrk}(A) + \operatorname{Columnrk}(B)$$

$$\Rightarrow \quad \operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B).$$

(EXERCISE: Suppose $U, V \subseteq W$ subspaces of a vector sp W over F. Then

$$\dim(U+V) \leq \dim(U) + \dim(V).\ )$$

$$U+V = \{\underline{u}+\underline{v} \mid \underline{u} \in U,\ \underline{v} \in V\} \subseteq W.$$

$$U \cap V = \{\underline{v} \in W \mid \underline{v} \in U \ \&\ \underline{v} \in V\} \subseteq W \text{ subsp.}$$

<u>Theorem</u>:- Let $A_{m\times n}$, $B_{n \times p}$ be matrices. Then

$$\dim\left(R(B) \cap N(A)\right) = \operatorname{rank}(B) - \operatorname{rank}(AB)$$

$\dim\left(R(B)\cap N(A)\right) = \text{rank}(B) - \text{rank}(AB)$.

$$\left(\begin{array}{ll} A: \mathbb{R}^n \to \mathbb{R}^m, & \underline{v} \mapsto A\underline{v}, \\ B: \mathbb{R}^p \to \mathbb{R}^n, & \underline{w} \mapsto B\underline{w} \\ R(B) \subseteq \mathbb{R}^n, & N(A) \subseteq \mathbb{R}^n. \end{array}\right) \left| \begin{array}{c} (AB)_{m\times p} \\ AB: \mathbb{R}^p \to \mathbb{R}^m \\ \underline{u} \mapsto AB\underline{u} \end{array} \right|$$

proof:- $\dim(N(B)) = p - \text{rank}(B) = q$ (by rank-nullity Thm)

& $\dim(N(AB)) = p - \text{rank}(AB) = k$ [say] "

Let $\{\underline{x_1}, \dots, \underline{x_q}\}$ be a basis of $N(B) \subseteq \mathbb{R}^p$.

i.e. $\underline{x_i} \in N(B)$ $\forall i = 1, 2, \dots, q$.

$\Rightarrow B\underline{x_i} = \underline{0}$ $\forall i$

$\Rightarrow AB\underline{x_i} = A\underline{0} = \underline{0}$

$\Rightarrow \underline{x_i} \in N(AB)$ $\forall i$.

Thus $N(B) \subseteq N(AB)$

& $\{\underline{x_1}, \dots, \underline{x_q}\}$ is a l.i set in $N(AB)$.

Expand this l.i set to a basis of $N(AB)$.

Say $\{\underline{x_1}, \dots, \underline{x_q}, \underline{x_{q+1}}, \dots, \underline{x_k}\}$ is a basis of $N(AB)$.

claim:- $\{B\underline{x_{q+1}}, \dots, B\underline{x_k}\}$ is a basis of $R(B)\cap N(A)$.

$(\Rightarrow \dim(R(B)\cap N(A)) = k - q$.

$= (p - rk(AB)) - (p - rk(B))$

$= rk(B) - rk(AB)$ $)$

what we want.)

<u>claim 1</u>:- $\{B\underline{x}_{q+1}, \ldots, B\underline{x}_k\}$ is l i

<u>Pf of claim 1</u>:-

Suppose $\alpha_1 B\underline{x}_{q+1} + \alpha_2 B\underline{x}_{q+2} + \cdots + \alpha_{k-q} B\underline{x}_k = \underline{0}$

for some scalars $\alpha_1, \ldots, \alpha_{k-q} \in F$.

$$\Rightarrow \quad B\left(\alpha_1 \underline{x}_{q+1} + \cdots + \alpha_{k-q}\underline{x}_k\right) = \underline{0}$$

$$\Rightarrow \quad \alpha_1 \underline{x}_{q+1} + \cdots + \alpha_{k-q}\underline{x}_k \in N(B).$$

$$\Rightarrow \quad \alpha_1 \underline{x}_{q+1} + \cdots + \alpha_{k-q}\underline{x}_k = \beta_1 \underline{x}_1 + \cdots + \beta_q \underline{x}_q$$

for some $\beta_1, \ldots, \beta_q \in F$.

$$\Rightarrow \quad \beta_1 \underline{x}_1 + \cdots + \beta_q \underline{x}_q - \alpha_1 \underline{x}_{q+1} - \cdots - \alpha_{k-q}\underline{x}_k = \underline{0}$$

But $\{\underline{x}_1, \ldots, \underline{x}_q, \ldots \underline{x}_k\}$ is l i

$$\Rightarrow \quad \beta_1 = \cdots = \beta_q = \alpha_1 = \cdots = \alpha_{k-q} = 0.$$

Then $\{B\underline{x}_{q+1}, \ldots, B\underline{x}_k\}$ is l.i.

<u>claim 2</u>:- $R(B) \cap N(A) = \text{span}\left(\{B\underline{x}_{q+1}, \ldots, B\underline{x}_k\}\right)$

i.e. $\{B\underline{x}_{q+1}, \ldots, B\underline{x}_k\}$ spans $R(B) \cap N(A)$.