Lecture 1

Field : $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$

reals — complex — rational numbers.

$F = \mathbb{R}$ or $\mathbb{C}$.

<u>Def</u>:- A vector space V over a field F is a set together with a binary operation $+$, a scalar multiplication $\cdot$, satisfying the following conditions:

(i) $\underline{v} + \underline{w} \in V$, $\forall\, \underline{v}, \underline{w} \in V$.

(ii) $(\underline{v} + \underline{w}) + \underline{u} = \underline{v} + (\underline{w} + \underline{u})$ $\forall\, \underline{u}, \underline{v}, \underline{w} \in V$

(iii) There exists a vector "zero vector" $\underline{0} \in V$ such that $\underline{v} + \underline{0} = \underline{0} + \underline{v} = \underline{v}$ $\forall\, \underline{v} \in V$.

(iv) For each $\underline{v} \in V$, there exists a vector $-\underline{v} \in V$ such that
$$\underline{v} + (-\underline{v}) = (-\underline{v}) + \underline{v} = \underline{0}$$

(v) $\underline{v} + \underline{w} = \underline{w} + \underline{v}$ $\forall\, \underline{v}, \underline{w} \in V$.

(vi) For $\underline{v} \in V$, $\lambda \in F$, $\lambda \cdot \underline{v} \in V$.

(vii) $1 \cdot \underline{v} = \underline{v}$, $\forall\, \underline{v} \in V$.

(viii) $(ab) \cdot \underline{v} = a \cdot (b \cdot \underline{v})$ $\forall\, a, b \in F$ $\forall\, \underline{v} \in V$.

(ix) $a \cdot (\underline{v} + \underline{w}) = a \cdot \underline{v} + a \cdot \underline{w}$, $\forall\, a \in F$

(x) $(a+b)\cdot \underline{v} = a\cdot\underline{v} + b\cdot\underline{v}$ $\forall \underline{v}, \underline{w} \in V$.

$\forall\, a, b \in F$

$\forall\, \underline{v} \in V$.

Elements of V are called vectors & elements of F are called scalars.

<u>Examples:-</u> ① $V = \mathbb{R}^n$ is a vector sp/$\mathbb{R}$.

② Let $V = M_{m\times n}(\mathbb{R})$

= the set of all $m\times n$ real matrices.

$= \left\{ A = [a_{ij}]_{m\times n} \;\middle|\; a_{ij}\text{'s} \in \mathbb{R} \right\}$.

③ $\mathbb{R}[x]$ = the set of all polynomials in the variable x with coefficients in $\mathbb{R}$.

<u>Def:-</u> Let V be a vector space over a field F. Let $W \subseteq V$ be a non-empty subset. We say W is a subspace of V, if W itself a vector space with respect to addition & scalar multiplication in V.

i.e.) It is enough to satisfy the following condition: (i) if $\underline{w_1}, \underline{w_2} \in W$, then

$$\underline{w_1} - \underline{w_2} \in W$$

(ii) For $\lambda \in F$, $\underline{w} \in W$, $\lambda . \underline{w} \in W$.

Examples:-

① $V = \mathbb{R}^2$. Let $W = \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} \middle| \, a \in \mathbb{R} \right\}$.

check that W is a subspace of V.

$$\begin{pmatrix} a \\ 0 \end{pmatrix} - \begin{pmatrix} b \\ 0 \end{pmatrix} = \begin{pmatrix} a-b \\ 0 \end{pmatrix} \in W.$$

&

$$\lambda \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda a \\ 0 \end{pmatrix} \in W, \forall \lambda \in \mathbb{R}$$

② $V = M_{2\times 2}(\mathbb{R})$.

Let $W = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}$.

W is a subspace of V.

Direct sum of two subspaces:

Let V be a vector space over F.

Let $U, W \subseteq V$ be subspaces of V.

Then the direct sum of U & W is defined as

$U \oplus W = \{ v \in V |$

$U \oplus W := \{ \underline{v} \in V \mid \underline{v} = \underline{u} + \underline{w}$ for some unique $\underline{u} \in U,\ \underline{w} \in W \}$

i.e., if $\underline{u} + \underline{w} = \underline{u}' + \underline{w}'$ for some $\underline{u}' \in U,\ \underline{w}' \in W$,

then $\underline{u} = \underline{u}'$ & $\underline{w} = \underline{w}'$.

Examples:-

① $V = \mathbb{R}^2$, Let $U = \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} \middle| a \in \mathbb{R} \right\}$

& $W = \left\{ \begin{pmatrix} 0 \\ b \end{pmatrix} \middle| b \in \mathbb{R} \right\}$

$$U \oplus W = \left\{ \underbrace{\begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix}}_{\text{unique}} \middle| a, b \in \mathbb{R} \right\}$$

Say $\begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix} = \begin{pmatrix} a' \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b' \end{pmatrix}$

$\Rightarrow$ $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a' \\ b' \end{pmatrix} \Rightarrow a = a'$ & $b = b'$

$$\therefore U \oplus W = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \middle| a, b \in \mathbb{R} \right\} = \mathbb{R}^2.$$

Result:- Let $U, W \subseteq V$ be two subspaces of V. Then, of V. then

$$U+W = U \oplus W$$ if and only if $U \cap W = \{0\}$.

EXERCISE: Sum of two subspaces is a subspace.

$$U+W = \{ \underline{u}+\underline{w} \,/\, \underline{u} \in U, \underline{w} \in W \}$$

need not be unique.

<u>Example</u>:- $V = \{\underline{0}\}$ is a vector space called "zero space".

<u>Defintion</u>:- Let V be a vector space over F. Let $S \subseteq V$ be any non-empty subset of V. Then S is said to be <u>linearly independent</u> (l.i) if

$$\lambda_1 \underline{v}_1 + \cdots + \lambda_r \underline{v}_r = \underline{0}$$

for some $\lambda_1, \ldots, \lambda_r \in F$, $\underline{v}_1, \ldots, \underline{v}_r \in S$

then $\lambda_1 = \lambda_2 = \cdots = \lambda_r = 0$.

<u>Examples</u>:- ① $S = \{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \}$ l.i in $\mathbb{R}^2$

② $S = \{1, x, x^2, \ldots, x^n, \ldots\}$ is l.i in $\mathbb{R}[x]$.

Def:- Let $S \subseteq V$ be a l.i ...

Then the linear span or span of S is defined as

$$\operatorname{span}(S) = L(S) = LS(S)$$
$$:= \left\{ \lambda_1 \underline{v}_1 + \cdots + \lambda_r \underline{v}_r \;\middle|\; \begin{array}{l} \lambda_i \in F \\ \underline{v}_i \in S \end{array} \right\}.$$

= the set of all linear combinations of elements of S.

Examples:- ① $\operatorname{span}\left(\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}\right) = \mathbb{R}^3$.

② $\operatorname{span}(\{1, x, x^2, \ldots\}) = \mathbb{R}[x]$.

Note: not l.i = linearly dependent (l.d).

Results: Let V be vector sp./F. Let $S \subseteq T \subseteq V$ be subsets. Then

① if T is l.i, then S is l.i.

② if S is l.d, then T is l.d.

*③ if S is l.i & $v \in V$ th...

$S \cup \{\underline{v}\}$ is l.i $\iff$ $\underline{v} \notin \mathrm{Span}(S)$.

__Def:-__ Let V be a vector sp./F. Let $S \subseteq V$ be a subset. Then S is said to be a __basis__ of V if

(i) S l.i

(ii) $\mathrm{span}(S) = V$.

i.e., S spans the vector space V.

__Remark:-__ Every vector space has a basis & the basis may not be unique.

__Examples:-__ ① $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is a basis of $\mathbb{R}^2$.

Also $T = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}$ is also a basis of $\mathbb{R}^2$.

T is l.i (EXERCISE)

$\mathrm{span}(T) = \mathbb{R}^2$.

Let $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2\lambda \\ \lambda - \mu \end{pmatrix}$$

$\Rightarrow$

$$\Rightarrow 2\lambda = a \quad \& \quad b = \lambda - \mu$$

$$\Rightarrow \boxed{\lambda = \frac{a}{2}} \quad \& \quad \mu = \lambda - b$$

$$\boxed{\mu = \frac{a}{2} - b}$$

$$\therefore \begin{pmatrix} a \\ b \end{pmatrix} = \frac{a}{2} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \left(\frac{a}{2} - b\right) \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$= a$ l.c of T.

$$\therefore \quad \mathbb{R}^2 \subseteq \mathrm{Span}(T) \subseteq \mathbb{R}^2$$

$$\therefore \quad \mathbb{R}^2 = \mathrm{Span}(T).$$