# SOME USEFUL THEOREMS:

Let $V$ be a vector space of finite dimension $n$.

Then,

i) Any $(n+1)$ or more vectors in $V$ are linearly dependent.

ii) Any linearly independent set $S=\{u_1,u_2,\dots,u_n\}$ with $n$ elements is a basis of $V$

iii) Any spanning set $\{v_1,v_2,\dots,v_n\}$ of $V$ with $n$ elements is a basis of $V$.

(iv) If a vector space has a finite basis, then all of its basis have the same number of elements.

(V) The number of elements in any basis of a vector space is called the dimension of that space.

**Theorem:** Suppose $S$ spans a vector space $V$. Then

i) Maximum number of linearly independent vectors in $S$ form a basis of $V$.

ii) Suppose one deletes from $S$ every vector that is a linear combination of preceding vectors in $S$. Then the remaining vectors form a basis of $V$.

**Theorem:** Let $V$ be a vector space of finite dimension and let $S=\{u_1, u_2, \dots, u_r\}$ be a set of linearly independent vectors in $V$. Then $S$ is a part of a basis of $V$, that is, $S$ may be extended to a basis of $V$.

Ex. 1: The following set in $\mathbb{R}^4$

$(1,1,1,1)^T, (0,1,1,1)^T, (0,0,1,1)^T, (0,0,0,1)^T$ forms a basis of $\mathbb{R}^4$. ($\dim(\mathbb{R}^4)=4$ & these vectors are linearly independent)

Ex. 2: Consider any four vectors in $\mathbb{R}^3$, say

$(1,2,1)^T, \ (0,1,3)^T, (3,4,0)^T, (6,1,8)^T$.

These vectors must be linearly dependent since $\dim(\mathbb{R}^3)=3$.

# MORE ON MATRICES:

Revisit: $Ax = 0$, where

$$A = \begin{bmatrix} 1 & 2 & -2 & -1 & 1 \\ 2 & 4 & -4 & 0 & 3 \\ -1 & -2 & 3 & 3 & 4 \\ 3 & 6 & -7 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{array}{c} \begin{array}{ccccc} x_1 & \phantom{0} & x_3 & x_4 & \phantom{0} \end{array} \\ \begin{bmatrix} \boxed{1} & 2 & -2 & -1 & 1 \\ 0 & 0 & \boxed{1} & 2 & 5 \\ 0 & 0 & 0 & \boxed{2} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \phantom{0} & x_2 & \phantom{0} & \phantom{0} & x_5 \end{array} \end{array} = \tilde{A}$$

Solution: $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \alpha_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -9.5 \\ 0 \\ -4 \\ -0.5 \\ 1 \end{bmatrix}$$

- Vectors that generates solutions of $Ax = 0$: $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} -9.5 \\ 0 \\ -4 \\ -0.5 \\ 1 \end{bmatrix}$
- Also note that these vectors are linearly independent.
- Therefore these vectors form basis of NULL SPACE of $A$.
- Dimension of NULL space of $A = 2$ (Called NULLITY)
- Number of pivots (usually denoted by $r$) is called the **rank** of the Matrix

- Clearly,

$$C_2 = 2C_1 \quad \& \quad C_5 = \frac{1}{2}C_4 + 4C_3 + 95C_1$$

where $C_i$'s are the columns of $A$ OR $\tilde{A}$.

$\Rightarrow$ $C_2$ & $C_5$ are <u>linearly</u> dependent in the set of Column vectors and <u>they</u> can be produced by linear combination of other column vectors.

Also note that $C_1, C_3, C_4$ are <u>linearly</u> independent.

- No of pivots = RANK = No. of <u>linearly</u> independent Columns

= No. of <u>linearly</u> independent rows.

$$= \dim(\underset{\substack{\uparrow\\ \text{Column space}\\ \text{(SPAN of column vectors of A)}}}{C(A)}) = \dim(\underset{\substack{\uparrow\\ \text{Row space.}\\ \text{(SPAN of row vectors of A)}}}{R(A)})$$

- <u>Nullity</u> = dim of Null space = No. of free variables

$= n - r$

- Rank $= r$ (no. of pivots)

- RANK-NULLITY THEOREM: $\boxed{\text{RANK}(A) + \text{Nullity}(A) = n}$

- The rank of a $m \times n$ Matrix cannot be greater than $n$ or $m$ i.e., $\boxed{\text{Rank}(A) \leq \min(m, n)}$

Example: Find the rank of

$$A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix}$$

$$\sim \begin{bmatrix} \boxed{3} & 0 & 2 & 2 \\ 0 & \boxed{42} & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{RANK}(A) = 2.$$

Example: Find the rank of the Matrix

$$A = \begin{bmatrix} 1 & 2 & -2 & -1 & 1 \\ 2 & 4 & -4 & 0 & 3 \\ -1 & -2 & 3 & 3 & 4 \\ 3 & 6 & -7 & 1 & 1 \end{bmatrix}$$

RANK(A) = 3 (No. of pivots)

Example: Find the rank of the Matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 6 & -3 & -3 \\ 3 & 10 & -6 & -5 \end{bmatrix}$$

Echelon form: $\sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ RANK = 2

## Rank in terms of determinant:

**Submatrix:** Suppose $A$ is any matrix of order $m \times n$ then a matrix obtained by leaving some rows and columns from $A$ is called a submatrix of $A$.

**Rank:** An $m \times n$ matrix $A$ has rank $r \geq 1$ iff $A$ has $r \times r$ submatrix with nonzero determinant, whereas the determinant of every square submatrix with $(r+1)$ or more rows is zero.

In particular, if $A$ is a square $n \times n$ matrix, it has rank $n$ iff $\det A \neq 0$.

Example: $A = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 2 & 4 \\ 3 & 1 & 2 \end{bmatrix}$

Clearly $|A| = 0$ (first two columns are identical)

Also, every $2 \times 2$ submatrices has $0$ determinant.

Therefore, the rank is $1$.

**Remark:** Rank of a zero matrix is zero.

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

Note that $|A| = 0 \Rightarrow \text{Rank}(A) < 3$

Also, $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \neq 0 \Rightarrow \text{Rank}(A) = 2$.

Example:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that

$$|A| = 1 \neq 0 \Rightarrow \text{Rank}(A) = 3$$

Question: Rank of a matrix of order $m \times n$ whose every element is unity.

ANS: 1.

PROPERTIES: Let $A, B$ be $m \times n$ matrices.

i) $A$ is nonsingular ($|A| \neq 0$) if $\text{rank}(A) = n$ and singular ($|A| = 0$) if $\text{rank}(A) < n$.

ii) If $B \sim A$ then $\text{rank}(B) = \text{rank}(A)$ $\left(A, B \in \mathbb{R}^{m \times n}\right)$

iii) If $\text{rank}(A) = n$ and $AB = AC$ then $B = C$

$\left(\text{If } A^{-1} \text{ exists then } A^{-1}AB = A^{-1}AC \Rightarrow B = C\right)$

(iv) If $\text{rank}(A) = n$ then $AB = 0 \Rightarrow B = 0$

Hence, if

$AB = 0$ but $A \neq 0$, $B \neq 0$ then

$\rho(A) < n$ and $\rho(B) < n$.

# CONSISTENCY OF SYSTEM OF LINEAR EQUATIONS USING RANK CONCEPT

Consider:

$$[A|b] = \left[\begin{array}{cccc|c} 1 & 2 & -1 & -1 & 0 \\ 2 & 5 & 1 & 1 & 2 \\ 3 & 7 & 2 & 2 & \beta \\ -1 & 0 & 1 & \alpha & 16 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -1 & -1 & 0 \\ 0 & 1 & 3 & 3 & 2 \\ 0 & 0 & 2 & 2 & \beta-2 \\ 0 & 0 & 0 & \alpha-1 & 3\beta+6 \end{array}\right]$$

Case I: UNIQUE SOLUTION $\Leftrightarrow \alpha \neq 1$

What does it mean in terms of Rank(A) & Rank(A|b)

$\Rightarrow$

$$\boxed{\text{Rank}(A) = \text{Rank}(A|b) = \text{Number of unknowns}}$$

Case II: NO SOLUTION $\Leftrightarrow \alpha = 1$ & $\beta \neq -2$

$$\boxed{\text{RANK}(A) \neq \text{RANK}(A|b)}$$

Case III: INFINITELY MANY SOLUTIONS $\Leftrightarrow \alpha = 1$ & $\beta = -2$

$$\boxed{\text{Rank}(A) = \text{Rank}(A|b) < \text{No. of unknowns}}$$

<u>Remark</u>: No. of free variable = No. of unknowns − rank(A)

$= n - r$

# FUNDAMENTAL THEOREM OF CONSISTENCY OF SYSTEM OF L.Es.

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ & $[A|b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$

The system of equations $Ax=b$ is

i) INCONSISTENT, i.e, there is no solution if $\text{rank}(A) \neq \text{rank}([A|b])$

ii) CONSISTENT WITH A UNIQUE SOLUTION if

$$\text{Rank}(A) = \text{Rank}([A|b]) = \text{No. of unknowns } (n)$$

(iii) CONSISTENT WITH INFINITELY MANY SOLUTIONS if,

$$\text{Rank}(A) = \text{Rank}([A|b]) < n$$

(giving arbitrary values to $(n-r)$ of unknowns we can express the other unknowns in terms of these.

FOR THE SYSTEM OF LINEAR HOMOGENEOUS EQUATIONS $Ax=0$

i) If $\text{rank}(A) = n$ (number of unknowns), the system will have only a trivial solution $x_1 = x_2 = \cdots = x_n = 0$. (UNIQUE SOL.)

ii) If $\text{rank}(A) < n$, the equations have $(n-r)$ linearly independent solutions. (INFINITY MANY SOL.)

NOTE:

(i) A homogeneous linear system is always consistent (it has atleast trivial solution)

(ii) A homogeneous linear system with fewer equations than unknowns always has nontrivial solutions.

## Solution of the system of equations $Ax = b$

$$\boxed{Ax = b}$$

$$\boxed{\rho(A) \neq \rho([A \mid b])}$$

Inconsistent
(no solution)

$$\boxed{\rho(A) = \rho([A \mid b])}$$ consistent

$$\boxed{\rho(A) = \rho([A \mid b]) < \text{no.of unknowns}}$$

Infinite solutions

$$\boxed{\rho(A) = \rho([A \mid b]) = \text{no.of unknowns}}$$

Unique solution

Miscellaneous Problems on Linear Algebra (vector space)

**Problem 1:**

Determine whether or not each of the following form a basis of $\mathbf{R}^3$:

(a) $(1,1,1), (1,0,1)$;

(b) $(1,2,3), (1,3,5), (1,0,1), (2,3,0)$;

(c) $(1,1,1), (1,2,3), (2,-1,1)$;

(d) $(1,1,2), (1,2,5), (5,3,4)$.

(a and b) No, since a basis of $\mathbf{R}^3$ must contain exactly 3 elements because $\dim \mathbf{R}^3 = 3$.

(c) The three vectors form a basis if and only if they are linearly independent. Thus form the matrix whose rows are the given vectors, and row reduce the matrix to echelon form:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

The echelon matrix has no zero rows; hence the three vectors are linearly independent, and so they do form a basis of $\mathbf{R}^3$.

(d) Form the matrix whose rows are the given vectors, and row reduce the matrix to echelon form:

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 5 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

The echelon matrix has a zero row; hence the three vectors are linearly dependent, and so they do not form a basis of $\mathbf{R}^3$.

**Problem 2:**

Determine whether $(1,1,1,1)$, $(1,2,3,2)$, $(2,5,6,4)$, $(2,6,8,5)$ form a basis of $\mathbf{R}^4$. If not, find the dimension of the subspace they span.

Form the matrix whose rows are the given vectors, and row reduce to echelon form:

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 5 & 6 & 4 \\ 2 & 6 & 8 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 4 & 2 \\ 0 & 4 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The echelon matrix has a zero row. Hence the four vectors are linearly dependent and do not form a basis of $\mathbf{R}^4$. Since the echelon matrix has three nonzero rows, the four vectors span a subspace of dimension 3.

**Problem 3:**

Extend $\{u_1 = (1,1,1,1), u_2 = (2,2,3,4)\}$ to a basis of $\mathbf{R}^4$.

First form the matrix with rows $u_1$ and $u_2$, and reduce to echelon form:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Then $w_1 = (1,1,1,1)$ and $w_2 = (0,0,1,2)$ span the same set of vectors as spanned by $u_1$ and $u_2$. Let $u_3 = (0,1,0,0)$ and $u_4 = (0,0,0,1)$. Then $w_1$, $u_3$, $w_2$, $u_4$ form a matrix in echelon form. Thus they are linearly independent, and they form a basis of $\mathbf{R}^4$. Hence $u_1$, $u_2$, $u_3$, $u_4$ also form a basis of $\mathbf{R}^4$.

Miscellaneous Problems on Linear Algebra (vector space)

**Problem 4:**

Find the dimension and a basis of the solution space $W$ of each homogeneous system:

$$\begin{aligned} x+2y+2z-s+3t&=0\\ x+2y+3z+s+\ \ t&=0,\\ 3x+6y+8z+s+5t&=0 \end{aligned} \qquad \begin{aligned} x+2y+\ \ z-2t&=0\\ 2x+4y+4z-3t&=0\\ 3x+6y+7z-4t&=0 \end{aligned} \qquad \begin{aligned} x+\ \ y+2z&=0\\ 2x+3y+3z&=0\\ x+3y+5z&=0 \end{aligned}$$

$(a)$ $\qquad$ $(b)$ $\qquad$ $(c)$

Reduce the system to echelon form:

$$\begin{aligned} x+2y+2z-\ \ s+3t&=0\\ z+2s-2t&=0\\ 2z+4s-4t&=0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x+2y+2z-\ \ s+3t&=0\\ z+2s-2t&=0 \end{aligned}$$

The system in echelon form has two (nonzero) equations in five unknowns. Hence the system has $5-2=3$ free variables, which are $y, s, t$. Thus $\dim W=3$. We obtain a basis for $W$:

(1) Set $y=1, s=0, t=0$ to obtain the solution $v_1=(-2,1,0,0,0)$.
(2) Set $y=0, s=1, t=0$ to obtain the solution $v_2=(5,0,-2,1,0)$.
(3) Set $y=0, s=0, t=1$ to obtain the solution $v_3=(-7,0,2,0,1)$.

The set $\{v_1, v_2, v_3\}$ is a basis of the solution space $W$.

(Here we use the matrix format of our homogeneous system.) Reduce the coefficient matrix $A$ to echelon form:

$$A=\begin{bmatrix} 1 & 2 & 1 & -2\\ 2 & 4 & 4 & -3\\ 3 & 6 & 7 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -2\\ 0 & 0 & 2 & 1\\ 0 & 0 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -2\\ 0 & 0 & 2 & 1\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This corresponds to the system

$$\begin{aligned} x+2y+2z-2t&=0\\ 2z+\ \ t&=0 \end{aligned}$$

The free variables are $y$ and $t$, and $\dim W=2$.

(i) Set $y=1, z=0$ to obtain the solution $u_1=(-2,1,0,0)$.
(ii) Set $y=0, z=2$ to obtain the solution $u_2=(6,0,-1,2)$.

Then $\{u_1, u_2\}$ is a basis of $W$.

Reduce the coefficient matrix $A$ to echelon form:

$$A=\begin{bmatrix} 1 & 1 & 2\\ 2 & 3 & 3\\ 1 & 3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2\\ 0 & 1 & -1\\ 0 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2\\ 0 & 1 & -1\\ 0 & 0 & 5 \end{bmatrix}$$

This corresponds to a triangular system with no free variables. Thus 0 is the only solution, that is, $W=\{0\}$. Hence $\dim W=0$.

Miscellaneous Problems on Linear Algebra (vector space)

**Problem 5:**

Find a basis and dimension of the subspace $W$ of $\mathbf{R}^3$ where:
(*a*) $W = \{(a, b, c) : a + b + c = 0\}$, (*b*) $W = \{(a, b, c) : (a = b = c)\}$

(*a*) Note that $W \neq \mathbf{R}^3$, since, e.g., $(1, 2, 3) \notin W$. Thus $\dim W < 3$. Note that $u_1 = (1, 0, -1)$ and $u_2 = (0, 1, -1)$ are two independent vectors in $W$. Thus $\dim W = 2$, and so $u_1$ and $u_2$ form a basis of $W$.

(*b*) The vector $u = (1, 1, 1) \in W$. Any vector $w \in W$ has the form $w = (k, k, k)$. Hence $w = ku$. Thus $u$ spans $W$ and $\dim W = 1$.

**Problem 6:**

Let $W$ be the subspace of $\mathbf{R}^4$ spanned by the vectors

$$u_1 = (1, -2, 5, -3), \qquad u_2 = (2, 3, 1, -4), \qquad u_3 = (3, 8, -3, -5)$$

(*a*) Find a basis and dimension of $W$. (*b*) Extend the basis of $W$ to a basis of $\mathbf{R}^4$.

(*a*) Apply Algorithm 4.1, the row space algorithm. Form the matrix whose rows are the given vectors, and reduce it to echelon form:

$$A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The nonzero rows $(1, -2, 5, -3)$ and $(0, 7, -9, 2)$ of the echelon matrix form a basis of the row space of $A$ and hence of $W$. Thus, in particular, $\dim W = 2$.

(*b*) We seek four linearly independent vectors, which include the above two vectors. The four vectors $(1, -2, 5, -3)$, $(0, 7, -9, 2)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$ are linearly independent (since they form an echelon matrix), and so they form a basis of $\mathbf{R}^4$, which is an extension of the basis of $W$.

**Problem 7:**

Show that $U = W$, where $U$ and $W$ are the following subspaces of $\mathbf{R}^3$:

$$U = \operatorname{span}(u_1, u_2, u_3) = \operatorname{span}(1, 1, -1), \ (2, 3, -1), \ (3, 1, -5)\}$$
$$W = \operatorname{span}(w_1, w_2, w_3) = \operatorname{span}(1, -1, -3), \ (3, -2, -8), \ (2, 1, -3)\}$$

Form the matrix $A$ whose rows are the $u_i$, and row reduce $A$ to row canonical form:

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -1 \\ 3 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Next form the matrix $B$ whose rows are the $w_j$, and row reduce $B$ to row canonical form:

$$B = \begin{bmatrix} 1 & -1 & -3 \\ 3 & -2 & -8 \\ 2 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since $A$ and $B$ have the same row canonical form, the row spaces of $A$ and $B$ are equal, and so $U = W$.

Miscellaneous Problems on Linear Algebra (vector space)

**Problem 8:**

Find the rank and basis of the row space of each of the following matrices:

$$(a)\quad A=\begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 6 & -3 & -3 \\ 3 & 10 & -6 & -5 \end{bmatrix}, \qquad (b)\quad B=\begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}.$$

(*a*) Row reduce $A$ to echelon form:

$$A \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & -3 & -1 \\ 0 & 4 & -6 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The two nonzero rows $(1, 2, 0, -1)$ and $(0, 2, -3, -1)$ of the echelon form of $A$ form a basis for $\operatorname{rowsp}(A)$. In particular, $\operatorname{rank}(A) = 2$.

(*b*) Row reduce $B$ to echelon form:

$$B \sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & -3 & -6 & -3 & 3 \\ 0 & -1 & -2 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The two nonzero rows $(1, 3, 1, -2, -3)$ and $(0, 1, 2, 1, -1)$ of the echelon form of $B$ form a basis for $\operatorname{rowsp}(B)$. In particular, $\operatorname{rank}(B) = 2$.

Example:

Consider the following two sets of vectors in $\mathbb{R}^4$.

$u_1 = (1,2,-1,3) \quad u_2 = (2,4,1,-2) \quad u_3 = (3,6,3,-7)$

$w_1 = (1,2,-4,11) \quad w_2 = (2,4,-5,14)$

Let $U = \text{span}(u_i)$ and $W = \text{span}(w_i)$

Show that $U = W$.

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 6 & -16 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & -4 & 11 \\ 2 & 4 & -5 & 14 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 3 & -8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \end{bmatrix}$$

$\Rightarrow \text{span}(u_1, u_2, u_3) = \text{span}(w_1, w_2)$.