Lecture 17

theorem:– Let A be any $n \times n$ matrix. Then the induced 2-norm of A is

$$\|A\|_2 = \max_j \left(\lambda_j(A^*A)\right)^{1/2}$$

where $\lambda_j(A^*A)$ are the eigenvalues of A^*A.

proof:–

$$\|A\|_2 = \max_{\substack{\underline{x} \in \mathbb{C}^n \\ \text{such that} \\ \|\underline{x}\|_2 = 1}} \left(\|A\underline{x}\|_2\right)$$

$$= \max_{\|\underline{x}\|_2 = 1} \sqrt{\langle A\underline{x}, A\underline{x}\rangle}$$ — standard inner product.

$$= \max_{\|\underline{x}\|_2 = 1} \sqrt{\langle A^*(A\underline{x}), \underline{x}\rangle}$$ $\left(\because \langle \underline{x}, A\underline{y}\rangle = \langle A^*\underline{x}, \underline{y}\rangle\right)$

$$= \sqrt{\max_{\|\underline{x}\|_2 = 1} \langle A^*A\underline{x}, \underline{x}\rangle}$$

$$= \sqrt{\max_{\substack{\underline{x} \neq 0 \\ \text{in } \mathbb{C}^n}} \frac{\langle A^*A\underline{x}, \underline{x}\rangle}{\langle \underline{x}, \underline{x}\rangle}}$$

$$= \sqrt{\max_j \lambda_j(A^*A)}$$ (by Reilgher quotient thm)

$$= \max_j \sqrt{\lambda_j(A^*A)}$$

Theorem:- Suppose $A_{n\times n}$ is an invertible matrix. Then
$$\|A^{-1}\| = \frac{1}{\min\limits_{\|\underline{x}\|=1}\left(\|A\underline{x}\|\right)}$$

proof:-

Let $A\underline{x} = \underline{y}$

$\Rightarrow \quad \underline{x} = A^{-1}\underline{y}$.

Now
$$\min_{\|\underline{x}\|=1} \|A\underline{x}\| = \min_{\substack{\underline{x}\neq\underline{0} \\ \text{in } \mathbb{C}^n}} \left(\frac{\|A\underline{x}\|}{\|\underline{x}\|}\right)$$

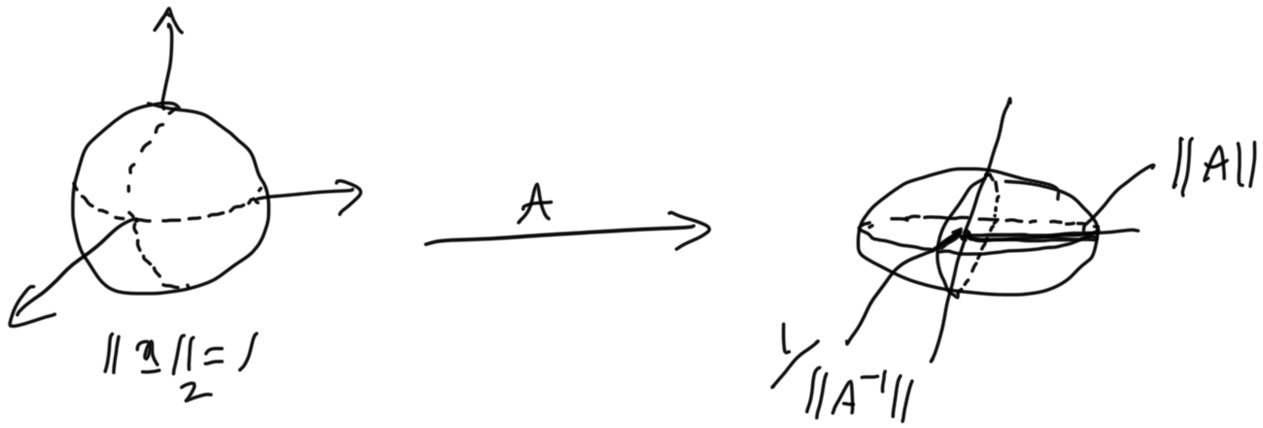
$$= \min_{\substack{\underline{y}\neq\underline{0} \\ \text{in } \mathbb{C}^n}} \left(\frac{\|\underline{y}\|}{\|A^{-1}\underline{y}\|}\right)$$

$$= \min_{\substack{\underline{y}\neq\underline{0} \\ \text{in } \mathbb{C}^n}} \left(\frac{1}{\left(\frac{\|A^{-1}\underline{y}\|}{\|\underline{y}\|}\right)}\right)$$

$$= \frac{1}{\max\limits_{\substack{\underline{y}\neq\underline{0} \\ \text{in } \mathbb{C}^n}} \left(\frac{\|A^{-1}\underline{y}\|}{\|\underline{y}\|}\right)}$$

$$= \frac{1}{\|A^{-1}\|}.$$

$$\Rightarrow \quad \|A^{-1}\| = \frac{1}{\min\limits_{\|\underline{x}\|=1} \left(\|A\underline{x}\| \right)}$$



An induced $\|A\|$ represents the maximum extend to which a vector on the unit sphere can be stretched by A & $\frac{1}{\|A^{-1}\|}$ measures the extend to which a non-singular matrix A can be shrink vectors on the unit sphere.

Decomposition of matrices.

- QR-decomposition.
- Cholesky decomposition
- Singular value decomposition.

Def:- <u>Householder transformation:-</u>

Let $\underline{w} \in \mathbb{C}^n$ be a unit vector.

i.e., $\|\underline{w}\| = 1$. Throughout $\|-\|$ is the vector 2-norm

i.e. $\|\underline{w}\| = \sqrt{|w_1|^2 + \cdots + |w_n|^2}$

A matrix $P_{n\times n}$ over $\mathbb{C}$ of the form

$$P = I - 2\, \underset{n\times 1}{\underline{w}}\, \underset{1\times n}{\underline{w}^*},$$

is called a <u>householder transformation</u>.

Note that $\underset{1\times n}{\underline{w}^*}\, \underset{n\times 1}{\underline{w}} = \|\underline{w}\|^2 = 1$

Another names: elementary Hermitian matrix

or elementary reflector.

<u>Properties</u>

① P is a Hermitian matrix.

i.e., $P^* = P$.

<u>proof:-</u>

$$P^* = (I - 2\underline{w}\,\underline{w}^*)^*$$

$$= I^* - 2(\underline{w}\,\underline{w}^*)^*$$

$$= I - 2(\underline{w}^*)^* \underline{w}^*$$

$$= I - 2\underline{w}\,\underline{w}^*$$
$$= P.$$

2. P is unitary. i.e., $P^*P = PP^* = I$.

<u>proof:-</u>
$$P^*P = PP = \left(I - 2\underline{w}\,\underline{w}^*\right)\left(I - 2\underline{w}\,\underline{w}^*\right)$$
$$= I - 2\underline{w}\,\underline{w}^* - 2\underline{w}\,\underline{w}^* + 4\left(\underline{w}\,\underline{w}^*\right)\left(\underline{w}\,\underline{w}^*\right)$$
$$= I - 4\underline{w}\,\underline{w}^* + 4\underline{w}\left(\underline{w}^*\underline{w}\right)\underline{w}^*$$
$$= I - 4\underline{w}\,\underline{w}^* + 4\underline{w}(1)\underline{w}^*$$
$$= I - \cancel{4\underline{w}\,\underline{w}^*} + \cancel{4\underline{w}\,\underline{w}^*}$$
$$= I.$$

3. $P^2 = I$ i.e. P is involutory.

4. $\|\underline{x}\| = \|P\underline{x}\| \quad \forall\, \underline{x} \in \mathbb{C}^n$.

<u>proof:-</u>
$$\|P\underline{x}\|^2 = \langle P\underline{x}, P\underline{x}\rangle$$
$$= (P\underline{x})^*(P\underline{x})$$
$$= \underline{x}^* P^* P \underline{x}$$
$$= \underline{x}^* I \underline{x}$$
$$= \|\underline{x}\|^2$$
$$\Rightarrow \quad \|P\underline{x}\| = \|\underline{x}\|.$$

Theorem:— Given a non-zero vector $\underline{x} = (x_1, \dots, x_n)$ in $\mathbb{R}^n$ (or in $\mathbb{C}^n$), there exists a householder transformation $P = I - 2\underline{w}\,\underline{w}^*$ of size n for some $\underline{w} \in \mathbb{R}^n$, $\underline{w}^*\underline{w} = 1$, such that

$$P\underline{x} = \alpha\,\underline{e_1} \quad \text{for some } \alpha \in \mathbb{R}$$

$$\& \quad \underline{e_1} = (1, 0, \dots, 0) \in \mathbb{R}^n.$$

proof:— Let $m = \|\underline{x}\| = \sqrt{|x_1|^2 + \dots + |x_n|^2}$

we need to find a unit vector $\underline{w}$ satisfying all the conditions of the thm.

Now $\|\underline{x}\|^2 = \|P\underline{x}\|^2 = \|\alpha\,\underline{e_1}\|^2 = |\alpha|^2 \|\underline{e_1}\| = |\alpha|^2$

$$\Rightarrow m^2 = |\alpha|^2$$

$$\Rightarrow \boxed{\alpha = \pm m. = \pm\|\underline{x}\|}$$

Let $\underline{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$ & $P = I - 2\,\underline{w}\,\underline{w}^*$.

Now $P\underline{x} = \alpha\,\underline{e_1}$

$$\Rightarrow (I - 2\underline{w}\,\underline{w}^*)\,\underline{x} = \alpha\,\underline{e_1}.$$

$$\Rightarrow \underline{x} - 2\,\underline{w}\,\underbrace{\underline{w}^*}_{1\times n}\underbrace{\underline{x}}_{n\times 1} = \pm m\,\underline{e_1}.$$

$$\Rightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} - 2 \begin{pmatrix} w_1 \underline{w}^* \underline{x} \\ w_2 \underline{w}^* \underline{x} \\ \vdots \\ w_n \underline{w}^* \underline{x} \end{pmatrix} = \begin{pmatrix} \pm m \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 - 2 w_1 \underline{w}^* \underline{x} \\ x_2 - 2 w_2 \underline{w}^* \underline{x} \\ \vdots \\ x_n - 2 w_n \underline{w}^* \underline{x} \end{pmatrix} = \begin{pmatrix} \pm m \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\Rightarrow \boxed{\begin{array}{l} x_1 - 2 w_1 \underline{w}^* \underline{x} = \pm m \\ \hline x_j - 2 w_j \underline{w}^* \underline{x} = 0 \quad \forall j = 2, \ldots, n. \end{array}}$$

Let $q = \underline{w}^* \underline{x}$.

Then

$$\left.\begin{array}{l} x_1 = \pm m + 2 w_1 q \\ 2 w_j q = x_j \quad \forall j = 2, \ldots, n. \end{array}\right\} \longrightarrow (*)$$

$$\Rightarrow \quad w_j = \frac{x_j}{2q} \quad \forall j = 2, 3, \ldots, n.$$

Now $\|\underline{w}\| = 1 \Rightarrow w_1^2 + \cdots + w_n^2 = 1$

$$\Rightarrow \left(\frac{x_1 \mp m}{2q}\right)^2 + \frac{x_2^2}{4q^2} + \cdots + \frac{x_n^2}{4q^2} = 1$$

(from $*$)

$$\Rightarrow (x_1 \mp m)^2 + x_2^2 + \cdots + x_n^2 = 4q^2$$

$$\Rightarrow x_1^2 \mp 2m x_1 + m^2 + x_2^2 + \cdots + x_n^2 = 4q^2$$

$$\Rightarrow \mp 2m x_1 + m^2 + m^2 = 4q^2$$

$$\Rightarrow \boxed{4q^2 = 2m(m \mp x_1)}$$

Among the two values admissible for $2q^2$ choose the largest one for numerical stability.

i.e., $$\boxed{2q^2 = m(m + \operatorname{sign}(x_1)\, x_1)}$$

where

$$\operatorname{sign}(x_1) = \begin{cases} 1 & \text{if } x_1 \geq 0 \\ -1 & \text{if } x_1 < 0. \end{cases}$$

$$\therefore \quad \underline{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} \frac{x_1 \mp m}{2q} \\ x_2/2q \\ \vdots \\ x_n/2q \end{pmatrix}$$

$$= \frac{1}{2q} \begin{pmatrix} x_1 \mp m \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Let $\underline{u} = \begin{pmatrix} x_1 \mp m \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$.

$\underline{w} = \frac{1}{2q} \underline{u}$.

Thus $P = I - 2 \underline{w}\,\underline{w}^*$

$= I - \frac{2}{4q^2} \underline{u}\,\underline{u}^t$

$\boxed{P = I - \frac{1}{2q^2} \underline{u}\,\underline{u}^t}$, $m = \| \underline{x} \|$

where $\underline{u} = \begin{pmatrix} x_1 \mp m \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

& $\boxed{2q^2 = m \left(m + \operatorname{sign}(x_1)\, x_1 \right)}$

$\boxed{\alpha = \pm m = -\operatorname{sign}(x_1) \| \underline{x} \|}$

check that $P\underline{x} = \alpha \underline{e}_1$.