Exercises 3

(1) Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Find its characteristic polynomial & minimal polynomial.

Sol: Char. poly of $A = \det(\lambda I - A)$

$$= \begin{vmatrix} \lambda & -1 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda-1 \end{vmatrix}$$

$$= \lambda(\lambda(\lambda-1)) + (-\lambda+1)$$

$$= \lambda^3 - \lambda^2 - \lambda + 1$$

$$= (\lambda-1)^2(\lambda+1).$$

possible divisors: $(\lambda-1)$, $\lambda+1$, $(\lambda-1)^2$, $(\lambda-1)(\lambda+1)$, $(\lambda-1)^2(\lambda+1)$.

$A - I \neq \underline{0}$

$A + I \neq \underline{0}$

$$(A-I)^2 = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \neq \underline{0}$$

$$(A-I)(A+I) = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \underline{0}$$

$\begin{bmatrix} & & & \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore$ minimal polynomial $= (\lambda-1)(\lambda+1)$
$= \lambda^2 - 1$
$\neq$ char. poly.

Yes: minimal poly $\big|$ char. poly.

<u>EXERCISE</u>: If λ is an eigenvalue of A, then λ is also a root of the minimal polynomial of A.

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(2) Let A be an $n\times n$ matrix with real entries. Suppose $a \pm ib$ are complex eigenvalues of A. Suppose $\underline{x} \pm i\underline{y}$ are eigenvectors corr. to $a \pm ib$ for A. where $a, b \in \mathbb{R}$

$\underline{x}, \underline{y} \in \mathbb{R}^n$.
$\underline{x} \neq \underline{0}$, $\underline{y} \neq \underline{0}$

Then $\{\underline{x}, \underline{y}\}$ is l.i in $\mathbb{R}^n$.

<u>proof</u>:- First notice that $\underline{x} + i\underline{y}$, $\underline{x} - i\underline{y}$ are l.i over $\mathbb{C}$ because they corresponding to distinct eigenvalues $a+ib$, $a-ib$.

<u>TO show</u>: $\{\underline{x}, \underline{y}\}$ is l.i in $\mathbb{R}^n$.

Suppose $c_1 \underline{x} + c_2 \underline{y} = \underline{0}$ for some $c_1, c_2 \in \mathbb{R}$.

To show: $c_1 = c_2 = 0$.

Let us choose $k_1 = \dfrac{c_1 - c_2 i}{2}$, $k_2 = \dfrac{c_1 + c_2 i}{2}$.

Consider

$$k_1(\underline{x} + i\underline{y}) + k_2(\underline{x} - i\underline{y})$$

$$= (k_1 + k_2)\underline{x} + (i)(k_1 - k_2)\underline{y}$$

$$= c_1 \underline{x} + (i)(-i c_2)\underline{y}$$

$$= c_1 \underline{x} + c_2 \underline{y}$$

$$= \underline{0}$$

But $\underline{x} + i\underline{y}$, $\underline{x} - i\underline{y}$ are l.i.

$\Rightarrow k_1 = k_2 = 0$.

$\Rightarrow \underline{c_1 = c_2 = 0}$.

$\therefore \{\underline{x}, \underline{y}\}$ is l.i in $\mathbb{R}^n$.

<u>EXERCISE:</u> Suppose $a \pm ib$, $c \pm id$ are complex eigenvectors of a real matrix $A_{n \times n}$ (but not real) & let $\underline{x} \pm i\underline{y}$, $\underline{z} \pm i\underline{w}$ be their corresponding eigenvectors of A.

Suppose $\{ \underline{x} \pm i\underline{y}, \ \underline{z} \pm i\underline{w} \}$ are l.i. in $\mathbb{C}^n$

Then $\{ \underline{x}, \underline{y}, \underline{z}, \underline{w} \}$ l.i in $\mathbb{R}^n$.

(Prove it!)

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(3) Find e^{At}, where $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

Sol:- Let $B = At = \begin{bmatrix} 3t & t & 0 \\ 0 & 3t & t \\ 0 & 0 & 3t \end{bmatrix}$

Eigenvalues of B are $3t, 3t, 3t$.

Let $f(B) = e^{B}$ | $r(B) = \alpha_0 I + \alpha_1 B + \alpha_2 B^2$

$f(\lambda) = e^{\lambda}$ | $r(\lambda) = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2$.

$f'(\lambda) = e^{\lambda}$ | $r'(\lambda) = \alpha_1 + 2\alpha_2 \lambda$

$f''(\lambda) = e^{\lambda}$ | $r''(\lambda) = 2\alpha_2$

Now $f(B) = r(B)$

At $\lambda = 3t$, $f(\lambda) = r(\lambda)$

$f'(\lambda) = r'(\lambda)$

$f''(\lambda) = r''(\lambda)$

$\Rightarrow f(3t) = r(3t)$

$\Rightarrow \boxed{e^{3t} = \alpha_0 + \alpha_1 3t + \alpha_2 9t^2}$

$$f'(3t) = r'(3t)$$

$$\Rightarrow \boxed{e^{3t} = \alpha_1 + 2\alpha_2 3t}$$

& $f''(3t) = r''(3t)$

$$\Rightarrow \boxed{e^{3t} = 2\alpha_2} \Rightarrow \boxed{\alpha_2 = \frac{e^{3t}}{2}}$$

$$\therefore \quad e^{3t} - 6t\left(\frac{e^{3t}}{2}\right) = \alpha_1$$

$$\Rightarrow \boxed{\alpha_1 = e^{3t} - 3te^{3t} = e^{3t}(1-3t)}$$

$$\therefore \quad e^{3t} = \alpha_0 + 3t(1-3t)e^{3t} + 9t^2\left(\frac{e^{3t}}{2}\right)$$

$$\alpha_0 = e^{3t}\left[1 - 3t(1-3t) - \frac{9}{2}t^2\right]$$

$$\boxed{\alpha_0 = e^{3t}\left(1 - 3t + \frac{9}{2}t^2\right)}$$

$$\therefore \quad e^B = r(B) = \alpha_0 I + \alpha_1 B + \alpha_2 B^2$$

$$B^2 = \begin{bmatrix} 3t & t & 0 \\ 0 & 3t & t \\ 0 & 0 & 3t \end{bmatrix} \begin{bmatrix} 3t & t & 0 \\ 0 & 3t & t \\ 0 & 0 & 3t \end{bmatrix}$$

$$= \begin{bmatrix} 9t^2 & 6t^2 & t^2 \\ 0 & 9t^2 & 6t^2 \\ 0 & 0 & 9t^2 \end{bmatrix}$$

$$\therefore e^{B} = e^{3t}\left(1-3t+\frac{9}{2}t^2\right)\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} + e^{3t}(1-3t)\begin{bmatrix}3t&t&0\\0&3t&t\\0&0&3t\end{bmatrix}$$

$$+ \frac{e^{3t}}{2}\begin{bmatrix}9t^2&6t^2&t^2\\0&9t^2&6t^2\\0&0&9t^2\end{bmatrix}$$

$$\therefore \boxed{e^{At} = e^{3t}\begin{bmatrix}1&t&t^2/2\\0&1&t\\0&0&1\end{bmatrix}}.$$

<u>Theorem</u>:- (Charatrization of orthogonal matrices)

A real $n\times n$ matrix A is <u>orthogonal</u> $\left(AA^t = I = A^tA\right)$

$$\Updownarrow$$

its column vectors $\underline{c}_1, \ldots, \underline{c}_n$ (& also row vectors $\underline{R}_1, \ldots, \underline{R}_n$) form an orthonormal system

i.e. $\langle \underline{c}_i, \underline{c}_j\rangle = c_j^t c_i = \begin{cases} 0 & \text{if } i\neq j \\ 1 & \text{if } i=j. \end{cases}$

$\forall\, i,j = 1, 2, \ldots, n.$

Theorem (Characterization for unitary matrices).

An $n \times n$ complex matrix A is unitary

$$\Updownarrow$$

its column vectors $\underset{\sim}{c_1}, \ldots, \underset{\sim}{c_n}$ (& also row vectors $\underset{\sim}{R_1}, \ldots, \underset{\sim}{R_n}$) form an orthonormal system

i.e.,

$$\langle \underset{\sim}{c_j}, \underset{\sim}{c_k} \rangle = c_k^* c_j = \overline{c_k}^t c_j = \begin{cases} 0 & \text{if } k \neq j \\ 1 & \text{if } k = j. \end{cases}$$

$\forall\, j, k = 1, 2, \ldots, n.$
