

(1)

Simple linear regression.

$$D = \{(x_i, y_i) \mid \begin{array}{l} x_i \text{ are non-stochastic} \\ y_i \text{ are stochastic} \end{array}\}$$

$$\epsilon_i \stackrel{\text{iid}}{\sim} N(0, \sigma^2).$$

Model : $y_i = \beta_0 + \beta_1 x_i + \epsilon_i \quad \forall i=1, 2, \dots, n.$

Gauss-Markov Model.

Unknown parameters are $\beta_0, \beta_1, \sigma^2$.

$$E(y_i) = \beta_0 + \beta_1 x_i \quad \text{as } E(\epsilon_i) = 0 \text{ and } x_i \text{ are non-stochastic.}$$

Hence.

$$\underline{Y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad \underline{\epsilon} = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{pmatrix} \quad \underline{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \quad X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_m \end{bmatrix} = \begin{pmatrix} 1 & \underline{x} \end{pmatrix}$$

$$\underline{Y} = X \underline{\beta} + \underline{\epsilon} \text{ where } \underline{\epsilon} \sim N(\underline{0}, I_n \sigma^2)$$

$$\Rightarrow \underline{Y} = X \underline{\beta} + \underline{\epsilon} \sim N(X \underline{\beta}, I_n \sigma^2)$$

• y_i 's are independent but not identically distributed.

→ Methods of estimation for $\beta_0, \beta_1, \sigma^2$

- Least square method.

- Maximum likelihood method.

LS condition: To minimize.

$$S(\beta_0, \beta_1) = \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i)^2$$

$$= \| \underline{Y} - X \underline{\beta} \|_2^2 = (\underline{Y} - X \underline{\beta})^T (\underline{Y} - X \underline{\beta})$$

$$\hat{\beta} = (\hat{\beta}_0, \hat{\beta}_1) = \arg \min_{(\beta_0, \beta_1)} S(\beta_0, \beta_1)$$

Solve $\frac{\partial S}{\partial \beta_0} \Big|_{\hat{\beta}_0} = 0$ and $\frac{\partial S}{\partial \beta_1} \Big|_{\hat{\beta}_1} = 0$.

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n y_i x_i - n \bar{x} \bar{y}}{\sum_{i=1}^n x_i^2 - n \bar{x}^2} = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

$$\Rightarrow \hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} \quad \text{and} \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

H.W. ① $\frac{\partial b^T A}{\partial b} = A^T. \quad \left. \begin{array}{l} \\ \end{array} \right\} = X^T Y = X^T \hat{B}$

② $\frac{\partial b^T A b}{\partial b} = b^T (A + A^T) \quad \text{Normal equation.}$

Show that $\hat{\beta} = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = (X^T X)^{-1} X^T Y. \quad \left. \begin{array}{l} \text{When} \\ \text{all } x_i \\ \text{are same} \\ (X^T X) \text{ is} \\ \text{not invertible.} \end{array} \right\}$

④ Show that regression line is passing through (\bar{x}, \bar{y}) .

Regression line is

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x.$$

$$= \bar{y} - \hat{\beta}_1 \bar{x} + \hat{\beta}_1 x.$$

$$= \bar{y} + \hat{\beta}_1 (x - \bar{x}).$$

If we replace $x = \bar{x}$ then $\hat{y} = \bar{y}$.

\Rightarrow Regression line is passing through (\bar{x}, \bar{y}) .

• We know that $\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}}$.

$$= \sum_{i=1}^n \frac{(x_i - \bar{x})}{S_{xx}} y_i$$

$$= \tilde{\alpha}^T \tilde{y}$$

Sax.
 $= \sum (x_i - \bar{x})(y_i - \bar{y})$
 $= \sum (x_i - \bar{x}) y_i$
 $= \sum (y_i - \bar{y}) x_i$

when $\alpha_i = \frac{(x_i - \bar{x})}{S_{xx}}$
 $i = 1, 2, \dots, n.$

$\Rightarrow \hat{\beta}_1$ is a linear estimator of β_1 .

$$\hat{\beta}_1 \sim N\left(\tilde{\alpha}^T \tilde{\beta}, \tilde{\alpha}^T I_n \tilde{\alpha}\right) \quad \tilde{y} \sim N\left(\tilde{\alpha}^T \tilde{\beta}, I_n \sigma^2\right)$$

$$\sigma^2 \tilde{\alpha}^T I_n \tilde{\alpha} = \sigma^2 \tilde{\alpha}^T \tilde{\alpha} = \sigma^2 \sum_{i=1}^n \alpha_i^2 = \sigma^2 \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{S_{xx}^2}$$

$$= \sigma^2 \frac{S_{xx}}{S_{xx}^2} = \sigma^2 / S_{xx} = \text{Var}(\hat{\beta}_1).$$

$$E(\hat{\beta}_1) = \tilde{\alpha}^T \tilde{\beta} = \tilde{\alpha}^T (\frac{1}{n} \tilde{x}) \tilde{\beta}.$$

$$= \begin{pmatrix} \tilde{\alpha}^T \frac{1}{n} & \tilde{\alpha}^T \tilde{x} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \cdot \frac{1}{n} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

$$= \beta_1.$$

$0 = \tilde{\alpha}^T \frac{1}{n} = \frac{\sum (x_i - \bar{x})}{S_{xx}}$
 $= \frac{\tilde{\alpha}^T \tilde{x}}{\sum_{i=1}^n (x_i - \bar{x}) \cdot x_i} = \frac{S_{xx}}{S_{xx}} = 1.$

$$\hat{\beta}_1 \sim N\left(\beta_1, \frac{\sigma^2}{S_{xx}}\right)$$

$\Rightarrow \hat{\beta}_1$ is an unbiased estimator of β_1 .

(4)

HW 2. Express $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$ as a linear estimator of β_0 . and show that.

$$\hat{\beta}_0 \sim N\left(\beta_0, \sigma^2 \left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)\right)$$

$$\begin{aligned}\hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x} \\ &= \frac{1}{n} \underbrace{\mathbf{1}^T \mathbf{y}}_{\sim} - \bar{x} \cdot \underbrace{\mathbf{x}^T \mathbf{y}}_{\sim} \\ &= \left(\frac{1}{n} \mathbf{1}^T - \bar{x} \mathbf{x}^T \right) \mathbf{y} \quad \theta_i = \left(\frac{1}{n} - \frac{\bar{x}(x_i - \bar{x})}{S_{xx}} \right) \\ &\equiv \underbrace{\boldsymbol{\theta}^T}_{\sim} \mathbf{y}.\end{aligned}$$

$$\hat{\beta}_0 \sim N\left(\boldsymbol{\theta}^T \mathbf{x}, \boldsymbol{\theta}^T \mathbf{I}_n \sigma^2 \boldsymbol{\theta}\right)$$

$$\left. \begin{aligned}E(\hat{\beta}_0) &= \beta_0 \\ \text{Var}(\hat{\beta}_0) &= \sigma^2 \left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}} \right)\end{aligned} \right\} \text{HW}.$$

• Estimation of σ^2 .

$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$. is the regression line.

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x \quad x \text{ is known to us.}$$

$$\Rightarrow \hat{y} = X \hat{\beta} \quad X = [1 \ x]$$

$$\Rightarrow \hat{y} = X(X^T X)^{-1} X^T \hat{y} \quad \hat{\beta} = (X^T X)^{-1} X^T \hat{y}.$$

Let us denote $P_x = X(X^T X)^{-1} X^T$

• P_x is an orthogonal projection matrix of $\ell(X)$.

• $P_x^T = P_x$. symmetric

• $P_x^2 = P_x$. idempotent.

\hat{y} is the orthogonal projection of y in $\ell(X)$.
and e is orthogonal to \hat{y} .

As $\hat{y} = P_x y$. \rightarrow prediction vector.

define. $e = y - \hat{y} = (I - P_x) y \rightarrow$ estimated error vector.

• ~~\mathcal{E}/var~~

• $(I - P_x)^T = (I - P_x)$ symmetric

• $(I - P_x)^2 = (I - P_x)$. idempotent.

$$\begin{aligned} e &= (I - P_x) y \\ &\sim N(0, (I - P_x) \sigma^2). \\ \hat{y} &= P_x y \\ &\sim N(X\beta, P_x \sigma^2) \\ y &\sim N(X\beta, \sigma^2 I_n) \end{aligned}$$

• $\text{cov}(\hat{y}, e)$.

$$= \text{cov}(P_x y, (I - P_x) y).$$

$$= \sigma^2 P_x I_n (I - P_x) = \sigma^2 O = 0 \text{ matrix.}$$

Also. $\hat{y}^T e = 0$ secon...

$$\boxed{\begin{aligned} \text{cov}(\hat{A}y, \hat{B}y) \\ = A \Sigma B^T \end{aligned}}$$

$$SSR = \text{Residual Sum of squares} = \sum_{i=1}^n (y_i - \hat{y})^2 \quad (7)$$

$$E(SSR) = E(e^T e) = E\left(\sum_{i=1}^n (y_i - \hat{y})^2\right) = E\left(y^T (I - P_{\hat{\beta}}) y\right) = (n-2)\sigma^2.$$

$$\Rightarrow E\left[\frac{\sum_{i=1}^n (y_i - \hat{y})^2}{n-2}\right] = \sigma^2.$$

$$\hat{\sigma}^2 = \frac{SSR}{n-2} \quad \text{Unbiased estimator.}$$

HW'3 ② Find the maximum likelihood estimator

of $\hat{\beta}_0, \hat{\beta}_1, \hat{\sigma}^2$.

③ Are they unbiased estimators of their corresponding parameter?

$$\hat{\beta}_{0, \text{MLE}} = \hat{\beta}_{OLS}$$

$$\hat{\beta}_{1, \text{MLE}} = \hat{\beta}_{OLS}.$$

$$\hat{\sigma}_{MLE}^2 = \frac{SSR}{n} \neq \frac{SSR}{n-2} = \hat{\sigma}_{OLS}^2.$$