Multivariate Analysis.

$$\underset{\sim}{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \underset{\sim}{\mu} = E(\underset{\sim}{X}) = \begin{bmatrix} E(x_1) \\ E(x_2) \\ \vdots \\ E(x_n) \end{bmatrix} \qquad E|x_i| < \infty \quad \forall\, i = 1, 2 \cdots n.$$

$$Y = ((Y_{ij}))_{m\times n} \qquad E(Y) = ((E\,Y_{ij}))_{m\times n}.$$

$$D(\underset{\sim}{X}) = ((\mathrm{cov}(X_i, X_j)))_{n\times n}.$$

$$= ((E[(x_i - E(x_i))(x_j - E(x_j))]))_{n\times n}.$$

$$= ((E(x_i x_j) - E(x_i)E(x_j)))_{n\times n}.$$

$$= ((E(x_i x_j) - \mu_i \mu_j))_{n\times n}.$$

$$= E\left[(\underset{\sim}{X} - \underset{\sim}{\mu})(\underset{\sim}{X} - \underset{\sim}{\mu})^T\right] = \Sigma_X$$

(1) $E(\underset{\sim}{X} + \underset{\sim}{b}) = E(\underset{\sim}{X}) + \underset{\sim}{b} = \underset{\sim}{\mu} + \underset{\sim}{b}.$

(2) $D(\underset{\sim}{X} + \underset{\sim}{b}) = D(\underset{\sim}{X}).$

(c) $\mathrm{cov}(\underset{\sim}{X} + \underset{\sim}{b}, \underset{\sim}{Y} + \underset{\sim}{c}) = \mathrm{cov}(\underset{\sim}{X}, \underset{\sim}{Y}) = \Gamma_{xy}.$

$\underset{\sim}{b}, \underset{\sim}{c}$ are constant vectors.

$$\Gamma_{xy} = ((\mathrm{cov}(X_i, Y_j)))_{p\times q}.$$

(1) $E(A\underset{\sim}{X}) = AE(\underset{\sim}{X}) = A\underset{\sim}{\mu}.$

If $A = l^T$ then $\cancel{E(l^T \mu)}$ $= \sum_{i=1}^{n} l_i\, E(X_i) = \sum_{i=1}^{n} l_i \mu_i$

$\underset{\sim}{l}^T \underset{\sim}{\mu} = E(\underset{\sim}{l}^T \underset{\sim}{X}) = E\left(\sum_{i=1}^{n} l_i X_i\right) =$

(2) $\mathrm{cov}(\underset{\sim}{U}, \underset{\sim}{V}) = \Gamma \Rightarrow \mathrm{cov}(AU, BV) = A\Gamma B^T.$

$\Rightarrow$ If $\underset{\sim}{U} = \underset{\sim}{V} = \underset{\sim}{X} \Rightarrow \mathrm{cov}(A\underset{\sim}{X}, B\underset{\sim}{X}) = A\Sigma B^T.$

$\Rightarrow \underset{\sim}{l}^T = A = B \Rightarrow \mathrm{cov}(\underset{\sim}{l}^T \underset{\sim}{X}, \underset{\sim}{l}^T \underset{\sim}{X}) = \cancel{D(X)} = \underset{\sim}{l}^T \Sigma \underset{\sim}{l}$

$\mathrm{Var}(\underset{\sim}{l}^T \underset{\sim}{X}) = D(\underset{\sim}{l}^T \underset{\sim}{X}).$

$\underset{\sim}{X}$ is a random vector.

$D(\underset{\sim}{X}) = \Sigma$. Then $D(\underset{\sim}{X}) = \Sigma$ is a p.s.d matrix.

Consider $\underset{\sim}{l}^T \neq \underset{\sim}{0}^T$ then.

$$D(\underset{\sim}{l}^T \underset{\sim}{X}) = V(\underset{\sim}{l}^T \underset{\sim}{X}) \geq 0 \quad \forall \; \underset{\sim}{l}^T \neq \underset{\sim}{0}^T.$$

$$\Rightarrow \quad \underset{\sim}{l}^T \Sigma \, l \geq 0 \quad \forall \; \underset{\sim}{l}^T \neq \underset{\sim}{0}^T.$$

$\Rightarrow$ Σ is a p.s.d or n.n.d matrix.

Also Σ is a symmetric matrix. $\Sigma = \Sigma^T$.

Th. Let $E(\underset{\sim}{X}) = \underset{\sim}{\mu}$, $D(\underset{\sim}{X}) = \Sigma$.

then. $P\left((\underset{\sim}{X} - \underset{\sim}{\mu}) \in \mathcal{C}(\Sigma)\right) = 1$

To prove this it is enough to show that

if $\underset{\sim}{l} \in (\mathcal{C}(\Sigma))^{\perp}$ then $\underset{\sim}{l}^T(\underset{\sim}{X} - \underset{\sim}{\mu}) = 0$.

If $l \in (\mathcal{C}(\Sigma))^{\perp}$.

$$\Leftrightarrow \quad \underset{\sim}{l}^T \Sigma = \underset{\sim}{0}^T.$$

$$\Rightarrow \quad \underset{\sim}{l}^T \Sigma \, l = \underset{\sim}{0}^T \underset{\sim}{l}.$$

$$\Rightarrow \quad l^T \Sigma \, l = 0$$

$$\Rightarrow \quad D(\underset{\sim}{l}^T \underset{\sim}{X}) = 0.$$

$$\Rightarrow \quad D\left(\underset{\sim}{l}^T(\underset{\sim}{X} - \underset{\sim}{\mu})\right) = 0.$$

① $D\left(\underset{\sim}{l}^T(\underset{\sim}{X} - \underset{\sim}{\mu})\right) = 0$.

② $E\left(\underset{\sim}{l}^T(\underset{\sim}{X} - \underset{\sim}{\mu})\right) = 0$

$$\Rightarrow P\left(\underset{\sim}{l}^T(\underset{\sim}{X} - \underset{\sim}{\mu}) = 0\right) = 1.$$

$\underset{\sim}{l}$ is orthogonal to $(\underset{\sim}{X} - \underset{\sim}{\mu})$.

$$\Rightarrow \quad P\left((\underset{\sim}{X} - \underset{\sim}{\mu}) \in \mathcal{C}(\Sigma)\right) = 1.$$

$$E(\underset{\sim}{x}^T A \underset{\sim}{x}) = tr(A\Sigma) + \underset{\sim}{\mu}^T A \underset{\sim}{\mu}. \qquad E(\underset{\sim}{x}) = \underset{\sim}{\mu}. \quad D(\underset{\sim}{x}) = \Sigma.$$

Special case.

- If $X \sim N(\underset{\sim}{\mu}, \Sigma)$

$$E(\underset{\sim}{x}^T\underset{\sim}{x}) = E(\underset{\sim}{x}^T I_n \underset{\sim}{x}) = tr(I\Sigma) + \underset{\sim}{\mu}^T\underset{\sim}{\mu}$$
$$= tr(\Sigma) + \underset{\sim}{\mu}^T\underset{\sim}{\mu}.$$

Non-central χ^2.

- If $\underset{\sim}{X} \sim N(\underset{\sim}{\mu}, I_n)$. $\quad \underset{\sim}{x}^T\underset{\sim}{x} \sim \chi^2_{n,\,ncp = \underset{\sim}{\mu}^T\underset{\sim}{\mu}}$.

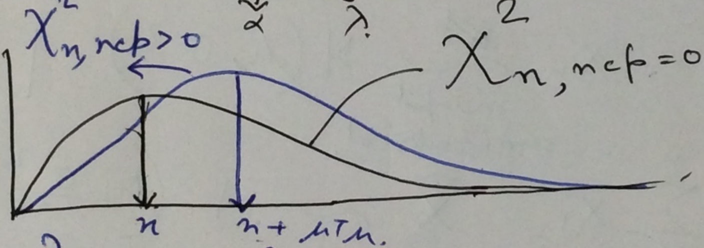
$$E(\underset{\sim}{x}^T\underset{\sim}{x}) = n + \underset{\sim}{\mu}^T\underset{\sim}{\mu}. = n + \sum_{i=1}^{n} \mu_i^2 = E\left(\sum x_i^2\right) = \sum_{i=1}^{n}(1+\mu_i^2)$$

- If $\underset{\sim}{x} \sim N(\underset{\sim}{0}, I_n)$ $\Rightarrow$ $x_1, x_2 \cdots x_n$ iid $N(0,1)$.

$$E(\underset{\sim}{x}^T\underset{\sim}{x}) = n + 0 = n. = E(\chi^2_n)$$

If $\underset{\sim}{X} \sim N(\underset{\sim}{0}, I_n)$ then $\underset{\sim}{x}^T\underset{\sim}{x} \sim \chi^2_n$ $\rightarrow$ (central χ^2)

$$\chi^2_n \equiv G\Big(\underbrace{\tfrac{n}{2}}_{\alpha}, \underbrace{\tfrac{1}{2}}_{\lambda}\Big). \qquad E(G(\alpha,\lambda)) = \frac{\alpha}{\lambda}.$$



$\sum x_i^2 = \underset{\sim}{x}^T\underset{\sim}{x}$.

$$E\left(\chi^2_{n,\,ncp=\mu^T\mu}\right) = n + \underset{\sim}{\mu}^T\underset{\sim}{\mu}.$$

$$E\left(\chi^2_{n,\,ncp=0}\right) = n + 0.$$

- $E(\underset{\sim}{x}^T A \underset{\sim}{x}) = tr(A\Sigma) + \underset{\sim}{\mu}^T A \underset{\sim}{\mu}$

proof: $E(\underset{\sim}{x}^T A \underset{\sim}{x}) = E\left[tr(\underset{\sim}{x}^T A \underset{\sim}{x})\right]$

$= E\ tr(A\underset{\sim}{x}\underset{\sim}{x}^T) = tr\left[A\,E(xx^T)\right]$

$= tr\left[A(\Sigma + \underset{\sim}{\mu}\underset{\sim}{\mu}^T)\right]$

$= tr\left[A\Sigma + A\underset{\sim}{\mu}\underset{\sim}{\mu}^T\right]$

$= tr(A\Sigma) + tr(A\underset{\sim}{\mu}\underset{\sim}{\mu}^T)$

$= tr(A\Sigma) + \underset{\sim}{\mu}^T A \underset{\sim}{\mu}.$

$tr(AB) = tr(BA)$.
$E(\underset{\sim}{x}) = \mu$
$D(\underset{\sim}{x}) = \Sigma$.
~~$D(x) =$~~
$E(xx^T) = \Sigma + \underset{\sim}{\mu}\underset{\sim}{\mu}^T$
$tr(A+B) = tr(A) + tr(B)$

$D(x) = \Sigma = E(\underset{\sim}{x}x^T) - \underset{\sim}{\mu}\underset{\sim}{\mu}^T$

- If $\underset{\sim}{X} \sim N(\underset{\sim}{\mu}, I_n)$ then $\underset{\sim}{X}^T A \underset{\sim}{X}$ follows a Chi-squared distribution iff A is an idempotent matrix.

- If A_1 & A_2 are square symmetric idempotent matrices and $\varrho = A_1 - A_2$ is p.s.d.

  then $\underset{\sim}{X}^T \varrho \underset{\sim}{X}$ and ~~$X^T A_2 X$~~ $\underset{\sim}{X}^T A_2 \underset{\sim}{X}$ are independently distributed.

- Let $\underset{\sim}{X} \sim N(\underset{\sim}{\mu}, I_n)$ A is symmetric and $C A = \underset{\text{matrix}}{O}$ then $\underset{\sim}{X}^T A \underset{\sim}{X}$ and $\underset{\sim}{C} \underset{\sim}{X}$ are independently distributed.

⊛ Construction of T-statistic.

$$x_1\, x_2 \cdots x_n \underset{\text{indept and identical.}}{\overset{\text{iid}}{\sim}} N(\mu, \sigma^2) \quad [\text{iid}]$$

then $\frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ &

$\sum_{i=1}^{n} (x_i - \bar{x})^2 = S^2 \sim \sigma^2 \chi^2_{n-1}$ &

> indepent.

Define.

$$\Rightarrow T = \frac{(\bar{x} - \mu)/(\sigma/\sqrt{n})}{\sqrt{\frac{S^2}{\sigma^2 (n-1)}}} = \frac{\sqrt{n}(\bar{x} - \mu)}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2/(n-1)}}$$

$T \sim t_{n-1}$, and $T^2 \sim F_{1,(n-1)}$.

$\bar{X} = \frac{1}{n} \underset{\sim}{1}^T \underset{\sim}{X} = \left(\frac{1}{n}\ \frac{1}{n} \cdots \frac{1}{n}\right)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underset{\sim}{l}^T \underset{\sim}{X}.$

where $l^T = \left(\frac{1}{n}, \frac{1}{n} \cdots \frac{1}{n}\right).$

$\underset{\sim}{X} \sim N\left(\underset{\sim}{\mu}, I_n \sigma^2\right)$ where $\underset{\sim}{\mu} = \mu \cdot \underset{\sim}{1} = \begin{pmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{pmatrix}.$

$\Rightarrow \underset{\sim}{l}^T \underset{\sim}{X} \sim N\left(\underset{\sim}{l}^T \underset{\sim}{\mu}, \underset{\sim}{l}^T \Sigma\, \underset{\sim}{l}\right) \equiv N\left(\mu, \frac{\sigma^2}{n}\right)$

$S^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 = \underset{\sim}{X}^T \left(I_n - \frac{1}{n} \underset{\sim}{1}\underset{\sim}{1}^T\right) \underset{\sim}{X}.$

$A = \left(I_n - \frac{1}{n} \underset{\sim}{1}\underset{\sim}{1}^T\right)$ hence $\underset{\text{Symmetric}}{A^T = A}, \underset{\text{Idempotent}}{A^2 = A}.$

$\underset{\sim}{Y} = \frac{\underset{\sim}{X}}{\sigma} \sim N\left(\frac{\underset{\sim}{\mu}}{\sigma}, I_n\right)$

$S^2 = \underset{\sim}{X}^T A \underset{\sim}{X}.$

$= \sigma^2 \left(\frac{\underset{\sim}{X}^T}{\sigma}\right) A \left(\frac{\underset{\sim}{X}}{\sigma}\right) = \sigma^2 \underset{\sim}{Y}^T A \underset{\sim}{Y}$

$S^2 \sim \sigma^2 \chi^2_{\text{rank}(A), \left(\frac{\underset{\sim}{\mu}}{\sigma}\right)^T A \left(\frac{\underset{\sim}{\mu}}{\sigma}\right)}.$

Note $\text{rank}(A) = \text{trace}(A) = n-1$

$\frac{1}{\sigma^2}\left(\underset{\sim}{\mu}^T A \underset{\sim}{\mu}\right) = \mu \cdot \left[\underset{\sim}{1}^T \left(I_n - \frac{1}{n} \underset{\sim}{1}\underset{\sim}{1}^T\right) \underset{\sim}{1}\right] \cdot \mu = \mu^2 \cdot 0 = \underline{\underline{0}}.$

$\Rightarrow S^2 \sim \sigma^2 \chi^2_{n-1}.$

$\underset{\sim}{l}^T A = \frac{1}{n} \underset{\sim}{1}^T \left(I_n - \frac{1}{n} \underset{\sim}{1}\underset{\sim}{1}^T\right)$

$= \frac{1}{n}\left(\underset{\sim}{1}^T - \frac{1}{n}\left(\underset{\sim}{1}^T \underset{\sim}{1}\right) \underset{\sim}{1}^T\right)$

$= \frac{1}{n}\left(\underset{\sim}{1}^T - \underset{\sim}{1}^T\right) = \underset{\sim}{0}^T.$

$\bar{X} = \underset{\sim}{l}^T \underset{\sim}{X}, \quad S^2 = \underset{\sim}{X}^T A \underset{\sim}{X}$

$\left\{ \begin{array}{l} A^T = A. \\ A^2 = A. \end{array} \right.$

$\bar{X}$ and S^2 are independently distributed.

$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \Rightarrow \sqrt{n}\left(\frac{\bar{X} - \mu}{\sigma}\right) \sim N(0,1).$

and $\frac{S^2}{\sigma^2} \sim \chi^2_{n-1}$

$\Big\}$ independent.