Multivariate Analysis.

$$\underset{\sim}{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \underset{\sim}{\mu} = E(\underset{\sim}{X}) = \begin{bmatrix} E(x_1) \\ E(x_2) \\ \vdots \\ E(x_n) \end{bmatrix} \qquad E|x_i| < \infty \quad \forall\, i = 1, 2 \cdots n.$$

$$Y = ((Y_{ij}))_{m \times n}. \qquad E(Y) = ((E Y_{ij}))_{m \times n}.$$

$$D(\underset{\sim}{X}) = ((\mathrm{cov}(X_i, X_j)))_{n \times n}.$$

$$= ((E[(X_i - E(X_i))(X_j - E(X_j))]))_{n \times n}.$$

$$= ((E(X_i X_j) - E(X_i) E(X_j)))_{n \times n}.$$

$$= ((E(X_i X_j) - \mu_i \mu_j))_{n \times n}.$$

$$= E\left[(\underset{\sim}{X} - \underset{\sim}{\mu})(\underset{\sim}{X} - \underset{\sim}{\mu})^T\right] = \Sigma_X$$

(1) $E(\underset{\sim}{X} + \underset{\sim}{b}) = E(\underset{\sim}{X}) + \underset{\sim}{b} = \underset{\sim}{\mu} + \underset{\sim}{b}$.

(2) $D(\underset{\sim}{X} + \underset{\sim}{b}) = D(\underset{\sim}{X})$.

(c) $\mathrm{cov}(\underset{\sim}{X} + \underset{\sim}{b}, \underset{\sim}{Y} + \underset{\sim}{c}) = \mathrm{cov}(\underset{\sim}{X}, \underset{\sim}{Y}) = \Gamma_{XY}$.

$\underset{\sim}{b}, \underset{\sim}{c}$ are constant vectors.

$$\Gamma_{XY} = ((\mathrm{cov}(X_i, Y_j)))_{p \times q}.$$

(1) $E(A\underset{\sim}{X}) = A E(\underset{\sim}{X}) = A\underset{\sim}{\mu}$.

If $A = l^T$ then $\cancel{E(l^T X)}$ $E\left(\sum_{i=1}^n l_i X_i\right) = \sum_{i=1}^n l_i E(X_i) = \sum_{i=1}^n l_i \mu_i$

$\underset{\sim}{l}^T \underset{\sim}{\mu} = E(\underset{\sim}{l}^T \underset{\sim}{X}) = E\left(\sum_{i=1}^n l_i X_i\right) = \sum_{i=1}^n l_i E(X_i) = \sum_{i=1}^n l_i \mu_i$

(2) $\mathrm{cov}(\underset{\sim}{U}, \underset{\sim}{V}) = \Gamma \Rightarrow \mathrm{cov}(AU, BV) = A \Gamma B^T$.

$\Rightarrow$ If $\underset{\sim}{U} = \underset{\sim}{V} = \underset{\sim}{X} \Rightarrow \mathrm{cov}(A\underset{\sim}{X}, B\underset{\sim}{X}) = A \Sigma B^T$.

$\Rightarrow \underset{\sim}{l}^T = A = B \Rightarrow \mathrm{cov}(\underset{\sim}{l}^T \underset{\sim}{X}, \underset{\sim}{l}^T \underset{\sim}{X}) = \cancel{D(X)} = \underset{\sim}{l}^T \Sigma \underset{\sim}{l}$

$\mathrm{Var}(\underset{\sim}{l}^T \underset{\sim}{X}) = D(\underset{\sim}{l}^T \underset{\sim}{X})$.

$\underset{\sim}{X}$ is a random vector.

$D(\underset{\sim}{X}) = \Sigma$. Then $D(\underset{\sim}{X}) = \Sigma$ is a p.s.d matrix.

Consider $\underset{\sim}{l}^T \neq \underset{\sim}{0}^T$ then.

$$D(\underset{\sim}{l}^T \underset{\sim}{X}) = V(\underset{\sim}{l}^T \underset{\sim}{X}) \geq 0 \quad \forall \; \underset{\sim}{l}^T \neq \underset{\sim}{0}^T.$$

$$\Rightarrow \quad \underset{\sim}{l}^T \Sigma\, l \geq 0 \quad \forall \; \underset{\sim}{l}^T \neq \underset{\sim}{0}^T.$$

$\Rightarrow$ Σ is a p.s.d or n.n.d matrix.

Also Σ is a symmetric matrix. $\Sigma = \Sigma^T$.

<u>Th</u>. Let $E(\underset{\sim}{X}) = \underset{\sim}{\mu}$, $D(\underset{\sim}{X}) = \Sigma$.

then $P\left((\underset{\sim}{X} - \underset{\sim}{\mu}) \in \mathcal{C}(\Sigma)\right) = 1$

To prove this it is enough to show that

if $\underset{\sim}{l} \in (\mathcal{C}(\Sigma))^{\perp}$ then $\underset{\sim}{l}^T(\underset{\sim}{X} - \underset{\sim}{\mu}) = 0$.

If $l \in (\mathcal{C}(\Sigma))^{\perp}$.

$$\Leftrightarrow \quad \underset{\sim}{l}^T \Sigma = \underset{\sim}{0}^T.$$

$$\Rightarrow \quad \underset{\sim}{l}^T \Sigma\, l = \underset{\sim}{0}^T \underset{\sim}{l}.$$

$$\Rightarrow \quad l^T \Sigma\, l = 0$$

$$\Rightarrow \quad D(\underset{\sim}{l}^T \underset{\sim}{X}) = 0.$$

$$\Rightarrow \quad D\left(\underset{\sim}{l}^T(\underset{\sim}{X} - \underset{\sim}{\mu})\right) = 0.$$

① $D\left(\underset{\sim}{l}^T(\underset{\sim}{X} - \underset{\sim}{\mu})\right) = 0.$

② $E\left(\underset{\sim}{l}^T(\underset{\sim}{X} - \underset{\sim}{\mu})\right) = 0$

$$\Rightarrow P\left(\underset{\sim}{l}^T(\underset{\sim}{X} - \underset{\sim}{\mu}) = 0\right) = 1.$$

$\therefore$ $\underset{\sim}{l}$ is orthogonal to $(\underset{\sim}{X} - \underset{\sim}{\mu})$.

$$\Rightarrow \quad P\left((\underset{\sim}{X} - \underset{\sim}{\mu}) \in \mathcal{C}(\Sigma)\right) = 1.$$

$$E(\underset{\sim}{x}^T A \underset{\sim}{x}) = tr(A\Sigma) + \underset{\sim}{\mu}^T A \underset{\sim}{\mu}.$$

$E(\underset{\sim}{x}) = \underset{\sim}{\mu}.$
$D(\underset{\sim}{x}) = \Sigma.$

Special case.

- If $X \sim N(\underset{\sim}{\mu}, \Sigma)$

$$E(\underset{\sim}{x}^T\underset{\sim}{x}) = E(\underset{\sim}{x}^T I_n \underset{\sim}{x}) = tr(I\Sigma) + \underset{\sim}{\mu}^T\underset{\sim}{\mu}$$

$$= tr(\Sigma) + \underset{\sim}{\mu}^T\underset{\sim}{\mu}.$$

Non-central χ^2.

- If $\underset{\sim}{X} \sim N(\underset{\sim}{\mu}, I_n)$. $\quad \underset{\sim}{x}^T\underset{\sim}{x} \sim \chi^2_{n,\, ncp = \underset{\sim}{\mu}^T\underset{\sim}{\mu}}$.

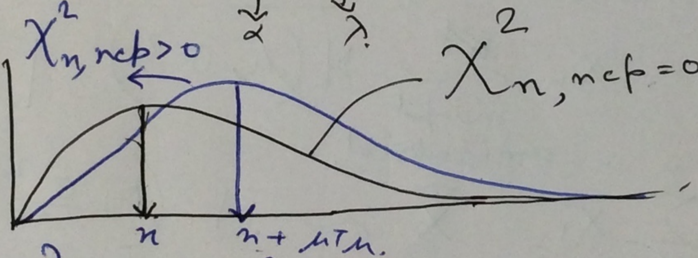
$$E(\underset{\sim}{x}^T\underset{\sim}{x}) = n + \underset{\sim}{\mu}^T\underset{\sim}{\mu}. = n + \sum_{i=1}^{n} \mu_i^2 = E\left(\sum x_i^2\right) = \sum_{i=1}^{n}(1+\mu_i^2)$$

- If $\underset{\sim}{x} \sim N(\underset{\sim}{0}, I_n)$. $\Rightarrow x_1, x_2 \cdots x_n$ iid $N(0,1)$.

$$E(\underset{\sim}{x}^T\underset{\sim}{x}) = n + 0 = n. = E(\chi^2_n)$$

If $\underset{\sim}{X} \sim N(\underset{\sim}{0}, I_n)$ then $\underset{\sim}{x}^T\underset{\sim}{x} \sim \chi^2_n$. $\rightarrow$ (central χ^2)

$$\chi^2_n \equiv G\left(\underbrace{\tfrac{n}{2}}_{\alpha}, \underbrace{\tfrac{1}{2}}_{\lambda}\right). \qquad E(G(\alpha,\lambda)) = \frac{\alpha}{\lambda}.$$



$\sum x_i^2 = \underset{\sim}{x}^T\underset{\sim}{x}$.

$$E\left(\chi^2_{n,\, ncp=\mu^T\mu}\right) = n + \underset{\sim}{\mu}^T\underset{\sim}{\mu}.$$

$$E\left(\chi^2_{n,\, ncp=0}\right) = n + 0.$$

- $E(\underset{\sim}{x}^T A \underset{\sim}{x}) = tr(A\Sigma) + \underset{\sim}{\mu}^T A \underset{\sim}{\mu}$

proof: $E(\underset{\sim}{x}^T A \underset{\sim}{x}) = E\left[tr(\underset{\sim}{x}^T A \underset{\sim}{x})\right]$

$= E\ tr(A\underset{\sim}{x}\underset{\sim}{x}^T) = tr\left[A\, E(xx^T)\right]$

$= tr\left[A\,(\Sigma + \underset{\sim}{\mu}\underset{\sim}{\mu}^T)\right]$

$= tr\left[A\Sigma + A\underset{\sim}{\mu}\underset{\sim}{\mu}^T\right]$

$= tr(A\Sigma) + tr(A\underset{\sim}{\mu}\underset{\sim}{\mu}^T)$

$= tr(A\Sigma) + \underset{\sim}{\mu}^T A \underset{\sim}{\mu}.$

$tr(AB) = tr(BA)$.
$E(\underset{\sim}{x}) = \mu$
$D(\underset{\sim}{x}) = \Sigma$.
$E(xx^T) = \Sigma + \underset{\sim}{\mu}\underset{\sim}{\mu}^T$
$tr(A+B) = tr(A) + tr(B)$

$D(x) = \Sigma = E(\underset{\sim}{x}x^T) - \underset{\sim}{\mu}\underset{\sim}{\mu}^T$

- If $\underset{\sim}{X} \sim N(\underset{\sim}{\mu}, I_n)$ then $\underset{\sim}{X}^T A \underset{\sim}{X}$ follows a Chi-squared distribution iff A is an idempotent matrix.

- If A_1 & A_2 are square symmetric idempotent matrices and $\wp = A_1 - A_2$ is p.s.d. then $\underset{\sim}{X}^T \wp \underset{\sim}{X}$ and $\cancel{X^T A_2 X}$ $\underset{\sim}{X}^T A_2 \underset{\sim}{X}$ are independently distributed.

* Let $\underset{\sim}{X} \sim N(\underset{\sim}{\mu}, I_n)$ A is symmetric and $CA = \underset{\text{matrix}}{O}$ then $\underset{\sim}{X}^T A \underset{\sim}{X}$ and $\underset{\sim}{C}\,\underset{\sim}{X}$ are independently distributed.

⊛ Construction of T-statistic.

$$x_1, x_2 \cdots x_n \underset{\text{indept and identical.}}{\overset{\cancel{iid}}{\sim}} N(\mu, \sigma^2) \quad [\,iid\,]$$

then $\frac{1}{n}\sum_{i=1}^{n} x_i = \bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

$\sum_{i=1}^{n} (x_i - \bar{x})^2 = S^2 \sim \sigma^2 \chi^2_{n-1}$

indept.

Define

$$\Rightarrow T = \frac{(\bar{x} - \mu)/(\sigma/\sqrt{n})}{\sqrt{\dfrac{S^2}{\sigma^2 (n-1)}}} = \frac{\sqrt{n}(\bar{x} - \mu)}{\sqrt{\sum_{i=1}^{n} (x_i - \bar{x})^2/(n-1)}}$$

$T \sim t_{n-1}$, and $T^2 \sim F_{1,(n-1)}$

$$\bar{X} = \frac{1}{n} \underset{\sim}{1}^T \underset{\sim}{X} = \left(\tfrac{1}{n}\ \tfrac{1}{n} \cdots \tfrac{1}{n}\right)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underset{\sim}{l}^T \underset{\sim}{X}.$$

where $l^T = \left(\frac{1}{n}, \frac{1}{n} \cdots \frac{1}{n}\right)$.

$\underset{\sim}{X} \sim N\left(\underset{\sim}{\mu}, I_n \sigma^2\right)$ where $\underset{\sim}{\mu} = \mu \cdot \underset{\sim}{1} = \begin{pmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{pmatrix}$.

$$\Rightarrow \underset{\sim}{l}^T \underset{\sim}{X} \sim N\left(\underset{\sim}{l}^T \underset{\sim}{\mu}, \underset{\sim}{l}^T \Sigma \underset{\sim}{l}\right) \equiv N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$S^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 = \underset{\sim}{X}^T \left(I_n - \frac{1}{n} \underset{\sim}{1}\underset{\sim}{1}^T\right) \underset{\sim}{X}.$$

$A = \left(I_n - \frac{1}{n} \underset{\sim}{1}\underset{\sim}{1}^T\right)$ hence $A^T = A$, $A^2 = A$.
Symmetric Idempotent.

$$\underset{\sim}{Y} = \frac{\underset{\sim}{X}}{\sigma} \sim N\left(\frac{\underset{\sim}{\mu}}{\sigma}, I_n\right)$$

$$S^2 = \underset{\sim}{X}^T A \underset{\sim}{X}.$$
$$= \sigma^2 \left(\frac{\underset{\sim}{X}^T}{\sigma}\right) A \left(\frac{\underset{\sim}{X}}{\sigma}\right) = \sigma^2 \underset{\sim}{Y}^T A \underset{\sim}{Y}$$

$$S^2 \sim \sigma^2 \chi^2_{\text{rank}(A),\ \left(\frac{\underset{\sim}{\mu}}{\sigma}\right)^T A \left(\frac{\underset{\sim}{\mu}}{\sigma}\right)}.$$

Note $\text{rank}(A) = \text{trace}(A) = n-1$

$$\frac{1}{\sigma^2}\left(\underset{\sim}{\mu}^T A \underset{\sim}{\mu}\right) = \mu \cdot \left[\underset{\sim}{1}^T \left(I_n - \frac{1}{n} \underset{\sim}{1}\underset{\sim}{1}^T\right) \underset{\sim}{1}\right] \cdot \mu = \mu^2 \cdot 0 = \underline{\underline{0}}.$$

$$\Rightarrow S^2 \sim \sigma^2 \chi^2_{n-1}.$$

$$\underset{\sim}{l}^T A = \frac{1}{n} \underset{\sim}{1}^T \left(I_n - \frac{1}{n} \underset{\sim}{1}\underset{\sim}{1}^T\right)$$
$$= \frac{1}{n}\left(\underset{\sim}{1}^T - \frac{1}{n}\left(\underset{\sim}{1}^T \underset{\sim}{1}\right) \underset{\sim}{1}^T\right)$$
$$= \frac{1}{n}\left(\underset{\sim}{1}^T - \underset{\sim}{1}^T\right) = \underset{\sim}{0}^T.$$

$\bar{X} = \underset{\sim}{l}^T \underset{\sim}{X}$, $S^2 = \underset{\sim}{X}^T A \underset{\sim}{X}$

$$\left\{ \begin{array}{l} A^T = A. \\ A^2 = A. \end{array} \right.$$

$\bar{X}$ and S^2 are independently distributed.

$$\left.\begin{array}{l} \bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \Rightarrow \sqrt{n}\left(\frac{\bar{X} - \mu}{\sigma}\right) \sim N(0,1). \\ \text{and } \frac{S^2}{\sigma^2} \sim \chi^2_{n-1} \end{array}\right\} \text{independent.}$$

$$\Sigma = \begin{bmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{bmatrix} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{bmatrix}$$

$$\underset{\sim}{x}^T A \underset{\sim}{x} = \sum_i^n \sum_j^n x_i x_j a_{ij}$$

$$|\Sigma| = \sigma_x^2 \sigma_y^2 (1-\rho^2)$$

$$\Sigma^{-1} = \begin{bmatrix} \sigma_x^{-1} & 0 \\ 0 & \sigma_y^{-1} \end{bmatrix} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \begin{bmatrix} \sigma_x^{-1} & 0 \\ 0 & \sigma_y^{-1} \end{bmatrix} \frac{1}{(1-\rho^2)}$$

$$g\begin{pmatrix} x \\ y \end{pmatrix} = \frac{e^{-\frac{1}{2}\begin{pmatrix} x-\mu_x \\ y-\mu_y \end{pmatrix}^T \Sigma^{-1} \begin{pmatrix} x-\mu_x \\ y-\mu_y \end{pmatrix}}}{2\pi \cdot \sqrt{|\Sigma|}}$$

$$= \frac{e^{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right\}}}{2\pi\, \sigma_x \sigma_y \sqrt{1-\rho^2}}$$

Bivariate Normal distribution.

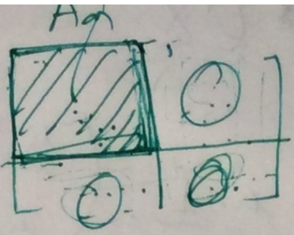
$$X \sim N(\mu_x, \sigma_x^2)$$
$$Y \sim N(\mu_y, \sigma_y^2)$$

$$X = (1\ 0)\begin{pmatrix} X \\ Y \end{pmatrix} \equiv \underset{\sim}{l}^T \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N(\underset{\sim}{\mu}, \Sigma)$$

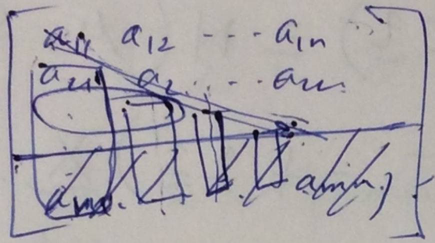
$$X = \underset{\sim}{l}^T \begin{pmatrix} X \\ Y \end{pmatrix} \sim N(l^T \underset{\sim}{\mu}, l^T \Sigma l)$$

$$= N\left((1\ 0)\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, (1\ 0)\begin{bmatrix} \sigma_x^2 & \sigma_x\sigma_y\rho \\ \sigma_x\sigma_y\rho & \sigma_y^2 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$



$$\rho\left(\underset{m\times n}{A}\right)=\rho\left(\underbrace{A A^{T}}_{\underbrace{m\times n \times n\times m}_{m\times m}}\right)$$

$A^{T}A$
$n\times n$



$$\underset{\sim}{X} \sim N(\underset{\sim}{\mu}, \Sigma) \qquad \underset{\sim}{x} \in \mathbb{R}^{n}$$

$$f(\underset{\sim}{x}) = \frac{e^{-\frac{1}{2}(\underset{\sim}{x}-\underset{\sim}{\mu})^{T}\Sigma^{-1}(\underset{\sim}{x}-\underset{\sim}{\mu})}}{(\sqrt{2\pi})^{n}\sqrt{|\Sigma|}} \qquad \begin{cases} |\Sigma|>0 \\ \Sigma^{T}=\Sigma \end{cases}$$

Get the pdf of bivariate normal distribution.

$$(X, Y) \sim N(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$$

Here X and Y both are random/stochastic.

~~$g(x,y)=$~~

$$E\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix} = \underset{\sim}{\mu}$$

$$D\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{bmatrix} \sigma_x^2 & \sigma_x\sigma_y\rho \\ \sigma_x\sigma_y\rho & \sigma_y^2 \end{bmatrix} = \Sigma$$

$$\Sigma = \begin{bmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{bmatrix} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} \sigma_x & 0 \\ 0 & \sigma_y \end{bmatrix}$$

$$\underset{\sim}{x}^T A \underset{\sim}{x} = \sum_i^n \sum_j^n x_i x_j a_{ij}$$

$$|\Sigma| = \sigma_x^2 \sigma_y^2 (1-\rho^2)$$

$$\Sigma^{-1} = \begin{bmatrix} \sigma_x^{-1} & 0 \\ 0 & \sigma_y^{-1} \end{bmatrix} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \begin{bmatrix} \sigma_x^{-1} & 0 \\ 0 & \sigma_y^{-1} \end{bmatrix} \frac{1}{(1-\rho^2)}$$

$$g\begin{pmatrix} x \\ y \end{pmatrix} = \frac{e^{-\frac{1}{2}\begin{pmatrix} x-\mu_x \\ y-\mu_y \end{pmatrix}^T \Sigma^{-1} \begin{pmatrix} x-\mu_x \\ y-\mu_y \end{pmatrix}}}{2\pi \cdot \sqrt{|\Sigma|}}$$

$$= \frac{e^{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right\}}}{2\pi\, \sigma_x \sigma_y \sqrt{1-\rho^2}}$$

Bivariate Normal distribution.

$$X \sim N(\mu_x, \sigma_x^2)$$
$$Y \sim N(\mu_y, \sigma_y^2)$$

$$X = (1\ 0)\begin{pmatrix} X \\ Y \end{pmatrix} \equiv \underset{\sim}{l}^T \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N(\underset{\sim}{\mu}, \Sigma)$$

$$X = \underset{\sim}{l}^T \begin{pmatrix} X \\ Y \end{pmatrix} \sim N(l^T \underset{\sim}{\mu}, l^T \Sigma l)$$

$$= N\left((1\ 0)\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, (1\ 0)\begin{bmatrix} \sigma_x^2 & \sigma_x\sigma_y\rho \\ \sigma_x\sigma_y\rho & \sigma_y^2 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$