Lecture 5

Theorem (Gram-Schmidt orthonormalization):—

Let $\{\underline{x_1}, \cdots, \underline{x_m}\}$ be a l.i set in an i.p.s $(V, \langle , \rangle)$. Then there exists an orthonormal set of vectors $\{\underline{v_1}, \cdots, \underline{v_m}\}$ in V associated to $\{\underline{x_1}, \cdots, \underline{x_m}\}$.

proof:—

Given $\{\underline{x_1}, \cdots, \underline{x_m}\}$ is l.i.

Let $\underline{y_1} = \underline{x_1} \neq \underline{0}$

Let $\underline{y_2} = \underline{x_2} + a_{21}\underline{y_1}$ for some scalars $a_{21} \in F$.

such that $\langle \underline{y_2}, \underline{y_1} \rangle = 0$.

$$\langle \underline{y_2}, \underline{y_1} \rangle = 0 \Rightarrow \langle \underline{x_2} + a_{21}\underline{y_1}, \underline{y_1} \rangle = 0$$

$$\Rightarrow \langle \underline{x_2}, \underline{y_1} \rangle + \langle a_{21}\underline{y_1}, \underline{y_1} \rangle = 0$$

$$\Rightarrow \langle \underline{x_2}, \underline{y_1} \rangle + a_{21}\langle \underline{y_1}, \underline{y_1} \rangle = 0$$

$$\Rightarrow \boxed{a_{21} = -\frac{\langle \underline{x_2}, \underline{y_1} \rangle}{\langle \underline{y_1}, \underline{y_1} \rangle}}$$

$$\therefore \boxed{\underline{y_2} = \underline{x_2} - \frac{\langle \underline{x_2}, \underline{y_1} \rangle}{\langle \underline{y_1}, \underline{y_1} \rangle}\underline{y_1}}$$

Let $\underline{y}_3 = \underline{x}_3 + a_{32}\,\underline{y}_2 + a_{31}\,\underline{y}_1$

for some scalars $a_{32}, a_{31} \in F$.

such that $\langle \underline{y}_3, \underline{y}_2 \rangle = 0$ & $\langle \underline{y}_3, \underline{y}_1 \rangle = 0$.

$\langle \underline{y}_3, \underline{y}_2 \rangle = 0 \Rightarrow \langle \underline{x}_3 + a_{32}\underline{y}_2 + a_{31}\underline{y}_1, \underline{y}_2 \rangle = 0$.

$\Rightarrow \langle \underline{x}_3, \underline{y}_2 \rangle + a_{32}\langle \underline{y}_2, \underline{y}_2 \rangle + a_{31}\underbrace{\langle \underline{y}_1, \underline{y}_2 \rangle}_{=0} = 0$

$$\Rightarrow \boxed{a_{32} = -\frac{\langle \underline{x}_3, \underline{y}_2 \rangle}{\langle \underline{y}_2, \underline{y}_2 \rangle}}.$$

Now

$\langle \underline{y}_3, \underline{y}_1 \rangle = 0 \Rightarrow \langle \underline{x}_3 + a_{32}\underline{y}_2 + a_{31}\underline{y}_1, \underline{y}_1 \rangle = 0$

$\Rightarrow \langle \underline{x}_3, \underline{y}_1 \rangle + a_{32}\underbrace{\langle \underline{y}_2, \underline{y}_1 \rangle}_{=0} + a_{31}\langle \underline{y}_1, \underline{y}_1 \rangle = 0$.

$$\Rightarrow \boxed{a_{31} = -\frac{\langle \underline{x}_3, \underline{y}_1 \rangle}{\langle \underline{y}_1, \underline{y}_1 \rangle}}.$$

$\therefore \underline{y}_3 = \underline{x}_3 - \frac{\langle \underline{x}_3, \underline{y}_1 \rangle}{\ldots}$

$$\langle y_1, y_1\rangle \qquad y_1 - \frac{\langle x_3, y_2\rangle}{\langle y_2, y_2\rangle}\, y_2 .$$

In this fashion, if we continue, we get,

$$\boxed{y_j = x_j - \frac{\langle x_j, y_1\rangle}{\langle y_1, y_1\rangle} y_1 - \frac{\langle x_j, y_2\rangle}{\langle y_2, y_2\rangle} y_2 - \cdots - \frac{\langle x_j, y_{j-1}\rangle}{\langle y_{j-1}, y_{j-1}\rangle} y_{j-1}}$$

$\forall\ j = 2, 3, \ldots, m.$

Notice that $\{y_1, \ldots, y_m\}$ is an orthogonal set.

Let $v_1 = \dfrac{y_1}{\|y_1\|}, \ldots, v_m = \dfrac{y_m}{\|y_m\|}$

Then $\{v_1, \ldots, v_m\}$ is an orthonormal set.

This process is known as the Gram-Schmidt orthonormalization process.

Problem (1) :- Using Gram-Schmidt orthonormaliz. process, find an orthonormal set for the l.i set $x_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 0 \\ 1 \\ \ldots \end{pmatrix}, x_3 = \begin{pmatrix} 1 \\ 0 \\ \ldots \end{pmatrix}$

Sol:

Let $\underline{y}_1 = \underline{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

$$\underline{y}_2 = \underline{x}_2 - \frac{\langle \underline{x}_2, \underline{y}_1 \rangle}{\langle \underline{y}_1, \underline{y}_1 \rangle} \underline{y}_1 .$$

$$\langle \underline{x}_2, \underline{y}_1 \rangle = 0(1) + 1(1) + 1(0) = 1$$

$$\langle \underline{y}_1, \underline{y}_1 \rangle = 1(1) + 1(1) + 0(0) = 2.$$

$$\therefore \quad \underline{y}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 \\ 1/2 \\ 1 \end{pmatrix}$$

$$\underline{y}_3 = \underline{x}_3 - \frac{\langle \underline{x}_3, \underline{y}_1 \rangle}{\langle \underline{y}_1, \underline{y}_1 \rangle} \underline{y}_1 - \frac{\langle \underline{x}_3, \underline{y}_2 \rangle}{\langle \underline{y}_2, \underline{y}_2 \rangle} \underline{y}_2 .$$

$$\langle \underline{x}_3, \underline{y}_1 \rangle = 1(1) + 0(1) + 1(0) = 1$$

$$\langle \underline{y}_1, \underline{y}_1 \rangle = 2$$

$$\langle \underline{x}_3, \underline{y}_2 \rangle = 1\left(-\frac{1}{2}\right) + 0\left(\frac{1}{2}\right) + 1(1) = \frac{1}{2}.$$

$$\langle \underline{y}_2, \underline{y}_2 \rangle = \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + 1(1)$$

$$= \frac{1}{2} + 1 = 3/2$$

$\frac{\ }{2} \quad\quad /2.$

$$\therefore \quad \underline{y_3} = \begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} - \frac{\left(\frac{1}{2}\right)}{\left(3/2\right)}\begin{pmatrix}-1/2\\1/2\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} - \frac{1}{3}\begin{pmatrix}-1/2\\1/2\\1\end{pmatrix}$$

$\frac{1}{2}+\frac{1}{6}$

$$= \begin{pmatrix}1-\frac{1}{2}+\frac{1}{6}\\0-\frac{1}{2}-\frac{1}{6}\\1-0-1/3\end{pmatrix} = \begin{pmatrix}2/3\\-2/3\\2/3\end{pmatrix}$$

$\{\underline{y_1}, \underline{y_2}, \underline{y_3}\}$ is an orthogonal set.

$$\text{Let } \underline{v_1} = \frac{\underline{y_1}}{\|\underline{y_1}\|} = \frac{\underline{y_1}}{\sqrt{\langle \underline{y_1}, \underline{y_1}\rangle}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$\underline{v_2} = \frac{\underline{y_2}}{\|\underline{y_2}\|} = \frac{\sqrt{2}}{\sqrt{3}}\begin{pmatrix}-1/2\\1/2\\1\end{pmatrix} = \begin{pmatrix}-1/\sqrt{6}\\1/\sqrt{6}\\\sqrt{2}/\sqrt{3}\end{pmatrix}$$

$$\underline{v_3} = \frac{\underline{y_3}}{\|\underline{y_3}\|} = \frac{\sqrt{3}}{2}\begin{pmatrix}2/3\\-2/3\\2/3\end{pmatrix} = \begin{pmatrix}1/\sqrt{3}\\-1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}$$

$$\langle \underline{y_3}, \underline{y_3}\rangle = \frac{4}{9}+\frac{4}{9}+\frac{4}{9} = \frac{4}{3}$$

$\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ is the required O.N set.

Eigenvalues & Eigenvectors.

Let A be an $n \times n$ matrix with entries in $F = \mathbb{R}$ or $\mathbb{C}$.

Def:- A scalar $\lambda \in F$ is called an *eigenvalue* of $A_{n \times n}$, if there exists a non-zero vector $\underline{v} \in F^n$ (i.e. $\mathbb{R}^n$ or $\mathbb{C}^n$) such that

$$A\underline{v} = \lambda \underline{v}.$$

$\underline{v}$ is called an *eigenvector* of A corresponding to the eigenvalue λ.
($a\underline{v}$ is also an *eigenvector* $\forall\, a \in F$, & $a \neq 0$).

Def:- The *eigenspace* of $A_{n \times n}$ corresponding to the eigenvalue λ of A, is defined as

$$E_\lambda(A) := \left\{ \underline{v} \in F^n \;\middle|\; A\underline{v} = \lambda \underline{v} \right\}$$

Check that $E_\lambda(A)$ is a subspace of F^n.

The $\dim(E_\lambda(A))$ is called the *geometric multiplicity* of A correspond to λ. (g.m).

Remarks

$A\underline{v} = \lambda \underline{v}$, $\underline{v} \neq \underline{0}$

$\Rightarrow (A - \lambda I)\underline{v} = \underline{0}$, $\underline{v} \neq \underline{0}$

$\Rightarrow \boxed{\det(A - \lambda I) = 0.}$

Def: The polynomial $p(\lambda) = \det(\lambda I - A)$ is called the <u>characteristic polynomial</u> of $A_{n\times n}$.

Eigenvalues of A are the roots of the char. poly.

Let the characteristic poly. of A

$$p(\lambda) = \det(\lambda I - A)$$

$$= (\lambda - \lambda_1)^{r_1} \cdots (\lambda - \lambda_s)^{r_s} \longrightarrow (*)$$

where $\lambda_1, \dots, \lambda_s$ are the distinct eigenvalues of A & $r_1 \geq 1, \dots, r_s \geq 1$.

Def: r_i in the above equation $(*)$ is called the <u>algebraic multiplicity</u> (a.m.) <u>corresponding to the eigenvalue</u> λ_i of A.

Def: Two matrices $A_{n\times n}$ & $B_{n\times n}$ are said to be similar

_______ if there exists an invertible matrix $P_{n\times n}$ such that $P^{-1}AP = B$.

Theorem: Similar matrices have the same char. poly & hence same eigenvalues

proof: Assume A is similar to B.

i.e. $A = P^{-1}BP$ for some invertible matrix P.

$$
\begin{aligned}
\det(\lambda I - A) &= \det(\lambda I - P^{-1}BP) \\
&= \det(\lambda P^{-1}IP - P^{-1}BP) \\
&= \det\left(P^{-1}(\lambda I - B)P\right) \\
&= \det(P^{-1})\det(\lambda I - B)\det(P) \\
&= \det(\lambda I - B)
\end{aligned}
$$