Lecture 11

Example:-

(1) $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

eigenvalues of A are $1, 2$.

Check that A is diagonizable.

$$AA^* = AA^t = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$$

$$A^*A = A^tA = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}$$

A is not normal.

Bilinear forms, Quadratic forms.

Def:- Let V, W be vector spaces over a field $F = \mathbb{R}$ or $\mathbb{C}$.

Let $m = \dim(W)$

$n = \dim(V)$.

For simplicity, let us take

$V = \mathbb{R}^n$, $W = \mathbb{R}^m$.

Then an expression of the form

$$\boxed{b(\underline{x}, \underline{y}) := \underline{x}^t A \underline{y} = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} x_i y_j}$$

where $A = [a_{ij}]_{n \times m}$

is called a <u>bilinear form</u> over $\mathbb{R}$.

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \underline{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}.$$

x_i, y_j are Variables.

<u>Examples:</u> (1) $b(\underline{x}, \underline{y}) = [x_1 \; x_2] \begin{bmatrix} 4 & 8 & 3 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

$$b(\underline{x}, \underline{y}) = [x_1 \; x_2] \begin{bmatrix} 4y_1 + 8y_2 + 3y_3 \\ -2y_1 + y_3 \end{bmatrix}$$

$$= 4x_1y_1 + 8x_1y_2 + 3x_1y_3 - 2x_2y_1 + x_2y_3$$

is a bilinear form.

(2) $b(\underline{x}, \underline{y}) = 2x_1y_1 - x_1y_2 + 2x_2y_1 - 4x_2y_2$

$$= [x_1 \; x_2] \begin{bmatrix} 2 & -1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

is a bilinear form.

More generally

In general,

<u>Def</u>:- A map $f: V \times W \to F$ is said to be a <u>bilinear map</u> or <u>bilinear function</u> if f satisfies the following conditions:

(i) $f(\underline{v_1} + \alpha \underline{v_2}, \underline{w}) = f(\underline{v_1}, \underline{w}) + \alpha f(\underline{v_2}, \underline{w})$

$\forall\, \underline{v_1}, \underline{v_2} \in V,\ \underline{w} \in W.$

$\forall\, \alpha \in F.$

(ii) $f(\underline{v}, \underline{w_1} + \alpha \underline{w_2}) = f(\underline{v}, \underline{w_1}) + \alpha f(\underline{v}, \underline{w_2})$

$\forall\, \underline{v} \in V,\ \forall\, \underline{w_1}, \underline{w_2} \in W,$

$\forall\, \alpha \in F.$

<u>Examples</u>:-

① $f: \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}$, $f(\underline{x}, \underline{y}) = \underline{x}^t A \underline{y}$

where A is an $n \times m$ matrix, fixed

$\forall\, \underline{x} \in \mathbb{R}^n,\ \forall\, \underline{y} \in \mathbb{R}^m.$

f is a bilinear map.

- Any bilinear form is a bilinear map.

② Let $g: \mathbb{P}_2(x) \times \mathbb{P}_2(x) \longrightarrow \mathbb{R}$

$$g\left(a_0+a_1x+a_2x^2,\ b_0+b_1x+b_2x^2\right)=a_0b_0+a_1b_1+a_2b_2$$

Then g is a bilinear map.

(3) $f: M_{m\times n}(\mathbb{R}) \times M_{m\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$. Let us fix $A_{m\times m}$

defined as $f(X,Y) = \operatorname{trace}(X^t A Y)$

$\forall\ X, Y \in M_{m\times n}(\mathbb{R})$.

Check that f is a bilinear map.

Defn A bilinear map $f: V\times V \to F$, is said to be symmetric, if

$$f(\underline{x},\underline{y}) = f(\underline{y},\underline{x}) \quad \forall\ \underline{x},\underline{y}\in V.$$

Examples:- (1) Let $f: \mathbb{R}^2\times\mathbb{R}^2 \longrightarrow \mathbb{R}$

$$f(\underline{x},\underline{y}) = x_1y_1 - 2x_1y_2 - 2x_2y_1 + 3x_2y_2.$$

$\forall\ \underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ \underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ in $\mathbb{R}^2$.

$$= \underline{x}^t \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix} \underline{y}.$$

Also f is a bilinear map.

$f(\underline{y}, \underline{x}) = f(\underline{x}, \underline{y})$; Thus f is a symmetric bilinear map.

Definition:- A bilinear form $\underline{x}^t A \underline{y}$ is said to be equivalent to $\underline{u}^t B \underline{v}$, where $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$, $\underline{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$, $\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}$ & A, B are $n \times m$ matrices, if there exists non-singular matrices, $C_{m \times m}$, $D_{n \times n}$ such that $B = D^t A C$.

Def:- Let $f : V \times W \to F$ be a bilinear map. Let $B = \{\underline{v}_1, \dots, \underline{v}_n\}$, $B' = \{\underline{w}_1, \dots, \underline{w}_m\}$ be bases of V & W respectively. Then the matrix representation of f w.r.t the bases B, B' is defined as

$$\big[\underbrace{f(\underline{v}_i, \underline{w}_j)}_{(i,j)^{th} \text{ entry}}\big]_{n \times m}.$$

Examples:- (1) Let $f : \mathbb{R}^3 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$,

$$f(\underline{x}, \underline{y}) = 2x_1y_1 - x_1y_2 + 3x_2y_1 + x_2y_2 - 2x_3y_2.$$

- The matrix repr. of f w.r.t std bases of $\mathbb{R}^3, \mathbb{R}^2$ is

$$\left\{ \underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \underline{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\& \left\{ \underline{e}_1' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \underline{e}_2' = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

$$f(\underline{e}_1, \underline{e}_1') = 2 \quad \Big| \quad f(\underline{e}_2, \underline{e}_1') = 3 \quad \Big| \quad f(\underline{e}_3, \underline{e}_1') = 0$$

$$f(\underline{e}_1, \underline{e}_2') = -1 \quad \Big| \quad f(\underline{e}_2, \underline{e}_2') = 1 \quad \Big| \quad f(\underline{e}_3, \underline{e}_2') = -2.$$

$$\therefore \text{ matrix repr.} = \begin{bmatrix} 2 & -1 \\ 3 & 1 \\ 0 & -2 \end{bmatrix}_{3\times 2}.$$

- Let $\mathcal{B} = \left\{ \underset{= \underline{x}_1}{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}, \underset{= \underline{x}_2}{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}, \underset{= \underline{x}_3}{\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}} \right\} \subseteq \mathbb{R}^3$ basis

$\mathcal{B}' = \left\{ \underset{= \underline{y}_1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}, \underset{= \underline{y}_2}{\begin{pmatrix} -1 \\ 2 \end{pmatrix}} \right\}$ basis for $\mathbb{R}^2$.

$$P = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ -2 & 1 & -1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

$$f(\underline{x}_1, \underline{y}_1) = -5 \quad \Big| \quad f(\underline{x}_2, \underline{y}_1) = 2 \quad \Big| \quad f(\underline{x}_3, \underline{y}_1) = 4$$

$$f(\underline{x}_1, \underline{y}_2) = -7 \quad \Big| \quad f(\underline{x}_2, \underline{y}_2) = -5 \quad \Big| \quad f(\underline{x}_3, \underline{y}_2) = 11.$$

$\therefore$ The matrix repr. of f w.r.t $\mathcal{B}, \mathcal{B}'$

is $\begin{bmatrix} -5 & -7 \\ 2 & -5 \\ 4 & 11 \end{bmatrix}$.

<u>Theorem</u>: Let $f : V \times W \to F$ be a bilinear map. Let $A_{n \times m}$ be the matrix repr of f w.r.t bases $X = \{\underline{x}_1, \ldots, \underline{x}_n\}$, $Y = \{\underline{y}_1, \ldots, \underline{y}_m\}$ for V, W respectively. And let X', Y' be new bases for V & W respectively. Then the matrix representation of f w.r.t the bases X', Y' is $P^t A Q$,

where $P =$ the Co-ordinate transformation matrix of V $(X \to X')$

$Q =$ " " " W. $(Y \to Y')$.

<u>Defn</u>: Let $x_1, \ldots, x_n$ be variables.

A <u>quadratic form</u> Q on $\mathbb{R}^n$

$$Q(\underline{x}) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} x_i x_j$$

$= \underline{x}^t A \underline{x}$, where $A = [a_{ij}]_{n \times n}$.

where $a_{ij} \in \mathbb{R}$

Remark 1) if b is a bilinear form on $\mathbb{R}^n$

$b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ bilinear map.

Then define $Q(\underline{x}) = b(\underline{x}, \underline{x})$, $\forall \underline{x} \in \mathbb{R}^n$.

Then Q is a quadratic form on $\mathbb{R}^n$.

Examples:-

(1) $Q(x_1, x_2) = 2x_1^2 - 3x_1 x_2 + x_2^2$. not symmetric

$$= [x_1, x_2] \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [x_1, x_2] \begin{bmatrix} 2 & -3/2 \\ -3/2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Symmetric.

If $Q(\underline{x})$ is a quadratic form, on $\mathbb{R}^n$,

then $Q(\underline{x}) = \underline{x}^t A \underline{x}$, for some symmetric matrix $A_{n \times n}$.