Exercises

Level - 1

(Problems Based on Fundamentals)

1. What conic does $\sqrt{ax}+\sqrt{by}=1$ represent ?

2. If the conic $x^2 - 4xy + ly^2 + 2x + 4y + 10 = 0$ represents a parabola, find the value of λ.

3. If the conic $16(x^2 + (y - 1)^2 = (x + \sqrt{3} - 5)^2$ represents a non-degenerate conic, write its name and also find its eccentricity.

4. If the focus and the directirix of a conic be (1, 2) and $x + 3y + 10 = 0$ respectively and the eccentricity be $\frac{1}{\sqrt{2}}$, then find its equation.

5. Find the equation of a parabola, whose focus is (1, 1) and the directrix is $x - y + 3 = 0$.

ABC OF PARABOLA

6. Find the vertex, the focus, the latus rectum, the directrix and the axis of the parabolas

(i) $y^2 = x + 2y + 2$

(ii) $y^2 = 3x + 4y + 2$

(iii) $x^2 = y + 4x + 2$

(iv) $x^2 + x + y = 0$

7. If the focal distance on a point to a parabola $y^2 = 12x$ is 6, find the co-ordinates of that point.

8. Find the equation of a parabola, whose focus (–6, –6) and the vertex is (–2, –2).

9. The parametric equation of a parabola is $x = t^2 + 1$ and $y = 2t + 1$. Find its directrix.

10. If the vertex of a parabola be (–3, 0) and the directrix is $x + 5 = 0$, find its equation.

11. Find the equation of the parabola whose axis is parallel to y-axis and which passes through the points (0, 2), (–1, 0) and (1, 6).

12. Find the equation of a parabola whose vertex is (1, 2) and the axis is parallel to x-axis and also passes through the point (3, 4).

13. If the axis of a parabola is parallel to y-axis, the vertex and the length of the latus rectum are (3, 2) and 12 respectively, find its equation.

PROPERTIES OF THE FOCAL CHORD

14. If the chord joining $P(at_1^2, 2at_1)$ and is the focal chord, prove that $t_1t_2 = 1$.

15. If the point $(at^2, 2at)$ be the extremity of a focal chord of the parabola $y^2 = 4ax$, prove that the length of the focal chord is $a\left(t+\frac{1}{t}\right)^2$.

16. If the length of the focal chord makes an angle θ with the positive direction of x-axis, prove that its length is $4a\operatorname{cosec}^2\theta$.

17. Prove that the semi-latus rectum of a parabola $y^2 = 4ax$ is the harmonic mean between the segments of any focal chord of the parabola.

18. Prove that the length of a focal chord of the parabola varies inversely as the square of its distance from the vertex.

19. Prove that the circle described on the focal chord as the diameter touches the tangent to the parabola.

20. Prove that the circle described on the focal chord as the diameter touches the directrix of the parabola.

POSITION OF A POINT RELATIVE TO A PARABOLA

21. If a point $(\lambda, -\lambda)$ lies in an interior point of the parabola $y^2 = 4x$, find the range of λ.

22. If a point $(\lambda, 2)$ is an exterior point of both the parabolas $y^2 = (x + 1)$ and $y^2 = -x + 1$, find the value of λ.

INTERSECTION OF A LINE AND A PARABOLA

23. If $2x + 3y + = 0$ is a tangent to the parabola $y^2 = 8x$, find the co-ordinates of the point of contact.

24. If $3x + 4y + \lambda = 0$ is a tangent to the parabola $y^2 = 12x$, find the value of λ.
25. Find the length of the chord intercepted by the parabola $y^2 = 4ax$ and the line $y = mx + c$.

TANGENT AND TANGENCY

26. Find the point of intersection of tangents at $P(t_1)$ and $Q(t_2)$ on the parabola $y^2 = 4ax$.
27. Find the equation of tangent to the parabola $y^2 = 2x + 5y - 8$ at $x = 1$.
28. Find the equation of the tangent to the parabola $y^2 = 8x$ having slope 2 and also find its point of contact.
29. Two tangents are drawn from a point (–1, 2) to a parabola $y^2 = 4x$. Find the angle between the tangents.
30. Find the equation of the tangents to the parabola $y = x^2 - 3x + 2$ from the point (1, –1).
31. Find the equation of the common tangent to the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.
32. Find the equation of the common tangent to the parabola $y^2 = 4ax$ and $x^2 = 4by$.
33. Find the equation of the common tangent to the parabola $y^2 = 16x$ and the circle $x^2 + y^2 = 8$.
34. Find the equation of the common tangents to the parabolas $y = x^2$ and $y = -(x-2)^2$.
35. Find the equation of the common tangents to the curves $y^2 = 8x$ and $xy = -1$.
36. Find the equation of the common tangent to the circle $x^2 + y^2 - 6y + 4 = 0$ and the parabola $y^2 = x$.
37. Find the equation of the common tangent touching the circle $x^2 + (y-3)^2 = 9$ and the parabola $y^2 = 4x$ above the x-axis.
38. Find the points of intersection of the tangents at the ends of the latus rectum to the parabola $y^2 = 4x$.
39. Find the angle between the tangents drawn from a point (1, 4) to the parabola $y^2 = 4x$.
40. Find the shortest distance between the line $y = x - 2$ and the parabola $y = x^2 + 3x + 2$.
41. Find the shortest distance from the line $x + y = 4$ and the parabola $y^2 + 4x + 4y = 0$.
42. If $y + b = m_1(x + a)$ and $y + b = m_2(x + a)$ are two tangents of the parabola $y^2 = 4ax$, find the value of m_1m_2.
43. The tangent to the curve $y = x^2 + 6$ at a point (1, 7) touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at Q. Find the co-ordinates of Q.
44. Two straight lines are perpendicular to each other. One of them touches the parabola $y^2 = 4a(x + a)$ and the other touches $y^2 = 4b(x + b)$. Prove that the point of intersection of the lines lie on the line $x + a + b = 0$.
45. Prove that the area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.
46. Prove that the circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.
47. Prove that the orthocentre of any triangle formed by three tangents to a parabola lies on the directrix.
48. Prove that the equation of the director circle to the parabola $y^2 = 4ax$ is $x + a = 0$.
49. Find the equation of the director circle to the following parabolas:
    (i) $y^2 = x + 2$
    (ii) $x^2 = 4x + 4y$
    (iii) $y^2 = 4x + 4y - 8$

NORMAL AND NORMALCY

50. Find the point of intersection of normals at $P(t_1)$ and $Q(t_2)$ on the parabola $y^2 = 4ax$.
51 Find the relation between t_1 and t_2, where the normal at t_1 to the parabola $y^2 = 4ax$ meets the parabola $y^2 = 4ax$ again at t_2.
52. If the normal at t_1 meets the parabola again at t_2, prove that the minimum value of t_2^2 is 8.
53. If two normals at t_1 and t_2 meet again the parabola $y^2 = 4ax$ at t_3, prove that $t_1t_2 = 2$.
54. Find the equation of the normal to the parabola $y^2 = 4x$ at the point (1, 2).
55. Find the equation of the normal to the parabola $y^2 = 8x$ at $m = 2$.
56. If $x + y = k$ is a normal to the parabola $y^2 = 12x$, find the value of k.
57. If the normal at $P(18, 12)$ to the parabola $y^2 = 8x$ cuts it again at Q, prove that $9PQ = 80\sqrt{10}$.
58. Find the locus of the point of intersection of two normals to the parabola $y^2 = 4ax$, which are at right angles to one another.
59. If $lx + my + n = 0$ is a normal to the parabola $y^2 = 4ax$, prove that $al^3 + 2alm^2 + m^2n = 0$.
60. If a normal chord subtends a right angle at the vertex of the parabola $y^2 = 4ax$, prove that it is inclined at an angle of $\tan^{-1}(\sqrt{2})$ to the axis of the parabola.
61. At what point on the parabola $y^2 = 4x$, the normal makes equal angles with the axes?
62. Find the length of the normal chord which subtends an angle of 90° at the vertex of the parabola $y^2 = 4x$.
63. Prove that the normal chord of a parabola $y^2 = 4ax$ at the point (p, p) subtends a right angle at the focus.
64. Show that the locus of the mid-point of the portion of the normal to the parabola $y^2 = 4ax$ intercepted between the curve and the axis is another parabola.
65. Find the shortest distance between the curves $y^2 = 4x$ and $x^2 + y^2 - 12x + 31 = 0$.
66. Find the shortest distance between the curves $x^2 + y^2 + 12y + 35 = 0$ and $y^2 = 8x$.

CO-NORMAL POINT

67. Prove that the algebraic sum of the three concurrent normals to a parabola is zero.
68. Prove that the algebraic sum of the ordinates of the feet of three normals drawn to a parabola from a given point is also zero.

69. Prove that the centroid of the triangle formed by the feet of the three normals lies on the axis of the parabola. Also find the centroid of the triangle.
70. If three normals drawn from a given point (h, k) to any parabola be real, prove that $h > 2a$.
71. If three normals from a given point (h, k) to any parabola $y^2 = 4ax$ be real and distinct, prove that $27ak^2 < 4(h - 2a)^3$.
72. If a normal to a parabola $y^2 = 4ax$ makes an angle θ with the axis of the parabola, prove that it will cut the curve again at an angle of $\tan^{-1}\left(\dfrac{\tan\theta}{2}\right)$.
73. Prove that the normal chord to a parabola $y^2 = 4ax$ at the point whose ordinate is equal to its abscissa, which subtends a right angle at the focus of the parabola.
74. Prove that the normals at the end-points of the latus rectum of a parabola $y^2 = 4ax$ intersect at right angle on the axis of the parabola and their point of intersection is $(3a, 0)$.
75. If S be the focus of the parabola and the tangent and the normal at any point P meet the axes in T and G respectively, prove that $ST = SG = SP$.
76. From any point P on the parabola $y^2 = 4ax$, a perpendicular PN is drawn on the axis meeting at N, the normal at P meets the axis in G. Prove that the sub-normal NG is equal to its semi-latus rectum.
77. The normal to the parabola $y^2 = 4ax$ at a point P on it, meets the x-axis in G, prove that P and G are equidistant from the focus S of the parabola.
78. The normal at P to the parabola $y^2 = 4ax$ meets its axis at G. Q is another point on the parabola such that QG is perpendicular to the axis of the parabola. Prove that $QG^2 - PG^2 =$ constant.

CHORD OF CONTACT

79. Find the equation of the chord of contact to the tangents from the point $(2, 3)$ to the parabola $y^2 = 4x$.
80. Find the chord of contact of the tangents to the parabola $y^2 = 12x$ drawn through the point $(-1, 2)$.
81. Prove that the locus of the point of intersection of two tangents to a parabola $y^2 = 4ax$ which make a given angle θ with one another is $y^2 - 4ax = (x + a)^2 \tan^2\theta$.
82. Prove that the length of the chord of contact of tangents drawn from (h, k) to the parabola $y^2 = 4ax$ is $\dfrac{1}{a}|(k^2 + 4a^2)(k^2 - 4ah)|^{1/2}$.
83. Prove that the area of the triangle formed by the tangents from the point (h, k) to the parabola $y^2 = 4ax$ and a chord of contact is $\dfrac{(k^2 - 4ah)^{3/2}}{2a}$.

CHORD BISECTED AT A POINT

84. Find the equation of the chord of the parabola $y^2 = 8x$ which is bisected at $(2, 3)$.
85. Prove that the locus of the mid-points of the focal chord of the parabola is another parabola.
86. Prove that the locus of the mid-points of the chord of a parabola passes through the vertex is a parabola.
87. Prove that the locus of the mid-points of a normal chords of the parabola $y^2 = 4ax$ is $y^4 - 2a(x - 2a)y^2 + 8a^4 = 0$
88. Prove that the locus of the mid-point of a chord of a parabola $y^2 = 4ax$ which subtends a right angle at the vertex is $y^2 = 2a(x - 4a)$.
89. Prove that the locus of the mid-points of chords of the parabola $y^2 = 4ax$ which touches the parabola $y^2 = 4bx$ is $y^2(2a - b) = 4a^2x$.
90. Find the locus of the mid-point of the chord of the parabola $y^2 = 4ax$, which passes through the point $(3b, b)$.
91. Prove that the locus of the mid-points of all tangents drawn from points on the directrix to the parabola $y^2 = 4ax$ is $y^2(2x + a) = a(3x + a)^2$.

DIAMETER OF A PARABOLA

92. Prove that the tangent at the extremity of a diameter of a parabola is parallel to the system of chords it bisects.
93. Prove that tangents at the end of any chord meet on the diameter which bisects the chords.

REFLECTION PROPERTY OF A PARABOLA

94. A ray of light moving parallel to the x-axis gets reflected from a parabolic mirror whose equation is $(y - 4)^2 = 8(x + 1)$. After reflection, the ray passes through the point (α, β), find the value of $\alpha + \beta + 10$.
95. A ray of light is moving along the line $y = x + 2$, gets reflected from a parabolic mirror whose equation is $y^2 = 4(x + 2)$. After reflection, the ray does not pass through the focus of the parabola. Find the equation of the line which containing the reflected ray.

Hints & Solutions

3. The given conic is

$$16(x^2+(y-1)^2)=(x+\sqrt{3}y-5)^2$$

$$\Rightarrow \quad (x^2+(y-1)^2)=\frac{1}{4}\left(\frac{x+\sqrt{3}y-5}{\sqrt{1+3}}\right)^2 \qquad \ldots\text{(i)}$$

which represents an ellipse.
Now, Eq. (i) can also be written as

$$SP^2=e^2\times PM^2$$

Thus, the eccentricity is $\frac{1}{2}$.

4. Let S be the focus, PM be the directrix and the eccentricity $= e$
From the definition of conic section, we get

$$\frac{SP}{PM}=e$$

$$\Rightarrow \quad SP=e\times PM$$

$$\Rightarrow \quad SP^2=e^2\times PM^2$$

$$\Rightarrow \quad (x-1)^2+(y-2)^2=\frac{1}{2}\times\left(\frac{x+3y+10}{\sqrt{1+9}}\right)^2$$

$$\Rightarrow \quad 20\{(x-1)^2+(y-2)^2\}=(x+3y+10)^2$$

$$\Rightarrow \quad 20\{x^2+y^2-2x-4y+5\}$$
$$=(x^2+9y^2+100+6xy+20x+6y)$$

5. Let S be the focus and PM be the directrix.
From the definition of conic section, it is clear that,

$$SP=PM$$

$$\Rightarrow \quad SP^2=PM^2$$

$$\Rightarrow \quad (x-1)^2+(y-1)^2=\left(\frac{x-y+3}{\sqrt{1+1}}\right)^2$$

$$\Rightarrow \quad 2\{(x-1)^2+(y-1)^2\}=(x-y+3)^2$$

$$\Rightarrow \quad 2(x^2+y^2-2x-2y+2)$$
$$=(x^2+y^2+9-2xy-6x-6y)$$

$$\Rightarrow \quad x^2+2xy+y^2+2x+2y+4=0$$

6. (i) The given equation is

$$y^2=x+2y+2$$
$$\Rightarrow \quad y^2-2y=x+2$$
$$\Rightarrow \quad (y-1)^2=x+3$$
$$\Rightarrow \quad Y^2=X, \text{ where } X=x+3,\ Y=y-1$$

Vertex: $V(0, 0)$

$$\Rightarrow \quad X=0, Y=0$$
$$\Rightarrow \quad x+3=0, y-1=0$$
$$\Rightarrow \quad x=-3, y=1$$

Hence, the vertex is $(-3, 1)$
Focus: $(a, 0)$

$$\therefore \quad X=a, Y=0$$
$$\Rightarrow \quad x+3=\frac{1}{4}, y-1=0$$
$$\Rightarrow \quad x=\frac{1}{4}-3, y=1$$
$$\Rightarrow \quad x=-\frac{11}{4}, y=1$$

Hence, the focus is $\left(-\frac{11}{4}, 1\right)$

Latus rectum: $4a = 1$
Directrix: $X + a = 0$

$$\Rightarrow \quad x+3=\frac{1}{4}$$
$$\Rightarrow \quad x=-\frac{11}{4}$$
$$\Rightarrow \quad 4x+11=0$$

Axis: $Y = 0$

$$\Rightarrow \quad y-1=0$$
$$\Rightarrow \quad y=1$$

(ii) The given equation is

$$y^2=3x+4y+2$$
$$\Rightarrow \quad y^2-4y=x^2+2$$
$$\Rightarrow \quad y^2-4y+4=2x+6$$
$$\Rightarrow \quad (y-2)^2=3(x+2)$$
$$\Rightarrow \quad Y^2=3X,$$

where $X = x + 2$ and $Y = y - 2$
Vertex: $(0, 0)$

$$\Rightarrow \quad X=0, Y=0$$
$$\Rightarrow \quad x+2=0 \text{ and } y-2=0$$
$$\Rightarrow \quad x=-2 \text{ and } y=2$$

Hence, the vertex is $(-2, 2)$
Focus: $(a, 0)$

$$\Rightarrow \quad X=a, Y=0$$
$$\Rightarrow \quad x+2=\frac{3}{4}, y-2=0$$
$$\Rightarrow \quad x=-\frac{5}{4} \text{ and } y=2$$

Hence, the focus is $\left(-\frac{5}{4}, 2\right)$
Latus rectum: $4a = 3$
Directrix: $X + a = 0$

$$\Rightarrow \quad x+2=3/4$$
$$\Rightarrow \quad x=-5/3$$
$$\Rightarrow \quad 3x+5=0$$

Axis: $Y = 0$

$$\Rightarrow \quad y-2=0$$
$$\Rightarrow \quad y=2$$

(iii) The given equation is

$$x^2=y+4x+2$$
$$\Rightarrow \quad x^2-4x=y+2$$
$$\Rightarrow \quad x^2-4x+4=y+6$$
$$\Rightarrow \quad (x-2)^2=y+6$$
$$\Rightarrow \quad X^2=Y,$$

where $X = x - 2$ and $Y = y + 6$
Vertex: $(0, 0)$

$$\Rightarrow \quad X=0, Y=0$$
$$\Rightarrow \quad x-2=0 \text{ and } y+6=0$$
$$\Rightarrow \quad x=2 \text{ and } y=-6$$

Hence, the vertex is $(2, -6)$
Focus: $(0, a)$

$$\Rightarrow \quad X=0, Y=a$$
$$\Rightarrow \quad x-2=0 \text{ and } y+6=1/4$$
$$\Rightarrow \quad x=2 \text{ and } y=-23/4$$

Hence, the focus is $\left(2, -\frac{23}{4}\right)$

Latus rectum: $4a = 1$

Directrix: $Y + a = 0$

$\Rightarrow \quad y + 6 = 1/4$

$\Rightarrow \quad 4y + 23 = 0$

Axis: $X = 0$

$\Rightarrow \quad x - 2 = 0$

$\Rightarrow \quad x = 2$

(iv) The given equation is

$$x^2 + x + y = 0$$

$\Rightarrow \quad x^2 + x = -y$

$\Rightarrow \quad \left(x + \frac{1}{2}\right)^2 = -y + \frac{1}{4} = -\left(y - \frac{1}{4}\right)$

$\Rightarrow \quad X^2 = -Y$, where

$$X = x + \frac{1}{2}, Y = y - \frac{1}{2}$$

Vertex: $(0, 0) \Rightarrow X = 0, Y = 0$

$\Rightarrow \quad x + \frac{1}{2} = 0, y - \frac{1}{2} = 0$

$\Rightarrow \quad x = -\frac{1}{2}, y = \frac{1}{2}$

Hence, the vertex is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Focus : $(0, -a)$

$\Rightarrow \quad X = 0, Y = -a$

$\Rightarrow \quad x + 1/2 = 0, y - 1/2 = 1/4$

$\Rightarrow \quad x = -1/2, y = 3/4$

Hence, the focus is $(-1/2, 3/4)$

Latus rectum: $4a = 1$

Directrix: $Y - a = 0$

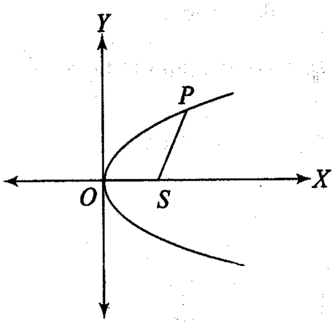
$\Rightarrow \quad y - 1/2 - 1/4 = 0$

$\Rightarrow \quad y - 3/4 = 0$

$\Rightarrow \quad 4y - 3 = 0$

Axis: $Y = 0 \Rightarrow y - 1/2 = 0 \Rightarrow 2y - 1 = 0$

7. Let the point be (x, y)



The given equation is

$$y^2 = 12x$$

$\Rightarrow \quad 4a = 12$

$\Rightarrow \quad a = 12/4 = 3$

Given focal distance = 6

$\therefore \quad x + a = 6$

$\Rightarrow \quad x + 3 = 6$

$\Rightarrow \quad x = 6 - 3 = 3$

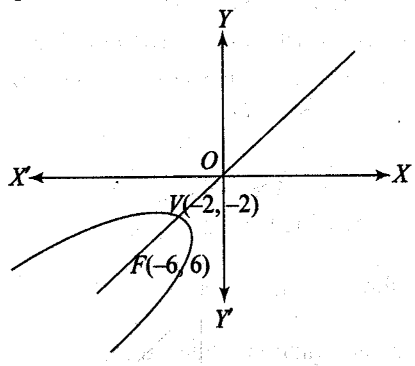
When $x = 3, y^2 = 12 \times 3 = 36$

$\Rightarrow \quad y = \pm 6$

Hence, the co-ordinates of the points are (3, 6) and (3, –6).

8. Let the vertex be V and the focus be S.

The equation of axis is $x - y = 0$.



Let the point Q is the point of intersection of the axis and the directrix.

Clearly, V is the mid-point of Q and S.

Then Q is (2. 2).

As we know that the directrix is perpendicular to the axis of the parabola. So, the equation of the directrix is

$$x + y - k = 0$$

which is passing through (2, 2).

Therefore, $k = 4$.

Hence, the equation of the directrix is

$$x + y - 4 = 0$$

Thus the equation of the parabola is

$$\sqrt{(x+6)^2 + (y+6)^2} = \left(\frac{x + y - 4}{\sqrt{1+1}}\right)$$

$\Rightarrow \quad 2((x+6)^2 + (y+6)^2) = (x + y - 4)^2$

$\Rightarrow \quad 2(x^2 + y^2 + 12x + 12y + 36)$

$\quad = (x^2 + y^2 + 16 + 2xy + 8x - 8y)$

$\Rightarrow \quad x^2 - 2xy + y^2 + 32x + 32y + 76 = 0$

9. The given equations are $x = t^2 + 1$ and $y = 2t + 1$

Eliminating t, we get

$$(y - 1)^2 = 4(x - 1)$$

$\Rightarrow \quad Y^2 = 4X$, where $X = (x - 1)$

and $\quad Y = (y - 1)$

Hence, the equation of the directrix is

$$X + a = 0$$

$\Rightarrow \quad x - 1 + 1 = 0$

$\Rightarrow \quad x = 0.$

10. Let the vertex be V and the focus be S.

Let Q be the point of intersection of the axis and the directrix.

Clearly, Q be $(-5, 0)$ and V be the mid-point of S and Q.

Then focus S is $(-1, 0)$.

Hence, the equation of the parabola is

$$\sqrt{(x+1)^2 + y^2} = \left(\frac{x+5}{\sqrt{1^2}}\right)$$

$\Rightarrow \quad (x + 1)^2 + y^2 = (x + 5)^2$

$\Rightarrow \quad y^2 = 8x + 24 = 8(x + 3)$

11. Let the equation of the parabola be

$$y = ax^2 + bx + c \quad \ldots(i)$$

which is passing through (0, 2), (–1, 0) and (1, 6). So

$$c = 2,\ a + c = b,\ a + b + c = 6$$

Solving, we get

$$a = 1,\ b = 3 \text{ and } c = 2$$

Hence, the equation of the parabola is $y = x^2 + 3x + 2$.

12. Let the equation of the parabola be $(y - k)^2 = 4a(x - h)$, where vertex is (h, k).

Then the equation becomes

$$(y - 2)^2 = 4a(x - 1)$$

which is passing through (3, 4).

Therefore, $8a = 4 \Rightarrow a = \frac{1}{2}$

Hence, the equation of the parabola is

$$(y - 2)^2 = 2(x - 1)$$

13. Let the equation of the parabola be $(x - H)^2 = 4a(y - k)$, where vertex is (h, k).

Thus the equation becomes

$$(x - 3)^2 = 4a(y - 2)$$

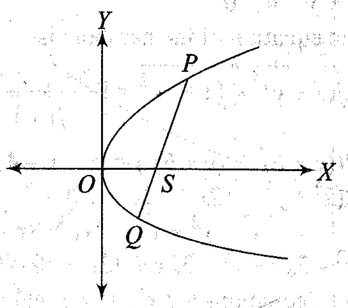
Also it given that, the length of the latus rectum = 12

$\Rightarrow \quad 4a = 12 \quad \Rightarrow a = 3.$

Hence, the equation of the parabola is

$$(x - 3)^2 = 12(y - 1).$$

14. Let $y^2 = 4ax$ be a parabola, if PQ be a focal chord.



Consider any two points on the parabola $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$.

Since PQ passes through the focus $S(a, 0)$, so P, S, Q are collinear.

Thus, $m(PS) = m(QS)$

$$\Rightarrow \quad \frac{2at_1 - 0}{at_1^2 - a} = \frac{0 - 2at_2}{a - at_2^2}$$

$$\Rightarrow \quad \frac{2at_1}{at_1^2 - a} = \frac{2at_2}{at_2^2 - a}$$

$$\Rightarrow \quad \frac{2t_1}{t_1^2 - 1} = \frac{2t_2}{t_2^2 - 1}$$

$$\Rightarrow \quad \frac{t_1}{t_1^2 - 1} = \frac{t_2}{t_2^2 - 1}$$

$$\Rightarrow \quad t_1(t_2^2 - 1) = t_2(t_1^2 - 1)$$

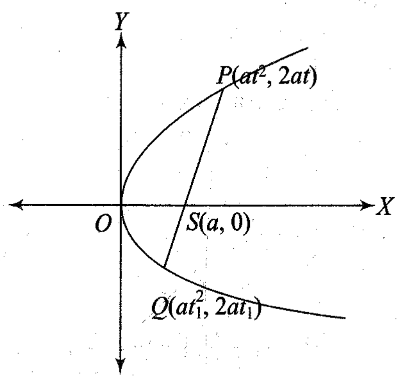
$$\Rightarrow \quad t_1t_2(t_2 - t_1) + (t_2 - t_1) = 0$$

$$\Rightarrow \quad t_1t_2 + 1 = 0$$

$$\Rightarrow \quad t_1t_2 = -1$$

which is the required relation.

15. Since one extremity of the focal chord is $P(at^2, 2at)$, then the other extremity will be $Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.



Thus, $PQ = SP + SQ$

$$= (at^2 + a) + \left(\frac{a}{t^2} + a\right)$$

$$= a\left(t^2 + \frac{1}{t^2} + 2\right) = a\left(t + \frac{1}{t}\right)^2$$

16. Now, slope of $PQ = \dfrac{2}{t - \frac{1}{t}} = \tan\theta$

$$\Rightarrow \quad 2\cot\theta = t - \frac{1}{t}$$

Thus, $PQ = a\left(t + \frac{1}{t}\right)^2$

$$= a\left[\left(t - \frac{1}{t}\right)^2 + 4\right]$$

$$= a(4\cot^2\theta + 4)$$

$$= 4a\operatorname{cosec}^2\theta$$

17. $S = (a, 0),\ P = (at^2, 2at)$ and $Q = \left(\frac{a}{t^2}, -\frac{2a}{t}\right)$

Thus, $SP = a + at^2,\ SQ = a + \frac{a}{t^2}$

Now, Harmonic mean of SP and SQ

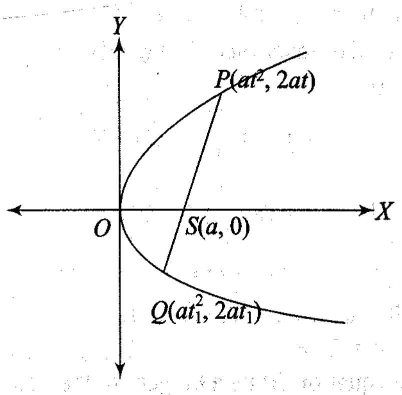
$$= \frac{2SP \cdot SQ}{SP + SQ} = \frac{2}{\frac{1}{SP} + \frac{1}{SQ}}$$

$$= \frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a + at^2} + \frac{1}{a + \frac{a}{t^2}} = \frac{1 + t^2}{a(1 + t^2)} = \frac{1}{a}$$

Thus, $\dfrac{2}{\frac{1}{SP} + \frac{1}{SQ}} = \dfrac{2}{\frac{1}{a}} = 2a$

= semi-latus rectum.

18.



The equation of the focal chord SP:

$$y - 0 = \frac{2at - 0}{at^2 - a}(x - a)$$

$\Rightarrow \quad y(t^2 - 1) = 2tx - 2at$

$\Rightarrow \quad 2tx - (t^2 - 1)y - 2at = 0$

Let d be the distance of the focal chord SP from the vertex (0, 0) to the parabola $y^2 = 4ax$.

$$\text{Then } d = \left|\frac{(0 - 0 - 2at)}{\sqrt{4t^2 + (t^2 - 1)^2}}\right|$$

$$= \frac{2at}{(t^2 + 1)} = \frac{2a}{\left(t + \frac{1}{t}\right)}$$

$$\text{Also, } PQ = a\left(t + \frac{1}{t}\right)^2 = a \times \frac{4a^2}{d^2} = \frac{4a^3}{d^2}$$

$$\text{Thus, } PQ \propto \frac{1}{d^2}$$

Hence the length of the focal chord varies inversely as the square of its distance from the vertex of the given parabola.

19. Let the circle described on the focal chord SP, where $S = (a, 0)$ and $P = (at^2, 2at)$.
The equation of the circle is

$$(x - at^2)(x - a) + (y - 2at)(y - 0) = 0$$

Solving it with y-axis, $x = 0$, we have

$$y^2 - 2aty + a^2t^2 = 0$$

Clearly, it has equal roots.
So the circle touches the y-axis.
Also, the point of contact is $(0, at)$.

20. The equation of the circle described on AB as diameter is

$$(x - at^2)\left(x - \frac{a}{t^2}\right) + (y - 2at)\left(y + \frac{2a}{t}\right) = 0$$

Put $x = a$, we have

$$y^2 - 2a\left(t - \frac{1}{t}\right)y + a^2\left(t - \frac{1}{t}\right)^2 = 0$$

Clearly, it has equal roots.
Hence the circle touches the directrix at $x = -a$.

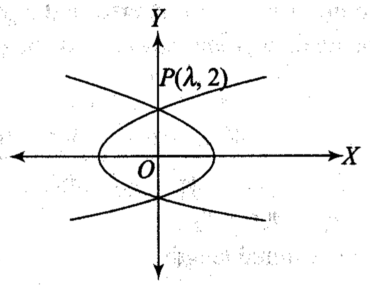
21. Since, the point $(\lambda, -\lambda)$ lies inside of the parabola $y^2 = 4x$, then $\lambda^2 - 4\lambda < 0$

$\Rightarrow \quad \lambda(\lambda - 4) < 0$

$\Rightarrow \quad 0 < \lambda < 4$

Hence, the range of λ is (0, 4).

22. Since the point $(\lambda, 2)$ is an exterior point of both the parabolas



$$y^2 = (x + 1) \text{ and } y^2 = -(x - 1),$$

So we have

$4 - x - 1 > 0$ and $4 + x - 1 > 0$

$\Rightarrow \quad 3 - x > 0$ and $3 + x > 0$

$\Rightarrow \quad x - 3 < 0$ and $x + 3 > 0$

$\Rightarrow \quad x < 3$ and $x > -3$

$\Rightarrow \quad -3 < x < 3$

23. The given line is

$2x + 3y + 5 = 0$ ...(i)

and the parabola is $y^2 = 8x$ ...(ii)

Since (i) is a tangent to the parabola $y^2 = 8x$, so

$$\left(\frac{-2x - 5}{3}\right)^2 = 8x$$

$\Rightarrow \quad (2x + 5)^2 = 72x$

$\Rightarrow \quad 4x^2 + 20x + 25 = 72x$

$\Rightarrow \quad 4x^2 - 52x + 25 = 0$

$\Rightarrow \quad 4x^2 - 2x - 50x + 25 = 0$

$\Rightarrow \quad 2x(2x - 1) - 25(2x - 1) = 0$

$\Rightarrow \quad (2x - 1)(2x - 25) = 0$

$\Rightarrow \quad x = 1/2, x = 25/2$

$$\text{When } x = 1/2, \text{ then } y = \left(\frac{-1 - 5}{3}\right) = -2$$

Also, when $x = 25/2$, then $y = -10$

Hence, the points of contact are

(1/2, –2) or (25/2, –10)

24. The given parabola is

$y^2 = 12x$ ...(i)

$\Rightarrow \quad 4a = 12$

$\Rightarrow \quad a = 3$

The given line is $3x + 4y + \lambda = 0$

$$\Rightarrow \quad y = -\frac{3}{4}x - \frac{\lambda}{4} \quad \text{...(ii)}$$

Since, the line (ii) is a tangent to the parabola (i), so

$$c = \frac{a}{m}$$

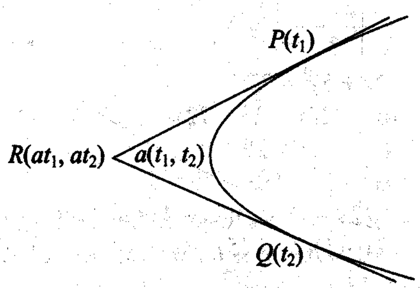
$$\Rightarrow \quad -\frac{\lambda}{4} = \frac{3}{\left(-\frac{3}{4}\right)} = -4$$

$\Rightarrow \quad \lambda = 16$

Hence, the value of λ is 16.

25. Let the equation of the parabola be
$$y^2 = 4ax$$
and the line be $y = mx + c$.
Solving the above equations, we get
$$(mx + c)^2 = 4ax$$
$$\Rightarrow \quad m^2x^2 + (2mc + 4a)x + c^2 = 0$$
Let the line $y = mx + c$ intersects the parabola in two real and distinct points, say (x_1, y_1) and (x_2, y_2).
Thus $(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2$
$$= \frac{4(mc-2a)^2}{m^4} - \frac{4c^2}{m^2} = \frac{16a(a-mc)}{m^4},$$
and $y_1 - y_2 = m(x_1 - x_2)$
Thus, the required length
$$= \sqrt{(y_1 - y_2)^2 + (x_1 - x_2)^2}$$
$$= \sqrt{(x_1 - x_2)^2 m^2 + (x_1 - x_2)^2}$$
$$= \sqrt{1 + m^2}\,(x_1 - x_2)$$
$$= \frac{4}{m^2}\sqrt{1+m^2}\sqrt{a(a-mc)}$$

26. Let the parabola be $y^2 = 4ax$ and the two points on the parabola are $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$, respectively.



The equation of the tangent at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are
$$t_1 y = x + at_1^2 \qquad \ldots\text{(i)}$$
and $$t_2 y = x + at_2 \qquad \ldots\text{(ii)}$$
Solving these equations, we get
$$x = at_1t_2, y = a(t_1 + t_2)$$
Hence the co-ordinates of the point of intersection of the tangents are $(at_1t_2, a(t_1 + t_2))$.

Notes:
1. x-co-ordinate is the geometric mean of the x-co-ordinates of P and Q.
2. y-co-ordinate is the arithmetic mean of the y-co-ordinates of P and Q.

27. The given parabola is $y^2 = 2x + 5y - 8$.
when $x = 1$, $y^2 = 5y - 6$
$$\Rightarrow \quad y^2 - 5y + 6 = 0$$
$$\Rightarrow \quad (y-2)(y-3)$$
$$\Rightarrow \quad y = 2, 3$$
Thus, the points are (1, 2) and (1, 3).
Hence, the equations of tangents can be at (1, 2) and (1, 3) be
$$2y = (x+1) + \frac{5}{2}(y+2) - 8$$
and $$3y = (x+1) + \frac{5}{2}(y+3) - 8$$
$$\Rightarrow \quad 2x + y - 4 = 0 \text{ and } 2x - y + 1 = 0$$

28. Let the equation of the tangent be
$$y = 2x + c \qquad \ldots\text{(i)}$$
If the equation (i) be a tangent to the parabola, then
$$c = \frac{a}{m} = \frac{2}{2} = 1.$$
Thus, the equation of the tangent is
$$y = 2x + 1 \qquad \ldots\text{(ii)}$$
The given parabola is $y^2 = 8x$ ...(iii)
Solving (ii) and (iii), we get
$$(2x+1)^2 = 8x$$
$$\Rightarrow \quad 4x^2 + 4x + 1 = 8x$$
$$\Rightarrow \quad (2x-1)^2 = 0$$
$$\Rightarrow \quad x = \frac{1}{2}$$
When $x = 1/2$, then $y = \pm 2$
Hence, the point of contacts are
$$\left(\frac{1}{2}, 2\right) \text{ or } \left(\frac{1}{2}, -2\right)$$

29. The equation of line from $(-1, 2)$ is
$$(y-2) = m(x+1)$$
$$\Rightarrow \quad mx - y + (m+2) = 0$$
$$\Rightarrow \quad y = mx + (m+2) \qquad \ldots\text{(i)}$$
The line (i) will be a tangent to the parabola $y^2 = 4x$, if
$$(m+2) = \frac{1}{m}$$
$$\Rightarrow \quad m^2 + 2m - 1 = 0$$
which is a quadratic in m.
Let its roots are m_1, m_2.
Thus, $m_1 + m_2 = -2$ and $m_1m_2 = -1$
Let θ be the angle between them.
Then, $$\tan(\theta) = \left|\frac{m_2 - m_1}{1 + m_1m_2}\right|$$
$$= \left|\frac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1 + m_1m_2}\right|$$
$$\Rightarrow \quad \tan(\theta) = \infty = \tan\frac{\pi}{2}$$
$$\Rightarrow \quad \theta = \frac{\pi}{2}$$
Hence, the angle between the tangents is $\theta = \frac{\pi}{2}$.

30. Let the equation of the tangent be

$$(y+1) = m(x-1)$$

$$\Rightarrow \quad y = mx - (m+1) \qquad \text{...(i)}$$

Equation (i) be a tangent to the parabola

$$y = x^2 - 3x + 2,$$

then $mx - m - 1 = x^2 - 3x + 2$

$$\Rightarrow \quad x^2 - (m+3)x + (m+3) = 0$$

Since it has equal roots, so

$$D = 0$$

$$\Rightarrow \quad (m+3)^2 - 4(m+3) = 0$$

$$\Rightarrow \quad (m+3)(m+3-4) = 0$$

$$\Rightarrow \quad (m+3)(m-1) = 0$$

$$\Rightarrow \quad m = 1, -3$$

Hence, the equation of the tangents are

$y = x - 2$ and $y = -3x + 2$

31. The given parabolas are

$y^2 = 4ax$ and $x^2 = 4ay$

Let the equation of the tangent be $y = mx + \dfrac{a}{m}$.

If it is a tangent to the parabola $x^2 = 4ay$, then

$$x^2 = 4a\left(mx + \frac{a}{m}\right)$$

$$\Rightarrow \quad mx^2 + 4am^2x + 4a^2$$

$$\Rightarrow \quad mx^2 - 4am^2x - 4a^2 = 0$$

Now $D = 0$ gives,

$$16a^2m^2 + 16a^2m = 0$$

$$\Rightarrow \quad 16a^2m\,(m^3 + 1) = 0$$

$$\Rightarrow \quad m(m^3 + 1) = 0$$

$$\Rightarrow \quad m = 0 \text{ and } -1$$

Since $m = 0$ will not satisfy the given tangent, so

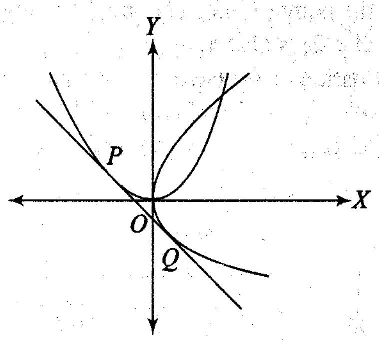
$$m = -1$$

Hence, the equation of the common tangent be

$$y = -x - a$$

$$\Rightarrow \quad x + y + a = 0$$

32. The given parabolas are $y^2 = 4ax$ and $x^2 = 4by$.



Let the equation of the tangent be

$$y = mx + \frac{a}{m} \qquad \text{...(i)}$$

Since the equation (i) is also a tangent to the parabola $x^2 = 4by$, so

$$x^2 = 4b\left(mx + \frac{a}{m}\right)$$

$$\Rightarrow \quad mx^2 = 5m^2x + 4ab$$

$$\Rightarrow \quad mx^2 - 4bm^2x - 4ab = 0$$

Since it has equal roots, so

$$D = 0$$

$$16b^2m^4 + 16abm = 0$$

$$\Rightarrow \quad 16bm(bm^3 + a) = 0$$

$$\Rightarrow \quad m^3 = -\frac{a}{b}$$

$$\Rightarrow \quad m = -\frac{a^{1/3}}{b^{1/3}}$$

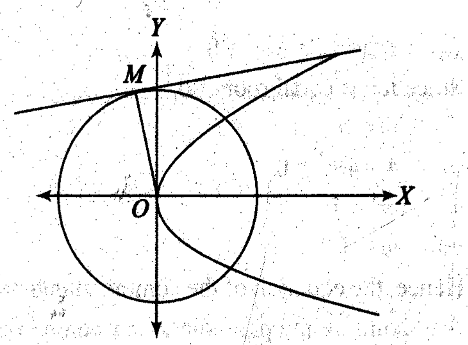
Hence, the equation of the common tangent be

$$y = \left(-\frac{a^{1/3}}{b^{1/3}}\right)x + a\left(-\frac{b^{1/3}}{a^{1/3}}\right)$$

33. Let the equation of the tangent to the parabola $y^2 = 16x$ is

$$y = mx + \frac{4}{m}$$

$$\Rightarrow \quad m^2x - my + 4 = 0 \qquad \text{...(i)}$$



If the Eq. (i) be a tangent to the circle $x^2 + y^2 = 8$, the length of the perpendicular from the centre to the tangent is equal to the radius of the circle.

Therefore,

$$\left|\frac{0 - 0 + 4}{\sqrt{m^4 + m^2}}\right| = 2\sqrt{2}$$

$$\Rightarrow \quad 8(m^4 + m^2) = 16$$

$$\Rightarrow \quad (m^4 + m^2 - 2) = 0$$

$$\Rightarrow \quad (m^4 + 2m^2 - m^2 - 2) = 0$$

$$\Rightarrow \quad (m^2 + 2)(m^2 - 1) = 0$$

$$\Rightarrow \quad (m^2 - 1) = 0$$

$$\Rightarrow \quad m = \pm 1$$

Hence, the common tangents are $y = \pm x \pm 4$.

34. Any point on the parabola $y = x^2$ is (t, t^2).

Now the tangent at (t, t^2) is

$$xx_1 = \frac{1}{2}(y + y_1)$$

$$\Rightarrow \quad tx = \frac{1}{2}(y + t^2)$$

$$\Rightarrow \quad 2tx - y - t^2 = 0$$

If it is a tangent to the parabola,

$$y = -(x-2)^2, \text{ then}$$

$$2tx - t^2 = -(x-2)^2$$

$$\Rightarrow \quad 2tx - t^2 = -x^2 + 4x - 4$$

$$\Rightarrow \quad x^2 + 2(2-t)x + (t^2 - 4) = 0$$

Since it has equal roots,

$$D = 0$$

$$4(2-t)^2 - 4(t^2-4) = 0$$

$$\Rightarrow \quad (2-t)^2 - (t^2-4) = 0$$

$$\Rightarrow \quad t = 2$$

Hence, the equation of the common tangent is

$$y = 4x - 4$$

35. Let the equation of the tangent to the parabola $y^2 = 8x$ is

$$y = mx + \frac{2}{m} \qquad \text{...(i)}$$

If it is a tangent to the curve $xy = -1$, then

$$x\left(mx + \frac{2}{m}\right) = -1$$

$$\Rightarrow \quad m^2x^2 + 2x + m = 0$$

Since it has equal roots, so,

$$D = 0$$

$$\Rightarrow \quad 4 - 4m^3 = 0$$

$$\Rightarrow \quad m^3 = 1$$

$$\Rightarrow \quad m = 1$$

Hence, the equation of the common tangent is $y = x + 2$.

36. Any point on the parabola $y^2 = x$ can be considered as (t^2, t).

The equation of the tangent to the parabola $y^2 = x$ at (t^2, t) is

$$yy_1 = \frac{1}{2}(x + x_1).$$

$$\Rightarrow \quad yt = \frac{1}{2}(x + t^2)$$

$$\Rightarrow \quad x + 2yt - t^2 = 0 \qquad \text{...(i)}$$

If it is a tangent to the circle $x^2 + y^2 - 6y + 4 = 0$, then

$$(2yt - t^2)^2 + y^2 - 6y + 4 = 0$$

$$\Rightarrow \quad 4y^2t^2 + t^4 - 4yt^3 + y^2 - 6y + 4 = 0$$

$$\Rightarrow \quad (4t^2 + 1)y^2 - 2(2t^3 + 3)y + (t^4 + 4) = 0$$

Since it has equal roots, so

$$D = 0$$

$$\Rightarrow \quad 4(2t^3 + 1)^2 - 4(4t^2 + 1)(t^4 + 4) = 0$$

$$\Rightarrow \quad (2t^3 + 1)^2 - (4t^2 + 1)(t^4 + 4) = 0$$

$$\Rightarrow \quad t^4 - 12t^3 + 16t^2 - 5 = 0$$

$$\Rightarrow \quad t = 1$$

Hence, the equation of the common tangent is $x + 2y = 1$.

37. Any tangent to the parabola $y^2 = 4x$ is

$$y = mx + \frac{a}{m}$$

$$\Rightarrow \quad y = mx + \frac{1}{m}$$

$$\Rightarrow \quad m^2x - my + 1 = 0 \qquad \text{...(i)}$$

If it is a tangent to the circle $x^2 + (y-3)^2 = 9$ the length of the perpendicular from the centre to the tangent is equal to the radius of the circle.

Therefore,

$$\left|\frac{3m^2 + 1}{\sqrt{m^4 + m^2}}\right| = 3$$

$$\Rightarrow \quad (3m^2 + 1)^2 = 9(m^4 + m^2)$$

$$\Rightarrow \quad (9m^4 + 6m^2 + 1) = 9(m^4 + m^2)$$

$$\Rightarrow \quad 3m^2 = 1$$

$$\Rightarrow \quad m = \pm\left(\frac{1}{\sqrt{3}}\right)$$

Since, the tangent touches the parabola above x-axis, so it will make an acute angle with x-axis, so that m is positive.

Thus $m = \dfrac{1}{\sqrt{3}}$.

Hence, the common tangent is $x - \sqrt{3}y + 3 = 0$.

38. The equation of the given parabola is $y^2 = 4x$.

We have, $4a = 4 \Rightarrow a = 1$

Let the end-points of the latus rectum are $L(a, 2a)$ and $L'(a, -2a)$.

Therefore $L = (1, 2)$ and $L' = (1, -2)$.

As we know that the point of intersection to the tangents at $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ to the parabola $y^2 = 4ax$ is

$$\left(at_1t_2, a\left(\frac{t_1 + t_2}{2}\right)\right)$$

Thus, the point of intersection of the tangents at $L(1, 2)$ and $L'(1, -2)$ is $(1, 0)$.

39. The equation of the tangent to the parabola

$$y^2 = 4x \qquad \text{...(i)}$$

from $(1, 4)$ is

$$y - 4 = m(x - 1)$$

$$\Rightarrow \quad y = mx + (4 - m) \qquad \text{...(ii)}$$

Since (ii) is a tangent to the parabola $y^2 = 4x$, so

$$c = \frac{a}{m}$$

$$\Rightarrow \quad (4 - m) = \frac{1}{m}$$

$$\Rightarrow \quad 4m - m^2 - 1 = 0$$

$$\Rightarrow \quad m^2 - 4m + 1 = 0$$

It has two roots, say m_1 and m_2.

Therefore, $m_1 + m_2 = 4$ and $m_1m_2 = 1$

Let θ be the angle between the tangents

Then $\tan(\theta) = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$

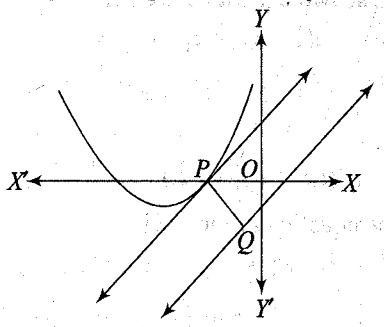
$$= \left|\frac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1 + m_1m_2}\right|$$

$$= \left|\frac{\sqrt{12}}{2}\right| = \sqrt{3} = \tan\frac{\pi}{3}$$

$\Rightarrow \quad \theta = \frac{\pi}{3}$

Hence, the angle between the tangents is $\frac{\pi}{3}$.

40. The shortest distance between a line and the parabola means the shortest distance between a line and a tangent to the parabola parallel to the given line.



Thus, the slopes of the tangent and the line will be the same.

Therefore,

$2x + 3 = 1$

$\Rightarrow \quad x = -1$

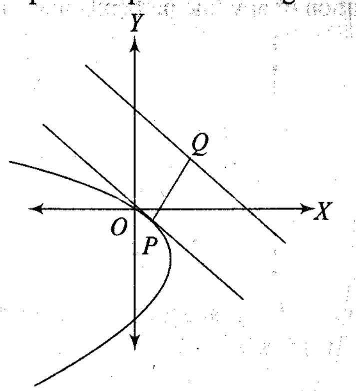
When $x = -1$, then $y = 0$.

Hence, the point on the parabola is $(-1, 0)$.

Thus, the required shortest distance

$$= \left|\frac{-1-0-2}{\sqrt{1+1}}\right| = \frac{3}{\sqrt{2}}$$

41. The shortest distance between a line and the parabola means the shortest distance between a line and a tangent to the parabola parallel to the given line.



Thus, the slopes of the tangent and the line will be the same.

Therefore, $-\frac{4}{2y+4} = -1 \Rightarrow y = 0$

When $y = 0$, then $x = 0$.

Thus, the point on the parabola is $(0, 0)$

Hence, the required shortest distance

$$= \left|\frac{0+0-4}{\sqrt{1+1}}\right| = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

42. The given tangents are $y + b = m_1(x + a)$ and $y + b = m_2(x + a)$

Therefore, both the tangents pass through $(-a, -b)$ which is a point lying on the directrix of the parabola. Thus, the angle between them is 90°.

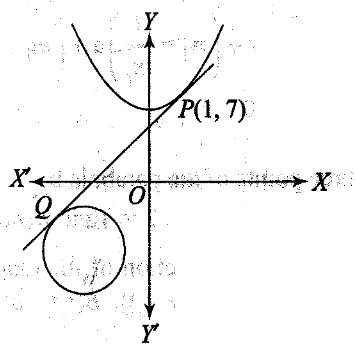
Hence, the value of m_1m_2 is -1.

43. The equation of the tangent to the curve $y = x^2 + 6$ at $(1, 7)$ is

$$\frac{1}{2}(y + y_1) = xx_1 + 6$$

$\Rightarrow \quad \frac{1}{2}(y + 7) = x + 6$

$\Rightarrow \quad 2x - y - 5 = 0 \quad$ ...(i)



The given circle is

$$(x+8)^2 + (y+6)^2 = (\sqrt{100-c})^2 \quad \text{...(ii)}$$

If the line (i) be a tangent to the circle (ii), the length of the perpendicular from the centre of the circle is equal to the radius of the circle.

Therefore, $\left|\frac{-16+6-5}{\sqrt{2^2+1}}\right| = \sqrt{100-c}$

$\Rightarrow \quad 45 = 100 - c$

$\Rightarrow \quad c = 100 - 45 = 5$

Thus, the equation of the circle is

$(x + 8)^2 + (y + 6)^2 = 45$

$\Rightarrow \quad x^2 + y^2 + 16x + 12y + 55 = 0 \quad$ ...(iii)

Solving Eqs (i) and (iii), we get

$x^2 + (2x - 5)^2 + 16x + 12(2x - 5) + 55 = 0$

$\Rightarrow \quad x^2 + 4x^2 + 25 - 20x + 16x + 24x - 60 + 55 = 0$

$\Rightarrow \quad 5x^2 + 20x + 20 = 0$

$\Rightarrow \quad x^2 + 4x + 4 = 0$

$\Rightarrow \quad (x + 2)^2 = 0$

$\Rightarrow \quad x = -2$

When $x = -2$, then $y = 2x - 5 = -4 - 5 = -9$.

Hence, the point Q is $(-2, -9)$.

44. Any tangent to $y^2 = 4(x + a)$ is

$$y = m_1(x+a) + \frac{a}{m_1} \quad \text{...(i)}$$

Also, any tangent to $y^2 = 4b(x + b)$ is

$$y = m_2(x+b) + \frac{b}{m_2} \quad \text{...(ii)}$$

Since, two tangents are perpendicular, so

$$m_1 m_2 = -1$$

$$\Rightarrow \quad m_2 = -\frac{1}{m_1}$$

From Eq. (ii), we get

$$y = -\frac{1}{m_1}(x+b) - bm_1 \quad \text{...(iii)}$$

Now subtracting Eq. (i) and Eq. (iii), we get

$$m_1(x+a) + \frac{1}{m_1}(x+b) + \frac{a}{m_1} + bm_1 = 0$$

$$\Rightarrow \quad \left(m_1 + \frac{1}{m_1}\right)x + \left(m_1 + \frac{1}{m_1}\right)a + \left(m_1 + \frac{1}{m_1}\right)b = 0$$

$$\Rightarrow \quad x + a + b = 0$$

Hence, the result.

45. Let the three points of the parabola be

$$P(at_1^2, 2at_1), Q(at_2^2, 2at_2) \text{ and } R(at_3^2, 2at_3)$$

and the points of intersection of the tangents at these points are $A(t_2 t_3, a(t_2 + t_3))$, $B(t_1 t_3, a(t_1 + t_3))$ and $A(t_1 t_2, a(t_1 + t_2)$

Now,

$$\text{ar}(\Delta PQR) = \frac{1}{2}\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix}$$

$$= a^2(t_1 - t_2)(t_2 - t_3)(t_3 - 1)$$

Also,

$$\text{ar}(\Delta ABC) = \frac{1}{2}\begin{vmatrix} at_2 t_3 & a(t_2+t_3) & 1 \\ at_3 t_1 & a(t_3+t_1) & 1 \\ at_1 t_2 & a(t_1+t_2) & 1 \end{vmatrix}$$

$$= \frac{1}{2}a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)$$

Hence, the result.

46. Let P, Q and R be the points on the parabola $y^2 = 4ax$ at which tangents are drawn and let their co-ordinates be

$$P(at_1^2, 2at_1), Q(at_2^2, 2at_2) \text{ and } R(at_3^2, 2at_3).$$

The tangents at Q and R intersect at the point

$$A[at_2 t_3, a(t_2 + t_3)]$$

Similarly, the other pairs of tangents at the points $B[at_1 t_3, a(t_1 + t_3)]$ and $C[at_1 t_2, a(t_1 + t_2)]$

Let the equation to the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

Since it passes through the above three points, we have

$$a^2 t_2^2 t_3^2 + a^2(t_2 + t_3)^2 + 2gat_2 t_3 + 2fa(t_2 + t_3) + c = 0 \quad \text{...(ii)}$$

$$a^2 t_1^2 t_3^2 + a^2(t_1 + t_3)^2 + 2gat_1 t_3 + 2fa(t_1 + t_3) + c = 0 \quad \text{...(iii)}$$

and

$$a^2 t_1^2 t_2^2 + a^2(t_1 + t_2)^2 + 2gat_1 t_2 + 2fa(t_1 + t_2) + c = 0 \quad \text{...(iv)}$$

Subtracting Eq. (iii) from Eq. (ii) and dividing by $a(t_2 - t_1)$, we get

$$a(t_3^2(t_1 + t_2) + t_1 + t_2 + 2t_3) + 2gt_3 + 2f = 0$$

similarly from Eqs (iii) and (iv), we get

$$a(t_1^2(t_3 + t_2) + t_3 + t_2 + 2t_1) + 2gt_1 + 2f = 0$$

From these two equations, we get

$$2g = -a(1 + t_2 t_3 + t_3 t_1 + t_1 t_2),$$

$$2f = -a(t_1 + t_2 + t_3 - t_1 t_2 t_3)$$

Substituting these values of g and f in Eq. (ii), we get

$$c = a^2(t_2 t_3 + t_3 t_1 + t_1 t_2).$$

Thus, the equation of the circle is

$$x^2 + y^2 - a(1 + t_2 t_3 + t_3 t_1 + t_1 t_2)x$$
$$- a(t_1 + t_2 + t_3 - t_1 t_2 t_3)y$$
$$+ a^2(t_2 t_3 + t_3 t_1 + t_1 t_2) = 0$$

47. Let the equations of the three tangents be

$$y = m_1 x + \frac{a}{m_1} \quad \text{...(i)}$$

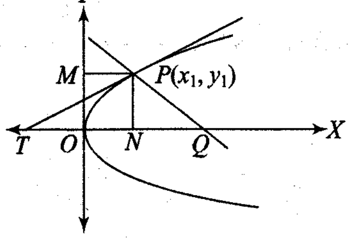
$$y = m_2 x + \frac{a}{m_2} \quad \text{...(ii)}$$

and $$y = m_3 x + \frac{a}{m_3} \quad \text{...(iii)}$$

The point of intersection of (ii) and (iii) is

$$\left(\frac{a}{m_2 m_3}, a\left(\frac{1}{m_2} + \frac{1}{m_3}\right)\right)$$

The equation of any line perpendicular to (i) and



The equation of any tangent to the parabola at (x_1, y_1) is

$$(y - y_1) = m(x - x_1),$$

where $m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

passes through the point of intersection of tangents (ii) and (iii) is

$$y - a\left(\frac{1}{m_2} + \frac{1}{m_3}\right) = -\frac{1}{m_1}\left(x - \frac{a}{m_2 m_3}\right)$$

i.e. $$y + \frac{x}{m_1} = a\left[\frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_1 m_2 m_3}\right] \quad \ldots\text{(iv)}$$

Similarly the equation to the straight line through the point of intersection of (iii) and (i) and perpendicular to (ii) is

$$y + \frac{x}{m_2} = a\left[\frac{1}{m_3} + \frac{1}{m_1} + \frac{1}{m_1 m_2 m_3}\right] \quad \ldots\text{(v)}$$

and the equation of the straight line through the point of intersection of (i) and (ii) and perpendicular to (iii) is

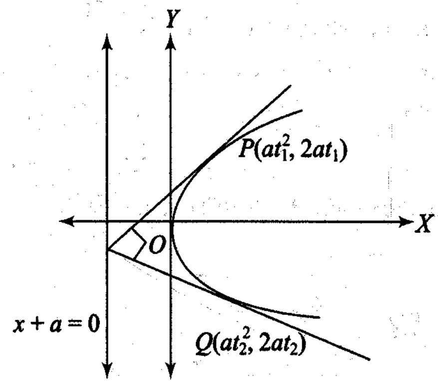
$$y + \frac{x}{m_3} = a\left[\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_1 m_2 m_3}\right] \quad \ldots\text{(vi)}$$

The point which is common to the straight lines (iv), (v) and (vi), i.e. the orthocentre of the triangle is

$$\left(-a, a\left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_1 m_2 m_3}\right)\right)$$

Hence, the point lies on the directrix.

48.



The equations of tangents at P and Q are $yt_1 = x + at_1^2$ and $yt_2 = x + at_2$.

The point of intersection of these tangents is

$[at_1t_2, a(t_1 + t_2)]$

Let this point be (h, k).

The slope of the tangents are $m_1 = \frac{1}{t_1}$ and $m_2 = \frac{1}{t_2}$.

Since these two tangents are perpendicular, so

$m_1 m_2 = -a$

$\Rightarrow \quad \frac{1}{t_1} \cdot \frac{1}{t_2} = -1$

$\Rightarrow \quad t_1 \cdot t_2 = -1$

$\Rightarrow \quad h = -a$

Thus the locus of the points of intersection is $x + a = 0$ which is the directrix of the parabola $y^2 = 4ax$.

49. (i) The given parabola is

$y^2 = x + 2$

$\Rightarrow \quad Y^2 = X,$

where $X = x + 2$ and $Y = y$

We have,

$4a = 1$

$\Rightarrow \quad a = \frac{1}{4}$

Hence, the equation of the director circle is

$X + a = 0$

$\Rightarrow \quad x + 2 + \frac{1}{4} = 0$

$\Rightarrow \quad 4x + 9 = 0$

(ii) The given parabola is

$x^2 = 4x + 4y$

$\Rightarrow \quad (x^2 - 4x + 4) = 4y + 4 = 4(y + 1)$

$\Rightarrow \quad (x - 2)^2 = 4(y + 1)$

$\Rightarrow \quad X^2 = 4Y,$

where $X = x - 2$

and $\quad Y = y + 1$

We have, $4a = 4$

$\Rightarrow \quad a = 1$

Hence, the equation of the director circle is

$Y + a = 0$

$\Rightarrow \quad y + 1 + 1 = 0$

$\Rightarrow \quad y + 2 = 0$

(iii) The given parabola is

$y^2 = 4x + 4y - 8$

$\Rightarrow \quad y^2 - 4y + 4 = 4x - 8 + 4$

$\Rightarrow \quad (y - 2)^2 = 4x - 4 = 4(x - 1)$

$\Rightarrow \quad Y^2 = 4X,$

where $X = (x - 1)$

and $\quad Y = (y - 2)$

We have $4a = 4 \Rightarrow a = 1$

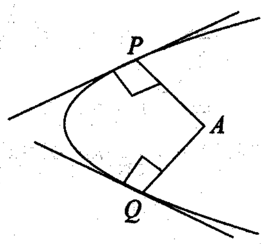
Hence, the equation of the director circle is

$X + a = 0$

$\Rightarrow \quad x - 1 + 1 = 0$

$\Rightarrow \quad x = 0$

50. Let the parabola be $y^2 = 4ax$ and two points on the parabola are $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$.



The equation of the normals at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are

$$y = -t_1 x + 2at_1 + at_1^3 \quad \ldots\text{(i)}$$

and

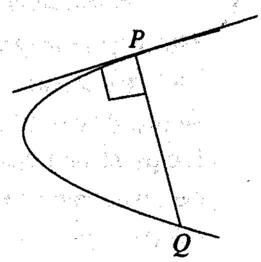
$$y = -t_2 x + 2at_2 + at_2^3 \quad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii), we get

$$x = 2a + a(t_1^2 + t_2^2 + t_1 t_2)$$

and $\quad y = -at_2t_2(t_1 + t_2)$.

51. Let the parabola be $y^2 = 4ax$ and the two points on the parabola are $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$.



The equation of the normal to the parabola at $P(at_1^2, 2at_1)$ is $y = -t_1x + 2at_1 + at_1^3$ which meets the parabola again at $Q(at_2^2, 2at_2)$.

Thus, $2at_2 = -at_1t_2^2 + 2at_1 + at_1^3$

$$\Rightarrow \quad 2a(t_2 - t_1) + at_1(t_2^2 - t_1^2) = 0$$

$$\Rightarrow \quad (t_2 - t_1)[2a + at_1(t_2 + t_1)] = 0$$

$$\Rightarrow \quad [2 + t_1(t_2 + t_1)] = 0$$

$$\Rightarrow \quad t_1^2 + t_1t_2 + 2 = 0$$

$$\Rightarrow \quad t_2 = -t_1 - \frac{2}{t_1}$$

which is the required condition.

52. As we know that if the normal at t_1 meets the parabola again at t_2, then

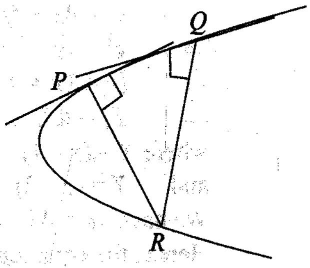
$$t_2 = -t_1 - \frac{2}{t_1}$$

$$\Rightarrow \quad t_2^2 = \left(-t_1 - \frac{2}{t_1}\right)^2$$

$$= t_1^2 + \frac{4}{t_1^2} + 4 \geq 4 + 4 = 8$$

Hence, the result.

53. Since the normal at t_1 meets the parabola at t_3, so $t_3 = -t_1 - \frac{2}{t_1}$.



Similarly, $t_3 = -t_2 - \frac{2}{t_2}$

Thus, $-t_1 - \frac{2}{t_1} = -t_2 - \frac{2}{t_2}$

$$\Rightarrow \quad (t_1 - t_2) = \left(\frac{2}{t_1} - \frac{2}{t_2}\right) = \frac{2(t_1 - t_2)}{t_1t_2}$$

$$\Rightarrow \quad t_1t_2 = 2$$

54. The equation of the normal is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1), \text{ where } a = 1$$

$$\Rightarrow \quad y - 2 = -\frac{2}{2}(x - 1) = -x + 1$$

$$\Rightarrow \quad x + y = 3$$

55. The equation of the normal to the parabola $y^2 = 4ax$ is

$$y = mx - 2am - am^3$$

Here, $a = 2$ and $m = 2$.

Therefore, $y = 2x - 8 - 16 = 2x - 24$

Hence, the equation of the normal is $y = 2x - 24$.

56. The given parabola is $y^2 = 12x$.

We have, $4a = 12 \Rightarrow a = 3$.

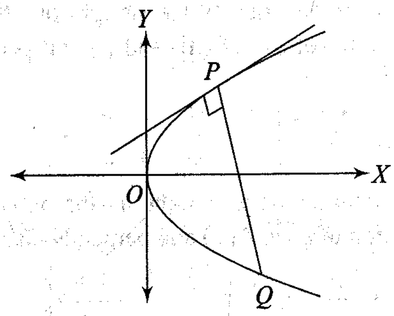
The given line $x + y = k \Rightarrow y = -x + k$ ...(i)

The line (i) will be a normal to the given parabola, if $k = -2am - am^3 = 6 + 3 = 9$.

Hence, the value of k is 9.

57. The equation of the parabola is $y^2 = 8x$.

We have, $4a = 8 \Rightarrow a = 2$



The equation of the normal to the given parabola at $P(8, 12)$ is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

$$\Rightarrow \quad y - 12 = -\frac{12}{4}(x - 18)$$

$$\Rightarrow \quad y - 12 = -3x + 54$$

$$\Rightarrow \quad 3x + y = 66$$

Solving $y = 66 - 3x$ and the parabola $y^2 = 8x$, we get

$$x = -\frac{44}{3} \text{ and } y = \frac{242}{9}$$

Hence, the co-ordinates of Q is $\left(-\frac{44}{3}, \frac{242}{9}\right)$.

Thus, $PQ = \sqrt{\left(\frac{242}{9} - 18\right)^2 + \left(-\frac{44}{3} - 12\right)^2}$

$$= \sqrt{\frac{6400}{81} + \frac{6400}{9}}$$

$$= 80\sqrt{\frac{1}{81} + \frac{1}{9}}$$

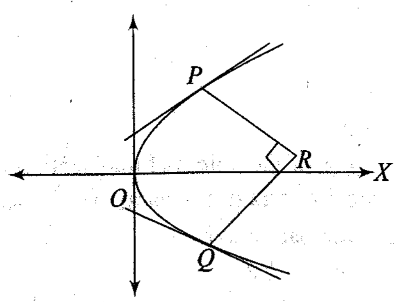
$$= \frac{80}{9} \times \sqrt{10}$$

$$\Rightarrow \quad 9PQ = 80\sqrt{10}$$

58. The equations of normals at $P(t_1)$ and $Q(t_2)$ are

$$y = -t_1x + 2at_1 + at_1^3$$

and $y = -t_2x + 2at_2 + at_2^3$



Since these two normals are at right angles, so $t_1t_2 = -1$.

Let $M(h, k)$ be the point of intersection of two normals.

Then, $h = 2a + a(t_1^2 + t_1t_2 + t_2^2)$

and $k = -at_1t_2(t_1 + t_2)$

$\Rightarrow \quad h = 2a + a\{(t_1 + t_2)^2 - 2t_1t_2\}$

and $k = -at_1t_2(t_1 + t_2)$

$\Rightarrow \quad h = 2a + a\{(t_1 + t_2)^2 + 2\}$

and $k = a(t_1 + t_2)$

Eliminating t_1 and t_2, we get,

$$k^2 = a(h - 3a)$$

Hence, the locus of $M(h, k)$ is $y^2 = a(x - 3a)$.

59. The given line is

$$lx + my = 0$$

$\Rightarrow \quad my = -lx - n$

$\Rightarrow \quad y = \left(-\frac{l}{m}\right)x + \left(-\frac{n}{m}\right)$

As we know that the line $y = mx + c$ will be a normal to the parabola $y^2 = 4ax$ if

$$c = -2am - am^3$$

$\Rightarrow \quad \left(-\frac{n}{m}\right) = -2a\left(-\frac{l}{m}\right) - a\left(-\frac{l}{m}\right)^3$

$\Rightarrow \quad al^3 + 2alm^2 + nm^2 = 0$

Hence, the result.

60. The equation of the parabola is $y^2 = 4ax$.

If the normal at $P(t_1)$ meets the parabola again at $Q(t_2)$, then

$$t_2 = -t_1 - \frac{2}{t_1}$$

$\Rightarrow \quad t_1t_2 = -t_1^2 - 2$

$\Rightarrow \quad t_1^2 + t_1t_2 + 2 = 0 \qquad \ldots\text{(i)}$

The chord joining t_1, t_2 subtends a right angle at the vertex, so the product of their slopes $= -1$

$\Rightarrow \quad \frac{2}{t_1} \cdot \frac{2}{t_2} = -1$

$\Rightarrow \quad t_1t_2 = -4 \qquad \ldots\text{(ii)}$

From Eqs (i) and (ii), we get

$$t_1^2 - 4 + 2 = 0$$

$\Rightarrow \quad t_1^2 = 2$

$\Rightarrow \quad t_1 = \sqrt{2}$

$\Rightarrow \quad \tan\theta = \sqrt{2}$

$\Rightarrow \quad \theta = \tan^{-1}(\sqrt{2})$

61. The given equation of the parabola is $y^2 = 4x$.

We have $4a = 4 \Rightarrow a = 1$.

The equation of the normal to the parabola

$y^2 = 4x$ at $(am^2, -2am)$ is

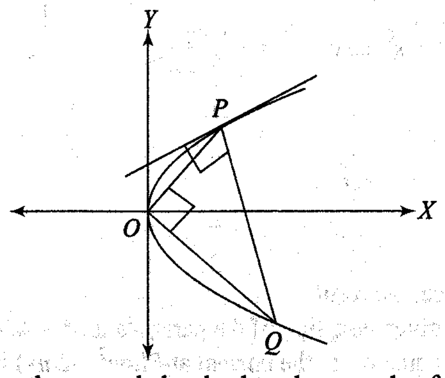
$y = mx - 2am - am^3 = mx - 2m - m^3$

Since, the normal makes equal angles with the axes, so $m = \pm 1$

Thus, the points are $(m^2, -2m) = (1, \mp 2)$

62. If the normal at $P(t_1)$ meets the parabola at $Q(t_2)$, then

$$t_2 = -t_1 - \frac{2}{t_1} \qquad \ldots\text{(i)}$$



Since the normal chord subtends an angle of 90° at the vertex, then

$$t_1t_2 = -4$$

From Eq. (i), we get

$$t_1^2 + t_1t_2 + 2 = 0$$

$\Rightarrow \quad t_1^2 - 4 + 2 = 0$

$\Rightarrow \quad t_1^2 - 2 = 0$

Also, $t_2^2 = \left(-t_1 - \frac{2}{t_1}\right)^2$

$$= t_1^2 + \frac{4}{t_1^2} + 4$$

$$= 2 + 2 + 4 = 8$$

Therefore,

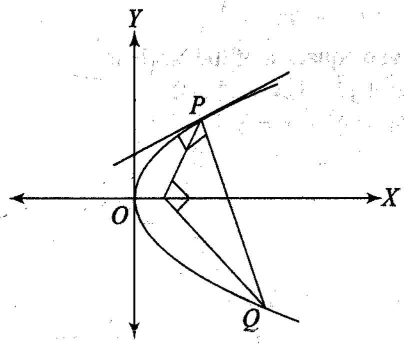
$$PQ^2 = a^2(t_1^2 - t_2^2)^2 + 4a^2(t_1 - t_2)^2$$

$$= 1 \cdot (2 - 8)^2 + 4(\sqrt{2} + 2\sqrt{2})^2$$

$$= 36 + 72 = 108$$

$\Rightarrow \quad PQ = 6\sqrt{3}$

63. The equation of the parabola is $y^2 = 4ax$.



Let the normal chord be PQ, where $P(t_1)$ and $Q(t_2)$.

Since the abscissa and ordinate of the point (p, p) are same, then

$$2at_1 = at_1^2$$

$\Rightarrow \quad t_1 = 2$

If the normal at $P(t_1)$ meets the parabola $Q(t_2)$, then

$$t_2 = -t_1 - \frac{2}{t_1}$$

$\Rightarrow \quad t_2 = -2 - 1 = -3$

Let $S(a, 0)$ be the focus of the parabola $y^2 = 4ax$.

Then the slope of $SP = \frac{2at_1}{at_1^2 - a} = \frac{4}{4-1} = \frac{4}{3}$

and the slope of $SQ = \frac{2at_2}{at_2^2 - a}$

$$= \frac{-6}{9-1} = \frac{-6}{8} = -\frac{3}{4}$$

It is clear that

$$m(SP) \times m(SQ) = \frac{4}{3} \times -\frac{3}{4} = -1$$

Hence, the result.

64. The given equation of the parabola is $y^2 = 4ax$.
The equation of the normal at $P(am^2, -2am)$ is

$$y = mx - 2am - am^3 \quad \text{...(i)}$$

Let Q be a point on the axis of the parabola.
Put $y = 0$ in Eq. (i), we get

$$x = 2a + am^2$$

Hence, the co-ordinates of the point Q is $(2a + am^2, 0)$.
Let $M(h, k)$ be the mid-point of the normal PQ.

Then, $h = \frac{am^2 + 2a + am^2}{2}$

and $k = -\frac{2am}{2} = -am$

Eliminating m, we get

$$h = a + \frac{k^2}{a}$$

$\Rightarrow \quad a^2 + k^2 = ah$

Hence, the locus of $M(h, k)$ is

$$a^2 + y^2 = ax$$

$\Rightarrow \quad y^2 = a(x - a)$

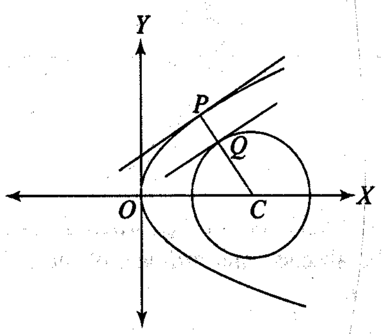
65. The given equation of the parabola is $y^2 = 4x$.
The equation of the normal at $P(m^2, -2m)$ to the parabola $y^2 = 4x$ is

$$y = mx - 2m - m^3 \quad \text{...(i)}$$

The given equation of the circle is

$$x^2 + y^2 - 12x + 31 = 0$$

$\Rightarrow \quad (x-6)^2 + y^2 = 5 \quad$ ...(ii)



The shortest distance between the parabola and the circle lies along the common normal.
Therefore, the centre of a circle passes through the normal, so we have

$$0 = 6m - 2m - m^3$$

$\Rightarrow \quad m^3 - 4m = 0$

$\Rightarrow \quad m = 0, -2, 2$

Therefore, P is $(4, -4)$ or $(4, 4)$ and let $C(6, 0)$ be the centre of the circle and Q be a point on the circle.
Therefore,

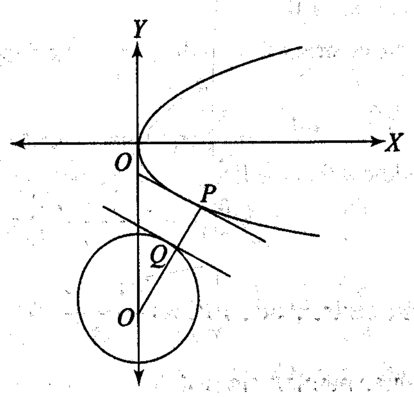
$$CP = \sqrt{(6-4)^2 + (4-0)^2} = \sqrt{20} = 2\sqrt{5}$$

and $CQ = \sqrt{5}$

Thus, the shortest distance $= CP - CQ$

$$= 2\sqrt{5} - \sqrt{5} = \sqrt{5}$$

66. The given equation of the parabola is $y^2 = 8x$.
We have, $4a = 8 \Rightarrow a = 2$.



The equation of the normal to the parabola

$y^2 = 8x$ at $P(4m^2, -4m)$ is

$$y = mx - 4m - 2m^3 \quad \text{...(i)}$$

The given equation of the circle is

$$x^2 + y^2 + 12y + 35 = 0$$

$\Rightarrow \quad x^2 + (y+6)^2 = 1$

Thus, the centre of the circle is $C(0, -6)$.
As we know that the shortest distance between a circle and the parabola lies along the common normal.
Therefore, the normal always passes through the centre of the circle. So

$$-6 = -4m - 2m^3$$

$\Rightarrow \quad m^3 + 2m - 3 = 0$

$\Rightarrow \quad m = 1$

Thus, the point P is $(4, -4)$.
Let Q be any point on the circle.
Then $CQ = 1$ and

$$CP = \sqrt{(4-0)^2 + (-4+6)^2} = \sqrt{20} = 2\sqrt{5}$$

Hence, the shortest distance $= PQ$

$$= CP - CQ = 2\sqrt{5} - 1.$$

67. The equation of any normal to a parabola $y^2 = 4ax$ is
$y = mx - 2am - am^3$
which meets at a point, say (h, k).
Thus, $am^3 + (2a - h)m + k = 0$.
which is a cubic equation in m.
So it has three roots, say m_1, m_2 and m_3.
Therefore, $m_1 + m_2 + m_3 = 0$
Hence the result.

68. Let the ordinates of A, B and C be y_1, y_2 and y_3 respectively.
Then, $y_1 = -2am_1, y_2 = -2am_2, y_3 = -2am_3$.
Thus,
$$y_1 + y_2 + y_3 = -2am_3 - 2am_2 - 2am_3$$
$$= -2a(m_1 + m_2 + m_3) = -2a.0 = 0$$

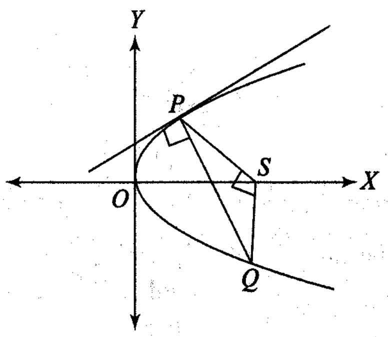
69. If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a ΔABC, then its centroid is
$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$
$$= \left(\frac{x_1 + x_2 + x_3}{3}, 0\right)$$
Hence the centroid lies on the axis of the parabola.
Also,
$$\left(\frac{x_1 + x_2 + x_3}{3}\right) = \frac{1}{3}(am_1^2 + am_2^2 + am_3^2)$$
$$= \frac{a}{3}\left\{0 - 2\left(\frac{2a-h}{a}\right)\right\} = \frac{2h-4a}{3}$$
Thus, the centroid of ΔABC is $\left(\frac{2h-4a}{3}, 0\right)$.

70. When three normals are real, then all the three roots of $am^3 + (2a - h)m + k = 0$ are real.
Let its three roots are m_1, m_2, m_3.
For any real values of m_1, m_2, m_3,
$$m_1^2 + m_2^2 + m_3^2 > 0$$
$$\Rightarrow \quad (m_1 + m_2 + m_3)^2 - 2(m_1m_2 + m_2m_3 + m_3m_1) > 0$$
$$\Rightarrow \quad 0 - 2\left(\frac{2a-h}{a}\right) > 0$$
$$\Rightarrow \quad h - 2a > 0$$
$$\Rightarrow \quad h > 2a$$

71. Let $f(m) = am^3 + (2a - h)m + k$
$$\Rightarrow \quad f'(m) = 3am^2 + (2a - h)\,2a$$
If $f(m)$ has three distinct roots, so $f'(m)$ has two distinct roots.
Let two distinct roots of $f'(m) = 0$ are α and β.
Thus, $\alpha = \sqrt{\left(\frac{h-2a}{3}\right)}$ and $\beta = -\sqrt{\left(\frac{h-2a}{3}\right)}$.
Now, $f(\alpha)f(\beta) = 0$
$$f(\alpha)\,f(-\alpha) = 0$$
$$\Rightarrow \quad (a\alpha^3 + (2a - h)\alpha + k)(-a\alpha^3 - (2a - h)\alpha + k) < 0$$
$$\Rightarrow \quad k^2 - ((a\alpha^2 + (2a - h))^2\,\alpha^2) < 0$$
$$\Rightarrow \quad k^2 - \left(\frac{h-2a}{3} + (2a-h)\right)^2\left(\frac{h-2a}{3}\right) < 0$$
$$\Rightarrow \quad k^2 - \left(\frac{4a-2h}{3}\right)^2\left(\frac{h-2a}{3}\right) < 0$$
$$\Rightarrow \quad k^2 - \frac{4(h-2a)^3}{27a} < 0$$
$$\Rightarrow \quad 27ak^2 < 4(h-2a)^3$$
Hence, the result.

72. Let the normal at $P(at_1^2, 2at_1)$ be $y = -t_1x + 2at_1 + at_1^3$
Thus slope of the normal $= \tan\theta = -t_1$
It meets the parabola again at $Q(at_2^2, 2at_2)$
Then $t_2 = -t_1 - \frac{2}{t_1}$.
Now the angle between the normal and the parabola = angle between the normal and the tangent at Q. If φ be the angle between them, then
$$\tan\varphi = \frac{m_1 - m_2}{1 + m_1m_2}$$
$$= \frac{-t_1 - \frac{1}{t_2}}{1 + (-t_1)\left(\frac{1}{t_2}\right)}$$
$$= -\frac{t_1t_2 + 1}{t_2 - t_1}$$
$$= -\frac{t_1\left(-t_1 - \frac{2}{t_1}\right) + 1}{-t_1 - \frac{2}{t_1} - t_1}$$
$$= -\frac{-t_1^2 - 1}{-2\left(\frac{1 + t_1^2}{t_1}\right)}$$
$$= -\frac{t_1}{2}$$
$$= \frac{\tan\theta}{2}$$
$$\Rightarrow \quad \varphi = \tan^{-1}\left(\frac{\tan\theta}{2}\right)$$

73. The equation of the normal to the parabola $y^2 = 4ax$ at $P(at^2, 2at)$ is
$$y = -tx + 2at + at^3$$



It is given that $at^2 = 2at \Rightarrow t = 2$.
Thus, the co-ordinates of P is $(4a, 4a)$ and focus is $S(a, 0)$.
Also, the normal chord meets the parabola at some point, say Q. Then the co-ordinates of Q is $(9a, -6a)$.
Now,

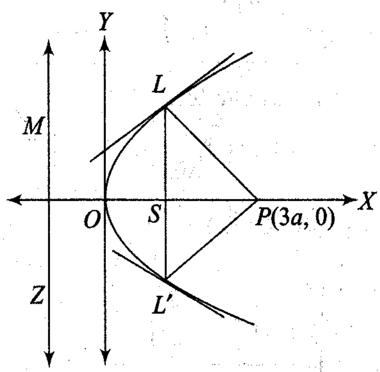
Slope of $SP = m_1 = \dfrac{4a}{3a} = \dfrac{4}{3}$

and the slope of $SQ = m_2 = \dfrac{-6a}{8a} = -\dfrac{3}{4}$.

Thus, $m_1 \times m_2 = \dfrac{4}{3} \times -\dfrac{4}{3} = -1$

Hence, the result.

74. Let the latus rectum be LSL', where $L = (a, 2a)$ and $L' = (a, -2a)$.

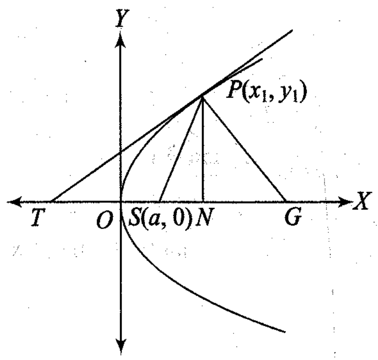


Normal at $L(a, 2a)$ is $x + y = 3a$ ...(i)
Normal at $L'(a, -2a)$ is $x - y = 3a$ ...(ii)
Clearly, (ii) is perpendicular to (i).
Solving, we get

$x = 3a$ and $y = 0$

Hence, the point of intersection is $(3a, 0)$.

75.



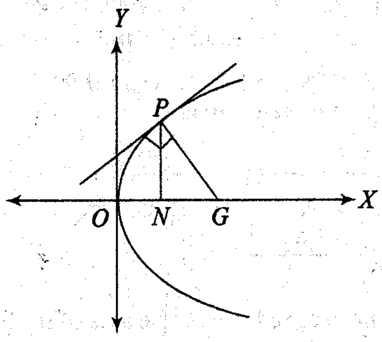
Let $P(x_1, y_1)$ be any point on the parabola $y^2 = 4ax$.
The equation of any tangent and any normal at $P(x_1, y_1)$ are

$$yy_1 = 2a(x + x_1) \text{ and } \frac{y - y_1}{y_1} = -\frac{x - x_1}{2a}$$

Since the tangent and the normal meet its axis at T and G, respectively, so the co-ordinates of T and G are $(-x_1, 0)$ and $(x_1 + 2a, 0)$, respectively.

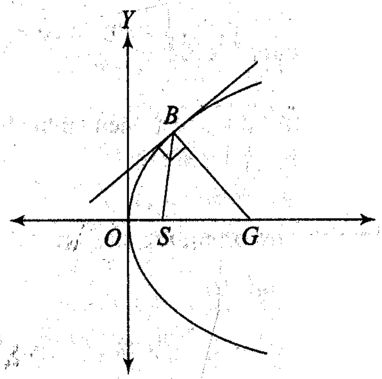
Thus, $SP = PM = x_1 + a$,
$SG = AG - AS = x_1 + 2x - a = x_1 + a$.
and $ST = AS + AT = a + x_1$.
Hence, $SP = SG = ST$.

76. The normal at $P(t)$ is $y = -tx + 2at + at^3$.



It meets the x-axis at G.
Thus the co-ordinates of G be $(2a + at^2, 0)$.
Also N is $(at^2, 0)$.
Thus $NG = 2a + at^2 - at^2 = 2a$ = semi-latus rectum.

77. The normal at $P(t)$ is $y = -tx + 2at + at^3$

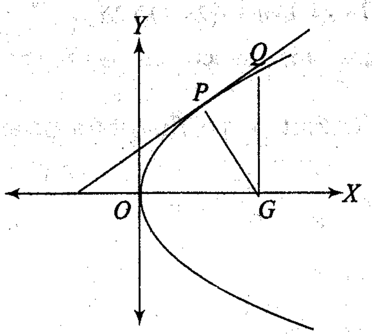


Thus, S is $(a, 0)$, G is $(2a + at^2, 0)$ and P is $(at^2, 2at)$.
Now, $SP = a + x = a + at^2 = a(1^2 + t^2)$

$SG = 2a + at^2 - a = a + at^2 = a(1 + t^2) = SP$

Thus, P and G are equidistant from the focus.

78.



The normal at $P(t)$ is $y = -tx + 2at + at^3$
Thus G is $(2a + at^2, 0)$ and P is $(at^2, 2at)$.
Now, $PG^2 = 4a^2 + 4a^2t^2$ ...(i)
Q is a point on the parabola such that QG is perpendicular to axis so that its ordinate is QG and abscissa is the same as of G.
Hence, the point Q is $(2a + at^2, QG)$.

But Q lies on the parabola $y^2 = 4ax$.

Now,

$$QG^2 = 4a(2a + at^2)$$
$$= 8a^2 + 4a^2t^2$$
$$= (4a^2 + 4a^2t^2) + 4a^2$$
$$= PG^2 + 4a^2$$
$$\Rightarrow \quad QG^2 - PG^2 = 4a^2 = \text{constant.}$$

Hence, the result.

79. The equation of the chord of contact of the tangents from the point (2, 3) to the parabola $y^2 = 4x$ is

$$yy_1 = 2(x + x_1)$$
$$\Rightarrow \quad 3y = 2(x + 2)$$
$$\Rightarrow \quad 2x - 3y + 4 = 0$$

80. The equation of the chord of contact of the tangents to the parabola $y^2 = 12x$ drawn through the point (–1, 2) is

$$2y = 6(x - 1)$$
$$\Rightarrow \quad y = 3x - 3$$

81. Let the point of intersection of the tangents be $R(\alpha, \beta)$. The equation of the tangent to the parabola at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are

Thus, the slopes of the tangents are

$$m_1 = \frac{1}{t_1} \text{ and } m_2 = \frac{1}{t_2}$$

Then the point of intersection of the two tangents be $[at_1t_2, a(t_1 + t_2)]$.

Therefore $\alpha = at_1t_2$ and $\beta = a(t_1 + t_2)$.

Let θ be the angle between the two tangents. Then

$$\tan(\theta) = \left|\frac{\frac{1}{t_1} - \frac{1}{t_2}}{1 + \frac{1}{t_1t_2}}\right| = \left|\frac{t_2 - t_1}{t_1t_2 + 1}\right|$$

$$\Rightarrow \quad (1 + t_1t_2)\tan(\theta) = \left(\sqrt{(t_1 + t_2)^2 - 4t_1t_2}\right)$$

$$\Rightarrow \quad (1 + t_1t_2)^2 \tan^2\theta = (t_1 + t_2)^2 - 4t_1t_2$$

$$\Rightarrow \quad \left(1 + \frac{\alpha}{a}\right)^2 \tan^2\theta = \frac{\beta^2}{a^2} - \frac{4\alpha}{a} = \frac{\beta^2 - 4a\alpha}{a^2}$$

$$\Rightarrow \quad (\alpha + a)^2 \tan^2\theta = (\beta^2 - 4a\alpha)$$

Hence, the locus of (α, β) is

$$(y^2 - 4ax) = (x + a)^2 \tan^2\theta$$

82. The equation of the parabola is $y^2 = 4ax$ and the point (h, k) be P.

Let the tangents from P touch the parabola at $Q(at_1^2, 2at_1)$ and $R(at_2^2, 2at_2)$, then P is the point of intersection of the tangents.

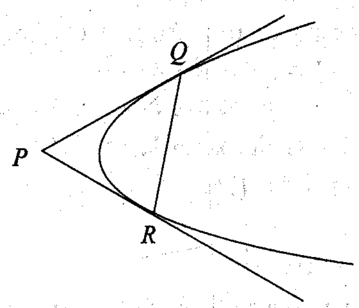
Therefore, $h = at_1t_2$ and $k = a(t_1 + t_2)$

$$\Rightarrow \quad t_1t_2 = \frac{h}{a} \text{ and } (t_1 + t_2) = \frac{k}{a}$$

Now,

$$QR = \sqrt{(at_1^2 - at_2^2)^2 + (2at_1 - 2at_2)^2}$$
$$= |a(t_1 - t_2)|\sqrt{(t_1 + t_2)^2 - 4}$$
$$= |a|\sqrt{\left(\frac{k^2}{a^2} - \frac{4h}{a}\right)}\sqrt{\left(\frac{k^2}{a^2} + 4\right)}$$
$$= \frac{1}{|a|} \times \sqrt{(k^2 - 4ah)(k^2 + 4a^2)}$$

83. Let tangents are drawn from $P(h, k)$ to the parabola $y^2 = 4ax$, intersects the parabola at Q and R.



Then the chord of contact of the tangents to the given parabola is QR.

Then QR is

$$yk = 2a(x + h)$$
$$\Rightarrow \quad 2ax - yk + 2ah = 0$$

Therefore PM = the length of perpendicular from $P(h, k)$ to QR

$$= \left|\frac{2ah - k^2 + 2ah}{\sqrt{k^2 + 4a^2}}\right|$$
$$= \left|-\frac{k^2 - 4ah}{\sqrt{k^2 + 4a^2}}\right|$$
$$= \left|\frac{k^2 - 4ah}{\sqrt{k^2 + 4a^2}}\right|$$

Thus, the area (ΔPQR)

$$= \frac{1}{2} \cdot QR \cdot PM$$
$$= \frac{1}{2} \times \frac{1}{|a|}\sqrt{(k^2 - 4ah)(k^2 + 4a^2)} \times \frac{(k^2 - 4ah)}{\sqrt{k^2 + 4a^2}}$$
$$= \frac{(k^2 - 4ah)^{3/2}}{2a} \text{ if } a > 0$$

84. The equation of the chord of the parabola $y^2 = 4x$, which is bisected at (2, 3) is

$$T = S_1$$
$$\Rightarrow \quad yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$$
$$\Rightarrow \quad 3y - 4(x + 2) = 9 - 8.2 = 7.$$
$$\Rightarrow \quad 3y - 4x - 1 = 0$$
$$\Rightarrow \quad 4x - 3y + 1 = 0$$

85. Let the equation of the parabola be $y^2 = 4ax$.

The equation of the chord of the parabola, whose mid-point (x_1, y_1) is

$T = S_1.$

$\Rightarrow \quad yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$

If it is a focal chord, then it will pass through the focus $(a, 0)$ of the parabola.

Therefore,

$0. y_1 - 2a(x + x_1) = y_1^2 - 4ax_1$

$\Rightarrow \quad y_1^2 = -2a^2 - 2ax_1 + 4ax_1 = 2a(x_1 - a)$

Hence, the locus of (x_1, y_1) is $y^2 = 2a(x - a)$.

86. Let the equation of the parabola be $y^2 = 4ax$.

Let VP be any chord of the parabola through the vertex and $M(h, k)$ be the mid-pont of it.

Then, the co-ordinates of P becomes $(2h, 2k)$.

Since P lies on the parabola, so

$(2k)^2 = 4a \cdot (2h)$

$\Rightarrow \quad 4k^2 = 8ah$

$\Rightarrow \quad k^2 = 2ah$

Hence, the locus of (h, k) is $y^2 = 2ax$.

87. Equation of the normal at any point $(at^2, 2at)$ of the parabola $y^2 = 4ax$ is

$y = -tx + 2at + at^3$ ...(i)

Lep PQ be the normal, whose mid-point is $M(\alpha, \beta)$.

Therefore,

$T = S_1$

$\Rightarrow \quad y\beta - 2a(x + \alpha) = \beta^2 - 3a\alpha$

$\Rightarrow \quad y\beta = 2a(x + \alpha) = (\beta^2 - 4a\alpha)$ ...(ii)

Equations (i) and (ii) are identical.

Therefore, $\dfrac{1}{\beta} = \dfrac{t}{-2a} = \dfrac{2at + at^3}{\beta^2 - 2a\alpha}$

$\Rightarrow \quad t = -\dfrac{2a}{\beta}$ and $\dfrac{t}{-2a} = \dfrac{2at + at^3}{\beta^2 - 2a\alpha}$

From the above two relations, eliminating t, we get

$$\frac{\left(-\dfrac{2a}{\beta}\right)}{-2a} = \frac{2a\left(\dfrac{-2a}{\beta}\right) + a\left(\dfrac{-2a}{\beta}\right)^3}{\beta^2 - 2a\alpha}$$

$$\Rightarrow \quad (\beta^2 - 2a\alpha) = -2a\left(2a + a\left(\frac{-2a}{\beta}\right)^2\right)$$

$\Rightarrow \quad \beta^2(\beta^2 - 2a\alpha) = -2a(2a\beta^2 + 4a^3)$

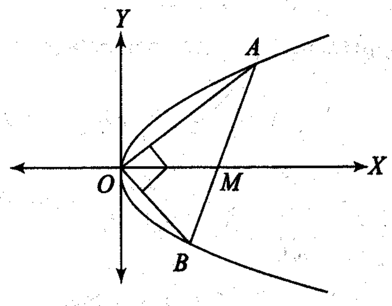
$\Rightarrow \quad \beta^2(\beta^2 - 2a\alpha) = -4a^2\beta^2 - 8a^4)$

$\Rightarrow \quad \beta^4 - 2a(\alpha - 2a)\beta^2 + 8a^4 = 0$

Hence the locus of $M(\alpha, \beta)$ is

$y^4 - 2a(x - 2a)y^2 + 8a^4 = 0$

88. Let QR be the chord and $M(\alpha, \beta)$ be the mid-point of it.



Then the equation of the chord of a parabola $y^2 = 4ax$ at $M(\alpha, \beta)$ is

$T = S_1$

$\Rightarrow \quad y\beta - 2a\,(x + \alpha) = \beta^2 - 2a\alpha$

$\Rightarrow \quad y\beta - 2ax = \beta^2 - 2a\alpha$ ...(i)

Let $V(0, 0)$ be the vertex of the parabola.

The combined equation of VQ and VR, making homogeneous by means of (i), we have

$$y^2 = 4ax \times \left(\frac{y\beta - 2ax}{\beta^2 - 2a\alpha}\right)$$

$\Rightarrow \quad y^2(\beta^2 - 2a\alpha) - 4a\beta xy + 8a^2x^2 = 0$

Since, the chord QR subtends a right angle at the vertex, so we have

co-oefficient of x^2 +co-efficient of $y^2 = 0$

$\Rightarrow \quad (\beta^2 - 2a\alpha) + 8a^2 = 0$

$\Rightarrow \quad \beta^2 = 2a(a - 4a)$

Hence, the locus of $M(\alpha, \beta)$ is

$y^2 = 2a(x - 4a)$

89. Let QR be the chord of the parabola and $M(\alpha, \beta)$ be its mid-point. Then the equation of the chord QR bisected at $M(\beta, \beta)$ is

$T = S_1$

$\Rightarrow \quad y\beta - 2a(x + a) = \beta^2 - 4a\alpha$

$\Rightarrow \quad y\beta = 2a(x + a) + (\beta^2 - 4a\alpha)$

$\Rightarrow \quad y\beta = 2ax + (\beta^2 - 2a\alpha)$ ...(i)

If the Eq. (i) be a tangent to the parabola $y^2 = 4bx$, then

$c = \dfrac{b}{m}$

$\Rightarrow \quad \left(\dfrac{\beta^2 - 2a\alpha}{\beta}\right) = \dfrac{b}{(2a/\beta)} = \dfrac{b\beta}{2a}$

$\Rightarrow \quad (b - 2a)\beta^2 + 4a^2\alpha = 0$

Hence, the locus of $M(\alpha, \beta)$ is

$(2a - b)y^2 = 4a^2x$

90. Let the mid-point of the chord be $M(h, k)$.

Then the equation of the chord at $M(h, k)$ is

$T = S_1$

$\Rightarrow \quad yk - 2a(x + h) = k^2 - 4ah$

which passes through the point $(3b, b)$.

Then, $bk - 2a(3b + h) = k^2 - 4ah$.

$\Rightarrow \quad k^2 - 2ah - bk + 6ab = 0$

Hence, the locus of $M(h, k)$ is

$$y^2 - 2ax - by + 6ab = 0$$

91. The equation of the given parabola is

$$y^2 = 4ax \quad \text{...(i)}$$

The equation of the tangent at $P(at^2, 2at)$ is

$$yt = x + at^2 \quad \text{...(ii)}$$

The equation of the directrix of the parabola

$$y^2 = 4ax \text{ is } x + a = 0 \quad \text{...(iii)}$$

Solving Eqs (ii) and (iii), we get

$$x = -a \text{ and } y = \frac{a(t^2-1)}{t}$$

Thus, the point on the directrix, say Q, whose co-ordinates are $\left(-a, \frac{a(t^2-1)}{t}\right)$.

Let $M(h, k)$ be the mid-point of P and Q. Then

$$h = \frac{at^2 - a}{2} \text{ and } k = \frac{a(t^2-1)}{2t} + \frac{2at}{2}$$

$$\Rightarrow \quad t^2 = \frac{2h+a}{a} \text{ and } 4k^2t^2 = a^2(3t^2-1)^2$$

Eliminating t, we get

$$4k^2\left(\frac{2h+a}{a}\right) = a^2\left(3\left(\frac{2h+a}{a}\right) - 1\right)^2$$

$$\Rightarrow \quad 4k^2(2h+a) = a(6h+3a-a)^2$$

$$\Rightarrow \quad k^2(2h+a) = a(3h+a)^2$$

Hence, the locus of $M(h, k)$ is

$$y^2(2x+a) = a(3x+a)^2$$

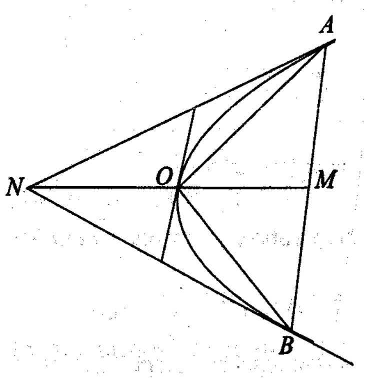
92. Let $y = mx + c$ represents the system of parallel chords. The equation of the diameter to the parabola $y^2 = 4ax$ is $y = \frac{2a}{m}$.

The diameter meets the parabola $y^2 = 4ax$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

The equation of the tangent to the parabola $y^2 = 4ax$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ is $y = mx + \frac{a}{m}$.

which is parallel to $y = mx + c$.

93.



Let AB be the chord, where $A = (at_1^2, 2at_1)$ and $B = (at_2^2, 2at_2)$.

Now,

Slope of AB

$$= m(AB) = \frac{2a(t_2 - t_1)}{a(t_2^2 - t_1^2)} = \frac{2}{t_1 + t_2}$$

The equation of the diameter is

$$y = \frac{2a}{m} \quad \Rightarrow \quad y = a(t_1 + t_2) \quad \text{...(i)}$$

Now the tangents at $A = (at_1^2, 2at_1)$ and $B = (at_2^2, 2at_2)$ meet at $N[at_1t_2, a(t_1 + t_2)]$.

Thus, N lies on $y = a(t_1 + t_2)$.

94. As we know that all rays of light parallel to x-axis of the parabola are reflected through the focus of the parabola.

The equation of the given parabola is

$$(y-4)^2 = 8(x+1)$$

$$\Rightarrow \quad Y^2 = 8X,$$

where $Y = y - 4$ and $X = x + 1$

Now the focus of the parabola is $(a, 0)$.

Therefore,

$$X = a \text{ and } Y = 0$$

$$\Rightarrow \quad x + 1 = 2 \text{ and } y - 4 = 0$$

$$\Rightarrow \quad x = 1 \text{ and } y = 4$$

Hence, the focus is $(1, 4)$.

Thus $\alpha = 1$ and $\beta = 4$

Now, $\alpha + \beta + 10 = 1 + 4 + 10 = 15$.

95. Let the line $y = x + 2$ intersects the parabola at P.

Solving the line $y = x + 2$ and the parabola $y^2 = 4(x + 2)$, we get

the point P is $(2, 4)$.

Now the equation of the tangent to the parabola

$y^2 = 4(x + 2)$ at $P(2, 4)$ is

$$yy_1 = 2(x + x_1) + 8$$

$$\Rightarrow \quad 4y = 2(x+2) + 8$$

$$\Rightarrow \quad x - 2y + 6 = 0$$

Let IP be the incident ray, PM be the reflected ray and PN be the normal

As we know that the normal is equally inclined with the incident ray as well as the reflected ray.

Now the slope of $IP = 1$, slope of normal $PN = -2$ and let the slope of the reflected ray $= m$. Then

$$\frac{1+2}{1-2} = \frac{-2-m}{1-2m}$$

$$\Rightarrow \quad m = \frac{1}{7}$$

Hence, the equation of the reflected ray is

$$y - 2 = \frac{1}{7}(x - 4)$$

$$\Rightarrow \quad 7y - 14 = x - 4$$

$$\Rightarrow \quad x - 7y + 10 = 0$$