Gravitational Waves Physics using Fermi Coordinates

PH-821 : Course Project

by

Sumit Kumar Adhya

(Roll No. 23B1806)

Course Instructor: Prof. Archana Pai

Department of Physics

Indian Institute of Technology Bombay

Mumbai - 400076, India

Contents

1 Introduction

The detection of gravitational waves (GWs) marked the beginning of a new era in multi-messenger astronomy, requiring a multidisciplinary understanding involving astrophysics, engineering, and quantum optics. Consequently, introducing the fundamental concepts of GW science into the undergraduate physics curriculum has become increasingly important. The standard pedagogical approach relies on the Transverse-Traceless (TT) gauge. While mathematically convenient for solving Einstein's field equations in a vacuum, TT coordinates lack direct physical meaning, often leading to misconceptions regarding the physical effects of waves on test masses.

In the TT gauge, the coordinates of a test mass at rest do not change upon the passage of a gravitational wave. The physical effect is only revealed by calculating the proper distance, which is a coordinate-independent scalar. To bridge the gap between mathematical formalism and physical intuition, this report analyzes an alternative approach proposed by Ruggiero(1),(2) using Fermi coordinates. This coordinate system is adapted to the world-line of an observer and relates directly to measurable quantities. Furthermore, this framework allows for a gravitoelectromagnetic (GEM) analogy, where the interaction between GWs and test masses is described by a Lorentz-like force equation involving gravitoelectric and gravitomagnetic fields. This report explores the mathematical derivation of these fields and their application to both static and rotating detectors.

2 Limitations of the Transverse-Traceless Gauge

To appreciate the utility of Fermi coordinates, we need to first understand the behavior of GWs in the standard linearized theory. Einstein's field equations are considered for a metric perturbation $h_{\mu\nu}$ on a Minkowski background $\eta_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1. \tag{1}$$

In the vacuum, the wave equation is $\Box \bar{h}_{\mu\nu} = 0$. The standard solution uses the TT gauge, where the metric takes the form(1):

$$ds^2 = -c^2dt^2 + dx^2 + (1 - h^+)dy^2 + (1 + h^+)dz^2 - 2h^\times dydz, \tag{2}$$

for a wave propagating along the x-axis.

The equation of motion for a test particle is the geodesic equation. For non-relativistic velocities ($v/c \ll 1$), the equation reduces to $d^2x^i/dt^2 \approx -c^2\Gamma^i_{00}$. In the TT gauge, calculation of the Christoffel symbols yields $\Gamma^i_{00} = 0$. Consequently, the coordinate acceleration vanishes:

$$\frac{d^2x^i}{dt^2} = 0. \tag{3}$$

This implies that test masses initially at rest remain at fixed coordinate positions. This result, in fact is true for any gravitational field in the form:

$$ds^2 = -c^2dt^2 + g_{ij}(x^\mu)dx^i dx^j \tag{4}$$

This result is often confusing for students, as it suggests no interaction occurs, but in reality, it's the TT coordinates (and not the physical coordinates) that do not change for particles, initially at rest. The physical reality of the wave is only manifested when integrating the proper distance between masses, which does oscillate with time. For example, for test masses located at $P_1 = (0, 0, 0)$ and $P_2 = (0, L, 0)$, the proper distance d_y between them is obtained using 2:

$$d_y = \int_0^L \sqrt{1 - h^+}dy \simeq \left(1 - \frac{h^+}{2}\right)L \tag{5}$$

which oscillates with time. While the TT gauge is useful for removing gauge-dependent ambiguities, it obscures the local physics experienced by an observer. For future convenience, we recall that in linear approximation in the perturbation:

$$R_{\alpha\mu\beta\nu} = \frac{1}{2}\left(h_{\alpha\nu,\mu\beta} + h_{\mu\beta,\nu\alpha} - h_{\mu\nu,\alpha\beta} - h_{\alpha\beta,\mu\nu}\right) \tag{6}$$

and for the particular TT metric:

$$R_{0\mu\beta\nu} = \frac{1}{2}\left(h_{\mu\beta,\nu 0} - h_{\mu\nu,0\beta}\right) \tag{7}$$

3 The Fermi Frame and Gravitoelectromagnetism

3.1 What are Fermi coordinates?

A Fermi frame is a mathematical realization of a laboratory frame in General Relativity.(2) Unlike global coordinate systems which view spacetime from an arbitrary external perspective, Fermi coordinates are adapted strictly to the world-line of a specific observer. The construction

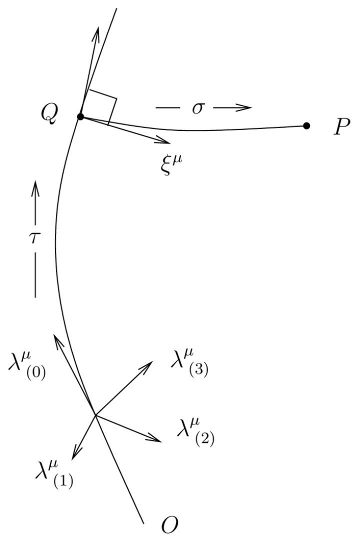

Figure 1: Schematic Construction of Fermi Coordinates

Image credit: Physical Review D. 74. 10.1103/PhysRevD.74.064019.

of this frame relies on an observer moving along a world-line with a specific proper time τ. To define the coordinates:

- **The Time Coordinate (T):** This is defined simply as the proper time measured by the observer's clock ($T = \tau$).
- **The Spatial Coordinates (X):** The position vector $\mathbf{X} = (X, Y, Z)$ is defined physically using space-like geodesics. From any event on the world-line, the observer constructs a set of locally orthogonal spatial axes (a tetrad). The spatial coordinates of a distant point are determined by the proper distance s along a geodesic that is orthogonal to the observer's world-line, such that $\mathbf{X} = s\mathbf{n}$, where $\mathbf{n}$ is the direction vector.

This construction allows for the derivation of a general line element that accounts for both the observer's motion and the background curvature. For an observer who is accelerating with four-acceleration $\mathbf{a}$ and whose spatial axes are rotating with angular velocity $\mathbf{\Omega}$, the spacetime metric in the Fermi frame is expanded up to quadratic order in the distance $|\mathbf{X}|$ as(2):

$$
\begin{aligned}
ds^2 = &-\left[\left(1+\frac{\mathbf{a}\cdot\mathbf{X}}{c^2}\right)^2 - \frac{1}{c^2}(\mathbf{\Omega}\times\mathbf{X})^2 + R_{0i0j}X^iX^j\right]c^2dT^2 \\
&+2\left[\frac{1}{c}(\mathbf{\Omega}\times\mathbf{X})_i - \frac{2}{3}R_{0jik}X^jX^k\right]c\,dT\,dX^i \\
&+\left(\delta_{ij}-\frac{1}{3}R_{ikjl}X^kX^l\right)dX^idX^j. \qquad (8)
\end{aligned}
$$

This general expression highlights the separation of physical effects detectable in the laboratory:

1. Terms containing $\mathbf{a}$ (acceleration) and $\mathbf{\Omega}$ (rotation) represent non-gravitational inertial forces, such as the relativistic redshift and Coriolis/centrifugal forces.
2. Terms containing the Riemann curvature tensor $R_{\alpha\beta\gamma\delta}$ represent the true tidal forces of gravity.

For the specific study of gravitational waves in this report, we simplify this general case to a geodesic observer ($\mathbf{a} = 0$) with non-rotating axes ($\mathbf{\Omega} = 0$), which reduces Eq. (8) to the specific form used in the subsequent gravitoelectromagnetic analysis Eq. (9). Fermi coordinates (cT, X, Y, Z) are constructed in the vicinity of an observer's world-line. They constitute a local inertial frame where the metric is Minkowskian along the reference geodesic, and deviations appear only at second order in the distance from the world-line.

3.2 The Metric in Fermi Coordinates

For an observer freely falling in a gravitational field, the metric in Fermi coordinates, up to quadratic displacements from the reference line (X^i), is given by(1):

$$
\begin{aligned}
ds^2 = &-(1+R_{0i0j}X^iX^j)c^2dT^2 - \frac{4}{3}R_{0jik}X^jX^kc\,dT\,dX^i \\
&+(\delta_{ij}-\frac{1}{3}R_{ikjl}X^kX^l)dX^idX^j, \qquad (9)
\end{aligned}
$$

where $R_{\alpha\beta\gamma\delta}$ is the Riemann curvature tensor evaluated along the reference world-line. This expression naturally incorporates the observer's measurements of space and time.

3.3 Gravitoelectromagnetic Potentials

Neglecting spatial curvature terms, the line element in Eq. (9) can be rewritten using a formal analogy with electromagnetism. We rewrite Eq. (9) as:

$$
ds^2 = -\left(1-2\frac{\Phi}{c^2}\right)c^2dT^2 - \frac{4}{c}(\mathbf{A}\cdot d\mathbf{X})c\,dT + \delta_{ij}dX^idX^j. \qquad (10)
$$

where we define the gravitoelectric potential $\Phi(T,\mathbf{X})$ and the gravitomagnetic vector potential $\mathbf{A}(T,\mathbf{X})$ as follows:

$$
\Phi(T,\mathbf{X}) = -\frac{c^2}{2}R_{0i0j}(T)X^iX^j, \qquad (11)
$$

$$
A_i(T,\mathbf{X}) = \frac{c^2}{3}R_{0jik}(T)X^jX^k. \qquad (12)
$$

This structure parallels the potential formulation of electromagnetism. Specifically, one can define the gravitoelectric field $\mathbf{E}$ and gravitomagnetic field $\mathbf{B}$ via standard relations:

$$
\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial}{\partial T}\left(\frac{1}{2}\mathbf{A}\right), \quad \mathbf{B} = \nabla\times\mathbf{A}. \qquad (13)
$$

Substituting the expressions for the potentials in terms of the curvature tensor, the components of these fields (to linear order in displacement) are derived as:

$$E_i(T, \mathbf{X}) = c^2 R_{0i0j}(T) X^j, \tag{14}$$

$$B_i(T, \mathbf{X}) = -\frac{c^2}{2}\epsilon_{ijk} R^{jk}{}_{0l}(T) X^l. \tag{15}$$

These equations establish that the interaction of the observer with the gravitational field is mediated by tidal forces dependent on the Riemann tensor.

3.4 The Lorentz-like Force Equation

The motion of a test particle in this Fermi frame is governed by the geodesic equation. When expressed using the fields defined above, the equation of motion closely resembles the Lorentz force law in electrodynamics. For a particle of mass m moving with velocity $\mathbf{V} = \frac{d\mathbf{X}}{dT}$, the equation is(1):

$$m\frac{d^2\mathbf{X}}{dT^2} = q_E \mathbf{E} + q_B \frac{\mathbf{V}}{c} \times \mathbf{B}. \tag{16}$$

Here, the "gravitoelectric charge" is $q_E = -m$ and the "gravitomagnetic charge" is $q_B = -2m$. The factor of 2 in the magnetic ratio ($q_B/q_E = 2$) arises because linearized gravity is a spin-2 field, distinguishing it from the spin-1 electromagnetic field.(1) The minus signs reflect the attractive nature of gravity.

Crucially, this formulation provides a direct interpretation of the equivalence principle. Since the gravitoelectromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ vanish along the reference world-line where $\mathbf{X} = 0$, the force vanishes, and a test mass located at the origin moves freely. This recovers the fundamental tenet that in a local freely falling frame, the effects of gravity are transformed away, and the physics reduces to that of Special Relativity.

This equation (16) is the central result utilized to analyze GW detectors in the Fermi frame.

4 Analysis of Plane Gravitational Waves

To apply the GEM formalism to gravitational waves, we utilize the fact that in the weak-field approximation, the Riemann tensor is gauge invariant. We can therefore calculate the Riemann components using the standard TT metric and substitute them into the Fermi field equations. We use in the approximation, following (1), that the extension of the reference frame is much smaller than the wavelength which allows us to neglect the spatial variation of the wave field, and thereby evaluating the components of the Riemann tensor at the origin of our frame, $\mathbf{X} = 0$

For a plane wave propagating along the X-axis with frequency ω, the non-vanishing Riemann components are $R_{0y0y} = -R_{0z0z}$ and R_{0y0z}. Using (2), (7) and utilizing the polarization amplitudes A^+ and $A^\times$, the resulting gravitoelectric field components are(1):

$$\begin{aligned} E_X &= 0, \\ E_Y &= -\frac{\omega^2}{2}[A^+ \sin(\omega T)Y + A^\times \cos(\omega T)Z], \\ E_Z &= -\frac{\omega^2}{2}[A^\times \cos(\omega T)Y - A^+ \sin(\omega T)Z]. \end{aligned} \tag{17}$$

Similarly, the gravitomagnetic field components are derived as(1):

$$\begin{aligned} B_X &= 0, \\ B_Y &= -\frac{\omega^2}{2}[-A^\times \cos(\omega T)Y + A^+ \sin(\omega T)Z], \\ B_Z &= -\frac{\omega^2}{2}[A^+ \sin(\omega T)Y + A^\times \cos(\omega T)Z]. \end{aligned} \tag{18}$$

Interestingly, these fields are transverse to the direction of propagation. Furthermore, it can be shown that $\mathbf{E} \cdot \mathbf{B} = 0$ and $|\mathbf{E}|^2 - |\mathbf{B}|^2 = 0$, surprisingly analogous to the properties of electromagnetic plane waves.

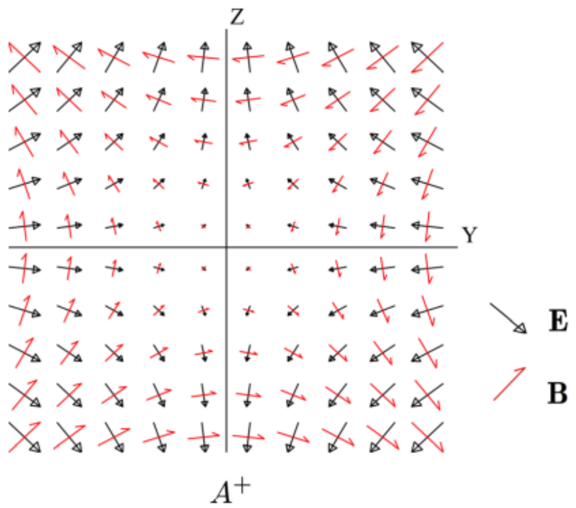

Figure 2: The gravitoelectric E and gravitomagnetic field **B** for a wave with A^+polarization: notice that the two fields orthogonal everywhere.

Image credit: Am. J. Phys. 89, 639–646 (2021)

5 Interaction with Gravitational Wave Detectors

The GEM approach provides a physically intuitive method for analyzing how GWs interact with detectors. We categorize detectors into two types based on the initial state of their test masses: static detectors (like interferometers) and rotating detectors (heterodyne antennas).

5.1 Static Detectors: Gravitoelectric Effects

Consider a standard interferometric detector modeled as two test masses. One mass is located at the origin of the Fermi frame ($X = 0$), and the other is at a position $\mathbf{X}_0 = (0, L, 0)$ along the Y-axis. Before the wave arrives, the masses are at rest, so $\mathbf{V} \approx 0$.

In the Lorentz-like force equation (Eq. 16), the velocity-dependent gravitomagnetic term vanishes. The dynamics are governed solely by the gravitoelectric field $\mathbf{E}$. For a wave with only $+$ polarization ($A^\times = 0$), the equation of motion is:

$$\frac{d^2Y}{dT^2} = -E_Y = \frac{\omega^2}{2} A^+ \sin(\omega T) L. \tag{19}$$

Integrating this equation twice with respect to time yields the trajectory of the test mass:

$$Y(T) = L\left[1 - \frac{A^+}{2}\sin(\omega T)\right]. \tag{20}$$

This result shows that the coordinate distance $Y(T)$ oscillates. Unlike the TT gauge, where the coordinate position is fixed, the Fermi coordinate describes the measurable physical distance. This derivation precisely matches the proper distance calculation from the standard theory, but is achieved through a force-based Newtonian analogy that is conceptually more accessible to students.

5.2 Rotating Detectors: Gravitomagnetic Effects

The pedagogical strength of the Fermi coordinate approach is most evident when analyzing moving masses. In this regime, the gravitomagnetic field **B** exerts a force that is absent in the static case.

Consider a "heterodyne antenna," a detector concept proposed by Braginskij in the 1970s(1). The model consists of two dumbbells crossed at a 90-degree angle, rotating in the YZ-plane with frequency ω_0. Let the arm length be R. The position of a mass m_1 at time T is $Y_1 = R\sin(\omega_0 T)$ and $Z_1 = R\cos(\omega_0 T)$.

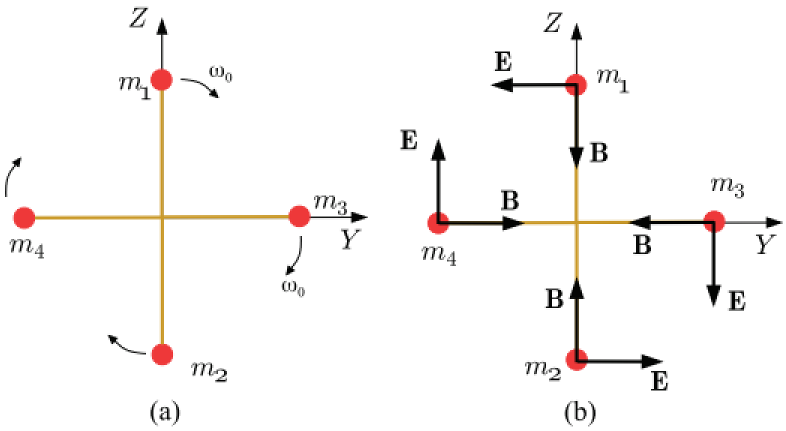

Figure 3: Two identical dumbbells are made by test masses $m_1 = m_2 = m_3 = m_4 = m$ at fixed distance R, and rotated by $\pi/2$ with respect each other. They independently rotate with frequency ω_0, before the passage of the wave. (b) The gravitoelectromagnetic fields acting on the test masses, due the passage of the wave.

Image credit: Am. J. Phys. 89, 639–646 (2021)

5.2.1 Gravitoelectric Resonance

First, we analyze the effect of the electric-like field. Assuming a circularly polarized wave ($A^+ = A^\times = A$) and the resonance condition $\omega_0 = \omega/2$, the gravitoelectric force components 18 acting on mass m_1 simplify to:

$$
\begin{aligned}
F^E_{1,Y} &= \frac{m\omega^2 AR}{2}\cos\left(\frac{\omega}{2}T\right), \\
F^E_{1,Z} &= -\frac{m\omega^2 AR}{2}\sin\left(\frac{\omega}{2}T\right). \qquad (21)
\end{aligned}
$$

This force is orthogonal to the dumbbell arm and has a constant magnitude. It produces a torque $\tau_{12} = -m\omega^2 AR^2 \mathbf{u}_X$ on the dumbbell that accelerates its rotation. Simultaneously, the second crossed dumbbell experiences a torque in the opposite direction but same magnitude. The observable effect is a change in the angular separation between the dumbbells $\Delta(\theta)(T) = (\pi/2) - \delta\theta(T) = (\pi/2) - (1/2)\omega^2 AT^2$, driven purely by the gravitoelectric (tidal) forces.

5.2.2 Gravitomagnetic Propulsion

The analysis extends beyond tidal torques when including the gravitomagnetic field. The mass m_1 has a velocity **V** due to its rotation. The gravitomagnetic force is given by $\mathbf{F}^B = -2m(\mathbf{V}/c)\times\mathbf{B}$. Since, we are just working in the linear order, we use the velocity of the system before the passage of the wave amplitude in this expression.

Using the resonance condition $\omega_0 = \omega/2$ and the field expressions in Eq. (**??**), the gravitomagnetic field at the mass's location is:

$$B_Y = -\frac{\omega^2 AR}{2}\sin\left(\frac{\omega}{2}T\right), \quad B_Z = -\frac{\omega^2 AR}{2}\cos\left(\frac{\omega}{2}T\right). \tag{22}$$

This field vector points toward the center of rotation. However, the cross product with the tangential velocity $\mathbf{V}$ yields a force directed along the X-axis (the direction of wave propagation). The mass m_1 undergoes the force $\mathbf{F}_1^B = \frac{m\omega^3 AR^2}{2c}\mathbf{u}_X$. The other mass experiences the same force. The total force on the first dumbbell therefore comes out to be:

$$\mathbf{F}_{12}^B = \frac{m\omega^3 AR^2}{c}\mathbf{u}_X. \tag{23}$$

Conversely, the second dumbbell, rotating with a phase difference, experiences a force in the opposite direction ($-\mathbf{u}_X$). This results in a relative linear acceleration between the two dumbbells along the propagation axis which results in the distance between them to change with time as $d(T) = (\omega^3 AR^2/c)T^2$. This effect is strictly gravitomagnetic and arises from the coupling of the wave's "magnetic" component with the mass current of the detector.

6 Physical Interpretation: Energy and Momentum

The interaction of the rotating detector with the GW highlights the capacity of the wave to transfer linear momentum. In the Fermi frame, this can be understood via the Poynting vector analogy. The gravitational Poynting vector $\mathbf{P}$ is defined as:

$$\mathbf{P} = \frac{c}{4\pi G}\mathbf{E} \times \mathbf{B}. \tag{24}$$

For the circular polarization considered in the heterodyne example, the Poynting vector acting on the first dumbbell aligns with the propagation direction, while it opposes it for the second dumbbell. This alignment explains the forces $\mathbf{F}_{12}^B$ and $\mathbf{F}_{34}^B$ derived in the previous section.

This derivation offers a concrete demonstration that gravitational waves carry linear momentum and energy, analogous to the famous "sticky bead" argument by Feynman and Bondi. While the sticky bead argument relies on friction (thermodynamics), the Fermi coordinate approach allows students to derive the momentum transfer using strictly mechanical equations of motion ($F = ma$) within a defined coordinate system.

7 Conclusion

The teaching of gravitational wave physics faces a hurdle in the abstract nature of the Transverse-Traceless coordinates. By adopting Fermi coordinates, we can map the complex tensorial interaction of General Relativity onto a gravitoelectromagnetic framework. This approach yields a Lorentz-like force equation that separates the interaction into tidal (electric) and velocity-dependent (magnetic) components.

As demonstrated in this report, this formalism successfully reproduces standard results for static interferometric detectors, describing the signal as a physical displacement caused by an electric-like force. Moreover, it allows for the intuitive analysis of rotating detectors, where gravitomagnetic forces reveal the transfer of momentum from the wave to the detector. By grounding the study of GWs in measurable quantities and electromagnetic analogies, the Fermi coordinate perspective offers a robust and accessible pedagogical tool for undergraduate physics education.

References

[1] Ruggiero, M. L. Gravitational waves physics using Fermi coordinates: A new teaching perspective. *American Journal of Physics* **89**(6), 639–646 (2021).

[2] Ruggiero, M. L., & Ortolan, A. Gravito-electromagnetic approach for the space-time of a plane gravitational wave. *Journal of Physics Communications* **4**(5), 055013 (2020).