Inflation and how it solves the Horizon and the Flatness Problems

Course Project, PH813: Advanced Topics in Astroparticle Physics

by

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January 8, 2026

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Chapter 1

Problems with Big-Bang Cosmology

1.1 The Standard Model: ΛCDM

The Standard Model of Cosmology, ΛCDM, describes the large-scale dynamics of the Universe with the Friedmann–Lemaître–Robertson–Walker (FLRW) metric and a stress-energy content dominated by radiation, non-relativistic matter and a cosmological constant. The Friedmann equation may be written as

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \qquad \rho = \rho_r + \rho_m + \rho_\Lambda, \tag{1.1}$$

where ρ_r, ρ_m and ρ_Λ denote the radiation, matter (baryons plus cold dark matter) and vacuum energy densities respectively, and k is the spatial curvature constant. The critical density and density parameters are

$$\rho_{\text{crit}} = \frac{3H^2}{8\pi G}, \qquad \Omega_i \equiv \frac{\rho_i}{\rho_{\text{crit}}} \quad (i = r, m, \Lambda), \qquad \Omega_k \equiv -\frac{k}{a^2H^2}, \tag{1.2}$$

giving the closure relation

$$1 = \Omega_r + \Omega_m + \Omega_\Lambda + \Omega_k. \tag{1.3}$$

The components scale as $\rho_r \propto a^{-4}$, $\rho_m \propto a^{-3}$, and ρ_Λ =const, which leads to the standard sequence of cosmic epochs: radiation domination, matter domination, and eventual Λ-domination. Observations are consistent with a spatially flat Universe ($\Omega_k \approx 0$) and $\Omega_{m,0} \simeq 0.3$, $\Omega_{\Lambda,0} \simeq 0.7$ (within current uncertainties). The ΛCDM model successfully accounts for the cosmic microwave background anisotropies, large-scale structure and the expansion history, while fails at addressing two key problems in cosmology- The Horizon and the Flatness Problems. Below we describe what they are and then go on describing how the Inflation Theory solves these problems.

1.2 The Horizon Problem

About 380,000 years after the Big Bang, the universe had cooled enough to allow the formation of the first hydrogen atoms. In this process, photons decoupled from the primordial plasma. We observe this event in the form of the cosmic microwave background (CMB), an afterglow of the hot Big Bang. Remarkably, this radiation is almost perfectly isotropic, with anisotropies in the CMB temperature being smaller than one part in ten thousand. A moment's thought will convince you that the finiteness of the conformal time elapsed between $t_i = 0$ and the time of the formation of the CMB, t_{rec}, implies a serious problem: it means that most parts of the CMB have non-overlapping past light cones and hence never were in causal contact. The Horizon problem is therefore basically the fact that two widely separated regions in sky looking so similar in CMB. An obvious solution would be to propose that the two different similar looking regions we observe in the sky today were once causally connected in the early universe and could influence each other causally, but as we will see below, in the standard ΛCDM cosmology, causality can explain only $1° - 2°$ of the CMB homogeneity we observe today. The decoupling, or the last scattering, is thought to have occurred about 380,000 years after the Big Bang, or at a redshift of about $z_{rec} \approx 1100$. The physical size of the universe then was around 380,000 ly. To compare how much that would amount to today, we use the comoving particle horizon (2.1) at t_{rec}. One can derive using the definition covered in the next chapter that the comoving particle horizon at t_{rec} was $\sim 1Gly$. The distance to the surface of last scattering today is $\sim 46Gly$. This gives the angular diameter of a causally connected region in the CMB at t_{rec} as observed today to be:

$$\theta \approx \frac{1}{46}\text{rad} \approx 1.3° \tag{1.4}$$

We therefore would expect any region of the CMB within $1° \sim 2°$ of angular separation to have been in causal contact, but at any scale larger than $2°$ there should have been no exchange of information. The horizon problem describes the fact that we see isotropy in the CMB temperature across the entire sky, despite the entire sky not being in causal contact to establish thermal equilibrium. The homogeneity of the CMB spans scales that are much larger than the particle horizon (2.1) at the time when the CMB was formed. If there wasn't enough time for these regions to communicate, why do they look so similar? This is the horizon problem.

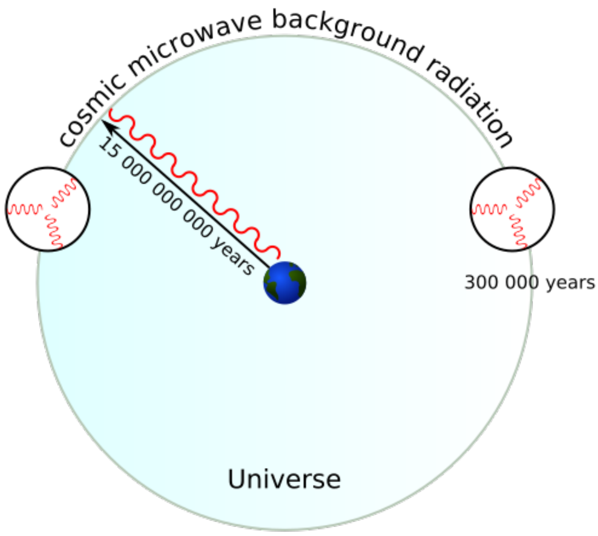

Figure 1.1: When we look at the CMB it comes from 46 billion comoving light-years away. However, when the light was emitted the universe was much younger (300,000 years old). In that time light would have only reached as far as the smaller circles. The two points indicated on the diagram would not have been able to contact each other because their spheres of causality do not overlap.

Image credit: Theresa Knott / Wikimedia Commons (public domain).

1.3 The Flatness Problem

The flatness problem is a cosmological fine-tuning problem within the Big Bang model of the universe. The problem was first mentioned by Robert Dicke in 1969. Measurements find the current universe close to perfectly flat and expansion of the universe increases flatness. In standard cosmology based on the Friedmann equations the density of matter and energy in the universe affects the curvature of space-time, with a very specific critical value being required for a flat universe. The current density of the universe is observed to be very close to this critical value. As we will see below, any departure of the total density from the critical value would increase rapidly over cosmic time, the early universe must have had a density even closer to the critical density, departing from it by one part in 10^{-62} or less. This leads cosmologists to question how the initial density came to be so closely fine-tuned to this 'special' value. We elaborate on the problem below: Starting with the Friedmann equation $H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$. Dividing both sides by H^2, we get

$$1 = \Omega - \frac{k}{a^2 H^2}, \quad \text{where} \quad \Omega \equiv \frac{8\pi G \rho}{3H^2} = \frac{\rho}{\rho_{\text{crit}}}. \tag{1.5}$$

We consider the derivative:

$$\frac{d\Omega}{d\ln a} = -\frac{2k}{a^2 H^4} \times \frac{\ddot{a}}{a} \tag{1.6}$$

We then use the second Friedmann equation $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) = -\frac{1}{2}(1+3w)\Omega H^2$, to write this as:

$$\frac{d\Omega}{d\ln a} = \frac{k}{a^2 H^2} \times (1+3w)\Omega \tag{1.7}$$

Using Eq.(1.5), we get:

$$\frac{d\Omega}{d\ln a} = (1+3w)\Omega(\Omega - 1) \tag{1.8}$$

Clearly, $\Omega = 1$ is a solution, but it turns out to be an unstable solution unless $1 + 3w < 0$. We can see this easily by perturbing the density parameter $\Omega(t) = 1 \pm \varepsilon(t)$, with $\varepsilon \ll 1$. At linear order in ε, we get:

$$\frac{d\varepsilon}{d\ln a} = (1+3w)\varepsilon \tag{1.9}$$

with solution

$$\varepsilon(a) = \varepsilon_i \left(\frac{a}{a_i}\right)^{1+3w} \tag{1.10}$$

If $1 + 3w > 0$ (e.g. matter or radiation) then the perturbation grows as the universe expands, a clear sign of an instability. - That's the flatness problem

1.3.1 How much fine-tuning?

In the standard thermal history, the Universe is radiation dominated ($w = 1/3$) until matter-radiation equality and matter dominated ($w = 0$) thereafter. Evolving ε from an initial time i through equality to today gives (using $a \propto 1/T$)

$$\varepsilon_0 = \varepsilon_i \frac{a_0\, a_{\text{eq}}}{a_i^2} = \varepsilon_i \frac{T_i^2}{T_0\, T_{\text{eq}}} \quad \Rightarrow \quad \varepsilon_i = \varepsilon_0 \frac{T_0\, T_{\text{eq}}}{T_i^2}. \tag{1.11}$$

Taking representative values $T_0 = 2.725$ K, $T_{\text{eq}} \simeq 9 \times 10^3$ K and a Planck-scale initial temperature $T_i \simeq 1.42 \times 10^{32}$ K yields

$$\frac{T_0 T_{\text{eq}}}{T_i^2} \simeq 1.2 \times 10^{-60}. \tag{1.12}$$

Hence, for a present-day curvature deviation $\varepsilon_0 \sim 10^{-2}$ one finds

$$\varepsilon_i \sim 10^{-2} \times 1.2 \times 10^{-60} \sim 1.2 \times 10^{-62}, \tag{1.13}$$

i.e. the early Universe must have been flat to one part in $\sim 10^{62}$ (for these assumptions). That's a very high level of fine-tuning!!!

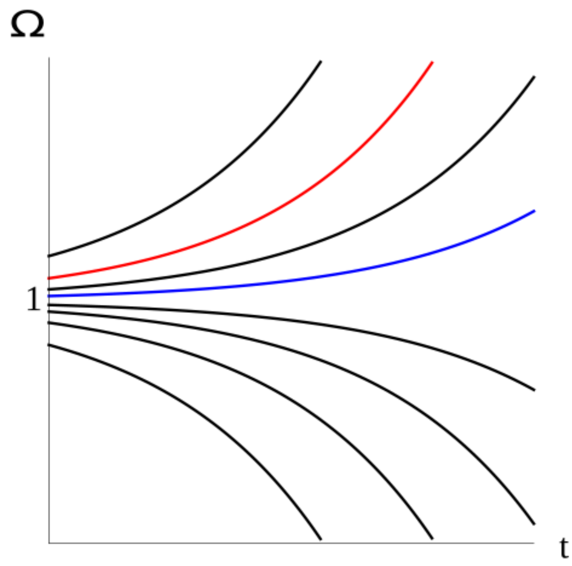

Figure 1.2: The relative density Ω against cosmic time t (neither axis to scale). Each curve represents a possible universe: note that Ω diverges rapidly from 1. The blue curve is a universe similar to our own, which at the present time (right of the graph) has a small $|\Omega - 1|$ and therefore must have begun with Ω very close to 1. The red curve is a hypothetical different universe in which the initial value of Ω differed slightly too much from 1: by the present day it has diverged extremely and would not be able to support galaxies, stars or planets.

Image credit: Olaf Davis / Wikimedia Commons (public domain).

1.3.2 Why is Fine-Tuning a problem?

Naturalness and the hierarchy problem: The notion of *naturalness* in theoretical physics captures the intuition that, in the absence of a symmetry or dynamical mechanism, dimensionless parameters should be "order unity". A common precise statement is the *technical naturalness* (or 't Hooft) criterion: a parameter is natural (or technically natural) if setting it to zero increases the symmetry of the theory. When no such symmetry exists, an extremely small parameter requires an unexplained cancellation or a special choice of initial conditions.

A central example is the *hierarchy problem* of particle physics. The Higgs mass (or its squared mass) is sensitive to ultraviolet physics: quantum corrections typically generate contributions of order the square of the high-energy cutoff. Keeping the measured weak scale much smaller than the Planck scale therefore requires delicate cancellations between bare parameters and radiative corrections. That perceived improbability — the need to tune parameters to enormous precision so that the low-energy Higgs mass stays small — is what is usually called the hierarchy problem.

Analogy with cosmological fine-tuning. The curvature (or "flatness") fine-tuning discussed above is the cosmological analogue of the hierarchy problem. With the representative numbers used above one obtains

$$\varepsilon_i \sim 10^{-62},$$

i.e. the early Universe must have been spatially flat to one part in $\sim 10^{62}$. This is directly analogous to the hierarchy problem: a quantity that naively could be "order unity" must instead be specified to absurd precision.

Chapter 2

The Solution- Inflation Theory

2.1 Some Terminologies

We start with some terminologies. The size of a causal patch of space is defined by how far light can travel in a certain amount of time. Since, photons travel along null geodesics and we can take them to travel along the radial direction (since spacetime is isotropic), their path is defined by:

$$\mathrm{d}s^2 = a^2(\tau)\left[\mathrm{d}\tau^2 - \mathrm{d}\chi^2\right] = 0 \implies \Delta\chi(\tau) = \pm\Delta\tau \tag{2.1}$$

where τ is the conformal time and we have used $\mathrm{d}\chi \equiv \mathrm{d}r/\sqrt{1-kr^2}$, where r is the radial co-ordinate and k is the curvature.

- *Particle Horizon* - It is defined as the maximum comoving distance from which light could have travelled to an observer since the big-bang. From Eq.(2.1), we see that this is just given by:

$$\chi_{\mathrm{ph}}(\tau) = \tau - \tau_i = \int_{t_i}^{t} \frac{\mathrm{d}t}{a(t)} = \int_{a_i}^{a} \frac{\mathrm{d}a}{a\dot{a}} = \int_{\ln a_i}^{\ln a} (aH)^{-1}\,\mathrm{d}\ln a \tag{2.2}$$

- *Hubble sphere/Hubble horizon* - It is the comoving distance light can travel in one Hubble time $t_H = 1/H$, $r_H = (aH)^{-1}$. Beyond this radius ($r > r_H$), the recessional velocity of galaxies ($v = \dot{a}r > \dot{a}r_H = aHr_H = 1$), due to expansion of the universe will be faster than light. Also, it is an approximate boundary of causal contact at a given cosmic time, since by comparing the comoving separation λ of two particles with $(aH)^{-1}$, one can determine whether the particles can communicate with each other at a given moment (i.e. within the next Hubble time).

Let's make the following distinction between these two definitions:

- if $\lambda > \chi_{\mathrm{ph}}$, then the particles could never have communicated.
- if $\lambda > (aH)^{-1}$, then the particles cannot talk to each other now.

where we take $t_i = 0$ for the Big-Bang singularity. Any causal influence to the observer has to come from within this. We see that the causal structure of spacetime can therefore be related to the evolution of the *Hubble radius* (aH^{-1}). From the Friedmann Equations, we can get the following relation for a universe dominated by a fluid with the equation of state $P = w\rho$: $(aH)^{-1} = H_0^{-1}a^{\frac{1}{2}(1+3w)}$ which using Eq.(2.2) then leads to the following relation:

$$\chi_{\mathrm{ph}}(t) = \frac{2H_0^{-1}}{(1+3w)}a(t)^{\frac{1}{2}(1+3w)} = \frac{2}{(1+3w)}(aH)^{-1} \tag{2.3}$$

taking $a_i = 0 \xrightarrow{w>-\frac{1}{3}} \tau_i = 0$. For $w > -1/3$, as with all familiar matter sources, the comoving Hubble radius and thereby the Particle horizon increases as the universe expands.

2.2 Solution to the Horizon Problem

From our previous discussion, we can conclude that a growing Hubble sphere corresponds to more and more regions coming in causal contact and a shrinking Hubble sphere would mean more and more regions getting causally disconnected. The horizon problem, as discussed earlier is basically the temperature of the CMB being almost same everywhere even in regions, which according to standard Big Bang expansion were never in causal contact. A simple solution to this problem would be to conjecture that these regions were in causal contact in the early universe and later got disconnected $\implies$ a decreasing Hubble sphere in the early universe:

$$\frac{d}{dt}(aH)^{-1} < 0 \tag{2.4}$$

The shrinking Hubble sphere would therefore require a fluid which obeys $1 + 3w < 0$(as will be shown below). τ_i is no longer 0. The Big Bang singularity is now pushed to negative conformal time.

$$\tau_i = \frac{2H_0^{-1}}{(1+3w)} a_i^{\frac{1}{2}(1+3w)} \xrightarrow{a_i \to 0, w < -\frac{1}{3}} -\infty \tag{2.5}$$

Consequences of a shrinking Hubble sphere:

- Accelerated expansion: We have the relation- $\frac{d}{dt}(aH)^{-1} = \frac{d}{dt}(\dot{a})^{-1} = -\frac{\ddot{a}}{(\dot{a})^2}$. We see clearly that a shrinking Hubble sphere $\frac{d}{dt}(aH)^{-1} < 0$ implies an accelerated expansion $\ddot{a} > 0$.

- Slowly-varying Hubble parameter: We can write-

$$\frac{d}{dt}(aH)^{-1} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\frac{1}{a}(1-\varepsilon) \tag{2.6}$$

  where $\varepsilon \equiv -\frac{\dot{H}}{H^2}$. A shrinking Hubble sphere would therefore imply $\varepsilon = -\frac{\dot{H}}{H^2} < 1 \implies$ a slowly varying Hubble parameter.

- Negative Pressure: Considering a perfect fluid with pressure P and density ρ, the Friedmann Equation, $H^2 = \frac{\rho}{3M_{pl}^2}$ and the continuity equation together imply:

$$\dot{H} + H^2 = -\frac{1}{6M_{\rm pl}^2}(\rho + 3P) = -\frac{H^2}{2}\left(1 + \frac{3P}{\rho}\right) \tag{2.7}$$

  Rearranging them gives, $\varepsilon = -\frac{\dot{H}}{H^2} = \frac{3}{2}\left(1 + \frac{P}{\rho}\right) < 1 \Leftrightarrow w \equiv \frac{P}{\rho} < -\frac{1}{3}$.

- Constant density: Combining the continuitiy equation, $\dot{\rho} = -3H(\rho + P)$, with Eq.(2.7), we find $\left|\frac{d\ln\rho}{d\ln a}\right| = 2\varepsilon < 1$. For small ε, the energy density is therefore nearly constant. Conventional matter sources all dilute with expansion, so we need to look for something more unusual.

2.2.1 Required duration of inflation

For the observable Universe to have originated from a single causal patch during inflation we require that the comoving Hubble radius at the beginning of inflation exceeded the comoving Hubble radius today,

$$(a_0 H_0)^{-1} < (a_I H_I)^{-1}. \tag{2.8}$$

Split the expansion history at the end of inflation (subscript E) and assume, for simplicity, instantaneous reheating followed by radiation domination and then matter domination.

$$\frac{a_0 H_0}{a_E H_E} \sim \frac{a_0 H_0}{a_{eq} H_{eq}} \frac{a_{eq} H_{eq}}{a_E H_E} \sim \frac{a_0 a_{eq}^{3/2}}{a_{eq} a_0^{3/2}} \frac{a_{eq} a_E^2}{a_E a_{eq}^2} \sim \frac{\sqrt{T_0 T_{eq}}}{T_E} \tag{2.9}$$

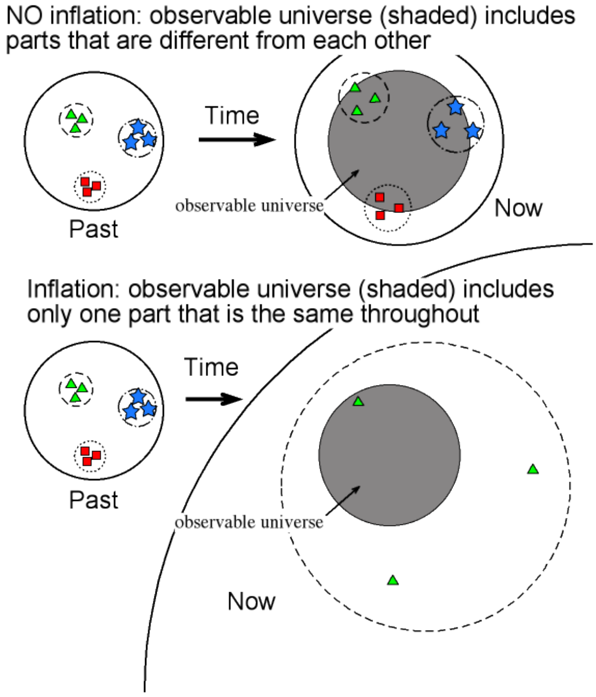

Figure 2.1: How inflation solves the Horizon problem
Image credit: Nick Strobel, Astronomy Notes (www.astronomynotes.com).

where "eq" stands for matter radiation equality. Hence, taking $T_{eq} = 0.7eV$, $T_0 = 10^{-3}eV$ and $T_E = 10^{15}GeV$ (GUT scale temperature)

$$(a_I H_I)^{-1} \gtrsim (a_0 H_0)^{-1} \sim 10^{28} \, (a_E H_E)^{-1} \tag{2.10}$$

and (for $H_I \approx H_E$)

$$\frac{a_E}{a_I} \gtrsim 10^{28} \quad \Longrightarrow \quad N \equiv \ln \frac{a_E}{a_I} \gtrsim \ln(10^{28}) \simeq 64 \tag{2.11}$$

where $N \equiv \ln(a_E/a_I)$ is the number of e-folds of inflation. Using a representative reheating temperature $T_E \sim 10^{15}$ GeV motivated the numerical factor 10^{28} above; different assumptions about the reheating history or T_E shift the required N by a few units but do not change the conclusion: inflation must last for of order 50–70 e-folds to solve the horizon problem.

2.3 Solution to the Flatness Problem

Starting from the relation derived in Chapter 1,

$$\frac{d\Omega}{d\ln a} = (1 + 3w)\,\Omega(\Omega - 1), \tag{2.12}$$

perturb the density parameter about flatness,

$$\Omega = 1 \pm \varepsilon, \qquad \varepsilon \ll 1,$$

to obtain at linear order

$$\frac{d\varepsilon}{d\ln a} = (1 + 3w)\,\varepsilon \quad \Longrightarrow \quad \varepsilon(a) = \varepsilon_i \left(\frac{a}{a_i}\right)^{1+3w}. \tag{2.13}$$

Thus any epoch with $1 + 3w < 0$ drives $\varepsilon \to 0$ as the scale factor increases. In particular, for $w \simeq -1$ after N e-folds,

$$\varepsilon_{\text{post}} = \varepsilon_{\text{pre}}\, e^{-2N}. \tag{2.14}$$

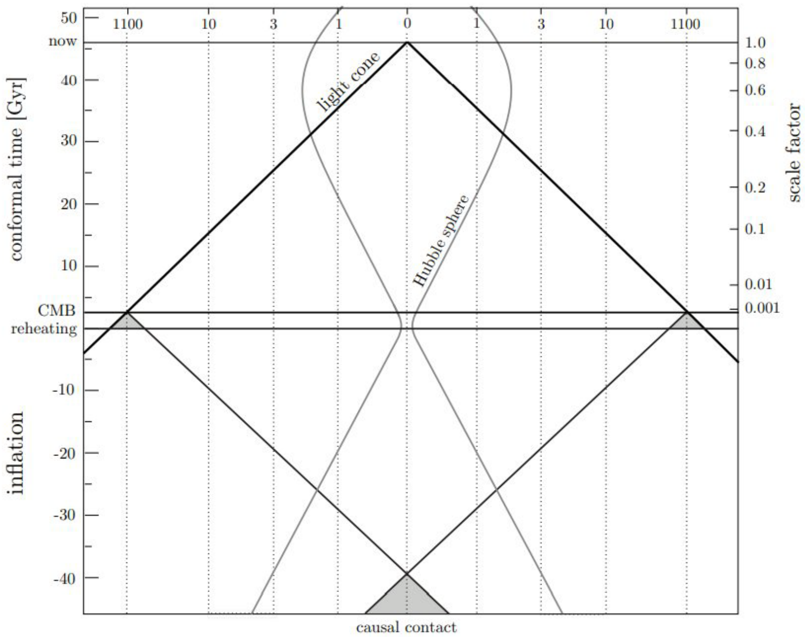

Figure 2.2: A pictorial representation to the inflationary solution.
Image credit: Daniel Baumann, *Cosmology: Lecture Notes*

Hence a prolonged early acceleration exponentially suppresses curvature. To reduce ε by a factor F requires

$$N > \tfrac{1}{2}\ln F. \tag{2.15}$$

For example, to erase a curvature deviation of order unity down to $\varepsilon \sim 10^{-60}$ one needs $N \gtrsim \frac{1}{2}\ln(10^{60}) \simeq 69$, while $N \sim 50$–60 is sufficient in more conservative (GUT-scale) estimates. The key point is temporal: an *early* epoch with $w < -1/3$ lasting many e-folds (inflation) dynamically drives $\Omega \to 1$, removing the extreme fine-tuning required in the standard radiation/matter history.

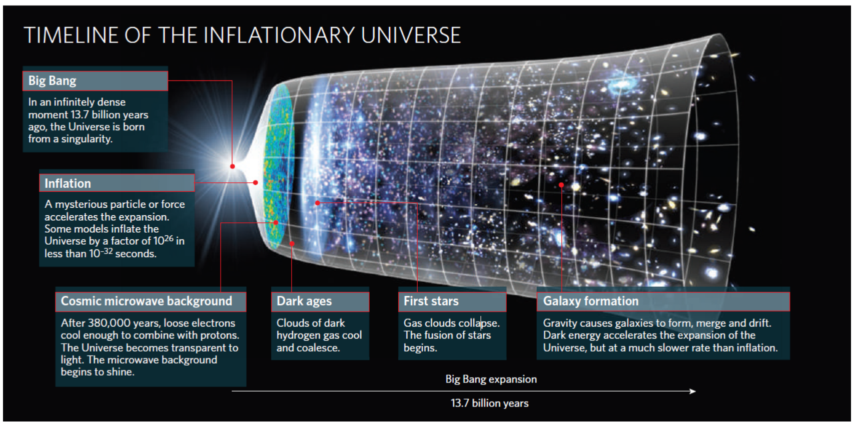

Figure 2.3: Timeline of the Inflationary Universe. Image credit: NASA/WMAP SCIENCE TEAM.

Chapter 3

Inflation as a Scalar field Theory

3.1 Dynamics of the field

To explain the physics of inflation in a much more rigorous manner, we develop a scalar field theory with a scalar field called the inflaton denoted by $\phi(t, \mathbf{x})$, We have a potential energy density $V(\phi)$ associated with each value of the field. The stress-energy tensor of this scalar field is given by:

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu}(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - V(\phi)) \tag{3.1}$$

Now due to the symmetries of FRW metric, the background value of the inflaton must depend only on time. From the stress-energy tensor we have:

$$T^0_0 = \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{3.2}$$

Hence the total energy density is simply the sum of the kinetic and potential energy densities. Also we have:

$$T^i_j = -P_\phi \delta^i_j = -\Big[\frac{1}{2}\dot{\phi}^2 - V(\phi)\Big]\delta^i_j \tag{3.3}$$

Now for inflation we want $P_\phi < -\frac{1}{3}\rho_\phi$, i.e., the potential energy should dominate over the kinetic energy. Substituting ρ_ϕ into the first Friedmann equation, we obtain:

$$H^2 = \frac{1}{3M_{pl}^2}\Big[\frac{1}{2}\dot{\phi}^2 + V(\phi)\Big] \tag{3.4}$$

Taking a time derivative we obtain:

$$2H\dot{H} = \frac{1}{3M_{pl}^2}\Big[\dot{\phi}\ddot{\phi} + \frac{dV}{d\phi}\dot{\phi}\Big] \tag{3.5}$$

Substituting ρ_ϕ and P_ϕ into the second Friedmann equation, we obtain:

$$\dot{H} = -\frac{1}{2}\frac{\dot{\phi}^2}{M_{pl}^2} \tag{3.6}$$

Combining the above two equations, we get the equation of motion for our scalar field (The klein-gordon equation):

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \tag{3.7}$$

We can interpret that the potential provides a force and the expansion of universe acts as a drag term in the evolution of the scalar field.

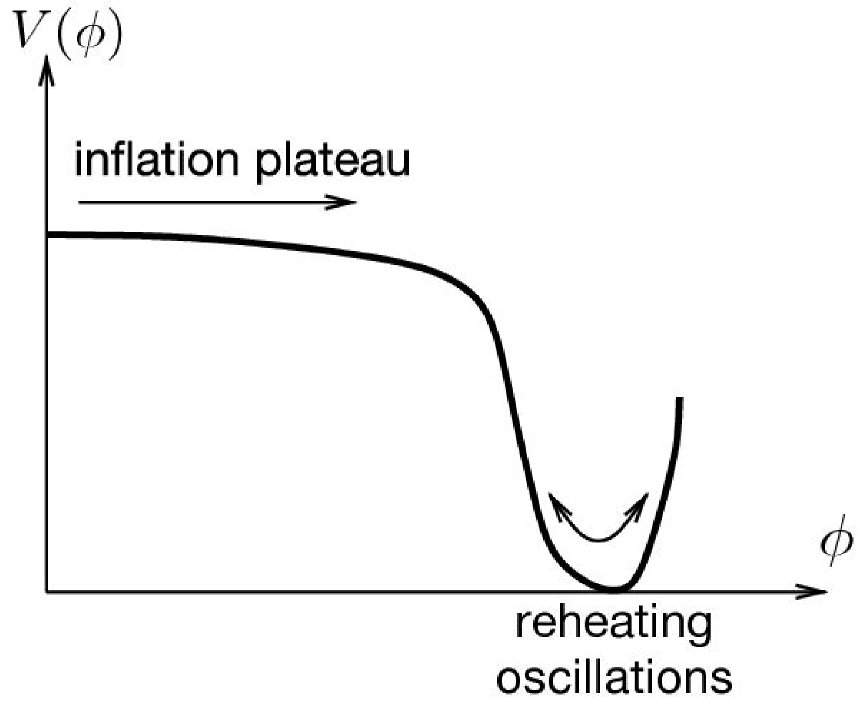

Figure 3.1: A slow roll potential

Image credit: Năstase, H. (2019). Slow-Roll Inflation. In: Cosmology and String Theory.

3.2 Slow roll inflation

We have $\epsilon = -\frac{\dot{H}}{H^2}$ as defined in the previous chapter. Substituting the expression for $\dot{H}$, we obtain:

$$\epsilon = \frac{\dot{\phi}^2}{2M_{pl}^2 H^2} \tag{3.8}$$

Since $\epsilon < 1$, inflation only occurs when the Kinetic energy $\frac{1}{2}\dot{\phi}^2$ makes only a small contribution to the total energy $\rho_\phi = 3M_{pl}^2 H^2$. This condition is called the slow roll condition. This means that the acceleration of the scalar field has to be small. We define another parameter δ as the following:

$$\delta = -\frac{\ddot{\phi}}{H\dot{\phi}} \tag{3.9}$$

Taking time derivative of Eq.(3.8), we have:

$$\dot{\epsilon} = \frac{\dot{\phi}\ddot{\phi}}{M_{pl}^2 H^2} - \frac{\dot{\phi}^2 \dot{H}}{M_{pl}^2 H^3} \tag{3.10}$$

Using this expression for $\dot{\epsilon}$, the parameter η can then be written as:

$$\eta = \frac{\dot{\epsilon}}{H\epsilon} = 2(\epsilon - \delta) \tag{3.11}$$

Hence $\epsilon, |\delta| \ll 1$ implies that $\epsilon, |\eta| \ll 1$. Also the conditions $\epsilon \ll 1$ (which means kinetic energy is much less than the potential energy) and $|\delta| \ll 1$ simplify the first Friedmann equation and the KG equation respectively as:

$$H^2 = \frac{V(\phi)}{3M_{pl}^2} \tag{3.12}$$

$$3H\dot{\phi} = -\frac{dV}{d\phi} \tag{3.13}$$

These equations provides a simple expression of ϵ as:

$$\epsilon \approx \frac{M_{pl}^2}{2}\Big(\frac{dV/d\phi}{V}\Big)^2 = \epsilon_v \tag{3.14}$$

Also taking the time derivative of the simplified KG equation gives:

$$3\dot{H}\dot{\phi} + 3H\ddot{\phi} = -\frac{d^2V}{d\phi^2}\dot{\phi} \tag{3.15}$$

Using which we can define a new parameter as the following:

$$\eta_v = \delta + \epsilon = -\frac{\ddot{\phi}}{H\dot{\phi}} - \frac{\dot{H}}{H^2} = M_{pl}^2\Big(\frac{d^2V/d\phi^2}{V}\Big) \tag{3.16}$$

Hence, in order to assess whether a given potential can lead to slow-roll inflation is to compute the potential slow-roll parameters ϵ_v and $|\eta_v|$ and check if they satisfy $\epsilon_v, |\eta_v| \ll 1$ or not.

3.3 Amount of Inflation

The total number of 'e-folds' of accelerated expansion are:

$$N_{\rm tot} \equiv \int_{a_S}^{a_E} d\ln a = \int_{t_S}^{t_E} H(t)\,dt \tag{3.17}$$

In the slow-roll regime, we can use:

$$H dt = \frac{H}{\dot{\phi}}\,d\phi = \frac{1}{\sqrt{2\epsilon}}\frac{|\mathrm{d}\phi|}{M_{\rm pl}} \approx \frac{1}{\sqrt{2\epsilon_V}}\frac{|\mathrm{d}\phi|}{M_{\rm pl}} \tag{3.18}$$

N_{tot} as an integral in the field space of the inflaton is thus given by:

$$N_{\rm tot} = \int_{\phi_S}^{\phi_E} \frac{1}{\sqrt{2\epsilon_V}}\frac{|d\phi|}{M_{\rm pl}} \tag{3.19}$$

where ϕ_S and ϕ_E are defined as the boundaries of the interval where $\epsilon_V < 1$. The largest scales observed in the CMB are produced about 60 e-folds before the end of inflation:

$$N_* = \int_{\phi_*}^{\phi_E} \frac{1}{\sqrt{2\epsilon_V}}\frac{|d\phi|}{M_{\rm pl}} \approx 60 \tag{3.20}$$

A successful solution to the horizon problem hence requires $N_{\rm tot} > N_* \approx 60$.

3.4 An example - $m^2\phi^2$ inflation

Consider a simple model of inflation, a mass term which acts as the potential given by:

$$V(\phi) = \frac{1}{2}m^2\phi^2 \tag{3.21}$$

The slow roll parameters are hence:

$$\epsilon_v = \eta_v = 2\Big(\frac{M_{pl}}{\phi}\Big)^2 \tag{3.22}$$

Hence for $\epsilon_v, |\eta_v| < 1$, we need $\phi > \sqrt{2}M_{pl} = \phi_E$ The number of e-folds before the end of inflation is hence:

$$N_{tot} = \int_{\phi_E}^{\phi} \frac{1}{\sqrt{2\epsilon_V}}\frac{d\phi}{M_{\rm pl}} = \frac{\phi^2}{4M_{pl}^2} - \frac{1}{2} \tag{3.23}$$

Hence the fluctuations in CMB were created at Super-Planckian values of ϕ given by:

$$\begin{aligned} N_{tot} = N_* = 60 \approx \frac{\phi_*^2}{4M_{pl}^2} \\ \implies \phi_* \approx 15M_{pl} \end{aligned} \tag{3.24}$$

3.5 Reheating

During inflation, most of the energy density of the Universe resides in the inflaton potential $V(\phi)$. Inflation ends when the potential steepens and the inflaton field acquires kinetic energy. The subsequent transfer of energy to the Standard Model particles initiates the **reheating** phase, marking the onset of the Hot Big Bang.

3.5.1 Scalar Field Oscillations

After inflation, the inflaton field ϕ oscillates around the minimum of its potential. We can assume the following form of the potential:

$$V(\phi) = \frac{1}{2}m^2\phi^2 \tag{3.25}$$

and for small amplitude homogeneous oscillations, the field satisfies:

$$\ddot{\phi} + 3H\dot{\phi} = -m^2\phi \tag{3.26}$$

When the expansion time scale becomes much longer than the oscillation period ($H^{-1} \gg m^{-1}$), the friction term can be neglected, and the oscillations occur with frequency m.
The energy continuity equation is

$$\dot{\rho}_\phi + 3H\rho_\phi = -3HP_\phi = -\frac{3}{2}H(m^2\phi^2 - \dot{\phi}^2) \tag{3.27}$$

Averaging over oscillations makes the right-hand side go to zero, giving $\rho_\phi \propto a^{-3}$, which implies that the oscillating inflaton behaves like pressureless matter.

3.5.2 Inflaton Decay

To prevent an empty universe, the inflaton must decay into Standard Model fields. If the decay is slow (for instance, when the inflaton can only decay into fermions), the inflaton energy density follows:

$$\dot{\rho}_\phi + 3H\rho_\phi = -\Gamma_\phi\rho_\phi \tag{3.28}$$

where Γ_ϕ is the inflaton decay rate. If decay occurs via bosonic channels, Bose enhancement can trigger rapid decay through **parametric resonance**, a process known as **preheating**.

3.5.3 Thermalisation

Particles produced from inflaton decay interact and thermalise to form a plasma with temperature $T_{\rm rh}$. This **reheating temperature** is related to the inflaton energy density at the end of reheating, $\rho_{\rm rh} < \rho_{\phi,E}$.
If thermalisation is efficient, the Universe becomes radiation dominated, setting the initial conditions for the standard Hot Big Bang. If it proceeds slowly, some relics (such as gravitinos) may never reach equilibrium due to their weak interactions.

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