GW Physics in Fermi Coordinates

PH 821: Course Project

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Outline

Introduction: The Pedagogical Gap

Teaching Gravitational Waves (GW) requires bridging the gap between abstract General Relativity and physically measurable effects.

Standard Approach (TT Gauge)

- **Focus:** Mathematical simplicity.
- **Metric:** Transverse-Traceless.
- **Result:** Test masses remain at fixed coordinate locations.
- Students confuse coordinates with physical position.

Proposed Approach (Fermi Frame)

- **Focus:** Operational measurement.
- **Metric:** Local expansion.
- **Result:** Coordinates represent proper distances.
- Intuitive "Force" analogy (GEM).

The "Motionless" Particle Paradox

Consider a test mass initially at rest in the Transverse-Traceless (TT) gauge. The geodesic equation is:

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} \approx -c^2\Gamma^i_{00} \tag{1}$$

when $|v_i|/c << 1$. In the TT gauge, $\Gamma^i_{00} = 0$. This leads to a confusing result:

Coordinate Acceleration Vanishes

$$\frac{d^2x^i}{dt^2} = 0 \implies x^i(t) = \text{constant}$$

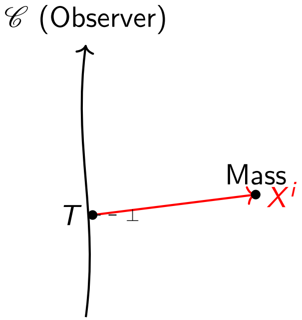
- **Misconception:** "Gravitational waves have no effect on test masses."
- **Reality:** The coordinates oscillate *with* the wave. To see the effect, one must calculate the **proper distance**, which is integral-dependent and non-local.

Fermi Coordinates: The Local Laboratory

To restore intuition, we use **Fermi Normal Coordinates** (T, X^i). This system is adapted to a specific observer, simulating a local inertial laboratory.

Construction:

1. **Origin:** The observer moves along a reference world-line $\mathscr{C}$ (a geodesic).
2. **Time (T):** The proper time measured by the observer's clock along $\mathscr{C}$.
3. **Space (X^i):** Defined by shooting spacelike geodesics orthogonal to $\mathscr{C}$ at time T.



Key Property: Coordinates X^i directly measure the **proper distance** from the observer (for $|X| \ll \mathscr{R}$).

Fermi Coordinates

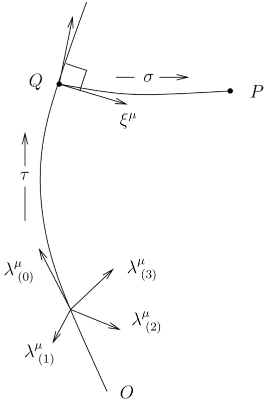


Figure: Schematic Construction of Fermi Coordinates

Image credit: Physical Review D. 74. 10.1103/PhysRevD.74.064019.

The Metric in Fermi Coordinates

We expand the metric $g_{\mu\nu}$ in a Taylor series around the observer's worldline ($|\mathbf{X}| \ll \mathscr{R}$).

For a freely falling, non-rotating observer, the first derivatives vanish ($g_{\mu\nu,\alpha} = 0$). The **second derivatives** (curvature) dominate[1]:

$$ds^2 \approx \underbrace{-(1 + R_{0i0j} X^i X^j) c^2 dT^2}_{\text{Scalar-like Potential}} \underbrace{- \frac{4}{3} c R_{0jik} X^j X^k (c dT) dX^i}_{\text{Vector-like Potential}} + \delta_{ij} dX^i dX^j \quad (2)$$

Why this form?

- It resembles the weak-field expansion of General Relativity.
- The R_{0i0j} and R_{0jik} terms can be mapped directly to **Gravito-Electric** and **Gravito-Magnetic** potentials.

Gravitoelectromagnetic (GEM) Analogy

We can rewrite the metric components using potentials Φ and $\mathbf{A}$[1]:

$$ds^2 = -\left(1 - 2\frac{\Phi}{c^2}\right) c^2 dT^2 - \frac{4}{c}(A \cdot dX)dt + \delta_{ij} dX^i dX^j \tag{3}$$

GEM Potentials

$$\Phi(T, \mathbf{X}) = -\frac{c^2}{2} R_{0i0j}(T) X^i X^j \quad \text{(Scalar Potential)}$$

$$A_i(T, \mathbf{X}) = \frac{c^2}{3} R_{0jik}(T) X^j X^k \quad \text{(Vector Potential)}$$

Defining fields via standard relations $\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial}{\partial T}(\frac{1}{2}\mathbf{A})$ and $\mathbf{B} = \nabla \times \mathbf{A}$:
$E_i = c^2 R_{0i0j} X^j, \quad B_i = -\frac{c^2}{2} \varepsilon_{ijk} R_{0l}^{jk} X^l$

Fermi Coordinates and the Metric

The spacetime metric $g_{\mu\nu}$ in Fermi coordinates is given by the expansion[2]:

$$g_{00}/c^2 = -1 - R_{0i0j}X^iX^j + \dots \qquad g_{0i}/c = -\frac{2}{3}R_{0jik}X^jX^k + \dots$$
$$g_{ij} = \delta_{ij} - \frac{1}{3}R_{ikjl}X^kX^l + \dots$$

Here, $R_{\alpha\beta\gamma\delta}$ represents the Riemann curvature tensor projected onto the observer's orthonormal tetrad $\lambda^{\mu}_{(\alpha)}$[2]:

$$R_{\alpha\beta\gamma\delta} = R_{\mu\nu\rho\sigma}\lambda^{\mu}_{(\alpha)}\lambda^{\nu}_{(\beta)}\lambda^{\rho}_{(\gamma)}\lambda^{\sigma}_{(\delta)} \tag{4}$$

Along the reference geodesic $T = \tau, \mathbf{X} = 0$ and $g_{\mu\nu} = \eta_{\mu\nu}$. These expansions as discussed earlier are only admissible within a cylindrical spacetime region of radius $\sim \mathscr{R}$. Here $\mathscr{R}$ is the curvature scale of the spacetime.

Deriving the Lorentz-like Force

Applying the standard geodesic equation to the Fermi metric yields:

$$\frac{d^2X^i}{dT^2} + \underbrace{c^2 R_{0i0j} X^j}_{\text{Gravitoelectric}} + \underbrace{2cR_{ikj0} V^k X^j}_{\text{Gravitomagnetic}} = 0 \tag{5}$$

By defining the GEM fields $E_i = c^2 R_{0i0j} X^j$ and $B_i = -\frac{c^2}{2}\varepsilon_{ijk} R^{jk}_{0l} X^l$, this equation transforms into the Lorentz-like force law:

$$\boxed{m\frac{d^2\mathbf{X}}{dT^2} = q_E \mathbf{E} + q_B \frac{\mathbf{V}}{c} \times \mathbf{B}} \tag{6}$$

- **Charges:** $q_E = -m$ and $q_B = -2m$.
- **Equivalence Principle:** At $\mathbf{X} = 0$, the fields vanish $\implies$ a mass at the origin moves freely ($F = 0$).

Analysis of Plane Gravitational Waves

Since upto linear order in $h_{\mu\nu}$, the Riemann tensor in invariant under cooordinate transformations, we use the TT values:
$h^{+} = A^{+} sin(\omega t - kx), \quad h^{\times} = A^{\times} cos(\omega t - kx)$
Gravitoelectric Field (E):

$$E_Y = -\frac{\omega^2}{2}[A^{+}\sin(\omega T)Y + A^{\times}\cos(\omega T)Z]$$
$$E_Z = -\frac{\omega^2}{2}[A^{\times}\cos(\omega T)Y - A^{+}\sin(\omega T)Z]$$

Gravitomagnetic Field (B):

$$B_Y = -\frac{\omega^2}{2}[-A^{\times}\cos(\omega T)Y + A^{+}\sin(\omega T)Z]$$
$$B_Z = -\frac{\omega^2}{2}[A^{+}\sin(\omega T)Y + A^{\times}\cos(\omega T)Z]$$

Note: Both fields are transverse to propagation and orthogonal to each other ($\mathbf{E}\cdot\mathbf{B} = 0$).
And also, $|\mathbf{E}| = |\mathbf{B}|$

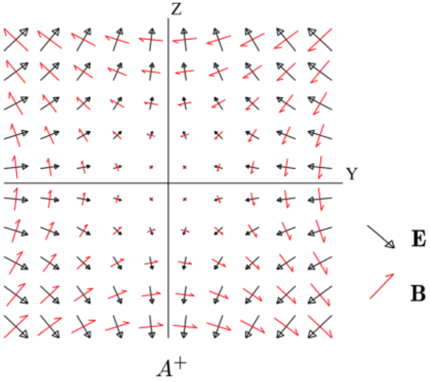


Figure: The gravitoelectric E and gravitomagnetic field **B** for a wave with A^+polarization: notice that the two fields orthogonal everywhere.

Image credit: Am. J. Phys. 89, 639–646 (2021)

Static Detectors: Gravitoelectric Effects

Consider an interferometer: Mass 1 at origin, Mass 2 at $\mathbf{X}_0 = (0, L, 0)$.
Initial velocity $\mathbf{V} = 0 \implies$ Magnetic term vanishes.

$$\frac{d^2 Y}{dT^2} = -E_Y = \frac{\omega^2}{2} A^+ \sin(\omega T) L$$

Integrating twice:

$$Y(T) = L \left[1 - \frac{A^+}{2} \sin(\omega T) \right] \tag{7}$$

- **Result:** The *coordinate* distance oscillates.
- This matches the physical proper distance calculation.

Note: The wave produces a change $\delta \mathbf{V}(T) = O(A)$ in the velocity, but in linear order we ignore it.

Rotating Detectors (Heterodyne Antenna)

- Proposed by Braginskij during 1970s.
- Two dumbbells crossed at 90°.
- Rotating in YZ-plane with frequency ω_0.
- Arm length R.

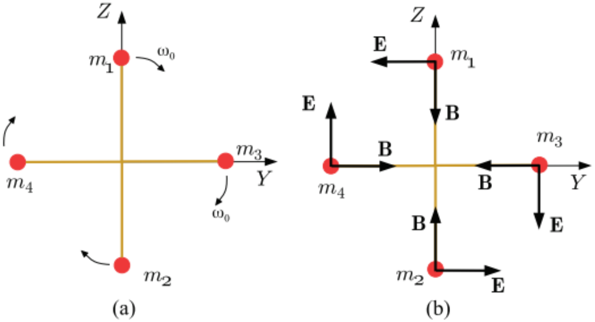


Figure: Rotating Detectors

Image credit: Am. J. Phys. 89, 639–646 (2021)

Effect 1: Gravitoelectric Resonance (Heterodyne Detector)

- For m_1, $\mathbf{X}_1(T) = (0, R\sin\omega_0 T, R\cos\omega_0 T)$
- Circular polarization ($A^+ = A^\times = A$).

$$F^E_{1,Y} = \frac{m\omega^2 AR}{2}\cos[(\omega-\omega_0)T], \quad F^E_{1,Z} = -\frac{m\omega^2 AR}{2}\sin[(\omega-\omega_0)T] \quad (8)$$

Resonance: Set $\omega_0 = \omega/2$.

- The force components now rotate *synchronously* with the mass:

$$\mathbf{F}^E_1 = \frac{m\omega^2 AR}{2}\left(0,\ \cos\frac{\omega}{2}T,\ -\sin\frac{\omega}{2}T\right) \quad (9)$$

- **Resulting Torque** $\tau_{12} = -m\omega^2 AR^2 \hat{u}_X$

This constant torque accelerates the dumbbell, while the other dumbell gets deccelerated, the angular separation $\Delta(\theta)(T) = \pi/2 - (1/2)\omega^2 AT^2$.

Effect 2: Gravitomagnetic Propulsion (Momentum Transfer)

The rotating mass m_1 also possesses a velocity $\mathbf{V}$, which couples to the Gravitomagnetic field $\mathbf{B}$ via the force $\mathbf{F}^B = -2m\frac{\mathbf{V}}{c} \times \mathbf{B}$.

1. **Tangential Velocity** (at $\omega_0 = \omega/2$): $\mathbf{V}_1 = \frac{d\mathbf{X}_1}{dT} = \frac{\omega R}{2}\left(0, \cos\frac{\omega}{2}T, -\sin\frac{\omega}{2}T\right)$
2. **Gravitomagnetic Field**: $\mathbf{B}_1 = -\frac{\omega^2 AR}{2}\left(0, \sin\frac{\omega}{2}T, \cos\frac{\omega}{2}T\right)$
3. **Lorentz Cross Product** $(\mathbf{V} \times \mathbf{B})$: Only the X-component survives.

Multiplying by the charge factor $(-2m/c)$ and summing for both masses $(m_1 + m_2)$ yields a **constant propulsive force**:

$$\boxed{\mathbf{F}_{12}^B = \frac{m\omega^3 AR^2}{c}\mathbf{u}_X} \tag{10}$$

The other dumbbell will experience the force in the opposite direction, thus changing their distance: $d(T) = (\omega^3 AR^2/c)T^2$

Physical Interpretation: Energy & Momentum

How do we explain the propulsion effect physically?
The Gravitational Poynting Vector:

$$\mathbf{P} = \frac{c}{4\pi G}\mathbf{E} \times \mathbf{B} \tag{11}$$

- For the first dumbbell, $\mathbf{P}$ aligns with propagation $\implies$ Momentum absorption.
- For the second dumbbell, $\mathbf{P}$ opposes propagation.

Pedagogical Value

This offers a mechanical derivation ($F = ma$) of the fact that **GWs carry linear momentum**, analogous to the famous thermodynamic "sticky bead" argument.

Conclusion

- **Fermi Coordinates** bridge the gap between abstract GR tensors and local laboratory measurements.
- The **GEM Analogy** provides a clean, Lorentz-like force equation:

$$m\ddot{\mathbf{X}} = q_E \mathbf{E} + q_B \frac{\mathbf{V}}{c} \times \mathbf{B}$$

- **Static Detectors:** Explained via Electric-like tidal forces.
- **Rotating Detectors:** Reveal Magnetic-like forces and momentum transfer.
- This framework is physically intuitive, and highly suitable for undergraduate physics education.

References

M. L. Ruggiero, "Gravitational waves physics using Fermi coordinates: A new teaching perspective", *Am. J. Phys.* **89**, 639–646 (2021).

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Thank You

Questions?