Inflation and how it solves the Horizon and the Flatness Problem

Sachin, Sumit Kumar Adhya, Yash Gupta,

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The Standard Model of Cosmology, ΛCDM, describes the large-scale dynamics of the Universe with the FLRW metric and a stress-energy content dominated by radiation, non-relativistic matter and a cosmological constant. The Friedmann equation may be written as

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \qquad \rho = \rho_r + \rho_m + \rho_\Lambda,$$

where $\rho_r, \rho_m, \rho_\Lambda$ denote radiation, matter and vacuum energy densities, and k is the curvature constant. The critical density and density parameters are

$$\rho_{\mathrm{crit}} = \frac{3H^2}{8\pi G}, \qquad \Omega_i \equiv \frac{\rho_i}{\rho_{\mathrm{crit}}}\,(i = r, m, \Lambda), \qquad \Omega_k \equiv -\frac{k}{a^2 H^2},$$

giving the closure relation

$$1 = \Omega_r + \Omega_m + \Omega_\Lambda + \Omega_k.$$

The components scale as $\rho_r \propto a^{-4}$, $\rho_m \propto a^{-3}$, $\rho_\Lambda =$const.

The Horizon Problem

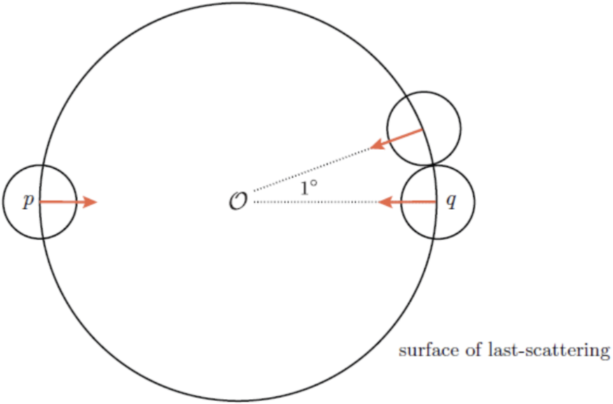
About 380,000 years after the Big Bang photons decoupled and the cosmic microwave background (CMB) was released. The CMB is almost perfectly isotropic (temperature anisotropies $\lesssim 10^{-4}$) as observed today. When we look at two widely separated regions of the sky, we observe the CMB temperatures to be almost same. But How? Were they causally connected at some the time when CMB was released ($t_{rec} = 380,000$ years or $z_{rec} \approx 1100$)? Let's figure it out.

- The physicle size of the particle horizon at t_{rec} 1.3 bly.
- Distance to the last scattering surface comes out to be 46 bly.
- Angle subtended by a causally connected patch of sky to us today $\approx (\frac{1.3}{46} \times \frac{180}{\pi})^\circ = 1.6^\circ \approx 2^\circ \implies$ that's the well known result.

So causality explains only 2° of CMB homogeneity, not the entire sky!!!

Why this is a problem

Regions of the CMB separated by angles larger than $\sim 2^\circ$ should not have been in causal contact at recombination, yet the temperature is uniform to high precision across the whole sky. The homogeneity of the CMB thus spans scales much larger than the particle horizon at decoupling. If there wasn't enough time for these regions to communicate, why do they look so similar? That's the horizon problem.



The Flatness Problem

The flatness problem is a fine-tuning issue: observations show the universe is very close to spatially flat today. Starting from the Friedmann equation

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2},$$

divide by H^2 to get

$$1 = \Omega - \frac{k}{a^2 H^2}, \qquad \Omega \equiv \frac{8\pi G\rho}{3H^2} = \frac{\rho}{\rho_{\mathrm{crit}}}.$$

Taking the derivative with respect to $\ln a$,

$$\frac{d\Omega}{d\ln a} = -\frac{2k}{a^2 H^4} \cdot \frac{\ddot{a}}{a}.$$

Using the second Friedmann equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) = -\tfrac{1}{2}(1+3w)\Omega H^2,$$

one obtains

$$\frac{d\Omega}{d\ln a} = (1+3w)\Omega(\Omega - 1).$$

How much fine-tuning?

Thus $\Omega = 1$ (spatially flat) is an unstable solution when $1 + 3w > 0$ (radiation/matter), that's the flatness problem. Perturb $\Omega = 1 \pm \varepsilon$ with $\varepsilon \ll 1$. At linear order,

$$\frac{d\varepsilon}{d\ln a} = (1+3w)\varepsilon \quad \Longrightarrow \quad \varepsilon(a) = \varepsilon_i \left(\frac{a}{a_i}\right)^{1+3w}.$$

With the standard thermal history (radiation $w = \frac{1}{3}$ until equality, then matter $w = 0$), evolving ε from an initial time i through equality to today (using $a \propto 1/T$) gives

$$\varepsilon_0 = \varepsilon_i \frac{a_0\, a_{\rm eq}}{a_i^2} = \varepsilon_i \frac{T_i^2}{T_0\, T_{\rm eq}} \quad \Rightarrow \quad \varepsilon_i = \varepsilon_0 \frac{T_0\, T_{\rm eq}}{T_i^2}.$$

Using $T_0 = 2.725$ K, $T_{\rm eq} \simeq 9 \times 10^3$ K and $T_i \simeq 1.42 \times 10^{32}$ K (Planck scale) yields

$$\frac{T_0\, T_{\rm eq}}{T_i^2} \simeq 1.2 \times 10^{-60}.$$

Hence for $\varepsilon_0 \sim 10^{-2}$ one finds $\varepsilon_i \sim 1.2 \times 10^{-62}$ — extreme fine-tuning!!!

Towards the solution

Inflation offers a dynamical solution to both the horizon and flatness problems by introducing an early epoch with $w < -1/3$ (accelerated expansion), which we will treat in the following presentation.

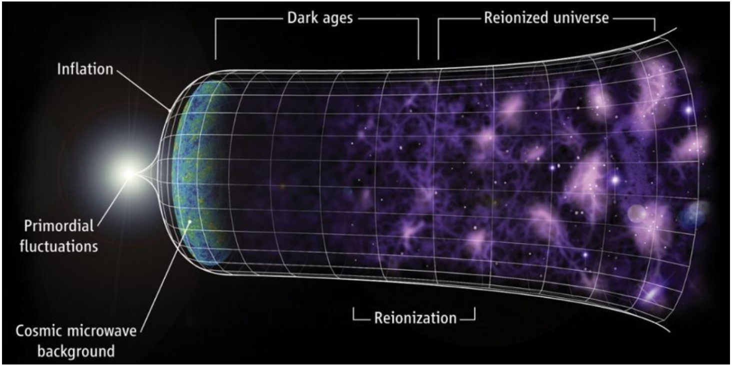


Figure: Ref: https://www.forbes.com/sites/startswithabang/2018/04/12/how-come-cosmic-inflation-doesnt-break-the-speed-of-light/

The Solution — Inflation Theory

The size of a causal patch of space is defined by how far light can travel in a certain amount of time. Since photons travel along null geodesics:

$$\mathrm{d}s^2 = a^2(\tau)\left[\mathrm{d}\tau^2 - \mathrm{d}\chi^2\right] = 0 \quad \Longrightarrow \quad \Delta\chi(\tau) = \pm\Delta\tau$$

where τ is the conformal time and we have used $\mathrm{d}\chi \equiv \mathrm{d}r/\sqrt{1-kr^2}$, where r is the radial co-ordinate and k is the curvature.

- *Particle Horizon* — maximum comoving distance from which light could have travelled to an observer since the Big-Bang:

$$\chi_{\mathrm{ph}}(\tau) = \tau - \tau_i = \int_{t_i}^{t} \frac{\mathrm{d}t}{a(t)} = \int_{a_i}^{a} \frac{\mathrm{d}a}{a\dot{a}} = \int_{\ln a_i}^{\ln a} (aH)^{-1}\,\mathrm{d}\ln a.$$

- *Hubble sphere/Hubble horizon* — comoving distance light can travel in one Hubble time $t_H = 1/H$:

$$r_H = (aH)^{-1}.$$

  Beyond r_H recession speeds exceed c and $(aH)^{-1}$ is an approximate causal boundary at a given time. Comparing the comoving separation λ of two particles with $(aH)^{-1}$, determines whether the particles can communicate with each other at a given moment (i.e. within the next Hubble time).

The Solution — Inflation Theory

- Distinction:
  - if $\lambda > \chi_{\rm ph}$, then the particles could never have communicated.
  - if $\lambda > (aH)^{-1}$, then the particles cannot talk to each other now.
- For a fluid with equation of state $P = w\rho$ one finds

$$(aH)^{-1} = H_0^{-1}\, a^{\frac{1}{2}(1+3w)}.$$

Using the particle horizon expression then gives

$$\chi_{\rm ph}(t) = \frac{2H_0^{-1}}{(1+3w)}\, a(t)^{\frac{1}{2}(1+3w)} \; = \; \frac{2}{(1+3w)}\,(aH)^{-1}.$$

Taking $a_i \to 0 \overset{w>-\frac{1}{3}}{\Longrightarrow} \tau_i \to 0$.
For $w > -1/3$ (familiar matter) the comoving Hubble radius and particle horizon grow with expansion.

Solution to the Horizon Problem

From the previous discussion: a growing Hubble sphere corresponds to regions coming into causal contact; a shrinking Hubble sphere would mean regions get causally disconnected. To make distant regions causally connected in the early universe we require a *decreasing* comoving Hubble radius:

$$\frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} < 0.$$

This requires a fluid with $1+3w<0$. In that case τ_i is no longer 0 and the Big-Bang singularity is pushed to negative conformal time:

$$\tau_i = \frac{2H_0^{-1}}{(1+3w)}\, a_i^{\frac{1}{2}(1+3w)} \xrightarrow{a_i \to 0,\ w < -\frac{1}{3}} -\infty.$$

Solution to the Horizon Problem

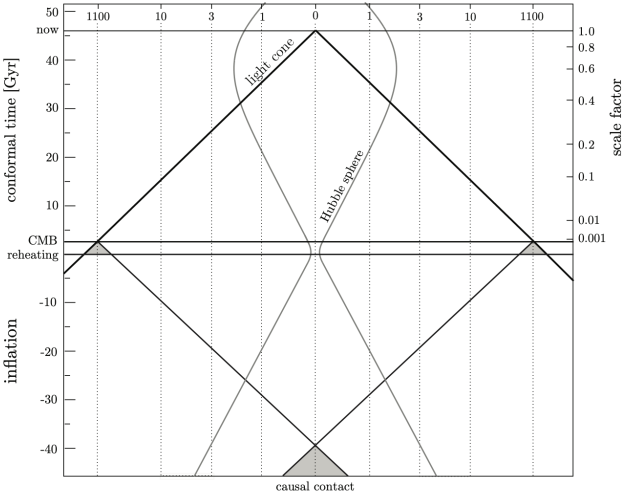


Figure: Solution to the horizon problem

Consequences of a shrinking Hubble sphere

- **Accelerated expansion:**

$$\frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{a})^{-1} = -\frac{\ddot{a}}{(\dot{a})^2}.$$

  So $\frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} < 0 \Rightarrow \ddot{a} > 0$.

- **Slowly-varying Hubble parameter:**

$$\frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\frac{1}{a}(1-\varepsilon),$$

  where $\varepsilon \equiv -\frac{\dot{H}}{H^2}$. Thus $\varepsilon < 1$.

Consequences of a shrinking Hubble sphere

- **Negative pressure:** From $H^2 = \frac{\rho}{3M_{pl}^2}$ and the continuity equation $\dot{\rho} = -3H(\rho + P)$,

$$\dot{H} + H^2 = -\frac{1}{6M_{pl}^2}(\rho + 3P) = -\frac{H^2}{2}\left(1 + \frac{3P}{\rho}\right).$$

  Rearranging gives

$$\varepsilon = -\frac{\dot{H}}{H^2} = \frac{3}{2}\left(1 + \frac{P}{\rho}\right) < 1 \iff w \equiv \frac{P}{\rho} < -\tfrac{1}{3}.$$

- **Nearly constant density:** Combining $\dot{\rho} = -3H(\rho + P)$ with the above,

$$\left|\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln a}\right| = 2\varepsilon < 1,$$

  so for small ε the energy density is approximately constant. Matter sources we know all dilute with expansion, we need something unusual!

Horizon Problem: Required Duration of Inflation

Let the observable Universe come from a single causal patch during inflation require

$$(a_0 H_0)^{-1} < (a_I H_I)^{-1}.$$

Assuming instantaneous reheating and $a \propto 1/T$:

$$\frac{a_0 H_0}{a_E H_E} \sim \frac{a_0 H_0}{a_{eq} H_{eq}} \frac{a_{eq} H_{eq}}{a_E H_E} \sim \frac{a_0 a_{eq}^{3/2}}{a_{eq} a_0^{3/2}} \frac{a_{eq} a_E^2}{a_E a_{eq}^2} \sim \frac{\sqrt{T_0 T_{eq}}}{T_E},$$

where "eq" stands for matter radiation equality. Hence, taking $T_{eq} = 0.7 eV$, $T_0 = 10^{-3} eV$ and $T_E = 10^{15} GeV$

$$(a_I H_I)^{-1} \gtrsim (a_0 H_0)^{-1} \sim 10^{28} (a_E H_E)^{-1},$$

and (for $H_I \approx H_E$)

$$\frac{a_E}{a_I} \gtrsim 10^{28} \quad \Longrightarrow \quad N \equiv \ln \frac{a_E}{a_I} \gtrsim \ln(10^{28}) \simeq 64.$$

Flatness Problem: Required Duration of Inflation

Starting point:

$$\frac{\mathrm{d}\Omega}{\mathrm{d}\ln a} = (1+3w)\,\Omega(\Omega-1).$$

Perturb about $\Omega = 1 \pm \varepsilon$ with $\varepsilon \ll 1$:

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}\ln a} = (1+3w)\,\varepsilon \quad \Longrightarrow \quad \varepsilon(a) = \varepsilon_i \left(\frac{a}{a_i}\right)^{1+3w}.$$

An epoch with $1+3w<0$ drives $\varepsilon \to 0$. For $(w \simeq -1)$:

$$\varepsilon_{\rm post} = \varepsilon_{\rm pre}\, e^{-2N},$$

where N is the number of e-folds.
Thus, large N exponentially suppresses curvature. For example, erasing a unity deviation down to:

$$\varepsilon \sim 10^{-60}$$

requires

$$N \gtrsim \tfrac{1}{2}\ln(10^{60}) \simeq 69.$$

Hence, inflation lasting $N \sim 50$–70 e-folds solves both the horizon and the flatness problem.

Inflation as a Scalar Field Theory

- Consider a scalar field: **Inflaton** $\implies \phi(\mathbf{x}, t)$
- Also, we have a **Potential energy** density associated with the field
- It determines the Dynamics of the field and hence how inflation proceeds
- The Lagrangian density of such a field is given by:

$$\mathscr{L} = \underbrace{\tfrac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi}_{\text{KE+PE}} - \underbrace{V(\phi)}_{\text{PE}}$$

Stress-Energy Tensor:

$$T_{\mu\nu} \;=\; \frac{\partial \mathscr{L}}{\partial(\partial^{\mu}\phi)}\,\partial_{\nu}\phi \;-\; g_{\mu\nu}\,\mathscr{L}$$

$$\Downarrow$$

$$T_{\mu\nu} \;=\; \partial_{\mu}\phi\,\partial_{\nu}\phi \;-\; g_{\mu\nu}\left[\tfrac{1}{2}g^{\alpha\beta}\partial_{\alpha}\phi\,\partial_{\beta}\phi - V(\phi)\right]$$

To stay consistent with the FLRW metric, the field should have no spatial dependence:

$$\phi(\mathbf{x},t) = \phi(t)$$

Hence we have:

$$\left.\begin{aligned} T^{0}{}_{0} &= \rho_{\phi} = \tfrac{1}{2}\dot{\phi}^{2} + V(\phi) \\ T^{i}{}_{j} &= -P_{\phi}\,\delta^{i}{}_{j} = -\left(\tfrac{1}{2}\dot{\phi}^{2} - V(\phi)\right)\delta^{i}{}_{j} \end{aligned}\right\} \qquad P_{\phi} < -\tfrac{1}{3}\rho_{\phi}$$

Substituting the expression for Energy density and Pressure into the **Friedmann Equations** we have:

$$H^2 = \frac{1}{3M_{\mathrm{pl}}^2}\rho_\phi \qquad \Rightarrow \qquad H^2 = \frac{1}{3M_{\mathrm{pl}}^2}\left(\tfrac{1}{2}\dot{\phi}^2 + V(\phi)\right)$$

$$\dot{H} = -\frac{1}{2M_{\mathrm{pl}}^2}\left(\rho_\phi + p_\phi\right) \qquad \Rightarrow \qquad \dot{H} = -\frac{1}{2M_{\mathrm{pl}}^2}\dot{\phi}^2$$

From this we get the **Equation of motion** of the field:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0$$

Slow Roll Inflation

As we discussed, for a slowly varying Hubble parameter
$\Rightarrow \varepsilon < 1$

Which in terms of the field translates to:

$$\varepsilon = \frac{\dot{\phi}^2}{2M_{\mathrm{pl}}^2 H^2} < 1$$

This means the contribution of kinetic energy is small in the total energy.

For inflation to last long enough, the acceleration of the field must also be small
$\Rightarrow \delta = -\frac{\ddot{\phi}}{H\dot{\phi}} < 1$

$$H^2 = \frac{1}{3M_{\mathrm{pl}}^2}\left(\tfrac{1}{2}\dot{\phi}^2 + V(\phi)\right) \implies H^2 = \frac{V(\phi)}{3M_{\mathrm{pl}}^2}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \implies 3H\dot{\phi} = -\frac{dV}{d\phi}$$

Slow Roll Inflation

This slow roll approximation gives us the slow roll parameters:

$$\varepsilon_V = \varepsilon_{\text{slow roll}} \approx \frac{M_{\text{pl}}^2}{2}\left(\frac{dV/d\phi}{V}\right)^2 \qquad \eta_V = \delta + \varepsilon = -\frac{\ddot{\phi}}{H\dot{\phi}} - \frac{\dot{H}}{H^2} = M_{\text{pl}}^2\left(\frac{d^2V/d\phi^2}{V}\right)$$

Hence for a successful inflationary theory, these two parameters must be small. This helps us know the nature of the Potential associated with the inflaton.

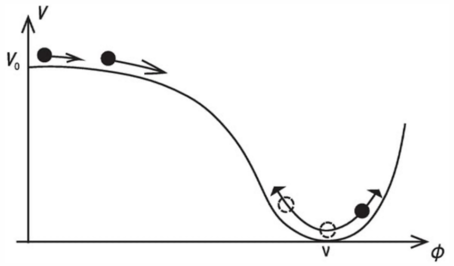


Figure: Example of a Slow-roll potential

An example of Slow roll inflation

Consider a simple potential of the form:

$$V(\phi) = \tfrac{1}{2} m^2 \phi^2$$

The slow roll parameters are given by:

$$\varepsilon_v = \eta_v = 2\left(\frac{M_{\rm pl}}{\phi}\right)^2 \quad \Longrightarrow \quad \phi > \sqrt{2}\, M_{\rm pl} = \phi_E$$

The amount of inflation in terms of the slow roll parameters can be written as:

$$N_{\rm tot} \equiv \int_{a_S}^{a_E} d\ln a = \int_{t_S}^{t_E} H(t)\, dt = \int_{\phi_S}^{\phi_E} \frac{H}{\dot{\phi}}\, d\phi = \int_{\phi_S}^{\phi_E} \frac{1}{\sqrt{2\varepsilon}} \frac{|d\phi|}{M_{\rm pl}} \approx \int_{\phi_S}^{\phi_E} \frac{1}{\sqrt{2\varepsilon_v}} \frac{|d\phi|}{M_{\rm pl}}$$

In this case:

$$N_{\rm tot} = \int_{\phi_E}^{\phi_*} \frac{1}{\sqrt{2\varepsilon_v}} \frac{d\phi}{M_{\rm pl}} = \frac{\phi_*^2}{4M_{\rm pl}^2} - \frac{1}{2}$$

For inflation to last at least 60 e-folds, i.e., $N > 60$:

$$\phi_* \approx 15.6\, M_{\rm pl}$$

Reheating — Oscillations

Again consider potential of the form $V(\phi) = \frac{1}{2}m^2\phi^2$.
Equation of motion is of the form:

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0.$$

Initially, the amplitude of oscillations decreases due to Hubble friction but after some time: $H^{-1} \gg m^{-1}$. Hence, the inflaton begins to oscillate with approximately constant frequency. The continuity equation for the inflaton is

$$\dot{\rho}_\phi + 3H\rho_\phi = -3HP_\phi = -\frac{3}{2}H(m^2\phi^2 - \dot{\phi}^2),$$

and the RHS goes to zero on average so the inflaton behaves like pressureless matter.

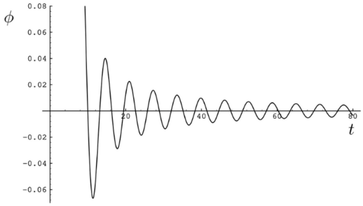


Figure: Field oscillations

As discussed before, when $\phi \sim \mathscr{O}(M_{\mathrm{pl}})$ inflation ends and this reheating era starts.

Inflaton

$$\begin{tikzcd} & \phi \arrow[dl, "-\frac{1}{2}g^2\phi^2\chi^2"'] \arrow[dr, "-h\bar{\psi}\psi\phi"] & \\ \textbf{Bosonic Field } \chi & & \textbf{Fermionic Field } \psi \end{tikzcd}$$

$m_{\mathrm{eff}} = g\phi$ $\qquad$ $m_{\mathrm{eff}} = |h\phi|$

Note that these masses disappear when the inflaton's field value goes to zero.

Things are actually a little different for bosons since many states can occupy the same state (forbidden in fermions due to Pauli's Exclusion Principle).

Reheating – Preheating and thermalisation

The equation for a particular k mode of bosonic field is given by:

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \left(\frac{k^2}{a^2(t)} + g^2\Phi^2 \sin^2(m_\phi t) \right) \chi_k = 0$$

Ignoring the friction term, the solution of the above *Mathieu equation* is of the form:

$$\chi_k \propto \exp(\mu_k^{(n)} z) \quad \Longrightarrow \quad n_k(t) \propto \exp(2\mu_k^{(n)} m_\phi t)$$

Where μ_k depends on some parameters in a set of resonance bands of frequencies labeled by n.
This mechanism is called **Parametric resonance** and causes exponential increase in number of bosons very quickly which is called **Preheating**.
Finally when

$$H \approx \Gamma_\phi ,$$

Reheating finishes. The particles thus produced then interact and the resulting particle soup will eventually reach **thermal equilibrium** with a reheating temperature given by:

$$T_{\rm rh} \approx 0.1 \sqrt{M_{\rm pl} \Gamma_\phi}$$

References

- Daniel Baumann, *TASI Lectures on Inflation*, Chapter 2, available in Baumann's lecture notes on cosmology.
- Ashoke Sen, *Lectures on Cosmology*, available on YouTube and lecture notes.
- Alan H. Guth, *Inflationary universe: A possible solution to the horizon and flatness problems*, Phys. Rev. D 23, 347 (1981).
- Lev Kofman, Andrei Linde, and Alexei Starobinsky, *Reheating after inflation*, Phys. Rev. Lett. 73, 3195 (1994).

Thank You!

Questions?