Reproducing $H(z)$-only constraints on ΛCDM, XCDM and ϕCDM

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August 15, 2025

Abstract

We reproduce constraints on the curved ΛCDM, XCDM, and ϕCDM models using the 28 independent $H(z)$ measurements compiled by Farooq & Ratra [1]. The analysis uses two Gaussian priors for H_0 (68 ± 2.8 and 73.8 ± 2.4 km s^{-1}Mpc^{-1}), evaluates likelihood grids for each model, and obtains best-fit parameters and 1–3σ confidence contours. Our recovered constraints closely match the published results; remaining small differences are attributable to grid resolution and numerical tolerances.

1 Introduction

In the standard cosmological model, the Universe evolved from an early decelerating phase dominated by nonrelativistic matter to a recent accelerating phase driven by dark energy. Observations from Type Ia supernovae, cosmic microwave background anisotropies, and baryon acoustic oscillations strongly support the existence of this negative-pressure component.

We consider three dark energy descriptions. In the ΛCDM model, dark energy is a cosmological constant with $p_\Lambda = -\rho_\Lambda$, giving

$$H^2(z) = H_0^2 \left[\Omega_{m0}(1+z)^3 + \Omega_\Lambda + (1 - \Omega_{m0} - \Omega_\Lambda)(1+z)^2\right]. \tag{1}$$

The XCDM parametrization treats dark energy as a homogeneous fluid with constant equation-of-state parameter $\omega_X < -1/3$:

$$H^2(z) = H_0^2 \left[\Omega_{m0}(1+z)^3 + (1 - \Omega_{m0})(1+z)^{3(1+\omega_X)}\right], \tag{2}$$

reducing to ΛCDM when $\omega_X = -1$. The ϕCDM model describes dark energy as a scalar field ϕ with inverse power-law potential $V(\phi) = \kappa m_p^2 \phi^{-\alpha}$, contributing [2]

$$\rho_\phi = \frac{m_p^2}{16\pi}\left(\frac{1}{2}\dot{\phi}^2 + \kappa m_p^2 \phi^{-\alpha}\right), \tag{3}$$

and yielding

$$H^2(z) = H_0^2 \left[\Omega_{m0}(1+z)^3 + \Omega_\phi(z, \alpha)\right]. \tag{4}$$

Here $\Omega_\phi(z, \alpha)$ is obtained by numerically solving the scalar field and Friedmann equations. In this work, we use 28 independent $H(z)$ measurements to constrain these models and reproduce the results of Farooq & Ratra [1].

2 Constraining the $H(z)$ Data

We use the same $H(z)$ dataset as in [1]. For a given H_0 and cosmological parameters p, the likelihood from $H(z)$ data is:

$$\mathcal{L}(p, H_0) \propto \exp\left[-\frac{\chi_H^2(p, H_0)}{2}\right], \tag{5}$$

where

$$\chi_H^2(p, H_0) = \sum_{i=1}^{N} \frac{\left[H_{\text{th}}(z_i; H_0, p) - H_{\text{obs}}(z_i)\right]^2}{\sigma_i^2}. \tag{6}$$

Following [1], two Gaussian priors are used:

$$H_0 = 68 \pm 2.8 \text{ km/s/Mpc}, \quad H_0 = 73.8 \pm 2.4 \text{ km/s/Mpc}.$$

The posterior before marginalization is:

$$\mathcal{L}_{\text{post}} \propto \exp\left[-\frac{\chi_H^2(p, H_0)}{2}\right] \cdot \exp\left[-\frac{(H_0 - \bar{H}_0)^2}{2\sigma_{H_0}^2}\right]. \tag{7}$$

Expanding χ_H^2 as a quadratic in H_0:

$$\chi_H^2(H_0, p) = A(p)H_0^2 - 2B(p)H_0 + C, \tag{8}$$

with:

$$A(p) = \sum_{i=1}^{N} \frac{E(z_i; p)^2}{\sigma_i^2}, \quad B(p) = \sum_{i=1}^{N} \frac{E(z_i; p)H_{\text{obs}}(z_i)}{\sigma_i^2}, \quad C = \sum_{i=1}^{N} \frac{H_{\text{obs}}(z_i)^2}{\sigma_i^2}.$$

Including the prior term:

$$\chi_{\text{total}}^2(H_0, p) = \alpha H_0^2 - 2\beta H_0 + \gamma, \tag{9}$$

where:

$$\alpha = A(p) + \frac{1}{\sigma_{H_0}^2}, \quad \beta = B(p) + \frac{\bar{H}_0}{\sigma_{H_0}^2}, \quad \gamma = C + \frac{\bar{H}_0^2}{\sigma_{H_0}^2}.$$

Marginalizing over H_0 analytically gives:

$$L_H(p) \propto \frac{1}{\sqrt{\alpha}} \exp\left[-\frac{1}{2}\left(\gamma - \frac{\beta^2}{\alpha}\right)\right] \cdot \left[1 + \text{erf}\left(\frac{\beta}{\sqrt{2\alpha}}\right)\right]. \tag{10}$$

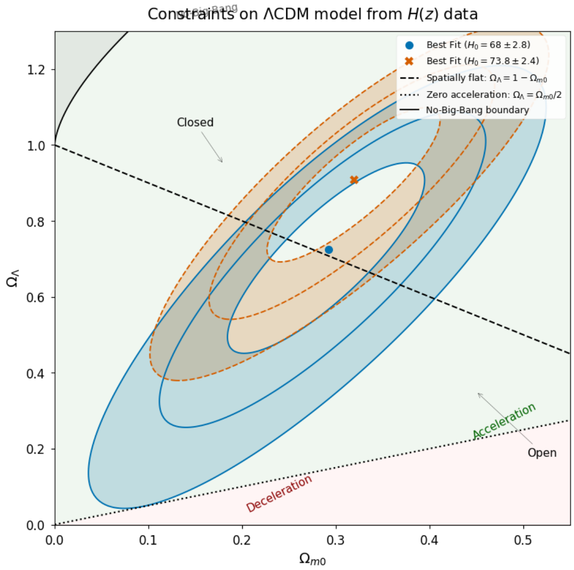
The best-fit parameters p_0 are found by maximizing $L_H(p)$ (or minimizing $\chi_H^2(p) = -2\ln L_H(p)$). Two-dimensional confidence contours are drawn using the usual $\Delta\chi^2$ thresholds for two parameters,

$$\Delta\chi^2 = \{2.30,\ 6.18,\ 11.83\} \quad (1,\ 2,\ 3\sigma \text{ levels}),$$

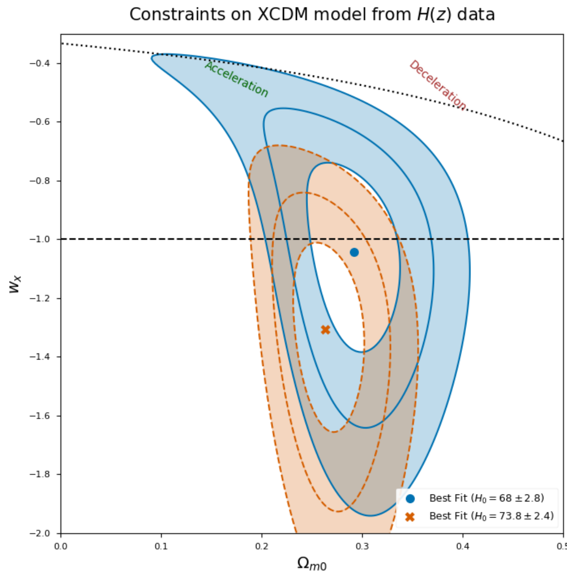
and one-dimensional marginalized intervals are obtained by minimizing the 2D χ^2 over the complementary parameter (e.g. $\chi_{1\text{D}}^2(\Omega_{m0}) = \min_{\Omega_\Lambda} \chi^2(\Omega_{m0}, \Omega_\Lambda)$) and selecting the region where $\chi_{1\text{D}}^2 - \chi_{\min}^2 < 4.0$ (approx. 95.4%, 2σ)

3 Results

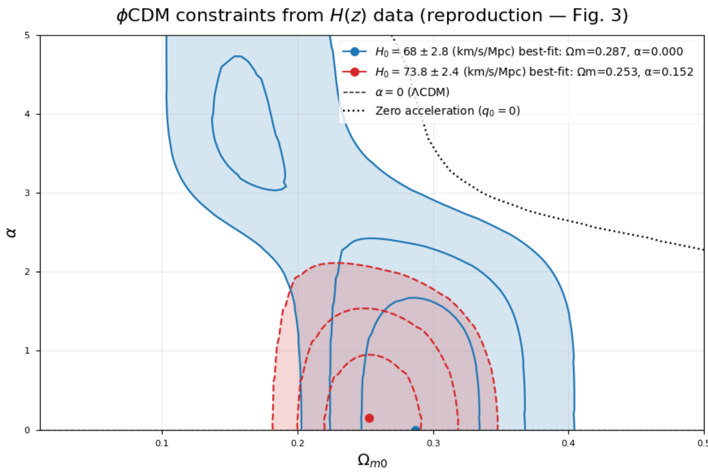
Below we present the constraint plots for each model. For each model, the solid [dashed] lines show 1,2 and 3 σ constraint contours.



(a) For ΛCDM



(b) For XCDM



(c) For ϕCDM

Table 1: For $\bar{H}_0 \pm \sigma_{H_0} = 68 \pm 2.8$ km s^{-1} Mpc^{-1}

Quantity	This Work	Farooq & Ratra(2013) [1]
ΛCDM best-fit $(\Omega_{m0}, \Omega_\Lambda)$	$0.29,\ 0.72$	$0.29,\ 0.72$
ΛCDM χ^2_{min}	16.38	18.24
ΛCDM 2σ intervals $([\Omega_{m0}], [\Omega_\Lambda])$	$[0.15, 0.43],\ [0.36, 1.02]$	$[0.15, 0.42],\ [0.35, 1.02]$
XCDM best-fit (Ω_{m0}, ω_x)	$0.29,\ -1.04$	$0.29,\ -1.04$
XCDM χ^2_{min}	16.34	18.18
XCDM 2σ intervals $([\Omega_{m0}], [\omega_x])$	$[0.23, 0.35],\ [-1.50, -0.65]$	$[0.23, 0.35],\ [-1.51, -0.64]$
ϕCDM best-fit (Ω_{m0}, α)	$0.29,\ 0.00$	$0.29,\ 0.00$
ϕCDM χ^2_{min}	19.59	18.24
ϕCDM 2σ intervals $([\Omega_{m0}], [\alpha])$	$[0.24, 0.35],\ [0.00, 2.02]$	$[0.17, 0.34],\ [0.00, 2.20]$

Table 2: For $\bar{H}_0 \pm \sigma_{H_0} = 73.8 \pm 2.4$ km s^{-1} Mpc^{-1}

Quantity	This Work	Farooq & Ratra (2013) [1]
ΛCDM best-fit $(\Omega_{m0}, \Omega_\Lambda)$	$0.32,\ 0.91$	$0.32,\ 0.91$
ΛCDM $\chi^2_{\rm min}$	17.75	19.30
ΛCDM 2σ intervals $([\Omega_{m0}], [\Omega_\Lambda])$	$[0.20, 0.44],\ [0.62, 1.15]$	$[0.20, 0.44],\ [0.62, 1.14]$
XCDM best-fit (Ω_{m0}, ω_x)	$0.26,\ -1.31$	$0.26,\ -1.30$
XCDM $\chi^2_{\rm min}$	16.60	18.15
XCDM 2σ intervals $([\Omega_{m0}], [\omega_x])$	$[0.22, 0.31],\ [-1.79, -0.93]$	$[0.22, 0.31],\ [-1.78, -0.92]$
ϕCDM best-fit (Ω_{m0}, α)	$0.253,\ 0.152$	$0.25,\ 0.00$
ϕCDM $\chi^2_{\rm min}$	21.919	20.64
ϕCDM 2σ intervals $([\Omega_{m0}], [\alpha])$	$[0.213, 0.302],\ [0.000, 1.212]$	$[0.16, 0.34],\ [0.00, 0.70]$

4 Discussion and Conclusion

We have successfully reproduced the constraints on the ΛCDM, XCDM, and ϕCDM models using the 28 independent $H(z)$ measurements from Farooq & Ratra [1]. For the ΛCDM and XCDM models, our resulting parameter estimates and confidence contours are in excellent agreement with the published values.

The most significant deviation is found in the ϕCDM model. While our best-fit parameters are similar, the shape of our confidence contours is notably different. This discrepancy arises from the different numerical methods used to solve the scalar field's evolution. In particular, the results presented here were obtained by initiating the field integration at $z_{\rm init} = 50$ with the field nearly at rest ($\phi_{\rm init} = 1,\ \dot{\phi}_{\rm init} = 0$), which tends to produce "thawing"-type evolution.

In contrast, the canonical analysis method, following the foundational work of Peebles & Ratra [3], begins the integration in the very early, radiation-dominated universe. This computationally intensive approach allows the scalar field to settle onto a stable "tracker" solution, where its evolution is an attractor. The nearly vertical contours in the original paper are a classic signature of this tracker behavior, which makes late-time observables largely insensitive to α. The discrepancy highlights the significant impact that the scalar field's early-time evolution has on observational constraints.

In conclusion, for all models, our results confirm the findings of Farooq & Ratra that the $H(z)$ data strongly support a currently accelerating Universe. Our work successfully validates their primary results while also revealing important nuances in the numerical modeling of dynamical dark energy.

References

[1] O. Farooq and B. Ratra, "Hubble parameter measurement constraints on the cosmological deceleration-acceleration transition redshift," Astrophys. J. Lett. **766**, L7 (2013), arXiv:1301.5243.

[2] O. Farooq, D. Mania, and B. Ratra, "Hubble parameter measurement constraints on dark energy," Astrophys. J. **764**, 138 (2013), arXiv:1211.4253.

[3] P. J. E. Peebles and B. Ratra, "Cosmology with a time-variable cosmological 'constant'," Astrophys. J. Lett. **325**, L17 (1988).