SOC QIQC Assignment 0

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1. i. The characteristic equation is:-

$$\begin{vmatrix} -\lambda & 5 & 0 & 4 \\ 5 & -\lambda & 4 & 0 \\ 0 & 3 & -\lambda & 2 \\ 3 & 0 & 2 & -\lambda \end{vmatrix} = 0$$

$\implies \lambda^4 - 53\lambda^2 + 4 = 0$ whose roots (and therefore the eigen values) are $\sqrt{\frac{53-7\sqrt{57}}{2}}$, $\sqrt{\frac{53+7\sqrt{57}}{2}}$, $-\sqrt{\frac{53-7\sqrt{57}}{2}}$, $-\sqrt{\frac{53+7\sqrt{57}}{2}}$.

1. ii. The characteristic equation is:-

$$\begin{vmatrix} -\lambda & 0 & 5 & 4 \\ 0 & -\lambda & 3 & 2 \\ 5 & 4 & -\lambda & 0 \\ 3 & 2 & 0 & -\lambda \end{vmatrix} = 0$$

$\implies \lambda^4 - 53\lambda^2 + 4 = 0$ whose roots (and therefore the eigen values) are $\sqrt{\frac{53-7\sqrt{57}}{2}}$, $\sqrt{\frac{53+7\sqrt{57}}{2}}$, $-\sqrt{\frac{53-7\sqrt{57}}{2}}$, $-\sqrt{\frac{53+7\sqrt{57}}{2}}$.

1. iii. The characteristic equation is:-

$$\begin{vmatrix} 25-\lambda & 20 & 20 & 16 \\ 15 & 10-\lambda & 12 & 8 \\ 15 & 12 & 10-\lambda & 8 \\ 9 & 6 & 6 & 4-\lambda \end{vmatrix} = 0$$

$\implies \lambda^4 - 49\lambda^3 - 204\lambda^2 - 196\lambda + 16 = 0$
$\implies (\lambda+2)(\lambda+2)(\lambda^2 - 53\lambda + 4) = 0$ whose roots (and therefore the eigen values) are -2, -2, $\frac{53-7\sqrt{57}}{2}$ and $\frac{53+7\sqrt{57}}{2}$

2. Let's denote the two eigen vectors of O having eigen values +1 and -1 to be $|+\rangle$ and $|-\rangle$. Then, O can be expressed as,

$$O = |+\rangle\langle+| - |-\rangle\langle-|$$

and

$$P_\pm = |\pm\rangle\langle\pm|; \quad I = |+\rangle\langle+| + |-\rangle\langle-|$$

From these relations it can be clearly seen that,

$$P_\pm = \frac{(|+\rangle\langle+| + |-\rangle\langle-|) \pm (|+\rangle\langle+| - |-\rangle\langle-|)}{2}$$

$$\therefore P_\pm = \frac{I \pm O}{2}$$

3. If the operator A is norm preserving,

$$\langle Ax|Ay\rangle = \langle x|y\rangle \quad \forall\, x, y \in V$$

$$\implies \langle x|A^\dagger A|y\rangle = \langle x|y\rangle \quad (as\ \langle Ax| = \langle x|A^\dagger)$$

$$\implies \langle x|A^\dagger A - I|y\rangle = 0$$

Since this must be true for all $|x\rangle$ and $|y\rangle$, $A^\dagger A = I$. Therefore A is unitary.