Contest design with a finite type-space

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- Vast literature: complete or incomplete information (infinite types)
- This paper: contest design an incomplete information environment with a finite type-space

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- **Incomplete information environment**: Glazer and Hassin [1988], Moldovanu and Sela [2001], Zhang [2024]

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- **Finite types**: Xiao [2018], Liu and Chen [2016], Szech [2011], Konrad [2004], Chen [2021]

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- Agents ranked according to effort, and awarded corresponding prizes
- If agent i is of type θ_k and wins prize v_i after exerting effort x_k, its payoff is

$$v_i - \theta_k x_k.$$

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- Design problem: find $v = (v_1, \ldots, v_n)$ given a budget to maximize

$$\mathbb{E}[X] = \sum_{k=1}^{K} p_k \mathbb{E}[X(\theta_k)].$$

Equilibrium structure

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- More efficient agents (those with lower θ) exert higher effort
- Applies to more general utility: $v_i - \theta_k c(x_k)$

Proof sketch

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6. if $b_k = a_j$, then $\theta_k \geq \theta_j$

Together, the properties imply the structure in the equilibrium.

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- Thus, it must be that for all $x_k \in [b_{k-1}, b_k]$,

$$\sum_{m=1}^{N-1} v_m H_{N-m}^{N-1}(P_{k-1} + p_k F_k(x_k)) - \theta_k x_k = u_k$$

Equilibrium characterization

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Theorem 2.

For any contest $v = \{v_1, v_2, \ldots, v_{N-1}, 0\}$*, the unique symmetric Bayes-Nash equilibrium is such that for any* $\theta_k \in \Theta$*, the distribution function* $F_k : [b_{k-1}, b_k] \to [0, 1]$ *is defined by*

$$\sum_{m=1}^{N-1} v_m H_{N-m}^{N-1}(P_{k-1} + p_k F_k(x_k)) - \theta_k x_k = u_k \text{ for all } x_k \in [b_{k-1}, b_k], \quad (1)$$

where $b = (b_1, \ldots, b_K)$ *and* $u = (u_1, \ldots, u_K)$ *are defined by*

$$b_k = \frac{\sum_{m=1}^{N-1} v_m H_{N-m}^{N-1}(P_k) - u_k}{\theta_k} \text{ for any } k \in \{1, 2, \ldots, K\}, \quad (2)$$

$$u_{k+1} - u_k = (\theta_k - \theta_{k+1}) b_k \text{ for any } k \in \{1, 2, \ldots, K-1\}. \quad (3)$$

Boundary points and utilities

Lemma 3.

The equilibrium boundary points $b = (b_1, b_2, \ldots, b_K)$ *and the equilibrium utilities* $u = (u_1, u_2, \ldots, u_K)$*, obtained by solving the system of Equations 2 and 3, together with the boundary condition* $u_1 = 0$*, are such that*

$$b_k = \sum_{m=1}^{N-1} v_m \left[\sum_{j=1}^{k} \frac{H_{N-m}^{N-1}(P_j) - H_{N-m}^{N-1}(P_{j-1})}{\theta_j} \right] \text{ for any } k \in \{1, 2, \ldots, K\}, \tag{4}$$

and

$$u_k = \theta_k \sum_{m=1}^{N-1} v_m \left[\sum_{j=1}^{k-1} H_{N-m}^{N-1}(P_j) \left[\frac{1}{\theta_{j+1}} - \frac{1}{\theta_j} \right] \right] \text{ for any } k \in \{2, \ldots, K\}. \tag{5}$$

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$$F_1(x_1) = \frac{-2v_2 + 2\sqrt{v_2^2 + (v_1 - 2v_2)2x_1}}{(v_1 - 2v_2)},$$

$$F_2(x_2) = \frac{-v_1 + \sqrt{v_1^2 + 4(v_1 - 2v_2)(x_2 - b_1)}}{v_1 - 2v_2},$$

where $b_1 = \dfrac{v_1 + 2v_2}{8}$.

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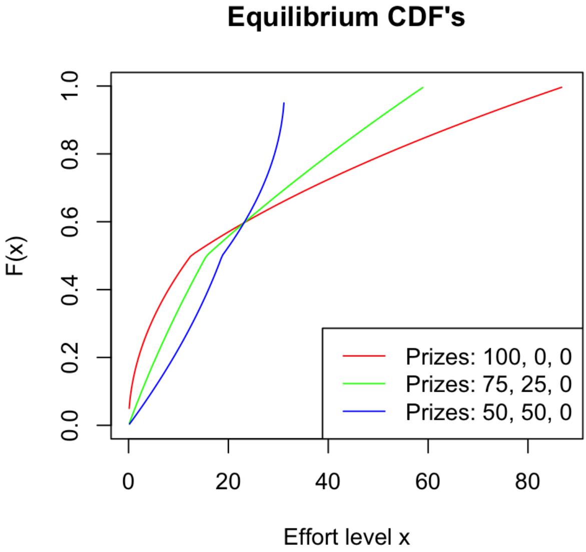
where $b_1 = \dfrac{v_1 + 2v_2}{8}$.

- We can find $\mathbb{E}[X_1] = \frac{v_1+4v_2}{24}$ and $\mathbb{E}[X_2] = \frac{11v_1+2v_2}{24}$ so that

$$\mathbb{E}[X] = \frac{1}{2}\mathbb{E}[X_1] + \frac{1}{2}\mathbb{E}[X_2] = \frac{12v_1 + 6v_2}{48}.$$

Plot


Equilibrium CDF's
F(x)
0.0
0.2
0.4
0.6
0.8
1.0
0
20
40
60
80
Effort level x
Prizes: 100, 0, 0
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Lemma 4.

For any contest $v = \{v_1, v_2, \ldots, v_{N-1}, 0\}$*, the expected equilibrium effort of an arbitrary agent is*

$$\mathbb{E}[X] = \sum_{m=1}^{N-1} v_m \alpha_m,$$

where

$$\alpha_m = \frac{1}{N}\left[\frac{1}{\theta_K} + \sum_{k=1}^{K-1}\left[H^N_{\geq N-m}(P_k) + (m-1)H^N_{N-m}(P_k)\right]\left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}}\right)\right].$$

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2. The winner-takes-all contest is optimal.
3. Complete info: $K = 1$

Two key ideas for finding expected effort

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- Use CDF equation to get

$$\mathbb{E}[X_k] = \frac{\mathbb{E}[v(\theta_k)] - u_k}{\theta_k},$$

where

$$\mathbb{E}[v(\theta_k)] = \mathbb{E}\left[\sum_{m=1}^{N-1} v_m H_{N-m}^{N-1}(P_{k-1} + p_k F_k(X_k))\right]$$

is simply the expected value of the prize an agent of type θ_k expects to receive in this contest (prior to exerting effort X_k).

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- Compute the total prize awarded to agents of type θ_k, and then use symmetry to find $\mathbb{E}[v(\theta_k)]$. In particular,

$$\mathbb{E}[V_k] = Np_k\mathbb{E}[v(\theta_k)] = \left[\sum_{m=1}^{N-1} v_m \left(H^N_{\geq N-m+1}(P_k) - H^N_{\geq N-m+1}(P_{k-1})\right)\right]$$

Convergence

Theorem 5.

Suppose there are N agents and consider a fixed contest $v = (v_1, v_2, \ldots, v_{N-1}, 0)$. Let $G : [\underline{\theta}, \overline{\theta}] \to [0,1]$ be a differentiable CDF and let $G^1, G^2, \ldots,$ be any sequence of CDF's, each with a finite support, such that for all $\theta \in [\underline{\theta}, \overline{\theta}]$,

$$\lim_{n\to\infty} G^n(\theta) = G(\theta).$$

Let $F^n : \mathbb{R} \to [0,1]$ denote CDF of the equilibrium effort under G^n, and let $F : \mathbb{R} \to [0,1]$ denote CDF of the equilibrium under G. Then, the sequence $F^1, F^2, \ldots,$ converges to F, i.e., for all $x \in \mathbb{R}$,

$$\lim_{n\to\infty} F^n(x) = F(x).$$

Summary

- Study the classic contest design problem with a finite type-space
- Provide a bridge between previous literature in the complete and the incomplete information (infinite type-space) settings
- Introduce new techniques for the study of contest design problems in finite type-space environment

Thank you!