Contest design with a finite type-space: A unifying apprach

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Introduction

- Contests: agents competing to win valuable prizes
- Examples: R&D, sports, education, politics
- Context: designer distributing a budget across various prizes
- Goal: understand how prize structure affects incentives
- Vast literature: complete or incomplete information (continuum type-space)
- This paper: contests with a finite type-space

Literature

- **Incomplete information**: Moldovanu and Sela [2001], Zhang [2024], Goel [2023]
- **Complete information**: Barut and Kovenock [1998], Fang, Noe, and Strack [2020], Letina, Liu, and Netzer [2023]
- **Both**: Glazer and Hassin [1988], Olszewski and Siegel [2016, 2020]
- Sisak [2009]: "The case of asymmetric individuals, where **types are private information but drawn from discrete, identical** or maybe even different distributions, has not been addressed so far. From the results discussed above, especially on asymmetric types with full information, one could **conjecture that multiple prizes might be optimal even with linear costs**."
- **Finite types**: Xiao [2018], Liu and Chen [2016], Szech [2011], Konrad [2004], Chen [2021]

Model

- N risk-neutral agents
- $\Theta = \{\theta_1, \ldots, \theta_K\}$: finite type-space with $\theta_1 > \theta_2 > \cdots > \theta_K$
- $p = (p_1, \ldots, p_K)$: distribution over Θ
- $c : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing cost function, with $c(0) = 0$
- If agent of type θ_k exerts effort x_k, it incurs a cost $\theta_k c(x_k)$.

Contest design problem

- $v = (v_1, \ldots v_N)$: prize vector with $v_1 \geq v_2 \geq \cdots \geq v_N$, with $v_N = 0$
- Given v, agents simultaneously choose their effort, are ranked according to effort, awarded corresponding prizes
- If agent i of type θ_k wins prize v_i after exerting effort x_k, its payoff is

$$v_i - \theta_k c(x_k).$$

- Symmetric Bayes-Nash equilibrium (potentially in mixed strategies):

$$X : \Theta \to \Delta\mathbb{R}_+$$

- Design problem: given a budget, find $v = (v_1, \ldots, v_n)$ to maximize

$$\mathbb{E}[X] = \sum_{k=1}^{K} p_k \mathbb{E}[X(\theta_k)].$$

Equilibrium structure

Lemma 1.

For any contest $v = (v_1, \ldots, v_{N-1}, 0)$, there is a unique symmetric Bayes-Nash equilibrium and it is such that there exist boundary points $b_1 < b_2 < \cdots < b_K$ so that for any $\theta_k \in \Theta$, an agent of type θ_k mixes between $[b_{k-1}, b_k]$ with $b_0 = 0$.

- $\theta_1 \to [0, b_1], \theta_2 \to [b_1, b_2], \ldots, \theta_K \to [b_{K-1}, b_K]$
- More efficient agents (those with lower θ) exert higher effort
- Exhibits both the mixed structure from complete information, and monotonic structure from continuum type-space environments

Proof sketch

Let F_k denote the equilibrium cdf's, $[a_k, b_k]$ denote the support of F_k, and u_k denote the equilibrium utility.

1. F_k cannot have atoms
2. $\min\{a_1, a_2, \ldots, a_K\} = 0$
3. $u_1 \leq u_2 \leq \cdots \leq u_K$
4. For any $j \neq k$, $|[a_k, b_k] \cap [a_j, b_j]| \leq 1$
5. If $b_k \neq \max\{b_1, b_2, \ldots, b_K\}$, then $b_k = a_j$ for some $j \in \{1, 2, \ldots, K\}$
6. if $b_k = a_j$, then $\theta_k \geq \theta_j$

Together, the properties imply the structure in the equilibrium.

Equilibrium distributions

- F_k on $[b_{k-1}, b_k]$ must be such that the marginal gain in reward from increasing effort equals the marginal cost.
- Formally, if type θ_k chooses $x_k \in [b_{k-1}, b_k]$, the probability it beats an arbitrary agent is $P_{k-1} + p_k F_k(x_k)$.
- By the indifference condition, for all $x_k \in [b_{k-1}, b_k]$,

$$\pi_v(P_{k-1} + p_k F_k(x_k)) - \theta_k c(x_k) = u_k,$$

  where $\pi_v(t) = \sum_{m=1}^{N} v_m H_{N-m}^{N-1}(t)$, and $H_{N-m}^{N-1}(t)$ is the probability $Y \sim Bin(N-1, t)$ takes the value $N - m$.
- Solve for u_k and b_k by using $u_1 = 0$, $F_k(b_k) = 1$ and $F_{k+1}(b_k) = 0$.

Equilibrium

Theorem 2.

For any v, the unique symmetric Bayes-Nash equilibrium is such that for any $\theta_k \in \Theta$, the distribution function $F_k : [b_{k-1}, b_k] \to [0,1]$ is defined by

$$\pi_v(P_{k-1} + p_k F_k(x_k)) - \theta_k c(x_k) = u_k \text{ for all } x_k \in [b_{k-1}, b_k],$$

where the points $b = (b_1, \dots, b_K)$ and utilities $u = (u_1, \dots, u_K)$ are

$$c(b_k) = \sum_{j=1}^{k} \frac{\pi_v(P_j) - \pi_v(P_{j-1})}{\theta_j} \text{ for any } k \in \{1, 2, \dots, K\},$$

and

$$u_k = \theta_k \left[\sum_{j=1}^{k-1} \pi_v(P_j) \left(\frac{1}{\theta_{j+1}} - \frac{1}{\theta_j} \right) \right] \text{ for any } k \in \{1, 2, \dots, K\}.$$

Utilities and expected effort

- For utilities, observe that for any $k \in \{1, \dots, K\}$ and any prize $m \in \{1, \dots, N-1\}$,

$$\frac{\partial u_k}{\partial v_m} = \theta_k \left[\sum_{j=1}^{k-1} H_{N-m}^{N-1}(P_j) \left(\frac{1}{\theta_{j+1}} - \frac{1}{\theta_j} \right) \right],$$

  which does not depend on the cost function c, or even the contest v.

- For expected effort, we can write

$$\pi_v(P_{k-1} + p_k F_k(X_k)) - \theta_k c(X_k) = u_k.$$

  This gives $\mathbb{E}[X_k]$, and since $\mathbb{E}[X] = \sum_{k=1}^K p_k \mathbb{E}[X_k]$, we get

$$\mathbb{E}[X] = \sum_{k=1}^{K} \int_{P_{k-1}}^{P_k} g\left(\frac{\pi_v(t) - u_k}{\theta_k} \right) dt.$$

Optimal contest: Linear costs

Lemma 3.

Suppose $c(x) = x$. *For any contest* $v = (v_1, \ldots, v_{N-1}, 0)$, *the expected equilibrium effort of an arbitrary agent is* $\mathbb{E}[X] = \sum_{m=1}^{N-1} \alpha_m v_m$ *where*

$$\alpha_m = \frac{1}{N}\left[\frac{1}{\theta_K} - \sum_{k=1}^{K-1}\left[H^N_{\geq N-m+1}(P_k) + mH^N_{N-m}(P_k)\right]\left(\frac{1}{\theta_{k+1}} - \frac{1}{\theta_k}\right)\right].$$

- If $K = 1$, $\alpha_b - \alpha_w = 0$ for any $b < w$.
- For $K > 1$, $\alpha_1 - \alpha_w > 0$ for any $w \in \{2, \ldots, N-1\}$.
- As soon as there is any *little* uncertainty, the winner-takes-all contest is strictly optimal.
- Effect of competition: compute $\alpha_b - \alpha_w$ (maybe < 0 when $b \neq 1$)

Effect of competition: Linear to general costs

Theorem 4.

For any pair of prizes $b, w \in \{1, \ldots, N-1\}$ with $b < w$ such that, either $b = 1$ or $\left(\frac{\partial u_K}{\partial v_b} - \frac{\partial u_K}{\partial v_w}\right) \leq 0$, the following hold:

1. *If $\alpha_b - \alpha_w \geq 0$ and c is concave, then for any v, $\frac{\partial \mathbb{E}[X]}{\partial v_b} - \frac{\partial \mathbb{E}[X]}{\partial v_w} > 0$.*
2. *If $\alpha_b - \alpha_w \leq 0$ and c is convex, then for any v, $\frac{\partial \mathbb{E}[X]}{\partial v_b} - \frac{\partial \mathbb{E}[X]}{\partial v_w} < 0$.*

- For some (N, Θ, p), the effect of increasing competition under general costs can be informed by those under linear costs.
- If $K = 1$, increasing competition encourages effort if costs are concave, and discourages effort if costs are convex.
- For $K > 1$, since $\alpha_1 - \alpha_w > 0$, the winner-takes-all contest is strictly optimal under concave costs.

Convergence to continuum

Theorem 5.

Suppose there are N agents and consider a fixed contest $v = (v_1, \ldots, v_{N-1}, 0)$. Let $G : [\underline{\theta}, \overline{\theta}] \to [0,1]$ be a differentiable CDF and let $G^1, G^2, \ldots,$ be any sequence of CDF's, each with a finite support, such that for all $\theta \in [\underline{\theta}, \overline{\theta}]$,

$$\lim_{n \to \infty} G^n(\theta) = G(\theta).$$

Let $F^n : \mathbb{R} \to [0,1]$ denote CDF of the equilibrium effort under the finite type-space distribution G^n, and let $F : \mathbb{R} \to [0,1]$ denote CDF of the equilibrium under continuum type-space distribution G. Then, the sequence of CDF's $F^1, F^2, \ldots,$ converges to the CDF F, i.e., for all $x \in \mathbb{R}$,

$$\lim_{n \to \infty} F^n(x) = F(x).$$

- Intuitively, as K increases, $[b_{k-1}, b_k]$ shrinks, and converges to the pure-strategy equilibrium effort under the continuum type-space.

Summary

- Study effect of increasing competition in contests with a finite type-space
- Provide a unifying approach to studying contests simultaneously in complete and incomplete information environments
- Identify effects under linear costs, and find conditions under which they extend to general costs, which pertain to how competition affects the equilibrium utility of the most efficient agent
- Generate insights into what drives some of the differences in the complete and incomplete information environments
- Solve the design problem under linear and concave costs, showing that the winner-takes-all contest is optimal with any limited uncertainty

Thank you!