The effect of competition in contests: A unifying approach

Andrzej Baranski   **Sumit Goel**

New York University Abu Dhabi

Contests

- Designed to motivate effort towards an objective:

Contests

- Designed to motivate effort towards an objective:
  - R&D: Millennium Prize Problems (2000)

Contests

- Designed to motivate effort towards an objective:
    - R&D: Millennium Prize Problems (2000)
    - Employee Performance: Sales Contests

Contests

- Designed to motivate effort towards an objective:
  - R&D: Millennium Prize Problems (2000)
  - Employee Performance: Sales Contests

Sales contest

Sales contest

- Prizes $v_0 \leq v_1 \leq \cdots \leq v_N$ awarded by rank

Sales contest

- Prizes $v_0 \leq v_1 \leq \cdots \leq v_N$ awarded by rank
- If an agent sells x units and outperforms m agents:

$$\text{Payoff} = v_m - \theta\, c(x)$$

Sales contest

- Prizes $v_0 \leq v_1 \leq \cdots \leq v_N$ awarded by rank
- If an agent sells x units and outperforms m agents:

$$\text{Payoff} = v_m - \theta\, c(x)$$

Research question

How should a budget be allocated across prizes to maximize total sales?

A puzzle in the literature

A puzzle in the literature

	Complete Info.	Finite types	Continuum of Types
Linear	(0, v, 100-v)	?	(0, 0, 100)
Concave	(0, 0, 100)	?	(0, 0, 100)
Convex	(0, 50, 50)	?	Depends

References: Barut & Kovenock [EJPE, 1998]; Fang et al. [JPE, 2020]; Moldovanu & Sela [AER, 2001]; Olszewski & Siegel [ECMA, 2016; TE, 2020]; Zhang [TE, 2024]

A puzzle in the literature

	Complete Info.	Finite types	Continuum of Types
Linear	(0, v, 100-v)	?	(0, 0, 100)
Concave	(0, 0, 100)	?	(0, 0, 100)
Convex	(0, 50, 50)	?	Depends

References: Barut & Kovenock [EJPE, 1998]; Fang et al. [JPE, 2020]; Moldovanu & Sela [AER, 2001]; Olszewski & Siegel [ECMA, 2016; TE, 2020]; Zhang [TE, 2024]

- Sisak [JES, 2009]'s conjecture:

A puzzle in the literature

	Complete Info.	Finite types	Continuum of Types
Linear	(0, v, 100-v)	?	(0, 0, 100)
Concave	(0, 0, 100)	?	(0, 0, 100)
Convex	(0, 50, 50)	?	Depends

References: Barut & Kovenock [EJPE, 1998]; Fang et al. [JPE, 2020]; Moldovanu & Sela [AER, 2001]; Olszewski & Siegel [ECMA, 2016; TE, 2020]; Zhang [TE, 2024]

- Sisak [JES, 2009]'s conjecture:

  The case of asymmetric individuals, where types are private information but drawn from discrete, identical or even different distributions, has not been addressed so far. From the results ..., one could conjecture that multiple prizes might be optimal even with linear costs.

Contribution

- The (most competitive) winner-takes-all contest is robustly optimal

Contribution

- The (most competitive) winner-takes-all contest is robustly optimal
  - Maximizes total effort of top q agents under linear or concave costs

Contribution

- The (most competitive) winner-takes-all contest is robustly optimal
  - Maximizes total effort of top q agents under linear or concave costs
  - Convex costs?

Contribution

- The (most competitive) winner-takes-all contest is robustly optimal
  - Maximizes total effort of top q agents under linear or concave costs
  - Optimality persists if cost is not too convex

Contribution

- The (most competitive) winner-takes-all contest is robustly optimal
  - Maximizes total effort of top q agents under linear or concave costs
  - Optimality persists if cost is not too convex
- Effort is not necessarily monotone increasing in competitiveness

Contribution

- The (most competitive) winner-takes-all contest is robustly optimal
  - Maximizes total effort of top q agents under linear or concave costs
  - Optimality persists if cost is not too convex
- Effort is not necessarily monotone increasing in competitiveness
  - Interior discouragement effect: shifting value to better prizes may reduce effort if relatively inefficient types are likely

Contribution

- The (most competitive) winner-takes-all contest is robustly optimal
  - Maximizes total effort of top q agents under linear or concave costs
  - Optimality persists if cost is not too convex
- Effort is not necessarily monotone increasing in competitiveness
  - Interior discouragement effect: shifting value to better prizes may reduce effort if relatively inefficient types are likely
- Experiment provides qualitative support for these findings

Contribution

- The (most competitive) winner-takes-all contest is robustly optimal
  - Maximizes total effort of top q agents under linear or concave costs
  - Optimality persists if cost is not too convex
- Effort is not necessarily monotone increasing in competitiveness
  - Interior discouragement effect: shifting value to better prizes may reduce effort if relatively inefficient types are likely
- Experiment provides qualitative support for these findings
- Novel approach for analyzing symmetric equilibrium in games

- Model
- Equilibrium characterization
- Effect of competition
  - Linear costs
  - General costs
- Experiment

Model

A contest environment is defined by $\{N+1, \Theta, p\}$:

Model

A contest environment is defined by $\{N + 1, \Theta, p\}$:

- $N + 1$: number of agents
- $\Theta = \{\theta_1, \ldots, \theta_K\}$: finite set of types

$$\theta_1 > \theta_2 > \cdots > \theta_K$$

- $p = (p_1, \ldots, p_K)$: distribution over Θ

Model

A contest environment is defined by $\{N+1, \Theta, p\}$:

- $N+1$: number of agents
- $\Theta = \{\theta_1, \ldots, \theta_K\}$: finite set of types

$$\theta_1 > \theta_2 > \cdots > \theta_K$$

- $p = (p_1, \ldots, p_K)$: distribution over Θ

A contest $v = (v_0, \ldots, v_N)$ assigns a prize for each rank:

$$v \in \mathcal{V} = \{v \in \mathbb{R}^{N+1} : v_0 \leq \cdots \leq v_N \text{ with } 0 = v_0 < v_N\}$$

- Given $(N+1, \Theta, p)$, $v \in \mathcal{V}$, the agents

Bayesian game

- Given $(N + 1, \Theta, p)$, $v \in \mathcal{V}$, the agents
    1. privately learn their types
    2. choose effort simultaneously
    3. and are awarded corresponding prizes

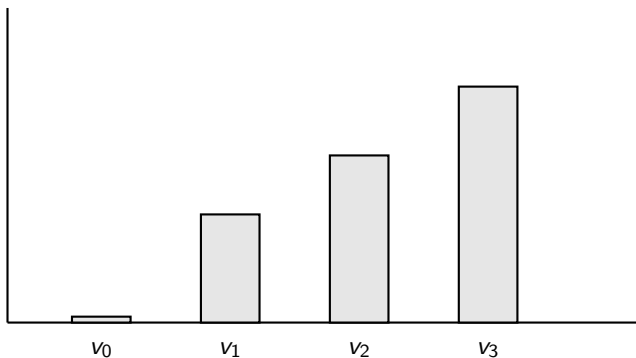# Bayesian game

- Given $(N+1, \Theta, p)$, $v \in \mathcal{V}$, the agents
  1. privately learn their types
  2. choose effort simultaneously
  3. and are awarded corresponding prizes

- Payoffs: If agent of type θ_k exerts effort x, and wins prize v_m (outperforms m out of remaining N agents), its vNM utility is

$$v_m - \theta_k \, x$$

Bayesian game

- Given $(N + 1, \Theta, p)$, $v \in \mathcal{V}$, the agents
  1. privately learn their types
  2. choose effort simultaneously
  3. and are awarded corresponding prizes

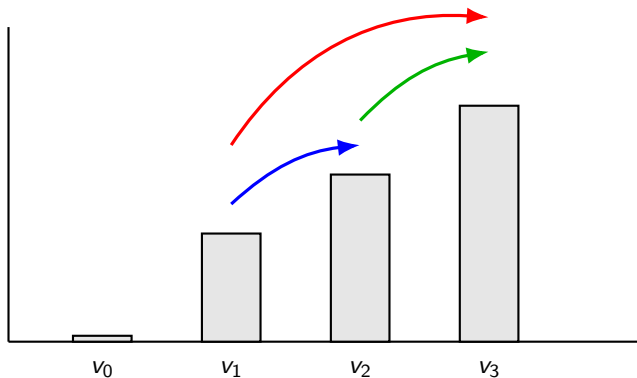- Payoffs: If agent of type θ_k exerts effort x, and wins prize v_m (outperforms m out of remaining N agents), its vNM utility is

$$v_m - \theta_k \, x$$

- Symmetric Bayes-Nash equilibrium: $(X_1, X_2, \ldots, X_K)$

Bayesian game

- Given $(N + 1, \Theta, p)$, $v \in \mathcal{V}$, the agents
  1. privately learn their types
  2. choose effort simultaneously
  3. and are awarded corresponding prizes
- Payoffs: If agent of type θ_k exerts effort x, and wins prize v_m (outperforms m out of remaining N agents), its vNM utility is

$$v_m - \theta_k\, x$$

- Symmetric Bayes-Nash equilibrium: $(X_1, X_2, \ldots, X_K)$
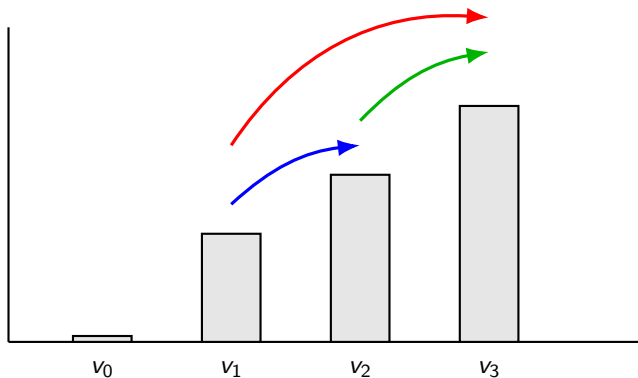- $\mathbb{E}[X] = \sum_{k=1}^{K} p_k \mathbb{E}[X_k]$: ex-ante expected effort

Increasing competition

Increasing competition

Research question

For prizes $m > m'$:

$$\frac{\partial \mathbb{E}[X]}{\partial v_m} - \frac{\partial \mathbb{E}[X]}{\partial v_{m'}}?$$

Equilibrium

Equilibrium structure

Equilibrium structure

Lemma.

Consider any $(N + 1, \Theta, p)$ and $v \in \mathcal{V}$. If $(X_1, \ldots, X_K)$ is an equilibrium, there exist boundary points $b_1 < \cdots < b_K$ so that X_k is continuously distributed on $[b_{k-1}, b_k]$.

Equilibrium structure

> **Lemma.**
>
> *Consider any $(N+1, \Theta, p)$ and $v \in \mathcal{V}$. If $(X_1, \ldots, X_K)$ is an equilibrium, there exist boundary points $b_1 < \cdots < b_K$ so that X_k is continuously distributed on $[b_{k-1}, b_k]$.*

Proof sketch

Proof sketch

1. Mixed strategies (no atoms):
   If type θ_k chooses x with positive probability, choosing $x + \epsilon$ leads to higher payoff

Proof sketch

1. Mixed strategies (no atoms):
   If type θ_k chooses x with positive probability, choosing $x + \epsilon$ leads to higher payoff

2. Disjoint supports (essentially):
   If types θ_j, θ_k have x, y in their support, it must be that
   $\theta_j(x - y) = \theta_k(x - y)$

Proof sketch

1. Mixed strategies (no atoms):
   If type θ_k chooses x with positive probability, choosing $x + \epsilon$ leads to higher payoff

2. Disjoint supports (essentially):
   If types θ_j, θ_k have x, y in their support, it must be that
   $\theta_j(x - y) = \theta_k(x - y)$

3. No gaps:
   If no type chooses in (d_1, d_2), the type choosing d_2 gets higher payoff by choosing d_1 than d_2

Proof sketch

1. Mixed strategies (no atoms):
   If type θ_k chooses x with positive probability, choosing $x + \epsilon$ leads to higher payoff

2. Disjoint supports (essentially):
   If types θ_j, θ_k have x, y in their support, it must be that
   $\theta_j(x - y) = \theta_k(x - y)$

3. No gaps:
   If no type chooses in (d_1, d_2), the type choosing d_2 gets higher payoff by choosing d_1 than d_2

4. Monotonicity:
   If more efficient type has x, y with $x < y$ in its support, then less efficient type gets a higher payoff from x than y
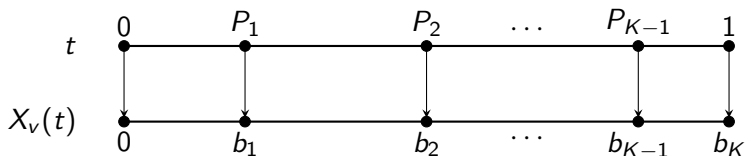
- $X_v(t)$: equilibrium effort in terms of
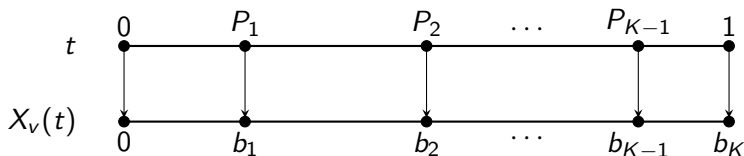 probability of outperforming an arbitrary agent

Equilibrium representation

- $X_v(t)$: equilibrium effort in terms of
  probability of outperforming an arbitrary agent (psuedo-type)

Equilibrium representation

- $X_v(t)$: equilibrium effort in terms of
  probability of outperforming an arbitrary agent (psuedo-type)

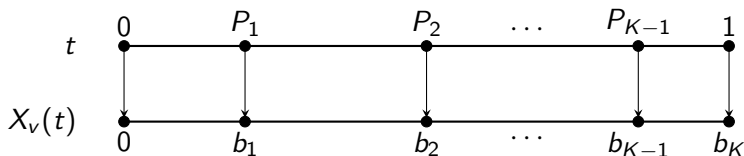# Equilibrium representation

- $X_v(t)$: equilibrium effort in terms of
  probability of outperforming an arbitrary agent (psuedo-type)

- $\pi_v(t)$: expected prize

Equilibrium representation

- $X_v(t)$: equilibrium effort in terms of
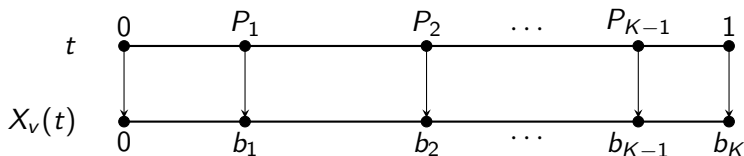 probability of outperforming an arbitrary agent (psuedo-type)

- $\pi_v(t)$: expected prize

$$\pi_v(t) = \sum_{m=0}^{N} v_m H_m^N(t) \qquad H_m^N(t) = \binom{N}{m} t^m (1-t)^{N-m}$$

Equilibrium representation

- $X_v(t)$: equilibrium effort in terms of
  probability of outperforming an arbitrary agent (psuedo-type)

- $\pi_v(t)$: expected prize

$$\pi_v(t) = \sum_{m=0}^{N} v_m H_m^N(t) \qquad H_m^N(t) = \binom{N}{m} t^m (1-t)^{N-m}$$

Example: for $v = (0, \ldots, 0, V)$, $\pi_v(t) = V t^N$

- Let $u_1, u_2, \ldots, u_K$ denote equilibrium utilities

- Let $u_1, u_2, \ldots, u_K$ denote equilibrium utilities
- For $t \in [P_{k-1}, P_k]$,

Indifference conditions

- Let $u_1, u_2, \ldots, u_K$ denote equilibrium utilities
- For $t \in [P_{k-1}, P_k]$,

$$\pi_v(t) - \theta_k \cdot X_v(t) = u_k$$

Indifference conditions

- Let $u_1, u_2, \ldots, u_K$ denote equilibrium utilities
- For $t \in [P_{k-1}, P_k]$,

$$X_v(t) = \frac{\pi_v(t) - u_k}{\theta_k}$$

- Let $u_1, u_2, \ldots, u_K$ denote equilibrium utilities
- For $t \in [P_{k-1}, P_k]$,

$$X_v(t) = \frac{\pi_v(t) - u_k}{\theta_k}$$

- Utilities:

Indifference conditions

- Let $u_1, u_2, \ldots, u_K$ denote equilibrium utilities
- For $t \in [P_{k-1}, P_k]$,

$$X_v(t) = \frac{\pi_v(t) - u_k}{\theta_k}$$

- Utilities:

$$u_1 = 0$$

Indifference conditions

- Let $u_1, u_2, \ldots, u_K$ denote equilibrium utilities
- For $t \in [P_{k-1}, P_k]$,

$$X_v(t) = \frac{\pi_v(t) - u_k}{\theta_k}$$

- Utilities:

$$u_1 = 0 \Rightarrow b_1 = \frac{\pi_v(P_1)}{\theta_1}$$

Indifference conditions

- Let $u_1, u_2, \ldots, u_K$ denote equilibrium utilities
- For $t \in [P_{k-1}, P_k]$,

$$X_v(t) = \frac{\pi_v(t) - u_k}{\theta_k}$$

- Utilities:

$$u_1 = 0 \Rightarrow b_1 = \frac{\pi_v(P_1)}{\theta_1}$$
$$\Rightarrow u_2 = \pi_v(P_1) - \theta_2 \cdot b_1$$

Indifference conditions

- Let $u_1, u_2, \ldots, u_K$ denote equilibrium utilities
- For $t \in [P_{k-1}, P_k]$,

$$X_v(t) = \frac{\pi_v(t) - u_k}{\theta_k}$$

- Utilities:

$$u_1 = 0 \Rightarrow b_1 = \frac{\pi_v(P_1)}{\theta_1}$$

$$\Rightarrow u_2 = \pi_v(P_1) - \theta_2 \cdot b_1$$

$$\Rightarrow b_2 = \frac{\pi_v(P_2) - u_2}{\theta_2}$$

Indifference conditions

- Let $u_1, u_2, \ldots, u_K$ denote equilibrium utilities
- For $t \in [P_{k-1}, P_k]$,

$$X_v(t) = \frac{\pi_v(t) - u_k}{\theta_k}$$

- Utilities:

$$u_1 = 0 \Rightarrow b_1 = \frac{\pi_v(P_1)}{\theta_1}$$

$$\Rightarrow u_2 = \pi_v(P_1) - \theta_2 \cdot b_1$$

$$\Rightarrow b_2 = \frac{\pi_v(P_2) - u_2}{\theta_2}$$

$$\Rightarrow u_3 = \ldots$$

Equilibrium characterization

Equilibrium characterization

Theorem 1.

For any $(N + 1, \Theta, p)$ and $v \in \mathcal{V}$, there is a unique symmetric equilibrium $(X_1, \ldots, X_K)$. It is such that for each $t \in [0, 1]$,

$$X_v(t) = \frac{\pi_v(t) - u_{k(t)}}{\theta_{k(t)}},$$

where $k(t) = \max\{k : P_{k-1} \leq t\}$ and the equilibrium utilities are

$$u_k = \theta_k \left[\sum_{j=1}^{k-1} \pi_v(P_j) \left(\frac{1}{\theta_{j+1}} - \frac{1}{\theta_j} \right) \right].$$

Competition: Linear Cost

Expected effort

- Since $t = F(X)$ is always $U[0, 1]$:

Expected effort

- Since $t = F(X)$ is always $U[0,1]$:

$$\mathbb{E}[X] = \int_0^1 X_v(t)\, dt$$

Expected effort

- Since $t = F(X)$ is always $U[0,1]$:

$$\mathbb{E}[X] = \int_0^1 \frac{\pi_v(t) - u_{k(t)}}{\theta_{k(t)}} \, dt$$

Expected effort

- Since $t = F(X)$ is always $U[0,1]$:

$$\mathbb{E}[X] = \int_0^1 \frac{\pi_v(t) - u_{k(t)}}{\theta_{k(t)}} \, dt$$

Lemma.

For any $(N+1, \Theta, p)$ and $v \in \mathcal{V}$,

$$\mathbb{E}[X] = \sum_{m=1}^{N} \alpha_m v_m, \text{ where}$$

Expected effort

- Since $t = F(X)$ is always $U[0,1]$:

$$\mathbb{E}[X] = \int_0^1 \frac{\pi_v(t) - u_{k(t)}}{\theta_{k(t)}} \, dt$$

Lemma.

For any $(N+1, \Theta, p)$ and $v \in \mathcal{V}$,

$$\mathbb{E}[X] = \sum_{m=1}^N \alpha_m v_m, \text{ where}$$

$$\alpha_m = \frac{1}{N+1} \left[\frac{1}{\theta_K} - \sum_{k=1}^{K-1} \left[H_{\geq m}^{N+1}(P_k) + (N-m) H_m^{N+1}(P_k) \right] \left(\frac{1}{\theta_{k+1}} - \frac{1}{\theta_k} \right) \right].$$

Competition

- For any $v \in \mathcal{V}$ and $m > m'$:

$$\frac{\partial \mathbb{E}[X]}{\partial v_m} - \frac{\partial \mathbb{E}[X]}{\partial v_{m'}} = \alpha_m - \alpha_{m'}$$

Competition

- For any $v \in \mathcal{V}$ and $m > m'$:

$$\frac{\partial \mathbb{E}[X]}{\partial v_m} - \frac{\partial \mathbb{E}[X]}{\partial v_{m'}} = \alpha_m - \alpha_{m'}$$

- Suppose $\Theta = \{\theta_1\}$:

$$\alpha_1 = \alpha_2 = \cdots = \alpha_N = \frac{1}{(N+1)\theta_1}$$

Complete information (Barut & Kovenock [1998])

For any $m > m'$, $\alpha_m - \alpha_{m'} = 0$.

Theorem 2.

Consider any $(N + 1, \Theta, p)$ with $|\Theta| > 1$. For any interior prize $m' \in \{1, \ldots, N - 1\}$,

$$\alpha_N - \alpha_{m'} > 0.$$

For the design problem, $v^* = (0, \ldots, 0, V)$ is uniquely optimal.

Theorem 2.

Consider any $(N + 1, \Theta, p)$ with $|\Theta| > 1$. For any interior prize $m' \in \{1, \ldots, N - 1\}$,

$$\alpha_N - \alpha_{m'} > 0.$$

For the design problem, $v^* = (0, \ldots, 0, V)$ is uniquely optimal.

Resolves Sisak [2009]'s conjecture in the negative

Monotonicity?

Monotonicity?

- For $m > m'$ but $m \neq N$:

$$\alpha_m - \alpha_{m'} > 0?$$

- For $m > m'$ but $m \neq N$:

$$\alpha_m - \alpha_{m'} > 0?$$

- Interior Discouragement Effect: Suppose $K = 2$. For any interior prize $m' \in \{1, \ldots, N-1\}$,

$$p_1 \geq \frac{m'+1}{N} \implies \alpha_{m'+1} - \alpha_{m'} \leq 0.$$

Monotonicity?

- For $m > m'$ but $m \neq N$:

$$\alpha_m - \alpha_{m'} > 0?$$

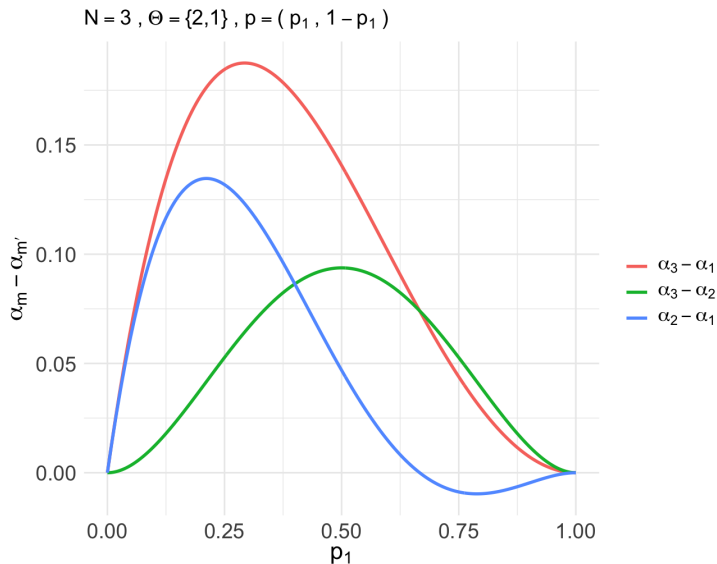- Interior Discouragement Effect: Suppose $K = 2$. For any interior prize $m' \in \{1, \ldots, N-1\}$,

$$p_1 \geq \frac{m'+1}{N} \implies \alpha_{m'+1} - \alpha_{m'} \leq 0.$$

- Extreme case: If $p_1 > \frac{N-1}{N}$, for any $m > m'$,

$$\alpha_m - \alpha_{m'} \geq 0 \iff m = N.$$

Competition (Linear Cost)

N = 3 , Θ = {2,1} , p = (p_1 , 1 − p_1)

Legend:
- $\alpha_3 - \alpha_1$ (red)
- $\alpha_3 - \alpha_2$ (green)
- $\alpha_2 - \alpha_1$ (blue)

y-axis: $\alpha_m - \alpha_{m'}$

x-axis: p_1

Competition: General Cost

General costs

- Given $(N+1, \Theta, p)$ and $v \in \mathcal{V}$, payoffs are

$$v_m - \theta_k \cdot c(x),$$

where $c(0) = 0$, $c'(x) > 0$ for $x > 0$, and $\lim_{x \to \infty} c(x) = \infty$.

General costs

- Given $(N+1, \Theta, p)$ and $v \in \mathcal{V}$, payoffs are

$$v_m - \theta_k \cdot c(x),$$

where $c(0) = 0$, $c'(x) > 0$ for $x > 0$, and $\lim_{x \to \infty} c(x) = \infty$.

- Key observation: The distribution of equilibrium effort costs $(c(X^*))$ coincides with the distribution of effort in the linear-cost case.

General costs

- Given $(N+1, \Theta, p)$ and $v \in \mathcal{V}$, payoffs are

$$v_m - \theta_k \cdot c(x),$$

where $c(0) = 0$, $c'(x) > 0$ for $x > 0$, and $\lim_{x \to \infty} c(x) = \infty$.

- Key observation: The distribution of equilibrium effort costs $(c(X^*))$ coincides with the distribution of effort in the linear-cost case.

- Equilibrium: For each $t \in [0, 1]$,

General costs

- Given $(N+1, \Theta, p)$ and $v \in \mathcal{V}$, payoffs are

$$v_m - \theta_k \cdot c(x),$$

where $c(0) = 0$, $c'(x) > 0$ for $x > 0$, and $\lim_{x \to \infty} c(x) = \infty$.

- Key observation: The distribution of equilibrium effort costs $(c(X^*))$ coincides with the distribution of effort in the linear-cost case.

- Equilibrium: For each $t \in [0, 1]$,

$$c(X_v^*(t)) = X_v(t)$$

where $X_v(t)$ is the equilibrium effort under $c(x) = x$ (Theorem 1)

General costs

- Given $(N+1, \Theta, p)$ and $v \in \mathcal{V}$, payoffs are

$$v_m - \theta_k \cdot c(x),$$

where $c(0) = 0$, $c'(x) > 0$ for $x > 0$, and $\lim_{x \to \infty} c(x) = \infty$.

- Key observation: The distribution of equilibrium effort costs $(c(X^*))$ coincides with the distribution of effort in the linear-cost case.

- Equilibrium: For each $t \in [0, 1]$,

$$X_v^*(t) = c^{-1}(X_v(t))$$

where $X_v(t)$ is the equilibrium effort under $c(x) = x$ (Theorem 1)

General costs

- Given $(N+1, \Theta, p)$ and $v \in \mathcal{V}$, payoffs are

$$v_m - \theta_k \cdot c(x),$$

where $c(0) = 0$, $c'(x) > 0$ for $x > 0$, and $\lim_{x \to \infty} c(x) = \infty$.

- Key observation: The distribution of equilibrium effort costs $(c(X^*))$ coincides with the distribution of effort in the linear-cost case.

- Equilibrium: For each $t \in [0, 1]$,

$$X_v^*(t) = g(X_v(t))$$

where $X_v(t)$ is the equilibrium effort under $c(x) = x$ (Theorem 1)

Competition

- Expected effort:

$$\mathbb{E}[X^*] = \int_0^1 g(X_v(t)) \ dt$$

Competition

- Expected effort:

$$\mathbb{E}[X^*] = \int_0^1 g(X_\nu(t)) \, dt$$

- For prizes $m > m'$:

$$\frac{\partial \mathbb{E}[X^*]}{\partial v_m} - \frac{\partial \mathbb{E}[X^*]}{\partial v_{m'}} = \int_0^1 g'(X_\nu(t)) \underbrace{\left(\frac{\partial X_\nu(t)}{\partial v_m} - \frac{\partial X_\nu(t)}{\partial v_{m'}} \right)}_{\text{effect on effort cost } c(X)} \, dt$$

where

$$\frac{\partial X_\nu(t)}{\partial v_m} = \frac{1}{\theta_{k(t)}} \left(H_m^N(t) - \frac{\partial u_{k(t)}}{\partial v_m} \right)$$

Competition

- Expected effort:

$$\mathbb{E}[X^*] = \int_0^1 g(X_v(t)) \, dt$$

- For prizes $m > m'$:

$$\frac{\partial \mathbb{E}[X^*]}{\partial v_m} - \frac{\partial \mathbb{E}[X^*]}{\partial v_{m'}} = \int_0^1 g'(X_v(t)) \underbrace{\left(\frac{\partial X_v(t)}{\partial v_m} - \frac{\partial X_v(t)}{\partial v_{m'}} \right)}_{\substack{\text{effect on effort cost } c(X) \\ \text{effect on effort if } c(x)=x}} dt$$

where

$$\frac{\partial X_v(t)}{\partial v_m} = \frac{1}{\theta_{k(t)}} \left(H_m^N(t) - \frac{\partial u_{k(t)}}{\partial v_m} \right)$$

$$\frac{\partial \mathbb{E}[X^*]}{\partial v_m} - \frac{\partial \mathbb{E}[X^*]}{\partial v_{m'}} = \int_0^1 \underbrace{g'(X_v(t))}_{\text{weights}} \underbrace{\left(\frac{\partial X_v(t)}{\partial v_m} - \frac{\partial X_v(t)}{\partial v_{m'}} \right)}_{\text{single-crossing?}} dt$$
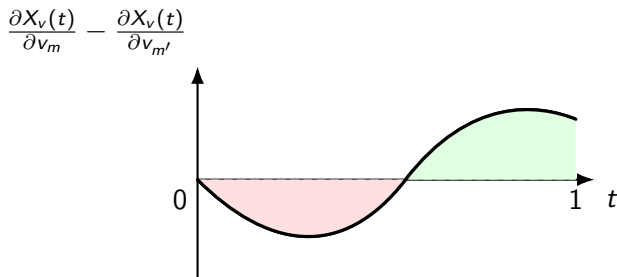
Approach: Linear to General

$$\frac{\partial \mathbb{E}[X^*]}{\partial v_m} - \frac{\partial \mathbb{E}[X^*]}{\partial v_{m'}} = \int_0^1 \underbrace{g'(X_v(t))}_{\text{weights}} \underbrace{\left(\frac{\partial X_v(t)}{\partial v_m} - \frac{\partial X_v(t)}{\partial v_{m'}} \right)}_{\text{single-crossing?}} dt$$
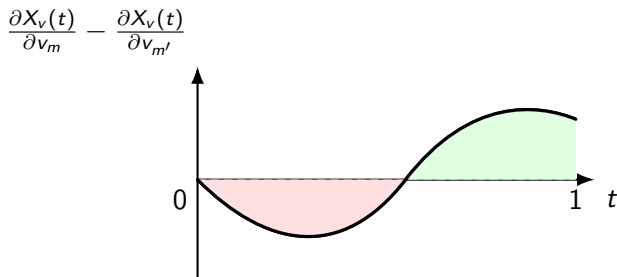
Approach: Linear to General

$$\frac{\partial \mathbb{E}[X^*]}{\partial v_m} - \frac{\partial \mathbb{E}[X^*]}{\partial v_{m'}} = \int_0^1 \underbrace{g'(X_v(t))}_{\text{weights}} \underbrace{\left(\frac{\partial X_v(t)}{\partial v_m} - \frac{\partial X_v(t)}{\partial v_{m'}} \right)}_{\text{single-crossing?}} dt$$

$$\frac{\partial X_v(t)}{\partial v_m} - \frac{\partial X_v(t)}{\partial v_{m'}}$$

- $\alpha_m - \alpha_{m'} \geq 0$, c concave $\implies$ for any $v \in \mathcal{V}$, $\dfrac{\partial \mathbb{E}[X^*]}{\partial v_m} - \dfrac{\partial \mathbb{E}[X^*]}{\partial v_{m'}} \geq 0$
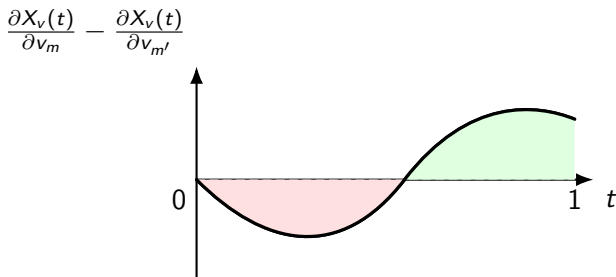
$$\frac{\partial \mathbb{E}[X^*]}{\partial v_m} - \frac{\partial \mathbb{E}[X^*]}{\partial v_{m'}} = \int_0^1 \underbrace{g'(X_v(t))}_{\text{weights}} \underbrace{\left(\frac{\partial X_v(t)}{\partial v_m} - \frac{\partial X_v(t)}{\partial v_{m'}} \right)}_{\text{single-crossing?}} dt$$

$\frac{\partial X_v(t)}{\partial v_m} - \frac{\partial X_v(t)}{\partial v_{m'}}$

- $\alpha_m - \alpha_{m'} \geq 0$, c concave $\implies$ for any $v \in \mathcal{V}$, $\frac{\partial \mathbb{E}[X^*]}{\partial v_m} - \frac{\partial \mathbb{E}[X^*]}{\partial v_{m'}} \geq 0$

- $\alpha_m - \alpha_{m'} \leq 0$, c convex $\implies$ for any $v \in \mathcal{V}$, $\frac{\partial \mathbb{E}[X^*]}{\partial v_m} - \frac{\partial \mathbb{E}[X^*]}{\partial v_{m'}} \leq 0$

Single-crossing?

- $m = N$: ✓

Single-crossing?

- $m = N$: ✓
- $m < N$: ✓ or ✗

Main Result

Main Result

Theorem 3.

Consider any $(N + 1, \Theta, p)$ with cost $c(\cdot)$. Let m, m' with $m > m'$ be such that either $m = N$ or $\left(\frac{\partial u_K}{\partial v_m} - \frac{\partial u_K}{\partial v_{m'}} \right) \leq 0$. Then, the following hold:

Main Result

Theorem 3.

Consider any $(N+1, \Theta, p)$ with cost $c(\cdot)$. Let m, m' with $m > m'$ be such that either $m = N$ or $\left(\frac{\partial u_K}{\partial v_m} - \frac{\partial u_K}{\partial v_{m'}} \right) \leq 0$. Then, the following hold:
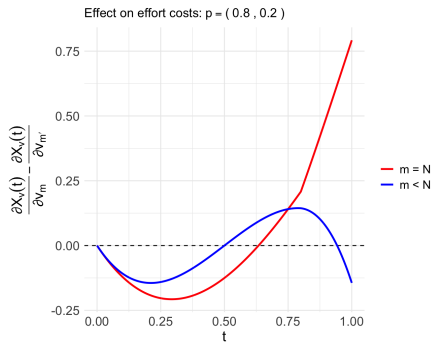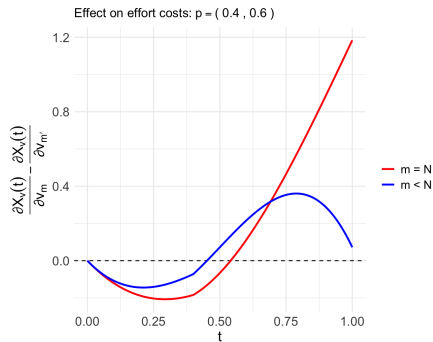
① $\alpha_m - \alpha_{m'} \geq 0$, c concave $\implies$ for any $v \in \mathcal{V}$, $\dfrac{\partial \mathbb{E}[X^*]}{\partial v_m} - \dfrac{\partial \mathbb{E}[X^*]}{\partial v_{m'}} \geq 0$

Main Result

Theorem 3.

Consider any $(N + 1, \Theta, p)$ with cost $c(\cdot)$. Let m, m' with $m > m'$ be such that either $m = N$ or $\left(\frac{\partial u_K}{\partial v_m} - \frac{\partial u_K}{\partial v_{m'}} \right) \leq 0$. Then, the following hold:

1. $\alpha_m - \alpha_{m'} \geq 0$, *c concave* $\implies$ *for any* $v \in \mathcal{V}$, $\dfrac{\partial \mathbb{E}[X^*]}{\partial v_m} - \dfrac{\partial \mathbb{E}[X^*]}{\partial v_{m'}} \geq 0$

2. $\alpha_m - \alpha_{m'} \leq 0$, *c convex* $\implies$ *for any* $v \in \mathcal{V}$, $\dfrac{\partial \mathbb{E}[X^*]}{\partial v_m} - \dfrac{\partial \mathbb{E}[X^*]}{\partial v_{m'}} \leq 0$

Main Result

Theorem 3.

Consider any $(N+1, \Theta, p)$ with cost $c(\cdot)$. Let m, m' with $m > m'$ be such that either $m = N$ or $\left(\frac{\partial u_K}{\partial v_m} - \frac{\partial u_K}{\partial v_{m'}} \right) \leq 0$. Then, the following hold:

1. $\alpha_m - \alpha_{m'} \geq 0$, c concave $\implies$ for any $v \in \mathcal{V}$, $\dfrac{\partial \mathbb{E}[X^*]}{\partial v_m} - \dfrac{\partial \mathbb{E}[X^*]}{\partial v_{m'}} \geq 0$

2. $\alpha_m - \alpha_{m'} \leq 0$, c convex $\implies$ for any $v \in \mathcal{V}$, $\dfrac{\partial \mathbb{E}[X^*]}{\partial v_m} - \dfrac{\partial \mathbb{E}[X^*]}{\partial v_{m'}} \leq 0$

Complete information (Fang, Noe, and Strack [2020])

Suppose $|\Theta| = 1$.

1. If c is concave, competition encourages effort.
2. If c is convex, competition discourages effort.

Other objectives

Other objectives

Theorem 3 can be readily generalized to other quantities of interest:

Other objectives

Theorem 3 can be readily generalized to other quantities of interest:

- X_{max}^*: highest effort across agents

Other objectives

Theorem 3 can be readily generalized to other quantities of interest:

- X^*_{max}: highest effort across agents

$$\frac{\partial \mathbb{E}[X^*_{max}]}{\partial v_m} - \frac{\partial \mathbb{E}[X^*_{max}]}{\partial v_{m'}} = \int_0^1 \underbrace{g'(X_v(t))\, t^N}_{\text{weights}} \underbrace{\left(\frac{\partial X_v(t)}{\partial v_m} - \frac{\partial X_v(t)}{\partial v_{m'}}\right)}_{\text{single-crossing?}} dt$$

Other objectives

Theorem 3 can be readily generalized to other quantities of interest:

- X^*_{max}: highest effort across agents

$$\frac{\partial \mathbb{E}[X^*_{max}]}{\partial v_m} - \frac{\partial \mathbb{E}[X^*_{max}]}{\partial v_{m'}} = \int_0^1 \underbrace{g'(X_v(t))\, t^N}_{\text{weights}} \underbrace{\left(\frac{\partial X_v(t)}{\partial v_m} - \frac{\partial X_v(t)}{\partial v_{m'}} \right)}_{\text{single-crossing?}} dt$$

- $\alpha_m - \alpha_{m'}$: informative under monotonic weights and single-crossing

Other objectives

Theorem 3 can be readily generalized to other quantities of interest:

- X^*_{max}: highest effort across agents

$$\frac{\partial \mathbb{E}[X^*_{max}]}{\partial v_m} - \frac{\partial \mathbb{E}[X^*_{max}]}{\partial v_{m'}} = \int_0^1 \underbrace{g'(X_v(t))\, t^N}_{\text{weights}} \underbrace{\left(\frac{\partial X_v(t)}{\partial v_m} - \frac{\partial X_v(t)}{\partial v_{m'}} \right)}_{\text{single-crossing?}} dt$$

- $\alpha_m - \alpha_{m'}$: informative under monotonic weights and single-crossing

Theorem 4.

Consider any $(N+1, \Theta, p)$ with cost $c(\cdot)$. If c is (weakly) concave, the winner-takes-all contest maximizes expected total effort of top q agents for any $q \in \{1, \ldots, N+1\}$.

Convex costs?

Convex costs?

Optimality of WTA persists if cost is not too convex:

Convex costs?

Optimality of WTA persists if cost is not too convex:

> **Proposition 5.**
>
> *Consider any $(N + 1, \Theta, p)$ with $|\Theta| > 1$ and cost $c(x) = x^\alpha$. For any q, there exists $\alpha^* > 1$ such that for all $\alpha \in [1, \alpha^*)$, the WTA maximizes expected total effort of top q agents.*

Convex costs?

Optimality of WTA persists if cost is not too convex:

Proposition 5.

Consider any $(N + 1, \Theta, p)$ with $|\Theta| > 1$ and cost $c(x) = x^\alpha$. For any q, there exists $\alpha^* > 1$ such that for all $\alpha \in [1, \alpha^*)$, the WTA maximizes expected total effort of top q agents.

Conjecture

The degree of convexity required to overturn the optimality of WTA shrinks as the contest environment approaches complete information.

Convergence

Theorem 6.

Fix any contest $v \in \mathcal{V}$ and cost function $c : \mathbb{R}_+ \to \mathbb{R}_+$. Let $G : [\underline{\theta}, \overline{\theta}] \to [0, 1]$ be a differentiable CDF and let $G^1, G^2, \ldots,$ be any sequence of CDF's, each with a finite support, such that for all $\theta \in [\underline{\theta}, \overline{\theta}]$,

$$\lim_{n \to \infty} G^n(\theta) = G(\theta).$$

Then, the corresponding sequence of finite-type space equilibrium CDF's, $F^1, F^2, \ldots,$ converges to the continuum type-space equilibrium CDF F, i.e., for all $x \in \mathbb{R}$,

$$\lim_{n \to \infty} F^n(x) = F(x).$$

Convergence

Theorem 6.

Fix any contest $v \in \mathcal{V}$ and cost function $c : \mathbb{R}_+ \to \mathbb{R}_+$. Let $G : [\underline{\theta}, \overline{\theta}] \to [0, 1]$ be a differentiable CDF and let $G^1, G^2, \ldots,$ be any sequence of CDF's, each with a finite support, such that for all $\theta \in [\underline{\theta}, \overline{\theta}]$,

$$\lim_{n \to \infty} G^n(\theta) = G(\theta).$$

Then, the corresponding sequence of finite-type space equilibrium CDF's, $F^1, F^2, \ldots,$ converges to the continuum type-space equilibrium CDF F, i.e., for all $x \in \mathbb{R}$,

$$\lim_{n \to \infty} F^n(x) = F(x).$$

- Experimental literature: Large but finite type-spaces

Experiment

Experiment

- Prolific : 445 participants
- Contest environment: 4 agents, $\Theta = \{2x, x\}$, $p = (0.8, 0.2)$
- Four contests which are progressively less competitive:
  WTA $= (0, 0, 0, 100)$
  High $= (0, 0, 25, 75)$
  Med $= (0, 0, 50, 50)$
  Low $= (0, 25, 25, 50)$
- Strategy method, contest order randomized

Expected effort

Treatment	$(0, v_1, v_2, v_3)$	Equilibrium Effort ($\mathbb{E}[X]$)			Observed Effort		
		$c_k(x) = x$	$c_k(x) = 2x$	Pooled	$c_k(x) = x$	$c_k(x) = 2x$	Pooled
WTA	$(0, 0, 0, 100)$	48.2	6.4	14.76	52.8	40.5	42.96
High	$(0, 0, 25, 75)$	37.0	8.0	13.80	46.9	37.4	39.30
Med	$(0, 0, 50, 50)$	25.8	9.6	12.84	43.3	36.5	37.86
Low	$(0, 25, 25, 50)$	24.6	10.2	13.08	42.9	35.9	37.30

Table: Equilibrium and observed efforts by treatment and cost type.

Expected effort

Treatment	$(0, v_1, v_2, v_3)$	Equilibrium Effort ($\mathbb{E}[X]$)			Observed Effort		
		$c_k(x) = x$	$c_k(x) = 2x$	Pooled	$c_k(x) = x$	$c_k(x) = 2x$	Pooled
WTA	$(0, 0, 0, 100)$	48.2	6.4	14.76	52.8	40.5	42.96
High	$(0, 0, 25, 75)$	37.0	8.0	13.80	46.9	37.4	39.30
Med	$(0, 0, 50, 50)$	25.8	9.6	12.84	43.3	36.5	37.86
Low	$(0, 25, 25, 50)$	24.6	10.2	13.08	42.9	35.9	37.30

Table: Equilibrium and observed efforts by treatment and cost type.

- Efficient type exerts higher effort than the less efficient type

Expected effort

Treatment	$(0, v_1, v_2, v_3)$	Equilibrium Effort ($\mathbb{E}[X]$)			Observed Effort		
		$c_k(x) = x$	$c_k(x) = 2x$	Pooled	$c_k(x) = x$	$c_k(x) = 2x$	Pooled
WTA	$(0, 0, 0, 100)$	48.2	6.4	14.76	52.8	40.5	42.96
High	$(0, 0, 25, 75)$	37.0	8.0	13.80	46.9	37.4	39.30
Med	$(0, 0, 50, 50)$	25.8	9.6	12.84	43.3	36.5	37.86
Low	$(0, 25, 25, 50)$	24.6	10.2	13.08	42.9	35.9	37.30

Table: Equilibrium and observed efforts by treatment and cost type.

- Efficient type exerts higher effort than the less efficient type
- Significant over-provision compared to equilibrium across treatments

Expected effort

Treatment	$(0, v_1, v_2, v_3)$	Equilibrium Effort ($\mathbb{E}[X]$)			Observed Effort		
		$c_k(x) = x$	$c_k(x) = 2x$	Pooled	$c_k(x) = x$	$c_k(x) = 2x$	Pooled
WTA	$(0, 0, 0, 100)$	48.2	6.4	14.76	52.8	40.5	42.96
High	$(0, 0, 25, 75)$	37.0	8.0	13.80	46.9	37.4	39.30
Med	$(0, 0, 50, 50)$	25.8	9.6	12.84	43.3	36.5	37.86
Low	$(0, 25, 25, 50)$	24.6	10.2	13.08	42.9	35.9	37.30

Table: Equilibrium and observed efforts by treatment and cost type.

- Efficient type exerts higher effort than the less efficient type
- Significant over-provision compared to equilibrium across treatments
- WTA remains optimal

Expected effort

Treatment	$(0, v_1, v_2, v_3)$	Equilibrium Effort ($\mathbb{E}[X]$)			Observed Effort		
		$c_k(x) = x$	$c_k(x) = 2x$	Pooled	$c_k(x) = x$	$c_k(x) = 2x$	Pooled
WTA	$(0, 0, 0, 100)$	48.2	6.4	14.76	52.8	40.5	42.96
High	$(0, 0, 25, 75)$	37.0	8.0	13.80	46.9	37.4	39.30
Med	$(0, 0, 50, 50)$	25.8	9.6	12.84	43.3	36.5	37.86
Low	$(0, 25, 25, 50)$	24.6	10.2	13.08	42.9	35.9	37.30

Table: Equilibrium and observed efforts by treatment and cost type.

- Efficient type exerts higher effort than the less efficient type
- Significant over-provision compared to equilibrium across treatments
- WTA remains optimal
- Going from Low to Med does not lead to a significant change in effort

Regression estimates

$$X = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \epsilon.$$

Prize	Equilibrium Weight			Estimated Coefficient		
	$c_k(x) = x$	$c_k(x) = 2x$	Pooled	$c_k(x) = x$	$c_k(x) = 2x$	Pooled
α_3	0.482	0.064	0.148	0.524	0.401	0.426
α_2	0.034	0.128	0.109	0.335	0.321	0.324
α_1	-0.014	0.152	0.119	0.334	0.314	0.318

Table: Expected Effort: Equilibrium and Regression Results

Summary

- The (most competitive) winner-takes-all is robustly optimal for maximizing effort under linear or concave costs.
- Despite this, increasing competition in the interior may discourage effort if inefficient types are relatively likely.
- An experiment provides qualitative support to these findings.
- The techniques we develop are broadly applicable, and may be valuable where mixed equilibria impede analysis.
- The noisy output environment provides an interesting avenue for future work.

- The (most competitive) winner-takes-all is robustly optimal for maximizing effort under linear or concave costs.
- Despite this, increasing competition in the interior may discourage effort if inefficient types are relatively likely.
- An experiment provides qualitative support to these findings.
- The techniques we develop are broadly applicable, and may be valuable where mixed equilibria impede analysis.
- The noisy output environment provides an interesting avenue for future work.

Thank you!