Luce contracts

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- How should a principal design a contract so as to incentivize agents to exert costly effort?
  - piece-rate contracts $\rightarrow$ high variability, possibility of large payments
  - assume principal is budget-constrained

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- Small budget assumption: $B < \min\{c_1'(1), c_2'(1), \ldots, c_n'(1)\}$

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- This paper: characterize $\mathcal{P}$, identify contracts that implement $\mathcal{P}$

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- Highest-priority successful agent takes all:

$$f_i(S) = \begin{cases} 1, & \text{if } i = \max\{j : j \in S\} \\ 0, & \text{otherwise} \end{cases}$$

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- For any $f \in \mathcal{F}_{FGN}$, agent i's best-response to p_{-i} is given by

$$\mathbb{E}[B \cdot f_i(S) | i \in S] = c_i'(b_i(f, p_{-i})).$$

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- Space of SGE contracts is large ($\Theta(n2^n)$-dimensional)

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- A contract f is a *Luce contract* if there exist weights $(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i > 0$ and a non-strict ordering $\succcurlyeq$ on the agents such that

$$f_i(S) = \begin{cases} \dfrac{\lambda_i}{\sum_{j \in \mathrm{Top}_{\succcurlyeq}(S)} \lambda_j}, & \text{if } i \in \mathrm{Top}_{\succcurlyeq}(S) \\ 0, & \text{otherwise} \end{cases}$$

where $\mathrm{Top}_{\succcurlyeq}(S) = \{i \in S \,:\, i \succcurlyeq j \ \forall j \in S\}$.

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- The principal can optimize over this $n - 1$ dimensional class of Luce contracts

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- From the structure of SGE contract,

$$\mathbb{E}\left[\sum_{i \in I} B \cdot f_i(S)\right] \leq B \cdot \mathbb{P}[S \cap I \neq \emptyset],$$

with equality for $I = [n]$,

$$\mathbb{E}\left[\sum_{i \in [n]} B \cdot f_i(S)\right] = B \cdot \mathbb{P}[S \neq \emptyset].$$

Luce implementable profiles

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- It follows that $p \in \mathcal{P}$ must be such that

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Proposition 2.

Suppose $p \in (0,1)^n$. There exists a Luce contract $f \in \mathcal{F}_{Luce}$ that implements p if and only if for all $I \subset [n]$,

$$\frac{\sum_{i \in I} p_i \cdot c_i'(p_i)}{\sum_{i \in [n]} p_i \cdot c_i'(p_i)} \leq \frac{\mathbb{P}[S \cap I \neq \emptyset]}{\mathbb{P}[S \neq \emptyset]}.$$

Luce contracts for standard environments

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- Without a budget constraint, plethora of contracts to implement any desired $q \in (0,1)^n$, including:
    1. piece-rate contract: pay $c_i'(q_i)$ to agent i if it succeeds
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- If q satisfies the inequalities in Proposition 2, the total payment under any FGN contract that implements q is a mean-preserving spread of the total payment under the Luce contract that implements q

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- Under LL constraints, FGN contracts offer the cheapest alternatives
- If q satisfies the inequalities in Proposition 2, the total payment under any FGN contract that implements q is a mean-preserving spread of the total payment under the Luce contract that implements q
- Thus, Luce contracts offer a desirable alternative for implementation in standard environments

Application: 2 agents, quadratic costs

Example 3.

Suppose $n = 2$, $c_i(p_i) = \frac{1}{2}C_i p_i^2$ with $C_i > 1$, and $V(p_1, p_2) = wp_1 + p_2$. Then, the optimal contract, defined by $\lambda_1(w)$, takes the form

$$f_i(S) = \begin{cases} 0, & \text{if } i \notin S \\ 1, & \text{if } S = \{i\} \\ \lambda_i(w), & \text{if } S = \{1,2\} \end{cases},$$

where $\lambda_2(w) = 1 - \lambda_1(w)$. Moreover, $\lambda_1(w)$ is increasing in w and in particular,

$$\lambda_1(w) = \begin{cases} 0, & \text{if } w \leq \frac{C_1C_2 - C_1}{C_1C_2 + C_2 - 1} \\ \frac{1}{2}, & \text{if } w = 1 \\ 1, & \text{if } w \geq \frac{C_1C_2 + C_1 - 1}{C_1C_2 - C_2} \end{cases}.$$

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  2. joint performance (Fleckinger [2012], Alchian and Demsetz [1972], Itoh [1991], Kambhampati [2024])

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  2. joint performance (Fleckinger [2012], Alchian and Demsetz [1972], Itoh [1991], Kambhampati [2024])
- Moral hazard in teams: Holmstrom [1982], Winter [2004], Battaglini [2006], Babaioff, Feldman, Nisan, and Winter [2012], Halac, Lipnowski, and Rappoport [2021], Dai and Toikka [2022]

Summary

- Study multi-agent contract design with budget-constrained principal
- Introduce a novel class of contracts, called Luce contracts
- Demonstrate their optimality in environments with budget considerations
- Illustrate their desirability for implementation in general contract design environments

Thank you!