Prizes and effort in contests with private information*

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Abstract

We consider contests where participants have private information about their ability and the contest designer can manipulate the values of different prizes to influence effort. We study the effect on effort of two different interventions: increasing the value of prizes and increasing competition (by transferring value from worse to better prizes). We find that the effect of the two interventions depend qualitatively on the distribution of abilities in the population, and in particular, identify two natural sufficient conditions under which both the interventions have opposite effects on effort. When productive agents are more likely than unproductive agents, these interventions discourage effort. And when unproductive agents are more likely, these interventions encourage effort. We also discuss applications to the design of optimal contests in three different environments: grading contests, contests where agents have concave utilities, and contests with homogeneous prizes.

1 Introduction

Contests are situations in which agents compete with one another by investing effort or resources to win prizes. Such competitive situations are common in many social and economic contexts, including college admissions, classroom settings, labor markets, R&D races, sporting events, politics, etc. While some of these situations arise naturally, there are many others where the contest designer can carefully design the rules of the contest so as to satisfy their objectives. The designer's objective, and the structural elements of the contest that it can and cannot manipulate may vary depending upon the situation.

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In this paper, we focus on situations where the contest participants have private information about their abilities (defined by their marginal costs of effort) and the designer can manipulate the values of the different prizes $v_1, \ldots, v_n$ to influence the effort exerted by the participants. Our goal is to understand how different contests (defined by the prizes $v_1, v_2, \ldots, v_n$) compare in terms of the effort they induce. In particular, we study the effect on effort of two different interventions. First, we study how the effort changes as the designer increases the values of the different prizes. Second, we study how effort changes as the designer increases the competitiveness of the contest (by transferring value from lower ranked prizes to better ranked prizes). Note that the first intervention requires the designer to put in more money into the contest, while the second one does not require any additional investment. We also illustrate the relevance of the exercise by discussing applications to the design of optimal contests in three different environments. More specifically, we study the design of grading contests which are widely used in classroom environments, the design of contests where agents have concave utility for prizes, and the design of contests where the designer can award any number of agents with a homogeneous prize of fixed value.

We find that the effect of the two interventions on effort, and thus the structure of optimal contests in the three applications, depend qualitatively on the distribution of abilities in the population. In particular, we identify two sufficient conditions on the distribution of abilities in the population under which increasing the values of prizes, and also increasing competition, has opposite effects on the effort exerted by the agents. The sufficient conditions are on the relative likelihood of highly productive (low marginal cost of effort) and less productive (high marginal cost of effort) agents in the population. More precisely, we measure an agent's ability by its marginal cost of effort $\theta \in [0, 1]$ and find that when the density of agents is increasing in marginal cost, so that inefficient agents are more likely than efficient agents, these interventions encourage effort. In contrast, when the density function is decreasing, so that efficient agents are more likely than inefficient agents, these interventions discourage effort. Consequently, the structure of optimal contests in our three applications also differ depending upon the distribution of abilities.

We characterize optimal contests in the three applications for the cases where the density function is monotone increasing or decreasing. We note here that we focus on a parametric class of distributions with monotone density functions ($F(\theta) = \theta^p$) for some of these results. For the design of grading contests, we assume that the value of a grade equals the expected productivity of the agent who gets it. Under this assumption, we find that more informative grading schemes lead to more competitive prize vectors, and thus, using our results on the effect of competition, we establish a link between the informativeness of a grading scheme and the effort induced by it. In short, we find that more informative grading schemes lead to greater effort when the density is increasing ($p > 1$) and lesser effort when the density is decreasing ($p < 1$). Thus, an effort-maximizing designer would want to reveal the rank when $p > 1$ and it would want to award only two different grades, say A and B, in some distribution when $p < 1$. For the second application where we consider agents with concave utilities for prizes, we find that when the density function is decreasing, an effort-maximizing budget-constrained designer would allocate the entire budget to the first prize irrespective of the concavity of the utility function. In comparison, when the density function is increasing, the optimal prize vector distributes the budget over $n-1$ prizes and the distribution becomes less competitive as the utility function becomes more concave. For the last application with homogeneous prizes, we find that the optimal contest awards either a single prize or $n-1$ prizes depending upon whether density function is increasing or decreasing.

Literature review

There is a vast literature studying the design of contests in incomplete information environments (Glazer and Hassin [34], Moldovanu and Sela [50, 51], Zhang [68], Liu and Lu [47], Liu, Lu, Wang, and Zhang [46]). The paper most closely related to ours is Moldovanu and Sela [50] who showed that in a model with linear utility and linear costs, an effort-maximizing budget-constrained designer would allocate the entire budget to the first prize, irrespective of the distribution of abilities. Moldovanu and Sela [51] and more recently, Zhang [68] showed that a winner-takes-all prize structure maximizes expected effort in a more general class of mechanisms. In summary, the literature has shown that increasing the value of first prize has a dominant effect in terms of encouraging effort as compared to increasing the value of other prizes. Our paper contributes to this literature by illustrating that if the value of the first prize was exogenously fixed, it is not always the case that an effort-maximizing designer would simply go down the ranks allocating as much prize money as possible until it runs out of budget, as perhaps the optimality of the winner-take-all contest would suggest. While this may be interesting in itself, we also illustrate its relevance through our applications, especially to the design of optimal grading schemes. We also note here that the distributional assumptions we make in this paper are disjoint from those made in the literature as the papers generally assume a lower bound on the marginal costs of effort while we allow for the possibility of genius agents with negligible marginal costs of effort.

The optimal contest design problem of allocating a budget across different prizes so as to maximize effort has also been considered in other contest environments. In a complete information environment with symmetric agents, Fang, Noe, and Strack [27] showed that increasing competition discourages effort which generalizes the idea that it is optimal for the designer to distribute the budget equally amongst the top $n-1$ prizes (Glazer and Hassin [34], Barut and Kovenock [2]). Clark and Riis [17] provides examples with asymmetric agents where splitting the budget into more than one prize might be optimal. Clark and Riis [18] considers Tullock form contest success functions and finds that increasing competition leads to an increase in total effort. Szymanski and Valletti [66] show that with asymmetric agents, a second prize might be optimal. Other related work that looks at the design of optimal contests under some different assumptions include Krishna and Morgan [42], Liu and Lu [48], Cohen and Sela [19]. Sisak [64] provides a more detailed survey of the literature on this problem. More general surveys of the theoretical literature in contest theory can be found in Corchón [21], Vojnović [67], Konrad et al. [38], Fu and Wu [30], Segev [59].

There is also a growing literature studying grading contests (Moldovanu, Sela, and Shi [52], Rayo [56], Popov and Bernhardt [55], Chan, Hao, and Suen [11], Dubey and Geanakoplos [24], Zubrickas [69], Rodina, Farragut et al. [57], Krishna, Lychagin, Olszewski, Siegel, and Tergiman [40]). The papers generally differ in whether they allow for relative or absolute grading schemes, and also in their assumptions about how the grades translate to prizes. Our paper contributes to the strand of literature in which the value associated with a grade is determined by the information it reveals about the agent's productivity to the market, and thus, the designer's problem of choosing a grading scheme is essentially one of determining how much information to disclose about the agent's type. Rayo [56], Zubrickas [69], Rodina, Farragut et al. [57] study information disclosure policies with absolute grading schemes and find conditions under which pooling types together with a common grade may be optimal. In recent work, Krishna, Lychagin, Olszewski, Siegel, and Tergiman [40] study information disclosure policies with relative grading schemes in a large contest framework and investigate how pooling intervals of performances together can improve the welfare of the agents in a Pareto sense. In other related work, Brownback [9] studies experimentally the effect on effort of increasing class size under a pass/fail grading scheme. Butcher et al. [10] uses data from Wellesley College to show that switching from letter grades to a pass/fail policy led to a decline in student effort. Assuming unproductive students are more likely than productive students, the empirical finding is consistent with our result that less informative grading schemes discourage effort.

The paper proceeds as follows. In section 2, we present the model of a contest in an incomplete-information environment. Section 3 characterizes the symmetric Bayes-Nash equilibrium of the contest game and studies the effect of prizes and competition on effort. In section 4, we discuss applications to the design of optimal contests in three natural environments. Section 5 concludes. All proofs are relegated to the appendix.

2 Model

There is a single contest designer and n agents. The designer chooses a vector of prizes $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ such that $v_i \geq v_{i+1}$ for all i. The agents compete for these prizes by exerting costly effort. Each agent i is privately informed about its marginal cost of effort $\theta_i \in [0,1]$ which is drawn independently from $[0,1]$ according to a distribution $F : [0,1] \to [0,1]$. The distribution F is common knowledge.

Assumption 1. The distribution of marginal costs F is twice-differentiable and it is such that $\lim_{\theta \to 0} f(\theta)F(\theta) = 0$.

The assumption ensures that there aren't too many highly productive agents in the population and is satisfied, in particular, by the parametric class of distributions $F(\theta) = \theta^p$ with $p > \frac{1}{2}$. Given a vector of prizes $\mathbf{v}$, marginal cost of effort θ_i, and belief F about the marginal costs of effort of other agents, each agent i simultaneously chooses an effort level e_i. The designer ranks the agents in order of the efforts they put in and awards them the corresponding prizes. The agent who puts in the maximum effort is awarded prize v_1. Agent with the second highest effort is awarded prize v_2 and so on. If agent i puts in effort e_i and wins prize v_i, its final payoff is

$$v_i - \theta_i e_i$$

Given a prize structure $\mathbf{v}$ and belief F, an agent's strategy $\sigma_i : [0,1] \to \mathbb{R}_+$ maps its marginal cost of effort to the level of effort it puts in. A strategy profile $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a Bayes-Nash equilibrium of the game if for all agents i and type $\theta_i \in [0,1]$, agent i's expected payoff from playing $\sigma_i(\theta_i)$ is at least as high as its payoff from playing anything else given that all other agents are playing σ_{-i}. We'll focus on the symmetric Bayes-Nash equilibrium of this contest game. This is a Bayes-Nash equilibrium where all agents are playing the same strategy $g_{\mathbf{v}} : [0,1] \to \mathbb{R}_+$.

Given a prior distribution F, we'll assume the designer's preferences over the different contests or prize vectors $\mathbf{v}$ is defined by a monotone utility function $U : \mathbb{R}_+ \to \mathbb{R}$ so that the designer prefers $\mathbf{v}$ over $\mathbf{v}'$ if and only if $\mathbb{E}[U(g_{\mathbf{v}}(\theta))] \geq \mathbb{E}[U(g_{\mathbf{v}'}(\theta))]$ where $g_{\mathbf{v}}$ represents the symmetric Bayes-Nash equilibrium function under prize vector $\mathbf{v}$. We'll impose conditions on U as required to illustrate our results. A standard objective for the designer in the literature is to maximize expected effort which is captured in our model by the utility function $U(x) = x$.

Notation

We will denote by $p_i(t)$ the probability that a random variable $X \sim Bin(n-1, t)$ takes the value $i-1$. That is,

$$p_i(t) = \binom{n-1}{i-1} t^{i-1}(1-t)^{n-i}$$

Also note that

$$p_i'(t) = \binom{n-1}{i-1} t^{i-2}(1-t)^{n-i-1}\left((i-1) - t(n-1)\right))$$

The Beta function is defined by

$$\beta(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$$

for $a, b > 0$. For integral a, b, we have that

$$\beta(a,b) = \frac{(a-1)!(b-1)!}{(a+b-1)!}.$$

3 Equilibrium

We begin by characterizing the unique symmetric Bayes-Nash equilibrium of the contest game for arbitrary prize vectors. While the equilibrium strategy function takes the same form as in Moldovanu and Sela [50], it satisfies an interesting property due to the presence of agents with negligible marginal costs of effort which we will discuss later. The following result displays the symmetric Bayes-Nash equilibrium strategy of the contest game (Moldovanu and Sela [50]).

Lemma 1. *The equilibrium function is given by*

$$g_{\mathbf{v}}(\theta) = \sum_{i=1}^{n} v_i m_i(\theta)$$

where

$$m_i(\theta) = -\int_{F(\theta)}^{1} \frac{p_i'(t)}{F^{-1}(t)} dt$$

The proof uses the standard approach of assuming $n-1$ agents are playing the same strategy and then getting conditions under which that strategy is also the best response for the last agent: $\sum_{i=1}^{n} v_i p_i'(F(\theta))f(\theta) - \theta g'(\theta) = 0$. The boundary condition $g_{\mathbf{v}}(1) = 0$ pins down the form of the equilibrium function as in Lemma 1. The full proof is in the appendix.

The equilibrium effort level of an agent of type θ is linear in the values of prizes v_i and the weights $m_i(\theta)$ depend on the distribution F. Observe that for any agent type $\theta \in [0,1]$, the sum of weights $\sum m_i(\theta) = 0$ (because $\sum_i p_i(t) = 1$ for all t). This makes sense because if all the prizes are equal, there is no incentive for any agent to put in any effort. Generally, studying the effects of manipulating different prizes on effort amounts to understanding properties of these marginal effect functions $m_i(\theta)$ and we'll discuss them in the next subsection.

In case an agent's value for prize v is given by some increasing utility function $u(v)$ and all agents share the same utility function u for prizes, the equilibrium is simply as if prize i was $u(v_i)$ instead of v_i in Lemma 1 . The following corollary states this formally.

Corollary 1. *If the agents have a common increasing utility function u for prizes, the equilibrium function is given by*

$$g_{\mathbf{v}}(\theta) = \sum_{i=1}^{n} u(v_i) m_i(\theta)$$

where

$$m_i(\theta) = -\int_{F(\theta)}^{1} \frac{p_i'(t)}{F^{-1}(t)} dt$$

3.1 Effect of prizes

Now we focus our attention on studying how the equilibrium effort changes as we vary the values of different prizes. Given the linearity of the equilibrium function in Lemma 1, we can do so by simply understanding the properties of the marginal effect functions $m_i(\theta)$.

The next result illustrates the differing effects of increasing the value of the first prize, an intermediate prize, or the last prize on effort.

Lemma 2. *The marginal effect functions* $m_i(\theta) = -\int_{F(\theta)}^{1} \frac{p_i'(t)}{F^{-1}(t)} dt$ *satisfy:*

1. $m_1(\theta) \geq 0$ *for all* $\theta \in [0, 1]$
2. $m_i(\theta) = \begin{cases} < 0 & \text{if } \theta \leq t_i \\ \geq 0 & \text{otherwise} \end{cases}$ *where* $t_i \in (0, 1)$ *for* $i \in \{2, \ldots, n-1\}$
3. $m_n(\theta) \leq 0$ *for all* $\theta \in [0, 1]$

The result follows from the fact that $p_1(t)$ is monotone decreasing, $p_i(t)$ is single peaked for $i \in \{2, \ldots, n-1\}$ and $p_n(t)$ is monotone increasing. The full proof is in the appendix.

In words, Lemma 2 says that increasing the value of the first prize encourages effort for all agent types and increasing the value of the last prize discourages effort for all agent types. In comparison, increasing the value of any intermediate prize $i \in \{2, \ldots, n-1\}$ has mixed effects in that it reduces the effort of highly efficient agents (those with low θ) and increases the effort of the less efficient agents (those with high θ). Intuitively, this is because the highly efficient agents are generally winning the best prizes and when the value of an intermediate prize increases, their value for the better prizes, and hence the incentive to exert effort, goes down. In contrast, the less efficient agents get lower ranked prizes and when the value of an intermediate prize increases, they are encouraged to exert greater effort for the better prize.

The next result identifies an interesting property of the aggregate effect of increasing the value of any prize.

Lemma 3. *The marginal effect functions* $m_i(\theta) = -\int_{F(\theta)}^{1} \frac{p_i'(t)}{F^{-1}(t)} dt$ *satisfy:*

1. $\int_0^1 m_1(\theta) d\theta = 1$
2. $\int_0^1 m_i(\theta) = 0$ *for* $i \in \{2, \ldots, n-1\}$
3. $\int_0^1 m_n(\theta) = -1$

Using assumption 1, we show that $\int_0^1 m_i(\theta) d\theta = p_i(0) - p_i(1)$ which implies the result. Note that the possibility of agents with negligible marginal costs of effort is essential for the equilibrium to satisfy this property. We believe ours is the first paper to consider this possibility and hence, identify this property of the equilibrium function. We'll make use of this property later to study the expected effect on effort of increasing the value of different prizes under some special distributions.

In words, Lemma 3 says that the aggregate impact of increasing the value of any prize (measured by area under its marginal effect curve) is balanced in that it does not depend on the distribution of abilities F. In particular, we know from Lemma 2 that increasing the value of an intermediate prize reduces effort of the most efficient agents while increasing the effort of the less efficient agents. Lemma 3 says that the reduction in the effort of the most efficient agents is exactly compensated for by the increase in effort of the less efficient agents in the sense that the area under the equilibrium function remains the same. Note that by compensate, we do not mean that the expected effort remains the same but that the area under the equilibrium function remains the same.

Corollary 2. *For any distribution F and prize vector $\mathbf{v}$,*

$$\int_0^1 g_{\mathbf{v}}(\theta)d\theta = v_1 - v_n$$

The properties of the marginal effect functions described in Lemmas 2, 3 are illustrated in figure 1 for the case of $n = 5$ and $F(\theta) = \theta^2$.

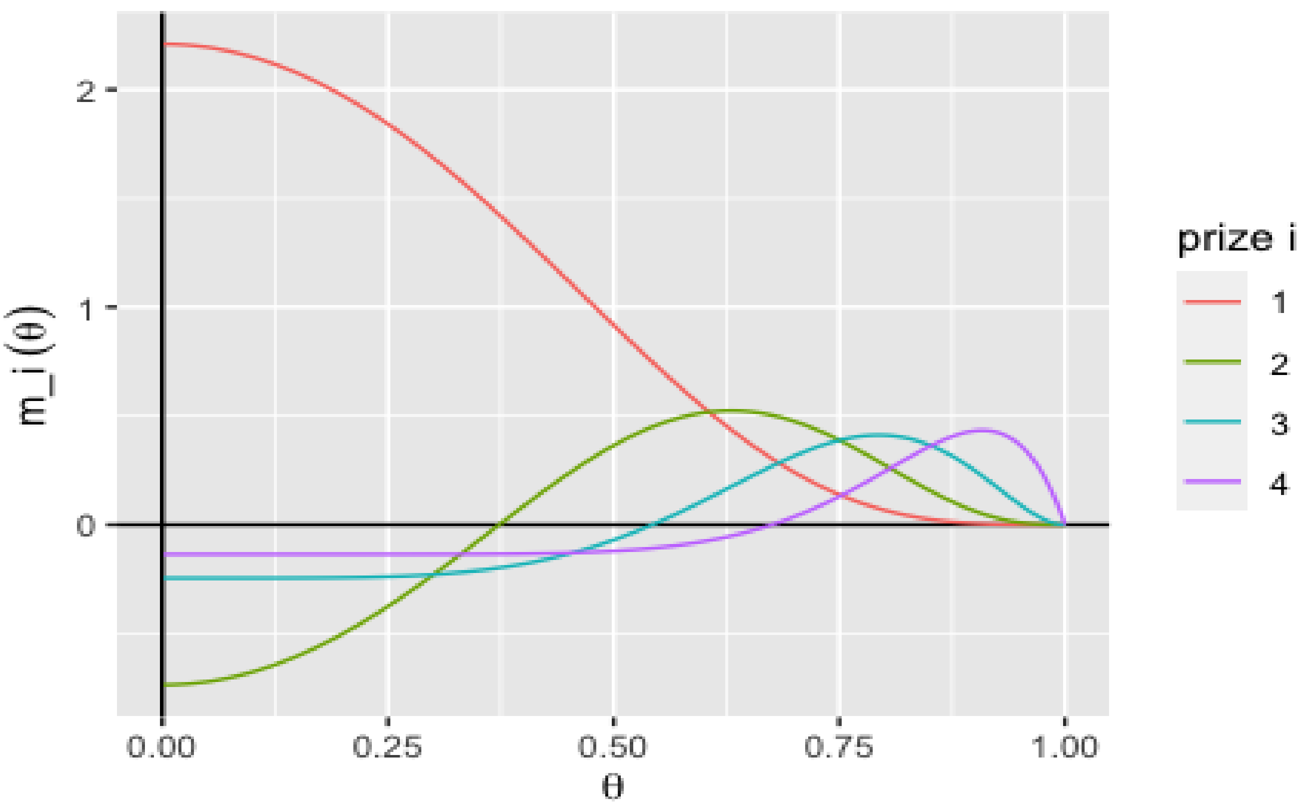

Figure 1: The marginal effect of prizes on effort for $n = 5$ and $F(\theta) = \theta^2$.

Now to study the overall effect of increasing the value of these prizes, we look at $\mathbb{E}[m_i(\theta)]$. Note that it is clear from Lemma 2 that $\mathbb{E}[m_1(\theta)] > 0$ and $\mathbb{E}[m_n(\theta)] < 0$ irrespective of the distribution of abilities F. Thus, increasing the value of first prize increases expected effort while increasing the value of last prize decreases expected effort. The next result identifies sufficient conditions on the distribution of abilities under which increasing the value of any intermediate prize $i \in \{2, \ldots, n-1\}$ has opposite effects on expected effort.

Theorem 1. *Suppose* $\mathbf{v}, \mathbf{w}$ *are two prize vectors such that* $v_i > w_i$ *for some intermediate prize* $i \in \{2, \ldots, n-1\}$ *and* $v_j = w_j$ *for* $j \neq i$.

1. *If the density* f *is increasing, then a designer with a concave utility* U *for effort prefers* $\mathbf{v}$ *over* $\mathbf{w}$.

2. *If the density* f *is decreasing, then a designer with a convex utility* U *for effort prefers* $\mathbf{w}$ *over* $\mathbf{v}$.

In particular, the theorem implies that a designer who cares about increasing expected effort may prefer to increase the value of intermediate prizes or decrease them depending on whether the density function is increasing or decreasing. Intuitively, we know from Lemmas 2 and 3 that increasing the value of any intermediate prize leads to a balanced transfer of effort from the highly efficient agents to the less efficient agents. The overall expected effect then depends on the relative likelihood of highly efficient and less efficient agents. When the density is increasing so that less efficient agents are dominant, increasing the value of intermediate prizes increases expected effort. When the density is decreasing so that highly efficient agents are dominant, increasing the value of intermediate prizes reduces expected effort.

Corollary 3. *Suppose* $\mathbf{v}, \mathbf{w}$ *are two prize vectors such that* $v_i > w_i$ *for some intermediate prize* $i \in \{2, \ldots, n-1\}$ *and* $v_j = w_j$ *for* $j \neq i$.

1. *If the density* f *is increasing, then* $\mathbb{E}[g_{\mathbf{v}}(\theta)] \geq \mathbb{E}[g_{\mathbf{w}}(\theta)]$.

2. *If the density* f *is decreasing, then* $\mathbb{E}[g_{\mathbf{v}}(\theta)] \leq \mathbb{E}[g_{\mathbf{w}}(\theta)]$.

Also, it follows from the Theorem that when F is uniform so that the density function is constant, the expected effort does not change as the value of intermediate prizes change. More precisely, we obtain the following corollary:

Corollary 4. *Suppose* $\mathbf{v}, \mathbf{w}$ *are two prize vectors such that* $v_i > w_i$ *for some intermediate prize* $i \in \{2, \ldots, n-1\}$ *and* $v_j = w_j$ *for* $j \neq i$. *If* F *is uniform, then* $g_{\mathbf{w}}(\theta)$ *is a mean-preserving spread of* $g_{\mathbf{v}}(\theta)$.

To prove the theorem, we use the fact that $\int_0^1 m_i(\theta)d\theta = 0$ (Lemma 3) and that it is initially negative and then positive (Lemma 2) to show that expected marginal effect on effort of prize i,

$$\mathbb{E}[m_i(\theta)] = \int_0^1 m_i(\theta)f(\theta)d\theta = -\int_0^1 \frac{p_i'(t)}{F^{-1}(t)}tdt$$

is positive if the density is increasing and negative if the density is decreasing. The more general comparison with respect to concave and convex utilities then follows from the single crossing property of the equilibrium functions $g_{\mathbf{v}}(\theta)$ and $g_{\mathbf{w}}(\theta)$. The full proof is in the appendix.

The effect of increasing prizes and the single crossing property of the equilibrium function is illustrated in figure 2 for the case of $n = 5$ and $F(\theta) = \theta^2$.

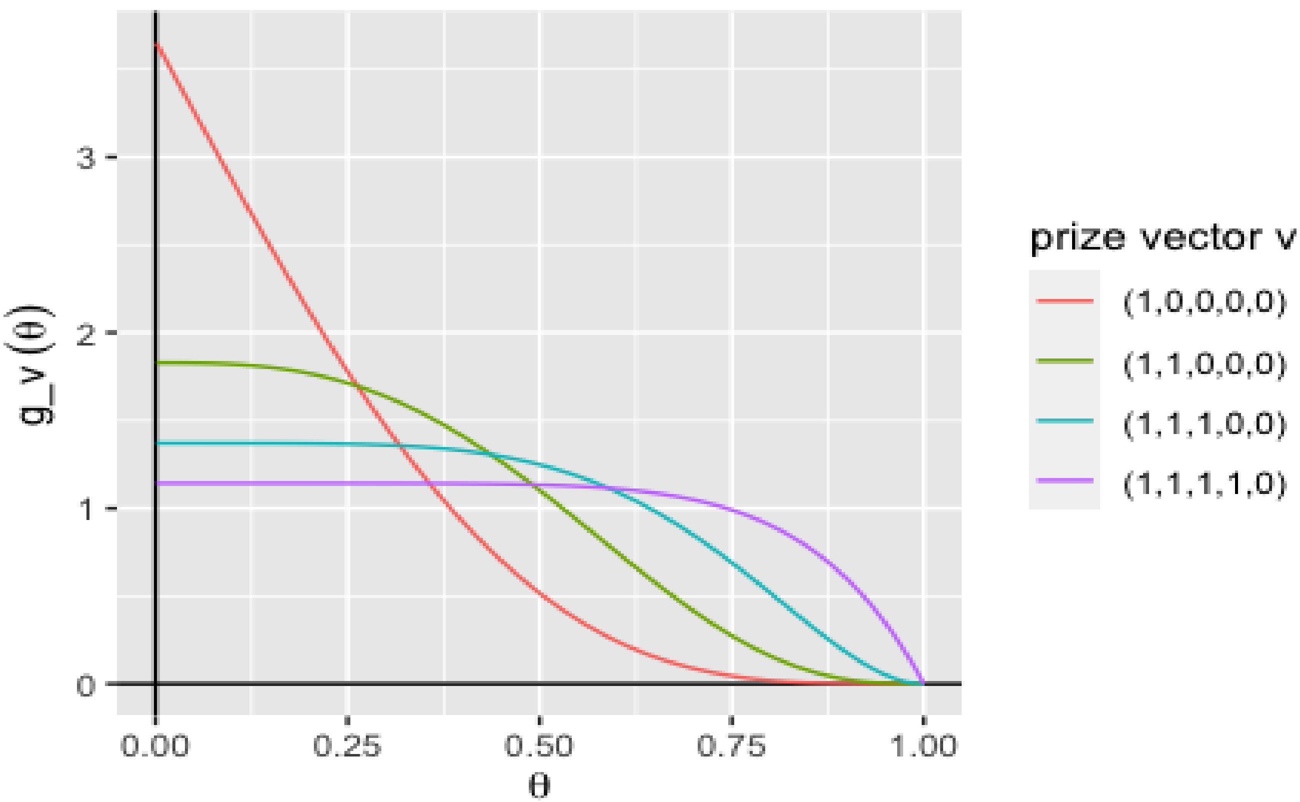

Figure 2: The effect of increasing prizes on equilibrium effort for $n = 5$ and $F(\theta) = \theta^2$.

3.2 Effect of competition

In this subsection, we study the effect of increasing the competitiveness of a contest on effort. In our framework, where a contest is defined by a prize vector $\mathbf{v}$, we say a contest $\mathbf{v}$ is more competitive than $\mathbf{w}$ if the prizes in $\mathbf{v}$ are more unequal than in $\mathbf{w}$. Formally:

Definition 3.1. A prize vector $\mathbf{v} = (v_1, v_2, \ldots, v_{n-1}, v_n)$ is more competitive than $\mathbf{w} = (w_1, w_2, \ldots, w_{n-1}, w_n)$ if $\mathbf{v}$ majorizes $\mathbf{w}$ (i.e. $\sum_{i=1}^k v_i \geq \sum_{i=1}^k w_i$ for all $k \in [n]$ and $\sum_{i=1}^n v_i = \sum_{i=1}^n w_i$).

In words, a more competitive prize vector $\mathbf{v}$ can be obtained from a less competitive prize vector $\mathbf{w}$ by transferring value from lower ranked prizes to better ranked prizes. This definition of competitiveness was also considered in Fang, Noe, and Strack [27] who showed that increasing competition has a discouraging effect on effort in contests under complete information environments.

To study the effect of competition on effort, we need to understand how the expected marginal effects vary across prizes. That is, we want to understand how $\mathbb{E}[m_i(\theta)]$ compares with $\mathbb{E}[m_j(\theta)]$. We know that

$$\mathbb{E}[m_i(\theta)] = \int_0^1 m_i(\theta) f(\theta) d\theta = -\int_0^1 \frac{p_i'(t)}{F^{-1}(t)} t dt$$

It is known from Moldovanu and Sela [50] that $\mathbb{E}[m_1(\theta)] > \mathbb{E}[m_i(\theta)]$ for any $i > 1$. That is, the effect of the first prize on expected effort dominates the effect of any other prize. But it is not clear how the effects of the intermediate prizes compare with each other. Does this idea generalize and do we have $\mathbb{E}[m_i(\theta)] \geq \mathbb{E}[m_j(\theta)]$ for all $i < j$ or do we get something else? While we are unable to say something in complete generality, we focus on a parametric class of distributions $F(\theta) = \theta^p$ with $p > \frac{1}{2}$ and show that the idea does not generalize. Note that $p > \frac{1}{2}$ ensures that assumption 1 is satisfied. Under these parametric assumptions, we show that the comparison of expected marginal effects of the intermediate prizes on effort depends on whether the density is increasing ($p > 1$) or decreasing ($p < 1$).

Theorem 2. *Suppose $F(\theta) = \theta^p$ and $\mathbf{v}$ and $\mathbf{w}$ are two prize vectors such that $\mathbf{v}$ is more competitive than $\mathbf{w}$.*

1. *If $p > 1$, then*
$$\mathbb{E}[g_{\mathbf{v}}(\theta)] \geq \mathbb{E}[g_{\mathbf{w}}(\theta)].$$
2. *If $\frac{1}{2} < p < 1$, and $v_1 = w_1, v_n = w_n$, then*
$$\mathbb{E}[g_{\mathbf{v}}(\theta)] \leq \mathbb{E}[g_{\mathbf{w}}(\theta)].$$
3. *If $p > \frac{1}{2}$ and $v_n = w_n$, then*
$$\mathbb{E}[g_{\mathbf{v}}(\theta_{max})] \leq \mathbb{E}[g_{\mathbf{w}}(\theta_{max})].$$

Here again, we find that the effect of increasing competition depends qualitatively on the relative likelihood of highly efficient and less efficient agents. When less efficient agents are more likely than highly efficient agents ($p > 1$), increasing competition by increasing prize inequality encourages effort. In contrast, when efficient agents are more likely ($\frac{1}{2} < p < 1$), increasing competition discourages effort.

The parametric assumptions allow us to compute the expected marginal effects for each prize i. More precisely, we get that for any $F(\theta) = \theta^p$ with $p > \frac{1}{2}$ and $i \in \{2, \ldots, n-1\}$,

$$\mathbb{E}[m_i(\theta)] = \binom{n-1}{i-1} \beta(i - \frac{1}{p}, n-i) \frac{(n-i)(p-1)}{np-1}$$

The ratio of expected marginal effects is then

$$\frac{\mathbb{E}[m_{i+1}(\theta)]}{\mathbb{E}[m_i(\theta)]} = \frac{n-i}{i} \frac{i - \frac{1}{p}}{n-i-1} \frac{n-i-1}{n-i} = \frac{i - \frac{1}{p}}{i} < 1$$

Thus, the ratio is always < 1 irrespective of p. In the case where $p > 1$ so that the density is increasing, the marginal effects are positive (Theorem 1) and therefore, the ratio being less than one implies that the expected effects are decreasing in the rank of the prize. That is, $\mathbb{E}[m_i(\theta)] > \mathbb{E}[m_j(\theta)]$ for all $i < j$. Thus, any transfer of value from lower ranked prize to better ranked prizes would lead to a net increase in expected effort. Since a more competitive prize vector $\mathbf{v}$ can be obtained from a less competitive prize vector $\mathbf{w}$ via a sequence of such transfers, we get that increasing competition encourages effort when $p > 1$. This is perhaps a bit surprising since these are distributions in which the designer puts more weight on the effort of the less efficient agents, who care more about the lower ranked prizes. While we do see that the relative benefit of prize $i+1$ over i increases as p increases, it remains ≤ 1 as $p \to \infty$. Also note that this is in contrast to the complete information case where increasing competition discourages effort (Fang, Noe, and Strack [27]).

An analogous argument holds for the case where $\frac{1}{2} < p < 1$. In this case, the density is decreasing and so the expected marginal effects are actually negative (Theorem 1). Thus, the ratio being less < 1 implies that the expected marginal effects are actually increasing in i. As a result, we get that conditional on the first and last prize being fixed, transfer of value from worse prizes to better prizes actually discourages effort. Thus, in this case we get that increasing competition discourages effort.

For the case of expected minimum effort, we again find the expected marginal effect of each prize on the effort of the least efficient agent. In the general case, we show that

$$\mathbb{E}[m_i(\theta_{max})] = -\int_0^1 \frac{p_i'(t)}{F^{-1}(t)} t^n dt$$

and then plug in $F(\theta) = \theta^p$ to get that for each $i \in \{1, 2, \ldots, n-1\}$,

$$\mathbb{E}[m_i(\theta_{max})] = \binom{n-1}{i-1} \beta\left(n+i-1-\frac{1}{p}, n-i\right) \frac{(n-i)(np-1)}{2np-p-1}$$

which implies

$$\frac{\mathbb{E}[m_{i+1}(\theta_{max})]}{\mathbb{E}[m_i(\theta_{max})]} = \frac{n+i-1-\frac{1}{p}}{i} > 1$$

By similar reasoning as above, we get here that conditional on the last prize being fixed, increasing competition discourages effort. In this case, the designer is putting a lot of weight on the effort of the least efficient agents, who care more about the value of the worse prizes (Lemma 2). Thus, we get that the lower ranked prizes induce greater effort in expectation from the least efficient agent than the top ranked prizes.

We note here that increasing competition by transferring value from a lower ranked intermediate prize to a better ranked intermediate prize leads to equilibrium functions that cross each other at two distinct points. Due to this double-crossing property, we do not believe that Theorem 2 generalizes to the case where the density is simply increasing or decreasing. Anyhow, the parametric assumption we make allow us to illustrate that it is not always the case that the expected marginal effects are decreasing in the rank of the prize, as one might suspect based on Moldovanu and Sela [50]'s result on the dominant effect of the first prize.

The effect of increasing competition and the double crossing property of the equilibrium function is illustrated in figure 3 for the case of $n = 5$ and $F(\theta) = \theta^2$.

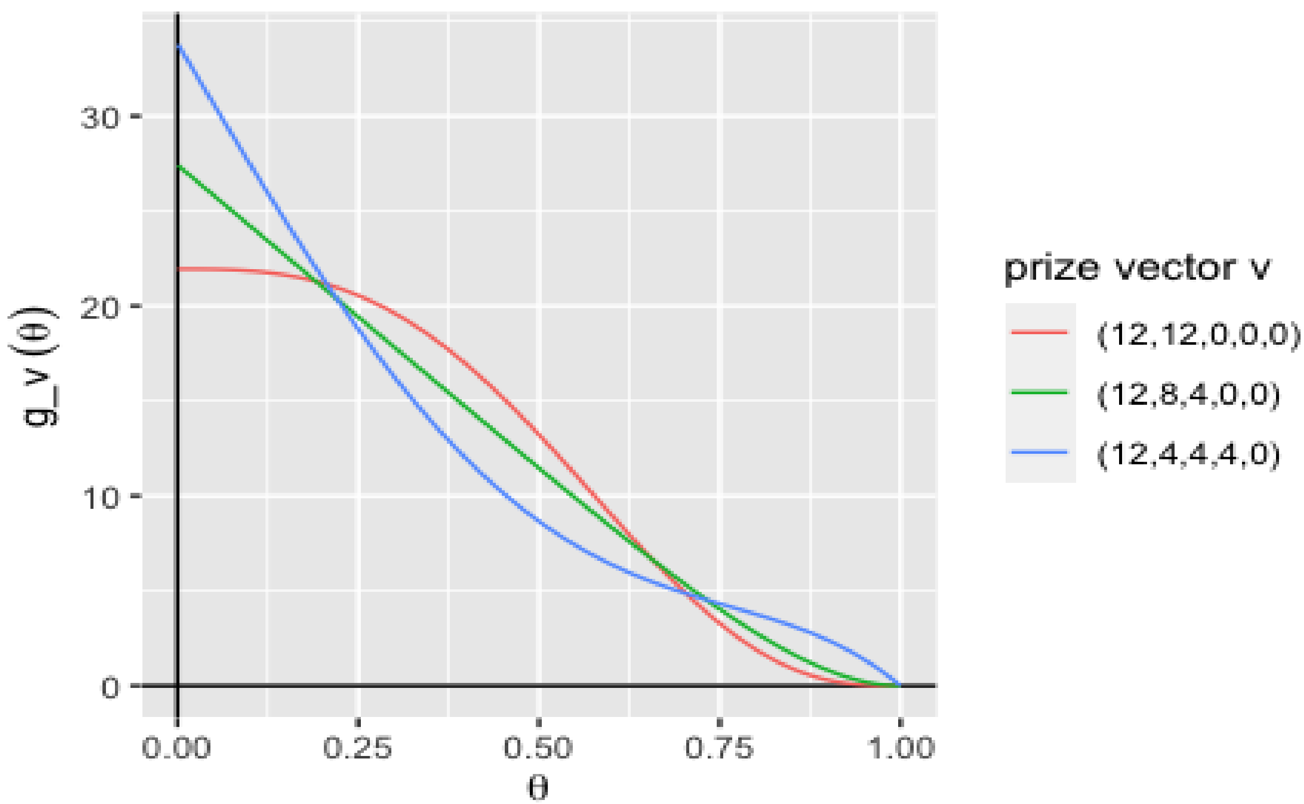

Figure 3: The effect of competition on equilibrium effort for $n = 5$ and $F(\theta) = \theta^2$.

4 Applications

In this section, we'll discuss applications of our results to the design of optimal contests in three different environments. First, we'll consider the design of grading schemes. Second, we'll consider settings where agents have concave utilities for prizes and the designer has a fixed budget that it can distribute arbitrarily across prizes. And last, we'll consider settings where the designer can costlessly award any number of agents with a homogeneous prize of a fixed value. In all of these environments, we'll find that the structure of the optimal contest depends in an important way on the distribution of abilities in the population.

4.1 Grading schemes

Our first applications looks at the design of grading schemes. These are generally used in classroom settings where the professor awards grades to students based on their performance in exams. While the grades may be assigned based on absolute scores as well, we focus here on the case where the grades can only be assigned based on relative performance. For instance, the professor may commit to giving grades A and B to the top 50% and bottom 50% respectively, or it may give $A+, A-, B+$, and $B-$ with distribution $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Formally, we define a grading contest as follows:

Definition 4.1. A grading contest with n agents is defined by a strictly increasing sequence of natural numbers $s = (s_1, s_2, \dots, s_k)$ such that $s_k = n$.

The interpretation of grading contest s is that the top s_1 agents get grade g_1, next $s_2 - s_1$ get grade g_2 and generally, $s_k - s_{k-1}$ agents get grade g_k. But how do these grades translate to prizes? In incomplete information environments like the one considered in this paper, the grade secured by an agent under any grading scheme reveals information about its rank, and thus, its type. We assume that the value of a grade is determined by the information it reveals about the type of the agent. More precisely, we assume that there is a publicly known wage (productivity) function $w : \Theta \to \mathbb{R}_+$ which maps an agent's marginal cost to its productivity and is monotone decreasing. The interpretation is that if the market could observe that an agent is of type θ, it would offer the agent a wage of $w(\theta)$. Given this wage function, we assume that the value of a grade in a grading contest s equals the expected productivity of the agent who gets the grade.

Observe that there is a natural partial order over grading contests in terms of how much information they reveal about the type of the agents. In the examples above, the grading contest that awards the grades $A+, A-, B+$, and $B-$ in equal proportion is more informative about the agents type then the one that awards just A and B in equal proportion. More generally, we can say the following:

Definition 4.2. A grading contest s is more informative than s' if s' is a subsequence of s.

Clearly, the rank revealing contest $s^* = (1, 2, \dots, n)$ is more informative than any other grading contest. Under our assumption for how grades translate to prizes, the rank revealing contest $s^* = (1, 2, \dots, n)$ induces the prize vector

$$v_i = \mathbb{E}[w(\theta)|\theta = \theta^n_{(i)}]$$

where $\theta^n_{(i)}$ is the ith order statistic in a random sample of n agents. This is because the rank revealing contest reveals the exact rank of the agent in a random sample of n observations. Note here that since $\theta^n_{(i)}$ is stochastically dominated by $\theta^n_{(j)}$ for all $i < j$ and w is monotone decreasing, the prize vector induced by the rank revealing contest is monotone decreasing $v_1 > v_2 \cdots > v_n$.

Now we can define the prize vectors induced by arbitrary grading contests s in terms of the v_i's as defined above. An arbitrary grading contest $s = (s_1, s_2, \dots, s_k)$ induces the prize vector $v(s)$ where

$$v(s)_i = \frac{v_{s_{j-1}+1} + v_{s_{j-1}+2} + \cdots + v_{s_j}}{s_j - s_{j-1}}$$

and j is such that $s_{j-1} < i \leq s_j$. This is because if an agent gets grade g_j in the grading contest $s = (s_1, s_2, \dots, s_k)$, then the market learns that the agent's rank must be one of $\{s_{j-1} + 1, \dots, s_j\}$ and further, it is equally likely to be ranked at any of these positions. The form of the prize vector above then follows from the assumption that the value of grade equals its expected productivity under the posterior induced by the grade. We state this formally in the assumption below:

Assumption 2. Given a monotone decreasing wage function $w : \Theta \to \mathbb{R}_+$, the rank revealing grading contest $s^* = (1, 2, \dots, n)$ induces prize vector $\mathbf{v} = (v_1, \dots, v_n)$ where

$$v_i = \mathbb{E}[w(\theta)|\theta = \theta^n_{(i)}].$$

A grading contest $s = (s_1, s_2, \dots, s_k)$ induces the prize vector $\mathbf{v}(s)$ where

$$\mathbf{v}(s)_i = \frac{v_{s_{j-1}+1} + v_{s_{j-1}+2} + \cdots + v_{s_j}}{s_j - s_{j-1}}$$

and j is such that $s_{j-1} < i \leq s_j$.

Given this framework, we can now ask how the different grading schemes compare in terms of the effort they induce. It turns out that under our assumption 2, if grading scheme s is more informative than s', then the prize vector $\mathbf{v}(s)$ induced by s is more competitive than the prize vector $\mathbf{v}(s')$ induced by s'. As a result, we can use our Theorem 2 describing the effects of competition on effort to say how informativeness of a grading scheme influences the effort they induce.

Corollary 5. *Suppose $F(\theta) = \theta^p$ and grading scheme s is more informative than s'.*

1. *If $p > 1$, then s induces greater expected effort than s'.*

2. *If $\frac{1}{2} < p < 1$, and $v(s)_1 = v(s')_1$, $v(s)_n = v(s')_n$, then s' induces greater expected effort than s.*

3. *If $p > \frac{1}{2}$ and $v(s)_n = v(s')_n$, then s' induces greater expected minimum effort than s.*

Moreover, we can also characterize optimal grading schemes. The following corollary describes the effort-maximizing grading contests.

Corollary 6. *Suppose $F(\theta) = \theta^p$.*

1. *If $p > 1$, the rank revealing contest $s = (1, 2, \ldots, n)$ maximizes expected effort among all grading contests.*

2. *If $\frac{1}{2} < p < 1$, the contest $s = (1, n-1, n)$ maximizes expected effort among all grading contests in which the last agent gets a unique grade.*

3. *If $p > \frac{1}{2}$, the contest $s = (n-1, n)$ maximizes expected minimum effort among all grading contests.*

Note that when the designer has a budget that it can distribute arbitrarily across prizes, the expected effort maximizing contest is a winner-take-all contest that allocates the entire budget to the first prize, irrespective of the distribution of abilities (Moldovanu and Sela [50]). When the designer can only choose a grading scheme, the set of feasible contests under our assumption 2 is actually a finite subset of these prize vectors that all add up to a constant sum. And as we see in the corollary, the optimal grading contest now depends on the prior distribution of abilities. If the density of agents is increasing in θ so that there is a greater proportion of inefficient agents ($p > 1$), the effort maximizing grading contest awards a unique grade to each agent. But when the density is decreasing ($\frac{1}{2} < p < 1$), the optimal grading contest, among those that award a unique grade to the last agent, awards a unique grade to the best agent and pools the rest of the agents by awarding them a common grade. And finally for the case where the designer wants to maximize expected minimum effort, which is perhaps a reasonable objective in a classroom environment, the optimal grading contest awards a common grade to everyone except the least efficient agent.

Next we characterize effort-minimizing grading contests. Note that a grading contest that pools all the agents together clearly minimizes effort among all grading contests as it leads to zero effort. So we focus on grading contests that reveal some information about the type of the agents. In other words, we exclude the trivial grading contest $s = (n)$ from consideration while referring to grading contests.

Corollary 7. *Suppose $F(\theta) = \theta^p$.*

1. *If $p > 1$, the effort-minimizing grading contest takes the form $s = (k, n)$ for some $k \in \{1, 2, \ldots, n-1\}$.*

2. *If $\frac{1}{2} < p < 1$, the effort-minimizing grading contest takes the form $s = (k, k+1, \ldots, n-1, n)$ for some $k \in \{1, 2, \ldots, n-1\}$.*

In words, when the density is increasing so that inefficient agents dominate the population, the effort-minimizing contest only awards two grades, say A and B, in some distribution. And when the density is decreasing so that highly efficient agents dominate the population, the effort-minimizing contest pools some of the top agents together by awarding them a common grade, and then awards a unique grade to each of the remaining agents.

4.2 Concave utilities

In this subsection, we consider a setting where the designer has a budget B that it can allocate across prizes $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ such that $v_i \geq v_{i+1}$. Again, with linear utility, the effort maximizing contest allocates the entire prize budget B to the first prize ([50]). We will consider the problem where agents have a common concave utility function. More precisely, we assume that under a prize vector $\mathbf{v}$, if agent i of type θ_i puts in effort e_i and wins prize j, its payoff equals $u(v_j) - \theta_i e_i$ where $u(v_j) = v_j^r$ for $r \in (0, 1)$. The next result characterizes the expected effort maximizing contest in this environment.

Theorem 3. *Suppose agents have utility $u(v) = v^r$ with $r \in (0, 1)$ and the distribution of abilities is $F(\theta) = \theta^p$.*

1. *If $p > 1$, the effort-maximizing contest awards $n-1$ prizes of decreasing values. Moreover, the optimal contest $v(r)$ becomes more competitive as r increases.*

2. *If $\frac{1}{2} < p < 1$, the effort-maximizing contest is a winner-take-all contest for any $r \in (0, 1)$.*

We use corollary 1 to identify the equilibrium function so that the design problem becomes $\max_{\mathbf{v}} \sum_{i=1}^{n-1} v_i^r \mathbb{E}[m_i(\theta)]$ such that $\sum v_i = B$. For $\frac{1}{2} < p < 1$, it follows from Theorem 1 that the marginal effects of all intermediate prizes is negative and so regardless of how concave the utilities are, the optimal contest awards the entire budget to the first prize. For $p > 1$, we know from Theorems 1 and 2 that the expected marginal effects for prizes $1, 2, \ldots, n-1$ are all positive and decreasing in rank. In this case, we solve the constrained optimization problem and characterize the optimal contest. To show that the optimal contest become more competitive as r increases, we define $f_k(r)$ as the sum of the first k prizes in the optimal contest under r and show that this sum is increasing in r. Thus, as the agents utility for prizes becomes less concave, the effort maximizing contest becomes more competitive.

4.3 Costless homogeneous prizes

For our last application, we consider a setting where the contest designer can award arbitrarily many prizes of a fixed value a. More precisely, the set of prize vectors available to the designer is given by

$$B = \{\mathbf{v} \in \mathbb{R}^n : \exists k \text{ such that } v_i = a \text{ if } i \leq k \text{ and } v_i = 0 \text{ if } i > k\}$$

This might be the case when the designer is awarding free trials or subscriptions to digital content or services. In these cases, the value of the prize for the winner does not diminish if it is awarded to many agents and further, the cost to awarding additional prizes is negligible for the designer. The designer wants to chose the number of prizes so as to maximize the expected effort. This contest design problem was also considered in Liu and Lu [48] but under different distributional assumptions. In their setting, the authors found that the expected effort was single-peaked in k. In our setting, we obtain the following as a corollary of Theorem 1.

Corollary 8. *Suppose a designer can award any number of homogeneous prizes of a fixed value.*

1. *If the density f is increasing, then a designer with concave and increasing utility U for effort awards $n-1$ prizes.*
2. *If the density f is decreasing, then a designer with convex and increasing utility U for effort awards only a single prize.*

5 Conclusion

We study the effect of increasing the value of prizes and increasing competition on effort in contests where agents have private information about their abilities. For prizes, we find that increasing the value of the first prize encourages effort for all agents, increasing the value of last prize discourages effort for all agent types, and increasing any intermediate prizes leads to a balanced transfer of effort from the more efficient agents to the less efficient agents. In expectation, the effects of prizes and competition depend qualitatively on the prior distribution of abilities in the population and we identify natural sufficient conditions on the distributions under which these interventions have opposite effects. If there is an increasing density of inefficient agents, increasing the value of prizes or competition encourages effort. If this density is decreasing, these interventions discourage effort.

We also discuss applications of these results to the design of optimal contests in three natural environments. First, we consider the design of grading contests under the assumption that the value of a grade is determined by the information it reveals about the type of the agent. We establish a connection between informativeness of a grading contest and the effort it induces and also derive effort-maximizing and effort-minimizing grading contests. Second, we consider a parametric setting where the designer has a budget that it must allocate across different prizes and the agents have concave utilities for prizes. Lastly, we consider settings where the designer can only choose the number of agents to award with a homogeneous prize and show that when the prior density is monotone, it is optimal to award either 1 or $n-1$ prizes depending on whether the density is increasing or decreasing. In summary, the structure of the optimal contest in all of these environments depends in an important way on the distribution of abilities in the population.

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A Proofs for Section 3 (Equilibrium)

Lemma 1. *The equilibrium function is given by*

$$g_{\mathbf{v}}(\theta) = \sum_{i=1}^{n} v_i m_i(\theta)$$

where

$$m_i(\theta) = -\int_{F(\theta)}^{1} \frac{p_i'(t)}{F^{-1}(t)} dt$$

Proof. Suppose $n-1$ agents are playing a strategy $g : [0,1] \to \mathbb{R}_+$ so that if the agent's type is θ, it exerts effort $g(\theta)$. Further, $g(\theta)$ is decreasing in θ. Now if an agent's type is θ and it imitates an agent of type $t \in [0,1]$, it's payoff is

$$\sum_{i=1}^{n} v_i p_i(F(t)) - \theta g(t)$$

where $p_i(x) = \binom{n-1}{i-1} x^{i-1}(1-x)^{n-i}$ is the probability that a random variable X following $Bin(n-1, x)$ takes the value $i-1$.

Taking the first order condition, we get

$$\sum_{i=1}^{n} v_i p_i'(F(t)) f(t) - \theta g'(t) = 0$$

Now we can plug in $t = \theta$ to get the condition for $g(\theta)$ to be a symmetric Bayes-Nash equilibrium:

$$\sum_{i=1}^{n} v_i p_i'(F(\theta)) f(\theta) - \theta g'(\theta) = 0$$

so that

$$-\sum_{i=1}^{n} v_i \int_{\theta}^{1} \frac{p_i'(F(t)) f(t)}{t} dt = g(\theta)$$

which can be equivalently written as

$$-\sum_{i=1}^{n} v_i \int_{F(\theta)}^{1} \frac{p_i'(t)}{F^{-1}(t)} dt = g(\theta)$$

Let's now make sure the second order condition is satisfied. Differentiating the lhs of the foc, we get

$$\sum_{i=1}^{n} v_i (p_i'(F(t)) f'(t) + f(t) p_i''(F(t)) f(t)) - \theta g''(t)$$

From the foc, we have that g satisfies $\sum_{i=1}^{n} v_i (p_i'(F(t)) f'(t) + f(t) p_i''(F(t)) f(t)) = t g''(t) + g'(t)$

Thus, when we plug in $t = \theta$ in the soc, we get $g'(\theta)$ which we know is < 0. Thus, the second order condition is satisfied.

∎

Lemma 2. *The marginal effect functions* $m_i(\theta) = -\int_{F(\theta)}^{1} \frac{p_i'(t)}{F^{-1}(t)}dt$ *satisfy:*

1. $m_1(\theta) \geq 0$ *for all* $\theta \in [0, 1]$
2. $m_i(\theta) = \begin{cases} < 0 & \textit{if } \theta \leq t_i \\ \geq 0 & \textit{otherwise} \end{cases}$ *where* $t_i \in (0, 1)$ *for* $i \in \{2, \ldots, n-1\}$
3. $m_n(\theta) \leq 0$ *for all* $\theta \in [0, 1]$

Proof. The first and third items follow from observing that $p_1'(t) < 0$ for all $t \in [0, 1]$ and $p_n'(t) > 0$ for all $t \in [0, 1]$.

For the second part, we'll make three observations that would imply the result.

First, observe that $m_i(1) = 0$.

Second, $m_i'(\theta) = \dfrac{p_i'(F(\theta))f(\theta)}{\theta}$ is initially positive because $p_i'(t) > 0$ for small t and then negative. More precisely, $m_i'(\theta) = \begin{cases} >= 0 & \text{if } \theta \leq t_i^2 \\ < 0 & \text{otherwise} \end{cases}$ where

$$t_i^2 = F^{-1}\left(\frac{i-1}{n-1}\right)$$

Third, $m_i(0) = -\int_0^1 \frac{p_i'(t)}{F^{-1}(t)}dt$. Since $\int_0^1 p_i'(t)dt = 0$, weighing these by $\frac{1}{F^{-1}(t)}$ puts more weight on small values of t (where $p_i'(t) > 0$) as compared to greater values of t (where $p_i'(t) < 0$). It follows then that $m_i(0) < 0$.

Together, these observations imply the $m_i(\theta)$ is initially negative, it increases and becomes positive and continues increasing till θ equals $F^{-1}\left(\dfrac{i-1}{n-1}\right)$. After this, it decreases and goes to 0 as $\theta \to 1$. □

Lemma 3. *The marginal effect functions* $m_i(\theta) = -\int_{F(\theta)}^{1} \frac{p_i'(t)}{F^{-1}(t)}dt$ *satisfy:*

1. $\int_0^1 m_1(\theta)d\theta = 1$
2. $\int_0^1 m_i(\theta) = 0$ *for* $i \in \{2, \ldots, n-1\}$
3. $\int_0^1 m_n(\theta) = -1$

Proof. First, we'll show that $\lim_{\theta\to 0} \theta m_i(\theta) = 0$. If $m_i(0)$ is finite, we are done. But if $\lim_{\theta\to 0} m_i(\theta) = \infty$, we have

$$\begin{aligned} \lim_{\theta\to 0} \theta m_i(\theta) &= \lim_{\theta\to 0} \frac{m_i(\theta)}{\frac{1}{\theta}} \\ &= \lim_{\theta\to 0} \frac{m_i'(\theta)}{\frac{-1}{\theta^2}} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{\theta \to 0} -\theta^2 \frac{p_i'(F(\theta))f(\theta)}{\theta} \\
&= \lim_{\theta \to 0} -\theta p_i'(F(\theta))f(\theta) \\
&= 0
\end{aligned}
$$

The last equality holds because we assume that $\lim_{\theta \to 0} F(\theta)f(\theta) = 0$. It implies the density function f is such that $\lim_{\theta \to 0} \theta f(\theta) = 0$. Using this, we have that

$$
\begin{aligned}
\int_0^1 m_i(\theta)d\theta &= -\int_0^1 m_i'(\theta)\theta d\theta \\
&= -\int_0^1 \frac{p_i'(F(\theta))}{\theta} f(\theta)\theta d\theta \\
&= -\int_0^1 p_i'(t)dt \\
&= p_i(0) - p_i(1)
\end{aligned}
$$

The result then follows from the definition of $p_i(x)$. □

Theorem 1. *Suppose* $\mathbf{v}, \mathbf{w}$ *are two prize vectors such that* $v_i > w_i$ *for some intermediate prize* $i \in \{2, \ldots, n-1\}$ *and* $v_j = w_j$ *for* $j \neq i$.

1. *If the density* f *is increasing, then a designer with a concave utility* U *for effort prefers* $\mathbf{v}$ *over* $\mathbf{w}$.

2. *If the density* f *is decreasing, then a designer with a convex utility* U *for effort prefers* $\mathbf{w}$ *over* $\mathbf{v}$.

Proof. First, we'll show that the expected marginal effects $\mathbb{E}[m_i(\theta)]$ are positive if the density is increasing and negative if the density is decreasing. We'll then use this along with the result of Lemma 2 to get the result in the theorem. The expected marginal effect of prize i is given by:

$$
\begin{aligned}
\mathbb{E}[m_i(\theta)] &= \int_0^1 m_i(\theta)f(\theta)d\theta \\
&= m_i(\theta)F(\theta)|_0^1 - \int_0^1 m_i'(\theta)F(\theta)d\theta \\
&= -\int_0^1 \frac{p_i'(F(\theta))}{\theta} f(\theta)F(\theta)d\theta \\
&= -\int_0^1 \frac{p_i'(t)}{F^{-1}(t)} t dt
\end{aligned}
$$

Note that Assumption 1 is sufficient to ensure that $\lim_{\theta \to 0} m_i(\theta)F(\theta) = 0$ and thus, the first term in the second line of the equation is just 0.

We know that $\int_0^1 p_i'(t)dt = 0$. Let $h(t) = \frac{t}{F^{-1}(t)}$ so that

$$h'(t) = \frac{F^{-1}(t) - \frac{t}{f(F^{-1}(t))}}{(F^{-1}(t))^2} = \frac{xf(x) - F(x)}{f(x)x^2}$$

where $x = F^{-1}(t)$.

The sign of $h'(t)$ is then determined by the numerator $xf(x) - F(x)$. Observe that $xf(x) - F(x) > 0$ for all x when the density f is increasing and it is < 0 for all x when the density is decreasing. That is, $h(t)$ is increasing in t when the density is increasing and decreasing in t when the density is decreasing.

Let $\alpha = \frac{i-1}{n-1}$ so that $p_i'(\alpha) = 0$ and $p_i'(t) > 0$ for $t < \alpha$ and $p_i'(t) < 0$ for $t > \alpha$. Going back to the expected marginal effects, suppose first that the density function f and thus $h(t)$ is increasing. Then, we have

$$\begin{aligned}
\mathbb{E}[m_i(\theta)] &= -\int_0^1 p_i'(t)h(t)dt \\
&= -\int_0^\alpha p_i'(t)h(t)dt - \int_\alpha^1 p_i'(t)h(t)dt \\
&\geq -\int_0^\alpha p_i'(t)h(\alpha)dt - \int_\alpha^1 p_i'(t)h(\alpha)dt \\
&= 0
\end{aligned}$$

When the density f is decreasing so that $h(t)$ is decreasing in t, we have

$$\begin{aligned}
\mathbb{E}[m_i(\theta)] &= -\int_0^1 p_i'(t)h(t)dt \\
&= -\int_0^\alpha p_i'(t)h(t)dt - \int_\alpha^1 p_i'(t)h(t)dt \\
&\leq -\int_0^\alpha p_i'(t)h(\alpha)dt - \int_\alpha^1 p_i'(t)h(\alpha)dt \\
&= 0
\end{aligned}$$

Thus, we have shown that the expected marginal effects are of opposite signs under increasing and decreasing density functions.

In addition, we know from lemma 2 that there exists t_i such that

$$g_{\mathbf{v}}(\theta) - g_{\mathbf{w}}(\theta) = \begin{cases} \leq 0 & \text{if } \theta < t_i \\ = 0 & \text{if } \theta = t_i \\ \geq 0 & \text{otherwise} \end{cases}$$

Let $G_{\mathbf{v}}(x) = \mathbb{P}[g_{\mathbf{v}}(\theta) \leq x]$ denote the cdf of effort under prize vector $\mathbf{v}$. Then, from above, we have that

$$G_{\mathbf{v}}(x) - G_{\mathbf{w}}(x) = \begin{cases} < 0 & \text{if } x < g_{\mathbf{v}}(t_i) \\ = 0 & \text{if } x = g_{\mathbf{v}}(t_i) \\ > 0 & \text{otherwise} \end{cases}$$

Thus, when the density f is increasing, we have that $\mathbb{E}[g_{\mathbf{v}}(\theta)] \geq \mathbb{E}[g_{\mathbf{w}}(\theta)]$ and also the sign of $G_{\mathbf{v}}(x) - G_{\mathbf{w}}(x)$ changes exactly once from $-$ to $+$ as x increases. It follows then from Theorem 4.A.22 in Shaked and Shanthikumar [62] that $g_{\mathbf{v}}(\theta)$ second order stochastically dominates $g_w(\theta)$.

The argument for the case of decreasing density is analagous.

□

Theorem 2. *Suppose $F(\theta) = \theta^p$ and $\mathbf{v}$ and $\mathbf{w}$ are two prize vectors such that $\mathbf{v}$ is more competitive than $\mathbf{w}$.*

1. *If $p > 1$, then*
$$\mathbb{E}[g_{\mathbf{v}}(\theta)] \geq \mathbb{E}[g_{\mathbf{w}}(\theta)].$$
2. *If $\frac{1}{2} < p < 1$, and $v_1 = w_1, v_n = w_n$, then*
$$\mathbb{E}[g_{\mathbf{v}}(\theta)] \leq \mathbb{E}[g_{\mathbf{w}}(\theta)].$$
3. *If $p > \frac{1}{2}$ and $v_n = w_n$, then*
$$\mathbb{E}[g_{\mathbf{v}}(\theta_{max})] \leq \mathbb{E}[g_{\mathbf{w}}(\theta_{max})].$$

Proof. For the parametric distribution $F(\theta) = \theta^p$, we can compute the expected marginal effects. We have that

$$\begin{aligned}
\mathbb{E}[m_i(\theta)] &= -\int_0^1 \frac{p_i'(t)}{F^{-1}(t)} t dt \\
&= -\int_0^1 p_i'(t) t^{1-\frac{1}{p}} dt \\
&= -\binom{n-1}{i-1} \int_0^1 t^{i-1-\frac{1}{p}} (1-t)^{n-i-1} \left((i-1) - t(n-1)\right) dt \\
&= \binom{n-1}{i-1} \left((n-1)\beta(i+1-\frac{1}{p}, n-i) - (i-1)\beta(i-\frac{1}{p}, n-i) \right) \\
&= \binom{n-1}{i-1} \left((n-1)\beta(i-\frac{1}{p}, n-i) \frac{i-\frac{1}{p}}{n-\frac{1}{p}} - (i-1)\beta(i-\frac{1}{p}, n-i) \right) \\
&= \binom{n-1}{i-1} \beta(i-\frac{1}{p}, n-i) \left((n-1) \frac{i-\frac{1}{p}}{n-\frac{1}{p}} - (i-1) \right)
\end{aligned}$$

$$= \binom{n-1}{i-1}\beta(i-\frac{1}{p}, n-i)\frac{(n-i)(p-1)}{np-1}$$

Observe that with $i \geq 2$ and $p > \frac{1}{2}$, $ip > 1$ and the expectation is well defined.
Now observe that

$$\frac{\mathbb{E}[m_{i+1}(\theta)]}{\mathbb{E}[m_i(\theta)]} = \frac{n-i}{i}\frac{i-\frac{1}{p}}{n-i-1}\frac{n-i-1}{n-i} = \frac{i-\frac{1}{p}}{i} < 1$$

For $p \geq 1$, the density f is increasing and we know from Theorem 1 that the marginal effects $\mathbb{E}[m_i(\theta)]$ are positive and thus, the effect of prize i on expected effort is decreasing in i. Since $\mathbf{w}$ can be obtained from $\mathbf{v}$ via a sequence of Robinhood operations which involve replacing v_i by $v_i - \epsilon$ and v_j by $v_j + \epsilon$ where $i < j$, each of which reduces expected effort, we get that the expected effort under $\mathbf{w}$ will be lesser than the expected effort under $\mathbf{v}$. So if v is more competitive than w and $p > 1$, then $\mathbb{E}[g_v(\theta)] \geq \mathbb{E}[g_w(\theta)]$. For the case where $\frac{1}{2} < p \leq 1$, the density f is decreasing and thus, the expected marginal effects $\mathbb{E}[m_i(\theta)]$ are negative. From above, we have that the ratio of effects is still < 1. Thus, the effect of prize i on expected effort is actually increasing in i. It follows that the Robinhood transfers would lead to an increase in expected effort.

Now let's prove the third result. In this case, we need to compute $\mathbb{E}[m_i(\theta_{max})]$.

$$\begin{aligned}
\mathbb{E}[m_i(\theta_{max})] &= \int_0^1 m_i(\theta)nF(\theta)^{n-1}f(\theta)d\theta \\
&= m_i(\theta)F(\theta)^n|_0^1 - \int_0^1 m_i'(\theta)F(\theta)^n d\theta \\
&= -\int_0^1 \frac{p_i'(F(\theta))}{\theta}f(\theta)F(\theta)^n d\theta \\
&= -\int_0^1 \frac{p_i'(t)}{F^{-1}(t)}t^n dt
\end{aligned}$$

For the case of $F(\theta) = \theta^p$, we get that

$$\begin{aligned}
\mathbb{E}[m_i(\theta_{max})] &= -\int_0^1 p_i'(t)t^{n-\frac{1}{p}}dt \\
&= -\binom{n-1}{i-1}\int_0^1 t^{n+i-2-\frac{1}{p}}(1-t)^{n-i-1}\left((i-1)-t(n-1)\right)dt \\
&= \binom{n-1}{i-1}\left((n-1)\beta\left(n+i-\frac{1}{p}, n-i\right) - (i-1)\beta\left(n+i-1-\frac{1}{p}, n-i\right)\right) \\
&= \binom{n-1}{i-1}\beta\left(n+i-1-\frac{1}{p}, n-i\right)\left((n-1)\frac{n+i-1-\frac{1}{p}}{2n-1-\frac{1}{p}} - (i-1)\right)
\end{aligned}$$

$$= \binom{n-1}{i-1} \beta \left(n+i-1-\frac{1}{p}, n-i \right) \frac{(n-i)(np-1)}{2np-p-1}$$

Observe that

$$\frac{\mathbb{E}[m_{i+1}(\theta_{max})]}{\mathbb{E}[m_i(\theta_{max})]} = \frac{n+i-1-\frac{1}{p}}{i} > 1$$

Thus, the marginal effect of prize any intermediate prize i positive and increasing in i. It follows that if v is more competitive than w and both have the same last prize, then $\mathbb{E}[g_v(\theta_{max})] \leq \mathbb{E}[g_w(\theta_{max})]$

□

B Proofs for Section 4 (Applications)

Theorem 3. *Suppose agents have utility $u(v) = v^r$ with $r \in (0,1)$ and the distribution of abilities is $F(\theta) = \theta^p$.*

1. *If $p > 1$, the effort-maximizing contest awards $n-1$ prizes of decreasing values. Moreover, the optimal contest $v(r)$ becomes more competitive as r increases.*

2. *If $\frac{1}{2} < p < 1$, the effort-maximizing contest is a winner-take-all contest for any $r \in (0,1)$.*

Proof. When $\frac{1}{2} < p < 1$, we know from Theorem 1 that the expected marginal effect of the intermediate prizes are negative. Thus, regardless of how concave the utilities are, it is best to allocate the entire budget to the first prize.

Now let's consider the case where $p > 1$ where we know from Theorems 1 and 2 that the expected marginal effects for prizes $1, 2, \ldots, n-1$ are all positive and decreasing in rank. From corollary 1, we know that the Bayes-Nash equilibrium function takes the form

$$g_{\mathbf{v}}(\theta) = \sum_{i=1}^{n} m_i(\theta) u(v_i)$$

Given this form of the equilibrium function, the problem is

$$\max_{\mathbf{v}} \sum_{i=1}^{n-1} u(v_i) \mathbb{E}[m_i(\theta)]$$

such that $\sum_{i=1}^{n-1} v_i = B$.

The solution will satisfy the equation

$$V_1(r) \left[1 + \sum_{i=2}^{n-1} c_i^{\frac{1}{1-r}} \right] = B$$

where $c_i = \frac{\mathbb{E}[m_i(\theta)]}{\mathbb{E}[m_1(\theta)]} < 1$ and $c_i > c_{i+1}$ for all i (Theorem 2). Note that c_i does not depend on r.

Let $f_k(r) = V_1(r)\left[1 + \sum_{i=2}^{k} c_i^{\frac{1}{1-r}}\right]$.

I want to show that $f_k'(r) > 0$ for all k.

If I can show $f_k'(r)$ is single peaked in k, that would imply the result since $f_n(r) = 0$.

Check that $V_1'(r) = \dfrac{-\left[\sum_{i=2}^{n-1} c_i^{\frac{1}{1-r}} \log(c_i)\right] V_1^2(r)}{(1-r)^2 B}$ Plugging it in, we get

$$
\begin{aligned}
f_k'(r) &= V_1(r)\left[\frac{1}{(1-r)^2}\sum_{i=2}^{k} c_i^{\frac{1}{1-r}} \log(c_i)\right] + V_1'(r)\left[1 + \sum_{i=2}^{k} c_i^{\frac{1}{1-r}}\right] \\
&= V_1(r)\left[\frac{1}{(1-r)^2}\sum_{i=2}^{k} c_i^{\frac{1}{1-r}} \log(c_i)\right] - \frac{\left[\sum_{i=2}^{n-1} c_i^{\frac{1}{1-r}} \log(c_i)\right] V_1^2(r)}{(1-r)^2 B}\left[1 + \sum_{i=2}^{k} c_i^{\frac{1}{1-r}}\right] \\
&= \frac{V_1(r)}{(1-r)^2}\sum_{i=2}^{k} c_i^{\frac{1}{1-r}} \log(c_i)\left[1 - \frac{V_1(r)}{B}\left(1 + \sum_{i=2}^{k} c_i^{\frac{1}{1-r}}\right)\right] - \frac{V_1^2(r)}{B(1-r)^2}\sum_{i=k+1}^{n-1} c_i^{\frac{1}{1-r}} \log(c_i)\left[1 + \sum_{i=2}^{k} c_i^{\frac{1}{1-r}}\right] \\
&= \frac{V_1(r)}{B(1-r)^2}\sum_{i=2}^{k} c_i^{\frac{1}{1-r}} \log(c_i)\left[B - f_k(r)\right] - \frac{V_1(r) f_k(r)}{B(1-r)^2}\sum_{i=k+1}^{n-1} c_i^{\frac{1}{1-r}} \log(c_i) \\
&= \frac{V_1(r)}{B(1-r)^2}\left(B\sum_{i=2}^{k} c_i^{\frac{1}{1-r}} \log(c_i) - f_k(r)\sum_{i=2}^{n-1} c_i^{\frac{1}{1-r}} \log(c_i)\right)
\end{aligned}
$$

To show that the term inside the bracket is positive, we basically need to show that for any decreasing sequence $1 \geq d_1 > d_2 > \ldots d_n > 0$, we have that

$$
h(k) = \sum_{i=1}^{n} d_i \sum_{i=1}^{k} d_i \log(d_i) - \sum_{i=1}^{k} d_i \sum_{i=1}^{n} d_i \log(d_i) \geq 0
$$

for any $k \in [n]$

Observe that

$$
\begin{aligned}
\Delta(k) &= h(k+1) - h(k) \\
&= d_{k+1}\log(d_{k+1})\sum_{i=1}^{n} d_i - d_{k+1}\sum_{i=1}^{n} d_i \log(d_i) \\
&= d_{k+1}\left(\log(d_{k+1})\sum_{i=1}^{n} d_i - \sum_{i=1}^{n} d_i \log(d_i)\right)
\end{aligned}
$$

Since d_k is a decreasing sequence, it follows that if $\Delta(k) < 0$, then $\Delta(j) < 0$ for all $j > k$. But observe that $h(n) = 0$. So we just need to show that $h(1) > 0$ which is obvious. $\square$