Contest design with a finite type-space

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Abstract

We study the classical contest design problem in an incomplete information environment with linear costs and a finite type-space. For any contest with an arbitrary finite type-space and distribution over this type-space, we characterize the unique symmetric Bayes-Nash equilibrium of the contest game. We find that the equilibrium is in mixed strategies, where agents of different types mix over disjoint but connected intervals, so that more efficient agents always exert greater effort than less efficient agents. Using this characterization, we solve for the expected equilibrium effort under any arbitrary contest, and find that a winner-takes-all contest maximizes expected effort among all contests feasible for a budget-constrained designer. Our results extend the optimality of the winner-takes-all contest under a continuum type-space (Moldovanu and Sela [19]) to the finite type-space environment, and our analysis introduces new techniques for the study of contest design problems in such environments.

1 Introduction

Contests are situations in which agents compete with one another by investing costly effort to win valuable prizes. Given their widespread prevalence across various domains, such as R&D, innovation, sports, it is important to understand how different contests influence the effort exerted by agents, and in particular, identify contest structures that are optimal from the designer's perspective. There is a vast literature studying variants of optimal contest design problems in different domains, including complete and incomplete information environments. In this paper, we study the classical contest design problem of finding an expected effort maximizing contest for a budget-constrained designer, in an incomplete information environment where agents have private information about their type (marginal cost of effort). In contrast to previous literature in this domain which assumes that the type space is a continuum and the prior distribution is smooth, we focus on the case where the type-space

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is finite and the distribution arbitrary.

For any arbitrary finite type-space $\Theta = \{\theta_1, \ldots, \theta_K\}$ with $\theta_1 > \theta_2 \cdots > \theta_K$, and any contest, defined by an allocation of budget to different prizes, we characterize the unique symmetric Bayes-Nash equilibrium of the contest game. We find that the equilibrium is in mixed strategies, and it is monotonic in the sense that agents who are more efficient always exert greater effort than those that are less efficient. More precisely, the equilibrium is such that there exist boundary points $b_1 < b_2 < \cdots < b_K$ so that an agent with the highest marginal cost (θ_1) mixes between $[0, b_1]$, the next type (θ_2) mixes between $[b_1, b_2]$, and more generally, an agent of type θ_k mixes between $[b_{k-1}, b_k]$. Moreover, since an agent of type θ_k must be indifferent between all actions in the interval $[b_{k-1}, b_k]$, the equilibrium distribution functions over these intervals are such that the marginal gain in prize won from increasing effort in the interval $[b_{k-1}, b_k]$ is equal to the marginal cost θ_k.

Given the equilibrium characterization, we solve for the expected effort of an arbitrary agent induced by any arbitrary contest. While an explicit calculation directly using the distribution functions appears complicated, we introduce techniques that exploit the monotone structure of the equilibrium and the additive separability of the reward and the costs in the agents' utility functions to obtain a fairly tractable representation for the expected effort. With this representation, we show that the marginal effect of increasing the value of the first prize is greater than that of increasing any other prize, and thus, we get that a winner-takes-all contest maximizes expected effort among all contests feasible for a budget constrained designer. Lastly, we show that for any continuous distribution and sequence of finite-type space distributions converging to this continuous distribution, the corresponding sequence of mixed-strategy equilibrium for the finite-type space distributions converges to the pure-strategy equilibrium of the continuous distribution.

Related literature

Our paper contributes to the vast literature studying the contest design problem of allocating a fixed budget across different prizes so as to maximize total effort. In the incomplete information environment (with a continuum type-space), the literature has generally shown that allocating the entire budget to the first prize (winner-takes-all) is optimal (Glazer and Hassin [11], Moldovanu and Sela [19], Zhang [30], Liu and Lu [16]). Our results illustrate the optimality of the winner-takes-all contest extends to the case where the type-space is finite, and we introduce new techniques to deal with the difficulties introduced by the mixed nature of the equilibrium. In comparison, in the complete information setting, distributing the budget arbitrarily amongst the top $N-1$ prizes has been shown to be optimal (Glazer and Hassin [11], Barut and Kovenock [1], Letina, Liu, and Netzer [14])[1]. Our results also

[1] Sisak [24], Vojnović [28] provide detailed surveys of the literature on this optimal contest design problem. More general surveys of the theoretical literature in contest theory can be found in Corchón [5], Vojnović

apply to the case where the type-space is just a singleton (which corresponds to a complete information setting), in which case we recover the result that any distribution of the budget among the top $N-1$ prizes is optimal. Thus, together with the convergence result, our analysis of the contest design problem with a finite type-space provides a bridge between the previous literature in the complete information environment and the incomplete information environment with a continuum type-space.

There is some related literature studying contests in an incomplete information with a finite type-space. Most of this literature focuses on problems with a small (or even binary) type-space and investigates properties of Bayes-Nash equilibrium. In particular, Szech [25] studies the value of information disclosure in a model with binary types and asymmetric distributions. Liu and Chen [17] allows for correlated types in an all-pay auction with two agents and binary types and shows that the symmetric Bayes-Nash equilibrium may be non-monotonic (have overlapping intervals) if the absolute correlation is large. Chen [2] considers a setting where players observe private signal about the types of their opponent and characterizes equilibrium for different degrees of informativeness of the signal structure. Xiao [29] assumes complete information (or perfect correlation) with heterogeneous agents and illustrates in a model with three agents and two prizes that the winner-takes-all contest maximizes the total expected effort if the top two players are similar, and two equal prizes maximize the total expected bid if the bottom two players are similar. In contrast to this literature, our paper studies the optimal contest design problem for arbitrarily general finite type-spaces.

The paper proceeds as follows. In section 2, we present the general model of a contest in an incomplete-information environment with a finite type-space. In section 3, we characterize the symmetric Bayes-Nash equilibrium of the contest game. In section 4, we study the design of effort-maximizing contest. In section 5, we discuss the convergence of finite-type space equilibrium to the equilibrium of the continuum type-space. Section 6 concludes. All proofs are relegated to the appendix.

2 Model

There is a set of N risk-neutral agents. Each agent has a privately known type θ (which represents its marginal cost of effort), drawn independently from type-space $\Theta = \{\theta_1, \theta_2, \dots, \theta_K\}$ according to distribution $p = (p_1, p_2, \dots, p_K)$ so that $\Pr[\theta = \theta_k] = p_k$. It is common knowledge that agents' types are independent and identically distributed according to p. Without loss of generality, we assume $\theta_1 > \theta_2 > \cdots > \theta_K > 0$ and $p_k > 0$ for all k.

There is a designer who designs a contest $v = (v_1, v_2, \dots, v_N)$ with $v_1 \geq \cdots \geq v_{N-1} \geq v_N$. Given the contest v, all agents simultaneously choose their effort. The agents are ranked

[28], Konrad [13], Chowdhury et al. [4].

according to their effort and awarded the corresponding prizes, with ties broken uniformly at random. If an agent of type θ_k wins prize v_i after exerting effort x_k, its payoff is

$$v_i - \theta_k x_k.$$

A contest v, together with the distribution p over type-space Θ, defines a Bayesian game between the N agents. We will focus on the symmetric Bayes-Nash equilibrium of this game. This is a strategy profile where all agents are using the same (potentially mixed) strategy, $X : \Theta \to \Delta\mathbb{R}_+$, mapping agent's type to a distribution over non-negative effort levels, so that for any agent of type θ_k, choosing any effort level within the support of $X(\theta_k)$ yields an expected payoff at least as high as that from choosing any other effort level, given that all other agents use the strategy $X(.)$.

The contest designer wants to maximize the expected equilibrium effort and has a fixed budget $V > 0$ that it can use to allocate to different prizes in order to incentivize agents to exert effort. We study this designer's problem of finding the contest that maximizes expected effort subject to the budget constraint.

Notation

Here, we introduce some notation that will be used in the rest of the paper.

We let $P_k = \sum_{i=1}^{k} p_i$. We will denote by

$$H_K^N(p) = \binom{N}{K} p^K (1-p)^{N-K}$$

the probability that a binomial random variable $Y \sim Bin(N, p)$ takes a value of exactly K. We also use

$$H_{\leq K}^N(p) = \sum_{k=0}^{K} \binom{N}{k} p^k (1-p)^{N-k} \text{ and } H_{\geq K}^N(p) = \sum_{k=K}^{N} \binom{N}{k} p^k (1-p)^{N-k}$$

to denote the probability that $Y \sim Bin(N, p)$ takes a value of at most K and at least K respectively.

3 Equilibrium

In this section, we characterize the symmetric Bayes-Nash equilibrium of contests with a finite type-space. The following result states the existence of equilibrium, its uniqueness, and identifies an important structural property of the equilibrium.

Lemma 1. *Suppose there are N agents, each with a private type drawn from a finite type-space $\Theta = \{\theta_1, \ldots, \theta_K\}$ according to distribution $p = (p_1, p_2, \ldots, p_K)$. For any contest $v = \{v_1, v_2, \ldots, v_{N-1}, 0\}$, there is a unique symmetric Bayes-Nash equilibrium. Moreover, the equilibrium is such that there exist boundary points $b_1 < b_2 < \cdots < b_K$ so that for any $\theta_k \in \Theta$, an agent of type θ_k mixes between $[b_{k-1}, b_k]$ with $b_0 = 0$.*

In words, Lemma 1 says that for any contest environment, the unique symmetric equilibrium is such that an agent who is least efficient (of type θ_1) mixes between $[0, b_1]$, an agent of type θ_2 mix between $[b_1, b_2]$, and so on, until we get to an agent who is most efficient (of type θ_K), who mixes between $[b_{K-1}, b_K]$. Thus, as in the case where Θ is a continuum (Moldovanu and Sela [19]), the symmetric equilibrium is unique and more efficient agents (those with lower θ) exert higher effort than less efficient agents (those with higher θ) with probability 1. However, unlike the continuum case, the equilibrium with a finite type-space is in mixed strategies. Next, we provide an informal sketch of the proof of the Lemma. The full proof is in the appendix.

To prove Lemma 1, we show that that a symmetric equilibrium $X : \Theta \to \Delta\mathbb{R}_+$ must satisfy the following properties (in order):

1. The equilibrium cannot have any atoms. This is because if an agent of type θ_k chose x_k with positive probability, there is a positive probability that all agents are tied at x_k, and an agent of type θ_k can instead chose $x_k + \epsilon$ and get strictly higher payoff.

2. The minimum effort in support of the mixed strategy equilibrium must be 0. This is because an agent who chooses the minimum effort level in the support of the mixed strategy wins the last prize $v_n = 0$ with probability 1. So if this minimum effort level is positive, the agent can deviate to $x = 0$ and get a strictly higher payoff.[2]

3. The equilibrium utility of more efficient agents should be higher than that of less efficient agents. This is because otherwise, a more efficient agent can simply imitate the strategy of a less efficient agent, in which case it gets the same expected reward as the less efficient agent, but it pays a lower cost leading to a higher payoff.

4. In equilibrium, the intersection of support for two different types cannot have more than one effort level. This is because going from one effort level to another, the change in expected reward is the same irrespective of type, but the change in cost depends on the type. Since an agent of any type must be indifferent between all actions in the support, it follows that agents of two different types cannot both be indifferent between two different effort levels.

5. In equilibrium, the supports of the different agent types must be connected. This is because of reasons similar to that in the second step.

6. In equilibrium, if the supports of two different types are connected at effort level x, then the more efficient type exerts effort greater than x and the less efficient type exerts effort less than x with probability one. This is because if the more efficient type is instead mixing in an interval $[a, x]$ and the less efficient is mixing in the interval $[x, b]$, then the less efficient agent can deviate to a and obtain a strictly higher payoff.

[2]The first two steps are analogous to the argument used to show that the Nash equilibrium in the complete information setting, which corresponds to the case where $|\Theta| = K = 1$, is in mixed strategies (Barut and Kovenock [1], Fang et al. [9]).

Together, the six properties imply that the equilibrium has the structure described in Lemma 1.

Using the structure of the equilibrium, we now obtain a more explicit description of the equilibrium. The following result identifies some important relationships between the boundary points $b = (b_1, b_2, \ldots, b_K)$, the equilibrium utilities $u = (u_1, u_2, \ldots, u_K)$, and the equilibrium distribution functions F_k over $[b_{k-1}, b_k]$, which allow us to complete characterize the unique Bayes-Nash equilibrium for any contest with a finite type-space.

Theorem 1. *Suppose there are N agents, each with a private type drawn from a finite type-space $\Theta = \{\theta_1, \ldots, \theta_K\}$ according to distribution $p = (p_1, p_2, \ldots, p_K)$. For any contest $v = \{v_1, v_2, \ldots, v_{N-1}, 0\}$, the unique symmetric Bayes-Nash equilibrium is such that for any $\theta_k \in \Theta$, the distribution function $F_k : [b_{k-1}, b_k] \to [0, 1]$ is defined by*

$$\sum_{m=1}^{N-1} v_m H_{N-m}^{N-1}(P_{k-1} + p_k F_k(x_k)) - \theta_k x_k = u_k \text{ for all } x_k \in [b_{k-1}, b_k], \tag{1}$$

where the boundary points $b = (b_1, \ldots, b_K)$ and equilibrium utilities $u = (u_1, \ldots, u_K)$ are defined by solutions to the equations

$$b_k = \frac{\sum_{m=1}^{N-1} v_m H_{N-m}^{N-1}(P_k) - u_k}{\theta_k} \text{ for any } k \in \{1, 2, \ldots, K\}, \tag{2}$$

and

$$u_{k+1} - u_k = (\theta_k - \theta_{k+1}) b_k \text{ for any } k \in \{1, 2, \ldots, K-1\}. \tag{3}$$

The proof of Theorem 1 proceeds as follows. We know from Lemma 1 that there exist boundary points $b_1 < b_2 < \cdots < b_K$ so that for any $\theta_k \in \Theta$, an agent of type θ_k mixes between $[b_{k-1}, b_k]$ with $b_0 = 0$. If F_k denotes the equilibrium distribution function for agent of type θ_k, observe that left hand side of Equation 1 is simply the agent's expected payoff from playing $x_k \in [b_{k-1}, b_k]$. This is because $P_{k-1} + p_k F_k(x_k)$ is the probability that this agent's effort x_k exceeds the effort of any arbitrary agent who is also playing according to the equilibrium, and thus, $H_{N-m}^{N-1}(P_{k-1} + p_k F_k(x_k))$ is the probability that the agent's effort x_k is greater than the effort exerted by exactly $N - m$ out of $N - 1$ other agents, in which case it is awarded a prize of v_m. Since the agent must be indifferent between all actions in the support, and in particular, every such action must lead to equilibrium utility u_k, we get that the equilibrium distribution functions must satisfy Equation 1. Now plugging in $x_k = b_k$ in Equation 1 gives Equation 2 (since $F_k(b_k) = 1$). And lastly, we use the fact that agents of both types θ_k and θ_{k+1} have b_k in the support to get the relationship between their equilibrium utilities in Equation 3. The full proof is in the appendix.

Intuitively, since an agent of type θ_k must be indifferent between all effort levels in the interval $[b_{k-1}, b_k]$, the distribution of effort F_k in this interval is such that the marginal gain in expected reward from increasing effort in the interval is equal to the marginal cost θ_k.

Thus, given the contest v, the distribution F_1 on $[0, b_1]$ is such that the marginal gain in reward from increasing effort equals θ_1, F_2 on $[b_1, b_2]$ is such that the marginal gain in reward equals θ_2, and more generally, for any $k \in \{1, 2, \ldots, K\}$, F_k on $[b_{k-1}, b_k]$ is such that the marginal gain in reward from increasing effort in the interval equals θ_k.

We note here that Equations 1, 2, and 3, together with the boundary conditions $b_0 = 0$ and $u_1 = 0$, are sufficient to derive exactly the unique symmetric Bayes-Nash equilibrium for any contest with a finite type-space. Since $u_1 = 0$, we can use Equation 2 to solve for b_1. And then using u_1 and b_1, we can use Equation 3 to solve for u_2. In general, using Equations 2 and 3 iteratively, we can solve explicitly for the equilibrium bounds and utilities for all $k \in \{1, 2, \ldots, K\}$. Once we have the equilibrium utilities u_k and boundary points b_k, Equation 1 provides a complete description of the equilibrium distribution function for any agent of type $\theta_k \in \Theta$. In the following result, we note the equilibrium bounds and utilities obtained by solving the system of Equations 2 and 3.

Lemma 2. *The equilibrium boundary points $b = (b_1, b_2, \ldots, b_K)$ and the equilibrium utilities $u = (u_1, u_2, \ldots, u_K)$, obtained by solving the system of Equations 2 and 3, together with the boundary condition $u_1 = 0$, are such that*

$$b_k = \sum_{m=1}^{N-1} v_m \left[\sum_{j=1}^{k} \frac{H_{N-m}^{N-1}(P_j) - H_{N-m}^{N-1}(P_{j-1})}{\theta_j} \right] \text{ for any } k \in \{1, 2, \ldots, K\}, \tag{4}$$

and

$$u_k = \theta_k \sum_{m=1}^{N-1} v_m \left[\sum_{j=1}^{k-1} H_{N-m}^{N-1}(P_j) \left[\frac{1}{\theta_{j+1}} - \frac{1}{\theta_j} \right] \right] \text{ for any } k \in \{2, \ldots, K\}. \tag{5}$$

We prove Lemma 2 by induction and the proof is in the appendix. Sometimes, we use an alternative representation for the boundary points and the utilities which we note here. The boundary points can be written as

$$b_k = \sum_{m=1}^{N-1} v_m \left[\frac{H_{N-m}^{N-1}(P_k)}{\theta_k} - \sum_{j=1}^{k-1} H_{N-m}^{N-1}(P_j) \left[\frac{1}{\theta_{j+1}} - \frac{1}{\theta_j} \right] \right] \text{ for any } k \in \{1, 2, \ldots, K\}, \tag{6}$$

and equilibrium utilities as

$$u_k = \sum_{m=1}^{N-1} v_m H_{N-m}^{N-1}(P_{k-1}) - \theta_k \sum_{m=1}^{N-1} v_m \left[\sum_{j=1}^{k-1} \frac{H_{N-m}^{N-1}(P_j) - H_{N-m}^{N-1}(P_{j-1})}{\theta_j} \right] \text{ for any } k \in \{2, \ldots, K\}, \tag{7}$$

Next, we illustrate the equilibrium for the special case of $N = 2$ agents as it reveals some additional insight into the structure of the equilibrium and how it depends on the various model parameters.

Corollary 1. *Suppose there are $N = 2$ agents, each with a private type drawn from a finite type-space $\Theta = \{\theta_1, \ldots, \theta_K\}$ according to distribution $p = (p_1, p_2, \ldots, p_K)$. For any contest $v = \{v_1, 0\}$, the unique symmetric Bayes-Nash equilibrium is such that the (random) level of effort exerted by an agent of type $\theta_k \in \Theta$ is*

$$X_k \sim U\left(v_1 \sum_{j=1}^{k-1} \frac{p_j}{\theta_j}, v_1 \sum_{j=1}^{k} \frac{p_j}{\theta_j}\right).$$

Thus, with two agents competing for a single prize, the distribution function over effort for any type θ_k is uniform. Notice that if $x \in [b_{k-1}, b_k]$ then the expected prize of an agent from choosing effort level x is

$$P_{k-1}v_1 + p_k \left[\frac{[x - b_{k-1}]\,\theta_k}{v_1 p_k}\right] v_1.$$

Differentiating with respect to x, we can check that the marginal gain in reward from increasing effort in the range $[b_{k-1}, b_k]$ is equal to the marginal cost θ_k.

4 Optimal contest

In this section, we solve the designer's problem of finding an allocation of prizes with a fixed budget so as to maximize expected equilibrium effort. Going forward, given a contest v with a finite type-space, we denote by X_k the (random) level of effort exerted in equilibrium by an agent of type θ_k and we denote by X the ex-ante (random) level of effort exerted in equilibrium by an arbitrary agent. Thus, if F_k is the equilibrium distribution function for an agent of type θ_k, then $X_k \sim F_k$. And we denote by $F : \mathbb{R} \to [0, 1]$ the distribution function of X, so that for any $x \in \mathbb{R}$,

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ P_{k-1} + p_k F_k(x) & \text{if } x \in [b_{k-1}, b_k] \\ 1 & \text{if } x \geq b_K \end{cases}. \tag{8}$$

The expected effort of an arbitrary agent is then

$$\mathbb{E}[X] = \sum_{k=1}^{K} p_k \mathbb{E}[X_k]. \tag{9}$$

The following result finds, perhaps surprisingly, a rather tractable representation for the expected effort of an arbitrary agent under any contest v and an arbitrary finite type-space.

Lemma 3. *Suppose there are N agents, each with a private type drawn from a finite type-space $\Theta = \{\theta_1, \ldots, \theta_K\}$ according to distribution $p = (p_1, p_2, \ldots, p_K)$. Under a contest $v = \{v_1, v_2, \ldots, v_{N-1}, 0\}$, the expected equilibrium effort of an arbitrary agent is*

$$\mathbb{E}[X] = \sum_{m=1}^{N-1} v_m \alpha_m$$

where

$$\alpha_m = \frac{1}{N}\left[\frac{1}{\theta_K} + \sum_{k=1}^{K-1}\left[H^N_{\geq N-m}(P_k) + (m-1)H^N_{N-m}(P_k)\right]\left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}}\right)\right]. \tag{10}$$

Note that the expected effort can alternatively be represented as

$$\mathbb{E}[X] = \sum_{k=1}^{K}\frac{\lambda_k}{\theta_k}$$

where

$$\lambda_k = \frac{1}{N}\left[\sum_{m=1}^{N-1} v_m\left(H^N_{\geq N-m}(P_k) - H^N_{\geq N-m}(P_{k-1}) + (m-1)\left(H^N_{N-m}(P_k) - H^N_{N-m}(P_{k-1})\right)\right)\right]. \tag{11}$$

To prove Lemma 3, we first find the expected equilibrium effort exerted by an arbitrary agent of type θ_k, and then use the representation in Equation 9 to get $\mathbb{E}[X]$. While an explicit calculation of $\mathbb{E}[X_k]$ using the distribution $F_k(x_k)$ (as described in Theorem 1) appears complicated, we exploit the monotone structure of the equilibrium to solve for $\mathbb{E}[X_k]$, and thus, $\mathbb{E}[X]$. Next, we discuss two important ideas that allow us to solve for $\mathbb{E}[X_k]$.

The first idea is to simply use Equation 1, which defines the distribution function $F_k(.)$ of X_k, to obtain a useful expression for $\mathbb{E}[X_k]$. Going into more detail, it follows from Theorem 1 that the (random) effort X_k satisfies

$$\sum_{m=1}^{N-1} v_m H^{N-1}_{N-m}(P_{k-1} + p_k F_k(X_k)) - \theta_k X_k = u_k.$$

Taking expectation on both sides and rearranging, we get

$$\mathbb{E}[X_k] = \frac{\mathbb{E}[v(\theta_k)] - u_k}{\theta_k},$$

where

$$\mathbb{E}[v(\theta_k)] = \mathbb{E}\left[\sum_{m=1}^{N-1} v_m H^{N-1}_{N-m}(P_{k-1} + p_k F_k(X_k))\right]$$

is simply the expected value of the prize an agent of type θ_k expects to receive in this contest (prior to exerting effort X_k). Since we already have u_k from Equation **??**, we just need to find $\mathbb{E}[v(\theta_k)]$ to solve for $\mathbb{E}[X_k]$. Again, computing $\mathbb{E}[v(\theta_k)]$ directly is non-trivial, especially since there might be other (random) agents of type θ_k.

The second idea, which allows us to circumvent this difficulty, is to instead compute the total prize awarded to agents of type θ_k, and then use symmetry to find $\mathbb{E}[v(\theta_k)]$. Let V_k denote the (random) ex-ante total prize awarded to agents of type θ_k. Clearly,

$$\mathbb{E}[V_k] = Np_k\mathbb{E}[v(\theta_k)].$$

By an alternative calculation, we also have that

$$\begin{aligned}\mathbb{E}[V_k] &= \sum_{m=1}^{N-1} v_m \Pr[\text{Prize m is awarded to agent of type } \theta_k] \\ &= \left[\sum_{m=1}^{N-1} v_m \left(H^N_{\geq N-m+1}(P_k) - H^N_{\geq N-m+1}(P_{k-1})\right)\right]\end{aligned}$$

Equating the two alternative expressions for $\mathbb{E}[V_k]$, we obtain $\mathbb{E}[v(\theta_k)]$, which subsequently allows us to find $\mathbb{E}[X_k]$, and ultimately, $\mathbb{E}[X]$. The full proof is in the appendix.

Given the equilibrium expected effort in Lemma 3, we are able to solve for the effort maximizing contest under an arbitrary finite type-space for a budget-constrained designer.

Theorem 2. *Suppose there are N agents, each with a private type drawn from a finite type-space $\Theta = \{\theta_1, \ldots, \theta_K\}$ according to distribution $p = (p_1, p_2, \ldots, p_K)$. Among all contests $v = (v_1, \ldots, v_N)$ such that $\sum_{i=1}^N v_i \leq V$, the winner-takes-all contest $v^* = (V, 0, 0, \ldots, 0)$ maximizes expected effort.*

From Lemma 3, the marginal effect of transferring value from mth prize to the first prize on expected equilibrium effort is

$$\alpha_1 - \alpha_m = \frac{1}{N}\left[\sum_{k=1}^{K-1}\left[H^N_{\geq N-1}(P_k) - H^N_{\geq N-m}(P_k) - (m-1)H^N_{N-m}(P_k)\right]\left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}}\right)\right]. \quad (12)$$

Since $\alpha_1 - \alpha_m > 0$ for any $m \in \{2, 3, \ldots, N-1\}$, it follows that the winner-takes-all contest maximizes expected effort. Thus, the optimality of the winner-takes-all contest in case where the type-space is a continuum (Moldovanu and Sela [19]) extends to the case where the type-space is finite. Note also that in case where the type-space is a singleton, so that $K = |\Theta| = 1$ and there is complete information, we have from Lemma 3 that $\alpha_1 = \alpha_2 = \cdots = \alpha_{N-1} = \frac{1}{N\theta_K}$ and thus, any allocation of the budget among the top $N-1$ prizes results in the same expected equilibrium effort (Barut and Kovenock [1], Fang et al. [9]). Thus, our analysis of the contest design problem with a finite type-space provides a bridge between previous literature in the complete information environment and the incomplete information environment with an infinite type-space.

Lastly, we illustrate our results using a specific example with $N = 3$ agents.

Example 1. *Suppose there are $N = 3$ agents, each with a private type drawn from a finite type-space $\Theta = \{2, 1\}$ according to distribution $p = (0.5, 0.5)$. For any contest $\mathbf{v} = (v_1, v_2, 0)$, the equilibrium distribution functions are*

$$F_1(x_1) = \frac{-2v_2 + 2\sqrt{v_2^2 + (v_1 - 2v_2)2x_1}}{(v_1 - 2v_2)} \text{ and } F_2(x_2) = \frac{-v_1 + \sqrt{v_1^2 + 4(v_1 - 2v_2)(x_2 - b_1)}}{v_1 - 2v_2},$$

where $b_1 = \dfrac{v_1 + 2v_2}{8}$*. And the expected efforts are*

$$\mathbb{E}[X_1] = \frac{v_1 + 4v_2}{24} \text{ and } \mathbb{E}[X_2] = \frac{11v_1 + 2v_2}{24}$$

so that the expected effort of an arbitrary agent is

$$\mathbb{E}[X] = \frac{1}{2}\mathbb{E}[X_1] + \frac{1}{2}\mathbb{E}[X_2] = \frac{12v_1 + 6v_2}{48}.$$

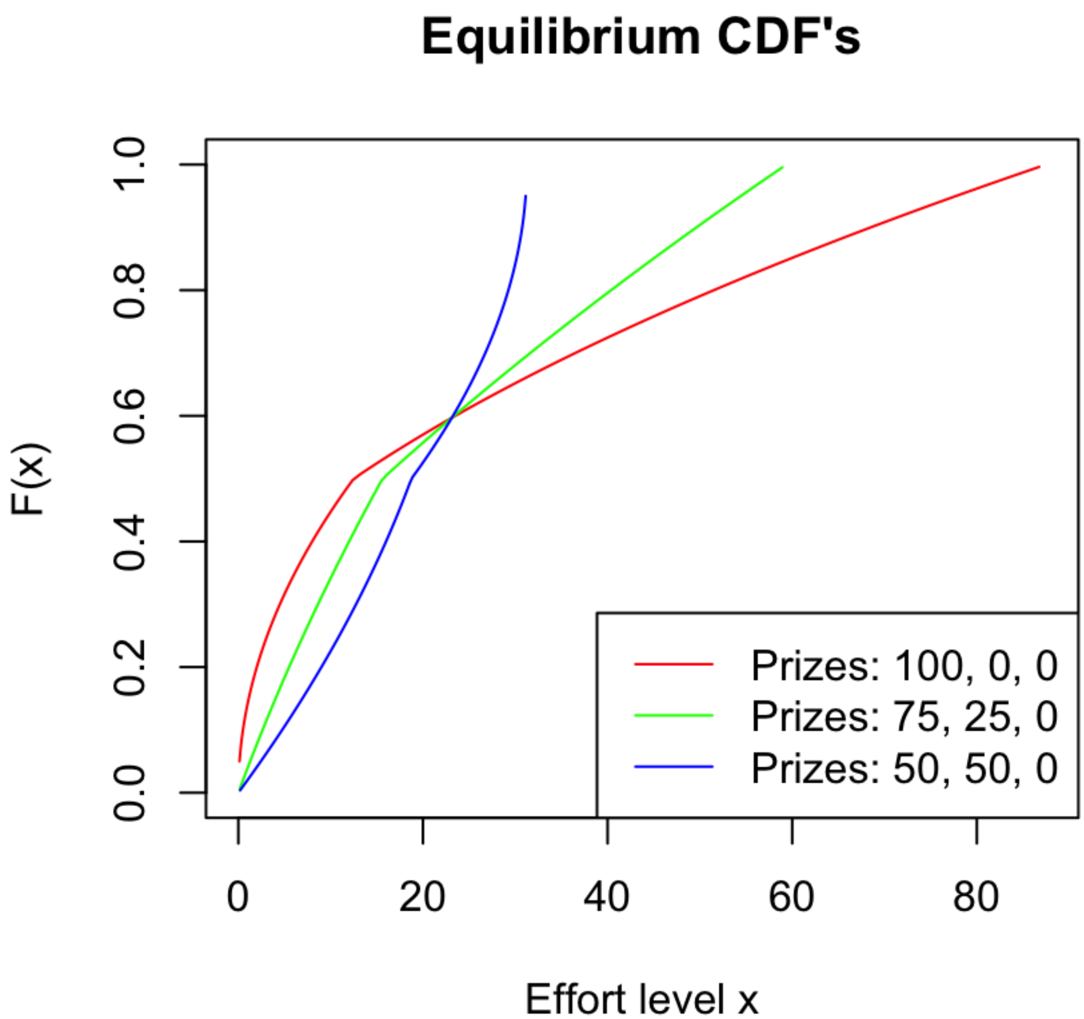

Figure 1: The equilibrium distribution functions, $F(.)$, under three different prize vectors for the environment in Example 1.

5 Continuum type-space

In this section, we study the limit properties of the symmetric equilibrium of finite type-space contest setting. We first note in the following Lemma the symmetric equilibrium under a continuum type-space, defined by a differentiable CDF G (Moldovanu and Sela [19]).

Lemma 4. *Suppose there are N agents, each with a private type drawn from $\Theta = [\underline{\theta}, \overline{\theta}]$ according to a differentiable CDF $G : [\underline{\theta}, \overline{\theta}] \to [0, 1]$. For any contest $v = \{v_1, v_2, \ldots, v_{N-1}, 0\}$, there is a unique symmetric Bayes-Nash equilibrium and it is such that for any $\theta \in \Theta$,*

$$X(\theta) = \sum_{m=1}^{N-1} v_m \lambda_m(\theta)$$

where

$$\lambda_m(\theta) = \int_{\theta}^{\overline{\theta}} \frac{H'^{N-1}_{N-m}(1 - G(t))g(t)}{t} dt.$$

Next, we show that for any sequence of finite type-space distributions that converges to a continuum distribution, the sequence of symmetric equilibrium distributions converges to the equilibrium distribution under the continuum distribution.

Theorem 3. *Suppose there are N agents and consider a fixed contest $v = (v_1, v_2, \ldots, v_{N-1}, 0)$. Let $G : [\underline{\theta}, \overline{\theta}] \to [0, 1]$ be a differentiable CDF and let $G^1, G^2, \ldots,$ be any sequence of CDF's, each with a finite support, such that for all $\theta \in [\underline{\theta}, \overline{\theta}]$,*

$$\lim_{n\to\infty} G^n(\theta) = G(\theta).$$

Let $F^n : \mathbb{R} \to [0, 1]$ denote CDF of the equilibrium effort under the finite type-space distribution G^n, and let $F : \mathbb{R} \to [0, 1]$ denote CDF of the equilibrium under continuum type-space distribution G. Then, the sequence of CDF's $F^1, F^2, \ldots,$ converges to the CDF F, i.e., for all $x \in \mathbb{R}$,

$$\lim_{n\to\infty} F^n(x) = F(x).$$

6 Conclusion

We study the canonical contest design problem in an incomplete information environment with a finite type-space. We characterize the unique symmetric Bayes-Nash equilibrium under any arbitrary contest with a finite type-space. We find that the equilibrium is in mixed strategies, and it is such that agents of adjacent types mix over disjoint but connected intervals so that more efficient agents always exert greater effort. Even though the equilibrium is in mixed strategies, we are able to exploit its monotonic structure to obtain a tractable representation for the expected equilibrium effort of an arbitrary agent. Using this representation, we find that a budget-constrained designer should allocate its entire budget to the first prize, and thus, run a winner-takes-all contest, in order to maximize expected equilibrium effort of an arbitrary agent. Our results extend the well-known optimality of winner-takes-all contest under a continuum type-space to the finite type-space environment.

We introduce some new techniques for the study of contest design problems in finite type-space environment and we hope that the results and methods in this paper will encourage further research in this fundamental domain. In particular, one could study the contest design problem with more general cost functions, and also perhaps other variants that have been previously explored in the literature dealing with the continuum type-space. Since our techniques rely on the separability of the reward and the costs in the utility function, we believe that the structure of equilibrium might be robust to some of these other variants, including for instance, convex cost functions. In addition, we believe that the finite type-space model presents a more convenient framework for experiments as compared to the continuum type-space, and thus, we hope to also inspire more experimental research investigating some of the theoretical predictions in the literature on contest design with incomplete information.

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A Proofs for Section 3 (Equilibrium)

Lemma 1. *Suppose there are N agents, each with a private type drawn from a finite type-space $\Theta = \{\theta_1, \ldots, \theta_K\}$ according to distribution $p = (p_1, p_2, \ldots, p_K)$. For any contest $v = \{v_1, v_2, \ldots, v_{N-1}, 0\}$, there is a unique symmetric Bayes-Nash equilibrium. Moreover, the equilibrium is such that there exist boundary points $b_1 < b_2 < \cdots < b_K$ so that for any $\theta_k \in \Theta$, an agent of type θ_k mixes between $[b_{k-1}, b_k]$ with $b_0 = 0$.*

Proof. Let F_k denote the equilibrium distribution function of agent of type θ_k for $k \in \{1, 2, \ldots, K\}$. Further, suppose F_k has support on the interval $[a_k, b_k]$ and u_k denotes the expected payoff of an agent of type θ_k when all agents play the equilibrium profile $F = (F_1, F_2, \ldots, F_K)$.

1. We first show that F_k cannot have any atoms. Suppose instead that F_k is such that an agent of type θ_k plays x_k with positive probability. Then, there is a positive probability that all agents are tied at effort x_k. But then, an agent of type θ_k will be strictly better off by bidding $x_k + \epsilon$ instead of x_k. This way, the agent earns the best prize among those that would have been otherwise split randomly between the tied agents and only pays an additional ϵ. Thus, F_k cannot have any atoms.

2. We now show that $\min\{a_1, a_2, \ldots, a_K\} = 0$. Suppose instead that this equals $a_k > 0$. Now when an agent of type θ_k plays a_k, it does not win any prize but it pays a positive cost of $\theta_k a_k$. So this agent can instead play 0 and while it still doesn't get a prize, it also doesn't pay any cost. Thus, it must be that $\min\{a_1, a_2, \ldots, a_K\} = 0$.

3. We now show that $u_1 \leq u_2 \leq \cdots \leq u_K$. Suppose instead that $u_k > u_{k+1}$ for some $k \in \{1, 2, \ldots, K-1\}$. Also let $x_k \in [a_k, b_k]$. Then, we have that

   $$u_k = \mathbb{E}_F[v(x_k)] - \theta_k x_k$$

   where $\mathbb{E}_F[v(x_k)]$ denotes the expected prize of an agent who plays x_k when all other agents are playing according to $F = (F_1, F_2, \ldots, F_K)$. Note that this depends only on x_k and not on the type of the agent θ_k. So suppose agent of type θ_{k+1} plays x_k. Its payoff will be

   $$\mathbb{E}_F[v(x_k)] - \theta_{k+1} x_k > \mathbb{E}_F[v(x_k)] - \theta_k x_k = u_k$$

   because $\theta_{k+1} < \theta_k$. Thus, this agent of type θ_{k+1} can imitate an agent of type θ_k and get strictly higher payoff. Thus, it must be that $u_1 \leq u_2 \leq \cdots \leq u_K$.

4. We now show that for any $j \neq k$, $|[a_k, b_k] \cap [a_j, b_j]| \leq 1$. Suppose instead that $x, y \in [a_k, b_k] \cap [a_j, b_j]$ and $x \neq y$. Since agents must be indifferent between all actions in their support, it must be that

   $$u_k = \mathbb{E}_F[v(x)] - \theta_k x = \mathbb{E}_F[v(y)] - \theta_k y$$

   and also

   $$u_j = \mathbb{E}_F[v(x)] - \theta_j x = \mathbb{E}_F[v(y)] - \theta_j y.$$

But this implies

$$\mathbb{E}_F[v(x)] - \mathbb{E}_F[v(y)] = \theta_k(x-y) = \theta_j(x-y)$$

which is a contradiction.

5. We now show that if $b_k \neq \max\{b_1, b_2, \ldots, b_K\}$, then $b_k = a_j$ for some $j \in \{1, 2, \ldots, K\}$. Suppose instead that there is a k such that $b_k \neq \max\{b_1, b_2, \ldots, b_K\}$ and $a_j \neq b_k$ for any $j \in \{1, 2, \ldots, K\}$. Let a_p denote the minimum of all a_j such that $a_j \geq b_k$. Now we can repeat the argument in (2) to show that a_p must be equal to b_k because otherwise, an agent of type θ_p would be strictly better off playing b_k instead of a_p. Thus, we have that the support intervals of the mixed strategies are connected.

6. We now show that if $b_k = a_j$, then $\theta_k \geq \theta_j$. Suppose instead that $\theta_k < \theta_j$. First note that,

$$u_k = \mathbb{E}_F[v(a_k)] - \theta_k a_k = \mathbb{E}_F[v(b_k)] - \theta_k b_k.$$

Since $b_k = a_j$, we have that

$$u_j = \mathbb{E}_F[v(b_k)] - \theta_j b_k = u_k + b_k(\theta_k - \theta_j).$$

Now the payoff of agent of type θ_j from playing $a_k < b_k = a_j$ will be

$$\mathbb{E}_F[v(a_k)] - \theta_j a_k = u_k + (\theta_k - \theta_j)a_k$$

which is greater than u_j if $\theta_k < \theta_j$. Thus, it must be that $\theta_k \geq \theta_j$.

Together, the above steps imply the result in the Lemma.

□

Theorem 1. *Suppose there are N agents, each with a private type drawn from a finite type-space $\Theta = \{\theta_1, \ldots, \theta_K\}$ according to distribution $p = (p_1, p_2, \ldots, p_K)$. For any contest $v = \{v_1, v_2, \ldots, v_{N-1}, 0\}$, the unique symmetric Bayes-Nash equilibrium is such that for any $\theta_k \in \Theta$, the distribution function $F_k : [b_{k-1}, b_k] \to [0,1]$ is defined by*

$$\sum_{m=1}^{N-1} v_m H_{N-m}^{N-1}(P_{k-1} + p_k F_k(x_k)) - \theta_k x_k = u_k \text{ for all } x_k \in [b_{k-1}, b_k], \tag{1}$$

where the boundary points $b = (b_1, \ldots, b_K)$ and equilibrium utilities $u = (u_1, \ldots, u_K)$ are defined by solutions to the equations

$$b_k = \frac{\sum_{m=1}^{N-1} v_m H_{N-m}^{N-1}(P_k) - u_k}{\theta_k} \text{ for any } k \in \{1, 2, \ldots, K\}, \tag{2}$$

and

$$u_{k+1} - u_k = (\theta_k - \theta_{k+1})b_k \text{ for any } k \in \{1, 2, \ldots, K-1\}. \tag{3}$$

Proof. We know from Lemma 1 that there exist boundary points $b_1 < b_2 < \cdots < b_K$ so that for any $\theta_k \in \Theta$, an agent of type θ_k mixes between $[b_{k-1}, b_k]$ with $b_0 = 0$. Let u_k denote the equilibrium utilities, and let F_k on $[b_{k-1}, b_k]$ denote the equilibrium distribution functions. If an agent of type θ_k exerts effort $x_k \in [b_{k-1}, b_k]$, its effort will be greater than that of an arbitrary agent with probability $P_{k-1} + p_k F_k(x_k)$. And the probability that this agent wins prize v_m after exerting effort $x_k \in [b_{k-1}, b_k]$ is the probability that its effort will be greater than exactly $N - m$ agents out of the remaining $N - 1$ agents. Since the agent beats an arbitrary agent with probability $P_{k-1} + p_k F_k(x_k)$, the probability that it beats exactly $N - m$ out of $N - 1$ agents is simply $H^{N-1}_{N-m}(P_{k-1} + p_k F_k(x_k))$. Thus, for an agent of type $\theta_k \in \Theta$, its expected payoff from playing $x_k \in [b_{k-1}, b_k]$ is

$$\sum_{m=1}^{N-1} v_m H^{N-1}_{N-m}(P_{k-1} + p_k F_k(x_k)) - \theta_k x_k.$$

And since this agent must be indifferent between all actions $x_k \in [b_{k-1}, b_k]$, we get that the equilibrium distribution function F_k and the equilibrium utility u_k must be such that for all $x_k \in [b_{k-1}, b_k]$,

$$\sum_{m=1}^{N-1} v_m H^{N-1}_{N-m}(P_{k-1} + p_k F_k(x_k)) - \theta_k x_k = u_k$$

This gives the distribution function in Equation 1 in the result. Now it remains to solve for the boundary points b_k and equilibrium utilities u_k. Clearly, we can plug in $x_k = b_k$ in Equation 1 to get that for any $k \in \{1, 2, \ldots, K\}$,

$$b_k = \frac{\sum_{m=1}^{N-1} v_m H^{N-1}_{N-m}(P_k) - u_k}{\theta_k}.$$

This is exactly Equation 2 in the result.

Lastly, for any $k \in \{1, 2, \ldots, K - 1\}$, observe that agents of both types θ_k and θ_{k+1} have b_k in the support. Thus, it must be that

$$u_k = \mathbb{E}_F[v(b_k)] - \theta_k b_k \text{ and } u_{k+1} = \mathbb{E}_F[v(b_k)] - \theta_{k+1} b_k$$

Taking a difference, we get that for any $k \in \{1, 2, \ldots, K - 1\}$,

$$u_{k+1} - u_k = (\theta_k - \theta_{k+1}) b_k.$$

This is exactly Equation 3 which completes the proof of the result.

□

Lemma 2. *The equilibrium boundary points $b = (b_1, b_2, \ldots, b_K)$ and the equilibrium utilities $u = (u_1, u_2, \ldots, u_K)$, obtained by solving the system of Equations 2 and 3, together with the boundary condition $u_1 = 0$, are such that*

$$b_k = \sum_{m=1}^{N-1} v_m \left[\sum_{j=1}^{k} \frac{H^{N-1}_{N-m}(P_j) - H^{N-1}_{N-m}(P_{j-1})}{\theta_j} \right] \text{ for any } k \in \{1, 2, \ldots, K\}, \tag{4}$$

and

$$u_k = \theta_k \sum_{m=1}^{N-1} v_m \left[\sum_{j=1}^{k-1} H_{N-m}^{N-1}(P_j) \left[\frac{1}{\theta_{j+1}} - \frac{1}{\theta_j} \right] \right] \text{ for any } k \in \{2, \ldots, K\}. \tag{5}$$

Proof. We prove the result by induction. First, we verify that the base case, u_2 and b_1 hold. Since the last prize $v_N = 0$ and an agent who is least efficient (of type θ_1) mixes between $[0, b_1]$, the equilibrium utility of this agent, u_1, must be 0. Now from Equation 2, we get that

$$b_1 = \frac{\sum_{m=1}^{N-1} v_m H_{N-m}^{N-1}(P_1)}{\theta_1}.$$

From Equation 3, we get that

$$u_2 = (\theta_1 - \theta_2) \frac{\sum_{m=1}^{N-1} v_m H_{N-m}^{N-1}(P_1)}{\theta_1} = \theta_2 \sum_{m=1}^{N-1} v_m \left[H_{N-m}^{N-1}(P_1) \left[\frac{1}{\theta_2} - \frac{1}{\theta_1} \right] \right].$$

Thus, the base case with b_1 and u_2 hold.

Before the induction step, we note that we can alternatively write

$$\begin{aligned}
u_k &= \sum_{m=1}^{N-1} v_m H_{N-m}^{N-1}(P_{k-1}) - \theta_k \sum_{m=1}^{N-1} v_m \left[\sum_{j=1}^{k-1} \frac{H_{N-m}^{N-1}(P_j) - H_{N-m}^{N-1}(P_{j-1})}{\theta_j} \right] \\
&= \sum_{m=1}^{N-1} v_m \left[H_{N-m}^{N-1}(P_{k-1}) - \theta_k \left[\sum_{j=1}^{k-1} \frac{H_{N-m}^{N-1}(P_j) - H_{N-m}^{N-1}(P_{j-1})}{\theta_j} \right] \right] \\
&= \theta_k \sum_{m=1}^{N-1} v_m \left[\sum_{j=1}^{k-1} H_{N-m}^{N-1}(P_j) \left[\frac{1}{\theta_{j+1}} - \frac{1}{\theta_j} \right] \right]
\end{aligned}$$

Now we do the induction step using this alternative representation of u_k. Suppose that the claim holds till b_k and u_{k+1}. Then, using Equation 2, we have that

$$\begin{aligned}
b_{k+1} &= \frac{\sum_{m=1}^{N-1} v_m H_{N-m}^{N-1}(P_{k+1}) - u_{k+1}}{\theta_{k+1}} \\
&= \frac{\sum_{m=1}^{N-1} v_m H_{N-m}^{N-1}(P_{k+1}) - \sum_{m=1}^{N-1} v_m H_{N-m}^{N-1}(P_k) + \theta_{k+1} \sum_{m=1}^{N-1} v_m \left[\sum_{j=1}^{k} \frac{H_{N-m}^{N-1}(P_j) - H_{N-m}^{N-1}(P_{j-1})}{\theta_j} \right]}{\theta_{k+1}} \\
&= \sum_{m=1}^{N-1} v_m \left[\sum_{j=1}^{k+1} \frac{H_{N-m}^{N-1}(P_j) - H_{N-m}^{N-1}(P_{j-1})}{\theta_j} \right].
\end{aligned}$$

Thus, the equilibrium boundary point b_{k+1} indeed takes the form given in the result. Now, for the equilibrium utility, we have from Equation 3 that

$$u_{k+1} = u_k + (\theta_k - \theta_{k+1}) b_k$$

$$= \sum_{m=1}^{N-1} v_m H_{N-m}^{N-1}(P_{k-1}) - \theta_k \sum_{m=1}^{N-1} v_m \left[\sum_{j=1}^{k-1} \frac{H_{N-m}^{N-1}(P_j) - H_{N-m}^{N-1}(P_{j-1})}{\theta_j} \right]$$
$$+ (\theta_k - \theta_{k+1}) \sum_{m=1}^{N-1} v_m \left[\sum_{j=1}^{k} \frac{H_{N-m}^{N-1}(P_j) - H_{N-m}^{N-1}(P_{j-1})}{\theta_j} \right]$$
$$= \sum_{m=1}^{N-1} v_m H_{N-m}^{N-1}(P_k) - \theta_{k+1} \sum_{m=1}^{N-1} v_m \left[\sum_{j=1}^{k} \frac{H_{N-m}^{N-1}(P_j) - H_{N-m}^{N-1}(P_{j-1})}{\theta_j} \right]$$

Thus, the equilibrium utility u_{k+1} also takes the form identified above. We can again simplify this form to get the expression in the result. This completes the proof of the result. □

B Proofs for Section 4 (Optimal contest)

Lemma 3. *Suppose there are N agents, each with a private type drawn from a finite type-space $\Theta = \{\theta_1, \dots, \theta_K\}$ according to distribution $p = (p_1, p_2, \dots, p_K)$. Under a contest $v = \{v_1, v_2, \dots, v_{N-1}, 0\}$, the expected equilibrium effort of an arbitrary agent is*

$$\mathbb{E}[X] = \sum_{m=1}^{N-1} v_m \alpha_m$$

where

$$\alpha_m = \frac{1}{N} \left[\frac{1}{\theta_K} + \sum_{k=1}^{K-1} \left[H_{\geq N-m}^{N}(P_k) + (m-1) H_{N-m}^{N}(P_k) \right] \left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}} \right) \right]. \tag{10}$$

Proof. To find expected equilibrium effort of an arbitrary agent, we first find the expected effort exerted in equilibrium by an agent of type θ_k. To find $\mathbb{E}[X_k]$, we have from Theorem 1 that for an agent of type θ_k, the (random) level of effort X_k it exerts in equilibrium satisfies

$$\sum_{m=1}^{N-1} v_m H_{N-m}^{N-1}(P_{k-1} + p_k F_k(X_k)) - \theta_k X_k = u_k.$$

Taking expectation on both sides, we get

$$\mathbb{E}\left[\sum_{m=1}^{N-1} v_m H_{N-m}^{N-1}(P_{k-1} + p_k F_k(X_k)) \right] - \theta_k \mathbb{E}[X_k] = u_k$$

which implies

$$\mathbb{E}[X_k] = \frac{\mathbb{E}[v(\theta_k)] - u_k}{\theta_k},$$

where

$$\mathbb{E}[v(\theta_k)] = \mathbb{E}\left[\sum_{m=1}^{N-1} v_m H_{N-m}^{N-1}(P_{k-1} + p_k F_k(X_k))\right]$$

is simply the expected value of the prize awarded to this agent of type θ_k. While an explicit calculation of this expectation using the definition of $F_k(.)$ in Equation 1 from Theorem 1 appears complicated, we use the following approach instead.
Let V_k denote the (random) ex-ante total prize awarded to agents of type θ_k. Clearly,

$$\mathbb{E}[V_k] = Np_k\mathbb{E}[v(\theta_k)].$$

By an alternative calculation, we also have that

$$\begin{aligned}
\mathbb{E}[V_k] &= \sum_{m=1}^{N-1} v_m \Pr[\text{Prize m is awarded to agent of type } \theta_k] \\
&= \left[\sum_{m=1}^{N-1} v_m \left(H_{\geq N-m+1}^{N}(P_k) - H_{\geq N-m+1}^{N}(P_{k-1})\right)\right]
\end{aligned}$$

The last equality holds because $H_{\geq N-m+1}^{N}(P_k)$ is the probability that there are at least $N-m+1$ agents out of N whose type is in $\{\theta_1, \theta_2, \ldots, \theta_k\}$, which is necessary and sufficient for prize m to be awarded to an agent whose type is in $\{\theta_1, \theta_2, \ldots, \theta_k\}$. Subtracting $H_{\geq N-m+1}^{N}(P_{k-1})$, which is the probability prize m is awarded to an agent whose type is in $\{\theta_1, \theta_2, \ldots, \theta_{k-1}\}$, we get the probability that prize m is awarded to an agent of type θ_k.[3]

Equating the two alternative expressions for $\mathbb{E}[V_k]$, we get that the expected value of the prize awarded to this agent of type θ_k is

$$\mathbb{E}[v(\theta_k)] = \frac{1}{Np_k}\left[\sum_{m=1}^{N-1} v_m \left(H_{\geq N-m+1}^{N}(P_k) - H_{\geq N-m+1}^{N}(P_{k-1})\right)\right].$$

Now we can plug this back in the equation for $\mathbb{E}[X_k]$ and use the definition of u_k from Equation 5 to get that

$$\begin{aligned}
\mathbb{E}[X_k] &= \frac{\mathbb{E}[v(\theta_k)] - u_k}{\theta_k} \\
&= \frac{1}{\theta_k}\left[\frac{1}{Np_k}\left[\sum_{m=1}^{N-1} v_m \left(H_{\geq N-m+1}^{N}(P_k) - H_{\geq N-m+1}^{N}(P_{k-1})\right)\right]\right] \\
&- \sum_{m=1}^{N-1} v_m \left[\sum_{j=1}^{k-1} H_{N-m}^{N-1}(P_j)\left[\frac{1}{\theta_{j+1}} - \frac{1}{\theta_j}\right]\right]
\end{aligned}$$

[3] An alternate representation is $\mathbb{E}[V_k] = \left[\sum_{m=1}^{N-1} v_m \left(H_{\geq m}^{N}(1-P_{k-1}) - H_{\geq m}^{N}(1-P_k)\right)\right]$

Now observe that we can write

$$\mathbb{E}[X_k] = \sum_{m=1}^{N-1} v_m \alpha_{mk}$$

where

$$\alpha_{mk} = \left[\sum_{j=1}^{k-1} H_{N-m}^{N-1}(P_j)\left(\frac{1}{\theta_j} - \frac{1}{\theta_{j+1}}\right)\right] + \frac{(H_{\geq N-m+1}^{N}(P_k) - H_{\geq N-m+1}^{N}(P_{k-1}))}{N\theta_k p_k}.$$

Then, we can write

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k=1}^{K} p_k \mathbb{E}[X_k] \\
&= \sum_{m=1}^{N-1} v_m \alpha_m
\end{aligned}$$

where

$$\begin{aligned}
\alpha_m &= \sum_{k=1}^{K} p_k \alpha_{mk} \\
&= \sum_{k=1}^{K} p_k \left[\left[\sum_{j=1}^{k-1} H_{N-m}^{N-1}(P_j)\left(\frac{1}{\theta_j} - \frac{1}{\theta_{j+1}}\right)\right] + \frac{(H_{\geq N-m+1}^{N}(P_k) - H_{\geq N-m+1}^{N}(P_{k-1}))}{N\theta_k p_k}\right] \\
&= \sum_{k=1}^{K} p_k \left[\sum_{j=1}^{k-1} H_{N-m}^{N-1}(P_j)\left(\frac{1}{\theta_j} - \frac{1}{\theta_{j+1}}\right)\right] + \sum_{k=1}^{K} \frac{(H_{\geq N-m+1}^{N}(P_k) - H_{\geq N-m+1}^{N}(P_{k-1}))}{N\theta_k} \\
&= \sum_{k=1}^{K-1} (1-P_k)\left[H_{N-m}^{N-1}(P_k)\left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}}\right)\right] + \frac{1}{N}\left[\frac{1}{\theta_K} + \sum_{k=1}^{K-1} H_{\geq N-m+1}^{N}(P_k)\left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}}\right)\right] \\
&= \frac{m}{N}\sum_{k=1}^{K-1}\left[H_{N-m}^{N}(P_k)\left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}}\right)\right] + \frac{1}{N}\left[\frac{1}{\theta_K} + \sum_{k=1}^{K-1} H_{\geq N-m+1}^{N}(P_k)\left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}}\right)\right] \\
&= \frac{1}{N}\left[\frac{1}{\theta_K} + \sum_{k=1}^{K-1}(H_{\geq N-m+1}^{N}(P_k) + mH_{N-m}^{N}(P_k))\left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}}\right)\right]
\end{aligned}$$

□

Theorem 2. *Suppose there are N agents, each with a private type drawn from a finite type-space $\Theta = \{\theta_1, \dots, \theta_K\}$ according to distribution $p = (p_1, p_2, \dots, p_K)$. Among all contests $v = (v_1, \dots, v_N)$ such that $\sum_{i=1}^{N} v_i \leq V$, the winner-takes-all contest $v^* = (V, 0, 0, \dots, 0)$ maximizes expected effort.*

Proof. For any contest v, we know from Lemma 3 that the expected equilibrium effort of an arbitrary agent is

$$\mathbb{E}[X] = \sum_{m=1}^{N-1} v_m \alpha_m$$

where

$$\alpha_m = \frac{1}{N}\left[\frac{1}{\theta_K} + \sum_{k=1}^{K-1}\left[H^N_{\geq N-m}(P_k) + (m-1)H^N_{N-m}(P_k)\right]\left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}}\right)\right].$$

Thus, α_m represents the change in expected equilibrium effort from increasing the value of the mth prize. And $\alpha_1 - \alpha_m$ represents the change in expected effort from transferring value from the mth prize to the first prize. We will now show that this change is always positive, so that transferring value from any arbitrary prize to the first prize always increases effort. To see this, observe that for any $m \in \{2, \dots, N-1\}$,

$$\alpha_1 - \alpha_m = \frac{1}{N}\left[\sum_{k=1}^{K-1}\left[H^N_{\geq N-1}(P_k) - H^N_{\geq N-m}(P_k) - (m-1)H^N_{N-m}(P_k)\right]\left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}}\right)\right]$$

Notice that for $m \in \{2, \dots, N-1\}$ and each $k \in \{1, 2, \dots, K-1\}$,

$$\left[H^N_{\geq N-1}(P_k) - H^N_{\geq N-m}(P_k) - (m-1)H^N_{N-m}(P_k)\right] < 0,$$

and also

$$\left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}}\right) < 0$$

since $\theta_k > \theta_{k+1}$. It follows then that for each $k \in \{1, 2, \dots, K-1\}$,

$$\left[H^N_{\geq N-1}(P_k) - H^N_{\geq N-m}(P_k) - (m-1)H^N_{N-m}(P_k)\right]\left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}}\right) > 0$$

which implies that

$$\frac{1}{N}\left[\sum_{k=1}^{K-1}\left[H^N_{\geq N-1}(P_k) - H^N_{\geq N-m}(P_k) - (m-1)H^N_{N-m}(P_k)\right]\left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}}\right)\right] > 0.$$

Thus, $\alpha_1 - \alpha_m > 0$ for each $m \in \{2, 3, \dots, N-1\}$. It follows that the winner-takes-all contest maximizes expected effort. □

C Proofs for Section 5 (Continuum type-space)

Lemma 4. *Suppose there are N agents, each with a private type drawn from $\Theta = [\underline{\theta}, \overline{\theta}]$ according to a differentiable CDF $G : [\underline{\theta}, \overline{\theta}] \to [0, 1]$. For any contest $v = \{v_1, v_2, \dots, v_{N-1}, 0\}$,* *there is a unique symmetric Bayes-Nash equilibrium and it is such that for any $\theta \in \Theta$,*

$$X(\theta) = \sum_{m=1}^{N-1} v_m \lambda_m(\theta)$$

where

$$\lambda_m(\theta) = \int_{\theta}^{\overline{\theta}} \frac{H'^{N-1}_{N-m}(1 - G(t))g(t)}{t} dt.$$

Proof. Suppose $N-1$ agents are playing a strategy $X : [\underline{\theta}, \overline{\theta}] \to \mathbb{R}_+$ so that if an agent's type is θ, it exerts effort $X(\theta)$. Further suppose that $X(\theta)$ is decreasing in θ. Now we want to find the remaining agent's best response to this strategy of the other agents. If the agent's type is θ and it pretends to be an agent of type $t \in [\underline{\theta}, \overline{\theta}]$, its payoff is

$$\sum_{m=1}^{N-1} v_m H^{N-1}_{N-m}(1 - G(t)) - \theta X(t).$$

Taking the first order condition, we get

$$\sum_{m=1}^{N-1} v_m H'^{N-1}_{N-m}(1 - G(t))(-g(t)) - \theta X'(t) = 0.$$

Now we can plug in $t = \theta$ to get the condition for $X(\theta)$ to be a symmetric Bayes-Nash equilibrium. Doing so, we get

$$\sum_{m=1}^{N-1} v_m H'^{N-1}_{N-m}(1 - G(\theta))(-g(\theta)) - \theta X'(\theta) = 0$$

so that

$$X(\theta) = \sum_{m=1}^{N-1} v_m \int_{\theta}^{\overline{\theta}} \frac{H'^{N-1}_{N-m}(1 - G(t))g(t)}{t} dt.$$

□

Theorem 3. *Suppose there are N agents and consider a fixed contest $v = (v_1, v_2, \ldots, v_{N-1}, 0)$. Let $G : [\underline{\theta}, \overline{\theta}] \to [0, 1]$ be a differentiable CDF and let $G^1, G^2, \ldots,$ be any sequence of CDF's, each with a finite support, such that for all $\theta \in [\underline{\theta}, \overline{\theta}]$,*

$$\lim_{n\to\infty} G^n(\theta) = G(\theta).$$

Let $F^n : \mathbb{R} \to [0, 1]$ denote CDF of the equilibrium effort under the finite type-space distribution G^n, and let $F : \mathbb{R} \to [0, 1]$ denote CDF of the equilibrium under continuum type-space distribution G. Then, the sequence of CDF's $F^1, F^2, \ldots,$ converges to the CDF F, i.e., for all $x \in \mathbb{R}$,

$$\lim_{n\to\infty} F^n(x) = F(x).$$

Proof. For the finite support CDF G^n, let $\Theta^n = (\theta_1^n, \theta_2^n, \ldots, \theta_{K(n)}^n)$ denote the support and $p^n = (p_1^n, p_2^n, \ldots, p_{K(n)}^n)$ denote the probability mass function. From Theorem 1, let $b^n = (b_1^n, b_2^n, \ldots, b_{K(n)}^n)$ denote the boundary points, $u^n = (u_1^n, u_2^n, \ldots, u_{K(n)}^n)$ denote the equilibrium utilities, and F_k^n denote the equilibrium CDF of agent of type θ_k^n on support $[b_{k-1}^n, b_k^n]$. Then, the CDF of the equilibrium effort, $F^n : \mathbb{R} \to [0,1]$, is such that for any $x \in \mathbb{R}$,

$$F^n(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ P_{k-1}^n + p_k^n F_k^n(x) & \text{if } x \in [b_{k-1}^n, b_k^n] \\ 1 & \text{if } x \geq b_{K(n)}^n \end{cases}. \tag{13}$$

For the continuum CDF $G : [\underline{\theta}, \overline{\theta}] \to [0,1]$, the CDF of the equilibrium effort, $F : \mathbb{R} \to [0,1]$, is such that for any $x \in \mathbb{R}$,

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 - G(\theta(x)) & \text{if } x \in [0, B] \\ 1 & \text{if } x \geq B \end{cases}. \tag{14}$$

where $\theta(x)$ is the inverse of $X(\theta)$ (from Lemma 4) and $B = X(\underline{\theta})$. The following Lemma will be the key to showing that $F^n(x)$ converges to $F(x)$ for all $x \in \mathbb{R}$.

Lemma 5. *Consider any $\theta \in (\underline{\theta}, \overline{\theta})$ and for any $n \in \mathbb{N}$, let $k(n) \in \{0, 1, 2, \ldots, K(n)\}$ be such that $\theta_{k(n)}^n > \theta \geq \theta_{k(n)+1}^n$ (where $\theta_0^n = \infty$ and $\theta_{K(n)+1}^n = 0$). Then,*

$$\lim_{n\to\infty} b_{k(n)}^n = X(\theta) \text{ and } \lim_{n\to\infty} F^n(b_{k(n)}^n) = 1 - G(\theta).$$

Proof. From Lemma 4 and Equation 6, we have

$$X(\theta) = \sum_{m=1}^{N-1} v_m \lambda_m(\theta) \text{ and } b_{k(n)}^n = \sum_{m=1}^{N-1} v_m \gamma_m^n$$

where

$$\lambda_m(\theta) = \int_\theta^{\overline{\theta}} \frac{H_{N-m}'^{N-1}(1-G(t))g(t)}{t} \, \mathrm{d}t \text{ and } \gamma_m^n = \left[\frac{H_{N-m}^{N-1}(P_{k(n)}^n)}{\theta_{k(n)}^n} - \sum_{j=1}^{k(n)-1} H_{N-m}^{N-1}(P_j^n) \left[\frac{1}{\theta_{j+1}^n} - \frac{1}{\theta_j^n} \right] \right].$$

Observe that

$$\begin{aligned} \gamma_m^n &= \left[\frac{H_{N-m}^{N-1}(P_{k(n)}^n)}{\theta_{k(n)}^n} - \sum_{j=1}^{k(n)-1} H_{N-m}^{N-1}(P_j^n) \left[\frac{1}{\theta_{j+1}^n} - \frac{1}{\theta_j^n} \right] \right] \\ &= \sum_{j=0}^{k(n)-1} \left[H_{N-m}^{N-1}(P_{k(n)}^n) - H_{N-m}^{N-1}(P_j^n) \right] \left[\frac{1}{\theta_{j+1}^n} - \frac{1}{\theta_j^n} \right] \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{1/\theta^n_{k(n)}} \left(H^{N-1}_{N-m}(P^n_{k(n)}) - H^{N-1}_{N-m}(1-G^n(1/x))\right) dx \\
&\xrightarrow{n\to\infty} \int_0^{\frac{1}{\theta}} H^{N-1}_{N-m}(1-G(\theta)) - H^{N-1}_{N-m}(1-G(1/x))dx \quad \text{(dominated convergence)} \\
&= \underbrace{[x(H^{N-1}_{N-m}(1-G(\theta)) - H^{N-1}_{N-m}(1-G(1/x)))]_0^{\frac{1}{\theta}}}_{\text{this is 0}} + \int_0^{\frac{1}{\theta}} \frac{H'^{N-1}_{N-m}(1-G(1/x))g(1/x)}{x} dx \\
&= \int_\theta^\infty \frac{H'^{N-1}_{N-m}(1-G(t))g(t)}{t}\, \mathrm{d}t \quad \text{(substitute } t = 1/x\text{)} \\
&= \int_\theta^{\overline{\theta}} \frac{H'^{N-1}_{N-m}(1-G(t))g(t)}{t}\, \mathrm{d}t \\
&= \lambda_m(\theta)
\end{aligned}
$$

It follows that

$$\lim_{n\to\infty} b^n_{k(n)} = X(\theta).$$

By definition, we have

$$
\begin{aligned}
\lim_{n\to\infty} F^n(b^n_{k(n)}) &= \lim_{n\to\infty} P^n_{k(n)} \\
&= \lim_{n\to\infty} [1-G^n(\theta)] \\
&= 1-G(\theta)
\end{aligned}
$$

□

Fix any $x \in (0,B)$ and let $\theta \in (\underline{\theta}, \overline{\theta})$ be such that $X(\theta) = x$. Then, we know that

$$F(x) = 1 - G(\theta).$$

We want to show that

$$\lim_{n\to\infty} F^n(x) = 1 - G(\theta).$$

Take $\epsilon > 0$. Find $\theta' < \theta$ and $\theta'' > \theta$ such that

$$0 < G(\theta) - G(\theta') = G(\theta'') - G(\theta) < \frac{\epsilon}{4}.$$

Let $x' = X(\theta')$, $x'' = X(\theta'')$, so that $x' > x > x''$. Let $\delta = \min\{x'-x, x-x''\}$. From Lemma 5, let $N(\epsilon)$ be such that for all $n > N(\epsilon)$,

$$\max\{|b^n_{k(n)} - x|, |b^n_{k'(n)} - x'|, |b^n_{k''(n)} - x''|\} < \frac{\delta}{2}$$

and

$$\max\{|F^n(b^n_{k'(n)}) - (1-G(\theta'))|, |F^n(b^n_{k''(n)}) - (1-G(\theta''))|\} < \frac{\epsilon}{4},$$

where $k(n), k'(n), k''(n)$ are sequences as defined in Lemma 5 for θ, θ' and θ'' respectively. Then, for all $n > N(\epsilon)$,

$$\begin{aligned}
F^n(x) &> F^n(b^n_{k''(n)}) \\
&> 1 - G(\theta'') - \frac{\epsilon}{4} \\
&> 1 - G(\theta) - \frac{\epsilon}{2}
\end{aligned}$$

and

$$\begin{aligned}
F^n(x) &< F^n(b^n_{k'(n)}) \\
&< 1 - G(\theta') + \frac{\epsilon}{4} \\
&< 1 - G(\theta) + \frac{\epsilon}{2}
\end{aligned}$$

so that $|F^n(x) - (1 - G(\theta))| < \epsilon$. Thus, $\lim_{n\to\infty} F^n(x) = 1 - G(\theta) = F(x)$ for all $x \in \mathbb{R}$. $\square$