Exchange economy (with indivisible goods)

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Exchange economy

- Economics: how to distribute scarce resources?
- A key model is exchange economies: agents are endowed with some goods, have preferences over consumption bundles, and can only redistribute goods amongst themselves (no production)
- Divisible goods: markets provide a nice mechanism (FWT)
- This presentation: indivisible goods (one per agent)
  1. Airplane seats, dining table, dorm rooms
  2. Kidney exchange (Al Roth)
  3. Context: Housing economy of Shapley-Scarf [1974]
- Results: Markets work well, and there is a nice algorithm to find good allocations

Example 1: Diverse preferences

- Consider a housing economy with three agents in which each agent i is endowed with a house h_i and has the following preferences:

Agent 1	Agent 2	Agent 3
h_2	h_3	h_1
h_1	h_2	h_3
h_3	h_1	h_2

- What allocations are desirable in this economy?
  1. $x = (h_1, h_2, h_3)$?
  2. $x = (h_2, h_3, h_1)$?
- What allocation would result if houses were traded via markets (prices)?
  1. $p = (1, 2, 3)$?
  2. $p = (2, 2, 2)$?

Example 2: Identical preferences

- Consider a housing economy with three agents in which each agent i is endowed with a house h_i and has the following preferences:

Agent 1	Agent 2	Agent 3
h_1	h_1	h_1
h_2	h_2	h_2
h_3	h_3	h_3

- What allocations are desirable in this economy?
  1. $x = (h_1, h_2, h_3)$?
  2. $x = (h_2, h_3, h_1)$?
- What allocation would result if houses were traded via markets (prices)?
  1. $p = (2, 2, 2)$?
  2. $x = (1, 2, 3)$?

Housing economy

- There are n agents, $N = \{1, 2, \ldots, n\}$.
- There are n houses, $X = \{h_1, h_2, \ldots, h_n\}$.
- Each agent $i \in N$ has its own preference $\succcurlyeq_i$ over the set of houses X.
- Each agent $i \in N$ is endowed with its own house $\omega_i \in X$, where $i \neq j$ implies $\omega_i \neq \omega_j$.
- We will assume, WLOG, that $\omega_i = h_i$.

Allocation

- An *allocation* in the housing economy is a function $x : N \to X$ such that
$$i \neq j \implies x_i \neq x_j.$$
In other words, x must be a bijection.
- Given any economy, one would want to find/implement allocations that are (at least)
  1. individually rational (agents are not worse off)
  2. efficient (make best use of available resources)
- We will now formalize these ideas.

1. Individually rational

- An allocation $x = (x_1, x_2, \ldots, x_n)$ is *individually rational* if for all $i \in N$,
$$x_i \succcurlyeq_i \omega_i.$$
- In words, x is individually rational if all agents weakly prefer their house under x than under the endowment ω.

2. Pareto efficient

- An allocation $y = (y_1, y_2, \dots, y_n)$ *Pareto dominates* allocation $x = (x_1, x_2, \dots, x_n)$ if for all $i \in N$,

$$y_i \succcurlyeq_i x_i$$

and for at least one $j \in N$,

$$y_j \succ_j x_j$$

- In words, y Pareto dominates x if all agents weakly prefer their house under y than under x, and at least some agent strictly prefers its house under y than x.
- An allocation $x = (x_1, x_2, \dots, x_n)$ is *Pareto efficient* if there does not exist another allocation y such that y Pareto dominates x.

3. Core

- An allocation $y = (y_1, y_2, \ldots, y_n)$ *blocks* allocation $x = (x_1, x_2, \ldots, x_n)$ if there is some coalition $S \subset N$ such that
  1. for all $i \in S$,
  $$y_i \succcurlyeq_i x_i$$
  and for at least one $j \in S$,
  $$y_j \succ x_j$$
  2. $\bigcup_{j \in S} y_j = \bigcup_{j \in S} \omega_j$
- In words, x is blocked by coalition S if it is possible for agents in S to just trade their endowment in a way such that all agents in S weakly prefer this trade to x and at least some agent in S strictly prefers this trade to x.
- An allocation $x = (x_1, x_2, \ldots, x_n)$ is in the *core* if there does not exist another allocation y that blocks x.

Example 3: Illustrating properties

1. Consider a housing economy with four agents in which each agent i is endowed with a house h_i and has the following preferences:

Agent 1	Agent 2	Agent 3	Agent 4
h_3	h_4	h_1	h_1
h_2	h_1	h_3	h_4
h_1	h_2	h_4	h_2
h_4	h_3	h_2	h_3

2. Allocations and their properties:

Allocation	IR	PE	Core
(h_1, h_2, h_3, h_4)			
(h_4, h_3, h_1, h_2)			
(h_3, h_4, h_1, h_2)			
(h_3, h_2, h_1, h_4)			

3. Market outcome? $p = (5, 2, 5, 3)$?

Market equilibrium

- Markets redistribute goods through prices.
- Let $p = (p_1, p_2, \ldots, p_n)$ denote the market prices for the n houses.
- Given market prices p, agent i (who is endowed with $\omega_i = h_i$) has a budget set
$$B_i(p) = \{h_j : p_j \leq p_i\}.$$
- Each agent $i \in N$ would demand its favorite house (according to $\succ_i$) among those in its budget set. Let $x_i(p) \in B_i(p)$ denote this demand.
- A market equilibrium occurs when market prices are such that the demand for houses equals supply of houses.
- A *market equilibrium* is a pair (p, x) where $p : H \to \mathbb{R}$ is a price function and $x : N \to H$ is a function such that for all $i \in N$,
    1. $x_i = x_i(p)$,
    2. x is an allocation

Questions

1. Does a market equilibrium always exist?
2. Can there be multiple market equilibrium allocations?
3. Which of the three properties do equilibrium allocations satisfy?

Some special cases

- Suppose all agents have the same preference: $h_1 \succ_i h_2 \succ_i \cdots \succ_i h_n$ for all $i \in N$. Then, (p, x) is a market equilibrium if and only if $p_1 > p_2 > \cdots > p_n$ and $x_i = h_i$.
- Suppose each agent has a different most-preferred house. Then, (p, x) where $p_1 = p_2 = \cdots = p_n$ and each agents gets its favorite house under x is a market equilibrium.
- In these two very special cases,
  1. market equilibrium exists,
  2. the equilibrium allocation is unique,
  3. and the equilibrium allocation is IR and PE.
- But what about more general cases, when agents may have other preferences?

First welfare theorem: Markets lead to efficient allocations

Theorem 1.

For any housing economy (with strict preferences),

1. *market equilibrium exists,*
2. *the equilibrium allocation is unique,*
3. *and the equilibrium allocation is Pareto efficient.*

Proof of efficiency

- Take any housing economy with n agents, n houses, and arbitrary (strict) preferences $\succ_i$ over $X = \{h_1, \ldots, h_n\}$ for $i \in N$.
- Suppose (p, x) is a market equilibrium in this economy
- Let $P = \sum_i p_i$ and suppose towards a contradiction that allocation y Pareto dominates x.
- But then, it must be that $p(y_i) \geq p_i \geq p(x_i)$ for all i. And for some j, $p(y_j) > p_j \geq p(x_j)$.
- But this implies $\sum_i p(y_i) > P$ which is a contradiction.

Proof of existence: TTC

- A proof of existence of market equilibrium is through the Top Trading Cycles (TTC) algorithm (due to David Gale)
- The algorithm works as follows:
  1. All agents point to the owner of their favorite house.
  2. Find a trading cycle $C = \{i_1, i_2, ..., i_k\}$ which has the property that
     1. i_j points to i_{j+1} for all $j \in \{1, 2, \ldots, k-1\}$,
     2. $i_1 = i_k$.
  3. Execute the trade suggested by the trading cycle and remove these agents and houses.
  4. If agents and houses remain, go back to step 1. Otherwise, terminate.
- The algorithm terminates and always results in a unique allocation x^{TTC} for any arbitrary preferences.

Example 4: Illustration of TTC

- Consider a housing economy with five agents in which each agent i is endowed with a house h_i and has the following preferences:

Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
h_4	h_1	h_1	h_2	h_2
h_3	h_4	h_3	h_3	h_4
h_1	h_3	h_5	h_4	h_3
h_2	h_2	h_2	h_1	h_1
h_5	h_5	h_3	h_5	h_5

- Step 1: Agents 1, 2, 4 trade
- Step 2: Agent 3 trades
- Step 3: Agent 5 trades

Prices

- If the algorithm terminates after m steps, it generates a partition of the agents as $N = C_1 \cup C_2 \cup \cdots \cup C_m$.
- We can assign prices $p_1 > p_2 \cdots > p_m$ to the houses belonging to the respective cycles.
- Then agent $i \in C_j$ can sell its house for p_j. It cannot afford a house owned by agents in $C_1 \cup C_2 \cup \cdots \cup C_{j-1}$. Thus, its utility is maximized from buying the house of its cyclic successor in C_j, which costs exactly p_j.
- Thus, the allocation x^{TTC} with the above prices constitute a market equilibrium of the housing economy.
- The allocation x^{TTC} is not just Pareto efficient, it is also in the core.

Second welfare theorem: Any efficient allocation can be supported in a market equilibrium

Theorem 2.

Consider a housing economy in which the endowment ω is Pareto optimal. Then, (p, x) is a market equilibrium only if $x = \omega$.

Multiple indivisible goods?

Example 3.

Suppose there are three agents endowed with a house and a parking spot $\omega_i = (h_i, p_i)$ and have the following preferences:

Agent 1	Agent 2	Agent 3
$(h_1, p_2)^1$	$(h_2, p_3)^2$	$(h_1, p_3)^3$
$(h_3, p_1)^3$	$(h_2, p_1)^1$	$(h_3, p_2)^2$
$(h_1, p_1)^2$	$(h_2, p_2)^3$	$(h_3, p_3)^1$
...	...	...

- The economy in this example has an empty core.
- Significant research in studying existence under natural preference domains.

Summary

- In economies with indivisible goods (one per agent), markets lead to efficient allocations.
- The TTC is a useful algorithm for finding desirable allocation in such economies.
- With multiple indivisible goods, challenges arise.