

Optimal grading contests ^{*}

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Abstract

We study the contest design problem in an incomplete information environment with linear effort costs and power distribution prior $F(\theta) = \theta^p$ on marginal cost of effort. We characterize the symmetric Bayes-Nash equilibrium strategy function for arbitrary prize vectors $v_1 \geq v_2 \cdots \geq v_n$ and find that the normalized equilibrium function is always a density function. To study the effects of competition, we compare the effort induced by prize vectors ordered in the majorization order and find that a more competitive prize vector leads to higher expected effort but lower expected minimum effort. We study the implications of these results for the design of grading contests where we assume that the value of a grade is determined by the information it reveals about the quality of the agent, and more precisely, equals its expected productivity. We find that more informative grading schemes induce more competitive prize vectors and hence lead to higher expected effort and lower expected minimum effort.

1 Introduction

There are many contest settings where the designer influences the effort exerted by the agents by choosing a distribution over grades or signals instead of a distribution over direct monetary rewards. But in these grading contests, how does one associate value with the different grades and then how do they compare in terms of the effort they induce? In settings where agents have private abilities, a grade reveals information about the ability of the agent who gets it. For instance, under the popular signalling theory of education, one can take the view of college as being a grading contest where students compete for better signals (grades or GPA) which they can then use to get better offers in the market. In this paper, assuming the value of grade equals the expected productivity of the agent who gets it, we consider the problem of comparing grading schemes in terms of the effort they induce.

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Towards this objective, we first study the standard contest design problem in an incomplete information environment with linear effort costs and power distribution prior $F(\theta) = \theta^p$ (with $p \geq 1$) on marginal cost of effort. We find that the symmetric Bayes-Nash equilibrium strategy function of this game always integrates to the difference of first and last prize ($v_1 - v_n$). This means that changing the prize structure by changing the value of the intermediate prizes v_2, \dots, v_{n-1} only redistributes the effort among the agents. We then study the effect of competition on expected effort and expected minimum effort by studying how these objects vary as prize vector becomes more unequal. We find that as the prize vector becomes more competitive, the expected effort increases while the expected minimum effort decreases. We then investigate the implications of these results for the design of grading contests. Assuming that there is a publicly known monotone decreasing function mapping agents marginal costs to their productivity, and that the prize vector induced by a grading contest is the expected productivity of the agent given the grade of the agent, we find that more informative grading contest induce prize vectors that are more competitive. And thus, we find that more informative grading schemes lead to higher expected effort but lower expected minimum effort. In particular, the rank revealing grading contest leads to highest expected effort while the grading contest that awards $n - 1$ A's and a single B leads to highest expected minimum effort

The paper contributes to a vast literature on design of optimal contests in an incomplete information environments (Moldovanu and Sela [31, 32], Liu and Lu [29, 30], Chawla et al. [7], Ales et al. [1], Fang et al. [15]). In a model similar to ours with linear effort costs but positive lower bounds on marginal costs, Moldovanu and Sela [31] find that allocating the entire prize budget to a single first prize maximizes expected effort. Liu and Lu [30] consider settings where the designer can award arbitrarily many homogeneous prizes without cost and find that the expected effort is single peaked in the number of prizes. The literature is vast in the complete information environment as well (Glazer and Hassin [21], Barut and Kovenock [2], Krishna and Morgan [25], Fang et al. [15]). Assuming convex effort costs, Fang et al. [15] find that more competitive prize vectors discourage effort. Surveys of the theoretical literature in contest theory can be found in Corchón [11], Vojnovic [41], Konrad et al. [24], Segev [39], Sisak [40].

The paper also relates to the literature on design of optimal grading schemes (Moldovanu et al. [33], Rayo [38], Popov and Bernhardt [37], Chan et al. [6], Dubey and Geanakoplos [13], Zubrickas [43]). Moldovanu et al. [33] consider a setting where the designer can associate grades with arbitrary monetary prizes subject to budget and individual rationality constraints and find that the optimal grading scheme awards the top grade to a unique agent and a single grade to all the remaining agents. Dubey and Geanakoplos [13] consider a complete information environment where agents care about relative ranks and find that absolute grading is generally better than relative grading and that it's better to clump scores into coarse categories. Other related papers look at the signalling value of grades under different models or assumptions (Costrell [12], Betts [4], Zubrickas [43], Boleslavsky and Cotton [5]).

The paper proceeds as follows. In section 2, we present the model of a contest in an incomplete information environment. Section 3 characterizes the symmetric Bayes-Nash equilibrium of this game and discusses some important properties of the equilibrium function. In section 4, we study the effects of competition on effort. In section 5, we present the model for studying grading schemes and use the results from sections 3 and 4 to compare grading schemes based on how much effort they induce in equilibrium.

2 Model

Consider a contest with n agents and n prizes described by prize vector $v = (v_1, v_2, \dots, v_n)$ with $v_i \geq v_{i+1}$ for all i . Each agent i simultaneously chooses to exert some effort e_i which leads to disutility $\theta_i e_i$. The agent's type θ_i is its private information. The types are drawn independently from $[0, 1]$ according to cdf F which is common knowledge. If agent i exerts effort e_i and wins prize v_j , its payoff is $v_j - \theta_i e_i$.

3 Equilibrium

Our model is basically same as the model in Moldovanu and Sela [31] except for the fact that we allow agents marginal cost to be 0 whereas Moldovanu and Sela [31] assume that the marginal costs are in an interval $[m, 1]$ where $m > 0$. We'll see that while the symmetric Bayes-Nash equilibrium strategy function takes the same form as in Moldovanu and Sela [31], it satisfies an interesting property when we allow for agents with 0 marginal costs. The following result displays the symmetric Bayes-Nash equilibrium strategy of the contest game.

Theorem 1. *In a contest with n agents, prizes $v = (v_1, v_2, \dots, v_{n-1}, v_n)$ and prior cdf F , the symmetric Bayes-Nash equilibrium strategy function is given by*

$$g_v(\theta) = \sum_{i=1}^n \alpha_i(\theta) v_i$$

where,

$$\alpha_i(\theta) = \binom{n-1}{i-1} \int_{F(\theta)}^1 \frac{[(1-t)^{n-i-1} t^{i-2}]}{F^{-1}(t)} ((n-1)t - (i-1)) dt$$

for $i \in [n-1]$ and $\alpha_n(\theta) = -\sum_{i=1}^{n-1} \alpha_i(\theta)$.

The proof proceeds by assuming that $n-1$ agents are playing $g_v(\theta)$ where g_v is a decreasing function. Then, we find a player's optimal effort level at type θ by taking the first order

condition. Plugging in $g_v(\theta)$ for the optimal level of effort in the condition gives the condition for $g_v(\theta)$ to be the symmetric Bayes-Nash equilibrium:

$$-f(\theta) \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} [(1-F(\theta))^{n-i-1} F(\theta)^{i-1}] = \theta g'_v(\theta)$$

Using the boundary condition $g_v(1) = 0$ pins down the form of the function

$$g_v(\theta) = \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_{F(\theta)}^1 \frac{[(1-t)^{n-i-1} t^{i-1}]}{F^{-1}(t)} dt$$

We can then rewrite the expression as in the theorem by combining the two terms with coefficient v_i . Lastly, we check that the second order condition is satisfied. The full proof is in the appendix.

Now we state an interesting property of the equilibrium function in the presence of agents with 0 marginal costs. If the prior F is such that the equilibrium effort cost of the most efficient agent goes to 0, then the normalized equilibrium function is a density function.

Lemma 1. *Consider a contest with n agents, prizes $v = (v_1, v_2, \dots, v_{n-1}, v_n)$ and prior cdf F . If the equilibrium function $g_v(\theta) = \sum_{i=1}^n \alpha_i(\theta) v_i$ is such that $\lim_{\theta \rightarrow 0} \theta \alpha_i(\theta) = 0$ for all $i \in [n-1]$, then*

$$\int_0^1 \alpha_1(\theta) d\theta = 1$$

and

$$\int_0^1 \alpha_i(\theta) d\theta = 0$$

for $i \in \{2, \dots, n-1\}$.

Under the given condition $\lim_{\theta \rightarrow 0} \theta \alpha_i(\theta) = 0$, we know that

$$\int_0^1 \theta \alpha'_i(\theta) d\theta = \theta \alpha_i(\theta) \Big|_0^1 - \int_0^1 \alpha(\theta) d\theta = - \int_0^1 \alpha(\theta) d\theta$$

Given the form of the equilibrium function, we can find

$$\int_0^1 \theta \alpha'_i(\theta) d\theta$$

and show that it is 0 for $i \in \{2, \dots, n-1\}$ and 1 for $i = 1$. The full proof is in the appendix.

A corollary of the above lemma is that

$$\int_0^1 g_v(\theta) d\theta = \sum_{i=1}^n \int_0^1 v_i \alpha_i(\theta) d\theta = v_1 - v_n$$

and so the normalized symmetric Bayes-Nash equilibrium function is a density function.

We now show that the condition $\lim_{\theta \rightarrow 0} \theta \alpha_i(\theta) = 0$ is satisfied for the power distribution prior $F(\theta) = \theta^p$ with $p \geq 1$. Note that this is the prior distribution we will focus on for the rest of the paper.

Lemma 2. *Consider a contest with n agents, prizes $v = (v_1, v_2, \dots, v_{n-1}, v_n)$ and prior cdf $F(\theta) = \theta^p$ with $p \geq 1$. The equilibrium function $g_v(\theta)$ is such that $\lim_{\theta \rightarrow 0} \theta \alpha_i(\theta) = 0$ for all i .*

Proof. We want to show that for any $i \in \{1, 2, \dots, n-1\}$, $\lim_{\theta \rightarrow 0} \theta \alpha_i(\theta) = 0$.

For the case of $F(\theta) = \theta^p$ with $p \geq 1$, we have

$$\begin{aligned} \alpha_i(\theta) &= \binom{n-1}{i-1} \int_{F(\theta)}^1 \frac{[(1-t)^{n-i-1} t^{i-2}]}{F^{-1}(t)} ((n-1)t - (i-1)) dt \\ &= \binom{n-1}{i-1} \int_{\theta^p}^1 (1-t)^{n-i-1} t^{i-2-\frac{1}{p}} ((n-1)t - (i-1)) dt \end{aligned}$$

This implies

$$\lim_{\theta \rightarrow 0} \theta \alpha_i(\theta) = \lim_{\theta \rightarrow 0} c_1 \theta \int_{\theta^p}^1 (1-t)^{n-i-1} t^{i-1-\frac{1}{p}} dt - c_2 \theta \int_{\theta^p}^1 (1-t)^{n-i-1} t^{i-2-\frac{1}{p}} dt$$

Observe that for $i \geq 1$, both the integrals are bounded above by $\int_{\theta^p}^1 \frac{1}{t} dt = -p \log \theta$ (Note that $c_2 = 0$ for $i = 1$) and we know that $\theta \log(\theta) \rightarrow 0$ as $\theta \rightarrow 0$. The result follows. \square

It follows from Lemma 2 that Lemma 1 applies for the case where $F(\theta) = \theta^p$ with $p \geq 1$. In particular, we get that for the uniform prior, the expected effort equals $v_1 - v_n$ and is independent of the value of the intermediate prizes v_i for $i \in \{2, \dots, n-1\}$.

Corollary 1. *In a contest with n agents, prizes $v = (v_1, v_2, \dots, v_{n-1}, v_n)$ and uniform prior cdf $F(\theta) = \theta$, the expected effort $\mathbb{E}[g_v(\theta)] = v_1 - v_n$.*

Now, we will find $\mathbb{E}[\alpha_i(\theta)]$ and $\mathbb{E}[\alpha_i(\theta_{max})]$ which are objects that will be useful in understand the effect of different prize structures on the expected effort of an arbitrary agent and that of the least efficient agent.

Lemma 3. *Consider a contest with n agents, prizes $v = (v_1, v_2, \dots, v_{n-1}, v_n)$ and prior cdf $F(\theta) = \theta^p$ with $p \geq 1$. If $g_v(\theta) = \sum_{i=1}^n \alpha_i(\theta) v_i$ is the equilibrium function, then*

$$\mathbb{E}[\alpha_i(\theta)] = \binom{n-1}{i-1} \beta\left(i - \frac{1}{p}, n-i\right) \frac{(n-i)(p-1)}{np-1}$$

for $i \in \{1, 2, \dots, n-1\}$.

Lemma 4. Consider a contest with n agents, prizes $v = (v_1, v_2, \dots, v_{n-1}, v_n)$ and prior cdf $F(\theta) = \theta^p$ with $p \geq 1$. If $g_v(\theta) = \sum_{i=1}^n \alpha_i(\theta) v_i$ is the equilibrium function, then

$$\mathbb{E}[\alpha_i(\theta_{max})] = \binom{n-1}{i-1} \beta\left(n+i-1-\frac{1}{p}, n-i\right) \frac{(n-i)(np-1)}{2np-p-1}$$

for $i \in \{1, 2, \dots, n-1\}$.

In the next sections, we will discuss the implications of the properties of the equilibrium function discussed in this section.

4 Competition

In this section, we aim to study the effect of having a more competitive prize structure on the expected effort induced in equilibrium.

Definition 4.1. We say prize vector $v = (v_1, v_2, \dots, v_{n-1}, v_n)$ is more competitive than $w = (w_1, w_2, \dots, w_{n-1}, w_n)$ if v majorizes w (i.e. $\sum_{i=1}^k v_i \geq \sum_{i=1}^k w_i$ for all $k \in [n]$ and $\sum_{i=1}^n v_i = \sum_{i=1}^n w_i$).

The following result shows that a more competitive prize vector leads to a higher expected effort in equilibrium.

Theorem 2. Suppose there are n agents and the prior cdf is $F(\theta) = \theta^p$ with $p \geq 1$. If prize vector v is more competitive than w , then

$$\mathbb{E}[g_v(\theta)] \geq \mathbb{E}[g_w(\theta)]$$

Proof. From Lemma 3, we know that the marginal effect of prize v_i is

$$\mathbb{E}[\alpha_i(\theta)] = \binom{n-1}{i-1} \beta\left(i-\frac{1}{p}, n-i\right) \frac{(n-i)(p-1)}{np-1}$$

Now observe that

$$\frac{\mathbb{E}[\alpha_{i+1}(\theta)]}{\mathbb{E}[\alpha_i(\theta)]} = \frac{n-i}{i} \frac{i-\frac{1}{p}}{n-i-1} \frac{n-i-1}{n-i} = \frac{i-\frac{1}{p}}{i} < 1$$

Thus, the marginal effect of prize i is decreasing in i . This implies that the change in expected effort from increasing any prize $i \in [n-1]$ is positive and the change from increasing v_i is greater than that from increasing v_j for any $i < j$. Since w can be obtained from v via a sequence of Robinhood operations which involve replacing v_i by $v_i - \epsilon$ and v_j by $v_j + \epsilon$, each of which reduces expected effort, we get that the expected effort under w will be lesser than the expected effort under v . So if v is more competitive than w , then $\mathbb{E}[g_v(\theta)] \geq \mathbb{E}[g_w(\theta)]$ \square

The fact that the expected marginal effect of prize v_i is decreasing in i is a bit surprising since we are looking at distributions where a large fraction of the agents are inefficient (the density is increasing in θ). Since the inefficient agents are generally competing for the intermediate prizes, one might expect that as their proportion increases (p increases), the expected marginal effect of the later prizes would increase relative to the earlier prizes. While we do see that the ratio $\frac{\mathbb{E}[a_{i+1}(\theta)]}{\mathbb{E}[\alpha_i(\theta)]} = \frac{i - \frac{1}{p}}{i}$ increases as p increases, it only goes to 1 as $p \rightarrow \infty$. Thus, the effect is not big enough to make later prizes more valuable than the earlier prizes in terms of the effort they induce.

Now we'll discuss the effect of competition on expected minimum effort. Note that in this case, the designer is putting even more weight on the effort of the least efficient agents. In contrast to the expected effort case, it turns out that the marginal effect of prize v_i on expected minimum effort is increasing in i .

Theorem 3. *Suppose there are n agents and the prior cdf is $F(\theta) = \theta^p$ with $p \geq 1$. If prize vector $v = (v_1, v_2, \dots, v_{n-1}, 0)$ is more competitive than prize vector $w = (w_1, w_2, \dots, w_{n-1}, 0)$, then*

$$\mathbb{E}[g_v(\theta_{max})] \leq \mathbb{E}[g_w(\theta_{max})]$$

Proof. From Lemma 4, we know that the marginal effect of prize v_i is

$$\mathbb{E}[\alpha_i(\theta_{max})] = \binom{n-1}{i-1} \beta \left(n + i - 1 - \frac{1}{p}, n - i \right) \frac{(n-i)(np-1)}{2np-p-1}$$

Now observe that

$$\frac{\mathbb{E}[\alpha_{i+1}(\theta_{max})]}{\mathbb{E}[\alpha_i(\theta_{max})]} = \frac{n + i - 1 - \frac{1}{p}}{i} > 1$$

Thus, the marginal effect of prize i is increasing in i . Again, since w can be obtained from v via a sequence of Robinhood operations which involve replacing v_i by $v_i - \epsilon$ and v_j by $v_j + \epsilon$, each of which increases expected minimum effort, we get that the expected minimum effort under w will be greater than that under v . It follows that if v is more competitive than w and both have the same last prize, then $\mathbb{E}[g_v(\theta_{max})] \leq \mathbb{E}[g_w(\theta_{max})]$ \square

The last result shows that when the designer cares about the expected effort of the least efficient agent, less competitive prize vectors are better. In the next section, we'll discuss the implications of our results for the design of grading schemes.

5 Grading schemes

In the last section, we studied the effect of competition on expected effort and assumed that the designer can arbitrarily choose prizes $v_1 \geq v_2 \geq \dots \geq v_n$ as long as they add up to V . But in many natural settings such as schools, the designer is constrained to choose a distribution over grades that it will assign to students based on their performance in exams.

For instance, the professor may commit to assigning grades A and B to the top 50% and bottom 50% respectively, or it may give $A+$, $A-$, $B+$, and $B-$ with distribution $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. How does the designer decide between these two schemes? One property of the latter scheme is that it is more informative about the quality of the students as compared to the former. So if the designer would like the contest to reveal information about the quality of agents, it would prefer the latter scheme. But the designer may also care about how much effort these schemes induce. So how do we determine the effort induced by the above grading schemes? And how does the induced effort differ across grading schemes such as above. In this section, we present a framework in which we can answer these questions and then make use of our results so far towards answering them.

Suppose there are n agents. Let us first formally define a grading contest with n agents.

Definition 5.1. A grading contest with n agents is defined by a strictly increasing sequence of natural numbers $s = (s_1, s_2, \dots, s_k)$ such that $s_k = n$.

The interpretation of grading contest s is that the top s_1 agents get grade g_1 , next $s_2 - s_1$ get grade g_2 and generally, $s_k - s_{k-1}$ get grade g_k .

There is a natural partial order over these grading contests in terms of how much information they reveal about the quality of the agents.

Definition 5.2. A grading contest s is more informative than s' if s' is a subsequence of s .

Clearly, the rank revealing contest $s^* = (1, 2, \dots, n)$ is more informative than any other grading contest.

To discuss the effort induced by grading contests, we want to find a way to think about the prize vectors that are induced by different grading contests. For this purpose, we suppose that there is a publicly known wage function $w : \Theta \rightarrow \mathbb{R}_+$ which is monotone decreasing. The wage function maps an agent's marginal cost to its productivity so that if an agent is revealed to be of type $\theta \in \Theta = [0, 1]$ in the market, its wage will be $w(\theta)$. More generally, if the market has a belief f over the type of the agent, then the agent will get a wage equal to its expected productivity $\int_0^1 w(\theta) f(\theta) d\theta$.

In this setup, we assume that a grading contest s induces the prize vector which is the expected productivity of the agent given the grade of the agent. In particular, we get that the rank revealing contest s^* induces the prize vector

$$v_i = \mathbb{E}[w(\theta) | \theta = \theta_{(i)}^n]$$

where $\theta_{(i)}^n$ is the i th order statistic in a random sample of n . This is because the rank revealing contest reveals the exact rank of the agent in a random sample of n observations. Note here that since $\theta_{(i)}^n$ is stochastically dominated by $\theta_{(j)}^n$ for all $i < j$ and w is monotone decreasing, the prize vector induced by s^* is monotone decreasing $v_1 > v_2 > \dots > v_n$.

Now we can exactly define the prize vectors induced by arbitrary grading contests in terms of the v_i 's as defined above.

An arbitrary grading contest $s = (s_1, s_2, \dots, s_k)$ induces the prize vector $v(s)$ where

$$v(s)_i = \frac{v_{s_{j-1}+1} + v_{s_{j-1}+2} + \dots + v_{s_j}}{s_j - s_{j-1}}$$

where j is such that $s_{j-1} < i \leq s_j$

Suppose an agent gets grade g_j in the grading contest $s = (s_1, s_2, \dots, s_k)$. Then, the market learns that the agent's type θ must be ranked at one of $\{s_{j-1} + 1, \dots, s_j\}$ and further, it is equally likely to be ranked at any of these positions. The form of the prize vector given above then follows from our assumption that the value of grade is determined by the information it reveals about the type of the agent and in particular, its expected productivity.

In this framework, we can now ask how do the different grading schemes compare in terms of effort.

Theorem 4. *Consider a setting with n agents and prior cdf $F(\theta) = \theta^p$ with $p > 1$. If grading scheme s is more informative than s' , then the expected effort under s is higher than that under s' .*

Proof. Observe that if s is more informative than s' , then the prize vector induced by s is more competitive than that induced by s' . It thus follows from Theorem 2 that

$$\mathbb{E}[g_{v(s)}(\theta)] \geq \mathbb{E}[g_{v(s')}(\theta)]$$

as required. □

Corollary 5. *The rank revealing contest $s^* = (1, 2, \dots, n)$ maximizes expected effort among all grading contests.*

We can do a similar analysis for expected minimum effort.

Theorem 6. *Consider a setting with n agents and prior cdf $F(\theta) = \theta^p$ with $p > 1$. If grading scheme s is more informative than s' and $v(s)_n = v(s')_n$, then the expected minimum effort under s is higher than that under s' .*

Proof. Observe that if s is more informative than s' and $v(s)_n = v(s')_n$, then the prize vector induced by s is more competitive than that induced by s' and in addition, they award the same last prize. It thus follows from Theorem 3 that

$$\mathbb{E}[g_{v(s)}(\theta_{min})] \leq \mathbb{E}[g_{v(s')}(\theta_{min})]$$

as required. □

Corollary 7. *The contest $s = (n-1, n)$ maximizes expected minimum effort among all grading contests that award a unique grade to the last agent.*

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A Proofs for Section 3 (Equilibrium)

Theorem 1. *In a contest with n agents, prizes $v = (v_1, v_2, \dots, v_{n-1}, v_n)$ and prior cdf F , the symmetric Bayes-Nash equilibrium strategy function is given by*

$$g_v(\theta) = \sum_{i=1}^n \alpha_i(\theta) v_i$$

where,

$$\alpha_i(\theta) = \binom{n-1}{i-1} \int_{F(\theta)}^1 \frac{[(1-t)^{n-i-1} t^{i-2}]}{F^{-1}(t)} ((n-1)t - (i-1)) dt$$

for $i \in [n-1]$ and $\alpha_n(\theta) = -\sum_{i=1}^{n-1} \alpha_i(\theta)$.

Proof. Suppose $n-1$ agents are playing a monotone decreasing strategy $g(\theta)$. Let $\theta_{(j)}^n$ denote the j th order statistic from n random draws with $\theta_{(0)}^n = 0$ and $\theta_{(n+1)}^n = 1$. Then, an agent of type θ 's utility from putting in x units of effort is given by:

$$\begin{aligned} u(\theta, x) &= \sum_{i=1}^n v_i \Pr[\theta_{(i-1)}^{n-1} \leq g^{-1}(x) \leq \theta_{(i)}^{n-1}] - \theta x \\ &= \sum_{i=1}^n v_i \binom{n-1}{i-1} F(g^{-1}(x))^{i-1} (1 - F(g^{-1}(x)))^{n-i} - \theta x \end{aligned}$$

Now, differentiating with respect to x gives:

$$\begin{aligned} \frac{\partial u(\theta, x)}{\partial x} &= \frac{f(g^{-1}(x))}{g'(g^{-1}(x))} \sum_{i=1}^n v_i \binom{n-1}{i-1} \\ &\quad [(1 - F(g^{-1}(x)))^{n-i} (i-1) F(g^{-1}(x))^{i-2} - F(g^{-1}(x))^{i-1} (n-i) (1 - F(g^{-1}(x)))^{n-i-1}] - \theta \end{aligned}$$

. Setting it to 0 and plugging in $g(\theta) = x$ gives the condition for $g(\theta)$ to be a symmetric Bayes-Nash equilibrium:

$$f(\theta) \sum_{i=1}^n v_i \binom{n-1}{i-1} [(1 - F(\theta))^{n-i} (i-1) F(\theta)^{i-2} - F(\theta)^{i-1} (n-i) (1 - F(\theta))^{n-i-1}] = \theta g'(\theta)$$

An alternate way to write this condition is:

$$-f(\theta) \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} [(1 - F(\theta))^{n-i-1} F(\theta)^{i-1}] = \theta g'(\theta)$$

Using the boundary condition $g(1) = 0$, we get that the symmetric Bayes-Nash equilibrium function is given by

$$\int_{\theta}^1 \frac{f(\theta)}{\theta} \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} [(1-F(\theta))^{n-i-1} F(\theta)^{i-1}] d\theta$$

Replacing $F(\theta) = t$, we get

$$g(\theta) = \int_{F(\theta)}^1 \frac{1}{F^{-1}(t)} \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} [(1-t)^{n-i-1} t^{i-1}] dt$$

Bringing the summation outside:

$$g_v(\theta) = \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_{F(\theta)}^1 \frac{[(1-t)^{n-i-1} t^{i-1}]}{F^{-1}(t)} dt$$

We can also write the equilibrium function as $g_v(\theta) = \sum_{i=1}^n \alpha_i(\theta) v_i$ where for $i \geq 2$,

$$\begin{aligned} \alpha_i(\theta) &= \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_{F(\theta)}^1 \frac{[(1-t)^{n-i-1} t^{i-1}]}{F^{-1}(t)} dt - \frac{(n-1)!}{(i-2)!(n-i)!} \int_{F(\theta)}^1 \frac{[(1-t)^{n-i} t^{i-2}]}{F^{-1}(t)} dt \\ &= \frac{(n-1)!}{(i-2)!(n-i-1)!} \int_{F(\theta)}^1 \left(\frac{[(1-t)^{n-i-1} t^{i-1}]}{(i-1)F^{-1}(t)} - \frac{[(1-t)^{n-i} t^{i-2}]}{(n-i)F^{-1}(t)} \right) dt \\ &= \frac{(n-1)!}{(i-2)!(n-i-1)!} \int_{F(\theta)}^1 \frac{[(1-t)^{n-i-1} t^{i-2}]}{F^{-1}(t)} \left(\frac{t}{(i-1)} - \frac{1-t}{n-i} \right) dt \\ &= \binom{n-1}{i-1} \int_{F(\theta)}^1 \frac{[(1-t)^{n-i-1} t^{i-2}]}{F^{-1}(t)} ((n-1)t - (i-1)) dt \end{aligned}$$

For $i = 1$, we have that

$$\alpha_1(\theta) = (n-1) \int_{F(\theta)}^1 \frac{[(1-t)^{n-2}]}{F^{-1}(t)} dt$$

Now we check that the second order condition is satisfied. □

Lemma 1. Consider a contest with n agents, prizes $v = (v_1, v_2, \dots, v_{n-1}, v_n)$ and prior cdf F . If the equilibrium function $g_v(\theta) = \sum_{i=1}^n \alpha_i(\theta) v_i$ is such that $\lim_{\theta \rightarrow 0} \theta \alpha_i(\theta) = 0$ for all $i \in [n-1]$, then

$$\int_0^1 \alpha_1(\theta) d\theta = 1$$

and

$$\int_0^1 \alpha_i(\theta) d\theta = 0$$

for $i \in \{2, \dots, n-1\}$.

Proof. We have

$$g_v(\theta) = \sum_{i=1}^n \alpha_i(\theta) v_i$$

where for $i \in \{2, \dots, n-1\}$,

$$\alpha_i(\theta) = \binom{n-1}{i-1} \int_{F(\theta)}^1 \frac{[(1-t)^{n-i-1} t^{i-2}]}{F^{-1}(t)} ((n-1)t - (i-1))$$

and

$$\alpha_1(\theta) = (n-1) \int_{F(\theta)}^1 \frac{[(1-t)^{n-2}]}{F^{-1}(t)} dt$$

We will show that if $\lim_{\theta \rightarrow 0} \theta \alpha_i(\theta) = 0$ then $\int_0^1 \alpha_i(\theta) d\theta = 0$ for $i \in \{2, \dots, n-1\}$ and $\int_0^1 \alpha_1(\theta) d\theta = 1$

By Leibniz rule, for $i \geq 2$, we have

$$\alpha'_i(\theta) = -\binom{n-1}{i-1} \frac{[(1-F(\theta))^{n-i-1} F(\theta)^{i-2}]}{\theta} ((n-1)F(\theta) - (i-1)) f(\theta)$$

When $\lim_{\theta \rightarrow 0} \theta \alpha_i(\theta) = 0$, we have that $\int_0^1 \theta \alpha'_i(\theta) d\theta = -\int_0^1 \alpha_i(\theta) d\theta$

From above, we have that

$$\begin{aligned} \int_0^1 \theta \alpha'_i(\theta) d\theta &= -\int_0^1 \theta \binom{n-1}{i-1} \frac{[(1-F(\theta))^{n-i-1} F(\theta)^{i-2}]}{\theta} ((n-1)F(\theta) - (i-1)) f(\theta) d\theta \\ &= -\binom{n-1}{i-1} \int_0^1 [(1-t)^{n-i-1} t^{i-2}] ((n-1)t - (i-1)) dt \\ &= 0 \end{aligned}$$

Thus, we get that $\int_0^1 \alpha_i(\theta) d\theta = 0$ for $i \geq 2$. For $i = 1$, we have that

$$\alpha_1(\theta) = (n-1) \int_{F(\theta)}^1 \frac{[(1-t)^{n-2}]}{F^{-1}(t)} dt$$

so that $\alpha'_1(\theta) = -(n-1) \frac{(1-F(\theta))^{n-2}}{\theta} f(\theta)$ and thus, $\int_0^1 \theta \alpha'_1(\theta) d\theta = -1$. This gives that $\int_0^1 \alpha_1(\theta) d\theta = 1$.

Together, we have that

$$\int_0^1 g_v(\theta) d\theta = \sum_{i=1}^n \int_0^1 v_i \alpha_i(\theta) d\theta = v_1 - v_n$$

□

Lemma 3. Consider a contest with n agents, prizes $v = (v_1, v_2, \dots, v_{n-1}, v_n)$ and prior cdf $F(\theta) = \theta^p$ with $p \geq 1$. If $g_v(\theta) = \sum_{i=1}^n \alpha_i(\theta) v_i$ is the equilibrium function, then

$$\mathbb{E}[\alpha_i(\theta)] = \binom{n-1}{i-1} \beta\left(i - \frac{1}{p}, n-i\right) \frac{(n-i)(p-1)}{np-1}$$

for $i \in \{1, 2, \dots, n-1\}$.

Proof.

$$\begin{aligned} \mathbb{E}[\alpha_i(\theta)] &= - \int_0^1 F(\theta) \alpha'_i(\theta) d\theta \\ &= \binom{n-1}{i-1} \int_0^1 \frac{[(1-F(\theta))^{n-i-1} F(\theta)^{i-1}]}{\theta} ((n-1)F(\theta) - (i-1)) f(\theta) d\theta \\ &= \binom{n-1}{i-1} \int_0^1 \frac{[(1-t)^{n-i-1} t^{i-1}]}{F^{-1}(t)} ((n-1)t - (i-1)) dt \end{aligned}$$

For $F(\theta) = \theta^p$,

$$\begin{aligned} \mathbb{E}[\alpha_i(\theta)] &= \binom{n-1}{i-1} \left((n-1)\beta\left(i+1 - \frac{1}{p}, n-i\right) - (i-1)\beta\left(i - \frac{1}{p}, n-i\right) \right) \\ &= \binom{n-1}{i-1} \left((n-1)\beta\left(i - \frac{1}{p}, n-i\right) \frac{i - \frac{1}{p}}{n - \frac{1}{p}} - (i-1)\beta\left(i - \frac{1}{p}, n-i\right) \right) \\ &= \binom{n-1}{i-1} \beta\left(i - \frac{1}{p}, n-i\right) \left((n-1) \frac{i - \frac{1}{p}}{n - \frac{1}{p}} - (i-1) \right) \\ &= \binom{n-1}{i-1} \beta\left(i - \frac{1}{p}, n-i\right) \frac{(n-i)(p-1)}{np-1} \end{aligned}$$

□

Lemma 4. Consider a contest with n agents, prizes $v = (v_1, v_2, \dots, v_{n-1}, v_n)$ and prior cdf $F(\theta) = \theta^p$ with $p \geq 1$. If $g_v(\theta) = \sum_{i=1}^n \alpha_i(\theta) v_i$ is the equilibrium function, then

$$\mathbb{E}[\alpha_i(\theta_{max})] = \binom{n-1}{i-1} \beta\left(n+i-1 - \frac{1}{p}, n-i\right) \frac{(n-i)(np-1)}{2np-p-1}$$

for $i \in \{1, 2, \dots, n-1\}$.

Proof.

$$\mathbb{E}[\alpha_i(\theta_{max})] = \int_0^1 \alpha_i(\theta) n F(\theta)^{n-1} f(\theta) d\theta$$

$$\begin{aligned}
&= \binom{n-1}{i-1} \int_0^1 \frac{[(1-F(\theta))^{n-i-1} F(\theta)^{i-2}]}{\theta} ((n-1)F(\theta) - (i-1)) F(\theta)^n f(\theta) d\theta \\
&= \binom{n-1}{i-1} \int_0^1 \frac{[(1-t)^{n-i-1} t^{n+i-2}]}{F^{-1}(t)} ((n-1)t - (i-1)) dt
\end{aligned}$$

For the case of $F(\theta) = \theta^p$, we get that

$$\begin{aligned}
\mathbb{E}[\alpha_i(\theta_{max})] &= \binom{n-1}{i-1} \left((n-1)\beta\left(n+i-\frac{1}{p}, n-i\right) - (i-1)\beta\left(n+i-1-\frac{1}{p}, n-i\right) \right) \\
&= \binom{n-1}{i-1} \beta\left(n+i-1-\frac{1}{p}, n-i\right) \left((n-1)\frac{n+i-1-\frac{1}{p}}{2n-1-\frac{1}{p}} - (i-1) \right) \\
&= \binom{n-1}{i-1} \beta\left(n+i-1-\frac{1}{p}, n-i\right) \frac{(n-i)(np-1)}{2np-p-1}
\end{aligned}$$

□