Multi-agent contract design with independent trials: Winners-take-all based on weight and priority

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Introduction

- Principal delegates individual tasks to multiple agents
- Each agent may succeed or fail depending upon the effort they exert
- Principal can only observe final outcomes and not effort
- How should a budget-constrained principal design contracts so as to encourage agents to exert costly effort?
- Potential applications: crowdsourcing contests, sales contests

Outline

- Model
- Main results
- Application
- Proofs

Model

- $A = \{1, 2, \ldots, n\}$: set of n risk-neutral agents, each attempts an independent task
- $O = \{0, 1\}$: possible outcomes for each agent, success or failure
- $p_i \in [0, 1]$: agent i's choice of probability of success
- $c_i(p_i)$: cost incurred by agent i for its choice of p_i
- Assume c_i is strictly convex, with $c_i(0) = 0, c_i'(0) = 0, c_i'(1) > 1$

Principal

- Principal's payoff: $V(p_1, \ldots, p_n) = \sum_{i=1}^{n} w_i p_i$
- Assume the principal
  - cannot observe p_i
  - can observe the outcome for each agent
  - is budget constrained ($B = 1$)
- A *contract* is a function $f : \mathcal{P}(A) \to \mathbb{R}_+^n$ such that for each $S \subseteq A$,

$$\sum_{i \in A} f_i(S) \leq 1.$$

Timing and payoffs

- Timing:
  - Principal commits to a contract $f : \mathcal{P}(A) \to \mathbb{R}^n_+$
  - Agents simultaneously choose $p = (p_1, \ldots, p_n)$
  - Each agent succeeds or fails, and is rewarded according to f
- Agent $i's$ payoff under profile $p = (p_1, \ldots, p_n)$ equals

$$u_i(p) = \sum_{S \subset A} f_i(S) \mathrm{Pr}^A_p(S) - c_i(p_i)$$

where

$$\mathrm{Pr}^A_p(S) = \prod_{i \in S} p_i \prod_{j \in A \setminus S} (1 - p_j).$$

Equilibrium

- Concave payoffs $\implies$ pure-strategy NE exists (Rosen [1965])
- If p is an equilibrium, then for all $i \in A$, either $p_i = 0$ or

$$\frac{\partial u_i(p)}{\partial p_i} = \sum_{S \subset A_{-i}} (f_i(S \cup \{i\}) - f_i(S)) \mathrm{Pr}_{p_{-i}}^{A_{-i}}(S) - c_i'(p_i) = 0$$

- Notation:
  - $\mathcal{F}$: set of all contracts
  - $E(f)$: NE under f
  - $\mathcal{E} := \cup_{f \in \mathcal{F}} E(f)$
  - $\mathcal{P}$:= Pareto frontier of $\mathcal{E}$
- Goal: characterize $\mathcal{P}$ and solve for some natural objectives $V(p)$

Some natural classes of contracts

- f is a *failures-get-nothing* (FGN) contract if $\forall i \in A$ and $\forall S$,

$$i \notin S \implies f_i(S) = 0.$$

- f is a *successful-get-everything* (SGE) contract if $\forall S$,

$$\sum_{i \in S} f_i(S) = 1.$$

- f is a *priority-based weighted* (PW) contract if there exists an ordered partition $(X_1, \ldots, X_l)$ of A and weights $\lambda = (\lambda_1, \ldots, \lambda_n)$ s.t. $\forall S$,

$$f_i(S) = \begin{cases} \dfrac{\lambda_i}{\sum_{j \in S \cap X_k} \lambda_j}, & \text{if } S \cap X_m = \emptyset \text{ for } m < k \text{ and } i \in S \cap X_k \\ 0, & \text{otherwise} \end{cases}$$

Some examples

- Constant: $f_i(S) = \frac{1}{n}$ for all $i \in A$
- Piece-rate ($\sum_{i \in A} \lambda_i \leq 1$):

$$f_i(S) = \begin{cases} \lambda_i, & \text{if } i \in S \\ 0, & \text{otherwise} \end{cases}$$

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- Weighted split among winners:

$$f_i(S) = \begin{cases} \frac{\lambda_i}{\sum_{j \in S} \lambda_j}, & \text{if } i \in S \\ 0, & \text{otherwise} \end{cases}$$

- Priority-based:

$$f_i(S) = \begin{cases} 1, & \text{if } i = \min\{j : j \in S\} \\ 0, & \text{otherwise} \end{cases}$$

Result 1: $\mathcal{P} = \mathcal{E}(SGE)$

Theorem 1.

Suppose $p \in E(f)$. Then, $p \in \mathcal{P}$ if and only if f is SGE.

Complexity reduction

Theorem 2.

For any $p \in \mathcal{P}$, there exists a unique PW contract f such that $p \in E(f)$.

Corollary 3.

Suppose the principal's objective $V(p_1, p_2, \ldots, p_n)$ is increasing in p_i. Then,

$$\max_{f \in \mathcal{F}} V(p) = \max_{f \in \mathcal{F}_{PW}} V(p).$$

Application

- $n = 2$ agents, $c_i(p_i) = \frac{1}{2} c_i p_i^2$ with $c_i > 1$
- $V(p_1, p_2) = wp_1 + p_2$
- Optimal contest design problem is just to find $\lambda = f_1(\{1, 2\})$.
- Unique pure-strategy Nash:

$$p_1(\lambda) = \frac{c_2 - (1 - \lambda)}{c_1 c_2 - \lambda(1 - \lambda)} \quad p_2(\lambda) = \frac{c_1 - \lambda}{c_1 c_2 - \lambda(1 - \lambda)}.$$

Optimal contract

Theorem 4.

$\lambda^*(w)$ *is increasing in* w. *In particular,*

$$\lambda^*(w) = \begin{cases} 0, & \text{if } w \le \frac{c_1 c_2 - c_1}{c_1 c_2 + c_2 - 1} \\ \frac{1}{2}, & \text{if } w = 1 \\ 1, & \text{if } w \ge \frac{c_1 c_2 + c_1 - 1}{c_1 c_2 - c_2} \end{cases}$$

SGE $\implies$ Pareto optimality

Claim 5.

If f is a SGE contract and $p \in E(f)$, then $p \in \mathcal{P}$.

1. Since p must satisfy the foc, we have

$$c_i'(p_i) = \sum_{S \subset A_{-i}} f_i(S \cup \{i\}) \mathrm{Pr}_{p_{-i}}^{A_{-i}}(S)$$

2. Multiplying by p_i and adding for all i gives

$$\sum_{i \in A} p_i \cdot c_i'(p_i) = 1 - \mathrm{Pr}_p^n(\phi).$$

which implies

$$\sum_{i \in A} p_i \cdot c_i'(p_i) + \Pi_{i=1}^n (1 - p_i) = 1.$$

3. Differentiating the lhs by p_i, we get

$$p_i c_i''(p_i) + c_i'(p_i) - \Pi_{j \neq i}(1 - p_j) > 0$$

Pareto optimality $\implies$ SGE

Lemma 6.

If p is Pareto optimal and $p \in E(f)$, then f must be a SGE contract.

Let

$$K_p := \{S \subseteq A : \sum_{i \in S} f_i(S) < 1 \text{ for some } f \in E^{-1}(p) \cap \mathcal{F}_{FGN}\},$$

1. Suppose $S \in K_p$. For any $T \subset S$, $T \in K_p$.
2. Suppose $S, T \in K_p$. Then, $S \cup T \in K_p$.
   It follows that $K_p = 2^{\kappa_p}$.
3. Suppose $f \in E^{-1}(p) \cap \mathcal{F}_{FGN}$. Then, for all $S \subset A$ such that $\kappa_p^C \cap S \neq \phi$, $f_i(S) = 0$ for all $i \in \kappa_p$.
4. Suppose $\kappa_p \neq \phi$. Then there is a p' that Pareto dominates p.

$\mathcal{P} = \mathcal{E}[F_{PW}]$

Given a SGE contract f and any profile p, define,

$$Z_p(f) := \max_{i \in A} c_i'(\Psi_i(p_{-i}, f)) - c_i'(p_i).$$

Lemma 7.

If $p \in \mathcal{P}$, then

$$\inf_{f \in \mathcal{F}_{PW}} Z_p(f) = 0.$$

- Let $z = \inf_{f \in \mathcal{F}_{PW}} Z_p(f) > 0$ and let $Z_p(f) = z$. Further, let i be the lowest priority agent such that

$$c_i'(\Psi_i(p_{-i}, f)) - c_i'(p_i) = z$$

- Group agents that have i's priority and those that succeed i.

Lit review

- **Multi-agent contract design:** Holmstrom [1982], Lazear and Rosen [1981], Green and Stokey [1983], Nalebuff and Stiglitz [1983], Malcomson [1986], Imhof and Kräkel [2014], Mookherjee [1984], Baiman and Rajan [1995], Castiglioni et al. [2023], Dütting et al. [2023]
- **Crowdsourcing contests:** Segev [2020], Taylor [1995], Halac et al. [2017], Gross [2020], Haggiag et al. [2022]

Summary

- Study a contract design problem between a principal and multiple agents with budget constraint
- Characterize the Pareto frontier of success probabilities that can be sustained in equilibrium as equilibrium of SGE contracts
- Identify a small class of PW contracts that can implement the Pareto frontier which provides a significant reduction in dimensionality of the optimal contract design problem
- Application to two agents suggests that the structure of optimal contract depends more on principal's bias than agents heterogeneity

Thank you!