Contest design with a finite type-space

Andrzej Baranski* Sumit Goel†

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Abstract

We study the classical contest design problem of allocating a budget across different prizes to maximize effort in an incomplete information environment with a finite type-space. For any contest with an arbitrary finite type-space and distribution over this type-space, we characterize the unique symmetric Bayes-Nash equilibrium of the contest game. We find that the equilibrium is in mixed strategies, where agents of different types mix over disjoint but connected intervals, so that more efficient agents always exert greater effort than less efficient agents. Using this characterization, we solve for the expected equilibrium effort under any arbitrary contest, and find that with linear costs, a winner-takes-all contest maximizes expected effort among all contests feasible for a budget-constrained designer. Our analysis introduces new techniques for the study of contests in a finite type-space and offers a unified approach to studying contest design simultaneously in the complete information environment, where the type-space is a singleton, and in the classical incomplete information setting with a continuum type-space, which we show can be well approximated by a sufficiently large finite type-space.

1 Introduction

Contests are situations in which agents compete with each other by investing costly effort or other resources to win valuable prizes. Given their widespread prevalence across various domains, such as business (e.g., firms' investments in innovation), sports (e.g., tournaments), and politics (e.g., securing positions of power), it is important to understand how different contests influence the effort exerted by agents and, in particular, identify contest structures that are optimal from the designer's perspective. Consequently, there is a vast literature studying variants of optimal contest design problems in different domains, including complete and incomplete information environments which differ in the assumptions they make

*NYU Abu Dhabi; a.baranski@nyu.edu
†NYU Abu Dhabi; sumitgoel58@gmail.com

about the information held by the agents about the types (abilities) of their opponents.

In this paper, we revisit the classical contest design problem of allocating a fixed budget across different prizes so as to maximize expected effort, focusing on a fundamental domain of an incomplete information environment with a finite type-space. Previous studies in the incomplete information domain typically assume a continuum type-space with smooth distributions, concluding that with linear costs, awarding the entire budget to only the best performing agent is strictly optimal (Glazer and Hassin [11], Moldovanu and Sela [19], Zhang [30]). In contrast, in the complete information domain with linear costs, any distribution of the budget in which the worst performing agent gets nothing yields the same expected effort, and is therefore, optimal (Glazer and Hassin [11], Barut and Kovenock [1]). And in fact, with complete information and convex costs, it becomes strictly optimal to award everyone except the worst-performing agent an equal share of the budget (Fang, Noe, and Strack [9]). These results highlight how the structure of the optimal contest can vary significantly depending on the domain[1]. Since our finite type-space domain encompasses the complete information domain as a special case (when the type-space is singleton) and can approximate any continuum type-space (with a sufficiently large finite-type space), the findings from the literature in these extreme cases do not directly inform what occurs in this intermediate domain. By analyzing the contest design problem in the finite type-space domain, our work not only provides a bridge between the literature on these extreme domains, it introduces new techniques that provide a unifying approach towards studying contest design problems in the classical complete and incomplete information domains, which have traditionally been explored separately using very different techniques.

For the finite type-space domain, we begin by characterizing the unique symmetric Bayes-Nash equilibrium of the contest game. Interestingly, even though the equilibrium is in mixed strategies, it always exhibits a neat and important monotonic structure. More specifically, the equilibrium is such that agents mix over disjoint but contiguous intervals so that agents who are more efficient always exert more effort than agents who are less efficient. Thus, the equilibrium in the finite type-space domain exhibits both the mixed nature of the equilibrium seen in complete information domains, and the monotonic structure seen in incomplete information domain with a continuum type-space. While this mixed and monotonic structure of the equilibrium is robust across type-spaces, cost functions and distributions of the budget, these elements of the model help pin down the specific equilibrium distribution over the intervals through the indifference condition.

Using this equilibrium characterization, we focus on solving the contest design problem. Restricting attention to linear costs, we first solve for the expected effort induced by any arbitrary contest. While an explicit calculation directly using the distribution functions

[1]More detailed surveys of the literature on this contest design problem can be found in Sisak [24], Vojnović [28]. For general surveys of the theoretical literature in contest theory, see Corchón [5], Vojnović [28], Konrad [13], Chowdhury, Esteve-González, and Mukherjee [4].

appears complicated, we introduce techniques that leverage the monotonic structure of the equilibrium and the additive separability of rewards and costs in the agents' utility functions to obtain a tractable representation for the expected effort. With this representation, we show that if there are at least two distinct types, the marginal effect of increasing the value of the first prize is greater than that of increasing any other prize, and thus, awarding the entire budget to the best performing agent maximizes expected effort. And in case the type-space is a singleton, we recover the result for the complete information domain that any distribution of the budget in which the worst-performing agents gets nothing is optimal. Perhaps interestingly, this suggests that as soon as there is any *little* uncertainty about the types, the winner-takes-all contest becomes strictly optimal.

Lastly, we establish an equilibrium convergence result that illustrates how the insights from studying contests in the finite type-space domain might apply to the classical domain with a continuum type-space. In particular, we study the limit behavior of the equilibrium of a sequence of finite type-spaces that converge to a continuum type-space, and show that the corresponding sequence of distributions of the mixed-strategy equilibrium converges to the distribution of the pure-strategy equilibrium under the continuum type-space. Essentially, the interval over which an agent of a certain type mixes shrinks as the finite type-space becomes large and converges to the effort prescribed by the pure-strategy equilibrium under the continuum type-space. Thus, for any continuum type-space, we can find a sufficiently large finite type-space so that the equilibrium behaviour under various contests in the finite type-space provides a reasonable approximation to the equilibrium behavior under these contests in the continuum type-space.[2] Since the finite type-space clearly includes the complete information domain as a special case, this convergence result formalizes the sense in which the finite-type space domain provides a framework to investigate simultaneously the classical complete and incomplete information environments.

There is some related literature studying contests in an incomplete information with a finite type-space. Most of this literature focuses on problems with a small (or even binary) type-space and investigates properties of Bayes-Nash equilibrium. In particular, Szech [25] studies the value of information disclosure in a model with binary types and asymmetric distributions. Liu and Chen [17] allows for correlated types in an all-pay auction with two agents and binary types and shows that the symmetric Bayes-Nash equilibrium may be non-monotonic (i.e., have overlapping intervals) if the absolute correlation is large. Chen [2] considers a setting where players observe private signal about the types of their opponent and characterizes equilibrium for different degrees of informativeness of the signal structure. Xiao [29] assumes complete information (or perfect correlation) with heterogeneous agents and illustrates in a model with three agents and two prizes that the winner-takes-all contest maximizes the total expected effort if the top two players are similar, and two equal prizes maximize the total expected bid if the bottom two players are similar. In contrast to this

[2]This is of particular importance for those interested in conducting experimental investigations of contest models as they can approximate the continuum of types setting with a finite type-space.

literature, our paper studies the optimal contest design problem for arbitrarily general finite type-spaces.

The paper proceeds as follows. In Section 2, we present the general model of a contest in an incomplete-information environment with a finite type-space. In Section 3, we characterize the symmetric Bayes-Nash equilibrium of the contest game. In Section 4, we study the design of effort-maximizing contest. In Section 5, we discuss the convergence of finite-type space equilibrium to the equilibrium of the continuum type-space. Section 6 concludes. Some technical proofs are relegated to the Appendix.

2 Model

There is a set of N risk-neutral agents. Each agent has a privately known type θ (which represents its marginal cost of effort), drawn independently from type-space $\Theta = \{\theta_1, \theta_2, \ldots, \theta_K\}$ according to distribution $p = (p_1, p_2, \ldots, p_K)$ so that $\Pr[\theta = \theta_k] = p_k$. It is common knowledge that agents' types are independent and identically distributed according to p. Without loss of generality, we assume $\theta_1 > \theta_2 > \cdots > \theta_K > 0$ and $p_k > 0$ for all k.

There is a designer who designs a contest $v = (v_1, v_2, \ldots, v_N)$ with $v_1 \geq \cdots \geq v_{N-1} \geq v_N$. Given the contest v, all agents simultaneously choose their effort. The agents are ranked according to their effort and awarded the corresponding prizes, with ties broken uniformly at random. If an agent of type θ_k wins prize v_i after exerting effort x_k, its payoff is

$$v_i - \theta_k x_k.$$

A contest v, together with the distribution p over type-space Θ, defines a Bayesian game between the N agents. We will focus on the symmetric Bayes-Nash equilibrium of this game. This is a strategy profile where all agents are using the same (potentially mixed) strategy, $X : \Theta \to \Delta\mathbb{R}_+$, mapping agent's type to a distribution over non-negative effort levels, so that for any agent of type θ_k, choosing any effort level within the support of $X(\theta_k)$ yields an expected payoff at least as high as that from choosing any other effort level, given that all other agents use the strategy $X(.)$.

The contest designer wants to maximize the expected equilibrium effort and has a fixed budget $V > 0$ that it can use to allocate to different prizes in order to incentivize agents to exert effort. We study this designer's problem of finding the contest that maximizes expected effort subject to the budget constraint.

Notation

Here, we introduce some notation that will be used in the rest of the paper.

We let $P_k = \sum_{i=1}^{k} p_i$. We will denote by

$$H_K^N(p) = \binom{N}{K} p^K (1-p)^{N-K}$$

the probability that a binomial random variable $Y \sim Bin(N, p)$ takes a value of exactly K. We also use

$$H_{\leq K}^N(p) = \sum_{k=0}^{K} \binom{N}{k} p^k (1-p)^{N-k} \text{ and } H_{\geq K}^N(p) = \sum_{k=K}^{N} \binom{N}{k} p^k (1-p)^{N-k}$$

to denote the probability that $Y \sim Bin(N, p)$ takes a value of at most K and at least K respectively. Given a contest $v = (v_1, v_2, \ldots, v_N)$, we let

$$\pi_v(p) = \sum_{m=1}^{N} v_m H_{N-m}^{N-1}(p)$$

denote the expected value of the prize that an agent gets if it beats any arbitrary agent with probability $p \in [0, 1]$.

3 Equilibrium

In this section, we characterize the symmetric Bayes-Nash equilibrium of contests with a finite type-space. Before providing a complete description of the equilibrium, we establish an important structural property of the equilibrium.

Lemma 1. *Suppose there are N agents, each with a private type drawn from a finite type-space $\Theta = \{\theta_1, \ldots, \theta_K\}$ according to distribution $p = (p_1, p_2, \ldots, p_K)$. For any contest $v = \{v_1, v_2, \ldots, v_{N-1}, 0\}$, there is a unique symmetric Bayes-Nash equilibrium. Moreover, the equilibrium is such that there exist boundary points $b_1 < b_2 < \cdots < b_K$ so that for any $\theta_k \in \Theta$, an agent of type θ_k mixes between $[b_{k-1}, b_k]$ with $b_0 = 0$.*

In words, Lemma 1 says that for any contest environment, the unique symmetric equilibrium is in mixed strategies and satisfies a useful monotonicity property. In particular, the equilibrium is such that an agent who is least efficient (of type θ_1) mixes between $[0, b_1]$, an agent of type θ_2 mix between $[b_1, b_2]$, and so on, until we get to an agent who is most efficient (of type θ_K), who mixes between $[b_{K-1}, b_K]$. Thus, there is monotonicity in that more efficient agents exert higher effort than less efficient agents with probability 1. And the possibility that an agent might face other agents of the same type creates incentives for them to mix and exert higher effort within their intervals. Thus, the symmetric equilibrium under a finite type-space domain exhibits both the mixed nature of the equilibrium seen in complete information domains (Barut and Kovenock [1]), and the monotonic structure of the equilibrium seen in incomplete information domains with a continuum type-space

(Moldovanu and Sela [19]). Next, we provide an informal sketch of the proof of the Lemma. The full proof is in the appendix.

To prove Lemma 1, we show that that a symmetric equilibrium $X : \Theta \to \Delta\mathbb{R}_+$ must satisfy the following properties (in order):

1. The equilibrium cannot have any atoms. This is because if an agent of type θ_k chose x_k with positive probability, there is a positive probability that all agents are tied at x_k, and an agent of type θ_k can instead chose $x_k + \epsilon$ and get strictly higher payoff.

2. The minimum effort in support of the mixed strategy equilibrium must be 0. This is because an agent who chooses the minimum effort level in the support of the mixed strategy wins the last prize $v_n = 0$ with probability 1. So if this minimum effort level is positive, the agent can deviate to $x = 0$ and get a strictly higher payoff.[3]

3. The equilibrium utility of more efficient agents should be higher than that of less efficient agents. This is because otherwise, a more efficient agent can simply imitate the strategy of a less efficient agent, in which case it gets the same expected reward as the less efficient agent, but it pays a lower cost leading to a higher payoff.

4. In equilibrium, the intersection of support for two different types cannot have more than one effort level. This is because going from one effort level to another, the change in expected reward is the same irrespective of type, but the change in cost depends on the type. Since an agent of any type must be indifferent between all actions in the support, it follows that agents of two different types cannot both be indifferent between two different effort levels.

5. In equilibrium, the supports of the different agent types must be connected. This is because of reasons similar to that in the second step.

6. In equilibrium, if the supports of two different types are connected at effort level x, then the more efficient type exerts effort greater than x and the less efficient type exerts effort less than x with probability one. This is because if the more efficient type is instead mixing in an interval $[a, x]$ and the less efficient is mixing in the interval $[x, b]$, then the less efficient agent can deviate to a and obtain a strictly higher payoff.

Together, the six properties imply that the equilibrium has the structure described in Lemma 1.

Using the structure of the equilibrium, we can now completely characterize the unique Bayes-Nash equilibrium for any contest in an arbitrary finite type-space.

[3]The first two steps are analogous to the argument used to show that the Nash equilibrium in the complete information setting, which corresponds to the case where $|\Theta| = K = 1$, is in mixed strategies (Barut and Kovenock [1], Fang, Noe, and Strack [9]).

Theorem 1. *Suppose there are N agents, each with a private type drawn from a finite type-space $\Theta = \{\theta_1, \dots, \theta_K\}$ according to distribution $p = (p_1, p_2, \dots, p_K)$. For any contest $v = \{v_1, v_2, \dots, v_{N-1}, 0\}$, the unique symmetric Bayes-Nash equilibrium is such that for any $\theta_k \in \Theta$, the distribution function $F_k : [b_{k-1}, b_k] \to [0, 1]$ is defined by*

$$\pi_v(P_{k-1} + p_k F_k(x_k)) - \theta_k x_k = u_k \text{ for all } x_k \in [b_{k-1}, b_k], \tag{1}$$

where the boundary points $b = (b_1, \dots, b_K)$ and equilibrium utilities $u = (u_1, \dots, u_K)$ are

$$b_k = \sum_{j=1}^{k} \frac{\pi_v(P_j) - \pi_v(P_{j-1})}{\theta_j} \text{ for any } k \in \{1, 2, \dots, K\}, \tag{2}$$

and

$$u_k = \theta_k \left[\sum_{j=1}^{k-1} \pi_v(P_j) \left(\frac{1}{\theta_{j+1}} - \frac{1}{\theta_j} \right) \right] \text{ for any } k \in \{1, 2, \dots, K\}. \tag{3}$$

Proof. We know from Lemma 1 that there exist boundary points $b_1 < b_2 < \cdots < b_K$ so that for any $\theta_k \in \Theta$, an agent of type θ_k mixes between $[b_{k-1}, b_k]$ with $b_0 = 0$. If F_k denotes the equilibrium distribution function for agent of type θ_k, observe that $\pi_v(P_{k-1} + p_k F_k(x_k))$ is simply the agent's expected payoff from playing $x_k \in [b_{k-1}, b_k]$. Since the agent must be indifferent between all actions in the support, and in particular, every such action must lead to equilibrium utility u_k, we get that the equilibrium distribution functions must satisfy Equation 1.

Now it remains to solve for the boundary points b_k and equilibrium utilities u_k. Notice that we can plug in $x_k = b_k$ in Equation 1 to get that

$$b_k = \frac{\pi_v(P_k) - u_k}{\theta_k} \text{ for any } k \in \{1, 2, \dots, K\}. \tag{4}$$

In addition, we use the fact that agents of both types θ_k and θ_{k+1} have b_k in the support to get a relationship between their equilibrium utilities. More precisely, for any $k \in \{1, 2, \dots, K-1\}$, we have that

$$u_k = \pi_v(P_k) - \theta_k b_k \text{ and } u_{k+1} = \pi_v(P_k) - \theta_{k+1} b_k.$$

Taking a difference, we get

$$u_{k+1} - u_k = (\theta_k - \theta_{k+1}) b_k \text{ for any } k \in \{1, 2, \dots, K-1\}. \tag{5}$$

Since $u_1 = 0$, we can use Equation 4 to solve for b_1. And then using u_1 and b_1, we can use Equation 5 to solve for u_2. In general, using Equations 4 and 5 iteratively, we can solve explicitly for the equilibrium bounds and utilities. Together with these equilibrium objects, as described in Equations 2 and 3, Equation 1 provides a complete description of the unique symmetric Bayes-Nash equilibrium of the contest game. □

Intuitively, since an agent of type θ_k must be indifferent between all effort levels in the interval $[b_{k-1}, b_k]$, the distribution of effort F_k in this interval is such that the marginal gain in expected reward from increasing effort in the interval is equal to the marginal cost θ_k. Thus, given the contest v, the distribution F_1 on $[0, b_1]$ is such that the marginal gain in reward from increasing effort equals θ_1, F_2 on $[b_1, b_2]$ is such that the marginal gain in reward equals θ_2, and more generally, for any $k \in \{1, 2, \ldots, K\}$, F_k on $[b_{k-1}, b_k]$ is such that the marginal gain in reward from increasing effort in the interval equals θ_k. We illustrate this in the following example which explicitly describes the equilibrium for the special case with $N = 2$ agents.

Corollary 1. *Suppose there are $N = 2$ agents, each with a private type drawn from a finite type-space $\Theta = \{\theta_1, \ldots, \theta_K\}$ according to distribution $p = (p_1, p_2, \ldots, p_K)$. For any contest $v = \{v_1, 0\}$, the unique symmetric Bayes-Nash equilibrium is such that the (random) level of effort exerted by an agent of type $\theta_k \in \Theta$ is*

$$X_k \sim U\left(v_1 \sum_{j=1}^{k-1} \frac{p_j}{\theta_j}, v_1 \sum_{j=1}^{k} \frac{p_j}{\theta_j}\right).$$

Thus, with two agents competing for a single prize, the distribution function over effort for any type θ_k is uniform. Notice that if $x \in [b_{k-1}, b_k]$ then the expected prize of an agent from choosing effort level x is

$$P_{k-1}v_1 + p_k \left[\frac{[x - b_{k-1}]\,\theta_k}{v_1 p_k}\right] v_1.$$

Differentiating with respect to x, we can check that the marginal gain in reward from increasing effort in the range $[b_{k-1}, b_k]$ is equal to the marginal cost θ_k.

4 Optimal contest

In this section, we solve the designer's problem of finding a distribution of budget across prizes so as to maximize expected equilibrium effort. Going forward, given a contest v with a finite type-space, we denote by $X_k \sim F_k$ the (random) level of effort exerted in equilibrium by an agent of type θ_k and we denote by $X \sim F$ the ex-ante (random) level of effort exerted in equilibrium by an arbitrary agent so that for any $x \in \mathbb{R}$,

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ P_{k-1} + p_k F_k(x) & \text{if } x \in [b_{k-1}, b_k] \\ 1 & \text{if } x \geq b_K \end{cases}. \tag{6}$$

The expected effort of an arbitrary agent is then

$$\mathbb{E}[X] = \sum_{k=1}^{K} p_k \mathbb{E}[X_k].$$

The following result finds, perhaps surprisingly, a rather tractable representation for the expected effort of an arbitrary agent under any contest v and an arbitrary finite type-space.

Lemma 2. *Suppose there are N agents, each with a private type drawn from a finite type-space $\Theta = \{\theta_1, \ldots, \theta_K\}$ according to distribution $p = (p_1, p_2, \ldots, p_K)$. Under a contest $v = \{v_1, v_2, \ldots, v_{N-1}, 0\}$, the expected equilibrium effort of an arbitrary agent is*

$$\mathbb{E}[X] = \sum_{m=1}^{N-1} v_m \alpha_m$$

where

$$\alpha_m = \frac{1}{N}\left[\frac{1}{\theta_K} + \sum_{k=1}^{K-1}\left[H^N_{\geq N-m}(P_k) + (m-1)H^N_{N-m}(P_k)\right]\left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}}\right)\right]. \tag{7}$$

To prove Lemma 2, we first find $\mathbb{E}[X_k]$. While an explicit calculation using the distribution $F_k(x_k)$ (as described in Theorem 1) appears complicated, we discuss two important ideas that allow us to solve for $\mathbb{E}[X_k]$.

First, we reinterpret Equation 1, which defines the distribution F_k of X_k, as the definition of X_k with

$$\pi_v(P_{k-1} + p_k F_k(X_k)) - \theta_k X_k = u_k,$$

where $F_k(X_k)$ is now just a uniformly distributed random variable. Taking expectation on both sides and rearranging, we get

$$\mathbb{E}[X_k] = \frac{\mathbb{E}[\pi_v(P_{k-1} + p_k F_k(X_k))] - u_k}{\theta_k},$$

where

$$\mathbb{E}[\pi_v(P_{k-1} + p_k F_k(X_k))] = \mathbb{E}\left[\sum_{m=1}^{N-1} v_m H^{N-1}_{N-m}(P_{k-1} + p_k F_k(X_k))\right]$$

is simply the ex-ante expected value of the prize an agent of type θ_k expects to receive in this contest (prior to exerting effort X_k). Again, computing this expectation directly is non-trivial, especially since there might be other (random) agents of type θ_k.

The second idea, which allows us to circumvent this difficulty, is to instead compute the total prize awarded to agents of type θ_k, and then exploit symmetry of the environment. Let V_k denote the (random) ex-ante total prize awarded to agents of type θ_k. By linearity of expectation,

$$\mathbb{E}[V_k] = N p_k \mathbb{E}[\pi_v(P_{k-1} + p_k F_k(X_k))].$$

By an alternative calculation, we also have that

$$\mathbb{E}[V_k] = \sum_{m=1}^{N-1} v_m \Pr[\text{Prize m is awarded to agent of type } \theta_k]$$

$$= \left[\sum_{m=1}^{N-1} v_m \left(H^N_{\geq N-m+1}(P_k) - H^N_{\geq N-m+1}(P_{k-1}) \right) \right]$$

Notice that $H^N_{\geq N-m+1}(P_k)$ is the probability that there are at least $N-m+1$ agents out of N whose type is in $\{\theta_1, \theta_2, \ldots, \theta_k\}$, which is both necessary and sufficient for prize m to be awarded to an agent whose type is in $\{\theta_1, \theta_2, \ldots, \theta_k\}$. From this, we subtract the probability that prize m is awarded to an agent whose type is in $\{\theta_1, \theta_2, \ldots, \theta_{k-1}\}$, given by $H^N_{\geq N-m+1}(P_{k-1})$, which leaves us with the probability that prize m is awarded to an agent of type θ_k. Equating the two alternative expressions for $\mathbb{E}[V_k]$, we obtain $\mathbb{E}[\pi_v(P_{k-1} + p_k F_k(X_k))]$, which subsequently allows us to find $\mathbb{E}[X_k]$, and ultimately, $\mathbb{E}[X]$. The full proof is in the Appendix.

Given the equilibrium expected effort in Lemma 2, we are able to solve for the effort maximizing contest under an arbitrary finite type-space for a budget-constrained designer.

Theorem 2. *Suppose there are N agents, each with a private type drawn from a finite type-space $\Theta = \{\theta_1, \ldots, \theta_K\}$ according to distribution $p = (p_1, p_2, \ldots, p_K)$. Among all contests $v = (v_1, \ldots, v_N)$ such that $\sum_{i=1}^N v_i \leq V$, the winner-takes-all contest $v^* = (V, 0, 0, \ldots, 0)$ maximizes expected effort.*

From Lemma 2, we know that if we increase the value of prize m by $\epsilon > 0$, the expected effort would change by exactly $\epsilon\alpha_m$. It follows that, subject to feasibility, allocating any remaining budget to the prize with the highest α_m maximizes effort. From Equation 7, we have that

$$\alpha_1 - \alpha_m = \frac{1}{N} \left[\sum_{k=1}^{K-1} \left[H^N_{\geq N-1}(P_k) - H^N_{\geq N-m}(P_k) - (m-1) H^N_{N-m}(P_k) \right] \left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}} \right) \right]. \quad (8)$$

Now as long as there are at least two distinct types in the finite type-space Θ, we can show that $\alpha_1 - \alpha_m > 0$ for any $m \in \{2, \ldots, N-1\}$, and it follows that allocating the entire budget to the first prize is strictly optimal for maximizing expected effort. In comparison, if the finite type-space contains only a single type, then we again have from Lemma 2 that $\alpha_1 = \alpha_2 = \cdots = \alpha_{N-1} = \frac{1}{N\theta_1}$ and thus, any allocation of the budget among the top $N-1$ prizes results in the same expected equilibrium effort. In particular, it follows that as soon as there is any little uncertainty (incomplete information) in the domain, allocation the entire budget to the first prize is strictly optimal.

Thus, Theorem 2 extends the optimality of the winner-takes-all contest in the incomplete information domain for the continuum type-space (Moldovanu and Sela [19]) to the finite-type space domain, while also simultaneously illustrating how the structure of the contest does not influence expected effort in the complete information domain (Barut and Kovenock [1]). In this sense, our analysis of the contest design problem in the finite type-space domain not only provides a bridge between the previous literature in complete information and incomplete information domains, it also provides a unifying approach to studying these domains simultaneously.

Next, we illustrate our results using a specific example with $N = 3$ agents.

Example 1. *Suppose there are $N = 3$ agents, each with a private type drawn from a finite type-space $\Theta = \{2, 1\}$ according to distribution $p = (0.5, 0.5)$. For any contest $\mathbf{v} = (v_1, v_2, 0)$, the equilibrium distribution functions are*

$$F_1(x_1) = \frac{-2v_2 + 2\sqrt{v_2^2 + (v_1 - 2v_2)2x_1}}{(v_1 - 2v_2)} \text{ and } F_2(x_2) = \frac{-v_1 + \sqrt{v_1^2 + 4(v_1 - 2v_2)(x_2 - b_1)}}{(v_1 - 2v_2)},$$

where $b_1 = \dfrac{v_1 + 2v_2}{8}$. And the expected efforts are

$$\mathbb{E}[X_1] = \frac{v_1 + 4v_2}{24} \text{ and } \mathbb{E}[X_2] = \frac{11v_1 + 2v_2}{24}$$

so that the expected effort of an arbitrary agent is

$$\mathbb{E}[X] = \frac{1}{2}\mathbb{E}[X_1] + \frac{1}{2}\mathbb{E}[X_2] = \frac{12v_1 + 6v_2}{48}.$$

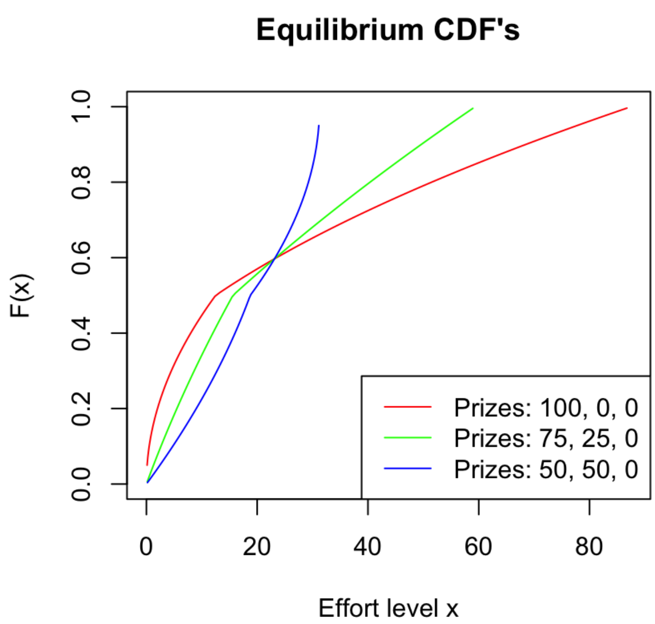

Figure 1: The equilibrium distribution functions, $F(.)$, under three different prize vectors for the environment in Example 1.

5 Continuum Type-Space Convergence

In this section, we study the limit properties of the symmetric equilibrium of finite type-space domain, focusing in particular on whether the equilibrium behavior under a continuum type-space is close to the equilibrium behavior under a sufficiently large finite type-space that approximates the continuum type-space. First, we note the symmetric equilibrium under a continuum type-space, defined by a differentiable distribution function G (Moldovanu and Sela [19]).

Lemma 3. *Suppose there are N agents, each with a private type drawn from $\Theta = [\underline{\theta}, \overline{\theta}]$ according to a differentiable CDF $G : [\underline{\theta}, \overline{\theta}] \to [0, 1]$. For any contest $v = \{v_1, v_2, \ldots, v_{N-1}, 0\}$, there is a unique symmetric Bayes-Nash equilibrium and it is such that for any $\theta \in \Theta$,*

$$X(\theta) = \int_{\theta}^{\overline{\theta}} \frac{\pi_v'(1 - G(t))g(t)}{t} dt.$$

Next, we show that for any continuum type space and distribution over this type-space, if we take a sequence of finite type-space distributions that converge to this distribution, the corresponding sequence of mixed-strategy equilibrium converges to the pure-strategy equilibrium under the continuum type-space.

Theorem 3. *Suppose there are N agents and consider a fixed contest $v = (v_1, v_2, \ldots, v_{N-1}, 0)$. Let $G : [\underline{\theta}, \overline{\theta}] \to [0, 1]$ be a differentiable CDF and let $G^1, G^2, \ldots,$ be any sequence of CDF's, each with a finite support, such that for all $\theta \in [\underline{\theta}, \overline{\theta}]$,*

$$\lim_{n\to\infty} G^n(\theta) = G(\theta).$$

Let $F^n : \mathbb{R} \to [0, 1]$ denote CDF of the equilibrium effort under the finite type-space distribution G^n, and let $F : \mathbb{R} \to [0, 1]$ denote CDF of the equilibrium under continuum type-space distribution G. Then, the sequence of CDF's $F^1, F^2, \ldots,$ converges to the CDF F, i.e., for all $x \in \mathbb{R}$,

$$\lim_{n\to\infty} F^n(x) = F(x).$$

Intuitively, as the finite type-space becomes large, the interval over which an agent of a certain type mixes shrinks, and essentially converges to the effort level prescribed by the pure-strategy equilibrium under the continuum type-space. Thus, the equilibrium strategy in an appropriate and sufficiently large finite-type space domain provides a reasonable approximation to the equilibrium strategy under the continuum type-space, and vice versa. This suggests that the insights derived from investigating equilibrium properties in one domain might extend to the other, as also observed with the optimality of the winner-takes-all contest in Theorem 2.

6 Conclusion

We study the canonical contest design problem in an incomplete information environment with a finite type-space. We characterize the unique symmetric Bayes-Nash equilibrium under any arbitrary contest with a finite type-space. We find that the equilibrium is in mixed strategies, and it is such that agents of adjacent types mix over disjoint but connected intervals so that more efficient agents always exert greater effort. Even though the equilibrium is in mixed strategies, we are able to exploit its monotonic structure to obtain a tractable representation for the expected equilibrium effort of an arbitrary agent. Using this representation, we find that a budget-constrained designer should allocate its entire budget to the first prize, and thus, run a winner-takes-all contest, in order to maximize expected equilibrium effort of an arbitrary agent. Our results extend the well-known optimality of winner-takes-all contest under a continuum type-space to the finite type-space environment.

In our analysis of contests in finite type-space environment, we introduce some new techniques and we hope that the results and methods in this paper will encourage further research in this fundamental domain. In particular, one could study the contest design problem with more general cost functions, and also perhaps other variants that have been previously explored in the literature dealing with the continuum type-space. Since our techniques rely on the separability of the reward and the costs in the utility function, we believe that the structure of equilibrium might be robust to some of these other variants, including for instance, convex cost functions. In addition, we believe that the finite type-space model presents a more convenient framework for experiments as compared to the continuum type-space, and thus, we hope to also inspire more experimental research investigating some of the theoretical predictions in the literature on contest design with incomplete information.

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A Proofs for Section 3 (Equilibrium)

Lemma 1. *Suppose there are N agents, each with a private type drawn from a finite type-space $\Theta = \{\theta_1, \ldots, \theta_K\}$ according to distribution $p = (p_1, p_2, \ldots, p_K)$. For any contest $v = \{v_1, v_2, \ldots, v_{N-1}, 0\}$, there is a unique symmetric Bayes-Nash equilibrium. Moreover, the equilibrium is such that there exist boundary points $b_1 < b_2 < \cdots < b_K$ so that for any $\theta_k \in \Theta$, an agent of type θ_k mixes between $[b_{k-1}, b_k]$ with $b_0 = 0$.*

Proof. Let F_k denote the equilibrium distribution function of agent of type θ_k for $k \in \{1, 2, \ldots, K\}$. Further, suppose F_k has support on the interval $[a_k, b_k]$ and u_k denotes the expected payoff of an agent of type θ_k when all agents play the equilibrium profile $(F_1, F_2, \ldots, F_K)$. Also let $F : \mathbb{R}_+ \to [0, 1]$ denote the CDF of the equilibrium effort under $(F_1, F_2, \ldots, F_K)$.

1. We first show that F_k cannot have any atoms. Suppose instead that F_k is such that an agent of type θ_k plays x_k with positive probability. Then, there is a positive probability that all agents are tied at effort x_k. But then, an agent of type θ_k will be strictly better off by bidding $x_k + \epsilon$ instead of x_k. This way, the agent earns the best prize among those that would have been otherwise split randomly between the tied agents and only pays an additional ϵ. Thus, F_k cannot have any atoms.

2. We now show that $\min\{a_1, a_2, \ldots, a_K\} = 0$. Suppose instead that this equals $a_k > 0$. Now when an agent of type θ_k plays a_k, it does not win any prize but it pays a positive cost of $\theta_k a_k$. So this agent can instead play 0 and while it still doesn't get a prize, it also doesn't pay any cost. Thus, it must be that $\min\{a_1, a_2, \ldots, a_K\} = 0$.

3. We now show that $u_1 \leq u_2 \leq \cdots \leq u_K$. Suppose instead that $u_k > u_{k+1}$ for some $k \in \{1, 2, \ldots, K-1\}$. Also let $x_k \in [a_k, b_k]$. Then, we have that

$$u_k = \pi_v(F(x_k)) - \theta_k x_k.$$

So suppose agent of type θ_{k+1} plays x_k. Its payoff will be

$$\pi_v(F(x_k)) - \theta_{k+1} x_k > \pi_v(t) - \theta_k x_k = u_k$$

because $\theta_{k+1} < \theta_k$. Thus, this agent of type θ_{k+1} can imitate an agent of type θ_k and get strictly higher payoff. Thus, it must be that $u_1 \leq u_2 \leq \cdots \leq u_K$.

4. We now show that for any $j \neq k$, $|[a_k, b_k] \cap [a_j, b_j]| \leq 1$. Suppose instead that $x, y \in [a_k, b_k] \cap [a_j, b_j]$ and $x \neq y$. Since agents must be indifferent between all actions in their support, it must be that

$$u_k = \pi_v(F(x)) - \theta_k x = \pi_v(F(y)) - \theta_k y$$

and also

$$u_j = \pi_v(F(x)) - \theta_j x = \pi_v(F(y)) - \theta_j y.$$

But this implies

$$\pi_v(F(x)) - \pi_v(F(y)) = \theta_k(x-y) = \theta_j(x-y)$$

which is a contradiction.

5. We now show that if $b_k \neq \max\{b_1, b_2, \dots, b_K\}$, then $b_k = a_j$ for some $j \in \{1, 2, \dots, K\}$. Suppose instead that there is a k such that $b_k \neq \max\{b_1, b_2, \dots, b_K\}$ and $a_j \neq b_k$ for any $j \in \{1, 2, \dots, K\}$. Let a_p denote the minimum of all a_j such that $a_j \geq b_k$. Now we can repeat the argument in (2) to show that a_p must be equal to b_k because otherwise, an agent of type θ_p would be strictly better off playing b_k instead of a_p. Thus, we have that the support intervals of the mixed strategies are connected.

6. We now show that if $b_k = a_j$, then $\theta_k \geq \theta_j$. Suppose instead that $\theta_k < \theta_j$. First note that,

$$u_k = \pi_v(F(a_k)) - \theta_k a_k = \pi_v(F(b_k)) - \theta_k b_k.$$

Since $b_k = a_j$, we have that

$$u_j = \pi_v(F(b_k)) - \theta_j b_k = u_k + b_k(\theta_k - \theta_j).$$

Now the payoff of agent of type θ_j from playing $a_k < b_k = a_j$ will be

$$\pi_v(F(a_k)) - \theta_j a_k = u_k + (\theta_k - \theta_j)a_k$$

which is greater than u_j if $\theta_k < \theta_j$. Thus, it must be that $\theta_k \geq \theta_j$.

Together, the above steps imply the result in the Lemma.

□

B Proofs for Section 4 (Optimal contest)

Lemma 2. *Suppose there are N agents, each with a private type drawn from a finite type-space $\Theta = \{\theta_1, \dots, \theta_K\}$ according to distribution $p = (p_1, p_2, \dots, p_K)$. Under a contest $v = \{v_1, v_2, \dots, v_{N-1}, 0\}$, the expected equilibrium effort of an arbitrary agent is*

$$\mathbb{E}[X] = \sum_{m=1}^{N-1} v_m \alpha_m$$

where

$$\alpha_m = \frac{1}{N}\left[\frac{1}{\theta_K} + \sum_{k=1}^{K-1}\left[H^N_{\geq N-m}(P_k) + (m-1)H^N_{N-m}(P_k)\right]\left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}}\right)\right]. \tag{7}$$

Proof. To find expected equilibrium effort of an arbitrary agent, we first find the expected effort exerted in equilibrium by an agent of type θ_k. To find $\mathbb{E}[X_k]$, we have from Theorem 1 that for an agent of type θ_k, the (random) level of effort X_k it exerts in equilibrium satisfies

$$\pi_v(P_{k-1} + p_k F_k(X_k)) - \theta_k X_k = u_k.$$

Taking expectation on both sides, we get

$$\mathbb{E}\left[\pi_v(P_{k-1} + p_k F_k(X_k))\right] - \theta_k \mathbb{E}[X_k] = u_k$$

which implies

$$\mathbb{E}[X_k] = \frac{\mathbb{E}[\pi_v(P_{k-1} + p_k F_k(X_k))] - u_k}{\theta_k}.$$

Notice that

$$\mathbb{E}[\pi_v(P_{k-1} + p_k F_k(X_k))] = \mathbb{E}\left[\sum_{m=1}^{N-1} v_m H_{N-m}^{N-1}(P_{k-1} + p_k F_k(X_k))\right]$$

is simply the ex-ante expected value of the prize awarded to this agent of type θ_k. While an explicit calculation of this expectation using the definition of $F_k(.)$ in Equation 1 from Theorem 1 appears complicated, we use the following approach instead.

Let V_k denote the ex-ante total prize awarded to agents of type θ_k. By linearity of expectation, we have that

$$\mathbb{E}[V_k] = N p_k \mathbb{E}[\pi_v(P_{k-1} + p_k F_k(X_k))].$$

By an alternative calculation, we also have that

$$\begin{aligned}
\mathbb{E}[V_k] &= \sum_{m=1}^{N-1} v_m \Pr[\text{Prize m is awarded to agent of type } \theta_k] \\
&= \left[\sum_{m=1}^{N-1} v_m \left(H_{\geq N-m+1}^{N}(P_k) - H_{\geq N-m+1}^{N}(P_{k-1})\right)\right]
\end{aligned}$$

Notice that $H_{\geq N-m+1}^{N}(P_k)$ is the probability that there are at least $N-m+1$ agents out of N whose type is in $\{\theta_1, \theta_2, \ldots, \theta_k\}$, which is both necessary and sufficient for prize m to be awarded to an agent whose type is in $\{\theta_1, \theta_2, \ldots, \theta_k\}$. From this, we subtract the probability that prize m is awarded to an agent whose type is in $\{\theta_1, \theta_2, \ldots, \theta_{k-1}\}$, given by $H_{\geq N-m+1}^{N}(P_{k-1})$, which leaves us with the probability that prize m is awarded to an agent of type θ_k.[4] Equating the two alternative expressions for $\mathbb{E}[V_k]$, we get that the required ex-ante expected value of the prize awarded to this agent of type θ_k is

$$\mathbb{E}[\pi_v(P_{k-1} + p_k F_k(X_k))] = \frac{1}{N p_k}\left[\sum_{m=1}^{N-1} v_m \left(H_{\geq N-m+1}^{N}(P_k) - H_{\geq N-m+1}^{N}(P_{k-1})\right)\right].$$

[4] An alternate representation is $\mathbb{E}[V_k] = \left[\sum_{m=1}^{N-1} v_m \left(H_{\geq m}^{N}(1-P_{k-1}) - H_{\geq m}^{N}(1-P_k)\right)\right]$

Now we can plug this back in the equation for $\mathbb{E}[X_k]$ and use the definition of u_k from Equation 3 to get that

$$\begin{aligned}
\mathbb{E}[X_k] &= \frac{\mathbb{E}[\pi_v(P_{k-1} + p_k F_k(X_k))] - u_k}{\theta_k} \\
&= \frac{1}{\theta_k}\left[\frac{1}{Np_k}\left[\sum_{m=1}^{N-1} v_m \left(H^N_{\geq N-m+1}(P_k) - H^N_{\geq N-m+1}(P_{k-1})\right)\right]\right] - \sum_{m=1}^{N-1} v_m \left[\sum_{j=1}^{k-1} H^{N-1}_{N-m}(P_j)\left[\frac{1}{\theta_{j+1}} - \frac{1}{\theta_j}\right]\right]
\end{aligned}$$

Observe that we can write

$$\mathbb{E}[X_k] = \sum_{m=1}^{N-1} v_m \alpha_{mk}$$

where

$$\alpha_{mk} = \left[\sum_{j=1}^{k-1} H^{N-1}_{N-m}(P_j)\left(\frac{1}{\theta_j} - \frac{1}{\theta_{j+1}}\right)\right] + \frac{(H^N_{\geq N-m+1}(P_k) - H^N_{\geq N-m+1}(P_{k-1}))}{N\theta_k p_k}.$$

It follows then that

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k=1}^{K} p_k \mathbb{E}[X_k] \\
&= \sum_{m=1}^{N-1} v_m \alpha_m
\end{aligned}$$

where

$$\begin{aligned}
\alpha_m &= \sum_{k=1}^{K} p_k \alpha_{mk} \\
&= \sum_{k=1}^{K} p_k \left[\left[\sum_{j=1}^{k-1} H^{N-1}_{N-m}(P_j)\left(\frac{1}{\theta_j} - \frac{1}{\theta_{j+1}}\right)\right] + \frac{(H^N_{\geq N-m+1}(P_k) - H^N_{\geq N-m+1}(P_{k-1}))}{N\theta_k p_k}\right] \\
&= \sum_{k=1}^{K} p_k \left[\sum_{j=1}^{k-1} H^{N-1}_{N-m}(P_j)\left(\frac{1}{\theta_j} - \frac{1}{\theta_{j+1}}\right)\right] + \sum_{k=1}^{K} \frac{(H^N_{\geq N-m+1}(P_k) - H^N_{\geq N-m+1}(P_{k-1}))}{N\theta_k} \\
&= \sum_{k=1}^{K-1} (1 - P_k)\left[H^{N-1}_{N-m}(P_k)\left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}}\right)\right] + \frac{1}{N}\left[\frac{1}{\theta_K} + \sum_{k=1}^{K-1} H^N_{\geq N-m+1}(P_k)\left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}}\right)\right] \\
&= \frac{m}{N}\sum_{k=1}^{K-1}\left[H^N_{N-m}(P_k)\left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}}\right)\right] + \frac{1}{N}\left[\frac{1}{\theta_K} + \sum_{k=1}^{K-1} H^N_{\geq N-m+1}(P_k)\left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}}\right)\right] \\
&= \frac{1}{N}\left[\frac{1}{\theta_K} + \sum_{k=1}^{K-1} (H^N_{\geq N-m+1}(P_k) + mH^N_{N-m}(P_k))\left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}}\right)\right]
\end{aligned}$$

$\square$

Theorem 2. *Suppose there are N agents, each with a private type drawn from a finite type-space $\Theta = \{\theta_1, \dots, \theta_K\}$ according to distribution $p = (p_1, p_2, \dots, p_K)$. Among all contests $v = (v_1, \dots, v_N)$ such that $\sum_{i=1}^{N} v_i \leq V$, the winner-takes-all contest $v^* = (V, 0, 0, \dots, 0)$ maximizes expected effort.*

Proof. For any contest v, we know from Lemma 2 that the expected equilibrium effort of an arbitrary agent is

$$\mathbb{E}[X] = \sum_{m=1}^{N-1} v_m \alpha_m$$

where

$$\alpha_m = \frac{1}{N}\left[\frac{1}{\theta_K} + \sum_{k=1}^{K-1}\left[H^N_{\geq N-m}(P_k) + (m-1)H^N_{N-m}(P_k)\right]\left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}}\right)\right].$$

Thus, α_m represents the change in expected equilibrium effort from increasing the value of the mth prize. And $\alpha_1 - \alpha_m$ represents the change in expected effort from transferring value from the mth prize to the first prize. We will now show that this change is always positive, so that transferring value from any arbitrary prize to the first prize always increases effort. To see this, observe that for any $m \in \{2, \dots, N-1\}$,

$$\alpha_1 - \alpha_m = \frac{1}{N}\left[\sum_{k=1}^{K-1}\left[H^N_{\geq N-1}(P_k) - H^N_{\geq N-m}(P_k) - (m-1)H^N_{N-m}(P_k)\right]\left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}}\right)\right]$$

Notice that for $m \in \{2, \dots, N-1\}$ and each $k \in \{1, 2, \dots, K-1\}$,

$$\left[H^N_{\geq N-1}(P_k) - H^N_{\geq N-m}(P_k) - (m-1)H^N_{N-m}(P_k)\right] < 0,$$

and also

$$\left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}}\right) < 0$$

since $\theta_k > \theta_{k+1}$. It follows then that for each $k \in \{1, 2, \dots, K-1\}$,

$$\left[H^N_{\geq N-1}(P_k) - H^N_{\geq N-m}(P_k) - (m-1)H^N_{N-m}(P_k)\right]\left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}}\right) > 0$$

which implies that

$$\frac{1}{N}\left[\sum_{k=1}^{K-1}\left[H^N_{\geq N-1}(P_k) - H^N_{\geq N-m}(P_k) - (m-1)H^N_{N-m}(P_k)\right]\left(\frac{1}{\theta_k} - \frac{1}{\theta_{k+1}}\right)\right] > 0.$$

Thus, $\alpha_1 - \alpha_m > 0$ for each $m \in \{2, 3, \dots, N-1\}$. It follows that the winner-takes-all contest maximizes expected effort. $\square$

C Proofs for Section 5 (Continuum Type-Space Convergence)

Lemma 3. *Suppose there are N agents, each with a private type drawn from $\Theta = [\underline{\theta}, \overline{\theta}]$ according to a differentiable CDF $G : [\underline{\theta}, \overline{\theta}] \to [0,1]$. For any contest $v = \{v_1, v_2, \ldots, v_{N-1}, 0\}$, there is a unique symmetric Bayes-Nash equilibrium and it is such that for any $\theta \in \Theta$,*

$$X(\theta) = \int_{\theta}^{\overline{\theta}} \frac{\pi_v'(1 - G(t))g(t)}{t} dt.$$

Proof. Suppose $N-1$ agents are playing a strategy $X : [\underline{\theta}, \overline{\theta}] \to \mathbb{R}_+$ so that if an agent's type is θ, it exerts effort $X(\theta)$. Further suppose that $X(\theta)$ is decreasing in θ. Now we want to find the remaining agent's best response to this strategy of the other agents. If the agent's type is θ and it pretends to be an agent of type $t \in [\underline{\theta}, \overline{\theta}]$, its payoff is

$$\pi_v(1 - G(t)) - \theta X(t).$$

Taking the first order condition, we get

$$\pi_v'(1 - G(t))(-g(t)) - \theta X'(t) = 0.$$

Now we can plug in $t = \theta$ to get the condition for $X(\theta)$ to be a symmetric Bayes-Nash equilibrium. Doing so, we get

$$\pi_v'(1 - G(\theta))(-g(\theta)) - \theta X'(\theta) = 0$$

so that

$$X(\theta) = \int_{\theta}^{\overline{\theta}} \frac{\pi_v'(1 - G(t))g(t)}{t} dt.$$

□

Theorem 3. *Suppose there are N agents and consider a fixed contest $v = (v_1, v_2, \ldots, v_{N-1}, 0)$. Let $G : [\underline{\theta}, \overline{\theta}] \to [0,1]$ be a differentiable CDF and let $G^1, G^2, \ldots,$ be any sequence of CDF's, each with a finite support, such that for all $\theta \in [\underline{\theta}, \overline{\theta}]$,*

$$\lim_{n\to\infty} G^n(\theta) = G(\theta).$$

Let $F^n : \mathbb{R} \to [0,1]$ denote CDF of the equilibrium effort under the finite type-space distribution G^n, and let $F : \mathbb{R} \to [0,1]$ denote CDF of the equilibrium under continuum type-space distribution G. Then, the sequence of CDF's $F^1, F^2, \ldots,$ converges to the CDF F, i.e., for all $x \in \mathbb{R}$,

$$\lim_{n\to\infty} F^n(x) = F(x).$$

Proof. For the finite support CDF G^n, let $\Theta^n = (\theta_1^n, \theta_2^n, \ldots, \theta_{K(n)}^n)$ denote the support and $p^n = (p_1^n, p_2^n, \ldots, p_{K(n)}^n)$ denote the probability mass function. From Theorem 1, let $b^n = (b_1^n, b_2^n, \ldots, b_{K(n)}^n)$ denote the boundary points, $u^n = (u_1^n, u_2^n, \ldots, u_{K(n)}^n)$ denote the equilibrium utilities, and F_k^n denote the equilibrium CDF of agent of type θ_k^n on support $[b_{k-1}^n, b_k^n]$. Then, the CDF of the equilibrium effort, $F^n : \mathbb{R} \to [0,1]$, is such that for any $x \in \mathbb{R}$,

$$F^n(x) = \begin{cases} 0 & \text{if } x \le 0 \\ P_{k-1}^n + p_k^n F_k^n(x) & \text{if } x \in [b_{k-1}^n, b_k^n] \\ 1 & \text{if } x \ge b_{K(n)}^n \end{cases}. \tag{9}$$

For the continuum CDF $G : [\underline{\theta}, \overline{\theta}] \to [0,1]$, the CDF of the equilibrium effort, $F : \mathbb{R} \to [0,1]$, is such that for any $x \in \mathbb{R}$,

$$F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 - G(\theta(x)) & \text{if } x \in [0, B] \\ 1 & \text{if } x \ge B \end{cases}. \tag{10}$$

where $\theta(x)$ is the inverse of $X(\theta)$ (from Lemma 3) and $B = X(\underline{\theta})$. The following Lemma will be the key to showing that $F^n(x)$ converges to $F(x)$ for all $x \in \mathbb{R}$.

Lemma 4. *Consider any $\theta \in (\underline{\theta}, \overline{\theta})$ and for any $n \in \mathbb{N}$, let $k(n) \in \{0, 1, 2, \ldots, K(n)\}$ be such that $\theta_{k(n)}^n > \theta \ge \theta_{k(n)+1}^n$ (where $\theta_0^n = \infty$ and $\theta_{K(n)+1}^n = 0$). Then,*

$$\lim_{n\to\infty} b_{k(n)}^n = X(\theta) \text{ and } \lim_{n\to\infty} F^n(b_{k(n)}^n) = 1 - G(\theta).$$

Proof. From Lemma 3 and Equation 2, we have

$$X(\theta) = \int_{\theta}^{\overline{\theta}} \frac{\pi_v'(1 - G(t))g(t)}{t} dt \text{ and } b_{k(n)}^n = \sum_{j=1}^{k(n)} \frac{\pi_v(P_j^n) - \pi_v(P_{j-1}^n)}{\theta_j^n}.$$

Observe that

$$\begin{aligned} b_{k(n)}^n &= \left[\frac{\pi_v(P_{k(n)}^n)}{\theta_{k(n)}^n} - \sum_{j=1}^{k(n)-1} \pi_v(P_j^n) \left[\frac{1}{\theta_{j+1}^n} - \frac{1}{\theta_j^n} \right] \right] \\ &= \int_0^{1/\theta_{k(n)}^n} \left[\pi_v(P_{k(n)}^n) - \pi_v(1 - G^n(1/x)) \right] dx \\ &\xrightarrow{n\to\infty} \int_0^{\frac{1}{\theta}} \left[\pi_v(1 - G(\theta)) - \pi_v(1 - G(1/x)) \right] dx \quad \text{(dominated convergence)} \\ &= \underbrace{\left[x(\pi_v(1 - G(\theta)) - \pi_v(1 - G(1/x))) \right]_0^{\frac{1}{\theta}}}_{\text{this is } 0} + \int_0^{\frac{1}{\theta}} \frac{\pi_v'(1 - G(1/x))g(1/x)}{x} dx \end{aligned}$$

$$= \int_{\theta}^{\infty} \frac{\pi_v'(1 - G(t))g(t)}{t} \text{ d}t \qquad \text{(substitute } t = 1/x\text{)}$$
$$= X(\theta)$$

By definition, we have

$$\begin{aligned}
\lim_{n\to\infty} F^n(b^n_{k(n)}) &= \lim_{n\to\infty} P^n_{k(n)} \\
&= \lim_{n\to\infty} [1 - G^n(\theta)] \\
&= 1 - G(\theta)
\end{aligned}$$

□

Returning to the proof of Theorem 3, fix any $x \in (0, B)$ and let $\theta \in (\underline{\theta}, \overline{\theta})$ be such that $X(\theta) = x$. Then, we know that

$$F(x) = 1 - G(\theta).$$

We want to show that

$$\lim_{n\to\infty} F^n(x) = 1 - G(\theta).$$

Take $\epsilon > 0$. Find $\theta' < \theta$ and $\theta'' > \theta$ such that

$$0 < G(\theta) - G(\theta') = G(\theta'') - G(\theta) < \frac{\epsilon}{4}.$$

Let $x' = X(\theta')$, $x'' = X(\theta'')$, so that $x' > x > x''$. Let $\delta = \min\{x' - x, x - x''\}$. From Lemma 4, let $N(\epsilon)$ be such that for all $n > N(\epsilon)$,

$$\max\{|b^n_{k(n)} - x|, |b^n_{k'(n)} - x'|, |b^n_{k''(n)} - x''|\} < \frac{\delta}{2}$$

and

$$\max\{|F^n(b^n_{k'(n)}) - (1 - G(\theta'))|, |F^n(b^n_{k''(n)}) - (1 - G(\theta''))|\} < \frac{\epsilon}{4},$$

where $k(n), k'(n), k''(n)$ are sequences as defined in Lemma 4 for θ, θ' and θ'' respectively. Then, for all $n > N(\epsilon)$,

$$\begin{aligned}
F^n(x) &> F^n(b^n_{k''(n)}) \\
&> 1 - G(\theta'') - \frac{\epsilon}{4} \\
&> 1 - G(\theta) - \frac{\epsilon}{2}
\end{aligned}$$

and

$$\begin{aligned}
F^n(x) &< F^n(b^n_{k'(n)}) \\
&< 1 - G(\theta') + \frac{\epsilon}{4} \\
&< 1 - G(\theta) + \frac{\epsilon}{2}
\end{aligned}$$

so that $|F^n(x) - (1 - G(\theta))| < \epsilon$. Thus, $\lim_{n\to\infty} F^n(x) = 1 - G(\theta) = F(x)$ for all $x \in \mathbb{R}$. □