TTC Domains

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January 26, 2025
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Abstract

We study the classical object reallocation problem under strict preferences, with a focus on characterizing "TTC domains" – preference domains on which the Top Trading Cycles (TTC) mechanism is the unique mechanism satisfying individual rationality, Pareto efficiency, and strategyproofness. We introduce a sufficient condition for a domain to be a TTC domain, which we call the *top-two condition*. This condition requires that, within any subset of objects, if two objects can each be most-preferred, they can also be the top-two most-preferred objects (in both possible orders). A weaker version of this condition, applying only to subsets of size three, is shown to be necessary. These results provide a complete characterization of TTC domains for the case of three objects, unify prior studies on specific domains such as single-peaked and single-dipped preferences, and classify several previously unexplored domains as TTC domains or not.

1 Introduction

This paper studies the object reallocation problem, first introduced by Shapley and Scarf [25], from a mechanism design perspective. There is a group of agents, each of whom owns an indivisible object. Each agent has a strict preference over the objects, which is their private information. A mechanism specifies how the objects are reallocated based on agents' reported preferences. The goal is to identify mechanisms that satisfy three key properties: individual rationality, Pareto efficiency, and strategyproofness.

In a seminal paper, Ma [13] showed that, on the unrestricted preference domain, the Top Trading Cycles (TTC) mechanism is the unique mechanism satisfying these three properties. This result sparked a substantial body of literature exploring the existence of other mechanisms satisfying the three properties under various natural domain restrictions, such

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as single-peaked and single-dipped domains. For the single-peaked domain, Bade [3] introduced a new mechanism called the "Crawler", which satisfies the three properties. However, when this single-peaked domain is further restricted to allow only two adjacent objects to be most-preferred, the Crawler actually coincides with the TTC, and TTC becomes the unique mechanism satisfying the three properties (Tamura [28]). For the single-dipped domain also, Tamura [29] showed that TTC remains the unique mechanism satisfying the three properties.

These findings on different specific domains raise a broader question: what underlying property of preference domains determines if they admit TTC as the only mechanism that satisfies the three properties? To address this question, we adopt a more abstract approach to studying restricted domains. In our framework, the preference domain - the set of admissible preferences - is any arbitrary subset of the set of all strict linear orders. It follows from results in the unrestricted domain that for any preference domain, TTC is individually rational, Pareto efficient, and strategyproof. We aim to characterize domains where TTC is the unique mechanism satisfying these properties, and going forward, refer to such domains as "TTC domains."

Our first main result (Theorem 1) provides a sufficient condition, termed the *top-two condition*, for a preference domain to be a TTC domain. A domain satisfies the top-two condition if, for any subset of objects and any pair of objects in that subset, if there exist preferences in which each object in the pair is the most-preferred among the subset, then there must also exist preferences in the domain where one object in the pair is the most-preferred and the other is the second most-preferred within the subset. It is straightforward to verify that the unrestricted domain, the single-peaked domain with only two adjacent peaks, and the single-dipped domain satisfy this condition. Thus, previous results regarding these domains being TTC domains follow as direct corollaries of Theorem 1.

Our second main result (Theorem 2) is a partial converse to Theorem 1: if a preference domain does not satisfy a weak version of the top-two condition, it is not a TTC domain. This weak version, which we call the *weak top-two condition*, requires the top-two property to hold only for subsets of size three. Thus, if there exists a triple of objects within which a pair of objects can be most-preferred but cannot be the top-two most-preferred objects in some order, the domain admits a non-TTC mechanism satisfying the three properties. We can verify that the single-peaked domain does not satisfy this weak top-two condition, and thus, is not a TTC domain.

Beyond offering a unifying perspective on previously studied domains, Theorems 1 and 2 also enable the classification of several previously unexplored domains as TTC domains or not. For instance, consider a preference domain defined by a partial order over the set of objects, so that a strict preference belongs to the domain if and only if it is consistent with the partial order. The partial order may represent a dominance relation between the objects, capturing the idea that it is inconceivable for an agent to hold a preference that contradicts this dominance relation. It can be verified that for any given partial order, the resulting preference domain satisfies the top-two condition, and by Theorem 1, is a TTC domain. Together, Theorems 1 and 2 fully characterize TTC domains for the special case of three objects, and more generally, offer a useful criterion to determine whether any given preference domain is a TTC domain or not.

Related work

The TTC mechanism has been shown to be fundamental to object reallocation problem from various different perspectives. Shapley and Scarf [25], who introduced the problem in the context of an exchange economy with indivisible goods, proposed the TTC algorithm (credited to David Gale) as a method for finding an allocation in the core. Roth and Postlewaite [21] later showed that the TTC allocation is, in fact, the unique such allocation. Focusing on incentives, Roth [20] showed that the TTC mechanism is strategyproof. Furthermore, by imposing the additional axioms of individual rationality and Pareto efficiency, Ma [13], in a seminal contribution, characterized TTC as the unique mechanism satisfying all three properties.[1] Subsequently, a substantial body of work has proposed alternative characterizations of the TTC mechanism using different axioms, including group strategyproofness (Bird [4], Takamiya [27]), independence of irrelevant rankings (Morrill [15]), non-bossiness (Miyagawa [14], Ehlers [6]), endowments-swapping-proofness (Fujinaka and Wakayama [10]), and more recently, pair-efficiency (Ekici [7], Ekici and Sethuraman [8]).

This paper contributes to the growing literature in object reallocation problems on characterizing mechanisms with desirable properties under restricted preference domains. As mentioned earlier, some domains that have been explored include the single-peaked domain (Bade [3], Tamura and Hosseini [30], Tamura [28]) and the single-dipped domain (Tamura [29]). In other related work, Nicolo and Rodriguez-Alvarez [17], Fujinaka and Wakayama [11] investigate the common ranking domain, restricting attention to allocations obtained through constrained exchanges. Many of these restricted domains originate in social choice theory, where they have been extensively studied as methods to circumvent impossibility results. A recent survey by Elkind, Lackner, and Peters [9] provides an overview of domain restrictions explored in social choice, some of which may also hold relevance in the context of object reallocation problems.[2]

[1] Svensson [26], Anno [2], Sethuraman [24], Bade [3] offer alternative proofs of this result.

[2] Other related streams of literature have focused on reallocation problems tailored to specific environments (Abdulkadiroğlu and Sönmez [1], Roth, Sönmez, and Ünver [22], Schummer and Vohra [23]) and object allocation problems (Carroll [5], Hylland and Zeckhauser [12], Pápai [18], Pycia and Ünver [19]). Morrill and Roth [16] provide a recent survey of this literature, highlighting the relevance of TTC in these environments.

2 Model

Preliminaries

Let $N = \{1, \ldots, n\}$ be a finite set of agents. Let $O = \{o_1, \ldots, o_n\}$ be a finite set of indivisible objects such that o_i denotes agent i's endowment. Agents have strict preferences over objects. We denote by $\mathcal{P}$ the set of all strict linear orders over O, and we let $\mathcal{D} \subset \mathcal{P}$ denote the preference domain. Let $P = (P_i)_{i \in N} \in \mathcal{D}^N$ denote a preference profile where $P_i \in \mathcal{D}$ denotes agent i's preference over O. Following standard convention, for $S \subset N$, we let $P_S = (P_i)_{i \in S}$, $P_{-S} = (P_i)_{i \in N \setminus S}$. For each $P_0 \in \mathcal{D}$, we denote by R_0 the "at least as desirable as" relation associated with P_0, i.e., for each pair $o, o' \in O$, $o \; R_0 \; o'$ if and only if either $o \; P_0 \; o'$ or $o = o'$. We refer to a set of agents, their endowments, and their preferences over these objects, as a *market*.

An *allocation* $x : N \to O$ is a bijection that assigns to each agent an object. Let $\mathcal{X}$ be the set of allocations. For each $x \in \mathcal{X}$ and each $i \in N$, we denote by $x_i \in O$ the assignment of agent i under the allocation x.

A *mechanism* $\varphi : \mathcal{D}^N \to \mathcal{X}$ associates with each preference profile $P \in \mathcal{D}^N$ an allocation $x \in \mathcal{X}$. For $P \in \mathcal{D}^N$, $\varphi(P) \in \mathcal{X}$ denotes the allocation under preference profile P.

Axioms

We now introduce some standard properties of allocations and mechanisms.

Given $P \in \mathcal{D}^N$, an allocation $x \in \mathcal{X}$ is *individually rational* at P if for each $i \in N$, $x_i \; R_i \; o_i$. A mechanism $\varphi : \mathcal{D}^N \to \mathcal{X}$ is *individually rational (IR)* if for each $P \in \mathcal{D}^N$, the allocation $\varphi(P)$ is individually rational at P.

Given $P \in \mathcal{D}^N$, an allocation $x \in \mathcal{X}$ is *Pareto efficient* at P if there is no other allocation $y \in \mathcal{X}$ such that for each $i \in N$, $y_i \; R_i \; x_i$ and for some $j \in N$, $y_j \; P_j \; x_j$. A mechanism $\varphi : \mathcal{D}^N \to \mathcal{X}$ is *Pareto efficient (PE)* if for each $P \in \mathcal{D}^N$, the allocation $\varphi(P)$ is Pareto efficient at P.

A mechanism $\varphi : \mathcal{D}^N \to \mathcal{X}$ is *strategyproof (SP)* if for any $P \in \mathcal{D}^N$, there is no $i \in N$ and $P'_i \in \mathcal{D}$ such that $\varphi_i(P'_i, P_{-i}) \; P_i \; \varphi_i(P)$.

Top Trading Cycles

We now describe the TTC mechanism. For any strict profile $P \in \mathcal{P}^N$, the TTC algorithm finds an allocation as follows:

1. Each agent points to the agent who owns their most-preferred object.

2. In the ensuing directed graph between the agents, there is at least one cycle. All agents in a cycle are assigned their most-preferred objects and leave the market.

3. The algorithm repeats with the remaining agents and their endowments.

We let $TTC(P) \in \mathcal{X}$ denote the allocation that results from running this algorithm at profile $P \in \mathcal{P}^N$. For any preference domain $\mathcal{D} \subset \mathcal{P}$, we define the *TTC mechanism* $\varphi : \mathcal{D}^N \to \mathcal{X}$ as the mechanism that selects for any preference profile $P \in \mathcal{D}^N$ the allocation $\varphi(P) = TTC(P)$.

From previous results in the literature, we know that under any preference domain, the TTC mechanism satisfies the three properties.

Corollary 1. *For any $\mathcal{D} \subset \mathcal{P}$, the TTC mechanism is individually rational, Pareto efficient, and strategyproof.*

The literature has identified specific preference domains where the TTC mechanism is the only mechanism that satisfies these properties, as well as domains where it is not the only such mechanism. In this paper, we aim to provide a general characterization of the domains where TTC is the unique mechanism satisfying individual rationality, Pareto efficiency, and strategyproofness. We refer to such domains as "TTC domains."

Notation

We will sometimes describe a preference $P_0 \in \mathcal{D}$ by listing the objects in O in the order specified by P_0. For example, a preference $P_0 \in \mathcal{D}$ such that $o_k \; P_0 \; o_{k+1}$ for all k can be succinctly represented as $o_1 o_2 \ldots o_n$.

For any preference $P_0 \in \mathcal{D}$ and subset of objects $O' \subset O$, we let $r_k(P_0, O')$ denote the object ranked k-th under the preference P_0, restricted to the subset of objects O'. In other words, $r_k(P_0, O')$ is the k-th object in the ordered list of O' according to P_0. For example, $r_1(P_0, O')$ represents the object that is most-preferred according to P_0 among the objects in O'.

Similarly, for any subdomain of preferences $\mathcal{D}' \subset \mathcal{D}$ and subset of objects $O' \subset O$, we let $r_k(\mathcal{D}', O')$ denote the set of objects than can be ranked k-th according to some preference $P_0 \in \mathcal{D}'$ restricted to the objects in O'. Formally,

$$r_k(\mathcal{D}', O') = \{o' \in O' : \text{there exists } P_0 \in \mathcal{D}' \text{ such that } r_k(P_0, O') = o'\}.$$

For example, $r_1(\mathcal{D}, O')$ represents the set of objects that can be most-preferred according to preferences in $\mathcal{D}$ among the objects in O'.

3 Results

In this section, we present our results. We begin by introducing our main richness condition on preference domains. We will then show that this condition is sufficient, and that a weak version of this condition is necessary, for a preference domain to be a TTC domain.

3.1 Top-two condition

Our key richness condition on preference domains requires that within any subset of objects, any two objects that can be most-preferred can also be the top-two most-preferred objects (in both possible orders).

Definition 1. A preference domain $\mathcal{D} \subset \mathcal{P}$ satisfies the *top-two condition* if for any $O' \subset O$ and distinct $a, b \in r_1(\mathcal{D}, O')$, there exists a $P_0 \in \mathcal{D}$ such that

1. $a = r_1(P_0, O')$,
2. $b = r_2(P_0, O')$.

In other words, if a and b can each be most-preferred within the objects in O', there must be a preference where a is most-preferred and b is second most-preferred, and also a preference where b is most-preferred and a is second most-preferred.

We now provide some examples of preference domains and examine whether they satisfy the top-two condition. Notice that if $n \leq 2$, any $\mathcal{D} \subset \mathcal{P}$ satisfies the top-two condition. In the following examples, the first four illustrate the definition of the top-two condition, the next four focus on domains that have been previously explored in the literature, and the final two introduce important domains that have not yet been studied.

1. Suppose $n = 3$ and $\mathcal{D} = \{o_1o_2o_3, o_2o_3o_1, o_2o_1o_3\}$: Observe that only o_1 and o_2 can be most-preferred, and there are also two preferences in which they are the top-two most-preferred objects, with each object ranked first in one of the preferences. Thus, this domain satisfies the top-two condition.

2. Suppose $n = 3$ and $\mathcal{D} = \{o_1o_2o_3, o_2o_3o_1, o_3o_1o_2\}$: Observe that o_1 and o_2 can be most-preferred, but there is no preference in $\mathcal{D}$ such that o_2 is most-preferred and o_1 is second most-preferred. Thus, this domain does not satisfy the top-two condition.

3. Suppose $n = 3$ and $\mathcal{D} \subset \mathcal{P}$ is such that $|\mathcal{D}| = 5$: Since $n = 3$, $|\mathcal{P}| = 6$. Without loss of generality (WLOG), say $\mathcal{D}$ is such that only the linear order $o_1o_2o_3 \notin \mathcal{D}$. Since $|\mathcal{D}| = 5$, there exist preferences such that o_1 and o_2 can be most-preferred. However, there is no preference in $\mathcal{D}$ such that o_1 is most-preferred, and o_2 is second most-preferred. Thus, this domain does not satisfy the top-two condition.

4. Suppose $n = 4$ and $\mathcal{D} = \{o_1o_2o_3o_4, o_1o_3o_2o_4, o_2o_1o_4o_3, o_2o_4o_3o_1\}$: Observe that within $O' = \{o_1, o_3, o_4\}$, the objects o_1 and o_4 can be most-preferred, but there is no preference in $\mathcal{D}$ such that o_4 is most-preferred, and o_1 is second most-preferred, among the objects in O'. Thus, this domain does not satisfy the top-two condition.

5. Suppose n is arbitrary and $\mathcal{D}$ is the **unrestricted domain** (Ma [13]): It is easy to verify that the domain $\mathcal{D} = \mathcal{P}$ satisfies the top-two condition.

6. Suppose $n \geq 3$ and $\mathcal{D}$ is a **single-peaked domain** (Bade [3]): This domain contains preferences that are single-peaked with respect to some underlying ordering of the objects. WLOG, say $\mathcal{D} = \mathcal{D}^{SP}$, where $\mathcal{D}^{SP}$ is single-peaked with respect to the ordering $o_1 \to o_2 \to \cdots \to o_n$ so that

$$\mathcal{D}^{SP} = \{P_0 \in \mathcal{P} : o_p = r_1(P_0, O) \implies o_{k+1}\ P_0\ o_k \text{ for } k < p \text{ and } o_k\ P_0\ o_{k+1} \text{ for } k \geq p\}.$$

Observe that within any adjacent triple, the two extreme objects can be most-preferred, but there is no preference in which they can be the top-two most-preferred objects. Thus, the single-peaked domain does not satisfy the top-two condition.

7. Suppose $n \geq 3$ and $\mathcal{D}$ is a **single-peaked domain with two adjacent peaks** (Tamura [28]): This domain further restricts the single-peaked domain, so that only two adjacent objects in the underlying ordering can be most-preferred. WLOG, say $\mathcal{D} = \mathcal{D}^{SP-2}(p)$ for some $p \in \{1, \dots, n-1\}$, where $\mathcal{D}^{SP-2}(p)$ is defined as

$$\mathcal{D}^{SP-2}(p) = \{P_0 \in \mathcal{D}^{SP} : r_1(P_0, O) \in \{o_p, o_{p+1}\}\}.$$

It can be verified that within any restricted subset of objects $O' \subset O$, only two objects can be most-preferred, and since these objects must be adjacent within the subset, there exist preferences where they are the top-two most-preferred objects (in both possible orders) within the subset. Thus, the single-peaked domain with two adjacent peaks satisfies the top-two condition.

8. Suppose $n \geq 3$ and $\mathcal{D}$ is a **single-dipped domain** (Tamura [29]): This domain contains preferences that are single-dipped with respect to some underlying ordering of the objects. WLOG, say $\mathcal{D} = \mathcal{D}^{SD}$, where $\mathcal{D}^{SD}$ is single-dipped with respect to the ordering $o_1 \to o_2 \to \cdots \to o_n$ so that

$$\mathcal{D}^{SD} = \{P_0 \in \mathcal{P} : o_d = r_n(P_0, O) \implies o_k\ P_0\ o_{k+1} \text{ for } k < d \text{ and } o_{k+1}\ P_0\ o_k \text{ for } k \geq d\}.$$

It can be verified that within any restricted subset of objects $O' \subset O$, only two objects can be most-preferred, and since these objects must be the extreme objects within the subset, there exist preferences where they are the top-two most-preferred objects (in both possible orders) within the subset. Thus, the single-dipped domain satisfies the top-two condition.

9. Suppose n is arbitrary and $\mathcal{D}$ is a **partial agreement domain**: This domain contains preferences that are consistent with some fixed partial dominance relation over the objects. Formally, say $\mathcal{D} = \mathcal{D}^{PA}(\succ)$ for some partial order $\succ$ on O, where $\mathcal{D}^{PA}(\succ)$ is defined as

$$\mathcal{D}^{PA}(\succ) = \{P_0 \in \mathcal{P} : \text{ for all } a, b \in O, a \succ b \implies a\ P_0\ b\}.$$

   Observe that for any restricted subset of objects $O' \subset O$, if a and b can be most-preferred within O', then there must not be any $c \in O'$ such that $c \succ a$ or $c \succ b$. It follows that there exist preferences where a and b are the top-two most preferred objects (with both possible orders) within O'. Thus, the partial agreement domain satisfies the top-two condition.

10. Suppose $n \geq 4$ and $\mathcal{D}$ is a **circular domain**: This domain contains preferences described by the choice of a most-preferred object, and a clockwise or counterclockwise traversal along some cyclic order on the set of objects. WLOG, say $\mathcal{D} = \mathcal{D}^C$ where $\mathcal{D}^C$ is circular with respect to the cyclic order $o_1 \to o_2 \to \cdots \to o_n \to o_1$ so that

$$D^C = \{P_0 \in \mathcal{P} : o_p = r_1(P_0, O) \implies P_0 \in \{o_p \ldots o_n o_1 \ldots o_{p-1}, o_p \ldots o_1 o_n \ldots o_{p+1}\}\}.$$

   Observe that o_1 and o_3 can be most-preferred, and with $n \geq 4$, there is no preference in $\mathcal{D}$ where these two objects can simultaneously be the top-two most-preferred objects. Thus, the circular domain does not satisfy the top-two condition.

Our results now will allow us to classify a broad family of preference domains, including all the domains discussed above (except for the circular domain, which we will elaborate on later), as either TTC domains or not TTC domains.

3.2 Top-two is sufficient

Our first main result establishes that the top-two condition is sufficient for a preference domain to qualify as a TTC domain.

Theorem 1. *If a preference domain $\mathcal{D} \subset \mathcal{P}$ satisfies the top-two condition, then TTC is the unique mechanism that is individually rational, Pareto efficient, and strategyproof on $\mathcal{D}$.*

Proof sketch. Suppose $\mathcal{D} \subset \mathcal{P}$ satisfies the top-two condition, and $\varphi : \mathcal{D}^N \to \mathcal{X}$ is a mechanism that is IR, PE, SP on $\mathcal{D}$. For any preference profile $P \in \mathcal{D}^N$, we show that $\varphi(P) = TTC(P) = x$.

Suppose $S \subset N$ denotes a subset of agents who would form a cycle and trade endowments in the first round of TTC at P. Notice that for any $i \in S$, it must be that both $x_i, o_i \in r_1(\mathcal{D}, O)$, and since $\mathcal{D}$ satisfies the top-two condition, we can find a $P_i' \in \mathcal{D}$ such that

1. $x_i = r_1(P_i', O)$,

2. $o_i = r_2(P'_i, O)$.

By IR and PE, it must be that for each $i \in S$, $\varphi_i(P'_S, P_{-S}) = x_i$. From here, we use SP repeatedly to show that for each $i \in S$, $\varphi_i(P_S, P_{-S}) = \varphi_i(P) = x_i$. And since this holds for any arbitrary P_{-S}, we can iteratively apply this argument to agents in the second cycle, third cycle, and so on, to get that for each $i \in N$, $\varphi_i(P) = x_i$. □

Theorem 1 identifies a broad family of TTC domains. In particular, as established above, the unrestricted domain ($\mathcal{D} = \mathcal{P}$), the single-dipped domain ($\mathcal{D} = \mathcal{D}^{SD}$), and the single-peaked domain with two adjacent peaks ($\mathcal{D} = \mathcal{D}^{SP-2}(p)$) all satisfy the top-two condition. Consequently, existing results regarding these preference domains being TTC domains follow directly from Theorem 1. Moreover, the result allows for the classification of several previously unexplored domains as TTC domains. For example, the partial agreement preference domain ($\mathcal{D} = \mathcal{D}^{PA}(\succ)$) satisfies the top-two condition, and therefore, is a TTC domain.

3.3 Weak top-two is necessary

In this subsection, we discuss if the top-two condition is necessary for a domain to qualify as a TTC domain. In the special case where there are only $n = 3$ agents, we are able to show that this is indeed the case.

Proposition 1. *Suppose $n = 3$. If a preference domain $\mathcal{D} \subset \mathcal{P}$ does not satisfy the top-two condition, there exists a non-TTC mechanism that is individually rational, Pareto efficient, and strategyproof on $\mathcal{D}$.*

Proof sketch. The proof is via construction. Since $n = 3$ and $\mathcal{D} \subset \mathcal{P}$ does not satisfy the top-two condition, there must be a pair of objects that can be most-preferred but cannot simultaneously be the top-two most-preferred objects (in some order). WLOG, suppose $\mathcal{D} \subset \mathcal{P}$ is such that $o_1, o_2 \in r_1(\mathcal{D}, O)$, and for any $P_0 \in \mathcal{D}$,

$$o_1 = r_1(P_0, O) \implies o_3 = r_2(P_0, O).$$

We now construct a mechanism on $\mathcal{D}$ that deviates from TTC on certain select profiles. Specifically, this mechanism essentially penalizes agent 2 for being unable to report its endowment o_2 as the second-most preferred object when it most prefers o_1.

Formally, we define a subset of preference profiles

$$Diff = \{(P_1, P_2, P_3) \in \mathcal{D}^N : r_1(P_1, O) = o_2, r_1(P_2, O) = o_1, \text{ and } o_1 \ P_3 \ o_3\},$$

and consider the mechanism $\varphi : \mathcal{D}^N \to \mathcal{X}$ so that for any $P \in \mathcal{D}^N$,

$$\varphi(P) = \begin{cases} TTC(P) & \text{if } P \notin Diff \\ (o_2, o_3, o_1) & \text{if } P \in Diff \end{cases}.$$

The mechanism φ is different from TTC on preference profiles in $Diff \subset \mathcal{D}^N$, as for any $P \in Diff$, $TTC(P) = (o_2, o_1, o_3)$. Moreover, it is easy to verify that φ is IR and PE. In the proof, we show that φ is SP as well.

□

Together, Theorem 1 and Proposition 1 provide a complete characterization of TTC domains for the case of $n = 3$.

Corollary 2. *Suppose $n = 3$. A preference domain $\mathcal{D} \subset \mathcal{P}$ admits TTC as the unique individually rational, Pareto efficient, and strategyproof mechanism if and only if $\mathcal{D}$ satisfies the top-two condition.*

In general, we know from Theorem 1 that the top-two condition is sufficient, and while we believe the top-two condition to be necessary as well, we are only able to establish the necessity of the following weak version of the top-two condition.

Definition 2. A preference domain $\mathcal{D} \subset \mathcal{P}$ satisfies the *weak top-two condition* if for any $O' \subset O$ where $|O'| = 3$ and distinct $a, b \in r_1(\mathcal{D}, O')$, there exists a $P_0 \in \mathcal{D}$ such that

1. $a = r_1(P_0, O')$,
2. $b = r_2(P_0, O')$.

While the top-two condition requires the preference domain to exhibit the top-two property for all subsets $O' \subset O$, this weak version only requires this property for subsets O' with $|O'| = 3$. Thus, if a preference domain does not satisfy the weak top-two condition, there exists a triple of objects such that two of these objects can each be most-preferred, but cannot simultaneously be the top-two most-preferred objects (in some order). Our second main result establishes that such a domain is not a TTC domain.

Theorem 2. *If a preference domain $\mathcal{D} \subset \mathcal{P}$ does not satisfy the weak top-two condition, there exists a non-TTC mechanism that is individually rational, Pareto efficient, and strategyproof on $\mathcal{D}$.*

The proof of Theorem 2 is again via construction, and proceeds in a way very similar to the proof of Proposition 1. Since $\mathcal{D}$ does not satisfy the weak top-two condition, there must be a triple where the top-two property is not satisfied. As before, we construct a subset of profiles $Diff$, and a mechanism φ which differs from TTC only on the subset $Diff$, and show that the mechanism satisfies the required properties. The only major difference from the proof of Proposition 1 is in the construction of $Diff$, where we now additionally ensure that agents who own the triple are allocated something in the triple, irrespective of their own reported preferences. The full proof is in the appendix.

Theorem 2 identifies a broad family of preference domains that are not TTC domains. In particular, we established above that the single-peaked domain ($\mathcal{D} = \mathcal{D}^{SP}$) does not satisfy the top-two condition, but our argument actually shows that it doesn't even satisfy the weak top-two condition. It follows that the single-peaked domain is not a TTC domain.

3.4 Example: weak top-two but not top-two

While Theorems 1 and 2 provide a useful criterion for determining whether a given preference domain is a TTC domain, the criterion is incomplete. Specifically, there exist preference domains that satisfy the weak top-two condition but do not satisfy the top-two condition, and our results do not determine whether such domains are TTC domains. Although we suspect that these domains are not TTC domains—and hence, that the top-two condition is necessary (as shown for $n = 3$)—we have not yet been able to prove this.

The primary challenge lies in extending our construction for the case of triples to cases where the top-two property is violated for subsets of sizes greater than three. A key aspect of our construction for triples is that an agent unable to report their endowment as the second most-preferred is penalized, while the interfering agent is rewarded. However, with more than three objects, there might be multiple agents who can interfere, which complicates the construction and the analysis.

To illustrate, we discuss below the example of circular preference domain ($\mathcal{D} = \mathcal{D}^C$), which satisfies the weak top-two condition but not the top-two condition.

Example 1. *Suppose $n = 4$ and $\mathcal{D} = \mathcal{D}^C$ which contains the following eight preferences:*

Rank								
1	o_1	o_1	o_2	o_2	o_3	o_3	o_4	o_4
2	o_2	o_4	o_3	o_1	o_4	o_2	o_1	o_3
3	o_3	o_3	o_4	o_4	o_1	o_1	o_2	o_2
4	o_4	o_2	o_1	o_3	o_2	o_4	o_3	o_1

Notice that

- *$\mathcal{D}$ does not satisfy the top-two condition: Both o_1 and o_3 can be most-preferred among O, but there is no preference in which o_1 is most-preferred, and o_3 is second-most preferred among O.*

- *$\mathcal{D}$ satisfies the weak top-two condition: For any O' with $|O'| = 3$, it is easy to verify that $\mathcal{D}$ allows for all the six possible linear orders over O'.*

Thus, we cannot tell from Theorems 1 and 2 if this circular domain is a TTC domain or not. We can try to adopt the idea behind our construction for the case of $n = 3$, and try to penalize agent 3 (relative to TTC) in profiles where it most-prefers o_1, but now there are multiple agents (agent 2 and agent 4) that interfere and can be rewarded instead. This complicates the construction, and the analysis of the designed mechanism.

4 Conclusion

We identify a broad family of preference domains, those satisfying the top-two condition, on which the Top Trading Cycles (TTC) is the unique mechanism satisfying the desirable properties of individual rationality, Pareto efficiency, and strategyproofness. It follows that the search for non-TTC mechanisms satisfying these three properties should focus on domains that do not satisfy the top-two condition, and in particular, we establish the existence of such mechanisms on domains that fail to satisfy even a weak version of the top-two condition. Our findings provide a unifying perspective on previously studied domain restrictions, such as single-peaked and single-dipped domains, while also allowing for the classification of some important and previously unexplored domains as TTC domains or not TTC domains.

Our results and analysis suggest several directions for future research. An immediate open question is to close the gap between our necessary and sufficient condition, and in particular, determine whether the top-two condition is also necessary for TTC to be the unique mechanism satisfying the three properties. Another natural direction for future work would be to explore the existence and identification of analogous conditions for other characterizations of the TTC mechanism, including those based on axioms such as group strategyproofness or pair efficiency.

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A Proofs for Section 3 (Results)

Theorem 1. *If a preference domain $\mathcal{D} \subset \mathcal{P}$ satisfies the top-two condition, then TTC is the unique mechanism that is individually rational, Pareto efficient, and strategyproof on $\mathcal{D}$.*

Proof. Suppose $\mathcal{D} \subset \mathcal{P}$ satisfies the top-two condition, and $\varphi : \mathcal{D}^N \to \mathcal{X}$ is a mechanism that is IR, PE, SP on $\mathcal{D}$. Consider any arbitrary preference profile $P \in \mathcal{D}^N$ and let $x = TTC(P)$. We will show that $\varphi(P) = x$.

Suppose $S \subset N$ denotes a subset of agents who would form a cycle and trade endowments in the first round of TTC at P. Notice that for any $i \in S$, it must be that both $x_i, o_i \in r_1(\mathcal{D}, O)$, and since $\mathcal{D}$ satisfies the top-two condition, we can find a $P'_i \in \mathcal{D}$ such that

1. $x_i = r_1(P'_i, O)$,
2. $o_i = r_2(P'_i, O)$.

By IR and PE, it must be that for each $i \in S$, $\varphi_i(P'_S, P_{-S}) = x_i$.

Fix any $j \in S$. By SP, $\varphi_j(P_j, P'_{S\setminus\{j\}}, P_{-S}) = x_j$, and by IR, for each $i \in S$,

$$\varphi_i(P_j, P'_{S\setminus\{j\}}, P_{-S}) = x_i.$$

Now suppose for any $T \subset S$ where $|T| \leq k < |S|$, we have that for each $i \in S$,

$$\varphi_i(P_T, P'_{S\setminus T}, P_{-S}) = x_i.$$

Fix any T of size $k+1$ and any $j \in T$. By SP, $\varphi_j(P_T, P'_{S\setminus T}, P_{-S}) = x_j$, and in fact, for any $i \in T$, $\varphi_i(P_T, P'_{S\setminus T}, P_{-S}) = x_i$. Now there must be some $i \in T$ such that $x_i = o_r$ where $r \in S \setminus T$. By IR, it must be that for each $i \in S$, $\varphi_i(P_T, P'_{S\setminus T}, P_{-S}) = x_i$.

It follows by induction that for each $i \in S$,

$$\varphi_i(P_S, P'_{S\setminus S}, P_{-S}) = \varphi_i(P) = x_i.$$

Now we can iteratively apply this argument to agents in the second cycle, third cycle, and so on, to get that for each $i \in N$, $\varphi_i(P) = x_i$.

It follows that $\varphi(P) = TTC(P)$ for all $P \in \mathcal{P}^N$, and thus, φ must be the TTC mechanism. □

Proposition 1. *Suppose $n = 3$. If a preference domain $\mathcal{D} \subset \mathcal{P}$ does not satisfy the top-two condition, there exists a non-TTC mechanism that is individually rational, Pareto efficient, and strategyproof on $\mathcal{D}$.*

Proof. Since $n = 3$ and $\mathcal{D} \subset \mathcal{P}$ does not satisfy the top-two condition, there must be a pair of objects that can be most-preferred but cannot simultaneously be the top-two most-preferred objects (in some order). WLOG, suppose $\mathcal{D} \subset \mathcal{P}$ is such that $o_1, o_2 \in r_1(\mathcal{D}, O)$, and for any $P_0 \in \mathcal{D}$,

$$o_1 = r_1(P_0, O) \implies o_3 = r_2(P_0, O).$$

To ease notation, we simply use $r_k(P_0)$ to denote $r_k(P_0, O)$ going forward.

Define a subset of preference profiles

$$Diff = \{(P_1, P_2, P_3) \in \mathcal{D}^N : r_1(P_1) = o_2, r_1(P_2) = o_1, \text{ and } o_1 \; P_3 \; o_3\},$$

and consider the mechanism $\varphi : \mathcal{D}^N \to \mathcal{X}$ so that for any $P \in \mathcal{D}^N$,

$$\varphi(P) = \begin{cases} TTC(P) & \text{if } P \notin Diff \\ (o_2, o_3, o_1) & \text{if } P \in Diff \end{cases}.$$

Observe that the mechanism φ is different from TTC on the non-empty subset of preference profiles in $Diff \subset \mathcal{D}^N$, as for any $P \in Diff$, $TTC(P) = (o_2, o_1, o_3)$. Further, it is straightforward to verify that for any $P \in \mathcal{D}^N$, $\varphi(P)$ is individually rational and Pareto efficient, and thus, the mechanism φ is IR and PE. We will now show that φ is SP.

Let $P \in \mathcal{D}^N$ be any arbitrary preference profile.

1. Suppose $P \notin Diff$. By definition, $\varphi(P) = TTC(P)$. Since the TTC mechanism is SP, the only cases to consider are those when there is an $i \in N$ and $P'_i \in \mathcal{D}$ such that $(P'_i, P_{-i}) \in Diff$.
   (a) Suppose $i = 1$. Since agent 1 is always assigned its TTC outcome, it has no incentive to misreport.
   (b) Suppose $i = 2$. By definition of $Diff$, it must be that $r_1(P_1) = o_2$, $r_1(P_2) \in \{o_2, o_3\}$ and $r_1(P_3) \in \{o_1, o_2\}$. It is easy to verify that in any of these cases, $\varphi_2(P) = r_1(P_2)$. Thus, there is no incentive for agent 2 to misreport.
   (c) Suppose $i = 3$. By definition of $Diff$, it must be that $r_1(P_1) = o_2$, $r_1(P_2) = o_1$ and $o_3 \; P_3 \; o_1$. Since $\varphi_3(P) = o_3$, there is no incentive for agent 3 to misreport.
2. Suppose $P \in Diff$. By definition, $\varphi(P) = (o_2, o_3, o_1)$. It follows that the only cases to consider are those when there is an $i \in N$ and $P'_i \in \mathcal{D}$ such that $(P'_i, P_{-i}) \notin Diff$.
   (a) Suppose $i = 1$. Since agent 1 is always assigned its TTC outcome, it has no incentive to misreport.
   (b) Suppose $i = 2$. By definition of $Diff$, it must be that $r_1(P_2) = o_1$, and more precisely, it must be that $o_1 \; P_2 \; o_3 \; P_2 \; o_2$. Note that $\varphi_2(P) = o_3$. The only potential misreport worth considering for agent 2 (if it exists in $\mathcal{D}$) is $o_3 \; P'_2 \; o_1 \; P'_2 \; o_2$. But since $r_1(P_1) = o_2$, and $r_1(P_3) \in \{o_1, o_2\}$, it follows that in any of these cases, $\varphi_2(P'_2, P_{-2}) = o_3$. Thus, there is no incentive for agent 2 to misreport.
   (c) Suppose $i = 3$. By definition of $Diff$, it must be that $r_1(P_1) = o_2$, $r_1(P_2) = o_1$ and $o_1 \; P_3 \; o_3$. Since $\varphi_3(P) = o_1$, there is no incentive for agent 3 to misreport.

Thus, we have constructed a non-TTC mechanism that is IR, PE, and SP on $\mathcal{D}$.

$\square$

Theorem 2. *If a preference domain $\mathcal{D} \subset \mathcal{P}$ does not satisfy the weak top-two condition, there exists a non-TTC mechanism that is individually rational, Pareto efficient, and strategyproof on $\mathcal{D}$.*

Proof. Since $\mathcal{D} \subset \mathcal{P}$ does not satisfy the weak top-two condition, there must be a triple of objects and two objects among them that can be most-preferred, but cannot simultaneously be the top-two most preferred. WLOG, suppose $\mathcal{D} \subset \mathcal{P}$ is such that for $S = \{1, 2, 3\}$ and $O_S = \{o_1, o_2, o_3\}$, we have $o_1, o_2 \in r_1(\mathcal{D}, O_S)$, and for any $P_0 \in \mathcal{D}$,

$$o_1 = r_1(P_0, O_S) \implies o_3 = r_2(P_0, O_S).$$

Define a subset of preference profiles

$$\begin{aligned} Diff = \{P \in \mathcal{D}^N : r_1(P_1, O_S) = o_2, r_1(P_2, O_S) = o_1, o_1\ P_3\ o_3, TTC_{1,2,3}(P) = (o_2, o_1, o_3), \\ \text{and for any } P'_S \in \mathcal{D}^S, TTC_{-S}(P'_S, P_{-S}) = TTC_{-S}(P)\}. \end{aligned}$$

Notice that the construction of $Diff$ is similar to that in the proof of Proposition 1, except we additionally require P_{-S} to be such that agents in S are allocated something in O_S, irrespective of their report $P'_S \in \mathcal{D}^S$. Observe that the set $Diff$ is non-empty because it contains, in particular, profiles where any agent $i \notin S$ has a preference in which it most prefers o_i whenever it can be most-preferred. Now consider the mechanism $\varphi : \mathcal{D}^N \to \mathcal{X}$ so that for any $P \in \mathcal{D}^N$,

$$\varphi(P) = \begin{cases} TTC(P) & \text{if } P \notin Diff \\ (o_2, o_3, o_1, TTC_{-S}(P)) & \text{if } P \in Diff \end{cases}.$$

Observe that the mechanism φ is different from TTC on preference profiles in $Diff \subset \mathcal{D}^N$, as for any $P \in Diff$, $TTC_{1,2,3}(P) = (o_2, o_1, o_3)$. Further, it is straightforward to verify that for any $P \in \mathcal{D}^N$, $\varphi(P)$ is individually rational and Pareto efficient, and thus, the mechanism φ is IR and PE. We will now show that φ is SP.

Let $P \in \mathcal{D}^N$ be any arbitrary preference profile.

1. Suppose $P \notin Diff$. By definition, $\varphi(P) = TTC(P) = x$. Since the TTC mechanism is SP, the only cases to consider are those when there is an $i \in N$ and $P'_i \in \mathcal{D}$ such that $(P'_i, P_{-i}) \in Diff$.
   (a) Suppose $i \neq 2, 3$. Since agent i is always assigned its TTC outcome, it has no incentive to misreport.
   (b) Suppose $i = 2$. By definition of $Diff$, it must be that $r_1(P_1, O_S) = o_2$, $r_1(P_3, O_S) \in \{o_1, o_2\}$. It is easy to verify that in any of these cases, $\varphi_2(P) = r_1(P_2, O_S)$, and deviating to P'_2 such that $(P'_2, P_{-2}) \in Diff$ only allows agent 2 to get $o_3 \in O_S$. Thus, there is no incentive for agent 2 to misreport.

(c) Suppose $i = 3$. By definition of $Diff$, it must be that $r_1(P_1, O_S) = o_2$, $r_1(P_2, O_S) = o_1$, and $o_3 \; P_3 \; o_1$. Since $x_3 = o_3$, there is no incentive for agent 3 to misreport.

2. Suppose $P \in Diff$. By definition, $\varphi(P) = (o_2, o_3, o_1, TTC_{-S}(P))$. It follows that the only cases to consider are those when there is an $i \in N$ and $P'_i \in \mathcal{D}$ such that $(P'_i, P_{-i}) \notin Diff$.

   (a) Suppose $i \neq 2, 3$. Since agent i is always assigned its TTC outcome, it has no incentive to misreport.

   (b) Suppose $i = 2$. By definition of $Diff$, it must be that $r_1(P_2, O_S) = o_1$, and more precisely, it must be that $o_1 \; P_2 \; o_3 \; P_2 \; o_2$. Note that $\varphi_2(P) = o_3$. The only potential misreport worth considering for agent 2 (if it exists in $\mathcal{D}$) is a P'_2 such that $o_3 \; P'_2 \; o_1 \; P'_2 \; o_2$. But since $r_1(P_1, O_S) = o_2$, and $r_1(P_3, O_S) \in \{o_1, o_2\}$, it follows that in any of these cases, $\varphi_2(P'_2, P_{-2}) = o_3$. Thus, there is no incentive for agent 2 to misreport.

   (c) Suppose $i = 3$. By definition of $Diff$, it must be that $r_1(P_1, O_S) = o_2$, $r_1(P_2, O_S) = o_1$ and $o_1 \; P_3 \; o_3$. Since $\varphi_3(P) = o_1$, there is no incentive for agent 3 to misreport.

Thus, we have constructed a non-TTC mechanism that is IR, PE, and SP on $\mathcal{D}$.

$\square$