Contest Design with a Finite Type-Space: A unifying approach *

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Abstract

We study the classical contest design problem of allocating a budget across different prizes to maximize effort in an incomplete information environment with a finite type-space. For any contest under an arbitrary finite type-space, we characterize the unique symmetric Bayes-Nash equilibrium of the contest game. We find that the equilibrium is in mixed strategies, where agents of different types mix over disjoint but connected intervals, so that more efficient agents always exert greater effort than less efficient agents. Using this characterization, we solve for the expected equilibrium effort under any arbitrary contest, and find that with linear or concave costs, a winner-takes-all contest maximizes expected effort among all contests feasible for a budget-constrained designer. Our analysis introduces new techniques for the study of contests in a finite type-space and offers a unified approach to studying contest design simultaneously in the complete information environment, where the type-space is a singleton, and in the classical incomplete information setting with a continuum type-space, which we show can be well approximated by a sufficiently large finite type-space.

1 Introduction

Contests are situations in which agents compete with each other by investing costly effort or other resources to win valuable prizes. Given their widespread prevalence across various domains, such as business (e.g., firms' investments in innovation), sports (e.g., tournaments), and politics (e.g., securing positions of power), it is important to understand how different

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contests influence the effort exerted by agents and, in particular, identify contest structures that are optimal from the designer's perspective. Consequently, there is a vast literature studying variants of optimal contest design problems in different domains, including complete and incomplete information environments which differ in the assumptions they make about the information held by the agents about the types (abilities) of their opponents.

In this paper, we revisit the classical contest design problem of allocating a fixed budget across different prizes so as to maximize expected effort, focusing on a fundamental domain of an incomplete information environment with a finite type-space. Previous studies in the incomplete information domain typically assume a continuum type-space with smooth distributions, concluding that with linear or concave costs, awarding the entire budget to only the best performing agent is strictly optimal (Glazer and Hassin [11], Moldovanu and Sela [19], Zhang [30]). In contrast, in the complete information domain with linear costs, any distribution of the budget in which the worst performing agent gets nothing yields the same expected effort, and is therefore, optimal (Glazer and Hassin [11], Barut and Kovenock [1]). And in fact, with complete information and convex costs, it becomes strictly optimal to award everyone except the worst-performing agent an equal share of the budget (Fang, Noe, and Strack [9]). These results highlight how the structure of the optimal contest can vary significantly depending on the domain.[1]. Since our finite type-space domain encompasses the complete information domain as a special case (when the type-space is singleton) and can approximate any continuum type-space (with a sufficiently large finite-type space), the findings from the literature in these extreme cases do not directly inform what occurs in this intermediate domain. By analyzing the contest design problem in the finite type-space domain, our work not only provides a bridge between the literature on these extreme domains, it introduces new techniques that provide a unifying approach towards studying contest design problems in the classical complete and incomplete information domains, which have traditionally been explored separately using very different techniques.

For the finite type-space domain, we begin by characterizing the unique symmetric Bayes-Nash equilibrium of the contest game. Interestingly, even though the equilibrium is in mixed strategies, it always exhibits a neat and important monotonic structure. More specifically, the equilibrium is such that agents mix over disjoint but contiguous intervals so that agents who are more efficient always exert more effort than agents who are less efficient. Thus, the equilibrium in the finite type-space domain exhibits both the mixed nature of the equilibrium seen in complete information domains, and the monotonic structure seen in incomplete information domain with a continuum type-space. While this mixed and monotonic structure of the equilibrium is robust across type-spaces, cost functions and distributions of the budget, these elements of the model help pin down the specific equilibrium distribution over the intervals through the indifference condition.

[1] More detailed surveys of the literature on this contest design problem can be found in Sisak [24], Vojnović [28] For general surveys of the theoretical literature in contest theory, see Corchón [5], Vojnović [28], Konrad [13], Chowdhury, Esteve-González, and Mukherjee [4].

Using this equilibrium characterization, we focus on solving the contest design problem. We first restrict attention to linear costs, and solve for the expected effort induced by any arbitrary contest. While an explicit calculation directly using the distribution functions appears complicated, we introduce techniques that leverage the monotonic structure of the equilibrium and the additive separability of rewards and costs in the agents' utility functions to obtain a tractable representation for the expected effort. With this representation, we show that if there are at least two distinct types, the marginal effect of increasing the value of the first prize is greater than that of increasing any other prize, and thus, awarding the entire budget to the best performing agent maximizes expected effort. And in case the type-space is a singleton, we recover the result for the complete information domain that any distribution of the budget in which the worst-performing agents gets nothing is optimal. Perhaps interestingly, this suggests that as soon as there is any *little* uncertainty about the types, the winner-takes-all contest becomes strictly optimal.

For the design problem with general costs, we identify sufficient conditions under which the effect of transferring value from one prize to another on equilibrium effort can be informed by the effect of such a transfer under linear costs. More precisely, we show that if a transfer from lower ranked prize to a better ranked prize has a positive impact on expected effort under linear costs, this effect extends to the equilibrium under concave costs as long as the prizes in consideration are such that the better ranked prize is actually the first prize, or the equilibrium utility of the most efficient agent is negatively impacted from this transfer. Analogously, if such a transfer has a negative impact under linear costs, this impact extends to the equilibrium under convex costs under the same conditions. The upshot is that we are able to establish the optimality of the winner-takes-all contest under arbitrary concave cost functions, while also simultaneously recovering the result about the differing effects of increasing competition on expected effort under concave and convex costs in complete information environments.

Lastly, we establish an equilibrium convergence result that illustrates how the insights from studying contests in the finite type-space domain might apply to the classical domain with a continuum type-space. In particular, we study the limit behavior of the equilibrium of a sequence of finite type-spaces that converge to a continuum type-space, and show that the corresponding sequence of distributions of the mixed-strategy equilibrium converges to the distribution of the pure-strategy equilibrium under the continuum type-space. Essentially, the interval over which an agent of a certain type mixes shrinks as the finite type-space becomes large and converges to the effort prescribed by the pure-strategy equilibrium under the continuum type-space. Thus, for any continuum type-space, we can find a sufficiently large finite type-space so that the equilibrium behaviour under various contests in the finite type-space provides a reasonable approximation to the equilibrium behavior under these contests in the continuum type-space.[2] Since the finite type-space clearly includes the complete

[2]This is of particular importance for those interested in conducting experimental investigations of contest

information domain as a special case, this convergence result formalizes the sense in which the finite-type space domain provides a framework to investigate simultaneously the classical complete and incomplete information environments.

There is some related literature studying contests in an incomplete information with a finite type-space. Most of this literature focuses on problems with a small (or even binary) type-space and investigates properties of Bayes-Nash equilibrium. In particular, Szech [25] studies the value of information disclosure in a model with binary types and asymmetric distributions. Liu and Chen [17] allows for correlated types in an all-pay auction with two agents and binary types and shows that the symmetric Bayes-Nash equilibrium may be non-monotonic (i.e., have overlapping intervals) if the absolute correlation is large. Chen [3] considers a setting where players observe private signal about the types of their opponent and characterizes equilibrium for different degrees of informativeness of the signal structure. Xiao [29] assumes complete information (or perfect correlation) with heterogeneous agents and illustrates in a model with three agents and two prizes that the winner-takes-all contest maximizes the total expected effort if the top two players are similar, and two equal prizes maximize the total expected bid if the bottom two players are similar. In contrast to this literature, our paper studies the optimal contest design problem for arbitrarily general finite type-spaces.

The paper proceeds as follows. In Section 2, we present the general model of a contest in an incomplete-information environment with a finite type-space. In Section 3, we characterize the symmetric Bayes-Nash equilibrium of the contest game. In Section 4, we study the design of effort-maximizing contest. In Section 5, we discuss the convergence of finite-type space equilibrium to the equilibrium of the continuum type-space. Section 6 concludes. Some technical proofs are relegated to the Appendix.

2 Model

There is a set of N risk-neutral agents. Each agent has a privately known type θ (which determines its marginal cost of effort), drawn independently from type-space $\Theta = \{\theta_1, \theta_2, \dots, \theta_K\}$ according to distribution $p = (p_1, p_2, \dots, p_K)$ so that $\Pr[\theta = \theta_k] = p_k$. It is common knowledge that agents' types are independent and identically distributed according to p. Without loss of generality, we assume $\theta_1 > \theta_2 > \cdots > \theta_K > 0$ and $p_k > 0$ for all k.

There is a designer who designs a contest $v = (v_1, v_2, \dots, v_N)$ with $v_1 \geq \cdots \geq v_{N-1} \geq v_N$. Given the contest v, all agents simultaneously choose their effort. The agents are ranked according to their effort and awarded the corresponding prizes, with ties broken uniformly at random. If an agent of type θ_k wins prize v_i after exerting effort $x_k \geq 0$, its payoff is

$$v_i - \theta_k c(x_k),$$

models as they can approximate the continuum of types setting with a finite type-space.

where $c : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing and differentiable cost function. We normalize the cost function so that $c(0) = 0$ and also let $g = c^{-1}$.

Given a finite type-space (N, Θ, p) and cost function c, a contest v defines a Bayesian game between the N agents. We will focus on the symmetric Bayes-Nash equilibrium of this game. This is a strategy profile where all agents are using the same (potentially mixed) strategy, $X : \Theta \to \Delta\mathbb{R}_+$, mapping agent's type to a distribution over non-negative effort levels, so that for any agent of type θ_k, choosing any effort level within the support of $X(\theta_k)$ yields an expected payoff at least as high as that from choosing any other effort level, given that all other agents use the strategy $X(.)$.

The contest designer wants to maximize the expected equilibrium effort and has a fixed budget $V > 0$ that it can use to allocate to different prizes in order to incentivize agents to exert effort. We study this designer's problem of finding the contest that maximizes expected effort subject to the budget constraint.

Notation

Here, we introduce some notation that will be used in the rest of the paper.

We let $P_k = \sum_{i=1}^{k} p_i$. We will denote by

$$H_K^N(t) = \binom{N}{K} t^K (1-t)^{N-K}$$

the probability that a binomial random variable $Y \sim Bin(N, t)$ takes a value of exactly K. We also use

$$H_{\leq K}^N(t) = \sum_{k=0}^{K} \binom{N}{k} t^k (1-t)^{N-k} \text{ and } H_{\geq K}^N(t) = \sum_{k=K}^{N} \binom{N}{k} t^k (1-t)^{N-k}$$

to denote the probability that $Y \sim Bin(N, t)$ takes a value of at most K and at least K respectively. Given a contest $v = (v_1, v_2, \dots, v_N)$, we let

$$\pi_v(t) = \sum_{m=1}^{N} v_m H_{N-m}^{N-1}(t)$$

denote the expected value of the prize that an agent gets if it beats any arbitrary agent with probability $t \in [0, 1]$.

3 Equilibrium

In this section, we characterize the symmetric Bayes-Nash equilibrium of contests with a finite type-space. Before providing a complete description of the equilibrium, we establish an important structural property of the equilibrium.

Lemma 1. *Consider an arbitrary finite type-space* (N, Θ, p) *and an increasing differentiable cost function* c *with* $c(0) = 0$*. For any contest* $v = (v_1, \dots, v_{N-1}, 0)$*, there is a unique symmetric Bayes-Nash equilibrium and it is such that there exist boundary points* $b_1 < b_2 < \cdots < b_K$ *so that for any* $\theta_k \in \Theta$*, an agent of type* θ_k *mixes between* $[b_{k-1}, b_k]$ *with* $b_0 = 0$*.*

In words, Lemma 1 says that for any contest environment, the unique symmetric equilibrium is in mixed strategies and satisfies a useful monotonicity property. In particular, the equilibrium is such that an agent who is least efficient (of type θ_1) mixes between $[0, b_1]$, an agent of type θ_2 mix between $[b_1, b_2]$, and so on, until we get to an agent who is most efficient (of type θ_K), who mixes between $[b_{K-1}, b_K]$. Thus, there is monotonicity in that more efficient agents exert higher effort than less efficient agents with probability 1. And the possibility that an agent might face other agents of the same type creates incentives for them to mix and exert higher effort within their intervals. Thus, the symmetric equilibrium under a finite type-space domain exhibits both the mixed nature of the equilibrium seen in complete information domains (Barut and Kovenock [1]), and the monotonic structure of the equilibrium seen in incomplete information domains with a continuum type-space (Moldovanu and Sela [19]). Next, we provide an informal sketch of the proof of the Lemma. The full proof is in the appendix.

To prove Lemma 1, we show that that a symmetric equilibrium $X : \Theta \to \Delta\mathbb{R}_+$ must satisfy the following properties (in order):

1. The equilibrium cannot have any atoms. This is because if an agent of type θ_k chose x_k with positive probability, there is a positive probability that all agents are tied at x_k, and an agent of type θ_k can instead chose $x_k + \epsilon$ and get strictly higher payoff.

2. The minimum effort in support of the mixed strategy equilibrium must be 0. This is because an agent who chooses the minimum effort level in the support of the mixed strategy wins the last prize $v_n = 0$ with probability 1. So if this minimum effort level is positive, the agent can deviate to $x = 0$ and get a strictly higher payoff.[3]

3. The equilibrium utility of more efficient agents should be higher than that of less efficient agents. This is because otherwise, a more efficient agent can simply imitate the strategy of a less efficient agent, in which case it gets the same expected reward as the less efficient agent, but it pays a lower cost leading to a higher payoff.

4. In equilibrium, the intersection of support for two different types cannot have more than one effort level. This is because going from one effort level to another, the change in expected reward is the same irrespective of type, but the change in cost depends on the type. Since an agent of any type must be indifferent between all actions in the support, it follows that agents of two different types cannot both be indifferent between two different effort levels.

[3]The first two steps are analogous to the argument used to show that the Nash equilibrium in the complete information setting, which corresponds to the case where $|\Theta| = K = 1$, is in mixed strategies (Barut and Kovenock [1], Fang, Noe, and Strack [9]).

5. In equilibrium, the supports of the different agent types must be connected. This is because of reasons similar to that in the second step.

6. In equilibrium, if the supports of two different types are connected at effort level x, then the more efficient type exerts effort greater than x and the less efficient type exerts effort less than x with probability one. This is because if the more efficient type is instead mixing in an interval $[a, x]$ and the less efficient is mixing in the interval $[x, b]$, then the less efficient agent can deviate to a and obtain a strictly higher payoff.

Together, the six properties imply that the equilibrium has the structure described in Lemma 1.

Using the structure of the equilibrium, we can now completely characterize the unique Bayes-Nash equilibrium for any contest in an arbitrary finite type-space.

Theorem 1. *Consider an arbitrary finite type-space (N, Θ, p) and an increasing differentiable cost function c with $c(0) = 0$. For any contest $v = (v_1, \ldots, v_{N-1}, 0)$, the unique symmetric Bayes-Nash equilibrium is such that for any $\theta_k \in \Theta$, the distribution function $F_k : [b_{k-1}, b_k] \to [0, 1]$ is defined by*

$$\pi_v(P_{k-1} + p_k F_k(x_k)) - \theta_k c(x_k) = u_k \text{ for all } x_k \in [b_{k-1}, b_k], \tag{1}$$

where the boundary points $b = (b_1, \ldots, b_K)$ and equilibrium utilities $u = (u_1, \ldots, u_K)$ are

$$c(b_k) = \sum_{j=1}^{k} \frac{\pi_v(P_j) - \pi_v(P_{j-1})}{\theta_j} \text{ for any } k \in \{1, 2, \ldots, K\}, \tag{2}$$

and

$$u_k = \theta_k \left[\sum_{j=1}^{k-1} \pi_v(P_j) \left(\frac{1}{\theta_{j+1}} - \frac{1}{\theta_j} \right) \right] \text{ for any } k \in \{1, 2, \ldots, K\}. \tag{3}$$

Proof. We know from Lemma 1 that there exist boundary points $b_1 < b_2 < \cdots < b_K$ so that for any $\theta_k \in \Theta$, an agent of type θ_k mixes between $[b_{k-1}, b_k]$ with $b_0 = 0$. If F_k denotes the equilibrium distribution function for agent of type θ_k, observe that $\pi_v(P_{k-1} + p_k F_k(x_k))$ is simply the agent's expected payoff from playing $x_k \in [b_{k-1}, b_k]$. Since the agent must be indifferent between all actions in the support, and in particular, every such action must lead to equilibrium utility u_k, we get that the equilibrium distribution functions must satisfy Equation 1.

Now it remains to solve for the boundary points b_k and equilibrium utilities u_k. Notice that we can plug in $x_k = b_k$ in Equation 1 to get that

$$c(b_k) = \frac{\pi_v(P_k) - u_k}{\theta_k} \text{ for any } k \in \{1, 2, \ldots, K\}. \tag{4}$$

In addition, we use the fact that agents of both types θ_k and θ_{k+1} have b_k in the support to get a relationship between their equilibrium utilities. More precisely, for any $k \in \{1, 2, \ldots, K-1\}$, we have that

$$u_k = \pi_v(P_k) - \theta_k c(b_k) \text{ and } u_{k+1} = \pi_v(P_k) - \theta_{k+1} c(b_k).$$

Taking a difference, we obtain

$$u_{k+1} - u_k = (\theta_k - \theta_{k+1}) c(b_k) \text{ for any } k \in \{1, 2, \ldots, K-1\}. \tag{5}$$

Since $u_1 = 0$, we can use Equation 4 to solve for b_1. And then using u_1 and b_1, we can use Equation 5 to solve for u_2. In general, using Equations 4 and 5 iteratively, we can solve explicitly for the equilibrium bounds and utilities. Together with these equilibrium objects, as described in Equations 2 and 3, Equation 1 provides a complete description of the unique symmetric Bayes-Nash equilibrium of the contest game. □

Intuitively, since an agent of type θ_k must be indifferent between all effort levels in the interval $[b_{k-1}, b_k]$, the distribution of effort F_k in this interval is such that the marginal gain in expected reward from increasing effort in the interval is equal to the marginal cost $\theta_k c'(x_k)$. And in particular, for linear cost $c(x) = x$, given the contest v, the distribution F_1 on $[0, b_1]$ is such that the marginal gain in reward from increasing effort equals θ_1, F_2 on $[b_1, b_2]$ is such that the marginal gain in reward equals θ_2, and more generally, for any $k \in \{1, 2, \ldots, K\}$, F_k on $[b_{k-1}, b_k]$ is such that the marginal gain in reward from increasing effort in the interval equals θ_k. We illustrate this in the following example which explicitly describes the equilibrium for the special case with $N = 2$ agents.

Example 1. *Consider an arbitrary finite type-space (N, Θ, p) with $N = 2$ and a linear cost function $c(x) = x$. For any contest $v = (v_1, 0)$, the unique symmetric Bayes-Nash equilibrium is such that the (random) level of effort exerted by an agent of type $\theta_k \in \Theta$ is*

$$X_k \sim U\left(v_1 \sum_{j=1}^{k-1} \frac{p_j}{\theta_j}, v_1 \sum_{j=1}^{k} \frac{p_j}{\theta_j}\right).$$

Thus, the equilibrium distribution function F_k for type θ_k is uniform. Notice that if $x \in [b_{k-1}, b_k]$ then the expected prize of an agent from choosing effort level x is

$$P_{k-1} v_1 + p_k \left[\frac{[x - b_{k-1}]\, \theta_k}{v_1 p_k}\right] v_1.$$

Differentiating with respect to x, we can check that the marginal gain in reward from increasing effort in the range $[b_{k-1}, b_k]$ is equal to the marginal cost θ_k.

An important consequence of Theorem 1 is that the equilibrium utilities of the agents only depend on the finite type-space (N, Θ, p) and the contest v, and is independent of the cost function c. Intuitively, one can redefine the Bayesian game as one in which the agents directly choose the effort costs $c(x_k)$ instead of the effort x_k. It follows that the expected equilibrium costs, and thus, the equilibrium utilities remain the same. Moreover, it follows from Equation 3 that the equilibrium utilities are linear in v_m, and in particular, for any $k \in \{1, \dots, K\}$ and $m \in \{1, \dots, N-1\}$,

$$\frac{\partial u_k}{\partial v_m} = \theta_k \left[\sum_{j=1}^{k-1} H_{N-m}^{N-1}(P_j) \left(\frac{1}{\theta_{j+1}} - \frac{1}{\theta_j} \right) \right]. \tag{6}$$

Thus, the change in equilibrium utilities from changing prizes is determined entirely by the finite type-space (N, Θ, p), and is independent of the cost function c and also the contest $v = (v_1, \dots, v_{N-1}, 0)$.

4 Optimal contest

In this section, we solve the designer's problem of finding a distribution of budget across prizes so as to maximize expected equilibrium effort. Going forward, given a finite type-space (N, Θ, p), cost function c, and contest v, we denote by $X_k \sim F_k$ the (random) level of effort exerted in equilibrium by an agent of type θ_k and we denote by $X \sim F$ the ex-ante (random) level of effort exerted in equilibrium by an arbitrary agent so that for any $x \in \mathbb{R}$,

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ P_{k-1} + p_k F_k(x) & \text{if } x \in [b_{k-1}, b_k] \\ 1 & \text{if } x \geq b_K \end{cases}.$$

The expected effort of an arbitrary agent is then

$$\mathbb{E}[X] = \sum_{k=1}^{K} p_k \mathbb{E}[X_k].$$

The following result finds a useful representation for this expected equilibrium effort.

Lemma 2. *Consider an arbitrary finite type-space (N, Θ, p) and an increasing differentiable cost function c with $c(0) = 0$. For any contest $v = (v_1, \dots, v_{N-1}, 0)$, the expected equilibrium effort of an arbitrary agent is*

$$\mathbb{E}[X] = \sum_{k=1}^{K} \int_{P_{k-1}}^{P_k} g\left(\frac{\pi_v(t) - u_k}{\theta_k} \right) dt. \tag{7}$$

To prove Lemma 2, we reinterpret Equation 1 as the definition of X_k with

$$\pi_v(P_{k-1} + p_k F_k(X_k)) - \theta_k c(X_k) = u_k,$$

where $F_k(X_k)$ is now just a uniformly distributed random variable. Rearranging and taking expectation on both sides gives $\mathbb{E}[X_k]$, which then allows us to obtain $\mathbb{E}[X]$.

4.1 Linear costs

In this subsection, we discuss the design problem for the special case where the cost function is linear. Using Lemma 2, we first obtain exactly the expected equilibrium effort in this case. Observe that both $\pi_v(t)$ and u_k are linear in v_m, and hence, it follows from Equation 7 that with linear costs $c(x) = x$, the expected effort must take the form $\mathbb{E}[X] = \sum_{m=1}^{N-1} \alpha_m v_m$ for some α_m. Moreover, since $\alpha_m = \frac{\partial \mathbb{E}[X]}{\partial v_m}$, we get from Equation 7 that

$$\alpha_m = \sum_{k=1}^{K} \int_{P_{k-1}}^{P_k} \left(\frac{H_{N-m}^{N-1}(t)}{\theta_k} - \frac{1}{\theta_k} \frac{\partial u_k}{\partial v_m} \right) dt.$$

Solving this integral leads to the following result.

Lemma 3. *Consider an arbitrary finite type-space (N, Θ, p) and the cost function $c(x) = x$. For any contest $v = (v_1, \dots, v_{N-1}, 0)$, the expected equilibrium effort of an arbitrary agent is*

$$\mathbb{E}[X] = \sum_{m=1}^{N-1} \alpha_m v_m$$

where

$$\alpha_m = \frac{1}{N} \left[\frac{1}{\theta_K} - \sum_{k=1}^{K-1} \left[H_{\geq N-m+1}^{N}(P_k) + m H_{N-m}^{N}(P_k) \right] \left(\frac{1}{\theta_{k+1}} - \frac{1}{\theta_k} \right) \right]. \tag{8}$$

Now to solve the design problem of maximizing expected effort, or more generally, to study the effect of increasing competition on effort, a standard approach in contest theory is to investigate the marginal effect of transferring value from one prize to another. For linear costs, it follows from Lemma 3 that the marginal effect of transferring value from a worse ranked prize $w \in \{2, \dots, N-1\}$ to a better ranked prize $b < w$ is captured by the difference

$$\alpha_b - \alpha_w = \frac{1}{N} \left[\sum_{k=1}^{K-1} \left[H_{\leq N-b}^{N}(P_k) - b H_{N-b}^{N}(P_k) - H_{\leq N-w}^{N}(P_k) + w H_{N-w}^{N}(P_k) \right] \left(\frac{1}{\theta_{k+1}} - \frac{1}{\theta_k} \right) \right].$$

Depending on the type-space (N, Θ, p), this difference may be strictly positive, suggesting that transferring value from w to b encourages effort; it may be zero, indicating that the transfer has no effect on effort; or it may be strictly negative, meaning the transfer discourages effort.

Observe that in case there are at least two distinct types in Θ, we have $\alpha_1 - \alpha_w > 0$ for any $w \in \{2, \dots, N-1\}$. It follows then that transferring value from any arbitrary prize w to the first prize leads to strictly higher expected equilibrium effort, and thus, it is strictly optimal for a budget-constrained designer to allocate the entire budget to the first prize. Thus, with an arbitrary finite type-space and linear costs, the winner-takes-all contest maximizes expected effort.

Also observe that in the special case when Θ contains only a single type, thus defining a complete information environment, it follows from Lemma 3 that $\alpha_1 = \alpha_2 = \cdots = \alpha_{N-1} = \frac{1}{N\theta_1}$. Since $\alpha_b - \alpha_w = 0$ for any b, w, transferring value from w to b has no effect on expected effort. Thus, we recover the result that in the complete information environment with linear costs, any allocation of the budget among the top $N-1$ prizes results in the same expected equilibrium effort (Barut and Kovenock [1]). However, as soon as there is any little uncertainty (incomplete information) in the environment with linear costs, allocating the entire budget to the first prize is strictly optimal.

4.2 General costs

In this subsection, we return to the design problem with general costs. Recall that for a general cost function c, the expected equilibrium effort under a contest v is defined by Equation 7. Unlike linear costs, in which case the expected effort depends linearly on the value of prizes, the precise relationship between expected effort and the value of the prizes under a generic cost function may be non-trivial. However, as the following result shows, the effect of transferring value from prize w to b on equilibrium expected effort under important classes of cost functions may be informed by the effect of such a transfer under linear costs $(\alpha_b - \alpha_w)$.

Theorem 2. *Consider an arbitrary finite type-space (N, Θ, p). For any pair of prizes $b, w \in \{1, \ldots, N-1\}$ with $b < w$ such that, either $b = 1$ or $\left(\frac{\partial u_K}{\partial v_b} - \frac{\partial u_K}{\partial v_w}\right) \leq 0$, the following hold:*

1. *If $\alpha_b - \alpha_w \geq 0$, then for any strictly concave cost c and contest v, $\dfrac{\partial \mathbb{E}[X]}{\partial v_b} - \dfrac{\partial \mathbb{E}[X]}{\partial v_w} > 0$.*
2. *If $\alpha_b - \alpha_w \leq 0$, then for any strictly convex cost c and contest v, $\dfrac{\partial \mathbb{E}[X]}{\partial v_b} - \dfrac{\partial \mathbb{E}[X]}{\partial v_w} > 0$.*

In words, Theorem 2 provides sufficient conditions on the type-space (N, Θ, p) under which the effect of transferring value from w to b on expected equilibrium effort under linear costs $(\alpha_b - \alpha_w)$ is informative about the effect of such a transfer under concave or convex cost functions. More precisely, if this transfer has a positive effect under linear costs, the effect persists under concave costs as well as long as the prizes in consideration are such that $b = 1$ or the transfer negatively impacts the equilibrium utility of the most efficient agent. Analogously, if the transfer has a negative impact under linear costs, it persists under convex costs under the same conditions. Next, we outline the proof of the result, and then use the result to solve the design problem in some important special cases.

For the proof of Theorem 2, observe that for any arbitrary finite type-space (N, Θ, p), cost function c, and contest v, it follows from Equation 7 that the marginal effect of transferring value from prize w to b is

$$\frac{\partial \mathbb{E}[X]}{\partial v_b} - \frac{\partial \mathbb{E}[X]}{\partial v_w} = \int_0^1 g'\left(\frac{\pi_v(t) - u_{k(t)}}{\theta_{k(t)}}\right)\left(\frac{H_{N-b}^{N-1}(t) - H_{N-w}^{N-1}(t)}{\theta_{k(t)}} - \frac{1}{\theta_{k(t)}}\left(\frac{\partial u_{k(t)}}{\partial v_b} - \frac{\partial u_{k(t)}}{\partial v_w}\right)\right) dt,$$

where $k(t) = \max\{k : P_{k-1} \leq t\}$. Under the assumption on b and w in the result, we can show that the second function in the integrand has the single-crossing property. More precisely, there is some $t^* \in [0, 1]$ so that this function is negative for $t \in [0, t^*]$ and positive for $t \in [t^*, 1]$. Now when c is concave, g' is increasing and so the integral is at least as high as (essentially) the integral of this second function itself, which equals $\alpha_b - \alpha_w$. And when c is convex, g' is decreasing and so the integral is at most (essentially) equal to the $\alpha_b - \alpha_w$. The result follows.

We now use Theorem 2 to solve the design problem in some important cases. First, we solve the design problem for a designer faced with an arbitrary finite type-space (N, Θ, p) and a concave cost function c. Fix an arbitrary contest v, and observe that if the designer transfers value from prize w to the first prize ($b = 1$), the marginal effect of such a transfer on equilibrium effort should always be non-negative since $\alpha_1 - \alpha_w \geq 0$ for any $w \in \{2, \ldots, N-1\}$. It follows then that for an arbitrary finite-type space and concave costs, the winner-takes-all contest maximizes expected effort.

Corollary 1. *Consider an arbitrary finite type-space (N, Θ, p) and a (weakly) concave cost function c. Among all contests $v = (v_1, \ldots, v_N)$ such that $\sum_{i=1}^{N} v_i \leq V$, the winner-takes-all contest $v^* = (V, 0, 0, \ldots, 0)$ maximizes expected effort.*

In addition to revealing the optimality of the winner-takes-all contest for concave costs, Theorem 2 also allows us to recover, more generally, the effect of increasing competition on equilibrium effort in the special case of a complete information environment. In this case, captured with type-space $\Theta = \{\theta_1\}$, the equilibrium utility of the most efficient agent is always $u_1 = 0$. Thus, transferring value from any worse ranked prize w to any better ranked prize b has no effect on its equilibrium utility, and we know from above that it also does not effect the expected effort under linear costs since $\alpha_b - \alpha_w = 0$. It follows from Theorem 2 then that the marginal effect of transferring value from w to b on expected equilibrium effort is positive under concave costs and negative under convex costs. While we already had the optimality of the winner-takes-all contest under concave costs, it follows that distributing the budget equally amongst $N-1$ prizes maximizes expected effort under convex costs. And more generally, we recover the result that in the complete information environment, increasing competition leads to an increase in expected effort under concave costs and a decrease in expected effort under convex costs (Fang, Noe, and Strack [9]).

Thus, Theorem 2 allows us to extend the optimality of the winner-takes-all contest under linear and concave costs in the incomplete information domain for the continuum type-space (Moldovanu and Sela [19]) to the finite-type space domain, while also simultaneously illustrating how increasing competition influences expected effort in the complete information domain (Fang, Noe, and Strack [9]). In this sense, our analysis of the contest design problem in the finite type-space domain not only provides a bridge between the previous literature in complete information and incomplete information domains, it also provides a unifying approach to studying these domains simultaneously.

4.3 Application: Binary Type-space

In this subsection, we discuss the application of Theorem 2 to the special case of a binary type space, focusing on the effect of transferring value between two adjacent prizes.

Lemma 4. *Consider a finite type-space* (N, Θ, p) *with* $\Theta = \{\theta_1, \theta_2\}$*. For* $m \in \{1, \ldots, N-2\}$,

$$\left(\frac{\partial u_K}{\partial v_m} - \frac{\partial u_K}{\partial v_{m+1}}\right) \leq 0 \iff p_2 \geq \frac{m}{N}.$$

and

$$\alpha_m - \alpha_{m+1} \geq 0 \iff p_2 \geq \frac{m-1}{N-1}.$$

Lemma 4 suggests that the effect of making the contest more competitive by transferring value from lower ranked prizes to better ranked prizes depends on the relative likelihood of efficient and inefficient agents. Intuitively, since efficient agents are more likely to win better prizes in equilibrium, this transfer encourages effort from efficient agents while discouraging effort from the inefficient agents. Lemma 4 then shows that the overall effect of such a transfer, under linear costs, is positive if there are sufficiently many efficient agents, and negative if efficient agents are unlikely. Moreover, we can generalize these effects to equilibrium under other cost functions when efficient agents are sufficiently likely by applying Theorem 2. However, Theorem 2 says nothing about whether these effects generalize when efficient agents are unlikely.

The following example describes the equilibrium under a specific binary type-space.

Example 2. *Consider the finite type-space domain* (N, Θ, p) *where* $N = 3$, $\Theta = \{2, 1\}$, $p = (0.5, 0.5)$*, and the cost function* $c(x) = x$*. For any contest* $\mathbf{v} = (v_1, v_2, 0)$*, the equilibrium distribution functions are*

$$F_1(x_1) = \frac{-2v_2 + 2\sqrt{v_2^2 + (v_1 - 2v_2)2x_1}}{(v_1 - 2v_2)} \text{ and } F_2(x_2) = \frac{-v_1 + \sqrt{v_1^2 + 4(v_1 - 2v_2)(x_2 - b_1)}}{(v_1 - 2v_2)},$$

where $b_1 = \dfrac{v_1 + 2v_2}{8}$*. And the expected efforts are*

$$\mathbb{E}[X_1] = \frac{v_1 + 4v_2}{24} \text{ and } \mathbb{E}[X_2] = \frac{11v_1 + 2v_2}{24}$$

so that the expected effort of an arbitrary agent is

$$\mathbb{E}[X] = \frac{1}{2}\mathbb{E}[X_1] + \frac{1}{2}\mathbb{E}[X_2] = \frac{12v_1 + 6v_2}{48}.$$

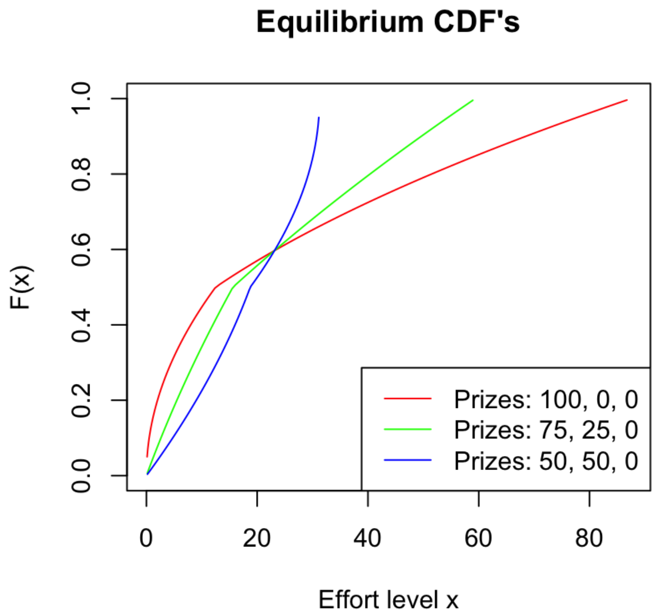

Figure 1: The equilibrium distribution functions, $F(.)$, for the environment in Example 2.

5 Continuum Type-Space Convergence

In this section, we study the limit properties of the symmetric equilibrium of finite type-space, focusing in particular on whether the equilibrium behavior under a continuum type-space is close to the equilibrium behavior under a sufficiently large finite type-space that approximates the continuum type-space. For this purpose, we restrict attention to the case where the cost function is linear $c(x) = x$, and note that this is without loss of generality because of the equivalence between convergence properties of equilibrium costs and equilibrium effort.

First, we note the symmetric equilibrium under a continuum type-space, defined by a differentiable distribution function G (Moldovanu and Sela [19]).

Lemma 5. *Suppose there are N agents, each with a private type drawn from $\Theta = [\underline{\theta}, \overline{\theta}]$ according to a differentiable CDF $G : [\underline{\theta}, \overline{\theta}] \to [0, 1]$. For any contest $v = \{v_1, v_2, \ldots, v_{N-1}, 0\}$, there is a unique symmetric Bayes-Nash equilibrium and it is such that for any $\theta \in \Theta$,*

$$X(\theta) = \int_{\theta}^{\overline{\theta}} \frac{\pi_v'(1 - G(t))g(t)}{t} dt.$$

Next, we show that for any continuum type space and distribution over this type-space, if we take a sequence of finite type-space distributions that converge to this distribution, the corresponding sequence of mixed-strategy equilibrium converges to the pure-strategy equilibrium under the continuum type-space.

Theorem 3. *Suppose there are N agents and consider a fixed contest $v = (v_1, v_2, \ldots, v_{N-1}, 0)$. Let $G : [\underline{\theta}, \overline{\theta}] \to [0, 1]$ be a differentiable CDF and let $G^1, G^2, \ldots,$ be any sequence of CDF's, each with a finite support, such that for all $\theta \in [\underline{\theta}, \overline{\theta}]$,*

$$\lim_{n\to\infty} G^n(\theta) = G(\theta).$$

Let $F^n : \mathbb{R} \to [0, 1]$ denote CDF of the equilibrium effort under the finite type-space distribution G^n, and let $F : \mathbb{R} \to [0, 1]$ denote CDF of the equilibrium under continuum type-space distribution G. Then, the sequence of CDF's $F^1, F^2, \ldots,$ converges to the CDF F, i.e., for all $x \in \mathbb{R}$,

$$\lim_{n\to\infty} F^n(x) = F(x).$$

Intuitively, as the finite type-space becomes large, the interval over which an agent of a certain type mixes shrinks, and essentially converges to the effort level prescribed by the pure-strategy equilibrium under the continuum type-space. Thus, the equilibrium strategy in an appropriate and sufficiently large finite-type space domain provides a reasonable approximation to the equilibrium strategy under the continuum type-space, and vice versa.

6 Conclusion

We study the canonical contest design problem in an incomplete information environment with a finite type-space. We characterize the unique symmetric Bayes-Nash equilibrium under any arbitrary contest with a finite type-space. We find that the equilibrium is in mixed strategies, and it is such that agents of adjacent types mix over disjoint but connected intervals so that more efficient agents always exert greater effort. Even though the equilibrium is in mixed strategies, we are able to exploit its monotonic structure to obtain a tractable representation for the expected equilibrium effort of an arbitrary agent. Using this representation, we find that a budget-constrained designer should allocate its entire budget to the first prize, and thus, run a winner-takes-all contest, in order to maximize expected equilibrium effort of an arbitrary agent. Our results extend the well-known optimality of winner-takes-all contest under a continuum type-space to the finite type-space environment.

In our analysis of contests in finite type-space environment, we introduce some new techniques and we hope that the results and methods in this paper will encourage further research in this fundamental domain. In particular, one could study the contest design problem with more general cost functions, and also perhaps other variants that have been previously explored in the literature dealing with the continuum type-space. Since our techniques rely on the separability of the reward and the costs in the utility function, we believe that the structure of equilibrium might be robust to some of these other variants, including for instance, convex cost functions. In addition, we believe that the finite type-space model presents a more convenient framework for experiments as compared to the continuum type-space, and thus, we hope to also inspire more experimental research investigating some of the theoretical predictions in the literature on contest design with incomplete information.

References

[1] Barut, Y. and D. Kovenock (1998): "The symmetric multiple prize all-pay auction with complete information," *European Journal of Political Economy*, 14, 627–644. 2, 6, 11

[2] Chen, Z. (2019): "Spying in contests," *Available at SSRN*, 2874296.

[3] ——— (2021): "All-pay auctions with private signals about opponents' values," *Review of Economic Design*, 25, 33–64. 4

[4] Chowdhury, S. M., P. Esteve-González, and A. Mukherjee (2023): "Heterogeneity, leveling the playing field, and affirmative action in contests," *Southern Economic Journal*, 89, 924–974. 2

[5] Corchón, L. C. (2007): "The theory of contests: a survey," *Review of Economic Design*, 11, 69–100. 2

[6] Dechenaux, E., D. Kovenock, and R. M. Sheremeta (2015): "A survey of experimental research on contests, all-pay auctions and tournaments," *Experimental Economics*, 18, 609–669.

[7] Doni, N. and D. Menicucci (2013): "Revenue comparison in asymmetric auctions with discrete valuations," *The BE Journal of Theoretical Economics*, 13, 429–461.

[8] Einy, E., M. P. Goswami, O. Haimanko, R. Orzach, and A. Sela (2017): "Common-value all-pay auctions with asymmetric information," *International Journal of Game Theory*, 46, 79–102.

[9] Fang, D., T. Noe, and P. Strack (2020): "Turning up the heat: The discouraging effect of competition in contests," *Journal of Political Economy*, 128, 1940–1975. 2, 6, 12

[10] Fu, Q. and Z. Wu (2019): "Contests: Theory and topics," in *Oxford Research Encyclopedia of Economics and Finance*.

[11] Glazer, A. and R. Hassin (1988): "Optimal contests," *Economic Inquiry*, 26, 133–143. 2

[12] Konrad, K. A. (2004): "Altruism and envy in contests: An evolutionarily stable symbiosis," *Social Choice and Welfare*, 22, 479–490.

[13] ——— (2009): "Strategy and dynamics in contests," . 2

[14] Letina, I., S. Liu, and N. Netzer (2023): "Optimal contest design: Tuning the heat," *Journal of Economic Theory*, 213, 105616.

[15] Liu, B., J. Lu, R. Wang, and J. Zhang (2018): "Optimal prize allocation in contests: The role of negative prizes," *Journal of Economic Theory*, 175, 291–317.

[16] Liu, X. and J. Lu (2014): "The effort-maximizing contest with heterogeneous prizes," *Economics Letters*, 125, 422–425.

[17] Liu, Z. and B. Chen (2016): "A symmetric two-player all-pay contest with correlated information," *Economics Letters*, 145, 6–10. 4

[18] Maskin, E. S. and J. G. Riley (1985): "Auction theory with private values," *The American Economic Review*, 75, 150–155.

[19] Moldovanu, B. and A. Sela (2001): "The optimal allocation of prizes in contests," *American Economic Review*, 91, 542–558. 2, 6, 12, 14

[20] Rentschler, L. and T. L. Turocy (2016): "Two-bidder all-pay auctions with interdependent valuations, including the highly competitive case," *Journal of Economic Theory*, 163, 435–466.

[21] Segev, E. and A. Sela (2014): "Multi-stage sequential all-pay auctions," *European Economic Review*, 70, 371–382.

[22] Siegel, R. (2011): "Asymmetric contests with interdependent valuations," *Available at SSRN 1836982*.

[23] ——— (2014): "Asymmetric all-pay auctions with interdependent valuations," *Journal of Economic Theory*, 153, 684–702.

[24] Sisak, D. (2009): "Multiple-prize contests–the optimal allocation of prizes," *Journal of Economic Surveys*, 23, 82–114. 2

[25] Szech, N. (2011): "Asymmetric all-pay auctions with two types," *University of Bonn, Discussion paper, January*. 4

[26] Tang, B., H. Fu, and Y. C. Wu (2023): "On reservation prices in the all-pay contest with correlated information," *Managerial and Decision Economics*, 44, 3932–3943.

[27] Vohra, R. V. (2012): "Optimization and mechanism design," *Mathematical programming*, 134, 283–303.

[28] Vojnović, M. (2015): *Contest theory: Incentive mechanisms and ranking methods*, Cambridge University Press. 2

[29] Xiao, J. (2018): "Equilibrium analysis of the all-pay contest with two nonidentical prizes: Complete results," *Journal of Mathematical Economics*, 74, 21–34. 4

[30] Zhang, M. (2024): "Optimal contests with incomplete information and convex effort costs," *Theoretical Economics*, 19, 95–129. 2

A Proofs for Section 3 (Equilibrium)

Lemma 1. *Consider an arbitrary finite type-space (N, Θ, p) and an increasing differentiable cost function c with $c(0) = 0$. For any contest $v = (v_1, \ldots, v_{N-1}, 0)$, there is a unique symmetric Bayes-Nash equilibrium and it is such that there exist boundary points $b_1 < b_2 < \cdots < b_K$ so that for any $\theta_k \in \Theta$, an agent of type θ_k mixes between $[b_{k-1}, b_k]$ with $b_0 = 0$.*

Proof. Let F_k denote the equilibrium distribution function of agent of type θ_k for $k \in \{1, 2, \ldots, K\}$. Further, suppose F_k has support on the interval $[a_k, b_k]$ and u_k denotes the expected payoff of an agent of type θ_k when all agents play the equilibrium profile $(F_1, F_2, \ldots, F_K)$. Also let $F : \mathbb{R}_+ \to [0, 1]$ denote the CDF of the equilibrium effort under $(F_1, F_2, \ldots, F_K)$.

1. We first show that F_k cannot have any atoms. Suppose instead that F_k is such that an agent of type θ_k plays x_k with positive probability. Then, there is a positive probability that all agents are tied at effort x_k. But then, an agent of type θ_k will be strictly better off by bidding $x_k + \epsilon$ instead of x_k. This way, the agent earns the best prize among those that would have been otherwise split randomly between the tied agents and only pays an additional ϵ. Thus, F_k cannot have any atoms.

2. We now show that $\min\{a_1, a_2, \ldots, a_K\} = 0$. Suppose instead that this equals $a_k > 0$. Now when an agent of type θ_k plays a_k, it does not win any prize but it pays a positive cost of $\theta_k c(a_k)$. So this agent can instead play 0 and while it still doesn't get a prize, it also doesn't pay any cost. Thus, it must be that $\min\{a_1, a_2, \ldots, a_K\} = 0$.

3. We now show that $u_1 \leq u_2 \leq \cdots \leq u_K$. Suppose instead that $u_k > u_{k+1}$ for some $k \in \{1, 2, \ldots, K-1\}$. Also let $x_k \in [a_k, b_k]$. Then, we have that

   $$u_k = \pi_v(F(x_k)) - \theta_k c(x_k).$$

   So suppose agent of type θ_{k+1} plays x_k. Its payoff will be

   $$\pi_v(F(x_k)) - \theta_{k+1} c(x_k) > \pi_v(F(x_k)) - \theta_k c(x_k) = u_k$$

   because $\theta_{k+1} < \theta_k$. Thus, this agent of type θ_{k+1} can imitate an agent of type θ_k and get strictly higher payoff. Thus, it must be that $u_1 \leq u_2 \leq \cdots \leq u_K$.

4. We now show that for any $j \neq k$, $|[a_k, b_k] \cap [a_j, b_j]| \leq 1$. Suppose instead that $x, y \in [a_k, b_k] \cap [a_j, b_j]$ and $x \neq y$. Since agents must be indifferent between all actions in their support, it must be that

   $$u_k = \pi_v(F(x)) - \theta_k c(x) = \pi_v(F(y)) - \theta_k c(y)$$

   and also

   $$u_j = \pi_v(F(x)) - \theta_j c(x) = \pi_v(F(y)) - \theta_j c(y).$$

But this implies

$$\pi_v(F(x)) - \pi_v(F(y)) = \theta_k(c(x) - c(y)) = \theta_j(c(x) - c(y))$$

which is a contradiction.

5. We now show that if $b_k \neq \max\{b_1, b_2, \ldots, b_K\}$, then $b_k = a_j$ for some $j \in \{1, 2, \ldots, K\}$. Suppose instead that there is a k such that $b_k \neq \max\{b_1, b_2, \ldots, b_K\}$ and $a_j \neq b_k$ for any $j \in \{1, 2, \ldots, K\}$. Let a_p denote the minimum of all a_j such that $a_j \geq b_k$. Now we can repeat the argument in (2) to show that a_p must be equal to b_k because otherwise, an agent of type θ_p would be strictly better off playing b_k instead of a_p. Thus, we have that the support intervals of the mixed strategies are connected.

6. We now show that if $b_k = a_j$, then $\theta_k \geq \theta_j$. Suppose instead that $\theta_k < \theta_j$. First note that,

$$u_k = \pi_v(F(a_k)) - \theta_k c(a_k) = \pi_v(F(b_k)) - \theta_k c(b_k).$$

Since $b_k = a_j$, we have that

$$u_j = \pi_v(F(b_k)) - \theta_j c(b_k) = u_k + c(b_k)(\theta_k - \theta_j).$$

Now the payoff of agent of type θ_j from playing $a_k < b_k = a_j$ will be

$$\pi_v(F(a_k)) - \theta_j c(a_k) = u_k + (\theta_k - \theta_j)c(a_k)$$

which is greater than u_j if $\theta_k < \theta_j$. Thus, it must be that $\theta_k \geq \theta_j$.

Together, the above steps imply the result in the Lemma.

□

B Proofs for Section 4 (Optimal contest)

Lemma 2. *Consider an arbitrary finite type-space (N, Θ, p) and an increasing differentiable cost function c with $c(0) = 0$. For any contest $v = (v_1, \ldots, v_{N-1}, 0)$, the expected equilibrium effort of an arbitrary agent is*

$$\mathbb{E}[X] = \sum_{k=1}^{K} \int_{P_{k-1}}^{P_k} g\left(\frac{\pi_v(t) - u_k}{\theta_k}\right) dt. \tag{7}$$

Proof. We first find the expected effort exerted in equilibrium by an agent of type θ_k. To find $\mathbb{E}[X_k]$, we have from Theorem 1 that for an agent of type θ_k, the (random) level of effort X_k it exerts in equilibrium satisfies

$$\pi_v(P_{k-1} + p_k F_k(X_k)) - \theta_k c(X_k) = u_k.$$

Rearranging and taking expectations on both sides, we obtain

$$\begin{aligned}
\mathbb{E}[X_k] &= \mathbb{E}\left[g\left(\frac{\pi_v(P_{k-1}+p_kF_k(x_k))-u_k}{\theta_k}\right)\right] \\
&= \int_{b_{k-1}}^{b_k} g\left(\frac{\pi_v(P_{k-1}+p_kF_k(x_k))-u_k}{\theta_k}\right) f_k(x_k)dx_k \\
&= \int_0^1 g\left(\frac{\pi_v(P_{k-1}+p_kt)-u_k}{\theta_k}\right) dt && \text{(Substituting } F_k(x_k)=t\text{)}.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k=1}^{K} p_k\mathbb{E}[X_k] \\
&= \sum_{k=1}^{K} p_k \int_0^1 g\left(\frac{\pi_v(P_{k-1}+p_kt)-u_k}{\theta_k}\right) dt \\
&= \sum_{k=1}^{K} \int_{P_{k-1}}^{P_k} g\left(\frac{\pi_v(p)-u_k}{\theta_k}\right) dp && \text{(Substituting } P_{k-1}+p_kt=p\text{)}.
\end{aligned}$$

as required. □

Lemma 3. *Consider an arbitrary finite type-space (N,Θ,p) and the cost function $c(x)=x$. For any contest $v=(v_1,\ldots,v_{N-1},0)$, the expected equilibrium effort of an arbitrary agent is*

$$\mathbb{E}[X] = \sum_{m=1}^{N-1}\alpha_m v_m$$

where

$$\alpha_m = \frac{1}{N}\left[\frac{1}{\theta_K} - \sum_{k=1}^{K-1}\left[H^N_{\geq N-m+1}(P_k)+mH^N_{N-m}(P_k)\right]\left(\frac{1}{\theta_{k+1}}-\frac{1}{\theta_k}\right)\right]. \tag{8}$$

Proof. With linear costs $c(x)=x$, since both $\pi_v(p)$ and u_k are linear in v_m, it follows from Equation 7 that the expected effort $\mathbb{E}[X]=\sum_{m=1}^{N-1}\alpha_m v_m$ for some α_m. Now to find α_m, we have that

$$\begin{aligned}
\alpha_m &= \frac{\partial\mathbb{E}[X]}{\partial v_m} \\
&= \sum_{k=1}^{K}\int_{P_{k-1}}^{P_k} g'\left(\frac{\pi_v(p)-u_k}{\theta_k}\right)\left(\frac{H^{N-1}_{N-m}(t)}{\theta_k}-\frac{1}{\theta_k}\frac{\partial u_k}{\partial v_m}\right)dt \\
&= \sum_{k=1}^{K}\int_{P_{k-1}}^{P_k} \left(\frac{H^{N-1}_{N-m}(t)}{\theta_k}-\frac{1}{\theta_k}\frac{\partial u_k}{\partial v_m}\right)dt
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{K} \left[\frac{H^N_{\geq N-m+1}(P_k) - H^N_{\geq N-m+1}(P_{k-1})}{N\theta_k} - \frac{p_k}{\theta_k}\frac{\partial u_k}{\partial v_m} \right] \\
&= \sum_{k=1}^{K} \left[\frac{H^N_{\geq N-m+1}(P_k) - H^N_{\geq N-m+1}(P_{k-1})}{N\theta_k} \right] - \sum_{k=1}^{K} p_k \left[\sum_{j=1}^{k-1} H^{N-1}_{N-m}(P_j) \left(\frac{1}{\theta_{j+1}} - \frac{1}{\theta_j} \right) \right] \\
&= \frac{1}{N} \left[\frac{1}{\theta_K} - \sum_{k=1}^{K-1} H^N_{\geq N-m+1}(P_k) \left(\frac{1}{\theta_{k+1}} - \frac{1}{\theta_k} \right) \right] - \sum_{k=1}^{K-1} (1-P_k) \left[H^{N-1}_{N-m}(P_k) \left(\frac{1}{\theta_k + 1} - \frac{1}{\theta_k} \right) \right] \\
&= \frac{1}{N} \left[\frac{1}{\theta_K} - \sum_{k=1}^{K-1} H^N_{\geq N-m+1}(P_k) \left(\frac{1}{\theta_{k+1}} - \frac{1}{\theta_k} \right) \right] - \frac{m}{N} \sum_{k=1}^{K-1} \left[H^{N}_{N-m}(P_k) \left(\frac{1}{\theta_{k+1}} - \frac{1}{\theta_k} \right) \right] \\
&= \frac{1}{N} \left[\frac{1}{\theta_K} - \sum_{k=1}^{K-1} \left[H^N_{\geq N-m+1}(P_k) + m H^N_{N-m}(P_k) \right] \left(\frac{1}{\theta_{k+1}} - \frac{1}{\theta_k} \right) \right]
\end{aligned}
$$

as required. $\square$

Theorem 2. *Consider an arbitrary finite type-space (N, Θ, p). For any pair of prizes $b, w \in \{1, \ldots, N-1\}$ with $b < w$ such that, either $b = 1$ or $\left(\frac{\partial u_K}{\partial v_b} - \frac{\partial u_K}{\partial v_w} \right) \leq 0$, the following hold:*

1. *If $\alpha_b - \alpha_w \geq 0$, then for any strictly concave cost c and contest v, $\frac{\partial \mathbb{E}[X]}{\partial v_b} - \frac{\partial \mathbb{E}[X]}{\partial v_w} > 0$.*

2. *If $\alpha_b - \alpha_w \leq 0$, then for any strictly convex cost c and contest v, $\frac{\partial \mathbb{E}[X]}{\partial v_b} - \frac{\partial \mathbb{E}[X]}{\partial v_w} > 0$.*

Proof. Fix any type-space (N, Θ, p), cost function c, and contest v. We know from Lemma 2 that

$$
\mathbb{E}[X] = \sum_{k=1}^{K} \int_{P_{k-1}}^{P_k} g\left(\frac{\pi_v(t) - u_k}{\theta_k} \right) dt.
$$

Differentiating with respect to v_m, we obtain

$$
\begin{aligned}
\frac{\partial \mathbb{E}[X]}{\partial v_m} &= \sum_{k=1}^{K} \int_{P_{k-1}}^{P_k} g'\left(\frac{\pi_v(t) - u_k}{\theta_k} \right) \left(\frac{H^{N-1}_{N-m}(t)}{\theta_k} - \frac{1}{\theta_k}\frac{\partial u_k}{\partial v_m} \right) dt \\
&= \int_0^1 g'\left(\frac{\pi_v(t) - u_{k(t)}}{\theta_{k(t)}} \right) \left(\frac{H^{N-1}_{N-m}(t)}{\theta_{k(t)}} - \frac{1}{\theta_{k(t)}}\frac{\partial u_{k(t)}}{\partial v_m} \right) dt \quad (\text{where } k(t) = \max\{k : P_{k-1} \leq t\}) \\
&= \int_0^1 \beta(t)\lambda_m(t) dt
\end{aligned}
$$

where

$$
\beta(t) = g'\left(\frac{\pi_v(t) - u_{k(t)}}{\theta_{k(t)}} \right)
$$

and

$$\lambda_m(t) = \left(\frac{H^{N-1}_{N-m}(t)}{\theta_{k(t)}} - \frac{1}{\theta_{k(t)}} \frac{\partial u_{k(t)}}{\partial v_m} \right).$$

It follows that for prizes $b, w \in \{1, \ldots, N-1\}$ with $b < w$,

$$\frac{\partial \mathbb{E}[X]}{\partial v_b} - \frac{\partial \mathbb{E}[X]}{\partial v_w} = \int_0^1 \beta(t) \left(\lambda_b(t) - \lambda_w(t)\right) dt.$$

Observe that

$$\begin{aligned}
\lambda_b(t) - \lambda_w(t) &= \left(\frac{H^{N-1}_{N-b}(t) - H^{N-1}_{N-w}(t)}{\theta_{k(t)}} \right) - \frac{1}{\theta_{k(t)}} \left(\frac{\partial u_{k(t)}}{\partial v_b} - \frac{\partial u_{k(t)}}{\partial v_w} \right) \\
&= \left(\frac{H^{N-1}_{N-b}(t) - H^{N-1}_{N-w}(t)}{\theta_{k(t)}} \right) - \left[\sum_{j=1}^{k(t)-1} \left(H^{N-1}_{N-b}(P_j) - H^{N-1}_{N-w}(P_j) \right) \left(\frac{1}{\theta_{j+1}} - \frac{1}{\theta_j} \right) \right]
\end{aligned}$$

where the last equality follows from Equation 6. From here, one can verify that

1. $\lambda_b(0) - \lambda_w(0) = 0$
2. $\lambda_b(1) - \lambda_w(1) = \begin{cases} \frac{1}{\theta_K} - \frac{1}{\theta_K} \left(\frac{\partial u_K}{\partial v_b} - \frac{\partial u_K}{\partial v_w} \right) & \text{if } b = 1 \\ -\frac{1}{\theta_K} \left(\frac{\partial u_K}{\partial v_b} - \frac{\partial u_K}{\partial v_w} \right) & \text{otherwise} \end{cases}$
3. $\lambda_b(t) - \lambda_w(t)$ is continuous in t
4. $\lambda_b(t) - \lambda_w(t)$ is differentiable at $t \in [0, 1]$ where $t \neq P_k$, and whenever this derivative exists, it has the same sign as the derivative of $H^{N-1}_{N-b}(t) - H^{N-1}_{N-w}(t)$ with respect to p.
   (a) For $2 \leq b < w$, $H^{N-1}_{N-b}(t) - H^{N-1}_{N-w}(t)$ is 0 at $t = 0$, and as t increases, it first decreases and is negative, then it increases and becomes positive, and then again decreases finishing at 0 at $t = 1$.
   (b) And for $1 = b < w$, $H^{N-1}_{N-b}(t) - H^{N-1}_{N-w}(t)$ is 0 at $t = 0$, and as t increases, it first decreases and is negative, then it keeps increasing and becomes positive and finishes at 1 when $t = 1$.

Since b, w is such that either $b = 1$ or $\left(\frac{\partial u_K}{\partial v_b} - \frac{\partial u_K}{\partial v_w} \right) \leq 0$, we have that $\lambda_b(1) - \lambda_w(1) \geq 0$. Together with the above properties, this implies that there is some $t^* \in [0, 1]$ such that $\lambda_b(t) - \lambda_w(t) \leq 0$ for $t \in [0, t^*]$, and $\lambda_b(t) - \lambda_w(t) \geq 0$ for $t \in [t^*, 1]$.

Now for the first claim, suppose that the cost function is weakly concave and $\alpha_b - \alpha_w \geq 0$. Then, the function $g = c^{-1}$ is weakly convex, and therefore, $\beta(t)$ is weakly increasing in t. From this, it follows that

$$\frac{\partial \mathbb{E}[X]}{\partial v_b} - \frac{\partial \mathbb{E}[X]}{\partial v_w} = \int_0^1 \beta(t) \left(\lambda_b(t) - \lambda_w(t)\right) dt$$

$$
\begin{aligned}
&= \int_0^{t^*} \beta(t)\left(\lambda_b(t) - \lambda_w(t)\right) dt + \int_{t^*}^1 \beta(t)\left(\lambda_b(t) - \lambda_w(t)\right) dt \\
&> \int_0^{t^*} \beta(t^*)\left(\lambda_b(t) - \lambda_w(t)\right) dt + \int_{t^*}^1 \beta(t^*)\left(\lambda_b(t) - \lambda_w(t)\right) dt \\
&= \beta(t^*) \int_0^1 \left(\lambda_b(t) - \lambda_w(t)\right) dt \\
&= \beta(t^*)(\alpha_b - \alpha_w) \\
&\geq 0
\end{aligned}
$$

The argument for the second claim with convex costs and $\alpha_b - \alpha_w \leq 0$ is analogous.

□

Lemma 4. *Consider a finite type-space* (N, Θ, p) *with* $\Theta = \{\theta_1, \theta_2\}$. *For* $m \in \{1, \ldots, N-2\}$,

$$\left(\frac{\partial u_K}{\partial v_m} - \frac{\partial u_K}{\partial v_{m+1}}\right) \leq 0 \iff p_2 \geq \frac{m}{N}.$$

and

$$\alpha_m - \alpha_{m+1} \geq 0 \iff p_2 \geq \frac{m-1}{N-1}.$$

Proof. From Equation 6, we have that the effect of transferring value from prize $w = m+1$ to prize $b = m$ on the equilibrium utility of the most efficient agent is

$$
\begin{aligned}
\left(\frac{\partial u_K}{\partial v_m} - \frac{\partial u_K}{\partial v_{m+1}}\right) &= \theta_K \left[\sum_{k=1}^{K-1} \left(H^{N-1}_{N-m}(P_k) - H^{N-1}_{N-(m+1)}(P_k)\right)\left(\frac{1}{\theta_{k+1}} - \frac{1}{\theta_k}\right)\right] \\
&= \theta_2 \left(H^{N-1}_{N-m}(P_1) - H^{N-1}_{N-m-1}(P_1)\right)\left(\frac{1}{\theta_2} - \frac{1}{\theta_1}\right).
\end{aligned}
$$

From here, it is easy to verify that for any $m \in \{1, 2, \ldots, N-2\}$,

$$\left(\frac{\partial u_K}{\partial v_m} - \frac{\partial u_K}{\partial v_{m+1}}\right) \leq 0 \iff P_1 \leq 1 - \frac{m}{N}.$$

From Equation 8, we also have that

$$
\begin{aligned}
\alpha_m - \alpha_{m+1} &= \frac{1}{N}\left[\sum_{k=1}^{K-1} \left[H^N_{\leq N-m}(P_k) - mH^N_{N-m}(P_k) - H^N_{\leq N-m-1}(P_k) + (m+1)H^N_{N-m-1}(P_k)\right]\left(\frac{1}{\theta_{k+1}} - \frac{1}{\theta_k}\right)\right] \\
&= \frac{1}{N}\left[H^N_{N-m}(P_1) - mH^N_{N-m}(P_1) + (m+1)H^N_{N-m-1}(P_1)\right]\left(\frac{1}{\theta_2} - \frac{1}{\theta_1}\right)
\end{aligned}
$$

From here, we can again verify that for any $m \in \{1, 2, \ldots, N-2\}$,

$$\alpha_m - \alpha_{m+1} \geq 0 \iff P_1 \leq 1 - \frac{m-1}{N-1}.$$

□

C Proofs for Section 5 (Continuum Type-Space Convergence)

Lemma 5. *Suppose there are N agents, each with a private type drawn from $\Theta = [\underline{\theta}, \overline{\theta}]$ according to a differentiable CDF $G : [\underline{\theta}, \overline{\theta}] \to [0, 1]$. For any contest $v = \{v_1, v_2, \ldots, v_{N-1}, 0\}$, there is a unique symmetric Bayes-Nash equilibrium and it is such that for any $\theta \in \Theta$,*

$$X(\theta) = \int_{\theta}^{\overline{\theta}} \frac{\pi_v'(1 - G(t))g(t)}{t} dt.$$

Proof. Suppose $N-1$ agents are playing a strategy $X : [\underline{\theta}, \overline{\theta}] \to \mathbb{R}_+$ so that if an agent's type is θ, it exerts effort $X(\theta)$. Further suppose that $X(\theta)$ is decreasing in θ. Now we want to find the remaining agent's best response to this strategy of the other agents. If the agent's type is θ and it pretends to be an agent of type $t \in [\underline{\theta}, \overline{\theta}]$, its payoff is

$$\pi_v(1 - G(t)) - \theta X(t).$$

Taking the first order condition, we get

$$\pi_v'(1 - G(t))(-g(t)) - \theta X'(t) = 0.$$

Now we can plug in $t = \theta$ to get the condition for $X(\theta)$ to be a symmetric Bayes-Nash equilibrium. Doing so, we get

$$\pi_v'(1 - G(\theta))(-g(\theta)) - \theta X'(\theta) = 0$$

so that

$$X(\theta) = \int_{\theta}^{\overline{\theta}} \frac{\pi_v'(1 - G(t))g(t)}{t} dt.$$

□

Theorem 3. *Suppose there are N agents and consider a fixed contest $v = (v_1, v_2, \ldots, v_{N-1}, 0)$. Let $G : [\underline{\theta}, \overline{\theta}] \to [0, 1]$ be a differentiable CDF and let $G^1, G^2, \ldots,$ be any sequence of CDF's, each with a finite support, such that for all $\theta \in [\underline{\theta}, \overline{\theta}]$,*

$$\lim_{n \to \infty} G^n(\theta) = G(\theta).$$

Let $F^n : \mathbb{R} \to [0, 1]$ denote CDF of the equilibrium effort under the finite type-space distribution G^n, and let $F : \mathbb{R} \to [0, 1]$ denote CDF of the equilibrium under continuum type-space distribution G. Then, the sequence of CDF's $F^1, F^2, \ldots,$ converges to the CDF F, i.e., for all $x \in \mathbb{R}$,

$$\lim_{n \to \infty} F^n(x) = F(x).$$

Proof. For the finite support CDF G^n, let $\Theta^n = (\theta_1^n, \theta_2^n, \ldots, \theta_{K(n)}^n)$ denote the support and $p^n = (p_1^n, p_2^n, \ldots, p_{K(n)}^n)$ denote the probability mass function. From Theorem 1, let $b^n = (b_1^n, b_2^n, \ldots, b_{K(n)}^n)$ denote the boundary points, $u^n = (u_1^n, u_2^n, \ldots, u_{K(n)}^n)$ denote the equilibrium utilities, and F_k^n denote the equilibrium CDF of agent of type θ_k^n on support $[b_{k-1}^n, b_k^n]$. Then, the CDF of the equilibrium effort, $F^n : \mathbb{R} \to [0,1]$, is such that for any $x \in \mathbb{R}$,

$$F^n(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ P_{k-1}^n + p_k^n F_k^n(x) & \text{if } x \in [b_{k-1}^n, b_k^n] \\ 1 & \text{if } x \geq b_{K(n)}^n \end{cases}. \tag{9}$$

For the continuum CDF $G : [\underline{\theta}, \overline{\theta}] \to [0,1]$, the CDF of the equilibrium effort, $F : \mathbb{R} \to [0,1]$, is such that for any $x \in \mathbb{R}$,

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 - G(\theta(x)) & \text{if } x \in [0, B] \\ 1 & \text{if } x \geq B \end{cases}. \tag{10}$$

where $\theta(x)$ is the inverse of $X(\theta)$ (from Lemma 5) and $B = X(\underline{\theta})$. The following Lemma will be the key to showing that $F^n(x)$ converges to $F(x)$ for all $x \in \mathbb{R}$.

Lemma 6. *Consider any $\theta \in (\underline{\theta}, \overline{\theta})$ and for any $n \in \mathbb{N}$, let $k(n) \in \{0, 1, 2, \ldots, K(n)\}$ be such that $\theta_{k(n)}^n > \theta \geq \theta_{k(n)+1}^n$ (where $\theta_0^n = \infty$ and $\theta_{K(n)+1}^n = 0$). Then,*

$$\lim_{n\to\infty} b_{k(n)}^n = X(\theta) \text{ and } \lim_{n\to\infty} F^n(b_{k(n)}^n) = 1 - G(\theta).$$

Proof. From Lemma 5 and Equation 2, we have

$$X(\theta) = \int_\theta^{\overline{\theta}} \frac{\pi_v'(1-G(t))g(t)}{t} dt \text{ and } b_{k(n)}^n = \sum_{j=1}^{k(n)} \frac{\pi_v(P_j^n) - \pi_v(P_{j-1}^n)}{\theta_j^n}.$$

Observe that

$$\begin{aligned} b_{k(n)}^n &= \left[\frac{\pi_v(P_{k(n)}^n)}{\theta_{k(n)}^n} - \sum_{j=1}^{k(n)-1} \pi_v(P_j^n) \left[\frac{1}{\theta_{j+1}^n} - \frac{1}{\theta_j^n} \right] \right] \\ &= \int_0^{1/\theta_{k(n)}^n} \left[\pi_v(P_{k(n)}^n) - \pi_v(1 - G^n(1/x)) \right] dx \\ &\xrightarrow{n\to\infty} \int_0^{\frac{1}{\theta}} \left[\pi_v(1-G(\theta)) - \pi_v(1-G(1/x)) \right] dx \quad \text{(dominated convergence)} \\ &= \underbrace{\left[x(\pi_v(1-G(\theta)) - \pi_v(1-G(1/x))) \right]_0^{\frac{1}{\theta}}}_{\text{this is 0}} + \int_0^{\frac{1}{\theta}} \frac{\pi_v'(1-G(1/x))g(1/x)}{x} dx \end{aligned}$$

$$= \int_{\theta}^{\infty} \frac{\pi_v'(1-G(t))g(t)}{t} \, \mathrm{d}t \qquad \text{(substitute } t = 1/x\text{)}$$
$$= X(\theta)$$

By definition, we have

$$\begin{aligned}
\lim_{n\to\infty} F^n(b^n_{k(n)}) &= \lim_{n\to\infty} P^n_{k(n)} \\
&= \lim_{n\to\infty} [1 - G^n(\theta)] \\
&= 1 - G(\theta)
\end{aligned}$$

□

Returning to the proof of Theorem 3, fix any $x \in (0, B)$ and let $\theta \in (\underline{\theta}, \overline{\theta})$ be such that $X(\theta) = x$. Then, we know that

$$F(x) = 1 - G(\theta).$$

We want to show that

$$\lim_{n\to\infty} F^n(x) = 1 - G(\theta).$$

Take $\epsilon > 0$. Find $\theta' < \theta$ and $\theta'' > \theta$ such that

$$0 < G(\theta) - G(\theta') = G(\theta'') - G(\theta) < \frac{\epsilon}{4}.$$

Let $x' = X(\theta')$, $x'' = X(\theta'')$, so that $x' > x > x''$. Let $\delta = \min\{x' - x, x - x''\}$. From Lemma 6, let $N(\epsilon)$ be such that for all $n > N(\epsilon)$,

$$\max\{|b^n_{k(n)} - x|, |b^n_{k'(n)} - x'|, |b^n_{k''(n)} - x''|\} < \frac{\delta}{2}$$

and

$$\max\{|F^n(b^n_{k'(n)}) - (1 - G(\theta'))|, |F^n(b^n_{k''(n)}) - (1 - G(\theta''))|\} < \frac{\epsilon}{4},$$

where $k(n), k'(n), k''(n)$ are sequences as defined in Lemma 6 for θ, θ' and θ'' respectively. Then, for all $n > N(\epsilon)$,

$$\begin{aligned}
F^n(x) &> F^n(b^n_{k''(n)}) \\
&> 1 - G(\theta'') - \frac{\epsilon}{4} \\
&> 1 - G(\theta) - \frac{\epsilon}{2}
\end{aligned}$$

and

$$\begin{aligned}
F^n(x) &< F^n(b^n_{k'(n)}) \\
&< 1 - G(\theta') + \frac{\epsilon}{4} \\
&< 1 - G(\theta) + \frac{\epsilon}{2}
\end{aligned}$$

so that $|F^n(x) - (1 - G(\theta))| < \epsilon$. Thus, $\lim_{n\to\infty} F^n(x) = 1 - G(\theta) = F(x)$ for all $x \in \mathbb{R}$. □