An efficiency ordering of k-price auctions under complete information

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Abstract

We study k-price auctions in a complete information environment and characterize all pure-strategy Nash equilibrium outcomes. In a setting with n agents having ordered valuations, we show that any agent, except those with the lowest $k-2$ valuations, can win in equilibrium. As a consequence, worst-case welfare increases monotonically as we go from $k = 2$ (second-price auction) to $k = n$ (lowest-price auction), with the first-price auction achieving the highest worst-case welfare.

1 Introduction

We study k-price sealed-bid auctions in a complete information environment with n agents who have strictly ordered valuations. In a k-price auction, all n agents submit their bids, the highest bidder wins the object (with ties broken in favor of the agent with the highest valuation), and pays the kth highest bid. We fully characterize the set of pure-strategy Nash equilibrium outcomes for every k-price auction.

For $k \in \{2, \ldots, n\}$, we show that any of the top $n-(k-2)$ valuation agents can win in equilibrium, while the bottom $(k-2)$ agents can never win. In other words, the second-price auction can be won by any of the n agents, the third-price auction by any of the top $n-1$ agents, and so on, until we reach the lowest-price auction ($k = n$), which can only be won by the top two agents. We further show that in the first-price auction ($k = n+1$), only the top agent can win. This equilibrium characterization reveals a natural ordering of k-price auctions in terms of their worst-case allocative efficiency: the worst-case equilibrium allocation becomes strictly more efficient as k increases from 2 to $n+1$.

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In closely related work, Tauman (2002) and Mathews and Schwartz (2017) also study k-price auctions in complete information environments. Under the restriction to pure strategies that are not weakly dominated, Tauman (2002) shows that for any k-price auction, only the top agent can win in equilibrium. Subsequently, Mathews and Schwartz (2017) construct an equilibrium in mixed-strategies where the top agent does not win. In comparison, we characterize all pure-strategy Nash equilibrium outcomes for all k-price auctions and obtain an ordering of these auctions based on their worst-case allocative efficiency.[1]

2 Model

A seller is selling an indivisible object to a set $N = \{1, \ldots, n\}$ of agents. Each agent $i \in N$ has a valuation $v_i > 0$ for the object, and we assume that

$$v_1 > v_2 > \cdots > v_n.$$

We further define $v_{n+1} = 0$. The valuations are assumed to be common knowledge.

The object is sold using a sealed-bid k-price auction. Each agent $i \in N$ simultaneously submits a non-negative bid $b_i \in \mathbb{R}_+$. The object is awarded to the agent who submits the highest bid, with ties resolved in favor of the agent with the highest valuation.[2] The winner pays the kth highest bid, denoted $b_{(k)}$, and all other agents pay zero. The utility of agent $i \in N$ at bid profile $b = (b_1, \ldots, b_n)$ is given by:

$$u_i(b) = \begin{cases} v_i - b_{(k)} & \text{if } i = \min\{j \in N : b_j = b_{(1)}\}, \\ 0 & \text{otherwise.} \end{cases}$$

We characterize pure-strategy Nash equilibrium outcomes of the k-price auction for all $k \in \{1, 2, \ldots, n\}$, where $k = 1$ denotes the first-price auction, $k = 2$ the second-price auction, and so on, with $k = n$ representing the lowest-price auction.[3] For notational convenience, we sometimes also use $k = n + 1$ to refer to the first-price auction.

[1] Other work on k-price auctions has focused on incomplete information settings (Kagel and Levin (1993), Monderer and Tennenholtz (2000, 2004), Mezzetti and Tsetlin (2009), Azrieli and Levin (2012), Mihelich and Shu (2020), Skitmore (2014)).

[2] Previous work has typically assumed that ties are broken uniformly at random (Tauman (2002), Mathews and Schwartz (2017)). We note that our results for k-price auctions with $k \in \{2, \ldots, n\}$ continue to hold under either tie-breaking rule. The choice of tie-breaking rule affects only the first-price auction and makes no substantive difference to the main conclusions. Our assumption allows us to discuss the first-price auction alongside the other k-price auctions in a unified way, aids exposition more generally, and is natural given our focus on welfare. Additionally, we note that from the seller's perspective, implementing this tie-breaking rule requires only knowledge of the ordinal ranking of agents' valuations, not their exact values.

[3] Since our analysis focuses on pure-strategy equilibria, our characterization results actually depend only on the ordinal ranking over bid profiles induced by the representation u_i. Specifically, we can define the outcome set for each agent as $X = \mathbb{R}_+ \cup \{-1\}$, where an outcome $x \in \mathbb{R}_+$ represents the payment made by the agent when it wins the object, and $x = -1$ denotes not winning. An agent $i \in N$ with valuation $v_i > 0$ is then represented by a preference relation $\succ_i$ over X, such that $x \succ_i y$ for all $x < y \in \mathbb{R}_+$, and $-1 \sim_i v_i$. Our results hold as long as the utility representation u_i is consistent with $\succ_i$ for all $i \in N$.

3 Results

We first characterize equilibrium outcomes from the seller's perspective. Specifically, we show that under the k-price auction, the seller's revenue must lie within the interval $[v_{n-(k-3)}, v_1]$, and any revenue within this interval is possible.

Proposition 1. *Consider a k-price auction with $k \in \{2, \ldots, n+1\}$. There exists a pure-strategy Nash equilibrium in which the seller's revenue is p if and only if*

$$p \in [v_{n-(k-3)}, v_1].$$

Proof. Fix any k-price auction. We first show that in equilibrium, the revenue $p \in [v_{n-(k-3)}, v_1]$. Suppose towards a contradiction that b is an equilibrium profile and $p = b_{(k)} \notin [v_{n-(k-3)}, v_1]$. Notice first that if $p > v_1$, the winner's utility is negative, which is not possible in equilibrium (since the winner can deviate by bidding 0). Thus, $p < v_{n-(k-3)}$. We analyze why this is not possible separately for different k.

1. $k = 2$: For the second-price auction, $p < v_{n-(k-3)}$ simplifies to $p < v_{n+1} = 0$. But $p = b_{(2)} \geq 0$, which is a contradiction.

2. $k \in \{3, \ldots, n\}$: For this k-price auction, $p < v_{n-(k-3)}$ and $p = b_{(k)}$ imply that

   (a) At least $(n-(k-3))$ agents have valuation $> p$ (as $v_1 > \cdots > v_{n-(k-3)} > p$).

   (b) At least k agents bid $\geq p$ at profile b.

   It follows then that there are at least three distinct agents with valuations strictly greater than p, each bidding at least p. At least two of these three agents receive utility 0. At least one of these two agents can deviate by bidding $> b_{(1)}$, and receive strictly positive utility. This contradicts b being an equilibrium.

3. $k = n+1$: For the first-price auction, $p < v_{n-(k-3)}$ simplifies to $p < v_2$. Notice that at least one of agents 1 and 2 receive utility 0, and this agent can deviate by bidding in $(b_{(1)}, v_2)$, and receive strictly positive utility. This contradicts b being an equilibrium.

It follows that in any equilibrium, $p \in [v_{n-(k-3)}, v_1]$.

Now we show that for any $p \in [v_{n-(k-3)}, v_1]$, there exists an equilibrium in which the revenue is p. We construct such an equilibrium separately for different k.

1. $k \in \{2, \ldots, n\}$: For this k-price auction, and $p \in [v_{n-(k-3)}, v_1]$, consider the bid profile

$$b = (v_1, \underbrace{p, \ldots, p}_{n-(k-1) \text{ agents}}, \underbrace{v_1, \ldots, v_1}_{(k-2) \text{ agents}}).$$

   At profile b, agent 1 wins the k-price auction (as ties are broken in favor of agent with highest valuation), and pays a price $b_{(k)} = p$ for the good. Agent 1's utility is $v_1 - p \geq 0$ and for $j \neq 1$, agent j's utility is 0. We now verify that b is indeed a Nash equilibrium. Consider agent $j \in N$.

(a) $j = 1$: If $b'_1 \geq v_1$, agent 1's utility does not change. If $b'_1 < v_1$, agent 1's utility is either 0 or does not change (possible when $k = 2$).

(b) $j \in \{2, \ldots, n-(k-2)\}$: If $b'_j > v_1$, agent j's utility will be $v_j - v_1 < 0$. If $b'_j \leq v_1$, agent j's utility remains 0.

(c) $j \in \{n-(k-3), \ldots, n\}$: If $b'_j > v_1$, agent j's utility will be $v_j - p \leq 0$. If $b'_j \leq v_1$, agent j's utility remains 0.

Thus, no agent has a profitable deviation, and b is an equilibrium in the k-price auction.

2. $k = n+1$: For the first-price auction, and $p \in [v_2, v_1]$, consider the bid profile

$$b = (p, \ldots, p).$$

At the profile b, agent 1 wins the first-price auction (as ties are broken in favor of agent with highest valuation), and pays a price $b_{(1)} = p$ for the good. Agent 1's utility is $v_1 - p \geq 0$ and for $j \neq 1$, agent j's utility is 0. It is straightforward to verify that b is a Nash equilibrium in the first-price auction.

Thus, for any $p \in [v_{n-(k-3)}, v_1]$, there exists an equilibrium in which the revenue is p.[4] □

We now characterize equilibrium outcomes from the buyers perspective. Under the k-price auction, we show that any agent whose valuation exceeds $v_{n-(k-3)}$ can win at any price between $v_{n-(k-3)}$ and their own valuation, and that no other outcomes are possible.

Proposition 2. *Consider a k-price auction with $k \in \{2, \ldots, n+1\}$. There exists a pure-strategy Nash equilibrium in which agent $i \in N$ wins and pays p if and only if*

$$v_i > v_{n-(k-3)} \text{ and } p \in [v_{n-(k-3)}, v_i].$$

Proof. Fix any k-price auction. We first show that in equilibrium, if agent i wins and pays p, it must be that $v_i > v_{n-(k-3)}$ and $p \in [v_{n-(k-3)}, v_i]$. Suppose towards a contradiction that b is an equilibrium profile in which agent $i \in N$ with $v_i \leq v_{n-(k-3)}$ wins. From Proposition 1, it must pay $p \geq v_{n-(k-3)}$. But since the winner's utility must be non-negative in equilibrium (as it can deviate by bidding 0 otherwise), the only possibility is that $v_i = p = v_{n-(k-3)}$. We analyze why this is not possible separately for different k.

1. $k = 2$: For the second-price auction, $v_i = p = v_{n-(k-3)}$ simplifies to $v_i = p = v_{n+1} = 0$. But $v_i > 0$ for all $i \in N$, which is a contradiction.

2. $k \in \{3, \ldots, n\}$: For this k-price auction, $v_i = p = v_{n-(k-3)}$ and $p = b_{(k)}$ imply that

(a) At least $(n-(k-2))$ agents have valuation $> p$ (as $v_1 > \cdots > v_{n-(k-2)} > p$).

[4] There are many other equilibrium bid profiles that induce the same outcome. In particular, for $k \in \{2, \ldots, n\}$, we can modify our b so that $b_1 > v_1$, and the resulting profile remains a Nash equilibrium with the same outcome, irrespective of the tie-breaking rule.

(b) At least k agents bid $\geq p$ at profile b.

It follows then that there are at least two distinct agents with valuations strictly greater than p, each bidding at least p. Further, both these agents receive utility 0, and at least one of these two agents can deviate by bidding $> b_{(1)}$, and receive strictly positive utility. This contradicts b being an equilibrium.

3. $k = n+1$: For the first-price auction, $v_i = p = v_{n-(k-3)}$ simplifies to $v_i = p = v_2$. But then, agent 1 receives a utility of 0, and it can deviate by bidding in the interval (v_2, v_1), and receive strictly positive utility. This contradicts b being an equilibrium.

Thus, it must be that $v_i > v_{n-(k-3)}$. Further, if agent i wins and pays p, it must be that $p \leq v_i$, and from Proposition 1, $p \geq v_{n-(k-3)}$. Thus, in any equilibrium where i wins and pays p, it must be that $v_i > v_{n-(k-3)}$ and $p \in [v_{n-(k-3)}, v_i]$.

Now we show that for any $i \in N$ such that $v_i > v_{n-(k-3)}$ and $p \in [v_{n-(k-3)}, v_i]$, there exists an equilibrium where i wins and pays p. We construct one separately for different k.

1. $k \in \{2, \ldots, n\}$: For this k-price auction, $i \in N$ such that $v_i > v_{n-(k-3)}$ and $p \in [v_{n-(k-3)}, v_i]$, consider the bid profile b where

$$b_i = v_1 \text{ and } b_{-i} = (\underbrace{p, \ldots, p}_{n-(k-1) \text{ agents}}, \underbrace{v_1, \ldots, v_1}_{(k-2) \text{ agents}}).$$

At profile b, agent i wins the k-price auction (as ties are broken in favor of agent with highest valuation), and pays a price $b_{(k)} = p$ for the good. Agent i's utility is $v_i - p \geq 0$ and for $j \neq i$, agent j's utility is 0. We now verify that b is indeed a Nash equilibrium. Consider agent $j \in N$.

(a) $j = i$: If $b'_i \geq v_1$, agent i's utility does not change. If $b'_i < v_1$, agent i's utility is either 0 or does not change (possible when $k = 2$).

(b) $j \in \{1, \ldots, n-(k-2)\} \setminus \{i\}$: If $b'_j > v_1$, agent j's utility will be $v_j - v_1 \leq 0$. If $b'_j \leq v_1$, agent j's utility will be either 0 or < 0.

(c) $j \in \{n-(k-3), \ldots, n\}$: If $b'_j > v_1$, agent j's utility will be $v_j - p \leq 0$. If $b'_j \leq v_1$, agent j's utility remains 0.

Thus, no agent has a profitable deviation, and b is an equilibrium in the k-price auction.

2. $k = n+1$: For the first-price auction, $i = 1$ and $p \in [v_2, v_1]$, consider the bid profile

$$b = (p, \ldots, p).$$

It is straightforward to verify that b is a Nash equilibrium in the first-price auction.

Thus, for any $i \in N$ such that $v_i > v_{n-(k-3)}$ and $p \in [v_{n-(k-3)}, v_i]$, there exists a Nash equilibrium where agent i wins and pays p. $\square$

Proposition 2 yields a natural ranking of k-price auctions in terms of their worst-case efficiency. Formally, we let

$$\underline{W}^k = \min\{v_i : \exists b \text{ such that } b \text{ is an equilibrium of the } k\text{-price auction where agent } i \text{ wins}\},$$

denote the worst-case equilibrium welfare under the k-price auction.

Corollary 1. *For $k \in \{2, \ldots, n+1\}$, the worst-case equilibrium welfare of k-price auction is*

$$\underline{W}^k = v_{n-(k-2)}.$$

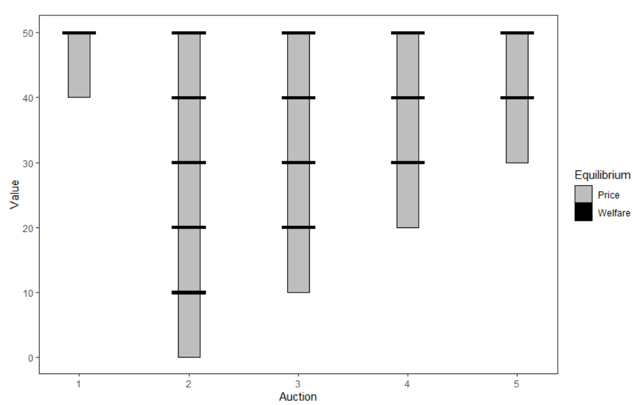
Hence,

$$\underline{W}^2 < \underline{W}^3 < \cdots < \underline{W}^n < \underline{W}^{n+1}.$$

Thus, the second-price auction is the worst, as even the lowest-valuation agent can win, while higher values of k progressively exclude lower-valuation agents from winning in equilibrium, culminating in the first-price auction, where only the highest-valuation agent can win.[5]

Lastly, we illustrate our results through an example with $n = 5$ agents whose valuations are $v_1 = 50, v_2 = 40, v_3 = 30, v_4 = 20$ and $v_5 = 10$. In figure 1 the black bars show the possible equilibrium welfare, while the grey vertical bars show the interval of possible equilibrium prices. In the first-price auction, only agent 1 can win at a price in $[40, 50]$. In the second-price auction, any agent i can win at a price in $[0, v_i]$. And so on, until we reach the lowest-price auction, where either agent 2 can win at a price in $[30, 40]$ or agent 1 can win at a price in $[30, 50]$. For example, under the lowest-price auction, $b = (35, 70, 65, 60, 55)$ is an equilibrium profile in which agent 2 wins (leading to welfare of 40) at price $p = 35$. Since the second lowest bid is above v_1, agent 1 is deterred from raising his bid, and since the price is above v_3, agents 3 through 5 are deterred from raising theirs.

Figure 1: k-Price Auctions With 5 Agents



[5]From Proposition 1, the same ordering also emerges when ranking k-price auctions by worst-case revenue.

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