Multi-agent contract design with independent trials: Winners-take-all based on weight and priority

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Abstract

We study a contract design problem between a principal and multiple agents. Each agent participates in an independent task, in which it may succeed or fail. The principal, who we assume has a fixed budget, can design a contract mapping observable outcomes to transfers for each agent, so as to incentivize the agents to expend effort into increasing their probability of success. We show that the Pareto frontier of success probabilities that can be sustained in equilibrium is characterized by equilibria of priority-based weighted contracts. These are contracts defined by a priority relation on the agents and a weight for each agent so that the entire budget is split among the highest-priority agents who are successful, with each agent's share proportional to her weight. We discuss applications to the design of optimal contracts for some special cases.

1 Introduction

Consider a scenario where a principal delegates individual tasks to agents, and the success or failure of these tasks depends stochastically on the effort exerted by the agents. The principal cares about increasing the likelihood of success and has a fixed budget that it can use to incentivize the agents to exert costly effort towards increasing their chances of success. Importantly, we assume that the principal cannot directly observe the effort exerted by the agents and thus, can only tie the rewards to the observed outcomes of success or failure. If the principal is dealing with just a solitary agent, the solution is straightforward—allocate the entire budget for success and none for failure. However, many real-world situations, like crowdsourcing events, involve multiple agents, making the challenge much more intricate. This paper studies the design of contracts between a principal and multiple agents, each

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tasked with an independent job that can result in either success or failure. The central question revolves around how the principal, operating within a fixed budget, can design a contract mapping observable outcomes to transfers for each agent, so as to incentivize the agents to exert costly effort towards increasing their chances of success.

Our main result is the characterization of the Pareto frontier of success probabilities that can be sustained in equilibrium as equilibria of priority-based weighted contracts. These are contracts defined by a priority relation on the agents and a weight for each agent so that the entire budget is split among the highest-priority agents who are successful, with each agent's share being proportional to her weight. In the process of obtaining this result, we derive several useful relationships between some natural classes of contracts and the equilibria they induce. First, we show that any feasible equilibrium can be induced by a contract in which agents who fail get nothing. Second, we show that the principal must allocate the entire budget among successful agents for the induced equilibrium to be Pareto optimal. And third, we show that the equilibria of priority-based weighted contracts actually coincide with equilibria of decreasing-returns contracts, in which the reward of any agent is decreasing in the set of agents that succeed. Thus, even though decreasing-returns contract would also be reasonable in that they induce Pareto-optimal equilibria, the principal can, without loss of generality, actually focus on a much smaller class of priority-based weighted contracts.

We believe our paper's findings have significant implications for crowdsourcing contests, where participants engage in independent tasks with uncertain outcomes. By utilizing priority-based weighted contracts, budget-constrained organizers can wisely utilize funds to motivate and reward participants. While the exact structure of the optimal contest may still depend in an important way on the application and the designer's objective, our results imply that the designer can restrict attention to priority-based weighted contracts, which is a significant reduction to the search space. More precisely, while the set of all contracts with n agents has dimension $n \cdot 2^n$, the set of all priority-based weighted contracts is only $n-1$-dimensional. We leverage this insight to solve the optimal contract design problem with $n = 2$ agents, where the principal values the weighted sum of the agents' success probabilities.

Below, we provide a review of the relevant literature. In section 2, we provide the model and definitions. In section 3, we describe and prove a characterization of the Pareto frontier. In section 4, we use our results to study an instance of the problem with two agents, quadratic costs, and a weighted sum objective.

Literature review

There is vast literature on principal-agent problems. Much of this literature has focused on a single agent case (Holmström [16], Grossman and Hart [12]), potentially with multiple tasks (Holmstrom and Milgrom [18], Dütting et al. [8], Bond and Gomes [4]). Our paper considers a contract design problem with multiple agents. In this domain, there has been a significant amount of work comparing the performance of rank order contracts, in which an agent's compensation depends on its relative output, with that of individual piece-rate contracts, in which the compensation depends only on its absolute output (Holmstrom [17], Lazear and Rosen [21], Green and Stokey [10], Nalebuff and Stiglitz [26], Malcomson [23], Imhof and Kräkel [19], Mookherjee [25]). In other related work, Baiman and Rajan [2] study discretionary bonus schemes, in which the total budget depends on the individual outcomes, and the allocation of this budget depends on some signal which is observable only to the principal.

There is a growing literature studying the behavior of both contestants and designers in crowdsourcing contests (for a recent survey, see e.g. Segev [32].) Our work fits most closely with a strain of this literature focusing on contests which involve search or innovation (Taylor [33], Halac et al. [14], Gross [11]). In Haggiag et al. [13], the authors also study a similar success/failure model of crowdsourcing, but one in which the relevant choice is whether to participate rather than how much effort to exert.

2 Model

There is a principal and n risk-neutral agents. Each agent $i \in [n]$ participates in an independent task, in which it may succeed or fail. The agents can choose their success probability, $p_i \in [0,1]$, and incur a cost of $c_i(p_i)$ in doing so, where the cost function $c_i : [0,1] \to \mathbb{R}_+$ is strictly convex with $c_i(0) = 0, c_i'(0) = 0, c_i'(1) > 1$. The principal cannot observe the chosen success probabilities, but it can observe whether each agent succeeds or fails in her respective task.

We assume that the principal has a budget of $B = 1$ that she can use to design a contract to incentivize the agents to improve their success probabilities.

Definition 2.1. A *contract* is a function $f = (f_1, \dots, f_n)$, where $f_i : 2^{[n]} \to \mathbb{R}_+$ such that for each $S \subseteq [n]$, $\sum_{i \in [n]} f_i(S) \leq 1$.

If the principal implements the contract f, then agent i receives a payment of $f_i(S)$ whenever S is the set of agents who are successful. We will denote by $\mathcal{F}$ the set of contracts. Note that it follows immediately from the definition of a contract that $\mathcal{F}$ is a compact, convex subset of $\mathbb{R}^{[n] \times 2^{[n]}}$.

Any contract $f \in \mathcal{F}$ defines a normal-form game between the n agents in which each agent chooses a $p_i \in [0,1]$ and agent $i's$ payoff under profile $p = (p_1, \dots, p_n)$ equals

$$u_i(p) = \sum_{S \subset [n]} f_i(S) \mathrm{Pr}_p^{[n]}(S) - c_i(p_i)$$

where

$$\mathrm{Pr}_p^{[n]}(S) = \Pi_{i:i \in S} p_i \Pi_{j:j \in [n] \setminus S} (1 - p_j).$$

It follows from the strict convexity of c_i that agent i's payoffs are strictly concave in p_i. Thus, for any contract f and p_{-i}, agent i has a unique best response. We denote by $\Psi_i(p_{-i}, f)$ this best response.

2.1 Equilibria

We first establish the existence of pure-strategy Nash equilibrium for any contract f.

Lemma 1. *For any contract $f \in \mathcal{F}$, there is a pure-strategy Nash equilibrium.*

Since the payoffs for each agent are concave in p_i, existence follows directly from Rosen [28][1]. Note that the marginal payoff of agent i at profile p is

$$\frac{\partial u_i(p)}{\partial p_i} = \sum_{S \subset [n]_{-i}} (f_i(S \cup \{i\}) - f_i(S)) \mathrm{Pr}_{p_{-i}}^{[n]_{-i}}(S) - c_i'(p_i)$$

and thus, if p is a pure-strategy Nash equilibrium, it must be that for all $i \in [n]$, either

$$p_i = 0 \text{ or } \frac{\partial u_i(p)}{\partial p_i} = 0 \iff c_i'(p_i) = \sum_{S \subset [n]_{-i}} (f_i(S \cup \{i\}) - f_i(S)) \mathrm{Pr}_{p_{-i}}^{[n]_{-i}}(S). \tag{1}$$

We denote by $E(f)$ the set of pure-strategy Nash equilibria under contract f, and by $E^{-1}(p)$ the set of contracts that induce p as an equilibrium. We denote by $\mathcal{E}$ the set of all equilibria that can be induced by some contract,

$$\mathcal{E} := \cup_{f \in \mathcal{F}} E(f),$$

and by $\mathcal{P}$ the Pareto frontier of $\mathcal{E}$,

$$\mathcal{P} := \{p \in \mathcal{E} \,:\, p' \in \mathcal{E},\, p' \geq p \implies p' = p\}.$$

We will say that $p \in \mathcal{E}$ is *Pareto optimal* if $p \in \mathcal{P}$.

We note here that $\mathcal{E}$ is compact, and for any $p \in \mathcal{E}$ there is some $p' \in \mathcal{P}$ such that $p' \geq p$. Thus, for any increasing continuous objective function for the principal $V(p)$, it is sufficient to find some $p^* \in argmax_{p \in \mathcal{P}} V(p)$ and some $f^* \in E^{-1}(p^*)$. In this paper, we will provide a characterization of $\mathcal{P}$ and a canonical contract for each $p \in \mathcal{P}$. We will use this characterization to solve for some natural objectives of the principal.

2.2 Contracts

Here, we define some important classes of contracts that will be useful in our analysis. Let $f \in \mathcal{F}$ be a contract.

Definition 2.2. f is a *failures-get-nothing* (FGN) contract if for every $i \in [n]$ and $S \subseteq [n]$,

$$i \notin S \implies f_i(S) = 0.$$

[1] While we haven't been able to establish uniqueness in general, we can use the diagonally strict concavity condition of Rosen [28] to show that equilibrium is unique if the cost functions are sufficiently convex.

Definition 2.3. f is a *successful-get-everything* (SGE) contract if for every $S \subseteq [n]$,

$$\sum_{i \in S} f_i(S) = 1.$$

Definition 2.4. f is a *diminishing-returns* (DR) contract if it is an SGE contract and for every $i \in [n]$ and every $S, T \subseteq [n]$ such that $S \subset T$,

$$f_i(S) \leq f_i(T).$$

Definition 2.5. f is a *priority-based weighted* (PW) contract if there exists an ordered partition $(X_1, X_2, \ldots, X_l)$ of $[n]$ and weights $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_i > 0$ such that for every $S \subseteq [n]$,

$$f_i(S) = \begin{cases} \dfrac{\lambda_i}{\sum_{j \in S \cap X_k} \lambda_j}, & \text{if } S \cap X_m = \emptyset \text{ for } m < k \text{ and } i \in S \cap X_k \\ 0, & \text{otherwise} \end{cases}$$

We denote these sets of contracts by $\mathcal{F}_{FGN}$, $\mathcal{F}_{SGE}$, $\mathcal{F}_{DR}$, and $\mathcal{F}_{PW}$, respectively. Observe that $\mathcal{F}_{PW} \subseteq \mathcal{F}_{DR} \subseteq \mathcal{F}_{SGE} \subseteq \mathcal{F}_{FGN}$.

There are several equivalent definitions of PW contracts that will be useful for our analysis. To state those, we introduce some objects. For any $f \in \mathcal{F}$, define $\succcurlyeq_f$ to be the relation on $[n]$ such that $i \succcurlyeq_f j$ if and only if $f_i(\{i, j\}) > 0$. Following standard notation for relations, we will write $i \succ_f j$ if $i \succcurlyeq_f j$ and $\neg(j \succcurlyeq_f i)$, and we will write $i \sim_f j$ if $i \succcurlyeq_f j$ and $j \succcurlyeq_f i$.

For any $S \subseteq [n]$, define

$$Top_f(S) = \{i \in S : i \succcurlyeq_f j \text{ for all } j \in S\}.$$

Lemma 2. *Let $f \in \mathcal{F}_{SGE}$. The following are equivalent.*

1. *f is a PW contract.*
2. *For all $S, T \subseteq [n]$, and $i, j \in S \cap T$,*

$$f_i(S) \cdot f_j(T) = f_i(T) \cdot f_j(S).$$

3. *The relation $\succcurlyeq_f$ is complete and transitive, and for each $i \in [n]$ and $S \subseteq [n]$,*

$$f_i(S) = \begin{cases} \dfrac{\lambda_i^f}{\sum_{j \in Top_f(S)} \lambda_j^f}, & \textit{if } i \in Top_f(S) \\ 0, & \textit{otherwise} \end{cases}$$

where

$$\lambda_i^f = f_i(\{j \in [n] : i \sim_f j\}).$$

We now note some important properties of the sets we have defined above.

Lemma 3.

1. *$\mathcal{F}_{FGN}$, $\mathcal{F}_{SGE}$, $\mathcal{F}_{DR}$ are compact, convex subsets of $\mathcal{F}$.*

2. *For any $p \in \mathcal{E}$, $E^{-1}(p) \cap \mathcal{F}_{FGN}$, $E^{-1}(p) \cap \mathcal{F}_{SGE}$, and $E^{-1}(p) \cap \mathcal{F}_{DR}$ are compact, convex subsets of $\mathcal{F}$.*

3. *$\mathcal{F}_{PW}$ is compact.*

The convexity and compactness of the sets follow directly from definitions. The convexity of set of contracts for a given p follows from the fact that agent's payoffs are linear in the contract. Thus, if f, g are such that $p \in E(f) \cap E(g)$, the first order condition in Equation 1 will continue to be satisfied at profile p under the contract $h = \lambda f + (1-\lambda)g$. It follows then that $p \in E(h)$.

We note here that the set $\mathcal{F}_{PW}$ is not convex. To see why, suppose f, g are two contracts in $\mathcal{F}_{PW}$ defined by relation $1 \succ_f 2 \succ_f 3$ and $3 \succ_g 2 \succ_g 1$ respectively. For $h = \lambda f + (1-\lambda)g$, we have $h_1(\{1,2,3\}) > 0$ while $h_2(\{1,2,3\}) = 0$. We also have $h_2(\{1,2\}) > 0$ and thus, $h \notin \mathcal{F}_{PW}$.

3 Pareto frontier

We now state our main results which is a characterization of contracts whose equilibria define the Pareto frontier.

Theorem 1. *If p is Pareto optimal, there is a PW contract f such that $p \in E(f)$.*

Theorem 2. *If f is a DR contract and $p \in E(f)$, then p is Pareto optimal.*

Theorem 1 says that $\mathcal{P} \subset E(\mathcal{F}_{PW})$ and Theorem 2 says that $E(\mathcal{F}_{DR}) \subset \mathcal{P}$. But we also have $E(\mathcal{F}_{PW}) \subset E(\mathcal{F}_{DR})$ because $\mathcal{F}_{PW} \subset \mathcal{F}_{DR}$. Thus, we obtain the following as a corollary of Theorems 1 and 2.

Corollary 1. $\mathcal{P} = E(\mathcal{F}_{DR}) = E(\mathcal{F}_{PW})$.

Now to prove Theorems 1 and 2, we first show that WLOG, we can restrict attention to FGN contracts.

Lemma 4. $E(\mathcal{F}_{FGN}) = \mathcal{E}$.

To prove Lemma 4, we consider any $p \in E(f)$ and construct a FGN contract

$$g_i(S) = \begin{cases} \lambda_i f_i(S), & \text{if } i \in S \\ 0, & \text{otherwise} \end{cases}$$

where

$$\lambda_i = \frac{c_i'(p_i)}{\sum_{S\subset[n]_{-i}} f_i(S\cup\{i\})\mathrm{Pr}_{p_{-i}}^{[n]_{-i}}(S)} \leq 1$$

and show that $p \in E(g)$. Going forward, we will restrict attention to FGN contracts.

Next, we show that for any $p \in \mathcal{P}$, $p_i > 0$.

Lemma 5. *If p is Pareto optimal, then $p_i > 0$ for all i.*

To prove Lemma 5, we take $p \in E(f)$ where f is a FGN contract and assume $p_k = 0$. Then, we construct a FGN contract

$$g_i(S) = f_i(S \setminus \{k\})$$

while

$$g_k(S) = \begin{cases} 1, & \text{if } S = \{k\} \\ 0, & \text{otherwise} \end{cases}$$

and show that $p' = (p'_k, p_{-k}) \in E(g)$ where $p'_k > 0$. It follows then that p cannot be on the Pareto frontier.

Next, we show that every contract for a Pareto optimal equilibrium is SGE.

Lemma 6. *If p is Pareto optimal and $p \in E(f)$, then f must be a SGE contract.*

To prove Lemma 6, we define the set

$$K_p := \{S \subseteq [n] \,:\, \sum_{i\in S} f_i(S) < 1 \text{ for some } f \in E^{-1}(p) \cap \mathcal{F}_{FGN}\},$$

and show that if $K_p \neq \{\phi\}$, we can find a p' that Pareto dominates p. First, we show that K_p is closed under taking subsets. The argument is that if $S \in K_p$ and $T \subset S$, we can pick an agent $i \in T$ and decrease its award under T while increasing its payoff in S so that p is still an equilibrium. Then, we get that $T \in K_p$. By a similar argument, we show that K_p is also closed under unions and thus, $K_p = 2^{\kappa_p}$ where $\kappa_p \subset [n]$. Next, we show that for any $f \in E^{-1}(p) \cap \mathcal{F}_{FGN}$, agents in κ_p^C must have priority over those in κ_p in that if any agent in κ_p^C succeeds, agents in κ_p get no reward. Using this, we construct a contract $h \in E^{-1}(p) \cap \mathcal{F}_{FGN}$ such that $\sum_{i\in S} h_i(S) < 1$ for any $S \subset \kappa_p$ and the reward for agent $i \notin \kappa_p$ do not depend on the success or failure of agents in κ_p. We then show that we can manipulate the awards for $S \subset \kappa_p$ to get a new contract h' such that $p' \in E(h')$ where $p'_i = p_i + \epsilon$ for $i \in \kappa_p$ and $\epsilon > 0$ while $p'_i = p_i$ for $i \notin \kappa_p$. Thus, it must be that $\kappa_p = \phi$ which complete the proof.

Going forward, we can focus on SGE contracts. Note that for $f \in \mathcal{F}_{FGN}$, the best response for agent i, $\Psi_i(p_{-i}, f)$, is defined by

$$c_i'(\Psi_i(p_{-i}, f)) = \sum_{S\subset[n]_{-i}} f_i(S\cup\{i\})\mathrm{Pr}_{p_{-i}}^{[n]_{-i}}(S). \tag{2}$$

Multiplying both sides of Equation 2 by p_i and adding up the resulting equations for the n agents, we obtain a useful aggregate relationship between p and the best responses $\Psi_i(p_{-i}, f)$ under any SGE contract f.

Lemma 7. *For any $p \in [0,1]^n$ and $f \in \mathcal{F}_{SGE}$,*

$$\sum_{i \in [n]} p_i \cdot c_i'(\Psi_i(p_{-i}, f)) = 1 - \Pr_p^{[n]}(\emptyset).$$

The reseult follows from the fact that $\sum_{i \in S} f_i(S) = 1$ for all $S \neq \phi$ when $f \in \mathcal{F}_{SGE}$.

Now given a SGE contract f and any profile p, let us define,

$$Z_p(f) := \max_{i \in [n]} c_i'(\Psi_i(p_{-i}, f)) - c_i'(p_i).$$

Then, we have the following.

Lemma 8. *If $p \in \mathcal{P}$ and $f \in \mathcal{F}_{SGE}$, then $Z_p(f) \geq 0$, and if $Z_p(f) = 0$ then $f \in E^{-1}(p)$.*

If $g \in E^{-1}(p)$ and $f \in \mathcal{F}_{SGE}$, we can apply Lemma 7 with both f, g to get that

$$\sum_{i \in [n]} p_i \cdot (c_i'(\Psi_i(p_{-i}, f)) - c_i'(p_i)) = 0 \tag{3}$$

which implies the result. Next, we establish another important property of PW contracts.

Lemma 9. *If $p \in \mathcal{P}$ and $f \in \mathcal{F}_{PW}$, then for every $i \in [n]$,*

$$\sum_{j \,:\, i \succcurlyeq_f j} p_j \cdot (c_j'(\Psi_j(p_{-j}, f)) - c_j'(p_j)) \leq 0.$$

Suppose $g \in E^{-1}(p)$ and $f \in \mathcal{F}_{PW}$. Multiplying Equation 2 by p_i and adding up the equations for agents j such that $i \succcurlyeq_f j$, we get that

$$\sum_{j \,:\, i \succcurlyeq_f j} p_j \cdot c_j'(\Psi_j(p_{-j}, f)) = \sum_{S \subseteq \{j \,:\, i \succcurlyeq_f j\}} \Pr_p^{[n]}(S) \leq \sum_{j \,:\, i \succcurlyeq_f j} \sum_{S \ni j} g_j(S) \Pr_p^{[n]}(S) = \sum_{j \,:\, i \succcurlyeq_f j} p_j \cdot c_j'(p_j)$$

which implies the result.

Lemma 10. *If $p \in \mathcal{P}$, then*

$$\inf_{f \in \mathcal{F}_{PW}} Z_p(f) = 0.$$

To prove Lemma 10, we suppose $z = \inf_{f \in \mathcal{F}_{PW}} Z_p(f) > 0$ and then show that we can find another *PW* contract g such that $Z_p(g) < z$. Informally, suppose $z = c_i'(\Psi_i(p_{-i}, f)) - c_i'(p_i) > 0$ where i is chosen so that $\iota(i) \geq \iota(j)$ among all agents j with this property. Then all agents j with lower priority than i must be such that $c_j'(\Psi_j(p_{-j}, f)) - c_j'(p_j) < z$. We can then basically construct g by grouping together agents with priority $\iota(i)$ and $\iota(i+1)$ giving a small weight ϵ to agents in $\iota(i+1)$. Then, the best response of agent $j \in \iota(i)$ at p_{-j} g will be smaller under g than f while for agents $j \in \iota(i+1)$, it will be greater under g than f. For $\epsilon > 0$ small enough, we will have $Z_p(g) < z$ which is a contradiction.

We are now ready to prove our two main results.

Proof of Theorem 1. Suppose $p \in \mathcal{P}$. By Lemma 10, $\inf_{f \in \mathcal{F}_{PW}} Z_p(f) = 0$. By Lemma 3, $\mathcal{F}_{PW}$ is compact. Since $Z_p(f)$ is continuous as a function of f, it follows that there is some $f \in \mathcal{F}_{PW}$ such that $Z_p(f) = 0$. By Lemma 8, it follows that $f \in E^{-1}(p)$. Thus, for any Pareto optimal p, there is a PW contract f such that $p \in E(f)$.

□

Proof of Theorem 2. Let $f \in \mathcal{F}_{DR}$ and $p \in E(f)$ and suppose towards a contradiction that $p \notin \mathcal{P}$. Then, there must be some $p' \in \mathcal{P}$ that Pareto dominates p. We know from Equation 3 that

$$\sum_{i \in [n]} p'_i(c'_i(\Psi_i(p'_{-i}, f)) - c'_i(p'_i)) = 0.$$

Since p'_{-i} Pareto dominates p_{-i}, we know that $\psi_i(p'_{-i}, f) \leq \psi_i(p_{-i}, f)$ for all $i \in [n]$. By convexity of c_i, $c'_i(\Psi_i(p'_{-i}, f)) \leq c'_i(\Psi_i(p_{-i}, f))$ and also, $c'_i(p'_i) \geq c'_i(p_i)$. It follows then that for all $i \in [n]$,

$$c'_i(\Psi_i(p'_{-i}, f)) - c'_i(p'_i) \leq c'_i(\Psi_i(p_{-i}, f)) - c_i(p_i) = 0$$

with the inequality being strict for at least one $i \in [n]$. But this implies

$$\sum_{i \in [n]} p'_i(c'_i(\Psi_i(p'_{-i}, f)) - c'_i(p'_i)) < 0.$$

which is a contradiction.

Thus, $p \in \mathcal{P}$. □

4 Application

In this section, we consider the case where there are two agents, each agent $i \in \{1, 2\}$ has a quadratic cost $c_i(p_i) = \frac{1}{2}c_i p_i^2$, and the designer's objective function is a weighted sum of the agents' probabilities of success:

$$V(p_1, p_2) = w_1 p_1 + w_2 p_2,$$

with $w_1, w_2 > 0$.

Since the objective V is strictly increasing in p_i,

$$\arg \max_{(p_1, p_2) \in \mathcal{E}} V(p_1, p_2) \subseteq \mathcal{P}.$$

Moreover, it follows from Theorem 1 that if $(p_1, p_2) \in \mathcal{P}$, then there is a contract $f \in E^{-1}((p_1, p_2))$ such that $f_1(\{1\}) = 1$, $f_2(\{2\}) = 1$, $f_1(\{1, 2\}) = \lambda$, and $f_2(\{1, 2\}) = 1 - \lambda$ for some $\lambda \in [0, 1]$. In words, the optimal contract design problem with two agents is now reduced to a one-dimensional problem of determining agent 1's share of the prize $\lambda \in [0, 1]$ when both agents 1 and 2 succeed in their task.

The equilibrium conditions for this contract are

$$c_1 p_1 = 1 - p_2 + \lambda p_2$$
$$c_2 p_2 = 1 - p_1 + (1 - \lambda) p_1.$$

For each λ, this system of equations has a unique solution:

$$p_1(\lambda) = \frac{c_2 - (1-\lambda)}{c_1 c_2 - \lambda(1-\lambda)} \quad p_2(\lambda) = \frac{c_1 - \lambda}{c_1 c_2 - \lambda(1-\lambda)}. \tag{4}$$

Hence, the designer's problem is equivalent to:

$$\max_{\lambda \in [0,1]} V(p_1(\lambda), p_2(\lambda)).$$

Using the first order condition, this is maximized either at $\lambda = 0$, or $\lambda = 1$, or where

$$\frac{p_2'(\lambda)}{p_1'(\lambda)} = -\frac{w_1}{w_2}.$$

We now note the following important property about the ratio $\frac{p_2'(\lambda)}{p_1'(\lambda)}$.

Lemma 11. *For $\lambda \in (0,1)$, $\frac{p_2'(\lambda)}{p_1'(\lambda)}$ is strictly decreasing.*

It follows from Lemma 11 that there is a function $\lambda^*(w)$ such that the unique optimal choice of λ is $\lambda^*\left(\frac{w_1}{w_2}\right)$. The next result identifies some important properties of this optimal contract with two agents.

Theorem 3. *$\lambda^*(w)$ is increasing in w. In particular,*

$$\lambda^*(w) = \begin{cases} 0, & \text{if } w \leq -\frac{p_2'(0)}{p_1'(0)} = \frac{c_1 c_2 - c_1}{c_1 c_2 + c_2 - 1} \\ \frac{1}{2}, & \text{if } w = 1 \\ 1, & \text{if } w \geq -\frac{p_2'(1)}{p_1'(1)} = \frac{c_1 c_2 + c_1 - 1}{c_1 c_2 - c_2} \end{cases}$$

For illustration purposes, let's assume $c_1 = c_2 = 2$. We already know that if only a single agent succeeds, it gets the entire budget. Theorem 3 says that if $w_1 \leq \frac{2}{3} w_2$, then the optimal contract awards the entire budget to agent 2 when both agents succeed. And if $w_1 \geq \frac{3}{2} w_2$, then it is best to award the entire budget to agent 1 when both agents succeed. And if the weights on the two agents are close enough so that $\frac{2}{3} w_2 < w_1 < \frac{3}{2} w_2$, then agent 1's share of the prize when both agents succeed is monotone increasing in its relative weigh w. Perhaps surprisingly, if the principal puts equal weight on the two agents so that $w_1 = w_2$, we get that the two agents should get equal share of the budget with $\lambda = \frac{1}{2}$, irrespective of the heterogeneity in their costs c_1, c_2.

5 Conclusion

We study a contract design problem between a principal and multiple agents. In a setting where each agent participates in an independent task, in which it may succeed or fail, we identify a simple and natural class of contracts that a budget-constrained principal can use to incentivize the agents to expend effort into increasing their probability of success. More precisely, we show that priority-based weighted contracts, in which the entire budget is split among the highest-priority agents who are successful according to their respective weight, characterize the Pareto frontier of equilibria that can be induced in equilibrium. The result provides a significant reduction in dimensionality of the optimal contract design problem for a principal whose objective is monotone increasing in the success probabilities of the agents. We illustrate this by applying our results to derive the optimal contract for a special parametric case with two agents. Our result for this case suggests that the structure of optimal contract depends more on the principal's bias towards agents than it does on their heterogeneity.

While it is difficult to give a closed form for the optimal contract for a given objective, the significant reduction in dimensionality and the simple structure of PW contracts suggests that approximating the optimal contract for given objective may not be computationally hard. We leave this and related questions for further work.

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A Proofs for Section 2 (Model)

Lemma 1. *For any contract $f \in \mathcal{F}$, there is a pure-strategy Nash equilibrium.*

Proof. Suppose $f \in \mathcal{F}$ is any contract. Then, the marginal utility for any agent i at profile p is

$$\frac{\partial u_i(p)}{\partial p_i} = \sum_{S \subset [n]_{-i}} (f_i(S \cup \{i\}) - f_i(S)) \Pr_{p_{-i}}^{[n]_{-i}}(S) - c_i'(p_i)$$

and thus,

$$\frac{\partial^2 u_i(p)}{\partial p_i^2} = -c_i''(p_i) < 0.$$

Since the payoffs $u_i(p)$ are concave in $p_i \in [0, 1]$, the induced game is a concave game and there exists a pure-strategy Nash equilibrium (Theorem 1 of Rosen [28]). □

Lemma 2. *Let $f \in \mathcal{F}_{SGE}$. The following are equivalent.*

1. *f is a PW contract.*
2. *For all $S, T \subseteq [n]$, and $i, j \in S \cap T$,*

$$f_i(S) \cdot f_j(T) = f_i(T) \cdot f_j(S).$$

3. *The relation $\succcurlyeq_f$ is complete and transitive, and for each $i \in [n]$ and $S \subseteq [n]$,*

$$f_i(S) = \begin{cases} \dfrac{\lambda_i^f}{\sum_{j \in Top_f(S)} \lambda_j^f}, & \textit{if } i \in Top_f(S) \\ 0, & \textit{otherwise} \end{cases}$$

where

$$\lambda_i^f = f_i(\{j \in [n] : i \sim_f j\}).$$

Proof. We will show $1 \implies 2$, $2 \implies 3$ and $3 \implies 1$.

1. Let's first show $1 \implies 2$. Suppose $f \in \mathcal{F}_{PW}$, and let $(X_1, \ldots, X_\ell)$ be the corresponding ordered partition and $\lambda_1, \ldots, \lambda_n$ be the corresponding weights. Define $\iota(i)$ to be the unique index such that $i \in X_{\iota(i)}$. Also let $Best_f(S) = \min\{i : S \cap X_i \neq \phi\}$.

   Suppose $S, T \subseteq [n]$ and $i, j \in S \cap T$.

   Note that $f_i(S) \cdot f_j(T) \neq 0$ if and only if $\iota(i) = Best_f(S)$ and $\iota(j) = Best_f(T)$. Since $\iota(i) = Best_f(S)$ implies $\iota(i) \leq \iota(j)$ and $\iota(j) = Best_f(T)$ implies $\iota(j) \leq \iota(i)$, it follows that $f_i(S) \cdot f_j(T) \neq 0$ if and only if $\iota(i) = \iota(j) = Best_f(S) = Best_f(T)$. Similarly, $f_i(T) \cdot f_j(S) \neq 0$ if and only if $\iota(i) = \iota(j) = Best_f(S) = Best_f(T)$.

Thus, we have $f_i(S)f_j(T) = 0$ if and only if $f_i(T) \cdot f_j(S) = 0$. And if $f_i(S)f_j(T) \neq 0$,

$$\begin{aligned}
f_i(S) \cdot f_j(T) &= \frac{\lambda_i}{\sum_{k \in X_{Best_f(S)}} \lambda_k} \cdot \frac{\lambda_j}{\sum_{k \in X_{Best_f(T)}} \lambda_k} \\
&= \frac{\lambda_i}{\sum_{k \in X_{Best_f(T)}} \lambda_k} \cdot \frac{\lambda_j}{\sum_{k \in X_{Best_f(S)}} \lambda_k} \\
&= f_i(T) \cdot f_j(S).
\end{aligned}$$

2. Now let's show $2 \implies 3$.

   Since $f \in \mathcal{F}_{SGE}$, $\succcurlyeq_f$ is complete. To see that $\succcurlyeq_f$ is transitive, suppose that $k \succ_f i$. Then for any j,

$$\begin{aligned}
f_i(\{i,j,k\}) &= f_k(\{i,k\}) \cdot f_i(\{i,j,k\}) \\
&= f_i(\{i,k\}) \cdot f_k(\{i,j,k\}) \\
&= 0.
\end{aligned}$$

   If $f_j(\{i,j,k\}) = 0$, then $f_k(\{i,j,k\}) = 1$ and

$$\begin{aligned}
f_j(\{j,k\}) &= f_j(\{j,k\}) \cdot f_k(\{i,j,k\}) \\
&= f_k(\{j,k\}) \cdot f_j(\{i,j,k\}) \\
&= 0,
\end{aligned}$$

   so $k \succ_f j$. If $f_j(\{i,j,k\}) \neq 0$, then

$$\begin{aligned}
0 &= f_i(\{i,j,k\}) \cdot f_j(\{i,j\}) \\
&= f_i(\{i,j\}) \cdot f_j(\{i,j,k\}),
\end{aligned}$$

   so $f_i(\{i,j\}) = 0$ and $j \succ_f i$.

   Thus, if $i \succcurlyeq_f j$ and $j \succcurlyeq_f k$, then $i \succcurlyeq_f k$.

   Now, suppose that $j \succ_f i$ and $i, j \in S$. Then

$$\begin{aligned}
f_i(S) &= f_j(\{i,j\}) \cdot f_i(S) \\
&= f_j(S) \cdot f_i(\{i,j\}) \\
&= 0.
\end{aligned}$$

   Hence, if $i \notin Top_f(S)$ then $f_i(S) = 0$.

   On the other hand, if $i \sim_f j$ and $i, j \in S$, then

$$f_i(S) \cdot f_j(\{i,j\}) = f_i(\{i,j\}) \cdot f_j(S),$$

   so $f_i(S) \neq 0$ if and only if $f_j(S) \neq 0$.

Since for every S there is some $i \in S$ such that $f_i(S) \neq 0$, it follows that $f_i(S) \neq 0$ if and only if $i \in Top_f(S)$, so

$$f_i(S) = \begin{cases} \dfrac{f_i(S)}{\sum_{j \in Top_f(S)} f_j(S)}, & \text{if } i \in Top_f(S) \\ 0, & \text{otherwise} \end{cases}$$

Now, if $i, j \in Top_f(S)$, then $i \sim_f j$, so

$$\begin{aligned} \lambda_i^f \cdot f_j(S) &= f_i(\{k \,:\, i \sim_f k\}) \cdot f_j(S) \\ &= f_i(S) \cdot f_j(\{k \,:\, i \sim_f k\}) \\ &= f_i(S) \cdot f_j(\{k \,:\, j \sim_f k\}) \\ &= f_i(S) \cdot \lambda_j^f, \end{aligned}$$

and thus

$$\frac{f_j(S)}{f_i(S)} = \frac{\lambda_j^f}{\lambda_i^f}.$$

It follows that if $i \in Top_f(S)$, then

$$\begin{aligned} f_i(S) &= \frac{f_i(S)}{\sum_{j \in Top_f(S)} f_j(S)} \\ &= \frac{1}{\sum_{j \in Top_f(S)} \frac{f_j(S)}{f_i(S)}} \\ &= \frac{1}{\sum_{j \in Top_f(S)} \frac{\lambda_j^f}{\lambda_i^f}} \\ &= \frac{\lambda_i^f}{\sum_{j \in Top_f(S)} \lambda_j^f}. \end{aligned}$$

3. Lastly, let's show $3 \implies 1$. Now, suppose $f \in \mathcal{F}$ satisfies the given conditions. Let $(X_1, \ldots, X_\ell)$ be the ordered partition in which each X_k is an equivalence class of $\succcurlyeq_f$ and $i \succ_f j$ whenever $i \in X_k$, $j \in X_{k'}$, and $k' > k$.

   Suppose $\lambda_i^f = 0$. Since $f \in \mathcal{F}_{SGE}$, there is some j such that $j \sim_f i$ and $\lambda_j^f > 0$. But then

$$f_i(\{i, j\}) = \frac{\lambda_i^f}{\lambda_i^f + \lambda_j^f} = 0,$$

   so $j \succ_f i$, contradiction. Thus, $\lambda_i^f > 0$ for all i.

Finally, observe that $i \in Top_f(S)$ if and only if there is a k such that $i \in X_k$ and $S \cap X_m = \emptyset$ for $m < k$, and if this holds then $Top_f(S) = S \cap X_k$. Hence,

$$f_i(S) = \begin{cases} \dfrac{\lambda_i^f}{\sum_{j \in S \cap X_k} \lambda_j^f}, & \text{if } S \cap X_m = \emptyset \text{ for } m < k \text{ and } i \in S \cap X_k \\ 0, & \text{otherwise} \end{cases}$$

Thus, $f \in \mathcal{F}_{PW}$.

□

Lemma 3.

1. *$\mathcal{F}_{FGN}$, $\mathcal{F}_{SGE}$, $\mathcal{F}_{DR}$ are compact, convex subsets of $\mathcal{F}$.*
2. *For any $p \in \mathcal{E}$, $E^{-1}(p) \cap \mathcal{F}_{FGN}$, $E^{-1}(p) \cap \mathcal{F}_{SGE}$, and $E^{-1}(p) \cap \mathcal{F}_{DR}$ are compact, convex subsets of $\mathcal{F}$.*
3. *$\mathcal{F}_{PW}$ is compact.*

Proof. We prove the statements in order.

1. The first statement is immediate from the definitions.
2. Suppose $p \in \mathcal{E}$ and let $f, g \in E^{-1}(p) \cap \mathcal{F}_{FGN}$. It follows from the first order condition in 1 that
$$\sum_{S \subset [n]_{-i}} f_i(S \cup \{i\}) \Pr_{p_{-i}}^{[n]_{-i}}(S) = c_i'(p_i) = \sum_{S \subset [n]_{-i}} (g_i(S \cup \{i\})) \Pr_{p_{-i}}^{[n]_{-i}}(S)$$
Now consider the contract $h = \lambda f + (1 - \lambda) g$. Clearly, $h \in \mathcal{F}_{FGN}$ and at the profile p, all the first order conditions continue to be satisfied so that $h \in E^{-1}(p)$. Thus, $E^{-1}(p) \cap \mathcal{F}_{FGN}$ is convex. The argument for $E^{-1}(p) \cap \mathcal{F}_{SGE}$ and $E^{-1}(p) \cap \mathcal{F}_{DR}$ being convex is analogous. The compactness of the sets follows from the fact that given a p, the sets are defined by solutions to the first order conditions which are simply linear equations.
3. For $i, j \in [n]$ and $S, T \subseteq [n]$, let
$$C_{i,j,S,T} := \{f \in \mathbb{R}^{[n] \times 2^{[n]}} \; : \; f_i(S) \cdot f_j(T) = f_i(T) \cdot f_j(S)\}.$$
Since $C_{i,j,S,T}$ is the set of solutions to a polynomial equation, it is closed, so
$$C := \bigcap_{S,T \subseteq [n],\, i,j \in S \cap T} C_{i,j,S,T}$$
is closed. We already know that $\mathcal{F}_{SGE}$ is compact. Finally, by Lemma 2,
$$\mathcal{F}_{PW} = \mathcal{F}_{SGE} \cap C,$$
so $\mathcal{F}_{PW}$ is compact.

□

B Proofs for Section 3 (Pareto frontier)

Lemma 4. $E(\mathcal{F}_{FGN}) = \mathcal{E}$.

Proof. Suppose $p \in E(f)$. Since $u_i(p)$ is concave in p_i and $c_i'(1) \geq 1$, it must be that for all $i \in [n]$, either

$$p_i = 0 \text{ or } \frac{\partial u_i(p)}{\partial p_i} = 0 \iff c_i'(p_i) = \sum_{S \subset [n]_{-i}} (f_i(S \cup \{i\}) - f_i(S)) \Pr_{p_{-i}}^{[n]_{-i}}(S).$$

For all $i \in [n]$, let $\lambda_i = 0$ if $p_i = 0$ and

$$\lambda_i = \frac{c_i'(p_i)}{\sum_{S \subset [n]_{-i}} f_i(S \cup \{i\}) \Pr_{p_{-i}}^{[n]_{-i}}(S)} \leq 1$$

otherwise. Now consider the FGN contract g such that for any outcome $S \subset [n]$,

$$g_i(S) = \begin{cases} \lambda_i f_i(S), & \text{if } i \in S \\ 0, & \text{otherwise} \end{cases}$$

Since $g_i(S) \leq f_i(S)$ for all i and S, the contract g is feasible. We want to show that p is a pure-strategy Nash equilibrium under g. Assume we are at the profile p under the contract g. Agent i's marginal payoff at p is given by

$$\begin{aligned} \frac{\partial u_i(p)}{\partial p_i} &= \sum_{S \subset [n]_{-i}} g_i(S \cup \{i\}) \Pr_{p_{-i}}^{[n]_{-i}}(S) - c_i'(p_i) \\ &= \lambda_i \sum_{S \subset [n]_{-i}} f_i(S \cup \{i\}) \Pr_{p_{-i}}^{[n]_{-i}}(S) - c_i'(p_i) \\ &= 0 \end{aligned}$$

Since the payoffs are strictly concave, it follows that p is a pure-strategy Nash equilibrium under g as well. □

Lemma 5. *If p is Pareto optimal, then $p_i > 0$ for all i.*

Proof. Suppose $p \in E(f)$ where f is a FGN contract. This is WLOG from Lemma 4. Let's assume $p_k = 0$. Consider another FGN contract g such that for $i \neq k$,

$$g_i(S) = f_i(S \setminus \{k\})$$

while

$$g_k(S) = \begin{cases} 1, & \text{if } S = \{k\} \\ 0, & \text{otherwise} \end{cases}$$

Let p_k' solve $c_k'(p_k') = \Pi_{i \in [n]_{-k}}(1 - p_i)$. Note that $p_k' > 0$ because in any equilibrium p, it must be that $p_i < 1$. We can now show that $p' = (p_k', p_{-k}) \in E(g)$. For agent k, $p_k' > 0$ is clearly the unique best response to p'_{-k} under g. For agent $i \neq k$, the payoff does not depend on agent k's success probability and thus, finding a best response to p'_{-i} is equivalent to finding the best response p_{-i} since $p_k = 0$. It follows that p_i is a best response to p'_{-i}. Thus, $p' \in E(g)$ and p' Pareto dominates p. Therefore, p cannot be on the Pareto frontier.

$\square$

Lemma 6. *If p is Pareto optimal and $p \in E(f)$, then f must be a SGE contract.*

Proof. Suppose $p \in \mathcal{P}$. Let

$$K_p := \{S \subseteq [n] \,:\, \sum_{i \in S} f_i(S) < 1 \text{ for some } f \in E^{-1}(p) \cap \mathcal{F}_{FGN}\},$$

and let

$$\kappa_p := \{i \in [n] \,:\, \{i\} \in K_p\}.$$

We now need to show that $K_p = \{\emptyset\}$. We do so in the following steps.

1. Step 1: Suppose $S \in K_p$. For any $T \subset S$, $T \in K_p$.

   Let $f \in E^{-1}(p) \cap \mathcal{F}_{FGN}$ be such that $\sum_{i \in S} f_i(S) < 1$. If $\sum_{i \in T} f_i(T) < 1$, we are done. Otherwise, pick an agent $i \in T$ such that $f_i(T) > 0$ and consider a contract g which differs from f only in its award for agent i at S and T. In particular, let g be such that $g_i(S) = f_i(S) + \epsilon$ and $g_i(T) = f_i(T) - \delta$ where $\epsilon, \delta > 0$ are chosen so that $p \in E(g)$. Note that we can do this because we know from Lemma 5 that for all $i \in [n]$, $0 < p_i < 1$ and therefore, $\Pr_{p_{-i}}^{[n]_{-i}}(S) > 0$ for all $i \in [n]$ and all $S \subset [n]_{-i}$. It follows then that $g \in E^{-1}(p) \cap \mathcal{F}_{FGN}$ and $\sum_{i \in T} g_i(T) < 1$. Thus, $T \in K_p$.

2. Step 2: Suppose $S, T \in K_p$. Then, $S \cup T \in K_p$.

   Let $f, g \in E^{-1}(p) \cap \mathcal{F}_{FGN}$ be such that $\sum_{i \in S} f_i(S) < 1$ and $\sum_{i \in T} g_i(T) < 1$. Consider the contract $h = \frac{1}{2}f + \frac{1}{2}g$. From Lemma 3, we know that $h \in E^{-1}(p) \cap \mathcal{F}_{FGN}$ and also $\sum_{i \in S} h_i(S) < 1$ and $\sum_{i \in T} h_i(T) < 1$. Now, if $\sum_{i \in S \cup T} h_i(S \cup T) < 1$, we are done. Otherwise, pick any agent $i \in S \cup T$ (WLOG, let $i \in S$) such that $h_i(S \cup T) > 0$ and consider a contract h' which differs from h only in its award for agent i at $S \cup T$ and S. In particular, let h' be such that $h'_i(S \cup T) = h_i(S \cup T) - \epsilon$ and $h'_i(S) = h_i(S) + \delta$ where $\epsilon, \delta > 0$ are chosen so that $p \in E(h')$. Again, we can do this because we know that for all $i \in [n]$, $0 < p_i < 1$. It follows then that $h' \in E^{-1}(p) \cap \mathcal{F}_{FGN}$ and $\sum_{i \in S \cup T} h'_i(S \cup T) < 1$. Thus, $S \cup T \in K_p$.

   Note that it follows from Steps 1 and 2 that $K_p = 2^{\kappa_p}$.

3. Step 3: Suppose $f \in E^{-1}(p) \cap \mathcal{F}_{FGN}$. Then, for all $S \subset [n]$ such that $\kappa_p^C \cap S \neq \phi$, $f_i(S) = 0$ for all $i \in \kappa_p$.

   Suppose towards a contradiction that there is an $S \subset [n]$ such that $\kappa_p^C \cap S \neq \phi$ and $f_i(S) > 0$ for some $i \in \kappa_p$. Let $g \in E^{-1}(p) \cap \mathcal{F}_{FGN}$ be such that $g_i(\{i\}) < 1$. Consider the contract $h = \frac{1}{2}f + \frac{1}{2}g$. From Lemma 3, we know that $h \in E^{-1}(p) \cap \mathcal{F}_{FGN}$ and also $h_i(S) > 0$ and $h_i(\{i\}) < 1$. Now, consider a contract h' which differs from h only in its award for agent i at S and $\{i\}$. In particular, let h' be such that $h'_i(\{i\}) = h_i(\{i\}) + \epsilon$ and $h'_i(S) = h_i(S) - \delta$ where $\epsilon, \delta > 0$ are chosen so that $p \in E(h')$. We can do this because for all $i \in [n]$, $0 < p_i < 1$. It follows then that $h' \in E^{-1}(p) \cap \mathcal{F}_{FGN}$ and $\sum_{i \in S} h'_i(S) < 1$. But this means that $S \subset \kappa_p$ which is a contradiction.

4. Step 4: Suppose $\kappa_p \neq \phi$. Then there is a p' that Pareto dominates p.

   For all $S \in K_p$, let $f^S \in E^{-1}(p) \cap \mathcal{F}_{FGN}$ be such that $\sum_{i \in S} f_i^S(S) < 1$. Consider the contract $g = \sum_{S \in K_p} \frac{1}{|K_p|} f^S$. From Lemma 3, we know that $g \in E^{-1}(p) \cap \mathcal{F}_{FGN}$ and also $\sum_{i \in S} g_i(S) < 1$ for all $S \in K_p$. Now, we can construct a contract $h \in E^{-1}(p) \cap \mathcal{F}_{FGN}$ such that

$$h_i(S) = \begin{cases} g_i(S), & \text{if } S \in K_p \\ h_i(S \setminus \kappa_p), & \text{if } S \notin K_p \end{cases}$$

   by averaging over the outcomes of agents in κ_p under g.

   Observe that if we manipulate h at any $S \subset \kappa_p$, it won't change the best responses for agents $i \notin \kappa_p$. We will now show that we can manipulate the awards for $S \subset \kappa_p$ so that under the new contract h', $p' \in E(h')$ where $p'_i > p_i$ for $i \in \kappa_p$ and $p'_i = p_i$ for $i \notin \kappa_p$. Towards this goal, let $A = \kappa_p$ and let $p' = (p_i + \epsilon)_{i \in A}$. For each $i \in A$, let $t_i(\epsilon)$ solve

$$c'_i(p'_i) = t_i(\epsilon) \sum_{S \subset A_{-i}} (h_i(S \cup \{i\})) \mathrm{Pr}_{p'_{-i}}^{A-i}(S)$$

   Observe that as $\epsilon \to 0$, $t_i(\epsilon) \to 1$ for all $i \in A$. Since $\sum_{i \in S} h_i(S) < 1$ for all $S \subset A$ and $t_i(\epsilon)$ is continuous in ϵ, we can find $\epsilon > 0$ small enough so that the contract $h'_i(S) = h_i(S) * t_i(\epsilon)$ for all $S \subset A$ and $i \in S$ is a feasible contract. By the definition of t_i, p' with $p'_i = p_i + \epsilon$ for $i \in \kappa_p$ and $p'_i = p_i$ for $i \notin \kappa_p$ will be an equilibrium under h'. Thus, we have that p is not Pareto optimal.

This is a contradiction and we get that $\kappa_p = \phi$ which implies the result. □

Lemma 7. *For any $p \in [0,1]^n$ and $f \in \mathcal{F}_{SGE}$,*

$$\sum_{i \in [n]} p_i \cdot c'_i(\Psi_i(p_{-i}, f)) = 1 - \mathrm{Pr}_p^{[n]}(\emptyset).$$

Proof. Note that for $f \in \mathcal{F}_{SGE}$, the best response for agent i, $\Psi_i(p_{-i}, f)$, is defined by

$$c'_i(\Psi_i(p_{-i}, f)) = \sum_{S \subset [n]_{-i}} g_i(S \cup \{i\}) \mathrm{Pr}_{p_{-i}}^{[n]-i}(S).$$

Multiplying both sides by p_i and adding up the n equations, we get

$$
\begin{aligned}
\sum_{i\in[n]} p_i \cdot c_i'(\Psi_i(p_{-i}, f)) &= \sum_{i\in[n]} \sum_{S\ni i} f_i(S)\mathrm{Pr}_p^{[n]}(S) \\
&= \sum_{\emptyset\neq S\subseteq[n]} \sum_{i\in S} f_i(S)\mathrm{Pr}_p^{[n]}(S) \\
&= \sum_{\emptyset\neq S\subseteq[n]} \mathrm{Pr}_p^{[n]}(S) \sum_{i\in S} f_i(S) \\
&= \sum_{\emptyset\neq S\subseteq[n]} \mathrm{Pr}_p^{[n]}(S) \\
&= \sum_{S\subseteq[n]} \mathrm{Pr}_p^{[n]}(S) - \mathrm{Pr}_p^{[n]}(\emptyset) \\
&= 1 - \mathrm{Pr}_p^{[n]}(\emptyset).
\end{aligned}
$$

□

Lemma 8. *If $p \in \mathcal{P}$ and $f \in \mathcal{F}_{SGE}$, then $Z_p(f) \geq 0$, and if $Z_p(f) = 0$ then $f \in E^{-1}(p)$.*

Proof. Let $g \in E^{-1}(p)$. Since p is Pareto optimal, we know from Lemma 6 that $g \in \mathcal{F}_{SGE}$. It then follows from Lemma 7 that

$$
\begin{aligned}
\sum_{i\in[n]} p_i \cdot c_i'(p_i) &= \sum_{i\in[n]} p_i \cdot c_i'(\Psi_i(p_{-i}, g)) \\
&= 1 - \mathrm{Pr}_p^{[n]}(\emptyset).
\end{aligned}
$$

Since $f \in \mathcal{F}_{SGE}$, it again follows from Lemma 7 that

$$\sum_{i\in[n]} p_i \cdot c_i'(\Psi_i(p_{-i}, f)) = 1 - \mathrm{Pr}_p^{[n]}(\emptyset).$$

Thus,

$$\sum_{i\in[n]} p_i \cdot (c_i'(\Psi_i(p_{-i}, f)) - c_i'(p_i)) = 0.$$

Hence, if $c_i'(\Psi_i(p_{-i}, f)) - c_i'(p_i) < 0$ for some i, there must be some j such that $c_j'(\Psi_j(p_{-j}, f)) - c_j'(p_j) > 0$, so $Z_p(f) \geq 0$, and $Z_p(f) = 0$ if and only if $c_i'(\Psi_i(p_{-i}, f)) - c_i'(p_i) = 0$ for all i if and only if $f \in E^{-1}(p)$. □

Lemma 9. *If $p \in \mathcal{P}$ and $f \in \mathcal{F}_{PW}$, then for every $i \in [n]$,*

$$\sum_{j\,:\,i \succcurlyeq_f j} p_j \cdot (c_j'(\Psi_j(p_{-j}, f)) - c_j'(p_j)) \leq 0.$$

Proof. First, observe that

$$\begin{aligned}
\sum_{j\,:\,i \succcurlyeq_f j} p_j \cdot c_j'(\Psi_j(p_{-j}, f)) &= \sum_{j\,:\,i \succcurlyeq_f j} \sum_{S \ni j} f_j(S) \Pr_p^{[n]}(S) \\
&= \sum_{j\,:\,i \succcurlyeq_f j} \sum_{Top_f(S) \ni j} f_j(S) \Pr_p^{[n]}(S) \\
&= \sum_{S\,:\,Top_f(S) \subseteq \{j\,:\,i \succcurlyeq_f j\}} \sum_{j \in Top_f(S)} f_j(S) \Pr_p^{[n]}(S) \\
&= \sum_{S\,:\,Top_f(S) \subseteq \{j\,:\,i \succcurlyeq_f j\}} \Pr_p^{[n]}(S) \sum_{j \in Top_f(S)} f_j(S) \\
&= \sum_{S\,:\,Top_f(S) \subseteq \{j\,:\,i \succcurlyeq_f j\}} \Pr_p^{[n]}(S) \\
&= \sum_{S \subseteq \{j\,:\,i \succcurlyeq_f j\}} \Pr_p^{[n]}(S)
\end{aligned}$$

Now, let $g \in E^{-1}(p)$. Then

$$\begin{aligned}
\sum_{j\,:\,i \succcurlyeq_f j} p_j \cdot c_j'(p_j) &= \sum_{j\,:\,i \succcurlyeq_f j} p_j \cdot c_j'(\Psi_j(p_{-j}, g)) \\
&= \sum_{j\,:\,i \succcurlyeq_f j} \sum_{S \ni j} g_j(S) \Pr_p^{[n]}(S) \\
&= \sum_{S} \sum_{j \in S\,:\,i \succcurlyeq_f j} g_j(S) \Pr_p^{[n]}(S) \\
&= \sum_{S \subseteq \{j\,:\,i \succcurlyeq_f j\}} \sum_{j \in S} g_j(S) \Pr_p^{[n]}(S) + \sum_{S \not\subseteq \{j\,:\,i \succcurlyeq_f j\}} \sum_{j \in S\,:\,i \succcurlyeq_f j} g_j(S) \Pr_p^{[n]}(S) \\
&\geq \sum_{S \subseteq \{j\,:\,i \succcurlyeq_f j\}} \sum_{j \in S} g_j(S) \Pr_p^{[n]}(S) \\
&= \sum_{S \subseteq \{j\,:\,i \succcurlyeq_f j\}} \Pr_p^{[n]}(S) \sum_{j \in S} g_j(S) \\
&= \sum_{S \subseteq \{j\,:\,i \succcurlyeq_f j\}} \Pr_p^{[n]}(S).
\end{aligned}$$

Thus,

$$\sum_{j\,:\,i \succcurlyeq_f j} p_j \cdot c_j'(p_j) \geq \sum_{j\,:\,i \succcurlyeq_f j} p_j \cdot c_j'(\Psi_j(p_{-j}, f)).$$

□

Lemma 10. *If $p \in \mathcal{P}$, then*

$$\inf_{f \in \mathcal{F}_{PW}} Z_p(f) = 0.$$

Proof. Suppose not, and let $z = \inf_{f \in \mathcal{F}_{PW}} Z_p(f)$.

Denote

$$\mathcal{C}(f) := \{i \in [n] : c_i'(\Psi_i(p_{-i}, f)) - c_i'(p_i) = Z_p(f)\}.$$

Let $f \in \mathcal{F}_{PW}$ such that $Z_p(f) = z$ and for any $g \in \mathcal{F}_{PW}$ such that $Z_p(g) = z$, $\mathcal{C}(g) \not\subset \mathcal{C}(f)$.

Observe that $\mathcal{C}(f) \neq \emptyset$. Let $(X_1, \ldots, X_\ell)$ be a partition and $\lambda_1, \ldots, \lambda_n$ be weights for f.

Let k be the maximum index such that $X_k \cap \mathcal{C}(f) \neq \emptyset$.

Suppose $k = \ell$. By Lemma 9, there must be some $i \in X_\ell$ such that $c_i'(\Psi_i(p_{-i}, f)) - c_i'(p_i) < 0$. Consider the PW contract g which is identical to f except that λ_i is increased by ε. Notice that for $j \notin X_\ell$,

$$c_j'(\Psi_j(p_{-j}, g)) - c_j'(p_j) = c_j'(\Psi_j(p_{-j}, f)) - c_j'(p_j)$$

and for $j \in X_\ell \setminus \{i\}$,

$$c_j'(\Psi_j(p_{-j}, g)) - c_j'(p_j) < c_j'(\Psi_j(p_{-j}, f)) - c_j'(p_j) \leq z.$$

Moreover, for ε sufficiently small,

$$c_i'(\Psi_i(p_{-i}, g)) - c_i'(p_i) < 0$$

by continuity. It follows that $Z_p(g) \leq z$ and $Z_p(g) \subseteq Z_p(f)$, and since $Z_p(g) \cap X_\ell = \emptyset$, it follows that $Z_p(g) = z$ and $\mathcal{C}(g) \subset \mathcal{C}(f)$, contradiction.

Thus, it must be that $k < \ell$. Now, consider the PW contract g which has the partition $(X_1, \ldots, X_{k-1}, X_k \cup X_{k+1}, X_{k+2}, \ldots, X_\ell)$ and weights λ_i' such that $\lambda_i' = \lambda_i$ for $i \notin X_{k+1}$ and $\lambda_i' = \varepsilon \lambda_i$ for $i \in X_{k+1}$.

Then for $j \notin X_k \cup X_{k+1}$,

$$c_j'(\Psi_j(p_{-j}, g)) - c_j'(p_j) = c_j'(\Psi_j(p_{-j}, f)) - c_j'(p_j)$$

and for $j \in X_k$,

$$c_j'(\Psi_j(p_{-j}, g)) - c_j'(p_j) < c_j'(\Psi_j(p_{-j}, f)) - c_j'(p_j) \leq z.$$

Finally, for ε sufficiently small and $j \in X_{k+1}$,

$$c_j'(\Psi_j(p_{-j}, g)) - c_j'(p_j) < z$$

by continuity. It follows that $Z_p(g) \leq z$ and $Z_p(g) \subseteq Z_p(f)$, and since $Z_p(g) \cap X_k = \emptyset$, it follows that $Z_p(g) = z$ and $\mathcal{C}(g) \subset \mathcal{C}(f)$, contradiction. $\square$

C Proofs for Section 4 (Application)

Lemma 11. *For $\lambda \in (0,1)$, $\frac{p_2'(\lambda)}{p_1'(\lambda)}$ is strictly decreasing.*

Proof. The equilibrium for any $\lambda \in (0,1)$ is given by

$$p_1(\lambda) = \frac{c_2 - (1-\lambda)}{c_1c_2 - \lambda(1-\lambda)} \quad p_2(\lambda) = \frac{c_1 - \lambda}{c_1c_2 - \lambda(1-\lambda)}.$$

Note that

$$p_1'(\lambda) = \frac{(c_1c_2 - \lambda(1-\lambda)) - (c_2 - (1-\lambda))(2\lambda - 1))}{(c_1c_2 - \lambda(1-\lambda))^2} = \frac{c_1c_2 - c_2(2\lambda-1) - (1-\lambda)^2}{(c_1c_2 - \lambda(1-\lambda))^2}$$

and

$$p_2'(\lambda) = \frac{-(c_1c_2 - \lambda(1-\lambda)) - (c_1 - \lambda)(2\lambda - 1)}{(c_1c_2 - \lambda(1-\lambda))^2} = \frac{-c_1c_2 - c_1(2\lambda-1) + \lambda^2}{(c_1c_2 - \lambda(1-\lambda))^2}$$

so

$$\frac{p_2'(\lambda)}{p_1'(\lambda)} = \frac{-c_1c_2 - c_1(2\lambda-1) + \lambda^2}{c_1c_2 - c_2(2\lambda-1) - (1-\lambda)^2} = -\frac{c_1}{c_2} \cdot \frac{-\lambda^2/c_1 + 2\lambda + c_2 - 1}{-(1-\lambda)^2/c_2 + 2(1-\lambda) + c_1 - 1}$$

Now observe that the numerator is increasing for $\lambda < c_1$ and the denominator is decreasing for $\lambda > -(c_2 - 1)$. In particular, the fraction is monotonically strictly increasing for $0 < \lambda < 1$, so $\frac{p_2'(\lambda)}{p_1'(\lambda)}$ is monotonically strictly decreasing. □

Theorem 3. *$\lambda^*(w)$ is increasing in w. In particular,*

$$\lambda^*(w) = \begin{cases} 0, & \text{if } w \le -\frac{p_2'(0)}{p_1'(0)} = \frac{c_1c_2 - c_1}{c_1c_2 + c_2 - 1} \\ \frac{1}{2}, & \text{if } w = 1 \\ 1, & \text{if } w \ge -\frac{p_2'(1)}{p_1'(1)} = \frac{c_1c_2 + c_1 - 1}{c_1c_2 - c_2} \end{cases}$$

Proof. From the equilibrium characterization in equation 4, the principal's objective is

$$\max_{\lambda \in [0,1]} w_1 p_1(\lambda) + w_2 p_2(\lambda).$$

Taking the derivative wrt λ, we get $w_1 p_1'(\lambda) + w_2 p_2'(\lambda)$. Thus, the optimal $\lambda^*(w)$ must be such that either $\lambda^*(w) = 0, \lambda^*(w) = 1$ or $\frac{w_1}{w_2} = -\frac{p_2'(\lambda)}{p_1'(\lambda)}$. We know from Lemma 11 that $\frac{p_2'(\lambda)}{p_1'(\lambda)}$ is strictly decreasing. In particular,

$$\frac{p_2'(\lambda)}{p_1'(\lambda)} \le \frac{p_2'(0)}{p_1'(0)} = -\frac{c_1}{c_2} \cdot \frac{c_2 - 1}{-1/c_2 + c_1 + 1} = -\frac{c_1c_2 - c_1}{c_1c_2 + c_2 - 1}$$

and

$$\frac{p_2'(\lambda)}{p_1'(\lambda)} \geq \frac{p_2'(1)}{p_1'(1)} = -\frac{c_1}{c_2} \cdot \frac{-1/c_1 + c_2 + 1}{c_1 - 1} = -\frac{c_1 c_2 + c_1 - 1}{c_1 c_2 - c_2}$$

Now if $\frac{w_1}{w_2} \leq -\frac{p_2'(0)}{p_1'(0)}$, the objective is decreasing in λ and thus $\lambda^*(w) = 0$.
And if $\frac{w_1}{w_2} \geq -\frac{p_2'(1)}{p_1'(1)}$, the objective is increasing in λ and thus $\lambda^*(w) = 1$.
Lastly, observe that

$$\frac{p_2'(\frac{1}{2})}{p_1'(\frac{1}{2})} = -1$$

irrespective of the costs c_1, c_2. And thus, if $w_1 = w_2$, we get that $\lambda^* = \frac{1}{2}$ no matter how heterogeneous the agents are.

$\square$