Luce contracts

Sumit Goel[1] Wade Hann-Caruthers[2]

[1]NYU Abu Dhabi
[2]Technion

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Introduction

- Principal delegates individual tasks to multiple agents
- Each agent may succeed or fail depending upon the effort they exert
- The principal can only observe final outcomes and not effort
- How should a principal design a contract so as to incentivize agents to exert costly effort?
    - piece-rate contracts $\rightarrow$ high variability, possibility of large payments
    - assume principal is budget-constrained

Model

- $[n]$: set of n risk-neutral agents, each attempts an independent task
- $p_i \in [0, 1]$: agent i's choice of probability of success
- $c_i(p_i)$: cost incurred by agent i for its choice of p_i
  - Strictly increasing and convex, with $c_i'(0) = 0$
- $V(p_1, \ldots, p_n)$: principal's objective, increasing in p_i
- The principal has an exclusive-use budget $B > 0$
- Small budget assumption: $B < \min\{c_1'(1), c_2'(1), \ldots, c_n'(1)\}$

Contract

- A *contract* is a function $f : 2^{[n]} \to \mathbb{R}_+^{[n]}$ such that for each $S \subseteq [n]$,
$$\sum_{i \in [n]} f_i(S) \leq 1.$$
- Under contract $f \in \mathcal{F}$, agent $i's$ expected payoff at $p = (p_1, \ldots, p_n)$ is
$$\begin{aligned} u_i(p) &= \mathbb{E}[B \cdot f_i(S)] - c_i(p_i) \\ &= p_i \cdot \mathbb{E}[B \cdot f_i(S) | i \in S] + (1 - p_i) \cdot \mathbb{E}[B \cdot f_i(S) | i \notin S] - c_i(p_i), \end{aligned}$$
- Concave payoffs $\implies$ pure-strategy Nash exist (Rosen [1965])

Design problem

- $E : \mathcal{F} \rightrightarrows [0,1]^n$ the equilibrium correspondence
- $\mathcal{E} := \{p \in [0,1]^n \,:\, p \in E(f) \text{ for some } f \in \mathcal{F}\}$
- $\mathcal{P} = \{p \in \mathcal{E} : \forall q \in \mathcal{E},\ q_i \geq p_i\ \forall i \in [n] \ \Rightarrow\ q = p\}$
- Principal's problem:

$$\max_{p \in \mathcal{P}} V(p).$$

- This paper: characterize $\mathcal{P}$, identify contracts that implement $\mathcal{P}$

Examples of contracts

- Constant: $f_i(S) = \frac{1}{n}$ for all $i \in [n]$
- Piece-rate:

$$f_i(S) = \begin{cases} \frac{1}{n}, & \text{if } i \in S \\ 0, & \text{otherwise} \end{cases}$$

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- Equal-split among successful:

$$f_i(S) = \begin{cases} \frac{1}{|S|}, & \text{if } i \in S \\ 0, & \text{otherwise} \end{cases}$$

- Highest-priority successful agent takes all:

$$f_i(S) = \begin{cases} 1, & \text{if } i = \max\{j : j \in S\} \\ 0, & \text{otherwise} \end{cases}$$

FGN contracts

- A contract f is *failures-get-nothing* (FGN) if $f_i(S) = 0$ whenever $i \notin S$.
- For any $p \in \mathcal{E}$, there is a FGN contract that implements p.

Proposition 1.

$\mathcal{E} = E(\mathcal{F}_{FGN})$.

- For any $f \in \mathcal{F}_{FGN}$, agent i's best-response to p_{-i} is given by

$$\mathbb{E}[B \cdot f_i(S) | i \in S] = c_i'(b_i(f, p_{-i})).$$

SGE contracts

- A contract f is *successful-get-everything* (SGE) if
  1. $f_i(\emptyset) = 0$ for all i
  2. $\sum_{i \in S} f_i(S) = 1$ for all $S \neq \emptyset$

Theorem 1.

Suppose $p \in E(f)$. Then $p \in \mathcal{P}$ if and only if $f \in \mathcal{F}_{SGE}$.

- For any monotone objective, an optimal contract must be SGE: piece-rate or bonus-pool contracts are never optimal in this setting
- Space of SGE contracts is large ($\Theta(n2^n)$-dimensional)

Luce contracts

- A contract f is a *weighted* (W) contract if there exist weights $(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i > 0$ such that

$$f_i(S) = \begin{cases} \dfrac{\lambda_i}{\sum_{j \in S} \lambda_j}, & \text{if } i \in S \\ 0, & \text{otherwise} \end{cases}$$

- A contract f is a *Luce contract* if there exist weights $(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i > 0$ and a non-strict ordering $\succcurlyeq$ on the agents such that

$$f_i(S) = \begin{cases} \dfrac{\lambda_i}{\sum_{j \in \mathrm{Top}_{\succcurlyeq}(S)} \lambda_j}, & \text{if } i \in \mathrm{Top}_{\succcurlyeq}(S) \\ 0, & \text{otherwise} \end{cases}$$

where $\mathrm{Top}_{\succcurlyeq}(S) = \{i \in S : i \succcurlyeq j \ \forall j \in S\}$.

Luce contracts are sufficient

Theorem 2.

If $p \in \mathcal{P}$, there is a unique Luce contract $f \in \mathcal{F}_{Luce}$ such that $p \in E(f)$.

- For any monotone objective, there is always a Luce contract that is optimal
- The principal can optimize over this $n-1$ dimensional class of Luce contracts

Proof sketch

- Suppose $p \in (0,1)^n$ is an equilibrium of $f \in \mathcal{F}_{SGE}$ with budget B.
- From the foc, the expected total payment to agents in $I \subset [n]$ is

$$\mathbb{E}\left[\sum_{i \in I} B \cdot f_i(S)\right] = \sum_{i \in I} p_i \cdot c_i'(p_i).$$

- From the structure of SGE contract,

$$\mathbb{E}\left[\sum_{i \in I} B \cdot f_i(S)\right] \leq B \cdot \mathbb{P}[S \cap I \neq \emptyset],$$

with equality for $I = [n]$,

$$\mathbb{E}\left[\sum_{i \in [n]} B \cdot f_i(S)\right] = B \cdot \mathbb{P}[S \neq \emptyset].$$

Luce implementable profiles

- It follows that $p \in \mathcal{P}$ must be such that

$$\frac{\mathbb{E}\left[\sum_{i \in I} B \cdot f_i(S)\right]}{\mathbb{E}\left[\sum_{i \in [n]} B \cdot f_i(S)\right]} = \frac{\sum_{i \in I} p_i \cdot c_i'(p_i)}{\sum_{i \in [n]} p_i \cdot c_i'(p_i)} \leq \frac{\mathbb{P}[S \cap I \neq \emptyset]}{\mathbb{P}[S \neq \emptyset]}.$$

Proposition 2.

Suppose $p \in (0,1)^n$. There exists a Luce contract $f \in \mathcal{F}_{Luce}$ that implements p if and only if for all $I \subset [n]$,

$$\frac{\sum_{i \in I} p_i \cdot c_i'(p_i)}{\sum_{i \in [n]} p_i \cdot c_i'(p_i)} \leq \frac{\mathbb{P}[S \cap I \neq \emptyset]}{\mathbb{P}[S \neq \emptyset]}.$$

Luce contracts for standard environments

- Without a budget constraint, plethora of contracts to implement any desired $q \in (0,1)^n$, including:
  1. piece-rate contract: pay $c_i'(q_i)$ to agent i if it succeeds
  2. bonus-pool contract: pay $\dfrac{q_i c_i'(q_i)}{\prod_{i \in [n]} q_i}$ to agent i if all agents succeed
- Under LL constraints, FGN contracts offer the cheapest alternatives
- If q satisfies the inequalities in Proposition 2, the total payment under any FGN contract that implements q is a mean-preserving spread of the total payment under the Luce contract that implements q
- Thus, Luce contracts offer a desirable alternative for implementation in standard environments

Application: 2 agents, quadratic costs

Example 3.

Suppose $n = 2$, $c_i(p_i) = \frac{1}{2}C_i p_i^2$ with $C_i > 1$, and $V(p_1, p_2) = wp_1 + p_2$. Then, the optimal contract, defined by $\lambda_1(w)$, takes the form

$$f_i(S) = \begin{cases} 0, & \text{if } i \notin S \\ 1, & \text{if } S = \{i\} \\ \lambda_i(w), & \text{if } S = \{1,2\} \end{cases},$$

where $\lambda_2(w) = 1 - \lambda_1(w)$. Moreover, $\lambda_1(w)$ is increasing in w and in particular,

$$\lambda_1(w) = \begin{cases} 0, & \text{if } w \leq \frac{C_1 C_2 - C_1}{C_1 C_2 + C_2 - 1} \\ \frac{1}{2}, & \text{if } w = 1 \\ 1, & \text{if } w \geq \frac{C_1 C_2 + C_1 - 1}{C_1 C_2 - C_2} \end{cases}.$$

Lit review

- Single-agent contract design: Holmström [1979], Grossman and Hart [1992], Mirrlees [1976], Georgiadis, Ravid, and Szentes [2022]
- Multi-agent contract design: payments based on
  1. relative performance (Green and Stokey [1983], Lazear and Rosen [1981], Malcomson [1986, 1984], Mookherjee [1984], Nalebuff and Stiglitz [1983], Imhof and Kräkel [2014])
  2. joint performance (Fleckinger [2012], Alchian and Demsetz [1972], Itoh [1991], Kambhampati [2024])
- Moral hazard in teams: Holmstrom [1982], Winter [2004], Battaglini [2006], Babaioff, Feldman, Nisan, and Winter [2012], Halac, Lipnowski, and Rappoport [2021], Dai and Toikka [2022]

Summary

- Study multi-agent contract design with budget-constrained principal
- Introduce a novel class of contracts, called Luce contracts
- Demonstrate their optimality in environments with budget considerations
- Illustrate their desirability for implementation in general contract design environments

Thank you!