Optimal tie-breaking rules*

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Abstract

We consider two-player contests with the possibility of ties and study the effect of different tie-breaking rules on effort. For contests with ratio-form and difference-form success functions that admit pure strategy Nash equilibrium, we find that the total effort is monotone decreasing in the probability that ties are broken in favor of the stronger player. Thus, the effort-maximizing tie-breaking rule commits to breaking ties in favor of the weaker agent. With symmetric agents, the optimal tie-breaking rule is generally unbiased and involves tossing a fair coin either before or after the contest to determine the winner. We identify sufficient conditions under which breaking ties before the contest actually leads to greater expected effort than the more commonly observed practice of breaking ties after the contest.

1 Introduction

Contests are situations in which agents exert costly effort to win one or more prizes. Examples of such competitive situations include sporting contests, promotional tournaments, political contests, R&D races, etc. In many of these situations, it is often the case that there is no outright winner and the contest ends in a draw or a tie. Moreover, a draw may not be an acceptable outcome for the designer. For instance, in sports competitions such as cricket, chess, and soccer, a significant fraction of the games end in a draw. But if these games happen to be knockout games of a world event, the designer must determine a single winner. In political contests, both candidates may end up with the same number of votes and the society must determine a single winning candidate.

Many different tie-breaking rules have been used to determine a winner in such situations when the contest ends in a tie. The result might be decided by chance (e.g. a toss of a coin

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after the contest ends in a tie)[1] or it might be pre-determined based on some statistic (like the win-loss record)[2]. Hypothetically, the designer may also toss a coin before the contest to pre-determine a winner in case it ends in a tie. In this paper, we consider contests between two agents and focus on understanding how different tie-breaking rules compare in terms of the effort they induce from the participating agents. We also discuss implications to the design of effort-maximizing tie-breaking rules. Since the designer may be constrained to be unbiased, we also derive effort-maximizing tie-breaking rules in the class of ex-ante unbiased rules.

We study the effect of tie-breaking rules in three contest environments that differ in how the effort exerted by the players determine the distribution over contest outcomes (player 1 wins, player 2 wins, or tie). We study ratio-form contest success functions (Tullock [44]), difference-form contest success functions (Hirshleifer [27]), and lastly, we discuss examples of contest success functions induced by single-shot and sudden death contests. The ratio-form and the difference-form success functions, in which the distribution over outcomes depend only on the ratio and difference of efforts of the two players respectively, have been prominently studied in contest theory[3]. These contest success functions have also been generalized in different ways to allow for the possibility of ties (Blavatskyy [7], Jia [30], Vesperoni and Yildizparlak [45]). A feature of the axiomatizations of contest success functions with ties in Vesperoni and Yildizparlak [45], Jia [30] is that the contest is more likely to end in a tie when the contest is more competitive. In particular, the probability of a tie is maximized when both agents exert equal efforts. Motivated by this, we assume that for contests with ratio-form and difference-form success functions, the probability of a tie increases as the contest becomes more equal (ratio of efforts goes to 1 or difference goes to 0). In addition, we make assumptions on the contest success functions that ensure existence and uniqueness of pure-strategy Nash equilibrium.

We make two primary contributions. First, we find that when the two agents are asymmetric (in that their valuations for the prize are different), the total effort in equilibrium is decreasing in the probability that the ties are broken in favor of the stronger player. As a result, an effort-maximizing contest designer would prefer to commit to breaking ties in favor of the weaker player. In this way, our results provide further evidence in favor of the idea of leveling the playing field and increasing the competitive balance of a contest for a

[1] The outcome of many elections that ended with ties have been determined via a coin toss (e.x. Kentucky city mayor race 2022). Even in sporting competitions where ties are broken by another short duration contest, like the super over in cricket or the penalty shootout in soccer, it can be argued that the outcome is almost a random draw as there is relatively little scope for skill or effort to have an impact.

[2] In the 1999 Cricket World Cup, the semi-final between Australia and South Africa ended in a tie. Australia went through because they had defeated South Africa earlier in the tournament.

[3] Contests with ratio-form contest success functions have been studied in Baik [4], Ewerhart [18], Wang et al. [46], Nti [40, 41] and contests with difference-form contest success functions have been studied in Baik [3], Skaperdas [43], Beviá and Corchón [6], Che and Gale [9], Ewerhart [19]. Surveys of this literature can be found in Garfinkel and Skaperdas [24], Jia et al. [31], Corchón and Serena [13], Chowdhury et al. [10], Mealem and Nitzan [36].

designer interested in increasing effort.

Second, we identify sufficient conditions under which breaking ties by tossing a fair coin before the contest would lead to greater expected effort than the standard practice of breaking ties after it ends in a tie. More generally, we find that with symmetric agents, the optimal tie-breaking rules are generally unbiased and involve tossing a fair coin either before or after the contest. For instance, we find that in a single-shot contest, where both players invest in the probability of being successful in one independent shot, the effort-maximizing tie-breaking rule breaks ties before the contest. In comparison, in a geometric contest, where the winner is the first player to land a successful shot, the designer is better-off breaking ties in a fair manner after the contest ends in a tie. For contests with difference-form success functions, we show that if the probability of ties is concave around zero, breaking ties before the contest is optimal in the class of unbiased tie-breaking rules.

The paper contributes to the growing literature on the relevance of ties and importance of tie-breaking rules in contest theory. There is significant work studying whether introducing ties in contests can improve designer's payoffs by lowering the probability that the prize will be paid while maintaining the effort exerted by the agents (Nalebuff and Stiglitz [38], Nti [39], Eden et al. [16], Chang et al. [8], Deng et al. [14], Minchuk [37], Gelder et al. [25], Imhof and Kräkel [28, 29], Cohen and Sela [11]). In comparison to this literature, a tie is a natural outcome in our contests and the designer can only choose how to break ties when they occur. In addition, the entire prize is awarded irrespective of the contest outcome which also prevents opportunistic behavior on the part of the designer, especially in cases of unverifiable performance measures (Malcomson [34, 35]). The paper also contributes to the literature studying the effect of leveling the playing field on effort in contests with heterogeneous agents. While there are many different mechanisms that have been studied like taxes, subsidies, gaps, multiplicative and additive biases on effort, quotas, etc., we believe ours is the first paper to discuss this idea in the context of tie-breaking rules. Surveys of the literature on the effects of leveling the playing field can be found in Mealem and Nitzan [36], Chowdhury et al. [10].

The paper proceeds as follows. In section 2, we present the model of a two-player contest with possibility of ties. In section 3 and 4, we discuss the ratio-form and difference-form contest success functions respectively. In section 5, we study examples of a single shot contest and a sudden death contest. Section 6 concludes. All the proofs are relegated to the appendix.

2 Model

There are two risk-neutral players $i = 1, 2$ competing in a contest. The contest designer has a prize and player i's valuation for the prize is denoted by V_i, where we assume $V_1 \geq V_2 > 0$. The contest has three potential outcomes: player 1 wins, player 2 wins, or it may be a tie.

The efforts $x_1 \geq 0$ for player 1 and $x_2 \geq 0$ for player 2 determine the distribution over the three outcomes. We let $p_i(x_1, x_2)$ denote the probability that player i wins the contest and $p_o(x_1, x_2)$ denotes the probability that the contest ends in a tie. In the event of a tie, we assume the designer tosses a potentially biased coin to determine who gets the prize. Suppose q denotes the probability that the coin lands in favor of player 1. Then, player 1's payoff is given by

$$\Pi_1(x_1, x_2) = V_1(p_1(x_1, x_2) + qp_0(x_1, x_2)) - c(x_1)$$

and player 2's payoff is given by

$$\Pi_2(x_1, x_2) = V_2(p_2(x_1, x_2) + (1-q)p_0(x_1, x_2) - c(x_2)$$

Here, $c : \mathbb{R}_+ \to \mathbb{R}_+$ is a cost function that is weakly increasing, convex, and satisfies $c(0) = 0$. We will assume either linear $c(x) = x$ or quadratic $c(x) = x^2$ costs in our analysis.

Given a contest $C = \{V_1, V_2, p_1(), p_2(), c(), q\}$, a mixed strategy for player i is a probability measure on $X_i = [0, c^{-1}(V_i)]$. A player's equilibrium effort will always belong to the set X_i as the probability of winning is bounded above by unity. An equilibrium is a pair of mixed strategies (μ_1^*, μ_2^*) satisfying

$$\Pi_i(\mu_i^*, \mu_{-i}^*) \geq \Pi_i(\mu_i, \mu_{-i}^*) \text{ for all } \mu_i \in \Delta X_i$$

for both agents $i = 1, 2$. We will impose conditions on the primitives of the model which ensure that an interior solution of the system of first-order conditions for players payoff maximization characterizes a pure strategy Nash equilibrium x_1^*, x_2^*. The first order condition for agent 1 is

$$\frac{\partial \Pi_1}{\partial x_1} = 0 \implies V_1 \left(\frac{\partial p_1}{\partial x_1} + q \frac{\partial p_0}{\partial x_1} \right) = c'(x_1) \tag{1}$$

and that for agent 2 is

$$\frac{\partial \Pi_2}{\partial x_2} = 0 \implies V_2 \left(\frac{\partial p_2}{\partial x_2} + (1-q) \frac{\partial p_0}{\partial x_2} \right) = c'(x_2) \tag{2}$$

Assumptions under which the objective function Π_i is concave and increasing at $x_i = 0$ would ensure that the solution $(x_1^*(C), x_2^*(C))$ to equations 1 and 2 characterizes a pure strategy Nash equilibrium of the contest game C.

Given the contest elements $\{V_1, V_2, p_1(), p_2(), c()\}$, a **tie-breaking rule** is defined by a distribution F on $[0, 1]$. In this paper, we focus on the designer's problem of choosing a tie-breaking rule so as to maximize expected total effort. Formally, letting $R(q)$ denote the total expected effort under the pure strategy Nash equilibrium in the contest $C = \{V_1, V_2, p_1(), p_2(), q\}$, the problem is

$$\max_{F : Q \sim F} \quad \mathbb{E}[R(Q)] \tag{P1}$$

Note that in this case, the problem is essentially to choose a value of q to maximize $R(q)$:

$$\max_{q\in[0,1]} \quad R(q) \tag{P1}$$

We also study the problem in case where the designer might be constrained to be unbiased.

$$\begin{aligned} \max_{F:Q\sim F} \quad & \mathbb{E}[R(Q)] \\ \text{s.t.} \quad & \mathbb{E}[Q] = \frac{1}{2} \end{aligned} \tag{P2}$$

3 Ratio-form contest success functions

In this section, we consider instances of our model where the distribution over the three outcomes depends only on the ratio of the efforts $\theta = \frac{x_1}{x_2}$ of the two players. Formally, we assume that there exist functions $p : \mathbb{R}_+ \to [0, 1]$ and $p_0 : \mathbb{R}_+ \to [0, 1]$ such that

- $p_1(x_1, x_2) = p(\theta)$ and $p_2(x_1, x_2) = p(\frac{1}{\theta})$
- $p_o(x_1, x_2) = p_0(\theta)$ and $p_0(\theta) = p_0(\frac{1}{\theta})$ for all $\theta \in \mathbb{R}_+$
- $c(x) = x$

We'll make the following assumption on these functions.

Assumption 1. Suppose $z_q(\theta) = p(\theta) + qp_0(\theta)$. For any $q \in [0, 1]$, $z_q'(\theta) > 0, z_q''(\theta) < 0$, and $2z_q'(\theta) + \theta z_q''(\theta) > 0$, for all θ in R_+.

Note that $z_q(\theta)$ denotes the probability that player 1 eventually wins the prize and $1 - z_q(\theta)$ is the probability that player 2 wins eventually wins the prize. The assumption then says that a player's probability of winning is increasing in its own effort, decreasing in the effort of the other player, and in addition, it is increasing at a decreasing rate in one's own effort.

The next assumption we make is on the probability of ties $p_0(\theta)$ and is motivated by the idea that the probability of a tie increases as the contest becomes more closely contested.

Assumption 2. The probability of tie $p_0(\theta)$ is increasing for $\theta \in (0, 1]$ and decreasing for $\theta \in [1, \infty)$.

Under these assumptions, we show that the effort-maximizing tie-breaking rule breaks ties in favor of the weaker agent.

Theorem 1. *Consider a ratio-form contest with $V_1 \geq V_2$.*

- *The optimal tie-breaking rule breaks ties in favor of the weaker player ($q = 0$).*
- *Any unbiased tie-breaking rule leads to the same expected total effort.*

The following lemma describes the total effort exerted by the agents in a ratio-form contest and is the key to the proof of Theorem 1.

Lemma 1. *Consider a ratio-form contest with $V_1 \geq V_2$ described by functions $p : \mathbb{R}_+ \to [0,1]$, $p_0 : \mathbb{R}_+ \to [0,1]$, and parameter $q \in [0,1]$. The total effort in Nash equilibrium is given by*

$$R(q) = V_1(1+\beta)(p'(\beta) + qp_0'(\beta))$$

where $\beta = \dfrac{V_1}{V_2}$.

Lemma 1 actually follows directly from the result of Baik [4] who showed that under assumption 1, the equilibrium effort level ratio equals the valuation ratio and in particular, the equilibrium effort profile is given by

$$x_1^* = V_1 \beta z_q'(\beta) \quad x_2^* = V_1 z_q'(\beta)$$

where $\beta = \dfrac{V_1}{V_2}$ and $z_q(\theta) = p(\theta) + qp_0(\theta)$. A complete proof of Lemma 1 using the first order conditions 1 and 2 is in the appendix. Theorem 1 then follows from the fact that the total effort $R(q)$ is linear in q and the coefficient $V_1(1+\beta)p_0'(\beta)$ is < 0 since $\beta = \frac{V_1}{V_2} > 1$.

Observe that with symmetric agents $V_1 = V_2$ and under assumption 2 which implies $p_0'(1) = 0$, it follows from Lemma 1 that $R(q) = V_1(1+\beta)p'(\beta)$. Interestingly, the total effort is independent of the tie-breaking rule.

Corollary 1. *In a ratio-form contest with symmetric agents, any tie-breaking rule leads to the same total expected effort.*

Let us now discuss some examples of ratio-form contest success functions that have been studied in the literature. Assuming zero probability of ties, these kind of ratio-form contest success functions were studied in Baik [4]. The well-known Tullock contests with the contest success function given by $p_i(x_i, x_{-i}) = \frac{x_i^r}{x_i^r + x_{-i}^r}$ is an important example of ratio-form contest success function. There has been significant work on generalizing the Tullock contests to allow for the possibility of ties. An important feature of some these generalizations is that the probability of ties $p_0(\theta)$ is maximized at $\theta = 1$. These models include the contest success function $p_i(x_i, x_{-i}) = \left(\frac{x_i}{x_i + x_{-i}}\right)^k$ with $k \geq 1$ which was proposed and axiomatized by Vesperoni and Yildizparlak [45]. In this model, the probability of ties is $p_0(x_1, x_2) = 1 - \frac{x_1^k + x_2^k}{(x_1+x_2)^k}$ which depends only on the ratio $\theta = \frac{x_1}{x_2}$ and is maximized at $\theta = 1$. Another version of the Tullock contests that allows for ties is given by the contest success function $p_i(x_i, x_{-i}) = \frac{x_i}{x_i + kx_{-i}}$ where $k \geq 1$. This model was axiomatized and proposed by Jia [30]. Again, the probability of a tie in this model is $p_0(x_1, x_2) = 1 - \frac{x_1}{x_1 + kx_2} - \dfrac{x_2}{x_2 + kx_1}$ which depends only on the ratio $\frac{x_1}{x_2}$ and is maximized when $x_1 = x_2$. Thus, we can apply Theorem 1 to characterize the optimal tie-breaking rule under both of these generalizations of Tullock contests.

We conclude this section with a discussion of a generalization of Tullock contest with ties that does not satisfy our assumptions. In this model, the contest success function is given by $p_i(x_i, x_{-i}) = \frac{x_i}{k+x_i+x_{-i}}$ where $k \geq 0$. This model was proposed by Blavatskyy [7]. The probability of a draw is $p_0(x_1, x_2) = \frac{k}{k+x_1+x_2}$ which depends on the sum of efforts and not on the ratio of efforts. Therefore, Theorem 1 does not apply to this case. We solve this model explicitly and find that even though the equilibrium itself depends on the parameter q, the total effort turns out to be independent of the choice of q, even when the agents are asymmetric.

Remark 2. *Suppose the contest success function is $p_i(x_i, x_{-i}) = \frac{x_i}{k+x_i+x_{-i}}$ with $k \geq 0$ and agents have linear costs $c(e) = e$. The equilibrium effort is given by*

$$x_1^*(q) = \frac{V_1^2 V_2}{(V_1 + V_2)^2} - qk \quad x_2^*(q) = \frac{V_1 V_2^2}{(V_1 + V_2)^2} - (1-q)k$$

Using the first order conditions in equations 1 and 2, we can get the Nash equilibrium (x_1^*, x_2^*). As we can see, the total effort takes the form $x_1^* + x_2^* = \frac{V_1 V_2}{V_1 + V_2} - k$ irrespective of the value of q. This is in contrast to the result under generalizations of Tullock contests that do not violate the ratio-form assumption where we showed that the optimal tie-breaking rule breaks ties in favor of the weaker player.

4 Difference-form contest success functions

In this section, we consider instances of our model where the distribution over the three outcomes depends only on the difference of the efforts $\theta = x_1 - x_2$ of the two players. Formally, we assume that there exist functions $p : \mathbb{R} \to [0, 1]$ and $p_0 : \mathbb{R} \to [0, 1]$ such that

- $p_1(x_1, x_2) = p(\theta)$ and $p_2(x_1, x_2) = p(-\theta)$
- $p_0(x_1, x_2) = p_0(\theta)$ and $p_0(\theta) = p_0(-\theta)$ for all $\theta \in \mathbb{R}$
- $c(x) = x^2/2$
- $1 \geq V_1 \geq V_2 > 0$

We'll make the following assumption on these functions.

Assumption 3. Suppose $z_q(\theta) = p(\theta) + qp_0(\theta)$. For any $q \in [0, 1]$, $z_q'(\theta) > 0$ and $z_q''(\theta) \in [-1, 1]$.

Here again, $z_q(\theta)$ denotes the probability that player 1 eventually wins the prize and $1 - z_q(\theta)$ is the probability that player 2 wins eventually wins the prize. The assumption then says that a player's probability of winning is increasing in its own effort, decreasing in the effort of the other player. The second assumption ensures that given the constraint on valuations $1 \geq V_1 \geq V_2 > 0$ and the cost function $c(x) = x^2$, the player's objective function is globally concave and a unique best response exists.

The next assumption we make is on the probability of ties $p_0(\theta)$ and is again motivated by the idea that the probability of a tie increases as the contest becomes more closely contested.

Assumption 4. The probability of tie $p_0(\theta)$ is increasing for $\theta \in (-\infty, 0]$ and decreasing for $\theta \in [0, \infty)$.

Under these assumptions, we have the following result about the optimal tie-breaking rules.

Theorem 2. *Consider a difference-form contest with $1 \geq V_1 > V_2 > 0$.*

- *The optimal tie-breaking rule breaks ties in favor of the weaker player ($q = 0$).*
- *The optimal unbiased tie-breaking rule depends on p_0. If $p_0''(x) < 0$ for all $x \in [0, \sqrt{2V_1}]$, then breaking ties before the contest is optimal ($q = 0, 1$ with equal probability).*

The following lemma describes the total effort exerted by the agents in a difference-form contest and is the key to the proof of Theorem 2.

Lemma 2. *Consider a difference-form contest with $V_1 > V_2$ described by functions $p : \mathbb{R} \to [0,1], p_0 : \mathbb{R} \to [0,1]$, and parameter $q \in [0,1]$. The total effort in Nash equilibrium is given by*

$$R(q) = \frac{V_1 + V_2}{V_1 - V_2}\beta(q)$$

where $\beta(q)$ is the unique solution to the equation $\theta = (V_1 - V_2)(p'(\theta) + qp_0'(\theta))$.

The lemma can be proved using the standard first order conditions in equations 1 and 2 and the full details are in the appendix. In particular, the conditions give us that the equilibrium effort profile is given by

$$x_1^* = \frac{\beta(q)V_1}{V_1 - V_2} \quad x_2^* = \frac{\beta(q)V_2}{V_1 - V_2}$$

To prove Theorem 2, we can use Lemma 2 and our assumptions on p_0 to see how the total effort changes as we increase q. We see that $R(q)$ is decreasing in q and under the additional condition that $p_0''(x) < 0$, the objective $R(q)$ is convex. Thus, the optimal tie-breaking rule would pre-determine a winner in a fair way by tossing a coin before the contest begins. The full proof is in the appendix.

Let us now discuss some examples of difference-form contest success functions that have been studied in the literature. These success functions were introduced in Hirshleifer [27]. A well-known example, with zero probability of ties, is the logit function $p_i(x_1, x_2) = \frac{\exp(x_i)}{\exp(x_i)+\exp(x_{-i})}$. As we saw in with ratio-form csfs, a couple of simple generalizations of the difference-form csf that allows for the possibility of ties would be $p_i(x_1, x_2) = \left(\frac{\exp(x_i)}{\exp(x_i)+\exp(x_{-i})}\right)^k$ and $p_i(x_1, x_2) = \frac{\exp(x_i)}{\exp(x_i)+k\exp(x_{-i})}$ with $k > 1$. We can apply Theorem 2 to identify optimal tie-breaking rules under both of these and other generalizations of difference-form contest success functions.

5 Examples

In this section, we consider some simple contests and illustrate how the timing of the tie-breaker may influence the total effort in equilibrium.

5.1 Single-shot contest

In this subsection, we consider an instance of our model where

- $p_i(x_i, x_{-i}) = x_i(1 - x_{-i})$
- $c(x) = x^2$
- $2 \geq V_1 \geq V_2 > 0$

A motivation for this specification is a situation where the two players get a single shot at a basketball hoop and a player wins if and only if it is the only one to make the shot. The players exert effort cost x_i^2 to improve their individual probability x_i of making the shot. The resulting probabilities of winning the contest and the cost of effort will then be as described above. The next result identifies the optimal tie-breaking rule for this instance.

Lemma 3. *In a single-shot contest, the optimal tie-breaking rule breaks ties in favor of the weaker player ($q = 0$). With symmetric agents, the optimal unbiased tie-breaking rule predetermines a winner in case of a tie by tossing a fair coin.*

To prove the result, we characterize the unique pure strategy Nash equilibrium of the single-shot contest game using equations 1 and 2 to get that total expected effort equals

$$R(b) = x_1^* + x_2^* = \frac{V_1(1-b) + V_2(1+b) + b^2 V_1 V_2}{4 + b^2 V_1 V_2}$$

where $b = 2q - 1$. We can then show that $R(b)$ is maximized at $b = -1$. With $V_1 = V_2$, $R(b)$ is maximized at both $b = -1, 1$ and so an unbiased tie-breaking rule would simply bias the contest in a fair way in favor of one the agents by tossing a fair coin before the contest. We believe this holds even in the case where $V_1 > V_2$ but haven't been able to prove it yet. But we can show that for any $V_1 > V_2$, breaking ties by tossing a fair coin before the contest leads to greater expected effort as compared to tossing a fair coin after the contest.

5.2 Sudden death contest

In this subsection, we consider an instance of our model where

- $p_i(x_i, x_{-i}) = \dfrac{x_i(1 - x_{-i})}{x_i + x_{-i} - x_i x_{-i}}$
- $c(x) = x$
- $2 \geq V_1 \geq V_2 > 0$

A motivation for this specification is a sudden death contest where players get multiple shots at a Basketball hoop and they can expend effort x_i to better their probability x_i of getting the shot. Player i wins if it is the first player to make the shot. The contest is drawn if both players make their first shot after missing the same number of times. The resulting probabilities of winning the contest and the cost of effort will then be as described above.

Lemma 4. *In a sudden death contest with symmetric agents $2 \geq V \geq \frac{3}{2}$, the expected effort from running a fair tie-breaker after the contest is higher than that from running it before the contest.*

To prove the result, we obtain conditions for an action profile to be a pure strategy Nash equilibrium of the sudden death contest game using equations 1 and 2. Using these conditions, we obtain the symmetric NE for the case where $q = \frac{1}{2}$. We find that the equilibrium effort for each agent takes the form $x^* = 1 - \sqrt{1 - \frac{V}{2}}$. While we don't obtain the equilibrium effort for $q = 0$ or $q = 1$, we use the conditions to obtain an upper bound on the total effort of 1. We then show that $2x^* \geq 1 \iff V \geq \frac{3}{2}$ which implies the result.

6 Conclusion

We study two-player contests with the possibility of ties under both ratio-form and difference-form contest success functions. In these contests, we study the effect different tie-breaking rules have on the effort exerted by the players. When players are heterogeneous, we find that the total effort decreases as the probability that ties are broken in favor of the stronger agent increases. Thus, an effort-maximizing designer would prefer to commit to breaking ties in favor of the weaker agent. The result lends further support to the encouraging effect of leveling the playing the field on effort in contests with heterogeneous agents.

With symmetric agents, we find that the optimal tie-breaking rules are unbiased. We illustrate through examples that the designer may prefer to pre-determine winners by breaking ties before the contest begins or may prefer to break ties after the contest. For our results, we do impose conditions on our model that ensure existence of pure strategy Nash equilibrium. It would be interesting to see how this idea extends to contests where pure strategy Nash may not exist. Another interesting extension would be to contests with more than two players.

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A Proofs for Section 3 (Ratio-form contest success functions)

Theorem 1. *Consider a ratio-form contest with $V_1 \geq V_2$.*

- *The optimal tie-breaking rule breaks ties in favor of the weaker player ($q = 0$).*
- *Any unbiased tie-breaking rule leads to the same expected total effort.*

Proof. From Lemma 1, the total effort $R(q)$ is linear in q with the coefficient $V_1(1+\beta)(p_0'(\beta))$. The sign of the coefficient is determined by $p_0'(\beta)$. Since $\beta = \frac{V_1}{V_2} > 1$ and by Assumption 2, $p_0(\theta)$ is decreasing for $\theta > 1$, we get that $p_0'(\beta) < 0$. It follows that the the total effort is decreasing in q and so the optimal tie-breaking rule breaks ties in favor of the weaker player ($q = 0$).

For the second part of the theorem, an unbiased tie-breaking rule is one that constrains the expected value of Q to be 0.5. For any random variable Q with the constraint $\mathbb{E}[Q] = 0.5$, we have that the expected total effort will be

$$\mathbb{E}[R(Q)] = V_1(1+\beta)(p'(\beta) + \frac{1}{2}p_0'(\beta))$$

□

Lemma 1. *Consider a ratio-form contest with $V_1 \geq V_2$ described by functions $p : \mathbb{R}_+ \to [0,1]$, $p_0 : \mathbb{R}_+ \to [0,1]$, and parameter $q \in [0,1]$. The total effort in Nash equilibrium is given by*

$$R(q) = V_1(1+\beta)(p'(\beta) + qp_0'(\beta))$$

where $\beta = \dfrac{V_1}{V_2}$.

Proof. Player 1's payoff is $V_1(p(\theta)+qp_0(\theta))-x_1$ and that of player 2 is $V_2(1-p(\theta)-qp_0(\theta))-x_2$. Using the first order conditions 1 and 2, we get

$$V_1(p'(\theta) + qp_0'(\theta)) = x_2$$

and

$$x_1 V_2 \left(p'(\theta) + qp_0'(\theta)\right) = x_2^2$$

This gives that the equilibrium ratio satisfies

$$\frac{x_2}{x_1} = \frac{V_2}{V_1}$$

As a result, we get that

$$x_1^* = V_1\beta(p'(\beta) + qp_o'(\beta)) \quad x_2^* = V_1(p'(\beta) + qp_o'(\beta))$$

where $\beta = \frac{V_1}{V_2}$ The total effort in equilibrium is then given by

$$x_1^* + x_2^* = V_1(1+\beta)(p'(\beta) + qp_o'(\beta))$$

where $\beta = \dfrac{V_1}{V_2}$ as required. □

B Proofs for Section 4 (Difference-form contest success functions)

Theorem 2. *Consider a difference-form contest with $1 \geq V_1 > V_2 > 0$.*

- *The optimal tie-breaking rule breaks ties in favor of the weaker player ($q = 0$).*
- *The optimal unbiased tie-breaking rule depends on p_0. If $p_0''(x) < 0$ for all $x \in [0, \sqrt{2V_1}]$, then breaking ties before the contest is optimal ($q = 0, 1$ with equal probability).*

Proof. From lemma 2, the total effort in Nash equilibrium is given by

$$R(q) = \frac{V_1 + V_2}{V_1 - V_2}\beta(q)$$

where $\beta(q)$ is the unique solution to the equation $\theta = (V_1 - V_2)(p'(\theta) + qp_0'(\theta))$.

Given that $V_1 > V_2$, $R'(q)$ has the same sign as $\beta'(q)$. From the characterizing equation, we know that

$$\beta'(q) = (V_1 - V_2)\left(p''(\beta(q))\beta'(q) + p_0'(\beta(q)) + qp_0''(\beta(q))\beta'(q)\right)$$

which implies

$$\beta'(q)\left(1 - (V_1 - V_2)\left(p''(\beta(q)) + qp_0''(\beta(q))\right)\right) = (V_1 - V_2)p_0'(\beta(q))$$

Given assumption 3 and that $1 \geq V_1 > V_2 > 0$, we know that $\beta'(q)$ has the same sign as $p_0'(\beta(q))$. We already know $\beta(q) > 0$ and therefore, from assumption 4, we get that $\beta'(q) < 0$. Therefore, the total expected effort $R(q)$ is decreasing in q and it follows that the optimal tie breaking rule breaks ties in favor of the weaker agent by setting $q = 0$.

Now let's discuss the optimal unbiased tie-breaking rule. We first note that the solution $x(t)$ of an implicit equation $F(x, t) = 0$ is convex iff

$$\frac{\partial F}{\partial t}\frac{\partial^2 F}{\partial x \partial t} - \frac{\partial F}{\partial x}\frac{\partial^2 F}{\partial t^2} \geq 0$$

In our case, $\beta(q)$ is the solution to the equation $(V_1 - V_2)(p'(\beta) + qp_0'(\beta)) - \beta = 0$ Using the above condition, we get that $\beta(q)$ is convex if and only if

$$(V_1 - V_2)^2 p_0'(\beta)p_0''(\beta) \geq 0$$

We know that $\beta \in [0, \sqrt{2V_1}]$ and $p_0'(\beta) < 0$. Thus, we get that $\beta(q)$, and consequently $R(q)$, is convex if $p_0''(x) < 0$ for all $x \in [0, \sqrt{2V_1}]$. If the objective is convex, the optimal unbiased tie-breaking rule would break ties in a fair manner before the contest begins. □

Lemma 2. *Consider a difference-form contest with $V_1 > V_2$ described by functions $p : \mathbb{R} \to [0, 1], p_0 : \mathbb{R} \to [0, 1]$, and parameter $q \in [0, 1]$. The total effort in Nash equilibrium is given by*

$$R(q) = \frac{V_1 + V_2}{V_1 - V_2}\beta(q)$$

where $\beta(q)$ is the unique solution to the equation $\theta = (V_1 - V_2)(p'(\theta) + qp_0'(\theta))$.

Proof. Player 1's payoff is $V_1(p(\theta) + qp_0(\theta)) - x_1^2/2$ and that of player 2 is $V_2(1 - p(\theta) - qp_0(\theta)) - x_2^2/2$. Using the first order conditions 1 and 2, we get

$$V_1(p'(\theta) + qp_0'(\theta)) = x_1$$

and

$$V_2(p'(\theta) + qp_0'(\theta)) = x_2$$

This again gives that the equilibrium ratio satisfies

$$\frac{x_2}{x_1} = \frac{V_2}{V_1}$$

In addition, we have that $x_1 - x_2 = (V_1 - V_2)(p'(x_1 - x_2) + qp_0'(x_1 - x_2))$ We know that $x_1 - x_2 = \beta(q)$ is the unique solution to this equation. Together, we get that

$$x_1^* = \frac{\beta(q)V_1}{V_1 - V_2} \quad x_2^* = \frac{\beta(q)V_2}{V_1 - V_2}$$

The total effort in equilibrium is then given by

$$x_1^* + x_2^* = \frac{V_1 + V_2}{V_1 - V_2}\beta(q)$$

as required. □

C Proofs for Section 5 (Examples)

Lemma 3. *In a single-shot contest, the optimal tie-breaking rule breaks ties in favor of the weaker player ($q = 0$). With symmetric agents, the optimal unbiased tie-breaking rule predetermines a winner in case of a tie by tossing a fair coin.*

Proof. In the single shot contest game, we have $\frac{\partial p_i}{\partial x_i} = (1 - x_{-i})$, $\frac{\partial p_0}{\partial x_i} = 2x_{-i} - 1$ and $c'(x) = 2x$. Plugging these into the two first order conditions in equations 1 and 2 and letting $q = \frac{1+b}{2}$ with bias $b \in [-1, 1]$, we get that the NE satisfies:

$$V_1(1 - q + x_2 b) = 2x_1$$
$$V_2(q - x_1 b) = 2x_2$$

Thus, we have two linear equations in two variables which we can solve to get the Nash equilibrium

$$x_1^* = \frac{V_1\,(1 - b + bqV_2)}{4 + b^2V_1V_2} \quad x_2^* = \frac{V_2\,(1 + b - b(1 - q)V_1)}{4 + b^2V_1V_2}$$

so that

$$R(b) = x_1^* + x_2^* = \frac{V_1(1-b) + V_2(1+b) + b^2V_1V_2}{4 + b^2V_1V_2}$$

Since $V_1 \geq V_2$, we have that for any $b > 0$, $R(b) \leq R(-b)$. And therefore, the optimal bias $b \in [-1, 0]$. In this domain, we can differentiate the objective with respect to b and show that $R(b)$ is decreasing in b. We have

$$R'(b) = \frac{(4 - b^2V_1V_2)(V_2 - V_1) - 2bV_1V_2(V_1 + V_2 - 4)}{(4 + b^2V_1V_2)^2}$$

Since $2 \geq V_1 \geq V_2 > 0$, $R'(b) < 0$ for $b \in [-1, 0]$ which implies $R(b)$ is decreasing in this domain. Thus, the solution to the designer's problem when it is unconstrained is to set $b = -1$. In other words, the optimal tie-breaking rule for the designer is to resolve ties in favor of the weaker agent.

Now let's suppose the designer is constrained in that it should be ex-ante fair. In the case where agents are symmetric so that $V_1 = V_2 = V$, it is clear that the objective is increasing for $b > 0$ and decreasing for $b < 0$. And also, it is symmetric around $b = 0$. Thus, the optimal tie-breaking rule among those that are ex-ante fair would bias the contest in favor of one of the two agents with equal probability.

□

Lemma 4. *In a sudden death contest with symmetric agents $2 \geq V \geq \frac{3}{2}$, the expected effort from running a fair tie-breaker after the contest is higher than that from running it before the contest.*

Proof. In the sudden death contest game, we have $\frac{\partial p_i}{\partial x_i} = \frac{(1-x_{-i})x_{-i}}{(x_i + x_{-i} - x_i x_{-i})^2}$, $\frac{\partial p_0}{\partial x_i} = \frac{x_{-i}^2}{(x_i + x_{-i} - x_i x_{-i})^2}$ and $c'(x) = 1$. Plugging these into the two first order conditions in equations 1 and 2 we get that the NE satisfies:

$$V_1 \left(\frac{x_2(1 - x_2 + qx_2)}{(x_1 + x_2 - x_1x_2)^2} \right) = 1$$

$$V_2 \left(\frac{x_1(1 - qx_1)}{(x_1 + x_2 - x_1x_2)^2} \right) = 1$$

Consider the case where $V_1 = V_2$ and $q = \frac{1}{2}$. In this case, the unique symmetric equilibrium takes the form

$$x^* = 1 - \sqrt{1 - \frac{V}{2}}$$

Now suppose $q = 0$. In this case, the equilibrium satisfies $x_1 = x_2(1 - x_2)$ and so the total effort in a pure strategy NE with $q = 0$ is bounded above by $\max_{x_2} x_2 + x_2(1 - x_2) =$

$\max_{x_2} x_2(2 - x_2) = 1$. Thus, the effort in a Nash equilibrium under a biased contest ($q = 0$) must be less than that under a fair contest ($q = \frac{1}{2}$) if

$$2 - 2\sqrt{1 - \frac{V}{2}} \geq 1 \iff V \geq \frac{3}{2}$$

Thus, we know that if the valuation is quite high, the principal would certainly prefer running a fair tie-breaker after the contest than biasing it by running a tie-breaker before the contest. $\square$