Exchange economies with indivisible goods

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What is exchange economy?

- An exchange economy consists of agents who are endowed with some goods and have some preferences over bundles they can consume.
- Some examples include:
  1. Students assigned to dorm rooms
  2. Passengers assigned airplane seats
  3. Kidney transplant patients with a family/friend donor
  4. Students assigned topics and dates for seminar series
- In this talk, we will focus on such exchange economies with indivisible goods.

Example

- Consider a housing economy with three agents in which each agent i is endowed with a house h_i and has the following preferences:

Agent 1	Agent 2	Agent 3
h_2	h_3	h_1
h_1	h_2	h_3
h_3	h_1	h_2

- In this economy, the agents can trade and be strictly better off. But doing so requires agents to know about the preferences of other agents.
- Or maybe we can use the market mechanism?

The market mechanism

- In economies with infinitely divisible goods, the market mechanism leads to allocations that are Pareto optimal (first welfare theorem).
- Thus, the market, through prices, serves as an efficient mechanism for aggregating agents preferences.
- Can we use markets for economies with indivisible goods as well?
- Let us investigate this in the context of the simplest exchange economy with indivisible goods: the housing economy [Shapley and Scarf [1974]]!

Housing economy

- A housing economy $\{N, H, (u_i)_{i \in N}\}$ consists of
  - $N = \{1, 2, \ldots, n\}$: set of agents
  - $H = \{h_1, h_2, \ldots h_n\}$: set of n houses, with h_i owned by i
  - $u_i : H \to \mathbb{R}$: agent i's utility function over houses
- An assignment in this economy is a function $x : N \to H$.
- An assignment $x : N \to H$ is an allocation if x is a bijection.

Market equilibrium

- A market equilibrium is a pair (p, x) where $p : H \to \mathbb{R}$ is a price function and $x : N \to H$ is an assignment such that
  - $x(i)$ maximizes agent i's utility over its budget set $\{h : p(h) \leq p(h_i)\}$
  - x is an allocation
- In the introductory example, observe that $p(h_i) = a$ for all i and x assigning each agent its favorite house constitutes a market equilibrium.
- In general with n agents and n houses, if each agent has a different favorite house, then there is an equilibrium that assigns the same price to every house.
- Observe that in the market, each individual only needs to know the prices p to make its choice.

Another example

- Each agent has same preference

Agent 1	Agent 2	Agent 3
h_1	h_1	h_1
h_2	h_2	h_2
h_3	h_3	h_3

- In this case, any p such that $p(h_1) > p(h_2) > p(h_3)$ and $x(i) = h_i$ constitutes a market equilibrium.
- In general with n agents and n houses, if each agent has the same preference over houses, then there is an equilibrium in which the price system reflects the common preference.

Efficient allocations

- In the above examples, the market implements what seem to be efficient allocations given the preferences.
- Formally, we would want the allocation to be *Pareto efficient*. That is, there shouldn't be another allocation that makes everyone better off.
- Does the market always lead to something that is Pareto efficient?

First welfare theorem

Theorem

For any housing economy, every market equilibrium allocation is Pareto optimal.

- Suppose (p, x) is a market equilibrium and y Pareto dominates x. so that $u_i(x(i)) \leq u_i(y(i))$ for all i with a strict inequality for at least one.
- But then, it must be that $p(y(i)) \geq p(x(i))$ for all i, with a strict inequality for at least one.
- But this contradicts $\sum_i p(x(i)) = \sum_i p(y(i)) = \sum_i p(h_i)$.

But, how often does the market equilibrium exist?

It always does!

Theorem (Shapley and Scarf [1974])

Every housing economy has a market equilibrium. Moreover, the equilibrium allocation is unique.

- Proof by construction: Top Trading Cycles (TTC) due to David Gale

Top trading cycle (TTC)

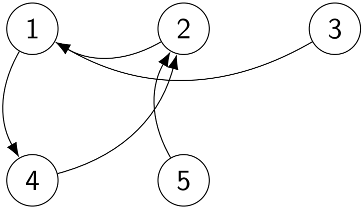
While agents remain:

1. all agents point to the owner of their favorite house
2. find a trading cycle $C = \{i_1, i_2, ..., i_k\}$ which has the property that
    1. i_j points to i_{j+1} for all $j \in \{1, 2, \ldots, k-1\}$
    2. $i_1 = i_k$
3. execute the trade suggested by the trading cycle and remove these agents and houses from the game

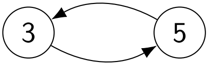
Illustration

Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
h_4	h_1	h_1	h_2	h_2
h_3	h_4	h_3	h_3	h_4
h_1	h_3	h_5	h_4	h_3
h_2	h_2	h_2	h_1	h_1
h_5	h_5	h_3	h_5	h_5

- Step 1:



- Step 2:



Prices

- If the algorithm terminates after m steps, it generates a partition of the agents as $N = C_1 \cup C_2 \cup \cdots \cup C_m$.
- We can assign prices $p_1 > p_2 \cdots > p_m$ to the houses belonging to the respective cycles.
- Then agent $i \in C_j$ can sell its house for p_j. It cannot afford a house owned by agents in $C_1 \cup C_2 \cup \cdots \cup C_{j-1}$. Thus, its utility is maximized from buying the house of its cyclic successor in C_j, which costs exactly p_j.
- Thus, the allocation x^{TTC} with the above prices constitute a market equilibrium of the housing economy.

Other nice properties

- TTC finds a Pareto-efficient allocation.
- TTC finds a core-stable allocation.
  1. No coalition $S \subset N$ can redistribute their houses amongst themselves so that all agents in S are better off.
- TTC mechanism is strategyproof.
  1. If agent's preferences are private information, it is in their best interest to reveal these preferences honestly to a designer who will implement the TTC allocation.
  2. In fact, TTC is the unique mechanism that is Pareto efficient, individually rational, and strategyproof.

Housing and parking spot market

Example

Suppose there are three agents endowed with a house and a parking spot $\omega_i = (h_i, p_i)$ and have the following preferences:

Agent 1	Agent 2	Agent 3
$(h_1, p_2)^1$	$(h_2, p_3)^2$	$(h_1, p_3)^3$
$(h_3, p_1)^3$	$(h_2, p_1)^1$	$(h_3, p_2)^2$
$(h_1, p_1)^2$	$(h_2, p_2)^3$	$(h_3, p_3)^1$
$\ldots$	$\ldots$	$\ldots$

- The economy in this example has an empty core.
- Significant research in studying existence under natural preference domains.

Thank you!