Luce contracts*

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Abstract

We study a multi-agent contract design problem with moral hazard. In our model, each agent exerts costly effort towards an individual task at which it may either succeed or fail, and the principal, who wishes to encourage effort, has an exclusive-use budget that it can use to reward the agents. We first show that any optimal contract must distribute the entire budget among the successful agents. Moreover, every such contract is optimal for some objective function. Our main contribution is then to introduce a novel class of contracts, which we call Luce contracts, and show that there is always a Luce contract that is optimal. A (generic) Luce contract assigns weights to the agents and distributes the entire budget among the successful agents in proportion to their weights. Lastly, we characterize effort profiles that can be implemented by Luce contracts, and note that Luce contracts offer a desirable alternative for implementation over commonly studied contracts, like piece-rate and bonus-pool contracts, on account of their reward variance-minimizing property.

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1 Introduction

The literature on multi-agent contract design has identified conditions under which contracts like piece-rate contracts or bonus-pool contracts are optimal. These studies, however, implicitly assume that the principal is unconstrained in its ability to make payments, leading to some potentially undesirable features of the optimal contracts, such as significant variability in total payments and the possibility of excessively large payments. In practice, the principal's capacity to pay may be limited due to liquidity constraints or organizational policies. In this paper, we model the contract design problem in the presence of such budgetary constraints and introduce a novel class of contracts called Luce contracts, demonstrating their optimality in such environments and also highlighting their relevance in more standard contract design environments.

In our model, each agent exerts effort towards an independent task, in which it may succeed or fail, and the principal can commit to a contract which defines the payment to each agent as a function of the set of agents who are successful. We capture the principal's budgetary restrictions by assuming it has access to an exclusive-use budget: the total payment to the agents can never exceed the budget, and any unused portion of the budget provides no additional value to the principal. This assumption is appropriate in situations where, for example, the budget is provided to the principal as a restricted-use fund, or where the principal can only reward the agents with probability shares for an indivisible prize. The principal's problem is then to find a contract, which describes how the budget is distributed among the agents for each outcome, so as to incentivize agents to exert higher effort towards being successful.

We first show that, even though there might be multiple contracts that are optimal, any such contract must be *successful-get-everything* (SGE): it rewards only the successful agents, and additionally, it distributes the entire budget among them. In particular, contracts such as piece-rate or bonus-pool contracts, which under certain outcomes would distribute only a fraction of the budget to successful agents, can never be optimal in this setting. Intuitively, the principal should be able to improve upon contracts that leave money on the table by using that leftover money to increase the rewards of some successful agent and thereby incentivize that agent to exert more effort. However, equilibrium forces imply that there isn't necessarily an obvious way to distribute this left-over budget and induce higher effort from all agents. Nevertheless, we show that there is always some way to do this, and this implies that any optimal contract must be SGE. We further show that any contract which is SGE is optimal for some monotone objective of the principal. Stated differently, the maximal set of implementable equilibria are characterized by equilibria of SGE contracts.

Our main contribution is then to introduce a subclass of SGE contracts, called Luce contracts, and to show that there is always a Luce contract that is optimal. A Luce contract is defined by assigning weights to and a (weak) priority order over the set of agents, and it distributes the entire budget among the highest-priority successful agents in proportion to their weights. We show that Luce contracts are minimally sufficient to implement the maximal set, and thus, the principal can always optimize over Luce contracts. This result provides for a significant reduction in the complexity of the design problem, narrowing the search space from a $\Theta(n2^n)$-dimensional space of complex and difficult-to-interpret contracts to an $(n-1)$-dimensional space of simple and easily interpretable contracts. We apply our results to the special case of two agents with quadratic costs, where our analysis uncovers an interesting robustness in the structure of the optimal contract to the heterogeneity in the agents' cost parameters.

Lastly, we discuss the desirability of Luce contracts for implementation in more standard contract design environments, where the principal incurs a direct cost from rewarding the agents but is not budget-constrained. In such environments, the principal has access to a wide range of contracts, including piece-rate and bonus-pool contracts, for the purpose of inducing any desired effort profile. We characterize the set of effort profiles that can be implemented via Luce contracts with some budget and show that, for any such profile, the total payment under any implementing contract is a mean-preserving spread of the total payment under the corresponding Luce contract. It follows that Luce contracts minimize both the maximum total payment the principal might need to make and the variability in total payment for these effort profiles. With these desirable properties, we believe Luce contracts might provide a useful alternative for inducing effort over commonly studied contracts, like piece-rate and bonus-pool contracts, for the purpose of implementation in general multi-agent contract design environments.

Related literature

There is a vast literature on principal-agent problems under moral hazard. In the canonical model with a single agent (Holmström [20], Grossman and Hart [16], Mirrlees [33]), the principal offers a wage contract that defines the agent's payment as a function of its observed output, and then the agent chooses some unobserved action (effort) that determines the distribution over outputs. A key finding is that optimal contracts reward the agent for output realizations that are informative about the target level of effort (informativeness principle), and may therefore be non-monotone in output. There has since been significant work studying variants of this single-agent model incorporating flexible actions (Georgiadis, Ravid, and Szentes [14]), multiple tasks (Holmstrom and Milgrom [23], Bond and Gomes [5]), bounded payments (Jewitt, Kadan, and Swinkels [26]), combinatorial actions (Dütting, Ezra, Feldman, and Kesselheim [9]), and informationally robust design (Carroll [6], Zhang [38]). See Georgiadis [13], Holmström [22] and Lazear [28] for recent surveys of this literature.

Our paper contributes to the literature studying a principal contracting with multiple agents. In these settings, agents may be rewarded not only based on their absolute performance but also on their relative performance or even the joint performance of the group. When agents' observed outputs are subject to a common shock or involve subjective evaluations, the optimal contract may involve pay for relative performance (Green and Stokey [15], Lazear and Rosen [29], Malcomson [32, 31], Mookherjee [34], Nalebuff and Stiglitz [35], Imhof and Kräkel [24]). And joint performance evaluation may be optimal in the presence of negatively correlated outputs (Fleckinger [12]), complementarities in production (Alchian and Demsetz [1]), incentives to help (Itoh [25]), and unknown production environment (Kambhampati [27]). Other related work on multi-agent contract design explores moral hazard in teams (Holmstrom [21], Winter [37], Battaglini [4], Babaioff, Feldman, Nisan, and Winter [2], Halac, Lipnowski, and Rappoport [19], Dai and Toikka [8]), computational challenges in finding optimal contracts (Castiglioni, Marchesi, and Gatti [7], Dütting, Ezra, Feldman, and Kesselheim [10], Ezra, Feldman, and Schlesinger [11]), and design problems tailored to more specific environments (Halac, Kartik, and Liu [18], Haggiag, Oren, and Segev [17], Baiman and Rajan [3]). In contrast to this literature, our work focuses on a principal contracting with multiple independent agents, where standard single-agent contracts would be optimal but are rendered infeasible due to budgetary constraints on the principal.

2 Model

There is a principal and n risk-neutral agents. Each agent $i \in [n]$ participates in an independent task, in which it may succeed or fail. The agents choose how much effort to exert towards succeeding in their respective tasks, captured by their choice of success probabilities: if agent i chooses success probability $p_i \in [0, 1]$, it incurs a cost $c_i(p_i)$ in doing so, where $c_i : [0, 1] \to \mathbb{R}$ is differentiable, strictly increasing, and strictly convex with $c_i'(0) = 0$. The cost functions are assumed to be common knowledge.

The principal would like to incentivize the agents to maximize the probability that they succeed in their tasks. Formally, we assume that the principal's preference over success probability profiles is represented by a continuous, strictly increasing objective function $V(p_1, \ldots, p_n)$. For example, if the principal obtains a profit of w_i if agent i succeeds in its task, and it cares about maximizing expected profit, its preference would be represented by the objective function

$$V(p) = \sum_{i=1}^{n} w_i p_i.$$

In order to incentivize the agents, the principal can design a contract mapping observed outcomes to a reward for each agent. We assume that the principal has an exclusive-use budget B that can be used for the purpose of rewarding the agents, and the principal gets no value from unused portions of the budget. Formally, the principal commits to a contract which specifies the fraction of the budget each agent receives depending on the observed outcome.

Definition 1. A *contract* is a function $f = (f_1, \ldots, f_n) : 2^{[n]} \to \mathbb{R}_+^{[n]}$ such that

- $f_i(S) \geq 0$ (limited liability) and
- $\sum_{j \in [n]} f_j(S) \leq 1$ (budget constraint)

for each $i \in [n]$ and $S \subseteq [n]$.

Under the contract f, if S is the set of agents who succeed, then each agent i receives the reward $f_i(S) \cdot B$. For instance, a natural contract in this environment would be to split the budget equally amongst agents who are successful:

$$f_i(S) = \begin{cases} \frac{1}{|S|}, & i \in S \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Note that, as in this contract, each agent's reward may depend not only on whether she succeeds, but also on the success of the other agents. We will denote by $\mathcal{F}$ the set of contracts. Throughout the paper, we will work with several subclasses of contracts, and we will in general denote by $\mathcal{F}_\phi$ the set of contracts with property ϕ.

A contract $f \in \mathcal{F}$ defines a normal-form game between the n agents, in which each agent chooses $p_i \in [0,1]$ and agent $i's$ payoff under the profile $p = (p_1, \dots, p_n)$ is its expected reward minus the cost of choosing p_i. Formally,

$$u_i(p) = \mathbb{E}[f_i(S) \cdot B] - c_i(p_i) = \sum_{S \subset [n]} \mathrm{Pr}_p^{[n]}(S) \cdot f_i(S) \cdot B - c_i(p_i)$$

where

$$\mathrm{Pr}_p^{[n]}(S) = \prod_{i \in S} p_i \prod_{j \in [n] \setminus S} (1 - p_j).$$

Observe that since agent i's expected reward is linear in p_i and its cost function is strictly convex, u_i is strictly concave in p_i. It follows then that for any contract $f \in \mathcal{F}$, a pure-strategy Nash equilibrium exists.

We will denote by $E : \mathcal{F} \rightrightarrows [0,1]^n$ the equilibrium correspondence, so $E(f)$ is the set of equilibria for the contract f and $E^{-1}(p)$ is the set of contracts for which p is an equilibrium. We further denote by $\mathcal{E}$ the set of implementable profiles:

$$\mathcal{E} := \{p \in [0,1]^n \,:\, p \in E(f) \text{ for some } f \in \mathcal{F}\}.$$

The principal's problem is to choose a contract f that induces an equilibrium $p \in E(f)$ that maximizes its objective V. As is standard, we assume that when a contract has multiple equilibria, the principal can select its most preferred equilibrium.[1] Hence, the principal's problem is to maximize its objective V over the set of implementable profiles $\mathcal{E}$. As we show, $\mathcal{E}$ is compact, and since V is strictly increasing, any maximizer of V must be a maximal element of $\mathcal{E}$ with respect to the standard coordinatewise ordering. We denote by $\mathcal{P}$ the set of maximal equilibrium profiles in $\mathcal{E}$. Formally,

$$\mathcal{P} = \{p \in \mathcal{E} : \forall q \in \mathcal{E},\ q_i \geq p_i\ \forall i \in [n] \ \Rightarrow\ q = p\}.$$

[1]For an illustration of multiplicity, suppose $B \geq \sum_{i \in [n]} c_i'(1)$ and consider the bonus-pool contract under which agent i's reward is $c_i'(1)$ if all agents succeed and 0 otherwise. For this contract, it is easy to verify that both $p = (1, 1, \dots, 1)$ and $p = (0, 0, \dots, 0)$ are Nash equilibria. That said, if the budget is sufficiently small and the cost functions are sufficiently convex, we can use the diagonally strict concavity condition of Rosen [36] to show that every contract has a unique equilibrium.

The principal's problem thus reduces to maximizing its objective V over the set of maximal profiles $\mathcal{P}$:

$$\max_{p \in \mathcal{P}} V(p).$$

Small budget

Notice that if $B \geq \sum_{i \in [n]} c_i'(1)$, the principal can induce any $q \in [0,1]^n$ using the following piece-rate contract:

$$f_i(S) = \begin{cases} \frac{1}{B} \cdot c_i'(q_i), & i \in S \\ 0, & \text{otherwise.} \end{cases}$$

Indeed, agent i's expected utility in this case is $p_i \cdot c_i'(q_i) - c_i(p_i)$ which is uniquely maximized at $p_i = q_i$. It follows that if the budget is sufficiently large, then $\mathcal{E} = [0,1]^n$, $\mathcal{P} = \{(1, 1, \ldots, 1)\}$, and the principal's problem is trivial.

Hence, we focus on the more interesting case where the principal's budget is small. In particular, we assume that the principal's budget is small enough that it could not induce even one of the agents to succeed with probability 1.

Assumption 1. The principal's budget $B < \min\{c_1'(1), c_2'(1), \ldots, c_n'(1)\}$.

Assumption 1 ensures that in equilibrium, no agent chooses $p_i = 1$. As we discuss later, our results extend to situations with a larger budget as long as we restrict attention to equilibria that are interior.

For convenience, we further normalize the budget to $B = 1$. Note that this is without loss of generality, since for any contract $f \in \mathcal{F}$, the game with budget B and costs c_i is strategically equivalent to the game with budget 1 and costs $\frac{1}{B} \cdot c_i$.

Best response

For any contract $f \in \mathcal{F}$, $u_i(p)$ is strictly concave in p_i. It follows that for any profile p_{-i}, agent i has a unique best response, which we denote by $b_i(f, p_{-i})$. To find this best response, notice that the marginal utility of agent i at profile p is

$$\frac{\partial u_i(p)}{\partial p_i} = r_i(f, p_{-i}) - c_i'(p_i),$$

where

$$r_i(f, p_{-i}) = \mathbb{E}[f_i(S) \,|\, i \in S] - \mathbb{E}[f_i(S) \,|\, i \notin S] = \sum_{S \subset [n]_{-i}} (f_i(S \cup \{i\}) - f_i(S)) \mathrm{Pr}_{p_{-i}}^{[n]_{-i}}(S)$$

denotes agent i's expected gain in reward from succeeding as compared to failing in her task. Since u_i is strictly concave in p_i, the best response is given by the solution to the first order condition. Moreover, since $r_i(f, p_{-i}) \leq \mathbb{E}[f_i(S) \,|\, i \in S] \leq 1$ and by Assumption 1, $c_i'(1) > B = 1$, this best response is strictly smaller than 1. In particular, we get that $b_i(f, p_{-i})$ is the solution to the equation

$$c_i'(p_i) = \max\{0, r_i(f, p_{-i})\}. \tag{1}$$

It follows that for any contract $f \in \mathcal{F}$, p is an equilibrium under f if and only if $p_i = b_i(f, p_{-i})$ for all $i \in [n]$.

In this paper, we identify contracts whose equilibria characterize $\mathcal{P}$, and further provide a subclass of contracts that is actually sufficient to implement $\mathcal{P}$. We use these results to solve for some natural objectives of the principal, and also briefly discuss the intermediate budget case towards the end of the paper.

3 Characterization of optimal contracts

In this section, we identify a class of contracts whose equilibria characterize the set of maximal equilibria.

3.1 FGN contracts

A natural property that one might expect any reasonable contract must have is that agents who fail get no reward.

Definition 2. A contract f is *failures-get-nothing* (FGN) if $f_i(S) = 0$ whenever $i \notin S$.

The following result shows that, in terms of implementing any given equilibrium $p \in \mathcal{E}$, one can restrict attention to FGN contracts.

Proposition 1. $\mathcal{E} = E(\mathcal{F}_{FGN})$.

To prove Proposition 1, we show that starting with any $p \in \mathcal{E}$ and any contract f which implements p, there is a scaling factor λ_i for each agent i so that the FGN contract

$$g_i(S) = \begin{cases} \lambda_i f_i(S), & \text{if } i \in S \\ 0, & \text{otherwise} \end{cases}$$

implements p.

Working with FGN contracts is thus without loss of generality, and is typically easier than working with general contracts.

To begin, it allows us to work with a simpler first order condition. Observe that if f is an FGN contract, then $r_i^f(p_{-i}) = \mathbb{E}[f_i(S) \mid i \in S] \geq 0$, and so by Equation (1), agent i's best response $b_i^f(p_{-i})$ satisfies the first order condition

$$c_i'(b_i^f(p_{-i})) = r_i^f(p_{-i}). \tag{2}$$

It follows then that for any FGN contract f, the expected total reward at $p \in [0,1]^n$ to agents in $I \subset [n]$ is

$$\begin{aligned} \mathbb{E}\left[\sum_{i \in I} f_i(S)\right] &= \sum_{i \in I} \mathbb{E}\left[f_i(S) | i \in S\right] \cdot \mathbb{P}[i \in S] \\ &= \sum_{i \in I} r_i^f(p_{-i}) \cdot p_i \\ &= \sum_{i \in I} p_i \cdot c_i'(b_i^f(p_{-i})). \end{aligned} \tag{3}$$

In particular, if $p \in \mathcal{E}$, and f is an FGN contract that implements p, then $b_i^f(p_{-i}) = p_i$. Using Equation (3), we obtain the following relationship between an implementable profile $p \in \mathcal{E}$ and any contract $f \in \mathcal{F}_{FGN}$ that implements p.

Lemma 1. *If $p \in \mathcal{E}$ and $f \in E^{-1}(p) \cap \mathcal{F}_{FGN}$, then*

$$\sum_{i \in [n]} p_i \cdot c_i'(p_i) = \mathbb{P}[S \neq \emptyset] \cdot \mathbb{E}\left[\sum_{i \in S} f_i(S) \mid S \neq \emptyset\right].$$

Moreover, since $\sum_{i \in S} f_i(S) \leq 1$, it follows from Lemma 1 that for any $p \in \mathcal{E}$,

$$\sum_i p_i \cdot c_i'(p_i) \leq \mathbb{P}[S \neq \emptyset] \iff \sum_i p_i \cdot c_i'(p_i) + \prod_i (1 - p_i) \leq 1.$$

Defining $z : [0,1)^n \to \mathbb{R}_+$ by

$$z(p) = \sum_i p_i \cdot c_i'(p_i) + \prod_i (1 - p_i),$$

we note this necessary condition for p to be implementable in the following Corollary.

Corollary 1. *If $p \in \mathcal{E}$, then $z(p) \leq 1$.*

3.2 SGE contracts

As we saw in Proposition 1, FGN contracts are sufficient to implement any implementable equilibrium $p \in \mathcal{E}$. However, some FGN contracts always lead to non-maximal equilibria, and so would be suboptimal for any objective. For instance, a contract which never gives any reward to any agent has a unique equilibrium $p = (0, \ldots, 0)$, which is clearly not maximal. In this subsection, we identify a subclass of FGN contracts whose equilibria characterize the set of maximal equilibria $\mathcal{P}$.

Consider first the special case where there is only a single agent. In this case, it is easy to see that $\mathcal{P}$ contains only a single element, which is the unique equilibrium of the contract in which the agent is rewarded with the entire budget if it succeeds and nothing if it fails. Thus, in addition to not giving any reward to the agent when it fails, it is optimal to reward the agent with the entire budget if it succeeds. As our first main result shows, this property of optimal contracts generalizes to more than one agent.

Definition 3. A contract f is *successful-get-everything* (SGE) if

- $f_i(S) = 0$ whenever $i \notin S$ and
- $\sum_{i \in S} f_i(S) = 1$ whenever $S \neq \emptyset$.

Theorem 1. *Suppose p is an equilibrium under contract f. Then, p is a maximal equilibrium if and only if f is a successful-get-everything contract.*

In other words, using SGE contracts is necessary and sufficient for implementing maximal equilibria. In particular, for any objective of the principal, any optimal contract must be SGE.

Observe that contracts like a piece-rate contract (where each agent i gets a fixed reward if she is successful and 0 otherwise), or a bonus pool contract (where each agent gets a nonzero share of the budget only if all agents succeed), are not successful-get-everything. It follows from Theorem 1 that such contracts never implement maximal equilibria, and so the choice of any such contract would be strictly suboptimal for any objective. In comparison, SGE contracts induce competition between the agents as they compete for a fixed budget, and from Theorem 1, always lead to maximal equilibria. Thus, the result suggests that in environments where the principal operates with an exclusive-use budget, fostering competition among the agents through successful-get-everything contracts creates stronger incentives than promoting teamwork through bonus-pool contracts or independent performance assessment through piece-rate contracts.

3.3 Application: two agents, quadratic costs

By illustrating the suboptimality of contracts that award agents when they fail or do not always exhaust the entire budget, Theorem 1 also reduces the complexity of the contract design problem. We illustrate this by solving the contract design problem for the special case where there are two agents with quadratic costs, $c_i(p_i) = \frac{1}{2}C_i p_i^2$.

With two agents, while a typical contract has eight parameters, since each agent's reward must be specified for each possible outcome, the set of SGE contracts is parameterized by a single parameter λ that represents agent 1's share of the budget when both agents succeed, $\lambda = f_1(\{1,2\})$. To see why, note that $f_i(S) = 0$ if $i \notin S$ and $f_i(S) = 1$ if $S = \{i\}$, so the only free variables are $f_i(\{1,2\})$, and since $f_2(\{1,2\}) = 1 - f_1(\{1,2\})$, f is entirely determined by $f_1(\{1,2\})$. The utilities of the agents under f are

$$u_1(p) = p_1 \cdot (1 - p_2) \cdot 1 + p_1 \cdot p_2 \cdot \lambda - \frac{1}{2}C_1 p_1^2$$
$$u_2(p) = (1 - p_1) \cdot p_2 \cdot 1 + p_1 \cdot p_2 \cdot (1 - \lambda) - \frac{1}{2}C_2 p_2^2$$

and using Equation (2), it follows that the unique equilibrium under f is

$$p_1(\lambda) = \frac{C_2 - (1-\lambda)}{C_1 C_2 - \lambda(1-\lambda)} \qquad p_2(\lambda) = \frac{C_1 - \lambda}{C_1 C_2 - \lambda(1-\lambda)}.$$

By Theorem 1, this provides the set of maximal equilibria,

$$\mathcal{P} = \left\{ \big(p_1(\lambda), p_2(\lambda)\big) \,:\, \lambda \in [0,1] \right\}.$$

The principal's problem is thus to find the λ that maximizes $V(p_1(\lambda), p_2(\lambda))$. In the following result, we consider the particular case where V is linear, and describe how λ might depend on the heterogeneity between the agents, and the principal's own bias.

Theorem 2. *Suppose $n = 2$, $c_i(p_i) = \frac{1}{2}C_i p_i^2$ with $C_i > 1$, and $V(p_1, p_2) = wp_1 + \cdot p_2$. Then, the optimal contract, defined by $\lambda_1(w)$, takes the form*

$$f_i(S) = \begin{cases} 0, & \text{if } i \notin S \\ 1, & \text{if } S = \{i\} \\ \lambda_i(w), & \text{if } S = \{1, 2\} \end{cases},$$

where $\lambda_2(w) = 1 - \lambda_1(w)$. Moreover, $\lambda_1(w)$ is increasing in w and in particular,

$$\lambda_1(w) = \begin{cases} 0, & \text{if } w \leq \frac{C_1C_2 - C_1}{C_1C_2 + C_2 - 1} \\ \frac{1}{2}, & \text{if } w = 1 \\ 1, & \text{if } w \geq \frac{C_1C_2 + C_1 - 1}{C_1C_2 - C_2} \end{cases}.$$

As Theorem 2 demonstrates, agent 1's share $\lambda_1(w)$ is increasing in w. In terms of comparative statics, this indicates that a principal whose objective places more weight on agent 1 will give a larger share of the reward to agent 1. Perhaps more surprisingly, $\lambda_1(1) = \frac{1}{2}$ regardless of the C_i, and since $\lambda_1(w)$ is increasing, it follows that the optimal contract will split the reward equally between the agents if the principal values their success equally, and in general will give a larger share of the reward to the agent whose success the principal values more. The robustness of this structure of the optimal contract to heterogeneity in the agents' costs is intriguing, and an interesting open question is whether this robustness holds more generally, and if so, under what conditions.

3.4 Proof of Theorem 1

In this subsection, we will discuss the ideas and techniques for proving Theorem 1. The result naturally splits into two directions, and we discuss each of these in turn.

First, we show that SGE contracts induce maximal equilibria. Notice that if f is a SGE contract, then $\mathbb{E}\left[\sum_{i \in S} f_i(S) \mid S \neq \emptyset\right] = 1$, and thus, it follows from Lemma 1 that for any $p \in E(f)$, $z(p) = 1$. To prove that p is maximal, we show that $z(x)$ is strictly increasing in x_i for $x \geq p$, and thus, for any $q \in [0, 1]^n$ which coordinate-wise dominates p, $z(q) > z(p)$. But since we know from Corollary 1 that for any $q \in \mathcal{E}$, $z(q) \leq 1$, it follows that there is no equilibrium q that dominates p.

Lemma 2. *If $f \in \mathcal{F}_{SGE}$, then $p \in E(f)$ is maximal.*

Proving the other direction that maximal equilibria can only be induced by SGE contracts is quite involved and nontrivial in comparison to the single-agent case. To see why, consider a contract f which is not SGE. We want to show that $p \in E(f)$ cannot be maximal. Intuitively, one might suspect that the principal can adjust the contract in such a way that $r_i(f, p_{-i})$ increases for some agent, either by decreasing its reward when it fails or by increasing its reward when it succeeds, and that this transformation should result in an equilibrium that dominates p. While this is true when there is only a single agent, the equilibrium effects with multiple agents mean that this transformation may simultaneously result in a decrease in the success probability chosen by another agent.

For an illustration, let us go back to the two agent example from Subsection 3.3. Consider the contract f such that

$$f_i(S) = \begin{cases} \beta_i, & \text{if } S = \{i\} \\ 0, & \text{otherwise} \end{cases}$$

That is, f awards $\beta_i \in [0, 1]$ to agent i if and only if i is the only successful agent. Using the first order conditions in Equation (2), one can verify that f has a unique equilibrium $p = (p_1, p_2)$ with

$$p_i = \frac{\beta_i(C_{-i} - \beta_{-i})}{C_1 C_2 - \beta_1 \beta_2}.$$

Now it is easy to check that increasing β_i would increase p_i, but it may also be accompanied by a decrease in p_{-i}. Thus, given a contract that is not SGE, simply increasing the reward of an agent when it succeeds does not necessarily lead to coordinate-wise dominant equilibrium.

To prove that non-SGE contracts lead to equilibria that are not maximal, we essentially demonstrate that there is always *some* adjustment that the principal can make that leads to a dominant equilibrium. More precisely, given a non-SGE contract f and $p \in E(f)$, we will illustrate transformations of f that work when f satisfies certain properties. And in case f does not satisfy those properties, we will find a non-SGE contract $g \in E^{-1}(p)$ that does satisfy them, thus proving the result.

First, we show that in any maximal equilibrium, every agent chooses a strictly positive probability of success.

Lemma 3. *If $p \in E(f)$ is such that $p_k = 0$ for some $k \in [n]$, then p is not maximal.*

We prove this by constructing a new contract g where $g_k(S) = \mathbb{1}_{S=\{k\}}$ and $g_i(S) = f_i(S \setminus \{k\})$ for $i \neq k$. For agents other than k, this contract induces a game that is strategically equivalent to the game induced by f when agent k chooses $p_k = 0$. In particular, switching from the contract f to the contract g induces agent k to choose a strictly positive probability of success without inducing any change in the other agents' behavior.

A useful consequence of Lemma 3 is that if p is maximal, it must be that $0 < p_i < 1$ for all $i \in [n]$, and therefore, for any $S \subset [n]$, there is positive probability that S is exactly the set of successful agents.

Next, we show that if $f \in \mathcal{F}_{FGN}$ never exhausts the entire budget, $p \in E(f)$ cannot be maximal.

Lemma 4. *If $f \in \mathcal{F}_{FGN}$ is such that $\sum_{i \in S} f_i(S) < 1$ for all $S \subset [n]$, then $p \in E(f)$ is not maximal.*

To prove this, for any $\varepsilon > 0$, let p' be the profile with $p'_i = p_i + \varepsilon$ for all i, let $t_i(\varepsilon) = \frac{c'_i(p'_i)}{\mathbb{E}_{p'}[f_i(S) \mid i \in S]}$, and let $g_i(S) = t_i(\varepsilon) \cdot f_i(S)$. Observe that if $\sum_{i \in S} g_i(S) \leq 1$ for all $S \subseteq [n]$, then g is a contract with equilibrium p' which dominates p. Since $\lim_{\varepsilon \to 0} t_i(\varepsilon) = 1$, g is a contract for all sufficiently small ε, and hence p is dominated.

Now, since Equation (2) is linear in f and $\mathcal{F}_{FGN}$ is closed and convex, the set of FGN contracts that implement p is closed and convex. This has the following immediate corollary: if for each nonempty $S \subseteq [n]$ there is an FGN contract g^S which implements p such that $\sum_{i \in S} g_i^S(S) < 1$, then there is an FGN contract f as in Lemma 4 that implements p. Moreover, a simple argument shows that if $p_i > 0$ for all i and there is some j such that $\sum_{i \in S} f_i(S) < 1$ for some $S \ni j$, then there is some FGN g that implements p such that $\sum_{i \in S} g_i(S) < 1$ for all $S \ni j$. Putting these observations together provides a larger class of FGN contracts which cannot be optimal.

Corollary 2. *If $f \in \mathcal{F}_{FGN}$ is such that for each j there is some $S \ni j$ such that $\sum_{i \in S} f_i(S) < 1$, then $p \in E(f)$ is not maximal.*

To prove Theorem 1, it remains to deal with contracts that are not SGE but do not satisfy the assumption of Corollary 2. This final step is involved, so we provide here only a high-level overview. Given an equilibrium p, consider the set of agents i for whom $\sum_{j \in S} f_j(S) = 1$ for every $S \ni i$ and every FGN contract f that implements p. Starting with an FGN contract for p, we can then construct a new contract for p such that the payoffs of these agents does not depend on the payoffs of the other agents. From this point, we can use the same techniques as above to modify the contract to induce the other agents to choose higher success probabilities. Hence, if f is not an SGE contract, then any equilibrium for f cannot be maximal.

Lemma 5. *If $f \notin \mathcal{F}_{SGE}$, then $p \in E(f)$ is not maximal.*

4 Implementing the maximal set

As Theorem 1 shows, maximal equilibria are characterized by the equilibria of SGE contracts, and so it is sufficient for the principal to optimize over SGE contracts. But observe that the set of SGE contracts is $\Theta(n2^n)$-dimensional. Given the high dimensionality of the set of SGE contracts, optimizing over SGE contracts may still present the principal with a computationally difficult problem, and the optimal contract may be difficult to understand and implement. In this section, we identify a subclass of SGE contracts that is actually sufficient to implement the maximal set.

4.1 Luce contracts

We begin by introducing a class of weighted contracts, which provide a natural generalization of the equal-split-amongst-successful contract by allowing for weighted splits.

Definition 4. A contract f is a *weighted* (W) contract if there exist weights $(\lambda_1, \dots, \lambda_n)$ with $\lambda_i > 0$ such that

$$f_i(S) = \begin{cases} \dfrac{\lambda_i}{\sum_{j \in S} \lambda_j}, & \text{if } i \in S \\ 0, & \text{otherwise} \end{cases}$$

Weighted contracts assigns weights to the agents and reward each successful agent with a fraction of the budget proportional to her weight. They are, however, insufficient to implement the maximal set. To see why, recall the structure of the optimal contract for the two agent problem from Theorem 2. For the objective $V(p) = wp_1 + p_2$ with w large, the optimal contract was such that if agent 1 succeeded, it was awarded the entire budget regardless of whether agent 2 succeeded or failed. And only in case agent 1 failed, agent 2 was awarded the entire budget if it was successful. It is easy to check that this optimal contract, while clearly implementing something in the maximal set, is not a weighted contract.

While this contract is not itself a weighted contract, it is a limit of weighted contracts: any sequence of weighted contracts with weights (λ_1, λ_2) such that $\frac{\lambda_1}{\lambda_2} \to \infty$ approaches this contract in $\mathcal{F}$. Observe that this contract can be naturally interpreted as assigning a priority order over the agents and rewarding the highest-priority successful agent with the entire budget. In the general context with n agents, any sequence of weighted contracts which converges must converge to a priority-based weighted contract, and we refer to such contracts as Luce contracts.

Definition 5. A contract f is a *Luce contract* if there exist weights $(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i > 0$ and a non-strict ordering $\succcurlyeq$ on the agents such that

$$f_i(S) = \begin{cases} \dfrac{\lambda_i}{\sum_{j \in \mathrm{Top}_{\succcurlyeq}(S)} \lambda_j}, & \text{if } i \in \mathrm{Top}_{\succcurlyeq}(S) \\ 0, & \text{otherwise} \end{cases}$$

where $\mathrm{Top}_{\succcurlyeq}(S) = \{i \in S \,:\, i \succcurlyeq j \ \forall j \in S\}$.

Luce contracts reward each *highest-priority* successful agent with a fraction of the budget proportional to her weight. The choice of terminology is motivated by the analogy between the structure of Luce contracts and the selection rules that satisfy Luce's choice axiom (Luce [30]). Just as Luce rules have the property that the relative odds of selecting one item over another from a pool is unaffected by which other items are in the pool, Luce contracts have the property that the reward of a successful agent relative to another successful agent is independent of which other agents succeed. While weighted contracts alone are insufficient, the following result shows that Luce contracts are indeed sufficient to implement the maximal set.

Theorem 3. *If $p \in \mathcal{P}$, there is a unique Luce contract $f \in \mathcal{F}_{Luce}$ such that $p \in E(f)$.*

In words, if p is maximal, then there is a unique Luce contract that implements p. Note that since this contract is unique, the set of Luce contracts is minimal among sets of contracts that implement the maximal set. As a consequence of Theorem 3, it follows that it is sufficient for the principal to optimize over the set of Luce contracts.

Corollary 3. *For any strictly increasing, continuous objective $V(p)$,*

$$\sup_{f \in \mathcal{F}} \sup_{p \in E(f)} V(p) = \max_{f \in \mathcal{F}_{Luce}} \max_{p \in E(f)} V(p).$$

Thus, Theorem 3 provides a significant reduction in the complexity of the principal's optimization problem, reducing the search space from a $(2^{n-1}(n-2)+1)$-dimensional space of difficult to interpret contracts to an $(n-1)$-dimensional space of easily interpretable contracts.

4.2 Proof of Theorem 3

To prove Theorem 3, we first characterize all profiles $p \in (0,1)^n$ that can be implemented by a Luce contract $f \in \mathcal{F}_{Luce}$ with some budget.

To begin, suppose $p \in (0,1)^n$ is an equilibrium of a Luce contract f with budget B. Since $f \in \mathcal{F}_{FGN}$, we have from Equation (3) that the expected total payment from principal to agents in $I \subset [n]$ is

$$\mathbb{E}\left[\sum_{i \in I} B \cdot f_i(S)\right] = \sum_{i \in I} p_i \cdot c_i'(p_i).$$

And since $f \in \mathcal{F}_{Luce}$, we have by an alternative calculation that

$$\mathbb{E}\left[\sum_{i \in I} B \cdot f_i(S)\right] \leq B \cdot \mathbb{P}[S \cap I \neq \emptyset],$$

and in particular, for $I = [n]$,

$$\mathbb{E}\left[\sum_{i \in [n]} B \cdot f_i(S)\right] = B \cdot \mathbb{P}[S \neq \emptyset].$$

Hence, for any subset I of the agents,

$$\frac{\sum_{i \in I} p_i \cdot c_i'(p_i)}{\sum_{i \in [n]} p_i \cdot c_i'(p_i)} \leq \frac{\mathbb{P}[S \cap I \neq \emptyset]}{\mathbb{P}[S \neq \emptyset]}. \tag{4}$$

This provides necessary conditions for any $p \in (0,1)^n$ to be implementable by a Luce contract. The following result shows that this is also sufficient: any $p \in (0,1)^n$ which satisfies (4) can be implemented by a Luce contract.

Proposition 2. *Suppose $p \in (0,1)^n$. There exists a Luce contract $f \in \mathcal{F}_{Luce}$ that implements p if and only if for all $I \subset [n]$,*

$$\frac{\sum_{i \in I} p_i \cdot c_i'(p_i)}{\sum_{i \in [n]} p_i \cdot c_i'(p_i)} \leq \frac{\mathbb{P}[S \cap I \neq \emptyset]}{\mathbb{P}[S \neq \emptyset]}.$$

In particular, for any $p \in (0,1)^n$ that satisfies (4), the principal can induce p by setting aside a budget of

$$B = \frac{\sum_{i\in[n]} p_i \cdot c_i'(p_i)}{\mathbb{P}[S \neq \emptyset]},$$

and committing to distributing this budget as per an appropriately defined Luce contract. The proof of Proposition 2 is technical and relegated to the appendix.

Now we turn to the proof of Theorem 3. The result states that under the small budget assumption (normalized to $B = 1$), if p is maximal, then there is a unique Luce contract that implements p. From Theorem 1, we know that p must be an equilibrium of a SGE contract. But then, by the same calculation as above, it follows that p must satisfy (4). From Proposition 2, there is a Luce contract that implements p. Thus, we get that every maximal equilibrium can be implemented by a Luce contract.

We now show that there is a unique Luce contract that implements p. To this end, suppose that f and g are distinct weighted contracts which both implement p, and let j be an agent whose ratio of weights $\frac{\lambda_j^f}{\lambda_j^g}$ is minimal. Then $f_j(S) \leq g_j(S)$ for all $S \subseteq [n]$ with strict inequality for some S, so the best response of agent j to p_{-j} must be strictly lower under f than under g, and hence p cannot be an equilibrium for both f and g. Though more technical, a similar argument extends to Luce contracts.

Lemma 6. *Suppose $f \in \mathcal{F}_{Luce}$ and $p \in E(f)$. Then for any $g \in \mathcal{F}_{Luce}$ such that $g \neq f$, $p \notin E(g)$.*

To conclude this section, in the following Corollary, we characterize the set of maximal equilibria in terms of the model parameters.

Corollary 4. *$p \in \mathcal{P}$ if and only if p satisfies (4) and $z(p) = 1$.*

Thus, the principal can potentially try to optimize its objective $V(p)$ over the set of solutions to the equation $z(p) = 1$, and then check if the solution satisfies (4).

5 Discussion

In this section, we briefly discuss how our results extend to environments where the small budget assumption may not hold, and also discuss the desirability of Luce contracts for implementation over other commonly-studied contracts like piece-rate contracts or bonus-pool contracts in standard multi-agent contract design environments.

Larger budgets

In this subsection, we discuss how our results extend to environments where the small budget assumption may not hold, so that $B > \min\{c_1'(1), \ldots, c_n'(1)\}$. Under the small budget assumption, we show that equilibria are maximal if and only if they are implementable exclusively by SGE contracts (Theorem 1). While the result does not exactly extend to environments where the budget is large, it extends as long as we restrict attention to equilibria that are interior. Indeed, the small budget assumption ensures that all $p \in \mathcal{E}$ are such that $p_i < 1$ for all $i \in [n]$, which is necessary for some of the arguments in the proof of Theorem 1. We can show that even with larger budgets, if we restrict attention to profiles $p \in [0,1)^n$, an equilibrium is maximal if and only if it is equilibrium of an SGE contract.

To illustrate why we need to restrict attention to interior equilibria in the larger budget case, we provide here an example where an SGE contract induces a non-maximal equilibrium, and an example where a maximal equilibrium is induced by a non-SGE contract. First, consider the SGE contract which pays the entire budget to the successful agent with the lowest index. If $B > c_1'(1)$, then the unique equilibrium is for agent 1 to choose $p_1 = 1$ and all other agents to choose $p_i = 0$. But this is dominated by the unique equilibrium of the piece-rate contract which pays $c_1'(1)$ to agent 1 if they succeed and $B - c_1'(1)$ to agent 2 if they succeed. Thus, in general, SGE contracts may lead to non-maximal equilibria, but in such a case, the equilibrium would necessarily involve some agent choosing $p_i = 1$. Now suppose $B > \sum_{i \in [n]} c_i'(1)$. In this case, the profile where all agents choose $p_i = 1$, which in this case is the unique maximal equilibrium, is implemented by the piece-rate contract which pays $c_i'(1)$ to agent i if they succeed and 0 if they fail. Thus, in general, maximal equilibria may be implemented by contracts that are not SGE, but again, any such equilibrium necessarily involves some agent choosing $p_i = 1$.

Implementation with Luce contracts

In this subsection, we note a useful property of Luce contracts and suggest their desirability for implementation in general multi-agent contract design environments. Our assumption that the principal has access to an exclusive-use budget departs from more standard contract design models, in which rewarding the agents is costly, and the principal trades off between the direct loss from rewarding the agents and the gain from inducing the agents to exert more effort. In such environments, where the principal is not budget-constrained, the principal has access to a larger set of contracts for the purpose of implementing any profile $q \in [0,1]^n$.

For instance, the principal could offer each agent $i \in [n]$ a simple piece-rate contract which pays $c_i'(q_i)$ to agent i if it succeeds, and nothing if it fails. Alternatively, the principal could also offer a bonus-pool contract which pays $\frac{q_i c_i'(q_i)}{\Pi_{i \in [n]} q_i}$ to agent $i \in [n]$ if all agents succeed, and nothing otherwise. And there is a plethora of other contracts, including a Luce contract in case q satisfies (4), that would implement q. It is easy to see that under limited liability constraints, FGN contracts offer the cheapest alternatives for implementing q, and each of these leads to an expected total payment of $\sum_{i \in [n]} q_i c_i'(q_i)$ from the principal to the agents.

If q satisfies (4), so that there is a Luce contract f that implements q (Proposition 2), it turns out that the total payment under any other FGN contract g that implements q is a mean-preserving spread of the total payment under f.[2] To see why, notice that under the Luce contract f, the total payment is 0 with probability $\Pi_{i \in [n]}(1 - q_i)$, and it is some $B > 0$ with the remaining probability. Since g is FGN, it must also have a total payment of 0 with probability $\Pi_{i \in [n]}(1 - q_i)$, and since the expected total payment from f and g is the same, with the remaining probability, the total payment must be spread around B in a way that the mean total payment is preserved. It follows that Luce contracts not only minimize the maximum payment the principal would have to make, but they also minimize the variance of the total payment among all FGN contracts that implement q. Thus, while Luce contracts are optimal in environments where the principal is budget-constrained (Theorem 3), we believe they also provide a useful alternative mechanism for inducing effort over well-studied contracts, like piece-rate and bonus-pool contracts, even in standard environments where the principal may not be budget-constrained.

6 Conclusion

We study a multi-agent contract design problem with a budget-constrained principal. In a model with independent and binary outcomes for each agent, we introduce a novel class of contracts, called Luce contracts, and demonstrate their sufficiency for implementing the maximal set of implementable equilibria. It follows that for any objective of the principal, there is always a Luce contract that is optimal. Additionally, we highlight the relevance of

[2] In the special case where all agents are homogeneous and q is such that $q_1 = q_2 = \cdots = q_n$, we can show that q satisfies (4), and an equal-split-amongst-successful contract with some budget B implements q.

Luce contracts in more standard contract design environments where the principal is not budget-constrained, as they provide a means for inducing effort while minimizing the maximum total payment and also the variance of the total payment.

Our results and analysis open several promising directions for future research. Within the framework of our model with binary outcomes, an intriguing question is how the structure of the optimal Luce contract varies with heterogeneity in agents' costs under different objectives of the principal. This is particularly compelling in light of the robustness to cost parameters observed in the special case of two agents with quadratic costs. Another valuable avenue would be to extend our model to allow for multiple outcomes per agent and to investigate potential generalizations of Luce contracts in these richer and more complex environments.

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A Proofs for Section 3 (Characterization of optimal contracts)

Proposition 1. $\mathcal{E} = E(\mathcal{F}_{FGN})$.

Proof. Let $p \in \mathcal{E}$ and let $f \in \mathcal{F}$ be a contract for which p is an equilibrium. Consider the contract g defined by

$$g_i(S) = \begin{cases} \lambda_i f_i(S), & \text{if } i \in S \\ 0, & \text{otherwise} \end{cases}$$

where

$$\lambda_i = \begin{cases} \dfrac{r_i(f, p_{-i})}{\mathbb{E}[f_i(S) \,|\, i \in S]} & \text{if } p_i > 0 \\ 0 & \text{if } p_i = 0. \end{cases}$$

Notice that it follows from the first-order condition in Equation 1 that $\lambda_i \le 1$ so that g is indeed an FGN contract. Now if $p_i > 0$, agent i's marginal utility at profile p under the contract g is

$$\begin{aligned} \frac{\partial u_i(p)}{\partial p_i} &= \mathbb{E}[g_i(S) \,|\, i \in S] - c_i'(p_i) \\ &= \lambda_i \mathbb{E}[f_i(S) \,|\, i \in S] - c_i'(p_i) \\ &= r_i(f, p_{-i}) - c_i'(p_i) \\ &= 0, && \text{(because } p \in E(f)) \end{aligned}$$

so by concavity of payoffs, $b_i(g, p_{-i}) = p_i$. If $p_i = 0$, then $g_i(S) = 0$ for all S, so $b_i(g, p_{-i}) = 0 = p_i$. Hence, $p \in E(g)$. $\square$

Lemma 1. *If $p \in \mathcal{E}$ and $f \in E^{-1}(p) \cap \mathcal{F}_{FGN}$, then*

$$\sum_{i \in [n]} p_i \cdot c_i'(p_i) = \mathbb{P}[S \neq \emptyset] \cdot \mathbb{E}\left[\sum_{i \in S} f_i(S) \,|\, S \neq \emptyset\right].$$

Proof. Since $p \in E(f)$ and $f \in \mathcal{F}_{FGN}$, it follows from Equation 3 that

$$\mathbb{E}\left[\sum_{i \in [n]} f_i(S)\right] = \sum_{i \in [n]} p_i c_i'(p_i)).$$

And since $f \in \mathcal{F}_{FGN}$,

$$\mathbb{E}\left[\sum_{i\in[n]} f_i(S)\right] = \mathbb{P}[S \neq \emptyset] \cdot \mathbb{E}\left[\sum_{i\in S} f_i(S) \mid S \neq \emptyset\right],$$

which gives the result. ◻

Theorem 2. *Suppose $n = 2$, $c_i(p_i) = \frac{1}{2}C_i p_i^2$ with $C_i > 1$, and $V(p_1, p_2) = wp_1 + \cdot p_2$. Then, the optimal contract, defined by $\lambda_1(w)$, takes the form*

$$f_i(S) = \begin{cases} 0, & \text{if } i \notin S \\ 1, & \text{if } S = \{i\} \\ \lambda_i(w), & \text{if } S = \{1,2\} \end{cases},$$

where $\lambda_2(w) = 1 - \lambda_1(w)$. Moreover, $\lambda_1(w)$ is increasing in w and in particular,

$$\lambda_1(w) = \begin{cases} 0, & \text{if } w \leq \frac{C_1C_2 - C_1}{C_1C_2 + C_2 - 1} \\ \frac{1}{2}, & \text{if } w = 1 \\ 1, & \text{if } w \geq \frac{C_1C_2 + C_1 - 1}{C_1C_2 - C_2} \end{cases}.$$

Proof. We know from Theorem 1 that we can restrict attention to SGE contracts. And with only two agents, the set of SGE contracts can be parametrized by a single parameter λ, where $f_i(S) = 0$ whenever $i \notin S$, $f_i(S) = 1$ whenever $S = \{i\}$, and $f_1(\{1,2\}) = 1 - f_2(\{1,2\}) = \lambda$. The equilibrium conditions for this contract from Equation (2) are

$$\begin{aligned} C_1 p_1 &= 1 - p_2 + \lambda p_2 \\ C_2 p_2 &= 1 - p_1 + (1-\lambda)p_1. \end{aligned}$$

For each λ, this system of equations has a unique solution:

$$p_1(\lambda) = \frac{C_2 - (1-\lambda)}{C_1C_2 - \lambda(1-\lambda)} \quad p_2(\lambda) = \frac{C_1 - \lambda}{C_1C_2 - \lambda(1-\lambda)}. \tag{5}$$

Hence, the principal's problem is equivalent to:

$$\max_{\lambda\in[0,1]} wp_1(\lambda) + p_2(\lambda).$$

Using the first order condition, this is maximized either at $\lambda = 0$, or $\lambda = 1$, or where

$$\frac{p_2'(\lambda)}{p_1'(\lambda)} = -w\cdot$$

Note that

$$p_1'(\lambda) = \frac{(C_1C_2 - \lambda(1-\lambda)) - (C_2 - (1-\lambda))(2\lambda - 1))}{(C_1C_2 - \lambda(1-\lambda))^2} = \frac{C_1C_2 - C_2(2\lambda-1) - (1-\lambda)^2}{(C_1C_2 - \lambda(1-\lambda))^2}$$

and

$$p_2'(\lambda) = \frac{-(C_1C_2 - \lambda(1-\lambda)) - (C_1 - \lambda)(2\lambda - 1)}{(C_1C_2 - \lambda(1-\lambda))^2} = \frac{-C_1C_2 - C_1(2\lambda-1) + \lambda^2}{(C_1C_2 - \lambda(1-\lambda))^2}$$

so

$$\frac{p_2'(\lambda)}{p_1'(\lambda)} = \frac{-C_1C_2 - C_1(2\lambda-1) + \lambda^2}{C_1C_2 - C_2(2\lambda-1) - (1-\lambda)^2} = -\frac{C_1}{C_2} \cdot \frac{-\lambda^2/C_1 + 2\lambda + C_2 - 1}{-(1-\lambda)^2/C_2 + 2(1-\lambda) + C_1 - 1}$$

Now observe that the numerator is increasing for $\lambda < C_1$ and the denominator is decreasing for $\lambda > -(C_2 - 1)$. In particular, the fraction is monotonically strictly increasing for $0 < \lambda < 1$, so $\frac{p_2'(\lambda)}{p_1'(\lambda)}$ is monotonically strictly decreasing. It follows then that there is a function $\lambda_1(w)$ such that the unique optimal choice of λ is $\lambda_1(w)$ and it is increasing in w. In particular,

$$\frac{p_2'(\lambda)}{p_1'(\lambda)} \leq \frac{p_2'(0)}{p_1'(0)} = -\frac{C_1}{C_2} \cdot \frac{C_2 - 1}{-1/C_2 + C_1 + 1} = -\frac{C_1C_2 - C_1}{C_1C_2 + C_2 - 1}$$

and

$$\frac{p_2'(\lambda)}{p_1'(\lambda)} \geq \frac{p_2'(1)}{p_1'(1)} = -\frac{C_1}{C_2} \cdot \frac{-1/C_1 + C_2 + 1}{C_1 - 1} = -\frac{C_1C_2 + C_1 - 1}{C_1C_2 - C_2}$$

Now if $w \leq -\frac{p_2'(0)}{p_1'(0)}$, the objective is decreasing in λ and thus $\lambda_1(w) = 0$. And if $w \geq -\frac{p_2'(1)}{p_1'(1)}$, the objective is increasing in λ and thus $\lambda_1(w) = 1$.

Lastly, observe that

$$\frac{p_2'(\frac{1}{2})}{p_1'(\frac{1}{2})} = -1$$

irrespective of the costs C_1, C_2. And thus, if $w_1 = w_2$, we get that $\lambda^* = \frac{1}{2}$ no matter how heterogeneous the agents are.

□

Lemma 2. *If $f \in \mathcal{F}_{SGE}$, then $p \in E(f)$ is maximal.*

Proof. Suppose p is not maximal. Then, there must be a $q \in \mathcal{E}$ that dominates p. Observe that since $p \in E(f)$ and $f \in \mathcal{F}_{SGE}$, we have from Lemma 1 that $z(p) = 1$. Also, since $q \in \mathcal{E}$, we have from Corollary 1 that $z(q) \leq 1$. Now, consider the following:

$$\frac{\partial z}{\partial x_i}\bigg|_{x \geq p} = x_i c_i''(x_i) + c_i'(x_i) - \prod_{j \neq i}(1 - x_j)$$

$$
\begin{aligned}
&> c_i'(x_i) - \prod_{j\neq i}(1-x_j) && \text{(because } c_i \text{ is convex)}\\
&\geq c_i'(p_i) - \prod_{j\neq i}(1-x_j) && \text{(because } c_i \text{ is convex)}\\
&\geq c_i'(p_i) - \prod_{j\neq i}(1-p_j) && \text{(because } x_j \geq p_j)\\
&\geq 0 && \text{(because } p \in E(f))
\end{aligned}
$$

Note that the last inequality holds because we know from the first order condition in Equation (2) that $c_i'(p_i) = \mathbb{E}[f_i(S)|i \in S] \geq \prod_{j\neq i}(1-p_j)$ since $f \in \mathcal{F}_{SGE}$. Now, since q dominates p, $q \geq p$ and it follows from above that $z(q) > z(p)$. But this is a contradiction since $z(p) = 1 \geq z(p)$. It follows then that $p \in E(f)$ is maximal. □

Lemma 3. *If $p \in E(f)$ is such that $p_k = 0$ for some $k \in [n]$, then p is not maximal.*

Proof. Let $f \in E^{-1}(p)$ be an FGN contract, and suppose $p_k = 0$. Consider the contract g where $g_i(S) = f_i(S \setminus \{k\})$ for $i \neq k$ and

$$
g_k(S) = \begin{cases} 1, & \text{if } S = \{k\} \\ 0, & \text{otherwise} \end{cases}
$$

Let $p_k' = b_k(g, p_{-i})$, and let $p' = (p_k', p_{-k})$. Observe that since $b_k(g, p_{-i}) > 0$, $p_k' > 0$, and for $i \neq k$, $r_i(g, p_{-i}') = r_i(f, p_{-i})$, so $b_i(g, p_{-i'}) = b_i(f, p_{-i}) = p_i$. Hence, $p' \in E(g)$ and p' dominates p. Thus, if $p_k = 0$ for some k, then $p \notin \mathcal{P}$, and the result follows. □

Lemma 4. *If $f \in \mathcal{F}_{FGN}$ is such that $\sum_{i\in S} f_i(S) < 1$ for all $S \subset [n]$, then $p \in E(f)$ is not maximal.*

Proof. If $p_i = 0$ for some $i \in [n]$, it follows from Lemma 3 that p is not maximal. So suppose $p_i > 0$ for all $i \in [n]$. Now let $p' = (p_1 + \epsilon, p_2 + \epsilon, \ldots, p_n + \epsilon)$. For each $i \in [n]$, let $t_i(\epsilon)$ solve

$$
c_i'(p_i') = t_i(\epsilon) \sum_{S\subset[n]_{-i}} (f_i(S \cup \{i\}))\mathrm{Pr}_{p_{-i}'}^{[n]_{-i}}(S).
$$

Observe that as $\epsilon \to 0$, $t_i(\epsilon) \to 1$ for all $i \in [n]$. Since $\sum_{i\in S} f_i(S) < 1$ for all $S \subset [n]$ and $t_i(\epsilon)$ is continuous in ϵ, we can find $\epsilon > 0$ small enough so that the contract $f_i'(S) = f_i(S) * t_i(\epsilon)$ for all $S \subset [n]$ and $i \in S$ is a feasible contract. By the definition of t_i, p' with $p_i' = p_i + \epsilon$ for $i \in [n]$ will be an equilibrium under f'. Thus, we have that p is not maximal.

□

Lemma 5. *If $f \notin \mathcal{F}_{SGE}$, then $p \in E(f)$ is not maximal.*

Proof. Suppose $p \in \mathcal{P}$. Let

$$K_p := \{S \subseteq [n] \,:\, \sum_{i \in S} f_i(S) < 1 \text{ for some } f \in E^{-1}(p) \cap \mathcal{F}_{FGN}\},$$

and let

$$\kappa_p := \{i \in [n] \,:\, \{i\} \in K_p\}.$$

We now show that $K_p = \{\emptyset\}$.

1. Step 1: Suppose $S \in K_p$. For any $T \subset S$, $T \in K_p$.

   Let $f \in E^{-1}(p) \cap \mathcal{F}_{FGN}$ be such that $\sum_{i \in S} f_i(S) < 1$. If $\sum_{i \in T} f_i(T) < 1$, we are done. Otherwise, pick an agent $i \in T$ such that $f_i(T) > 0$ and consider a contract g which differs from f only in its award for agent i at S and T. In particular, let g be such that $g_i(S) = f_i(S) + \epsilon$ and $g_i(T) = f_i(T) - \delta$ where $\epsilon, \delta > 0$ are chosen so that $p \in E(g)$. Note that we can do this because we know from Lemma 3 that for all $i \in [n]$, $0 < p_i < 1$ and therefore, $\Pr_{p_{-i}}^{[n]_{-i}}(S) > 0$ for all $i \in [n]$ and all $S \subset [n]_{-i}$. It follows then that $g \in E^{-1}(p) \cap \mathcal{F}_{FGN}$ and $\sum_{i \in T} g_i(T) < 1$. Thus, $T \in K_p$.

2. Step 2: Suppose $S, T \in K_p$. Then, $S \cup T \in K_p$.

   Let $f, g \in E^{-1}(p) \cap \mathcal{F}_{FGN}$ be such that $\sum_{i \in S} f_i(S) < 1$ and $\sum_{i \in T} g_i(T) < 1$. Consider the contract $h = \frac{1}{2}f + \frac{1}{2}g$. Since Equation (2) is linear in f and $\mathcal{F}_{FGN}$ is closed and convex, the set of FGN contracts that implement p is closed and convex. Thus, $h \in E^{-1}(p) \cap \mathcal{F}_{FGN}$ and also $\sum_{i \in S} h_i(S) < 1$ and $\sum_{i \in T} h_i(T) < 1$. Now, if $\sum_{i \in S \cup T} h_i(S \cup T) < 1$, we are done. Otherwise, pick any agent $i \in S \cup T$ (WLOG, let $i \in S$) such that $h_i(S \cup T) > 0$ and consider a contract h' which differs from h only in its award for agent i at $S \cup T$ and S. In particular, let h' be such that $h'_i(S \cup T) = h_i(S \cup T) - \epsilon$ and $h'_i(S) = h_i(S) + \delta$ where $\epsilon, \delta > 0$ are chosen so that $p \in E(h')$. Again, we can do this because we know that for all $i \in [n]$, $0 < p_i < 1$. It follows then that $h' \in E^{-1}(p) \cap \mathcal{F}_{FGN}$ and $\sum_{i \in S \cup T} h'_i(S \cup T) < 1$. Thus, $S \cup T \in K_p$.

   Note that it follows from Steps 1 and 2 that $K_p = 2^{\kappa_p}$.

3. Step 3: Suppose $f \in E^{-1}(p) \cap \mathcal{F}_{FGN}$. Then, for all $S \subset [n]$ such that $\kappa_p^C \cap S \neq \phi$, $f_i(S) = 0$ for all $i \in \kappa_p$.

Suppose towards a contradiction that there is an $S \subset [n]$ such that $\kappa_p^C \cap S \neq \phi$ and $f_i(S) > 0$ for some $i \in \kappa_p$. Let $g \in E^{-1}(p) \cap \mathcal{F}_{FGN}$ be such that $g_i(\{i\}) < 1$. Consider the contract $h = \frac{1}{2}f + \frac{1}{2}g$. Then $h \in E^{-1}(p) \cap \mathcal{F}_{FGN}$ and also $h_i(S) > 0$ and $h_i(\{i\}) < 1$. Now, consider a contract h' which differs from h only in its award for agent i at S and $\{i\}$. In particular, let h' be such that $h'_i(\{i\}) = h_i(\{i\}) + \epsilon$ and $h'_i(S) = h_i(S) - \delta$ where $\epsilon, \delta > 0$ are chosen so that $p \in E(h')$. We can do this because for all $i \in [n]$, $0 < p_i < 1$. It follows then that $h' \in E^{-1}(p) \cap \mathcal{F}_{FGN}$ and $\sum_{i \in S} h'_i(S) < 1$. But this means that $S \subset \kappa_p$ which is a contradiction.

4. Step 4: Suppose $\kappa_p \neq \phi$. Then there is a p' that Pareto dominates p.

   For all $S \in K_p$, let $f^S \in E^{-1}(p) \cap \mathcal{F}_{FGN}$ be such that $\sum_{i \in S} f_i^S(S) < 1$. Consider the contract $g = \sum_{S \in K_p} \frac{1}{|K_p|} f^S$. Then $g \in E^{-1}(p) \cap \mathcal{F}_{FGN}$ and also $\sum_{i \in S} g_i(S) < 1$ for all $S \in K_p$. Now, we can construct a contract $h \in E^{-1}(p) \cap \mathcal{F}_{FGN}$ such that

$$h_i(S) = \begin{cases} g_i(S), & \text{if } S \in K_p \\ h_i(S \setminus \kappa_p), & \text{if } S \notin K_p \end{cases}$$

   by averaging over the outcomes of agents in κ_p under g.

   Observe that if we manipulate h at any $S \subset \kappa_p$, it won't change the best responses for agents $i \notin \kappa_p$. We will now show that we can manipulate the awards for $S \subset \kappa_p$ so that under the new contract h', $p' \in E(h')$ where $p'_i > p_i$ for $i \in \kappa_p$ and $p'_i = p_i$ for $i \notin \kappa_p$. Towards this goal, let $A = \kappa_p$ and let $p' = (p_i + \epsilon)_{i \in A}$. For each $i \in A$, let $t_i(\epsilon)$ solve

$$c'_i(p'_i) = t_i(\epsilon) \sum_{S \subset A_{-i}} (h_i(S \cup \{i\})) \Pr_{p'_{-i}}^{A_{-i}}(S)$$

   Observe that as $\epsilon \to 0$, $t_i(\epsilon) \to 1$ for all $i \in A$. Since $\sum_{i \in S} h_i(S) < 1$ for all $S \subset A$ and $t_i(\epsilon)$ is continuous in ϵ, we can find $\epsilon > 0$ small enough so that the contract $h'_i(S) = h_i(S) * t_i(\epsilon)$ for all $S \subset A$ and $i \in S$ is a feasible contract. By the definition of t_i, p' with $p'_i = p_i + \epsilon$ for $i \in \kappa_p$ and $p'_i = p_i$ for $i \notin \kappa_p$ will be an equilibrium under h'. Thus, we have that p is not Pareto optimal.

It follows then that $K_p = \{\emptyset\}$. By definition of K_p, this means that for any $f \in E^{-1}(p)$, either $f \notin \mathcal{F}_{FGN}$ or $f \in \mathcal{F}_{SGE}$. Now suppose there exists an f such that $f \notin \mathcal{F}_{FGN}$ and $p \in E(f)$. From Proposition 1, we can find $g \in \mathcal{F}_{FGN}$ such that $p \in E(g)$. Moreover, we know from the construction of g in the argument of Proposition 1 that $g \notin \mathcal{F}_{SGE}$. Thus, we have that $g \in \mathcal{F}_{FGN}, g \notin \mathcal{F}_{SGE}$, and $p \in E(g)$. But this means that $K_p \neq \emptyset$ which is a contradiction. Therefore, it must be that for any $f \in E^{-1}(p)$, $f \in \mathcal{F}_{SGE}$.

$\square$

B Proofs for Section 4 (Implementing the maximal set)

Proposition 2. *Suppose $p \in (0,1)^n$. There exists a Luce contract $f \in \mathcal{F}_{Luce}$ that implements p if and only if for all $I \subset [n]$,*

$$\frac{\sum_{i \in I} p_i \cdot c_i'(p_i)}{\sum_{i \in [n]} p_i \cdot c_i'(p_i)} \leq \frac{\mathbb{P}[S \cap I \neq \emptyset]}{\mathbb{P}[S \neq \emptyset]}.$$

Proof. Assume there is a Luce contract $f \in \mathcal{F}_{Luce}$ with budget $B > 0$ that implements p. Then for any $I \subseteq [n]$, the expected total reward to agents in I is

$$\sum_{i \in I} p_i \cdot c_i'(p_i) = \mathbb{E}\left[\sum_{i \in I} B \cdot f_i(S)\right] = B \cdot \mathbb{E}\left[\sum_{i \in I} f_i(S) \mid S \cap I \neq \emptyset\right] \cdot \mathbb{P}[S \cap I \neq \emptyset],$$

and since f is a Luce contract with budget B,

$$\mathbb{E}\left[\sum_{i \in I} f_i(S) \mid S \cap I \neq \emptyset\right] \leq 1$$

with equality if $I = [n]$. Hence, if there is a Luce contract with budget B that implements p, then (4) must hold.

Now, suppose that $p \in (0,1)^n$ satisfies (4). Observe that for any Luce contract that uses a budget of $B = \dfrac{\sum_{i \in [n]} p_i \cdot c_i'(p_i)}{\mathbb{P}[S \neq \emptyset]}$,

$$\sum_{j \in [n]} p_j \cdot c_j'\left(b_j\left(f, p_{-j}\right)\right) = B \cdot \mathbb{E}[\sum_{j \in [n]} f_j(S)] = B \cdot \mathbb{P}[S \neq \emptyset] = \sum_{j \in [n]} p_j \cdot c_j'(p_j),$$

so

$$\sum_{i \in [n]} p_i \cdot [c_i'(b_i(f, p_{-i})) - c_i'(p_i)] = 0.$$

Moreover, for any agent i,

$$\sum_{j:j\succ i} p_j \cdot c_j'\left(b_j\left(f, p_{-j}\right)\right) = B \cdot \mathbb{E}[\sum_{j:j\succ i} f_j(S)] = B \cdot \mathbb{P}[S \cap \{j : j \succ i\} \neq \emptyset] \geq \sum_{j:j\succ i} p_j \cdot c_j'(p_j),$$

so

$$\sum_{j:i\not\succ j} p_j \cdot \left[c_j'\left(b_j\left(f, p_{-j}\right)\right) - c_j'\left(p_j\right)\right] \leq 0.$$

Let

$$Z(f) := \max_{i\in[n]} \left(c_i'(b_i(f, p_{-i})) - c_i'(p_i)\right)$$

and

$$\mathcal{C}(f) := \{i \in [n] \ : \ c_i'(b_i(f, p_{-i})) - c_i'(p_i) = Z(f)\}$$

From above, we know that $Z(f) \geq 0$ for all $f \in \mathcal{F}_{Luce}$ and so

$$z = \inf_{f\in\mathcal{F}_{Luce}} Z(f) \geq 0.$$

We will now show that $z = 0$.

Suppose towards a contradiction that $z > 0$. Let $f \in \mathcal{F}_{Luce}$ be such that

1. $Z(f) = z$ and
2. for any other $g \in \mathcal{F}_{Luce}$ such that $Z(g) = z$, $\mathcal{C}(g) \not\subset \mathcal{C}(f)$.

Let $(X_1, \ldots, X_\ell)$ be the ordered partition corresponding to $\succcurlyeq$ and $\lambda_1, \ldots, \lambda_n$ be the weights that define f.

Let k be the maximum index such that $X_k \cap \mathcal{C}(f) \neq \emptyset$.

First suppose $k = \ell$. From above, we know that there must be some $i \in X_l$ such that $c_i'(b_i(f, p_{-i})) - c_i'(p_i) < 0$. Now consider another Luce contract g which is the same as f, except that the weight for agent i is $\lambda_i' = \lambda_i + \epsilon$. Thus, when all agents in $[n] \setminus X_l$ have failed, agent i gets a slightly higher share of the reward if it succeeds under g than it did under f. Notice that for $j \notin X_\ell$,

$$c_j'(b_j(g, p_{-j})) - c_j'(p_j) = c_j'(b_j(f, p_{-j})) - c_j'(p_j)$$

and for $j \in X_\ell \setminus \{i\}$,

$$c_j'(b_j(g, p_{-j})) - c_j'(p_j) < c_j'(b_j(f, p_{-j})) - c_j'(p_j) \leq z.$$

Moreover, for ε sufficiently small,

$$c_i'(b_i(g, p_{-i})) - c_i'(p_i) < 0$$

by continuity. It follows that $Z(g) \leq z$. In particular, we either have $Z(g) < z$ or $Z(g) = z$ and $\mathcal{C}(g) \subset \mathcal{C}(f)$. In either case, we have a contradiction.

Thus, it must be that $k < \ell$. Now, consider another Luce contract g which has the partition $(X_1, \ldots, X_{k-1}, X_k \cup X_{k+1}, X_{k+2}, \ldots, X_\ell)$ and weights

$$\lambda_i' = \begin{cases} \lambda_i & \text{if } i \notin X_{k+1} \\ \varepsilon \lambda_i & \text{if } i \in X_{k+1} \end{cases}.$$

Then for $j \notin X_k \cup X_{k+1}$,

$$c_j'(b_j(g, p_{-j})) - c_j'(p_j) = c_j'(b_j(f, p_{-j})) - c_j'(p_j)$$

and for $j \in X_k$,

$$c_j'(b_j(g, p_{-j})) - c_j'(p_j) < c_j'(b_j(f, p_{-j})) - c_j'(p_j) \leq z.$$

Finally, for ε sufficiently small and $j \in X_{k+1}$,

$$c_j'(b_j(g, p_{-j})) - c_j'(p_j) < z$$

by continuity. It follows that $Z(g) \leq z$. In particular, we either have $Z(g) < z$ or $Z(g) = z$ and $\mathcal{C}(g) \subset \mathcal{C}(f)$. In either case, we have a contradiction. Thus, we have that $\inf_{f \in \mathcal{F}_{Luce}} Z(f) = 0$. By compactness of $\mathcal{F}_{Luce}$, there exists an $f \in \mathcal{F}_{Luce}$ such that $Z(f) = 0$ which means

$$\left[\max_{i \in [n]} \left(c_i'(b_i(f, p_{-i})) - c_i'(p_i) \right) \right] = 0.$$

But from above, we also know that

$$\sum_{i \in [n]} p_i \left[c_i'(b_i(f, p_{-i})) - c_i'(p_i) \right] = 0.$$

Therefore, it must be that for all $i \in [n]$,

$$\left(c_i'(b_i(f, p_{-i})) - c_i'(p_i) \right) = 0$$

which implies $f \in E^{-1}(p)$. Thus, for any p that satisfies (4), there is a Luce contract f and a budget B such that $p \in E(f)$.

$\square$

Lemma 6. *Suppose $f \in \mathcal{F}_{Luce}$ and $p \in E(f)$. Then for any $g \in \mathcal{F}_{Luce}$ such that $g \neq f$, $p \notin E(g)$.*

Proof. Consider the case where f, g are weighted contracts and suppose towards a contradiction that $p \in E(f) \cap E(g)$. Let $i \in [n]$ denote the agent with the smallest weight ratio $\frac{\lambda_i^g}{\lambda_i^f}$, where λ^f, λ^g represent the weights that define the contracts $f, g \in \mathcal{F}_W$. It follows from the definition of weighted contracts that for any $S \subset [n]$ such that $i \in S$, agent i's reward is weakly lower under g than f. As a result, it must be that $r_i(g, p_{-i}) < r_i(f, p_{-i})$ and so it can't be that $b_i(g, p_{-i}) = b_i(f, p_{-i})$. It follows that two different weighted contracts cannot have the same equilibrium.

Now suppose f, g are arbitrary Luce contracts that implement p. If $\succcurlyeq_f = \succcurlyeq_g$, then the same argument works. So suppose that $\succcurlyeq_f \neq \succcurlyeq_g$. Then there is some i such that $T = \{k : k \succcurlyeq_f i\} \neq \{k : k \succcurlyeq_g i\} = T'$, and we may assume without loss of generality that $T' \not\subseteq T$ by exchanging f and g if necessary. Observe that for any S,

$$\sum_{k \in T} g_k(S) \leq \mathbb{1}_{S \cap T \neq \emptyset} = \sum_{k \in T} f_k(S).$$

Moreover, for any $j \in T' \setminus T$,

$$\sum_{k \in T} g_k(\{i, j\}) = g_i(\{i, j\}) < 1 = f_i(\{i, j\}) = \sum_{k \in T} f_k(\{i, j\}).$$

Thus,

$$\sum_{k \in T} \mathbb{P}[k \in S] \cdot \mathbb{E}[g_k(S) \,|\, k \in S] = \mathbb{E}[\sum_{k \in T} g_k(S)] < \mathbb{E}[\sum_{k \in T} f_k(S)] = \sum_{k \in T} \mathbb{P}[k \in S] \cdot \mathbb{E}[f_k(S) \,|\, k \in S].$$

However, since p is an equilibrium for f and g, both of the outside expressions must be equal to $\sum_{k \in T} p_k \cdot c_k'(p_k)$, contradiction. $\square$