Prizes and effort in contests with private information *

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Abstract

We consider contests where participants have private information about their ability and study the effect of different prizes on the effort exerted by the agents. We first characterize the symmetric Bayes-Nash equilibrium strategy function for arbitrary prize vectors $v_1 \geq v_2 \cdots \geq v_n$ and find that increasing value of the first prize increases effort for all types of agents, the last prize reduces effort for all agents, and any intermediate prize leads to a balanced transfer of effort from high ability participants to low ability participants. In expectation, the effect of any intermediate prize depends on the prior distribution of abilities. If there is an increasing density of less efficient agents, the expected effect on effort is positive and if this density is decreasing, the expected effect is negative. Lastly, we discuss applications of these results to the design of effort-maximizing contests in environments that impose natural constraints on feasible contests including grading contests, contests with risk-averse agents, and contests where the designer can only award homogeneous prizes of a fixed value.

1 Introduction

Contests are situations in which agents compete with one another by investing effort or resources to win prizes. Such competitive situations are common in many social and economic contexts, including college admissions, labor markets, R&D races, sporting events, rent-seeking activities, politics, competition-based crowdsourcing, wars etc. While some of these situations like wars arise naturally, there are many others where the contest designer can design the rules of the contest. In sporting competitions, the designer can decide how to split a monetary budget across different prizes. In classroom settings where the students are graded on a curve, the professor has a choice over the distribution of grades. Many firms reward their best-performing salesmen in sales contests and can choose exactly how they want to reward them. In all of these situations, the contest designer can optimally design

*I am grateful to Federico Echenique, Omer Tamuz, Thomas Palfrey, Jingfeng Lu, Wade Hann-Caruthers, and seminar audience at the 8th Annual Conference on Contests: Theory and Evidence (2022) for helpful comments and suggestions.

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the contest so as to satisfy various objectives.

The designer's objective and the set of available prize vectors $\mathbf{v} = (v_1, \ldots, v_n)$ may vary across applications. In sports competitions, the organizers may want to distribute a budget across prizes to maximize the total effort of the agents. In classroom environments, professors may wish to choose a grading scheme to maximize the learning effort of the least efficient student. In many online contests and games, the designer is unconstrained in its ability to award digital medals or badges to winners. Motivated by the diversity of applications, our focus in this paper is to understand how each prize influences the effort exerted by the agents and then discuss its implications for the design of optimal contest in various applications.

The optimal contest design problem was first posed by Galton in 1902 and has since been studied in various different environments. In this paper, our focus will be on environments where the agents are privately informed about their abilities, defined by their marginal cost of effort $\theta \in [0, 1]$ which are independent and identically distributed random variables, and given the prize vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and knowledge of the prior distribution F, the agents simultaneous choose their effort levels and are consequently awarded based on their perfectly observable relative performance. We note here that while there is a significant literature that studies contest design in incomplete information environments, our focus on obtaining a more complete comparison of how prize vectors compare in terms of the effort they induce, and also the distributional assumptions we make allowing for agents with almost zero marginal costs of effort, allow us to differentiate our work from the literature and contribute to it. We will now discuss our results.

We first characterize the symmetric Bayes-Nash equilibrium of the contest game with incomplete information for arbitrary prior F and prize vectors $\mathbf{v} = (v_1, \ldots, v_n)$. The equilibrium effort function is linear in prizes and more precisely, takes the form $g_{\mathbf{v}}(\theta) = \sum_{i=1}^{n} m_i(\theta) v_i$ where $m_i(\theta)$ represents the marginal effect of prize i on the effort of agent of type θ and depends on the prior distribution F. This characterization is the same as in Moldovanu and Sela [44] who study a similar model but with a different type domain.

With the equilibrium function in hand, we focus on understanding how the prizes influences the effort exerted by the agents, which amounts to understanding the marginal effect functions $m_i(\theta)$. We show that the effect of the first prize $m_1(\theta)$, under any prior distribution F, is positive for all agent types. That is, increasing the value of the first prize encourages effort from all types of agents. It follows then that if we fix the values of the remaining prizes, the equilibrium effort function with a higher first prize first order stochastically dominates the equilibrium with a smaller first prize. In contrast, the effect of any intermediate prize $m_i(\theta)$ for $i \in \{2, \ldots, n-1\}$ depends on the agent's type θ. More precisely, we find that increasing any intermediate prize discourages effort from the more efficient agents (agents with low marginal costs of effort θ) and encourages effort from the less efficient agents (those with high θ). Intuitively, this is because increasing any intermediate prize i essentially reduces the value of the better prizes ($j < i$) for the more efficient agents as they would now be willing to settle for prize i as well. On the other hand, the less efficient agents who are generally competing for the worse prizes ($j > i$) are now encouraged to put in extra effort for the better prize i. Interestingly, we get that the negative effect on the more efficient agents and the positive effect on the less efficient agents balances out since $\int_0^1 m_i(\theta) = 0$ for all $i \in \{2, \ldots, n-1\}$. In other words, for any prior distribution F, increasing the value of an intermediate prize i leads to a balanced transfer of effort from the more efficient agents to the less efficient agents. The overall expected effect of these prizes then depends on the prior distribution. In the special case where the density function is monotone increasing (decreasing) so that there is an increasing (decreasing) density of less efficient agents, the expected effects of prize i for $i \in \{2, \ldots, n-1\}$ is positive (negative) and in fact, if we fix the values of the remaining prizes, the (negative of) equilibrium effort function with a higher value of prize i second order stochastically dominates the (negative of) equilibrium with a smaller value of prize i.

Next, we study the effect of competition on expected effort. For this purpose, we focus on the parametric subclass of distributions $F(\theta) = \theta^p$ and compare the expected effort induced by prize vectors ordered in the majorization partial order. We show that for $p \geq 1$, more competitive prize vectors induce higher expected effort which is in contrast to the case of $\frac{1}{2} \leq p \leq 1$, where conditional on the first and last prize being fixed, more competitive prize vectors induce smaller expected effort. Thus, as is the case in the effect of prizes of effort, the effect of competition on effort differs qualitatively depending on whether the prior density of less-efficient agents is increasing ($p \geq 1$) or decreasing ($p \leq 1$). As a corollary, we get that for the uniform prior ($p = 1$), the intermediate prizes do not matter for the expected effort. That is, the expected effort under the uniform prior for any prize vector $\mathbf{v}$ equals $v_1 - v_n$. We also show that for any $p \geq \frac{1}{2}$, the expected minimum effort decreases as contest becomes more competitive. Intuitively, this is because the designer puts a lot of weight on the effort of the less efficient agents who care more about the lower ranked prizes. Therefore, we get that the expected marginal effects $\mathbb{E}[m_i(\theta_{max})]$ are increasing in i which implies the result.

We next discuss applications of our results to the design of effort maximizing contests in three different environments. First, we consider the design of grading contests. Assuming the value of a grade in a grading contest is determined by the information it reveals about the type of the agent, and more precisely, equals its expected productivity, we show that a more informative grading scheme induces a more competitive prize vector. The optimal grading contests can then be derived from from our results on the effects of competition on effort. For the case of $F(\theta) = \theta^p$ with $p \geq 1$, the rank revealing contest maximizes expected effort. But for $\frac{1}{2} \leq p \leq 1$, the optimal grading contest awards a unique grade to the best agent and pools the bottom $n-1$ agents together by awarding them a common grade. If the designer cares about the effort of the least efficient agent, as would be reasonable in a classroom environment, the optimal grading contest awards a unique grade to the least efficient agent while pooling the top $n-1$ agents together with a common grade. For our second application, we consider a setting where the contest designer has a budget that it can split across n prizes and the agents utility for prizes is represented by $u(v) = v^r$ for $r \in (0, 1]$. We derive the effort-maximizing contest and show that as r increases (so that the risk aversion decreases), the optimal prize vector becomes more competitive. For our third and last application, we derive the optimal contest in settings where the designer can choose the number of winners to award with a costless homogeneous prize.

1.1 Literature review

There is a vast literature on contest theory studying the effect of manipulating various structural elements of a contest on the equilibrium behaviour of the agents and consequently designing these elements so as to satisfy certain objective. The various design elements that have been considered include winner selection mechanisms, introduction of dynamics with sequential decision making, introduction of asymmetry via head starts, information disclosure at intermediate stages, entry constraints, introducing multiple contests or group contests, etc. Fu and Wu [26] provides a survey of the theoretical literature on contests from these different perspectives. This paper focuses on the simple case where the designer can manipulate the value of the prizes it awards. In particular, the version of the optimal contest design problem that is closely related to this work is to *distribute a given budget B among N prizes $(v_1, v_2, \ldots, v_n)$ so as to maximize the effort of the n competing agents.* This problem was first posed by Galton in 1902 and has since been studied in various different environments. The environments generally differ in the assumptions they make about the *contest success function* which is a mapping from agents effort levels to their probability of winning a prize, the *abilities or valuations of the agents* which may be symmetric or asymmetric, and the *information* agents have about the abilities of other agents which may be complete or incomplete. We briefly discuss what is known about this problem for the perfectly discriminatory contest success function under which agents are simply ranked according to their effort levels and awarded the corresponding prizes.

In the complete information setting with symmetric agents, Glazer and Hassin [30] find that with concave utility and linear costs, awarding $N-1$ equal prizes worth $\frac{B}{N-1}$ is optimal. When utility is also linear, Barut and Kovenock [2] show that any equilibrium induces the same expected aggregate effort of $B - Nv_n$ and so the only restriction posed by optimality is that $v_n = 0$. More recently, Fang et al. [23] generalize these results and find that with linear utility and convex costs, increasing prize inequality reduces each agent's effort in the sense of second order stochastic dominance. In contrast, our work shows that in the incomplete information setting, the effect of competition depends on the prior distribution of abilities. It encourages effort if the density of inefficient agents is increasing and discourages effort if this density is decreasing.

In an incomplete environment with ex-ante symmetric agents, which is the focus of this paper, Glazer and Hassin [30] show that with concave utility and linear costs $c(x) = \theta x$

where $\frac{1}{\theta} \sim U[0,1]$, the optimal prize vector involves setting all the later half of the prizes to 0. Moldovanu and Sela [44] consider the case with linear utility and costs $c(x) = \theta\gamma(x)$ where θ is drawn from the interval $[m, 1]$ with $m > 0$. For linear costs $\gamma(x) = x$, they show that awarding only a single prize is optimal. For convex costs, they identify conditions under which awarding more than one prize might be optimal. In comparison, our paper obtains a more complete ordering of prize vectors in terms of the effort they induce and under different distributional assumptions. The analysis, which may be useful in itself, also allows us to find optimal contests in other natural environments where the set of feasible prize vectors may be constrained or different. There has been relatively little work on the problem with asymmetric agents. In a complete information setting with linear utility and linear costs, Clark and Riis [14] study the problem of finding the number of prizes in which to split the budget. While they don't fully solve the problem, they provide examples where splitting the budget into more than one prize might be optimal. We are not aware of any work on the problem in an incomplete information environment with ex-ante asymmetric agents.

There has also been work on this problem for other contest success functions. Perhaps the most popular alternative is the ratio-form contest success function. Clark and Riis [15] consider a complete information setting with symmetric agents having linear utility and costs and find that under the Tullock csf with discriminatory power r ($\frac{e_i^r}{\sum e_j^r}$), increasing prize inequality leads to an increase in total effort. Szymanski and Valletti [59] show that with asymmetric agents, a second prize might be optimal. We are not aware of work on the ratio-form csf in the incomplete information environment. Other related work that looks at the design of optimal contests under some different assumptions include Krishna and Morgan [37], Liu and Lu [42], Cohen and Sela [16]. Sisak [57] provides a more detailed survey of the literature on this problem. More general surveys of the theoretical literature in contest theory can be found in Corchón [18], Vojnović [60], Konrad et al. [34], Segev [52].

The settings we focus on for our applications have also been studied in the literature. In particular, there has been significant work on the design of optimal grading schemes (Moldovanu et al. [46], Rayo [50], Popov and Bernhardt [49], Chan et al. [8], Dubey and Geanakoplos [20], Zubrickas [62]). Moldovanu et al. [46] consider a setting where the designer can associate grades with arbitrary monetary prizes subject to budget and individual rationality constraints and find that the optimal grading scheme awards the top grade to a unique agent and a single grade to all the remaining agents. Dubey and Geanakoplos [20] consider a complete information environment where agents care about relative ranks and find that absolute grading is generally better than relative grading and that it's better to clump scores into coarse categories. Other related papers look at the signalling value of grades under different models or assumptions (Costrell [19], Betts [5], Zubrickas [62], Boleslavsky and Cotton [6]). The setting of our last application where the designer can only choose the number of winners to receive a fixed homogeneous prize was also considered in Liu and Lu [42] under different distributional assumptions. They find that the expected effort is single peaked in the number of prizes. In comparison, we identify natural conditions on the prior distribution under which awarding a single prize or $n-1$ prizes is optimal.

The paper proceeds as follows. In section 2, we present the model of a contest in an incomplete-information environment. Section 3 characterizes the symmetric Bayes-Nash equilibrium of this game and discusses some important properties of the equilibrium function. In section 4, we discuss applications of our results to the design of optimal contest in three different settings. Section 5 concludes. All proofs are relegated to the appendix.

2 Model

Consider a contest with n agents and n prizes described by prize vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ with $v_i \geq v_{i+1}$ for all i. The agents compete for these prizes by exerting costly effort. Each agent i is privately informed about its marginal cost of effort $\theta_i \in [0, 1]$. The marginal cost for each agent is drawn independently from $[0, 1]$ according to cdf F which is common knowledge. Given a vector of prizes $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, marginal cost θ_i, and belief F, each agent i simultaneously chooses an effort level e_i. The agents are ranked according to how much effort they put in and agent i is awarded prize v_j if it puts in the jth highest effort. Agent i's final payoff in that case is $v_j - \theta_i e_i$.

3 Equilibrium

In this section, we first characterize the symmetric Bayes-Nash equilibrium of the game for arbitrary prize vectors and then discuss how the equilibrium changes as we vary different prizes.

Note that our model is the same as the model studied in Moldovanu and Sela [44] except for the fact that we allow agents marginal cost to be 0 whereas Moldovanu and Sela [44] assume that the marginal costs are in an interval $[m, 1]$ where $m > 0$. It turns out that the symmetric Bayes-Nash equilibrium strategy function takes the same form as in Moldovanu and Sela [44], but it also satisfies an interesting property due to the presence of agents with negligible marginal costs of effort as we will see later. The following result displays the symmetric Bayes-Nash equilibrium strategy of the contest game (Moldovanu and Sela [44]).

Theorem 1. *In a contest with n agents, prizes $\mathbf{v} = (v_1, v_2, \ldots, v_{n-1}, v_n)$ and prior cdf F, the symmetric Bayes-Nash equilibrium strategy function is given by*

$$g_v(\theta) = \sum_{i=1}^{n} m_i(\theta) v_i$$

where,

$$m_i(\theta) = \binom{n-1}{i-1} \int_{F(\theta)}^{1} \frac{[(1-t)^{n-i-1} t^{i-2}]}{F^{-1}(t)} \left((n-1)t - (i-1)\right) dt$$

for all $i \in \{1, 2, \ldots, n\}$*.*

The proof proceeds by assuming that $n-1$ agents are playing the strategy $g_{\mathbf{v}}(.)$ where $g_{\mathbf{v}}$ is a decreasing function. Given the strategy of other agents, we find an agent's optimal effort level at type θ by taking the first order condition. Plugging in $g_v(\theta)$ for the optimal level of effort in the condition gives the condition for $g_v(\theta)$ to be the symmetric Bayes-Nash equilibrium:

$$-f(\theta)\sum_{i=1}^{n-1}(v_i - v_{i+1})\frac{(n-1)!}{(i-1)!(n-i-1)!}\left[(1-F(\theta))^{n-i-1}F(\theta)^{i-1}\right] = \theta g_v'(\theta)$$

Using the boundary condition $g_v(1) = 0$ pins down the form of the function

$$g_v(\theta) = \sum_{i=1}^{n-1}(v_i - v_{i+1})\frac{(n-1)!}{(i-1)!(n-i-1)!}\int_{F(\theta)}^{1}\frac{[(1-t)^{n-i-1}t^{i-1}]}{F^{-1}(t)}dt$$

We can then rewrite the expression as in the theorem by combining the two terms with coefficient v_i. Note that $\sum_{i=1}^{n} m_i(\theta) = 0$ for all $\theta \in [0,1]$. Lastly, we check that the second order condition is satisfied. The full proof is in the appendix.

In case an agent's value for prize v is given by some utility function $u(v)$ and all agents share the same utility function u for prizes (upto affine transformations), the equilibrium in Theorem 1 is simply as if prize i was $u(v_i)$ instead of v_i. The following corollary states this formally.

Corollary 1. *In a contest with* n *agents, each with utility function* u *for prizes, prizes* $\mathbf{v} = (v_1, v_2, \ldots, v_{n-1}, v_n)$ *and prior cdf* F*, the symmetric Bayes-Nash equilibrium strategy function is given by*

$$g_v(\theta) = \sum_{i=1}^{n} m_i(\theta)u(v_i)$$

where,

$$m_i(\theta) = \binom{n-1}{i-1}\int_{F(\theta)}^{1}\frac{[(1-t)^{n-i-1}t^{i-2}]}{F^{-1}(t)}\left((n-1)t - (i-1)\right)dt$$

for $i \in [n-1]$ *and* $m_n(\theta) = -\sum_{i=1}^{n-1} m_i(\theta)$*.*

We will use this when we discuss the effort-maximizing contest for risk-averse agents.

Now that we know what the equilibrium function looks like, we focus our attention on studying how the equilibrium changes as we vary the values of different prizes.

Theorem 2. *Consider a setting with* n *agents and prior* F*. Suppose* $\mathbf{v} = (v_1, \ldots, v_n)$ *and* $\mathbf{w} = (w_1, \ldots, w_n)$ *are two prize vectors. Then, we have the following:*

1. *if* $v_1 > w_1$ *and* $v_{-1} = w_{-1}$*, then* $g_v(\theta)$ *first order stochastically dominates* $g_w(\theta)$

2. *if* $v_j > w_j$ *for* $j \in \{2, \dots, n-1\}$, $v_{-j} = w_{-j}$, *and* F *is such that the density* f *is increasing, then* $g_v(\theta)$ *second order stochastically dominates* $g_w(\theta)$

3. *if* $v_j > w_j$ *for* $j \in \{2, \dots, n-1\}$, $v_{-j} = w_{-j}$, *and* F *is such that the density* f *is decreasing, then* $-g_v(\theta)$ *second order stochastically dominates* $-g_w(\theta)$

In words, the first prize always encourages effort irrespective of the prior distribution of abilities. The effect of the intermediate prizes depends on the prior distribution of abilities. If the prior distribution is such that the density of less efficient agents is increasing, then higher values of intermediate prizes encourage effort. But if this density is decreasing, then these prizes discourage effort in expectation. This is partly because any intermediate prize i has a different effect on different types of agents. The more efficient agents who are fighting for the top prizes put in lesser effort as the gain from winning these better prizes has gone down. In contrast, the less efficient agents who generally get lower prizes now put in more effort to get the increased prize i. Importantly, the decrease in effort of the more efficient agents and the increase for the less efficient agents cancel out so that there is basically a *transfer* of effort from the more efficient agents to the less efficient agents as any intermediate prize is increased. Note that the existence of agents with negligible marginal costs of effort is important for the equilibrium to have this property. This property is formally stated in the following lemma which is the key to proving Theorem 2.

Lemma 1. *For any number of agents* n *and prior* F*, the following hold:*

1. $\theta m_i(\theta) \leq 1$ *for all* $i \in \{1, \dots, n-1\}, \theta \in [0,1]$

2. $\lim_{\theta \to 0} \theta m_i(\theta) = 0$ *for all* $i \in \{1, \dots, n-1\}$

3. $\int_0^1 m_1(\theta) = 1$ *and* $\int_0^1 m_i(\theta) = 0$ *for* $i \in \{2, n-1\}$

4. $m_1(\theta) > 0$ *for all* θ *and monotone decreasing*

5. *For* $i \in \{2, n-1\}$*, there exist* $t_i^1 < t_i^2$ *such that*

$$m_i(\theta) = \begin{cases} <= 0 & \text{if } \theta \leq t_i^1 \\ > 0 & \text{otherwise} \end{cases}$$

$$m_i'(\theta) = \begin{cases} >= 0 & \text{if } \theta \leq t_i^2 \\ < 0 & \text{otherwise} \end{cases}$$

The first property provides an upper bound on the marginal effect of prize i on the effort of agent of type θ and is fairly straightforward. The second property says that something stronger is true for the most efficient agents. It says that the effort cost of an agent goes to zero as its marginal cost of effort goes to 0. The third property says that the overall effect of any prize i other than the first prize is zero. So increasing any prize i just transfers the effort from some set of agents to another set of agents. The first prize is special in that its overall effect is positive and equals 1 irrespective of the prior distribution. To prove this, we apply integration by parts to integrate $\int_0^1 m_i(\theta)d\theta$ and using the second property, we get that it equals $-\int_0^1 \theta m_i'(\theta)d\theta$. By Leibniz rule, we know

$$m_i'(\theta) = -\binom{n-1}{i-1} \frac{[(1-F(\theta))^{n-i-1}F(\theta)^{i-2}]}{\theta} \left((n-1)F(\theta) - (i-1)\right) f(\theta)$$

and so we are able to integrate $\theta m_i'(\theta)$ to get the result. Moving on, the fourth property says the marginal effect of the first prize is positive for all types of agents and also decreasing in their type. And finally, the last property describes exactly how the effort transfers from one set of agents to another as we increase the value of some prize. In particular, for any prize i that is not the first prize, increasing its value leads to a transfer of effort from more efficient agents ($\theta \leq t_i^1$) to less efficient agents ($\theta > t_i^1$). These properties are illustrated in Figure 1 for the case of $n = 5$ agents and prior cdf $F(\theta) = \theta^3$.

The overall expected effect of each prize i as described in Theorem 2 depends on the prior distribution. If there is a higher proportion of less efficient agents who are positively influenced by intermediate prizes, the overall effect of these intermediate prizes is positive. On the other hand, if there is a higher proportion of more efficient agents who are discouraged by these intermediate prizes, the overall effect is negative. The full proofs of Lemma 1 and Theorem 2 are in the appendix.

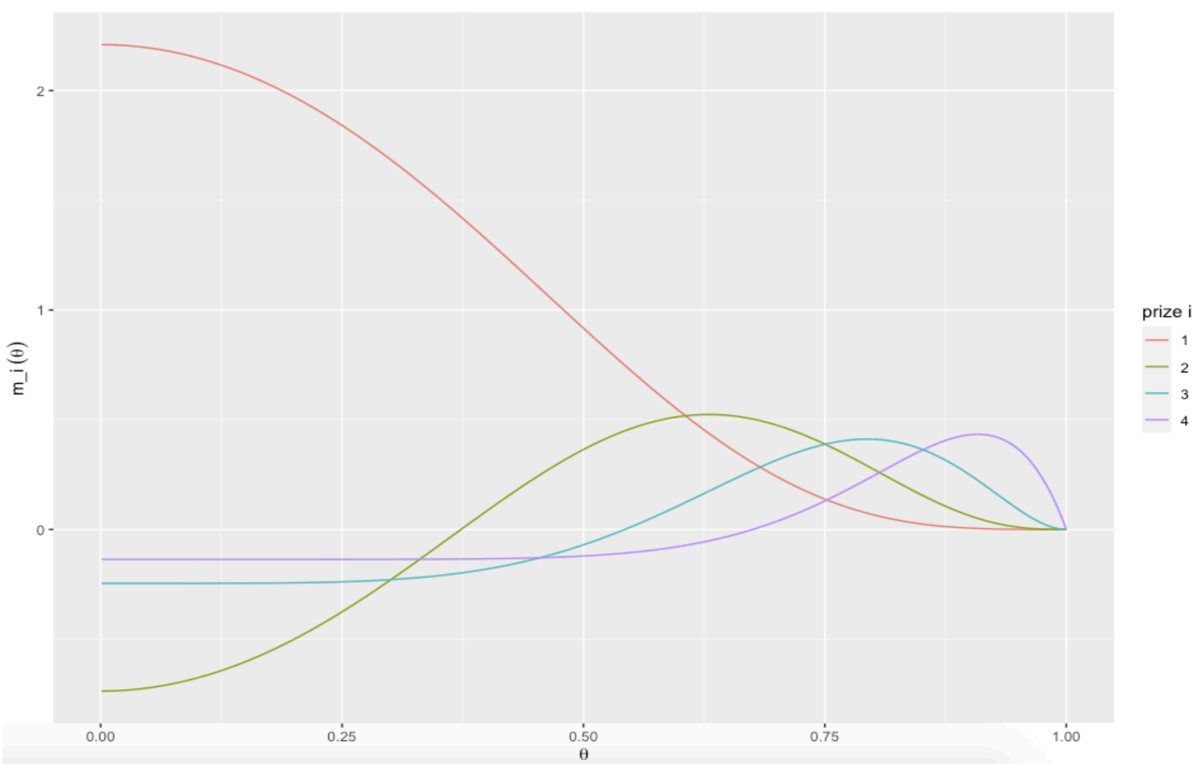

Figure 1: The marginal effect of prizes on effort for $n = 5$ and $F(\theta) = \theta^3$.

Theorem 2 identifies natural conditions on the prior distributions under which the expected marginal effects are positive or negative. Next, we consider a natural parametric class of the priors for which we can compute and compare the expected marginal effects of prizes. We then discuss what this comparison implies for the effects of prize inequality or competition in contests on the effort exerted by the agents under these priors. First, let us formally define what it means for a prize vector to be more competitive than another.

Definition 3.1. A prize vector $\mathbf{v} = (v_1, v_2, \ldots, v_{n-1}, v_n)$ is more competitive than $\mathbf{w} = (w_1, w_2, \ldots, w_{n-1}, w_n)$ if $\mathbf{v}$ majorizes $\mathbf{w}$ (i.e. $\sum_{i=1}^{k} v_i \geq \sum_{i=1}^{k} w_i$ for all $k \in [n]$ and $\sum_{i=1}^{n} v_i = \sum_{i=1}^{n} w_i$).

This is the definition that was also considered in Fang et al. [23]. The next result describes the effect of competition on expected effort and expected minimum effort and how it depends on the prior distribution.

Theorem 3. *Consider a setting with n agents and prior $F(\theta) = \theta^p$. Suppose $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ are two prize vectors such that $\mathbf{v}$ is more competitive than $\mathbf{w}$. Then, we have the following:*

1. *if $p \geq 1$, then*
$$\mathbb{E}[g_v(\theta)] \geq \mathbb{E}[g_w(\theta)]$$
2. *if $\frac{1}{2} \leq p \leq 1$, $v_1 = w_1, v_n = w_n$, then*
$$\mathbb{E}[g_v(\theta)] \leq \mathbb{E}[g_w(\theta)]$$
3. *if $p \geq \frac{1}{2}$ and $v_n = w_n$, then*
$$\mathbb{E}[g_v(\theta_{max})] \leq \mathbb{E}[g_w(\theta_{max})]$$

To prove this, we first show that for $F(\theta) = \theta^p$,

$$\mathbb{E}[m_i(\theta)] = \binom{n-1}{i-1} \beta(i - \frac{1}{p}, n-i) \frac{(n-i)(p-1)}{np-1}$$

which implies that

$$\frac{\mathbb{E}[m_{i+1}(\theta)]}{\mathbb{E}[m_i(\theta)]} = \frac{n-i}{i} \frac{i - \frac{1}{p}}{n-i-1} \frac{n-i-1}{n-i} = \frac{i - \frac{1}{p}}{i} < 1$$

It follows then that for $p \geq 1$, the expected marginal effects $\mathbb{E}[m_i(\theta)]$ are positive and decreasing in i. Since $\mathbf{w}$ can be obtained from $\mathbf{v}$ by a sequence of Robinhood transfers which involve a transfer of value from a top prize to a bottom prize, each of which reduces expected effort, we get that the expected effort goes down as prize vector becomes less competitive. An analagous argument holds for $\frac{1}{2} \leq p \leq 1$.

For the case of expected minimum effort, we again find that

$$\mathbb{E}[m_i(\theta_{max})] = \binom{n-1}{i-1}\beta\left(n+i-1-\frac{1}{p}, n-i\right)\frac{(n-i)(np-1)}{2np-p-1}$$

which implies

$$\frac{\mathbb{E}[m_{i+1}(\theta_{max})]}{\mathbb{E}[m_i(\theta_{max})]} = \frac{n+i-1-\frac{1}{p}}{i} > 1$$

It follows then that less competitive prize vectors lead to higher expected minimum effort.

The fact that the expected marginal effect of prize v_i is decreasing in i for $p \geq 1$ is perhaps a bit surprising since these are distributions where a large proportion of the agents are inefficient (the density is increasing in θ). Since these inefficient agents are generally competing for the intermediate prizes, one might expect that as their proportion increases (p increases), the expected marginal effect of the later prizes would increase relative to the earlier prizes. While we do get that the ratio $\frac{\mathbb{E}[m_{i+1}(\theta)]}{\mathbb{E}[m_i(\theta)]} = \frac{i-\frac{1}{p}}{i}$ increases as p increases, it only goes to 1 as $p \to \infty$. Thus, the effect is not big enough to make later prizes more valuable than the earlier prizes in terms of the effort they induce. In contrast, for the case of expected minimum effort, the designer is putting even more weight on the effort of the least efficient agents. And it turns out in this case that the marginal effect of prize v_i on expected minimum effort is increasing in i. We conclude this section with the following corollary about the case of the uniform prior.

Corollary 2. *In a contest with n agents, prizes $v = (v_1, v_2, \ldots, v_{n-1}, v_n)$ and uniform prior cdf $F(\theta) = \theta$, the expected effort $\mathbb{E}[g_v(\theta)] = v_1 - v_n$.*

4 Applications

In this section, we discuss some applications to the design of optimal contests in environments where the set of feasible prize vectors available to the designer may be constrained. In particular, we will consider three different environments. First, we'll consider settings where the designer can commit to a grading scheme and the value of these grades is determined by the information they reveal about the type of the agents. Second, we'll consider settings where agents are risk-averse and we'll derive the effort-maximizing prize structure under the standard constraint that the designer has a budget that it must allocate across prizes. We'll also discuss how the optimal prize structure changes as the degree of risk-aversion increases in the population. At last, we'll consider settings where the contest designer is constrained to award homogeneous prizes of a fixed value and only needs to decide how many prizes it must award to maximize effort.

4.1 Optimal grading contests

First, we focus on the design of grading schemes where the contest designer doesn't have an explicit budget that it can distribute across prizes but instead, can choose a distribution of grades that it can award based on the rank of the agents. This is generally the case in classroom settings where the professor awards grades to students based on their performance in exams. For instance, the professor may commit to giving grades A and B to the top 50% and bottom 50% respectively, or it may give $A+, A-, B+$, and $B-$ with distribution $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Formally, we define a grading contest as follows:

Definition 4.1. A grading contest with n agents is defined by a strictly increasing sequence of natural numbers $s = (s_1, s_2, \ldots, s_k)$ such that $s_k = n$.

The interpretation of grading contest s is that the top s_1 agents get grade g_1, next $s_2 - s_1$ get grade g_2 and generally, $s_k - s_{k-1}$ get grade g_k. There is a natural partial order over these grading contests in terms of how much information they reveal about the quality of the agents. From above, the grading contest that awards the grades $A+, A-, B+$, and $B-$ in equal proportion is more informative about the agents type then the one that awards just A and B in equal proportion. More generally, we can say the following:

Definition 4.2. A grading contest s is more informative than s' if s' is a subsequence of s.

Clearly, the rank revealing contest $s^* = (1, 2, \ldots, n)$ is more informative than any other grading contest.

To discuss how these grading contests compare in terms of the effort they induce, we need to define how the agents assign value to these grades. We assume that the value of a grade is determined by the information it reveals about the type of the agent. More precisely, we suppose that there is a publicly known wage function $w : \Theta \to \mathbb{R}_+$ which maps an agent's marginal cost to its productivity and is monotone decreasing. So if the market could observe the type of the agent to be θ, the agent would be offered a wage of $w(\theta)$. Given this wage function, we assume that a grading contest s induces the prize vector which is the expected productivity or wage of the agent given its grade. That is, if the market has a posterior belief f over the type of the agent, then the agent will get a wage equal to $\int_0^1 w(\theta)f(\theta)d\theta$.

Under this assumption, we get that the rank revealing contest $s^* = (1, 2, \ldots, n)$ induces the prize vector

$$v_i = \mathbb{E}[w(\theta)|\theta = \theta^n_{(i)}]$$

where $\theta^n_{(i)}$ is the ith order statistic in a random sample of n observations. This is because the rank revealing contest reveals the exact rank of the agent in a random sample of n observations. Note here that since $\theta^n_{(i)}$ is stochastically dominated by $\theta^n_{(j)}$ for all $i < j$ and w is monotone decreasing, the prize vector induced by s^* is monotone decreasing $v_1 > v_2 \cdots > v_n$.

Now we can define the prize vectors induced by arbitrary grading contests s in terms of the v_i's as defined above. An arbitrary grading contest $s = (s_1, s_2, \ldots, s_k)$ induces the prize vector $v(s)$ where

$$v(s)_i = \frac{v_{s_{j-1}+1} + v_{s_{j-1}+2} + \cdots + v_{s_j}}{s_j - s_{j-1}}$$

and j is such that $s_{j-1} < i \leq s_j$

This is because if an agent gets grade g_j in the grading contest $s = (s_1, s_2, \dots, s_k)$, then the market learns that the agent's type θ must be ranked at one of $\{s_{j-1}+1, \dots, s_j\}$ and further, it is equally likely to be ranked at any of these positions. The form of the prize vector above then follows from the assumption that the value of grade equals its expected productivity under the posterior induced by the grade.

Given this framework, we can now ask how the different grading schemes compare in terms of the effort they induce.

Theorem 4. *Consider a setting with n agents and prior cdf $F(\theta) = \theta^p$. Suppose grading scheme s is more informative than s'. Then, we have the following:*

- *if $p \geq 1$, then $\mathbb{E}[g_{v(s)}(\theta)] \geq \mathbb{E}[g_{v(s')}(\theta)]$*
- *if $\frac{1}{2} \leq p \leq 1$, $v(s)_1 = v(s')_1$, and $v(s)_n = v(s')_n$, then $\mathbb{E}[g_{v(s)}(\theta)] \leq \mathbb{E}[g_{v(s')}(\theta)]$*
- *if $p \geq \frac{1}{2}$ and $v(s)_n = v(s')_n$, then $\mathbb{E}[g_{v(s)}(\theta_{max})] \geq \mathbb{E}[g_{v(s')}(\theta_{max})]$*

Proof. Observe that if s is more informative than s', then it induces a prize vector $v(s)$ that is more competitive than the prize vector $v(s')$. The result then follows directly from Theorem 3.

□

With this comparison, we can now find the optimal grading contest in these settings.

Corollary 3. *Consider a setting with n agents and prior cdf $F(\theta) = \theta^p$. Then, we have the following:*

- *if $p \geq 1$, the rank revealing contest $s = (1, 2, \dots, n)$ maximizes expected effort among all grading contests.*
- *if $\frac{1}{2} \leq p \leq 1$, the contest $s = (1, n)$ in which the best agent gets a unique grade and all other agents get a common grade maximizes expected effort among all grading contests.*
- *if $p \geq \frac{1}{2}$, the contest $s = (n-1, n)$ in which the worst agent gets a unique grade and all other agents get a common grade maximizes expected minimum effort among all grading contests.*

Note that when the designer has a budget that it can distribute across prizes, the expected effort maximizing prize vector allocates the entire budget to the first prize. But as we see in the corollary, the optimal grading contest depends on the prior distribution of abilities. If the density of agents is increasing in θ so that there is a greater proportion of inefficient agents, the effort maximizing grading contest awards a unique grade to each agent. But when the density is decreasing, the optimal grading contest awards a unique grade to the best agent and pools the rest of the agents by awarding them a common grade. And for the case where the designer wants to maximize expected minimum effort, which is perhaps a reasonable objective in a classroom environment, the optimal grading contest awards a common grade to everyone except the least efficient agent.

4.2 Optimal contests with risk-averse agents

Now we consider the contest design problem in a typical environment where the designer has a budget of B that it can allocate across prizes $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ such that $v_i \geq v_{i+1}$. It is known that when the participants are risk neutral $u(v) = v$, the effort maximizing contest awards the entire prize budget B to the first prize ([44]). In this section, we consider the contest design problem with risk-averse agents. More precisely, we assume that under a prize vector $\mathbf{v}$, if agent i of type θ_i puts in effort e_i and wins prize j, its payoff equals $u(v_j) - \theta_i e_i$ where $u(v_j) = v_j^r$. The parameter $r \in (0, 1]$ measures the degree of risk aversion so that if r increases, the agents are less risk-averse. The next result characterizes the expected effort maximizing contest in this environment.

Theorem 5. *Suppose there are n agents with utility $u(v) = v^r$ and the prior cdf on marginal costs is $F(\theta) = \theta^p$ with $p \geq 1$. The prize structure that maximizes expected effort is*

$$v(r) = (v_1(r), c_2^{\frac{1}{1-r}} v_1(r), \ldots c_{n-1}^{\frac{1}{1-r}} v_1(r), 0)$$

where

$$c_i = \frac{\mathbb{E}[m_i(\theta)]}{\mathbb{E}[m_1(\theta)]} < 1$$

and $v_1(r)$ is such that

$$\sum v_i(r) = B$$

Further, if $1 \geq r > r' > 0$, then $v(r)$ is more competitive than $v(r')$.

The proof proceeds by using corollary 1 to identify the Bayes-Nash equilibrium function so that the problem becomes $\max_{\mathbf{v}} \sum_{i=1}^{n-1} v^r \mathbb{E}[m_i(\theta)]$ such that $\sum v_i = B$. Solving this constrained optimization problem characterizes the optimal contest. To show that the optimal contest become more competitive as r increases, we define $f_k(r)$ as the sum of the first k prizes in the optimal contest under r and show that this sum is increasing in r.

4.3 Optimal contests with costless homogeneous prizes

For our last application, we consider a setting where the contest designer can award arbitrarily many prizes of a fixed value a. More precisely, the set of prize vectors available to the designer is given by

$$B = \{\mathbf{v} \in \mathbb{R}^n : \exists k \text{ such that } v_i = a \text{ if } i \leq k \text{ and } v_i = 0 \text{ if } i > k\}$$

This might be the case in online contests run on platforms like Leetcode and Kaggle where the platforms essentially have an unlimited supply of digitial certificates or medals that they can award to participants in the contest. The designer wants to chose the number of prizes so as to maximize the expected effort. Note that this problem was also considered in [42] but under different distributional assumptions. In their setting, the authors found that the expected effort was single-peaked in k. For our setting, we have the following as a corollary of Theorem 2.

Corollary 4. *Consider a setting with n agents and prior F. Then, we have the following:*

1. *if F is such that the density f is increasing, then the contest that awards $k = n - 1$ prizes maximizes expected effort among the contests in B.*
2. *if F is such that the density f is decreasing, then the contest that awards $k = 1$ prize maximizes the expected effort among the contests in B.*

In fact, in the first case where the density is increasing, we actually have something stronger. In this case, awarding $k = n - 1$ second order stochastically dominates awarding fewer number of prizes. In other words, for any concave increasing utility function U, the contest designer's optimal decision would be to award $k = n - 1$ prizes.

5 Conclusion

In contests where agents have private information about their abilities, we study the effect different prizes have on the effort exerted by the agents in equilibrium. While increasing the value of first prize encourages effort for all agents, increasing any intermediate prize leads to a balanced transfer of effort from the more efficient agents to less efficient agents. The overall expected effect then depends on the prior distribution of abilities. If the density of agents is increasing in inefficiency, the expected effects are positive. If this density is decreasing, the expected effects are negative. For a parametric subclass of priors with monotone density functions, we also study the effects of competition on expected effort and expected minimum effort.

We also discuss the application of these results to the design of effort-maximizing contests in settings where the set of feasible prize vectors may be constrained due to various reasons. First, we consider the design of grading contests under the assumption that the value of a grade is determined by the information it reveals about the type of the agent. We find that when the density function is increasing, more informative contests lead to higher expected effort and therefore, the rank revealing contest maximizes expected effort among all grading contests. Second, we consider the setting where the designer has a budget that it must allocate across different prizes and the agents are risk-averse. We find the optimal contest in this setting and show that it becomes more competitive as the degree of risk aversion decreases. Lastly, we consider settings where the designer can only choose the number of homogeneous prizes to award and show that when the prior density is monotone, it is optimal to award either 1 or $n-1$ prizes depending on whether the density is decreasing or increasing.

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A Proofs for Section 3 (Equilibrium)

Theorem 1. *In a contest with n agents, prizes $\mathbf{v} = (v_1, v_2, \ldots, v_{n-1}, v_n)$ and prior cdf F, the symmetric Bayes-Nash equilibrium strategy function is given by*

$$g_v(\theta) = \sum_{i=1}^{n} m_i(\theta) v_i$$

where,

$$m_i(\theta) = \binom{n-1}{i-1} \int_{F(\theta)}^{1} \frac{\left[(1-t)^{n-i-1} t^{i-2}\right]}{F^{-1}(t)} \left((n-1)t - (i-1)\right) dt$$

for all $i \in \{1, 2, \ldots, n\}$.

Proof. Suppose $n-1$ agents are playing a monotone decreasing strategy $g(\theta)$. Let $\theta^n_{(j)}$ denote the jth order statistic from n random draws with $\theta^n_{(0)} = 0$ and $\theta^n_{n+1} = 1$. Then, an agent of type θ's utility from putting in x units of effort is given by:

$$\begin{aligned} u(\theta, x) &= \sum_{i=1}^{n} v_i \Pr[\theta^{n-1}_{(i-1)} \leq g^{-1}(x) \leq \theta^{n-1}_{(i)}] - \theta x \\ &= \sum_{i=1}^{n} v_i \binom{n-1}{i-1} F(g^{-1}(x))^{i-1} (1 - F(g^{-1}(x)))^{n-i} - \theta x \end{aligned}$$

Now, differentiating with respect to x gives:

$$\frac{\partial u(\theta, x)}{\partial x} = \frac{f(g^{-1}(x))}{g'(g^{-1}(x))} \sum_{i=1}^{n} v_i \binom{n-1}{i-1}$$
$$\left[(1 - F(g^{-1}(x)))^{n-i} (i-1) F(g^{-1}(x))^{i-2} - F(g^{-1}(x))^{i-1} (n-i)(1 - F(g^{-1}(x)))^{n-i-1}\right] - \theta$$

. Setting it to 0 and plugging in $g(\theta) = x$ gives the condition for $g(\theta)$ to be a symmetric Bayes-Nash equilibrium:

$$f(\theta) \sum_{i=1}^{n} v_i \binom{n-1}{i-1} \left[(1 - F(\theta))^{n-i} (i-1) F(\theta)^{i-2} - F(\theta)^{i-1} (n-i)(1 - F(\theta))^{n-i-1}\right] = \theta g'(\theta)$$

An alternate way to write this condition is:

$$-f(\theta) \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} \left[(1 - F(\theta))^{n-i-1} F(\theta)^{i-1}\right] = \theta g'(\theta)$$

Using the boundary condition $g(1) = 0$, we get that the symmetric Bayes-Nash equilibrium function is given by

$$\int_\theta^1 \frac{f(\theta)}{\theta} \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} \left[(1-F(\theta))^{n-i-1} F(\theta)^{i-1}\right] d\theta$$

Replacing $F(\theta) = t$, we get

$$g(\theta) = \int_{F(\theta)}^1 \frac{1}{F^{-1}(t)} \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} \left[(1-t)^{n-i-1} t^{i-1}\right] dt$$

Bringing the summation outside:

$$g_v(\theta) = \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_{F(\theta)}^1 \frac{\left[(1-t)^{n-i-1} t^{i-1}\right]}{F^{-1}(t)} dt$$

We can also write the equilibrium function as $g_v(\theta) = \sum_{i=1}^n m_i(\theta) v_i$ where for $i \geq 2$,

$$\begin{aligned}
m_i(\theta) &= \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_{F(\theta)}^1 \frac{\left[(1-t)^{n-i-1} t^{i-1}\right]}{F^{-1}(t)} dt - \frac{(n-1)!}{(i-2)!(n-i)!} \int_{F(\theta)}^1 \frac{\left[(1-t)^{n-i} t^{i-2}\right]}{F^{-1}(t)} dt \\
&= \frac{(n-1)!}{(i-2)!(n-i-1)!} \int_{F(\theta)}^1 \left(\frac{\left[(1-t)^{n-i-1} t^{i-1}\right]}{(i-1)F^{-1}(t)} - \frac{\left[(1-t)^{n-i} t^{i-2}\right]}{(n-i)F^{-1}(t)} \right) dt \\
&= \frac{(n-1)!}{(i-2)!(n-i-1)!} \int_{F(\theta)}^1 \frac{\left[(1-t)^{n-i-1} t^{i-2}\right]}{F^{-1}(t)} \left(\frac{t}{(i-1)} - \frac{1-t}{n-i} \right) dt \\
&= \binom{n-1}{i-1} \int_{F(\theta)}^1 \frac{\left[(1-t)^{n-i-1} t^{i-2}\right]}{F^{-1}(t)} \left((n-1)t - (i-1)\right) dt
\end{aligned}$$

For $i = 1$, we have that

$$m_1(\theta) = (n-1) \int_{F(\theta)}^1 \frac{\left[(1-t)^{n-2}\right]}{F^{-1}(t)} dt$$

Now we check that the second order condition is satisfied. To simplify calculations, let $g^{-1}(x) = t$ so the agent of type θ is imitating an agent of type t. Then, the foc can be written as:

$$f(t) \sum_{i=1}^n v_i \binom{n-1}{i-1} \left[(1-F(t))^{n-i}(i-1)F(t)^{i-2} - F(t)^{i-1}(n-i)(1-F(t))^{n-i-1}\right] - \theta g'(t) = 0$$

or alternatively

$$-\sum_{i=1}^{n-1} (v_i - v_{i+1}) \binom{n-1}{i-1} (n-i) f(t) \left[(1-F(t))^{n-i-1} F(t)^{i-1}\right] - \theta g'(t) = 0$$

Let $V(t) = -\sum_{i=1}^{n-1}(v_i - v_{i+1})\binom{n-1}{i-1}(n-i)f(t)\left[(1-F(t))^{n-i-1}F(t)^{i-1}\right]$

Then, the foc is that $V(t) = tg'(t)$ and so $V'(t) = tg''(t) + g'(t)$. Taking the derivative of lhs of the foc wrt t gives

$$V'(t) - \theta g''(t) = V'(t) - \theta\frac{(V'(t) - g'(t))}{t}$$

At $t = \theta$, we get that this equals $g'(\theta)$ which we know is < 0. Thus, the second order condition is satisfied.

$\square$

Theorem 2. *Consider a setting with n agents and prior F. Suppose $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ are two prize vectors. Then, we have the following:*

1. *if $v_1 > w_1$ and $v_{-1} = w_{-1}$, then $g_v(\theta)$ first order stochastically dominates $g_w(\theta)$*
2. *if $v_j > w_j$ for $j \in \{2, \ldots, n-1\}$, $v_{-j} = w_{-j}$, and F is such that the density f is increasing, then $g_v(\theta)$ second order stochastically dominates $g_w(\theta)$*
3. *if $v_j > w_j$ for $j \in \{2, \ldots, n-1\}$, $v_{-j} = w_{-j}$, and F is such that the density f is decreasing, then $-g_v(\theta)$ second order stochastically dominates $-g_w(\theta)$*

Proof. Let's prove each of the claims in order.

For the first claim, we know from lemma 1 that $g_v(\theta) > g_w(\theta)$ for all $\theta \in [0,1]$ since $m_1(\theta) > 0$ for all $\theta \in [0,1]$. As a result,

$$\mathbb{P}[g_w(\theta) \le x] = \mathbb{P}[\theta \ge g_w^{-1}(x)] \ge \mathbb{P}[\theta \ge g_v^{-1}(x)] = \mathbb{P}[g_v(\theta) \le x]$$

Note that the result also follows from Theorem 1.A.17 in [55].

Now let's prove the second claim. First we'll show that $\mathbb{E}[m_j(\theta)] \ge 0$. Note that $\mathbb{E}[m_j(\theta)] = \int_0^1 m_j(\theta)f(\theta)d\theta$. For any $j \in \{2, \ldots, n-1\}$, we know from Lemma 1 that there exists t_j^1 such that $m_j(\theta) = \begin{cases} <= 0 & \text{if } \theta \le t_j^1 \\ > 0 & \text{otherwise} \end{cases}$

Using this, we have that

$$\begin{aligned}
\mathbb{E}[m_j(\theta)] &= \int_0^1 m_j(\theta)f(\theta)d\theta \\
&= \int_0^{t_j^1} m_j(\theta)f(\theta)d\theta + \int_{t_j^1}^1 m_j(\theta)f(\theta)d\theta \\
&\ge \int_0^{t_j^1} m_j(\theta)f(t_j^1)d\theta + \int_{t_j^1}^1 m_j(\theta)f(t_j^1)d\theta \\
&= f(t_j^1)\int_0^1 m_j(\theta)d\theta
\end{aligned}$$

$$= 0$$

. It follows then that $\mathbb{E}[g_v(\theta)] \geq \mathbb{E}[g_w(\theta)]$. In addition, we know from lemma 1 that there exists t_j^1 such that $g_v(\theta) - g_w(\theta) = \begin{cases} < 0 & \text{if } \theta < t_j^1 \\ = 0 & \text{if } \theta = t_j^1 \\ > 0 & \text{otherwise} \end{cases}$

Let $G_v(x) = \mathbb{P}[g_v(\theta) \leq x]$ denote the cdf of effort under prize vector $\mathbf{v}$. Then, from above, we have that

$$G_v(x) - G_w(x) = \begin{cases} < 0 & \text{if } x < g_v(t_j^1) \\ = 0 & \text{if } x = g_v(t_j^1) \\ > 0 & \text{otherwise} \end{cases}$$

Thus, we have that $\mathbb{E}[g_v(\theta)] \geq \mathbb{E}[g_w(\theta)]$ and also the sign of $G_v(x) - G_w(x)$ changes exactly once from $-$ to $+$ as x increases. It follows then from Theorem 4.A.22 in [55] that $g_v(\theta)$ second order stochastically dominates $g_w(\theta)$.

The argument for the case of decreasing density is analagous. □

Lemma 1. *For any number of agents n and prior F, the following hold:*

1. $\theta m_i(\theta) \leq 1$ *for all* $i \in \{1, \ldots, n-1\}, \theta \in [0, 1]$
2. $\lim_{\theta \to 0} \theta m_i(\theta) = 0$ *for all* $i \in \{1, \ldots, n-1\}$
3. $\int_0^1 m_1(\theta) = 1$ *and* $\int_0^1 m_i(\theta) = 0$ *for* $i \in \{2, n-1\}$
4. $m_1(\theta) > 0$ *for all* θ *and monotone decreasing*
5. *For* $i \in \{2, n-1\}$*, there exist* $t_i^1 < t_i^2$ *such that*

$$m_i(\theta) = \begin{cases} <= 0 & \textit{if } \theta \leq t_i^1 \\ > 0 & \textit{otherwise} \end{cases}$$

$$m_i'(\theta) = \begin{cases} >= 0 & \textit{if } \theta \leq t_i^2 \\ < 0 & \textit{otherwise} \end{cases}$$

Proof. We'll prove the properties one by one.

The first property provides an upper bound on the marginal effect of any prize i on the effort of agent of type θ: $m_i(\theta) \leq \frac{1}{\theta}$. To prove this, consider a case where prize i increases by Δ. Then from the characterization in Theorem 1, it follows that an agent of type θ will increase its effort by $\Delta m_i(\theta)$. This would correspond to an increase in cost of $\Delta \theta m_i(\theta)$. But the overall gain from the increased prize is $\leq \Delta$. Since the change in cost must be less than the gain in prize, we get a simple bound of $\theta m_i(\theta) \leq 1$ for all θ.

The second property essentially says that the cost of the most efficient agent goes to zero. We'll first prove this property for $i \in \{2, n-1\}$ by squeeze theorem. First let's obtain an upper bound:

$$
\begin{aligned}
\theta m_i(\theta) &= \binom{n-1}{i-1}\theta \int_{F(\theta)}^{1} \frac{\left[(1-t)^{n-i-1}t^{i-2}\right]}{F^{-1}(t)}\left((n-1)t-(i-1)\right)dt \\
&\leq \binom{n-1}{i-1}\theta \int_{F(\theta)}^{1} \frac{\left[(1-t)^{n-i-1}t^{i-2}\right]}{\theta}\left((n-1)t-(i-1)\right)dt \\
&= \binom{n-1}{i-1} \int_{F(\theta)}^{1} (1-t)^{n-i-1}t^{i-2}\left((n-1)t-(i-1)\right)dt
\end{aligned}
$$

At $\theta = 0$, the integral equals $(n-1)\beta(i, n-i) - (i-1)\beta(i-1, n-i) = 0$

For the lower bound, we have

$$
\theta m_i(\theta) \geq \binom{n-1}{i-1}\theta \int_{F(\theta)}^{1} (1-t)^{n-i-1}t^{i-2}\left((n-1)t-(i-1)\right)dt
$$

which goes to 0 as $\theta \to 0$. Thus, $\lim_{\theta\to 0} \theta m_i(\theta) = 0$ for $i \in \{2, 3, \ldots, n-1\}$

Now we'll prove the limit is 0 for $i = 1$. We have

$$
\theta m_1(\theta) = \theta(n-1) \int_{F(\theta)}^{1} \frac{\left[(1-t)^{n-2}\right]}{F^{-1}(t)}dt
$$

. If the integral is finite, we are done. If it is infinite, we can apply L-Hospital's rule to get

$$
\lim_{\theta\to 0} \frac{m_1(\theta)}{\frac{1}{\theta}} = \lim_{\theta\to 0} -\theta^2 m_1'(\theta) = \lim_{\theta\to 0} \theta^2 (n-1) f(\theta) \frac{(1-F(\theta))^{n-2}}{\theta} = 0
$$

Now we prove the third property. By Leibniz rule, for $i \geq 2$, we have

$$
m_i'(\theta) = -\binom{n-1}{i-1} \frac{\left[(1-F(\theta))^{n-i-1}F(\theta)^{i-2}\right]}{\theta}\left((n-1)F(\theta)-(i-1)\right)f(\theta)
$$

Since $\lim_{\theta\to 0} \theta m_i(\theta) = 0$, we have that $\int_0^1 \theta m_i'(\theta) = -\int_0^1 m_i(\theta)d\theta$

From above, we have that

$$
\begin{aligned}
\int_0^1 \theta m_i'(\theta)d\theta &= -\int_0^1 \theta \binom{n-1}{i-1} \frac{\left[(1-F(\theta))^{n-i-1}F(\theta)^{i-2}\right]}{\theta}\left((n-1)F(\theta)-(i-1)\right)f(\theta)d\theta \\
&= -\binom{n-1}{i-1} \int_0^1 \left[(1-t)^{n-i-1}t^{i-2}\right]\left((n-1)t-(i-1)\right)dt \\
&= 0
\end{aligned}
$$

Thus, we get that $\int_0^1 m_i(\theta)d\theta = 0$ for $i \geq 2$. For $i = 1$, we have that

$$
m_1(\theta) = (n-1) \int_{F(\theta)}^{1} \frac{\left[(1-t)^{n-2}\right]}{F^{-1}(t)}dt
$$

so that $m_1'(\theta) = -(n-1)\dfrac{(1-F(\theta))^{n-2}}{\theta}f(\theta)$ and thus, $\int_0^1 \theta m_1'(\theta)d\theta = -1$. This gives that $\int_0^1 m_1(\theta)d\theta = 1$.

The fourth property follows from the fact that $m_1'(\theta) < 0$ and $m_1(1) = 0$.

For the last property, we can use the expression for $m_i'(\theta)$ to get that $t_i^2 = F^{-1}(\frac{i-1}{n-1})$. The claim on existence of $t_i^1 < t_i^2$ then follows from the fact that $\int_0^1 m_i(\theta)d\theta = 0$ for $i \in \{2, \dots, n-1\}$.

$\square$

Theorem 3. *Consider a setting with n agents and prior $F(\theta) = \theta^p$. Suppose $\mathbf{v} = (v_1, \dots, v_n)$ and $\mathbf{w} = (w_1, \dots, w_n)$ are two prize vectors such that $\mathbf{v}$ is more competitive than $\mathbf{w}$. Then, we have the following:*

1. *if $p \geq 1$, then*
$$\mathbb{E}[g_v(\theta)] \geq \mathbb{E}[g_w(\theta)]$$

2. *if $\frac{1}{2} \leq p \leq 1$, $v_1 = w_1, v_n = w_n$, then*
$$\mathbb{E}[g_v(\theta)] \leq \mathbb{E}[g_w(\theta)]$$

3. *if $p \geq \frac{1}{2}$ and $v_n = w_n$, then*
$$\mathbb{E}[g_v(\theta_{max})] \leq \mathbb{E}[g_w(\theta_{max})]$$

Proof. First, we show that the expected marginal effect of prize v_i is

$$\mathbb{E}[m_i(\theta)] = \binom{n-1}{i-1}\beta(i-\frac{1}{p}, n-i)\frac{(n-i)(p-1)}{np-1}$$

This follows from the following calculations:

$$\begin{aligned}
\mathbb{E}[m_i(\theta)] &= -\int_0^1 F(\theta)m_i'(\theta)d\theta \\
&= \binom{n-1}{i-1}\int_0^1 \frac{[(1-F(\theta))^{n-i-1}F(\theta)^{i-1}]}{\theta}\left((n-1)F(\theta)-(i-1)\right)f(\theta)d\theta \\
&= \binom{n-1}{i-1}\int_0^1 \frac{[(1-t)^{n-i-1}t^{i-1}]}{F^{-1}(t)}\left((n-1)t-(i-1)\right)dt
\end{aligned}$$

For $F(\theta) = \theta^p$,

$$\mathbb{E}[m_i(\theta)] = \binom{n-1}{i-1}\left((n-1)\beta(i+1-\frac{1}{p}, n-i) - (i-1)\beta(i-\frac{1}{p}, n-i)\right)$$

$$= \binom{n-1}{i-1}\left((n-1)\beta(i-\frac{1}{p}, n-i)\frac{i-\frac{1}{p}}{n-\frac{1}{p}} - (i-1)\beta(i-\frac{1}{p}, n-i)\right)$$
$$= \binom{n-1}{i-1}\beta(i-\frac{1}{p}, n-i)\left((n-1)\frac{i-\frac{1}{p}}{n-\frac{1}{p}} - (i-1)\right)$$
$$= \binom{n-1}{i-1}\beta(i-\frac{1}{p}, n-i)\frac{(n-i)(p-1)}{np-1}$$

Now observe that

$$\frac{\mathbb{E}[m_{i+1}(\theta)]}{\mathbb{E}[m_i(\theta)]} = \frac{n-i}{i}\frac{i-\frac{1}{p}}{n-i-1}\frac{n-i-1}{n-i} = \frac{i-\frac{1}{p}}{i} < 1$$

Note that for $p \geq 1$, these marginal effects are positive and thus, the marginal effect of prize i is decreasing in i. This implies that the change in expected effort from increasing any prize $i \in [n-1]$ is positive and the change from increasing v_i is greater than that from increasing v_j for any $i < j$. Since w can be obtained from v via a sequence of Robinhood operations which involve replacing v_i by $v_i - \epsilon$ and v_j by $v_j + \epsilon$, each of which reduces expected effort, we get that the expected effort under w will be lesser than the expected effort under v when $p \geq 1$. So if v is more competitive than w, then $\mathbb{E}[g_v(\theta)] \geq \mathbb{E}[g_w(\theta)]$. For $\frac{1}{2} \leq p \leq 1$, the expected marginal effects are < 0 but the ratio is still < 1. Thus, the expected marginal effects are actually increasing in i and so we get the inequality in the second item.

For the inequality in the third item, we first show that

$$\mathbb{E}[m_i(\theta_{max})] = \binom{n-1}{i-1}\beta\left(n+i-1-\frac{1}{p}, n-i\right)\frac{(n-i)(np-1)}{2np-p-1}$$

for $i \in \{1, 2, \ldots, n-1\}$.

$$\begin{aligned}
\mathbb{E}[m_i(\theta_{max})] &= \int_0^1 m_i(\theta) nF(\theta)^{n-1} f(\theta) d\theta \\
&= \binom{n-1}{i-1}\int_0^1 \frac{[(1-F(\theta))^{n-i-1}F(\theta)^{i-2}]}{\theta}\left((n-1)F(\theta) - (i-1)\right)F(\theta)^n f(\theta) d\theta \\
&= \binom{n-1}{i-1}\int_0^1 \frac{[(1-t)^{n-i-1}t^{n+i-2}]}{F^{-1}(t)}\left((n-1)t - (i-1)\right) dt
\end{aligned}$$

For the case of $F(\theta) = \theta^p$, we get that

$$\mathbb{E}[m_i(\theta_{max})] = \binom{n-1}{i-1}\left((n-1)\beta\left(n+i-\frac{1}{p}, n-i\right) - (i-1)\beta\left(n+i-1-\frac{1}{p}, n-i\right)\right)$$

$$= \binom{n-1}{i-1}\beta\left(n+i-1-\frac{1}{p}, n-i\right)\left((n-1)\frac{n+i-1-\frac{1}{p}}{2n-1-\frac{1}{p}} - (i-1)\right)$$
$$= \binom{n-1}{i-1}\beta\left(n+i-1-\frac{1}{p}, n-i\right)\frac{(n-i)(np-1)}{2np-p-1}$$

Now observe that
$$\frac{\mathbb{E}[m_{i+1}(\theta_{max})]}{\mathbb{E}[m_i(\theta_{max})]} = \frac{n+i-1-\frac{1}{p}}{i} > 1$$
Thus, the marginal effect of prize i is increasing in i. Again, since w can be obtained from v via a sequence of Robinhood operations which involve replacing v_i by $v_i - \epsilon$ and v_j by $v_j + \epsilon$, each of which increases expected minimum effort, we get that the expected minimum effort under w will be greater than that under v. It follows that if v is more competitive than w and both have the same last prize, then $\mathbb{E}[g_v(\theta_{max})] \leq \mathbb{E}[g_w(\theta_{max})]$ □

B Proofs for Section 4 (Applications)

Theorem 5. *Suppose there are n agents with utility $u(v) = v^r$ and the prior cdf on marginal costs is $F(\theta) = \theta^p$ with $p \geq 1$. The prize structure that maximizes expected effort is*

$$v(r) = (v_1(r), c_2^{\frac{1}{1-r}} v_1(r), \dots c_{n-1}^{\frac{1}{1-r}} v_1(r), 0)$$

where
$$c_i = \frac{\mathbb{E}[m_i(\theta)]}{\mathbb{E}[m_1(\theta)]} < 1$$
and $v_1(r)$ is such that
$$\sum v_i(r) = B$$
Further, if $1 \geq r > r' > 0$, then $v(r)$ is more competitive than $v(r')$.

Proof. From corollary 1, we know that the Bayes-Nash equilibrium function takes the form

$$g_v(\theta) = \sum_{i=1}^{n} m_i(\theta) u(v_i)$$

where,
$$m_i(\theta) = \binom{n-1}{i-1}\int_{F(\theta)}^{1} \frac{[(1-t)^{n-i-1}t^{i-2}]}{F^{-1}(t)}\left((n-1)t - (i-1)\right) dt$$
for $i \in [n-1]$ and $m_n(\theta) = -\sum_{i=1}^{n-1} m_i(\theta)$. Given this form of the equilibrium function, the problem is

$$\max_{\mathbf{v}} \sum_{i=1}^{n-1} u(v_i)\mathbb{E}[m_i(\theta)]$$

such that $\sum_{i=1}^{n-1} v_i = B$.

Check that the solution will satisfy the equation

$$V_1(r)\left[1 + \sum_{i=2}^{n-1} c_i^{\frac{1}{1-r}}\right] = B$$

where $c_i = \frac{\mathbb{E}[m_i(\theta)]}{\mathbb{E}[m_1(\theta)]} < 1$ and $c_i > c_{i+1}$ for all i. Note that c_i does not depend on r.

Let $f_k(r) = V_1(r)\left[1 + \sum_{i=2}^{k} c_i^{\frac{1}{1-r}}\right]$.

I want to show that $f_k'(r) > 0$ for all k.

If I can show $f_k'(r)$ is single peaked in k, that would imply the result since $f_n(r) = 0$.

Check that $V_1'(r) = \dfrac{-\left[\sum_{i=2}^{n-1} c_i^{\frac{1}{1-r}}\log(c_i)\right]V_1^2(r)}{(1-r)^2 B}$ Plugging it in, we get

$$\begin{aligned}
f_k'(r) &= V_1(r)\left[\frac{1}{(1-r)^2}\sum_{i=2}^{k} c_i^{\frac{1}{1-r}}\log(c_i)\right] + V_1'(r)\left[1 + \sum_{i=2}^{k} c_i^{\frac{1}{1-r}}\right] \\
&= V_1(r)\left[\frac{1}{(1-r)^2}\sum_{i=2}^{k} c_i^{\frac{1}{1-r}}\log(c_i)\right] - \frac{\left[\sum_{i=2}^{n-1} c_i^{\frac{1}{1-r}}\log(c_i)\right]V_1^2(r)}{(1-r)^2 B}\left[1 + \sum_{i=2}^{k} c_i^{\frac{1}{1-r}}\right] \\
&= \frac{V_1(r)}{(1-r)^2}\sum_{i=2}^{k} c_i^{\frac{1}{1-r}}\log(c_i)\left[1 - \frac{V_1(r)}{B}\left(1 + \sum_{i=2}^{k} c_i^{\frac{1}{1-r}}\right)\right] - \frac{V_1^2(r)}{B(1-r)^2}\sum_{i=k+1}^{n-1} c_i^{\frac{1}{1-r}}\log(c_i)\left[1 + \sum_{i=2}^{k} c_i^{\frac{1}{1-r}}\right] \\
&= \frac{V_1(r)}{B(1-r)^2}\sum_{i=2}^{k} c_i^{\frac{1}{1-r}}\log(c_i)\left[B - f_k(r)\right] - \frac{V_1(r)f_k(r)}{B(1-r)^2}\sum_{i=k+1}^{n-1} c_i^{\frac{1}{1-r}}\log(c_i) \\
&= \frac{V_1(r)}{B(1-r)^2}\left(B\sum_{i=2}^{k} c_i^{\frac{1}{1-r}}\log(c_i) - f_k(r)\sum_{i=2}^{n-1} c_i^{\frac{1}{1-r}}\log(c_i)\right)
\end{aligned}$$

To show that the term inside the bracket is positive, we basically need to show that for any decreasing sequence $1 \geq d_1 > d_2 > \dots d_n > 0$, we have that

$$h(k) = \sum_{i=1}^{n} d_i \sum_{i=1}^{k} d_i\log(d_i) - \sum_{i=1}^{k} d_i \sum_{i=1}^{n} d_i\log(d_i) \geq 0$$

for any $k \in [n]$

Observe that

$$\begin{aligned}
\Delta(k) &= h(k+1) - h(k) \\
&= d_{k+1}\log(d_{k+1})\sum_{i=1}^{n} d_i - d_{k+1}\sum_{i=1}^{n} d_i \log(d_i) \\
&= d_{k+1}\left(\log(d_{k+1})\sum_{i=1}^{n} d_i - \sum_{i=1}^{n} d_i \log(d_i)\right)
\end{aligned}$$

Since d_k is a decreasing sequence, it follows that if $\Delta(k) < 0$, then $\Delta(j) < 0$ for all $j > k$. But observe that $h(n) = 0$. So we just need to show that $h(1) > 0$ which is obvious. $\square$