Prizes and effort in contests with private information

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Introduction

- Contests involve agents exerting costly effort to win valuable prizes
- Examples: sporting events, classroom settings, politics, etc.
- How does the structure of the contest influence effort?
    - What happens if designer puts in more money into the contest?
    - What happens if designer increases competitiveness of the contest?

Model

- n agents
- $\theta_i \in [0,1]$: marginal cost of effort for agent i
- F: distribution of marginal costs ($\lim_{\theta \to 0} F(\theta) f(\theta) = 0$)
- Designer chooses prize vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ such that $v_1 \geq v_2 \geq \cdots \geq v_n$
- Agents simultaneously choose their effort level $e_1, e_2, \ldots, e_n$
- Agents ranked according to their efforts, awarded corresponding prizes
- If agent i wins prize v_j, its payoff is

$$v_j - \theta_i e_i$$

Model contd.

- Symmetric Bayes-Nash equilibrium function $g_{\mathbf{v}} : [0,1] \to \mathbb{R}_+$
- Designer's preferences over contests represented by $\succeq$ so that

$$\mathbf{v} \succeq \mathbf{w} \iff \mathbb{E}[g_{\mathbf{v}}(\theta)] \geq \mathbb{E}[g_{\mathbf{w}}(\theta)]$$

Equilibrium

- $p_i(t)$: probability that $X \sim Bin(n-1,t)$ takes the value $i-1$

Lemma 1 (Moldovanu and Sela [2001]).

The contest $(n, F, \mathbf{v})$ has a unique symmetric equilibrium and it is given by

$$g_{\mathbf{v}}(\theta) = \sum_{i=1}^{n} v_i m_i(\theta)$$

where

$$m_i(\theta) = -\int_{F(\theta)}^{1} \frac{p_i'(t)}{F^{-1}(t)} dt$$

Some observations

1. For any agent type θ, the sum of marginal effects is 0

$$\sum_{i=1}^{n} m_i(\theta) = -\int_{F(\theta)}^{1} \frac{\sum_{i=1}^{n} p_i'(t)}{F^{-1}(t)} dt = 0$$

2. Equilibrium is monotone decreasing

$$g_{\mathbf{v}}'(\theta) = \frac{f(\theta)}{\theta} \sum_{i=1}^{n} v_i p_i'(F(\theta)) \leq 0$$

3. Equilibrium characterization holds more generally

$$g_{\mathbf{v}}(\theta) = \sum_{i=1}^{n} u(v_i) m_i(\theta)$$

Preview

- Effect of putting more money into contest: depends on F
- Effect of increasing competition: depends on F
- Application: grading contests

Effect of putting more money into the contest

Theorem 2.

Suppose $\mathbf{v}, \mathbf{w}$ *are two prize vectors such that* $v_i > w_i$ *for some prize* i *and* $v_j = w_j$ *for* $j \neq i$*.*

1. *If* $i = 1$*, then* $\mathbf{v} \succeq \mathbf{w}$*.*
2. *If* $i = n$*, then* $\mathbf{w} \succeq \mathbf{v}$*.*
3. *If* $i \in \{2, \ldots, n-1\}$*, then the comparison depends on* f*.*
   - *If* f *is increasing,* $\mathbf{v} \succeq \mathbf{w}$*.*
   - *If* f *is decreasing,* $\mathbf{w} \succeq \mathbf{v}$*.*

Since $g_{\mathbf{v}}(\theta) = \sum_{i=1}^{n} v_i m_i(\theta)$, effect of increasing prize i depends on

$$m_i(\theta) = -\int_{F(\theta)}^{1} \frac{p_i'(t)}{F^{-1}(t)} dt$$

Marginal effect functions

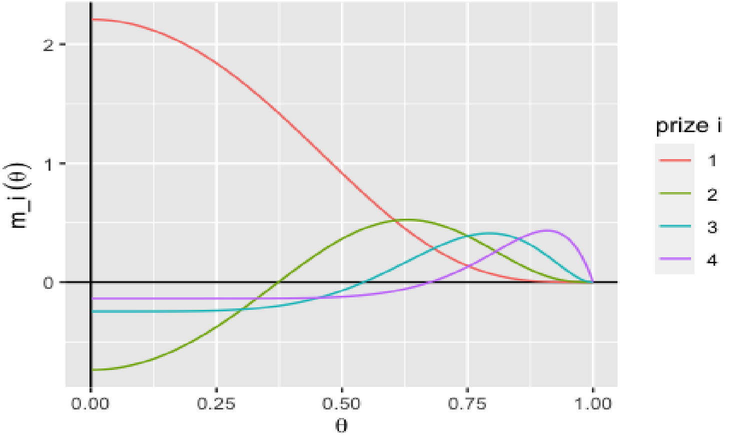

Figure: The marginal effect of prizes on effort for $n = 5$ and $F(\theta) = \theta^3$.

Property 1

Lemma 3.

The marginal effect functions $m_i(\theta)$ satisfy:

1. $m_1(\theta) \geq 0$ *for all* $\theta \in [0,1]$
2. $m_n(\theta) \leq 0$ *for all* $\theta \in [0,1]$
3. $m_i(\theta) = \begin{cases} < 0 & \text{if } \theta \leq t_i \\ \geq 0 & \text{otherwise} \end{cases}$ *where* $t_i \in (0,1)$ *for* $i \in \{2, \ldots, n-1\}$

The lemma follows from properties of $p_i'(t)$.

Property 2

Lemma 4.

The marginal effect functions $m_i(\theta)$ satisfy:

1. $\int_0^1 m_1(\theta)d\theta = 1$
2. $\int_0^1 m_n(\theta)d\theta = -1$
3. $\int_0^1 m_i(\theta)d\theta = 0$ *for* $i \in \{2, \dots, n-1\}$

- $\int_0^1 m_i(\theta)d\theta = p_i(0) - p_i(1)$
- Relies on possibility of agents with negligible marginal costs of effort
- For any F, we have $\int_0^1 g_{\mathbf{v}}(\theta)d\theta = v_1 - v_n$

Effect of increasing prizes

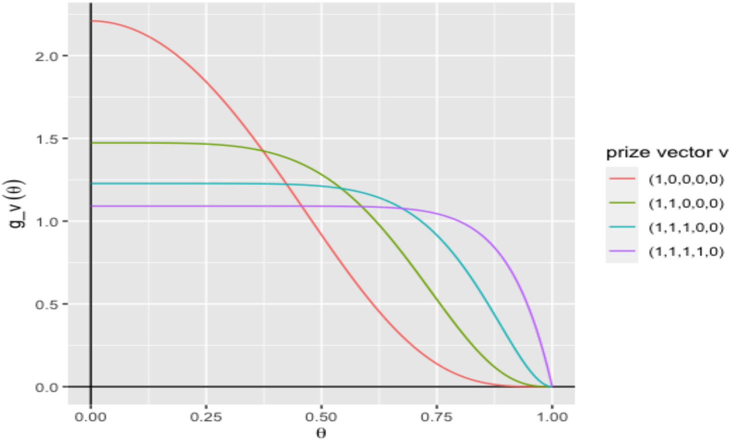

Figure: The effect of increasing prizes on equilibrium effort for $n = 5$ and $F(\theta) = \theta^3$.

Competition

- $\mathbf{v}$ is more competitive than $\mathbf{w}$ if $\mathbf{v}$ majorizes $\mathbf{w}$ i.e.
  - $\sum_{i=1}^{k} v_i \geq \sum_{i=1}^{k} w_i$ for all $k \in [n]$ and
  - $\sum_{i=1}^{n} v_i = \sum_{i=1}^{n} w_i$
- $\mathbf{w}$ obtained from $\mathbf{v}$ by sequence of value transfers ($i \to j$ where $i < j$)
- Compare $\mathbb{E}[m_i(\theta)]$ with $\mathbb{E}[m_j(\theta)]$
- $\mathbb{E}[m_1(\theta)] \geq \mathbb{E}[m_j(\theta)]$ for all $j \in \{2, \ldots, n\}$ and all F (Moldovanu and Sela [2001])
- Focus on a parametric class of distributions $F(\theta) = \theta^p$ with $p > \frac{1}{2}$

Effect of competition on effort

Theorem 5.

Suppose $F(\theta) = \theta^p$ and $\mathbf{v}$ and $\mathbf{w}$ are two prize vectors such that $\mathbf{v}$ is more competitive than $\mathbf{w}$.

1. *If $p > 1$, then $\mathbf{v} \succeq \mathbf{w}$.*
2. *If $\frac{1}{2} < p < 1$, and $v_1 = w_1, v_n = w_n$, then $\mathbf{w} \succeq \mathbf{v}$.*
3. *If $p > \frac{1}{2}$ and $v_n = w_n$, then $\mathbb{E}[g_{\mathbf{v}}(\theta_{max})] \leq \mathbb{E}[g_{\mathbf{w}}(\theta_{max})]$.*

Effect of competition

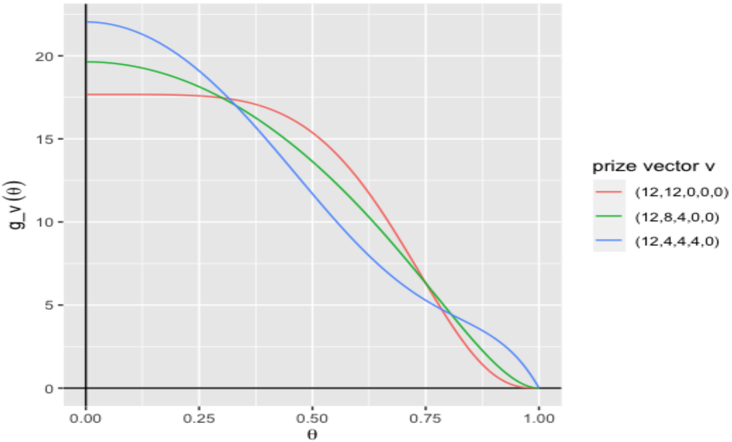

Figure: The effect of competition on equilibrium effort for $n = 5$ and $F(\theta) = \theta^3$.

Application 1: Grading contests

- Defined by an increasing sequence of numbers $s = (s_1, s_2, \ldots, s_k)$ with $s_k = n$ so that top s_1 agents get A, next $s_2 - s_1$ get B and so on
- Value of a grade determined by the information it reveals about the agent's type (productivity)
- Given a decreasing wage function $w : [0,1] \to \mathbb{R}$, the contest $s^* = (1, 2, \ldots, n)$ induces prize vector $\mathbf{v} = (v_1, \ldots, v_n)$ where

$$v_i = \mathbb{E}[w(\theta) | \theta = \theta^n_{(i)}].$$

- Contest $s = (s_1, s_2, \ldots, s_k)$ induces prize vector $v(s)$ where

$$v(s)_i = \frac{v_{s_{j-1}+1} + v_{s_{j-1}+2} + \cdots + v_{s_j}}{s_j - s_{j-1}}$$

and j is such that $s_{j-1} < i \leq s_j$

Relation between information and effort

- s is more informative than s' if s' is a subseq. of s.
- $\mathbf{v}(s)$ is more competitive than $\mathbf{v}(s')$

Corollary 6.

Suppose $F(\theta) = \theta^p$ and grading scheme s is more informative than s'.

1. *If $p > 1$, then s induces greater expected effort than s'.*
2. *If $\frac{1}{2} < p < 1$, and $v(s)_1 = v(s')_1$, $v(s)_n = v(s')_n$, then s' induces greater expected effort than s.*
3. *If $p > \frac{1}{2}$ and $v(s)_n = v(s')_n$, then s' induces greater expected minimum effort than s.*

Summary

The overall ranking of contests, and thus, the structure of optimal contests in many environments, depends in an important way on the distribution of abilities. Two natural sufficient conditions illustrate this dependence:

	Unprod. likely	Prod. likely
$\uparrow$ value of prizes $\uparrow$ competition	$\uparrow$ effort $\uparrow$ effort	$\downarrow$ effort $\downarrow$ effort
Effort max. G.C. Effort min. G.C.	$(A, B, C, \dots)$ $(A, \dots, A, B, \dots, B)$	$(A, B, B \dots, B)$ $(A, \dots, A, B, C, \dots)$
Homogeneous prizes	$n-1$ prizes	one prize
Concave utilities	$\uparrow$ concavity implies $\downarrow$ competition	winner-take-all

Literature review

- **Incomplete information environment**: Glazer and Hassin [1988], Moldovanu and Sela [2001], Zhang [2019]
- **Complete information environment**: Glazer and Hassin [1988], Barut and Kovenock [1998], Fang, Noe, and Strack [2020]
- **Grading contests**: Moldovanu and Sela [2006], Rayo [2013], Dubey and Geanakoplos [2010], Zubrickas [2015], Liu and Lu [2017], Krishna, Lychagin, Olszewski, Siegel, and Tergiman [2022]

Thank you!