Optimal number of costless prizes in contests

Sumit Goel*

April 21, 2022

Abstract

We consider an incomplete-information optimal contest design problem in which n players invest effort to win one of k costless homogeneous prizes. Assuming private marginal costs of putting in effort, we find the symmetric Bayes Nash equilibrium strategy function and show that it is a density function as long as the equilibrium effort of the most efficient agents does not grow too quickly. As corollary, we get that the expected under uniform prior is independent of number of prizes or agents. Under the power distribution prior $F(\theta) = \theta^p$ with $p > 1$, the optimal contest for maximizing both expected effort and expected minimum effort awards $k = n - 1$ prizes. The solution for maximizing expected maximum effort is typically interior. We also consider contest as a screening device and find the optimal number of prizes for maximizing the information revealed (measured by variance of posterior means) per unit expected effort. For the uniform case, we get that $k = \frac{n}{2}$ prizes is optimal and for the power distribution $F(\theta) = \theta^p$, we conjecture that the optimal number of prizes is decreasing in p so that eventually, awarding just a single prize is optimal.

1 Introduction

Contests are situations where agents compete by exerting costly effort to win one or more prizes. Examples of such contests include sports competitions, crowdsourcing, R&D race between firms, school admissions, labor markets etc. While some of these contests are natural, there are others in which the designer can carefully choose the prize structure so as to satisfy various objectives such as inducing effort or screening participants. In addition, in some of these settings, the designer may be unconstrained in its ability to award prizes. For instance, Kaggle, an online community of data scientists, organizes competitions to solve data science challenges and awards digital medals to a certain fraction of the top solutions. The gain in status and improved future career prospects make these medals valuable for the participating community and incentivize them to exert costly effort. In this paper, we consider the optimal contest design problem for such environments where the designer can

*California Institute of Technology; sgoel@caltech.edu; 0000-0003-3266-9035

costlessly award any number of prizes.

More precisely, we consider a setting where there are n agents, each with a privately known marginal cost $\theta_i \in [0,1]$ of putting in effort. The contest designer chooses the number of prizes k, the agents simultaneously decide how much effort they put in and the k agents who put in the highest effort are awarded with a prize each. In this setup, we find the symmetric Bayes-Nash equilibrium strategy function and show that it is actually a density function as long as $\lim_{\theta \to 0} g(\theta)\theta = 0$. We then consider the designer's problem of finding the optimal number of prizes k for two different objectives.

First, we consider the objective of maximizing the effort put in by the participants which is standard in the optimal contest design literature. For the uniform prior, it follows from the density property of the equilibrium function that the expected effort is independent of the number of prizes or even agents. For the power distribution prior $F(\theta) = \theta^p$ with $p \geq 1$, we find that awarding a prize to all but the agent who puts in the least effort is optimal for maximizing both expected effort and expected minimum effort. This is perhaps not very intuitive as one would expect that the optimal number of prizes shouldn't be too high or too low. While it is true that the marginal gain in effort is decreasing in the number of prizes, we find that it is always positive. For maximizing expected maximum effort, we find that the optimal number of prizes is typically in the interior.

We also take the view of contest as a screening device and consider a designer who wants to choose the number of prizes to maximize the information revealed by the contest per unit expected effort put in by the agents. As motivation, consider a university that awards pass-fail certificates to students (based on their test scores) which signal their quality to the market. The university would like to award certificates so as to be informative about the quality of the students. At the same time, the university understands that the effort put in by the students is otherwise wasteful and therefore, would like to design the test so that this wasteful effort is minimized (perhaps to encourage more participation). Even though we do not formally justify the exact form of the objective, we take the above ideas as motivation for considering a contest designer who wants to choose the number of prizes to maximize the variance of posterior means (measure of information revealed) per unit expected effort put in by the agents. For this objective, we find that under the uniform prior with n agents, awarding $\frac{n}{2}$ prizes is optimal. For the power distribution $F(\theta) = \theta^p$ with $p > 1$, we conjecture that the optimal number of prizes is decreasing in p and eventually, awarding just a single prize is optimal.

The problem of finding the optimal prize structure has been extensively studied in the literature. The paper most closely related to ours is Liu and Lu [25] who also consider the problem of finding the optimal number of prizes for maximizing effort but under different distributional assumptions. Under their assumptions, they find that both expected effort and expected maximum effort are single peaked in the number of prizes. They further find that maximizing expected highest effort requires a smaller number of prizes as compared to maximization of expected effort. In comparison, we find that expected effort is monotone increasing in the number of prizes though expected maximum effort continues to be single peaked in our model.

The contest design problem has been studied under various other environments. A vast literature considers the allocation of a fixed budget across multiple prizes in complete or incomplete information environments (Glazer and Hassin [16], Barut and Kovenock [2], Krishna and Morgan [20], Moldovanu and Sela [26, 27], Liu and Lu [24], Chawla et al. [4], Ales et al. [1]). Under linear effort costs, Moldovanu and Sela [26] find that it is optimal to allocate the entire budget to a single first prize. In contrast, our results suggest that when prizes are costless, it is optimal to award a prize to everyone except the agent who puts in the least effort. Fang et al. [10] obtain similar results in a complete information all-pay contest setting as they show that increased competition discourages effort. Surveys of the theoretical literature in contest theory can be found in Corchón [8], Vojnovic [33], Konrad et al. [19], Segev [31], Sisak [32].

The paper proceeds as follows. In section 2, we present the model. Section 3 contains the analysis of the Bayes-Nash equilibrium strategy function. In sections 4 and 5, we find the optimal number of prizes for maximizing effort and information per unit effort respectively. Section 6 concludes.

2 Model

There are n risk-neutral agents competing for k homogeneous prizes. The value of each prize is normalized to 1. All agents exert efforts simultaneously and the k agents who exert the highest efforts get the prizes. The cost of effort e_i for agent i is given by $\theta_i e_i$ where θ_i is the agent's private information. We assume θ_i's are independent and identically distributed random variables with cumulative distribution function $F(.)$ on $[0, 1]$ and density $f(.) > 0$.

In this setup, we are interested in finding the optimal number of prizes for two contrasting objectives. First, we'll consider a standard contest design problem where the designer wants to maximize the effort put in by the participants. In this case, we'll find the optimal number of prizes that maximizes the agents expected effort, expected maximum effort and expected minimum effort. Second, we'll consider contest as a screening device and find the number of prizes that maximizes the information revealed (measured by variance of posterior means) per unit expected effort put in by the agents. To solve these problems, we first find the Bayes-Nash equilibrium strategy of the agents.

3 Equilibrium

First, we find the symmetric Bayes-Nash equilibrium of the above game for arbitrary n, k.

Theorem 1. *In an incomplete information contest with n agents, k prizes and marginal costs θ_i drawn according to $F(.)$ on $[0,1]$, the symmetric Bayes-Nash equilibrium effort function is given by* [1]

$$g(\theta) = \binom{n-1}{k-1}(n-k)\int_{F(\theta)}^{1} \frac{t^{k-1}(1-t)^{n-k-1}}{F^{-1}(t)}dt$$

The proof proceeds by assuming that $n-1$ agents are playing $g(\theta)$ where g is a decreasing function. Then, we find a player's optimal effort level at type θ by taking the first order condition. Plugging in $g(\theta)$ in the condition gives the condition for $g(\theta)$ to be the symmetric Bayes-Nash equilibrium. Using the boundary condition $g(1) = 0$ pins down the form of the function. We then check that the second order condition is satisfied. The full proof is in the appendix.

For our optimality results, we'll focus on the class of distributions $F(\theta) = \theta^p$ with $p \geq 1$. In this case, we have $F^{-1}(t) = t^{\frac{1}{p}}$ and thus, we can use the above result to get that the equilibrium effort function takes the form:

$$\begin{aligned}
g(\theta) &= \binom{n-1}{k-1}(n-k)\int_{\theta^p}^{1} t^{k-1-\frac{1}{p}}(1-t)^{n-k-1}dt \\
&= \binom{n-1}{k-1}(n-k)\int_{0}^{1-\theta^p} t^{n-k-1}(1-t)^{k-1-\frac{1}{p}}dt \\
&= \binom{n-1}{k-1}(n-k)\mathrm{B}\left(1-\theta^p; n-k, k-\frac{1}{p}\right)
\end{aligned}$$

where $\mathrm{B}(x;a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt$ is the incomplete beta function.

For our results, we need to understand how the equilibrium function behaves as $\theta \to 0$. A simple observation is that $\lim_{\theta\to 0} \theta g(\theta) \leq 1$ holds for all n, k and F. This is because we know for sure that $g(\theta) \leq \frac{1}{\theta}$ for all θ given the form of the utility function. The next lemma shows that something stronger is true for the case where $F(\theta) = \theta^p$. We'll use this property later on when we find the optimal number of prizes for this case.

Lemma 1. *In an incomplete information contest with n agents, k prizes and marginal costs θ_i drawn according to cdf $F(\theta) = \theta^p$ with $p \geq 1$ on $[0,1]$, the symmetric Bayes-Nash equilibrium effort function satisfies* $\lim_{\theta\to 0} \theta g(\theta) = 0$.

Proof. First consider the case where $k \geq 2$. In this case, we have that

$$\theta g(\theta) = \theta\binom{n-1}{k-1}(n-k)\int_{0}^{1-\theta^p} t^{n-k-1}(1-t)^{k-1-\frac{1}{p}}dt$$

[1]More generally, if the payoff from being in top k is v more than the payoff from being in the bottom $n-k$, then the symmetric Bayes-Nash equilibrium function is simply $vg(\theta)$.

$$
\begin{aligned}
&\leq \theta \binom{n-1}{k-1}(n-k)(1-\theta^p) \\
&\to 0 \text{ as } \theta \to 0
\end{aligned}
$$

Now for $k = 1$, we have

$$
\begin{aligned}
\theta g(\theta) &= \theta \binom{n-1}{k-1}(n-k)\int_0^{1-\theta^p} t^{n-k-1}(1-t)^{k-1-\frac{1}{p}}dt \\
&= \theta(n-1)\int_0^{1-\theta^p} \frac{t^{n-2}}{(1-t)^{\frac{1}{p}}}dt \\
&\leq \theta(n-1)\int_0^{1-\theta^p} \frac{1}{(1-t)}dt \\
&= -p\theta\ln(\theta)(n-1) \\
&\to 0 \text{ as } \theta \to 0
\end{aligned}
$$

□

Figure 1 illustrates the equilibrium function for $n = 6$, $F(\theta) = \theta^{1.4}$ and $k \in \{1, 2, 3, 4, 5\}$. It suggests that as we increase the number of prizes, the most efficient agents (with low θ) put in a lower amount of effort while the least efficient agents (high θ) put in a higher amount of effort. The following lemma identifies an interesting property of the equilibrium effort function which suggests how the efforts translate as we change not just the number of prizes, but also the number of agents or the distribution $F(.)$.

Lemma 2. *For any n, k and F such that $\lim_{\theta\to 0}\theta g(\theta) = 0$, the equilibrium effort function $g(\theta)$ is a density function on $[0, 1]$. That is,*

$$
\int_0^1 g(\theta)d\theta = 1
$$

The proof proceeds by showing that $\int_0^1 \theta g'(\theta)d\theta$ always equals -1. This is fairly straightforward to show since we know

$$
\theta g'(\theta) = -\binom{n-1}{k-1} f(\theta)(n-k)F(\theta)^{k-1}(1-F(\theta))^{n-k-1}
$$

Then, using integration by parts together with the condition that $\lim_{\theta\to 0}\theta g(\theta) = 0$ gives the result.

Importantly, the above lemma leads to the following corollary:

Corollary 1. *Under the uniform distribution $F(\theta) = \theta$ on $[0, 1]$ and any number of agents n and prizes k, the expected effort put in by an agent equals 1.*

This follows from the fact that in the uniform case, the expected effort $\int_0^1 g(\theta)f(\theta)d\theta$ equals $\int_0^1 g(\theta)d\theta$ which we know from the lemma is 1. Thus, under the uniform distribution, the choice of k or even n does not matter for the expected effort.

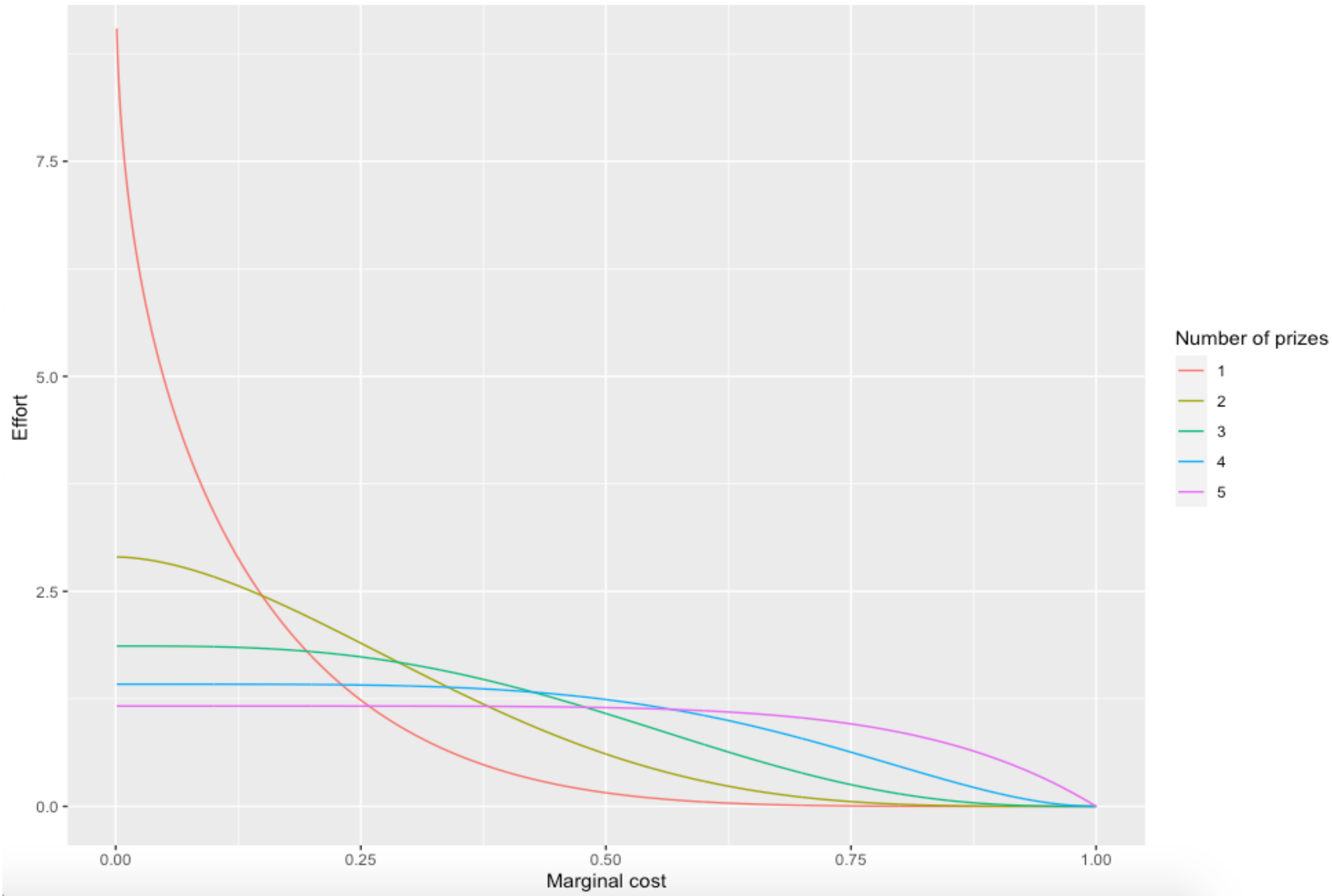

Figure 1: The equilibrium effort function for $n = 6$, $F(\theta) = \theta^{1.4}$ under different no. of prizes

4 Maximizing effort

In this section, we find the optimal number of prizes that maximizes expected effort, expected maximum effort, expected minimum effort. To do so, we will first derive general expressions for our three objectives and then focus on the case of the power distribution $F(\theta) = \theta^p$ and find the optimal number of prizes for these objectives in that case.

4.1 General F

First, let's use the definition of the Bayes-Nash equilibrium from Theorem 1 to derive expressions for expected effort, expected minimum effort and expected maximum effort.

Theorem 2. *Suppose* n, $k \in \{1, 2, \ldots, n-1\}$ *and* F *on* $[0, 1]$ *are such that* $\lim_{\theta \to 0} F(\theta)g(\theta) = 0$. *Let* f, f_l, f_m *denote the density functions of* θ, $\min_i \theta_i, \max_i \theta_i$ *respectively. Then we have that*

- *the expected effort equals*

$$\int_0^1 g(\theta)f(\theta)d\theta = \binom{n-1}{k-1}(n-k)\int_0^1 \frac{t^k(1-t)^{n-k-1}}{F^{-1}(t)}dt$$

- *the expected maximum effort equals*

$$\int_0^1 g(\theta)f_l(\theta)d\theta = \binom{n-1}{k-1}(n-k)\left[\int_0^1 \frac{t^{k-1}(1-t)^{n-k-1}}{F^{-1}(t)}dt - \int_0^1 \frac{t^{k-1}(1-t)^{2n-k-1}}{F^{-1}(t)}dt\right]$$

- *the expected minimum effort equals*

$$\int_0^1 g(\theta)f_m(\theta)d\theta = \binom{n-1}{k-1}(n-k)\int_0^1 \frac{t^{n+k-1}(1-t)^{n-k-1}}{F^{-1}(t)}dt$$

The expressions above are obtained by finding the integrals on the left hand side using integration by parts. The proof is in the appendix.

4.2 $F(\theta) = \theta^p$ with $p > 1$

In this subsection, we'll find the optimal choice of k for the case where $F(\theta) = \theta^p$ with $p > 1$.

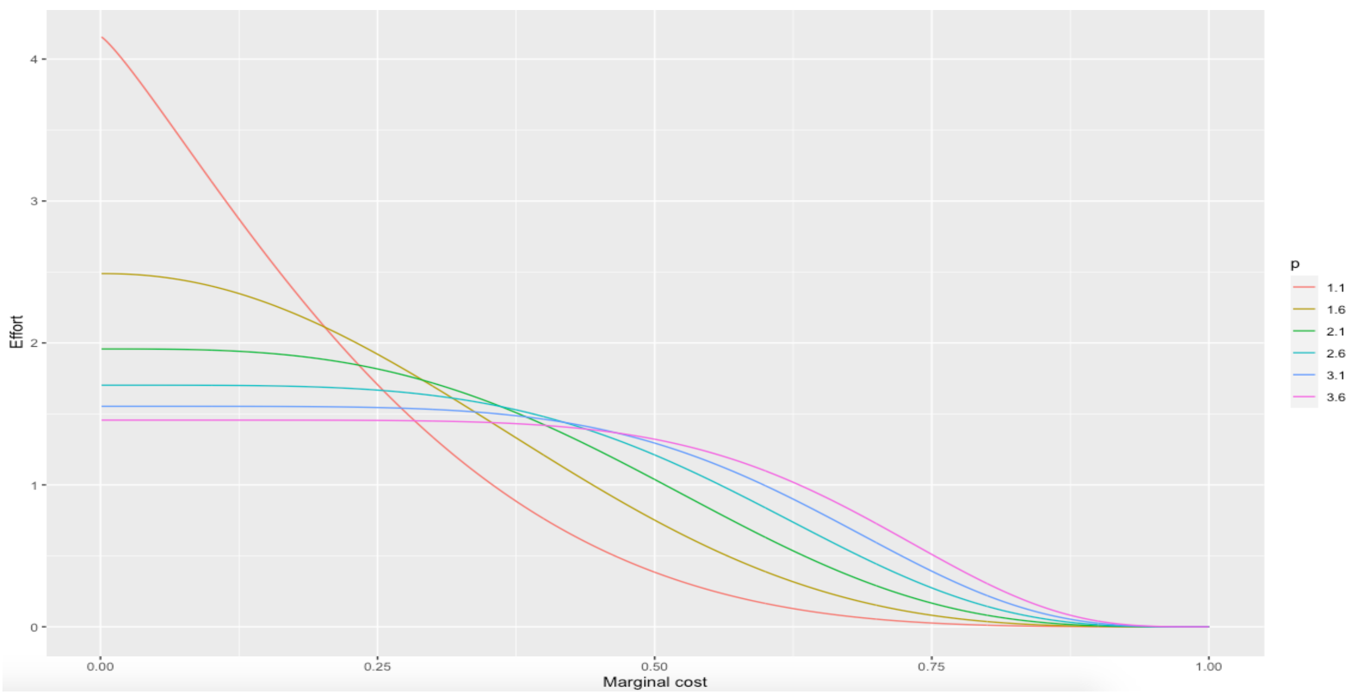

Figure 2: The equilibrium effort function for $n = 6$, $k = 2$ under different values of p

Figure 2 illustrates the equilibrium effort function when there are 6 agents, 2 prizes under different values of p. We can see that the effort put in by the most efficient agents goes down as the proportion of more inefficient agents increases while the effort of the inefficient agents goes up. Similar patterns are observed for other values of k as well.

By lemma 1, we know that the condition $\lim_{\theta \to 0} F(\theta)g(\theta) = 0$ is satisfied for $F(\theta) = \theta^p$ with $p \geq 1$. Thus, we will now use the above theorem to get the optimal number of prizes for the power distribution.

Theorem 3. *Suppose $F(\theta) = \theta^p$ for $\theta \in [0,1]$ with $p > 1$. Then,*

- *the expected effort equals*

$$\binom{n-1}{k-1}(n-k)\beta\left(k+1-\frac{1}{p}, n-k\right)$$

  and it is maximized at $k = n-1$.

- *the expected maximum effort equals*

$$\binom{n-1}{k-1}(n-k)\left[\beta(k-\frac{1}{p}, n-k) - \beta(k-\frac{1}{p}, 2n-k)\right]$$

  and it is maximized at some interior k

- *the expected minimum effort equals*

$$\binom{n-1}{k-1}(n-k)\left[\beta(n+k-\frac{1}{p}, n-k)\right]$$

  and it is maximized at $k = n-1$.

The proof proceeds by plugging in $F(\theta) = \theta^p$ in the expressions from Theorem 2 and then looking at the ratio of the objective for $k+1$ and k. For the case of expected effort and expected minimum effort, it turns out that this ratio is always bigger than 1 and thus, these objectives are maximized when $k = n-1$. Note that the ratio is decreasing in k and so the relative gain decreases as we increase k but it is always positive. The analysis for maximizing expected maximum effort is more complicated and we get that the optimal number of prizes typically admits an interior solution. Figure 3 plots the expected maximum effort as a function of number of prizes for the case of $n = 15$ agents under different values of p. We can see that the optimal number of prizes is interior and actually increases as we increase p. We believe that is generally true but we haven't been able to prove it yet.

5 Optimal screening contest

In this section, we consider a designer who wants to choose k to maximize the information revealed by the contest per unit expected effort put in by the agents. As motivation, consider

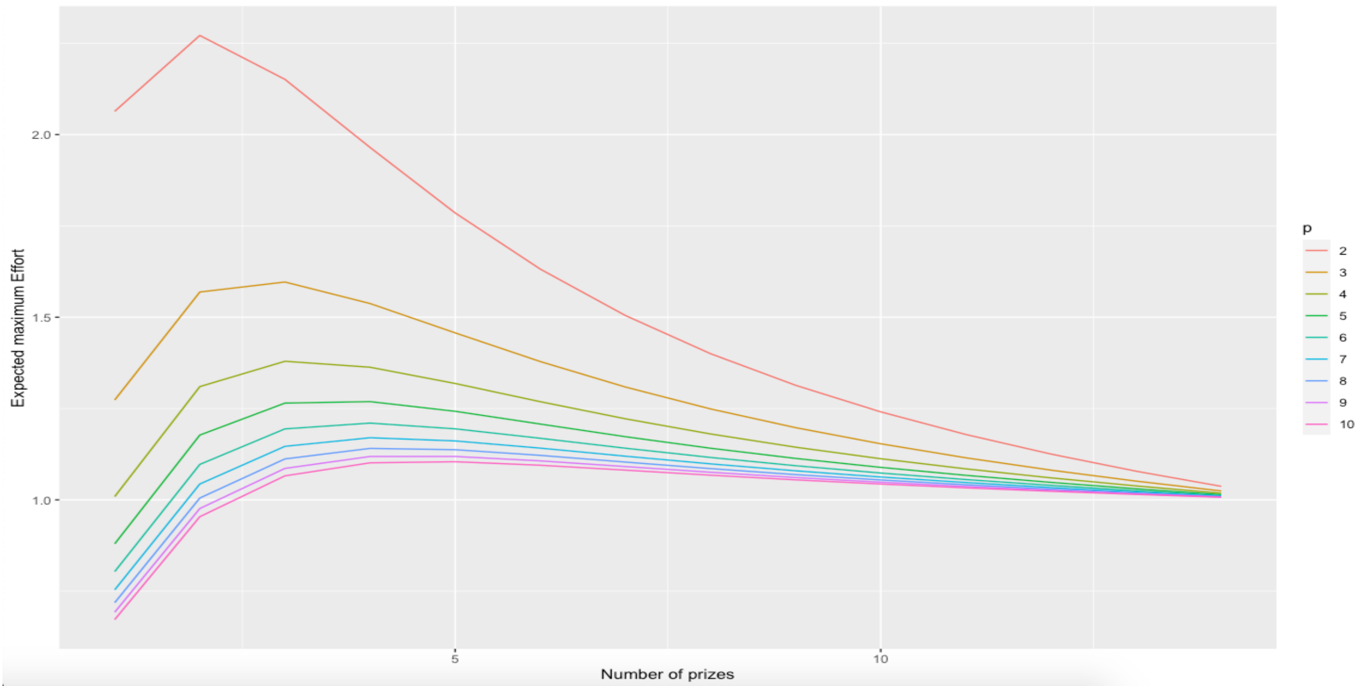

Figure 3: The expected maximum effort as a function of number of prizes for different values of p

a university that organizes pass-fail exams that students can take to signal their quality to the market. The university would like to design the test so as to be informative about the quality of the students. At the same time, the university understands that the effort put in by the students is otherwise wasteful and therefore, would like to design the test so that this wasteful effort is minimized (perhaps to encourage more participation). Even though we do not formally justify the objective, we'll take the above ideas as motivation for considering a contest designer who wants to choose the number of prizes k for a contest with n agents to maximize the variance of posterior means (measure of information revealed) per unit expected effort put in by the agents.

Formally, we have n agents so that the state $\theta \in [0,1]^n$. Each θ_i is iid according to cdf F and density f. Let $\theta^n_{(k)}$ be the kth order statistic in n samples so that $\theta^n_{(1)} < \theta^n_{(2)} \cdots < \theta^n_{(n)}$. Let $S = \{0,1\}^n$. Then, a contest that awards prizes to the top k agents generates the signal $s : \Theta \to S$ such that

$$s_i(\theta) = \begin{cases} 1 & \text{if } \theta_i \leq \theta^n_{(k)} \\ 0 & \text{otherwise} \end{cases}$$

Upon observing an agent with $s_i = 1$, the updated posterior density about the agent's type will be:

$$f_1(t) = \frac{\Pr[s_i = 1|\theta_i = t]f(t)}{\Pr[s_i = 1]} = \frac{n}{k}\Pr[\theta_{(k)}^{n-1} > t]f(t) = \frac{n}{k}\Pr[X_{(k)}^{n-1} > F(t)]f(t)$$

and the posterior density upon observing $s_i = 0$ will be

$$f_0(t) = \frac{\Pr[s_i = 0|\theta_i = t]f(t)}{\Pr[s_i = 0]} = \frac{n}{n-k}\Pr[\theta_{(k)}^{n-1} \leq t]f(t) = \frac{n}{n-k}\Pr[X_{(k)}^{n-1} \leq F(t)]f(t)$$

where X_i are $U[0,1]$ random variables. Let $\mu_i = \int_0^1 t f_i(t)dt$ denote the expected posteriors and again, let $g(\theta)$ denote the equilibrium bidding function. Then, the contest designer's objective is to find the number of prizes to maximize the variance of posterior means (a measure of information revealed) per unit expected effort induced. Formally, it is

$$\max_{k\in\{1,2,\ldots n-1\}} \frac{\frac{k}{n}\left(\mu_1 - \mu\right)^2 + \frac{n-k}{n}\left(\mu_0 - \mu\right)^2}{\int_0^1 g(\theta)f(\theta)d\theta}$$

5.1 $F(\theta) = \theta^p$

Now we focus on the case where $F(\theta) = \theta^p$ with $p \geq 1$.

Theorem 4. *Suppose $F(\theta) = \theta$ for $\theta \in [0,1]$. Then, the optimal screening contest with n agents awards $k = \lfloor \frac{n}{2} \rfloor$ prizes.*

The proof is in the appendix. We write it under the prior $F(\theta) = \theta^p$ with $p \geq 1$ and plug in $p = 1$ towards the end to get the result for the uniform case. The proof proceeds by first computing the posteriors using the fact that order statistics of uniform follow the beta distribution. We then find the posterior means and use the expected effort from Theorem 3 to get the value of the objective function. The problem then becomes:

$$\max_{k\in\{1,2,\ldots,n-1\}} \frac{\frac{\mu^2}{k(n-k)}\left(k - n\frac{\beta(k+1+\frac{1}{p}, n-k)}{\beta(k,n-k)}\right)^2}{\binom{n-1}{k-1}(n-k)\beta\left(k+1-\frac{1}{p}, n-k\right)}$$

For $p = 1$, we get that the optimal is $k = \frac{n}{2}$ which gives us the result. The full proof is in the appendix.

Figure 4 illustrates the variance of posterior means per unit expected effort as a function of k under $n = 15$ agents for different p values. We can see that the optimal number of prizes for $p = 1$ is at $k = 8 = \frac{n}{2}$ and for p large, the objective is monotone decreasing in k. In addition, the figure suggests that the optimal number of prizes is weakly decreasing from $n/2$ for $p = 1$ to 1 for p large. We conjecture that this is true more generally.

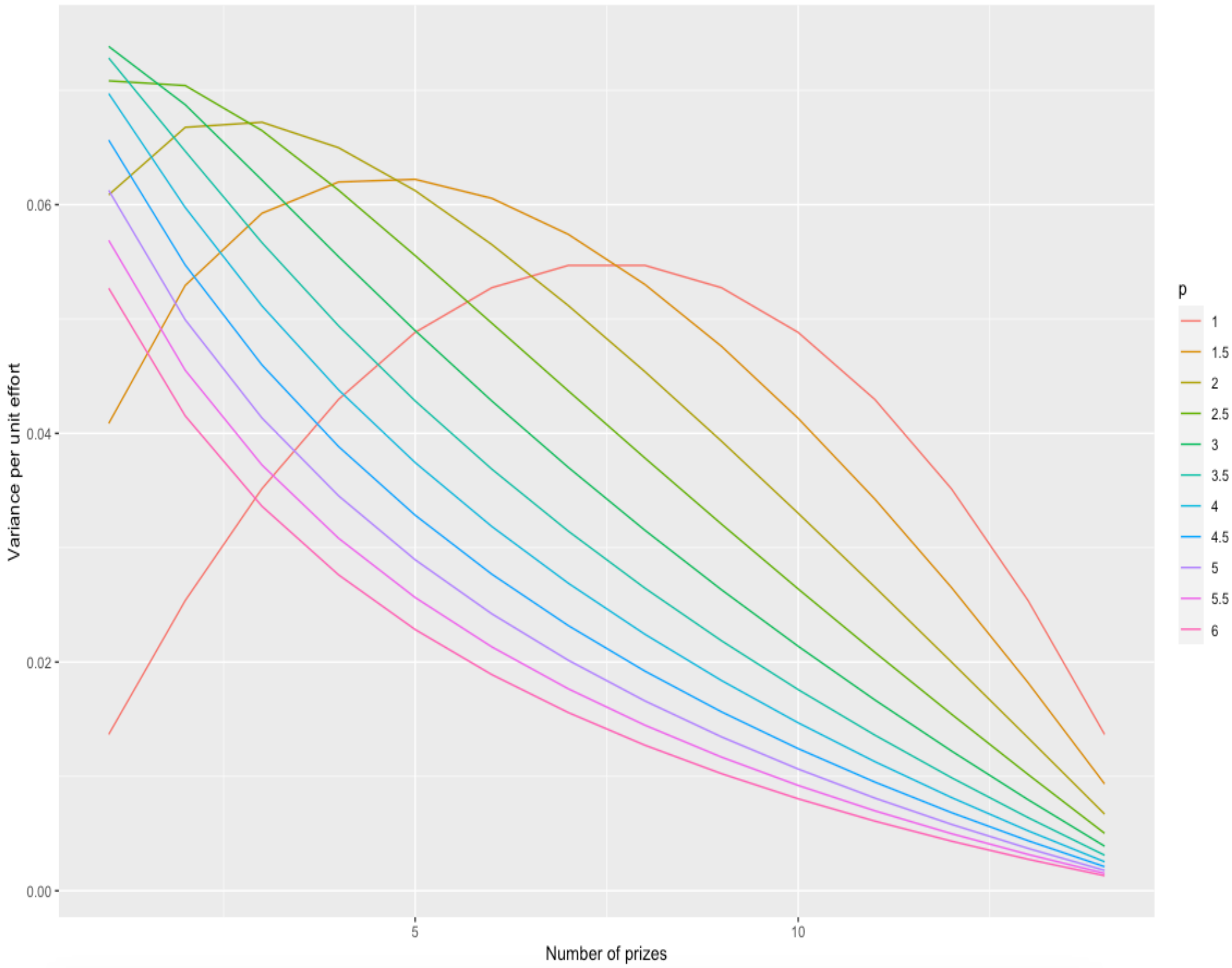

Figure 4: The information revealed per unit effort as a function of number of prizes for different values of p

6 Conclusion

We consider an optimal contest design problem with costless homogeneous prizes. Working in the incomplete information environment with private marginal costs, we find the symmetric Bayes-Nash equilibrium strategy. When the effort levels of the most efficient agents does not grow too quickly, the equilibrium strategy turns out to be a density function. This includes the cases where the prior is a power distribution $F(\theta) = \theta^p$ with $p \geq 1$. As a corollary, we get that the expected effort under uniform prior is independent of the number or prizes or even the number of players.

For the power distribution prior with $p \geq 1$, we study the optimal contest design problem for two different objectives. First, we find that for maximizing expected effort and expected minimum effort, awarding a prize to all but the agent who puts in the least effort is optimal. Then, we study contest as a screening device and consider a designer interested in designing a contest to maximize the information revealed (by the signal of whether or not a prize was awarded) per unit of expected effort put in by the participants. For the uniform case, we find that awarding $k = \frac{n}{2}$ awards is optimal and for p large, we conjecture that awarding just a single prize is optimal.

References

[1] Ales, L., S.-H. Cho, and E. Körpeoğlu (2017): "Optimal Award Scheme in Innovation Tournaments," *Operations Research*, 65, 693–702. 3

[2] Barut, Y. and D. Kovenock (1998): "The symmetric multiple prize all-pay auction with complete information," *European Journal of Political Economy*, 14, 627–644. 3

[3] Bastani, S., T. Giebe, and O. Gürtler (2019): "A General Framework for Studying Contests," SSRN Scholarly Paper ID 3507264, Social Science Research Network, Rochester, NY.

[4] Chawla, S., J. D. Hartline, and B. Sivan (2019): "Optimal crowdsourcing contests," *Games and Economic Behavior*, 113, 80–96. 3

[5] Chowdhury, S. M. (2021): "The Economics of Identity and Conflict," in *Oxford Research Encyclopedia of Economics and Finance*, Oxford University Press.

[6] Chowdhury, S. M., P. Esteve-Gonzalez, and A. Mukherjee (2020): "Heterogeneity, Leveling the Playing Field, and Affirmative Action in Contests," SSRN Scholarly Paper ID 3655727, Social Science Research Network, Rochester, NY.

[7] Chowdhury, S. M. and S.-H. Kim (2017): ""Small, yet Beautiful": Reconsidering the optimal design of multi-winner contests," *Games and Economic Behavior*, 104, 486–493.

[8] Corchón, L. C. (2007): "The theory of contests: a survey," *Review of Economic Design*, 11, 69–100. 3

[9] Ewerhart, C. and F. Quartieri (2020): "Unique equilibrium in contests with incomplete information," *Economic Theory*, 70, 243–271.

[10] Fang, D., T. Noe, and P. Strack (2020): "Turning up the heat: The discouraging effect of competition in contests," *Journal of Political Economy*, 128, 1940–1975. 3

[11] Faravelli, M. and L. Stanca (2012): "When less is more: Rationing and rent dissipation in stochastic contests," *Games and Economic Behavior*, 74, 170–183.

[12] Fu, Q. and Z. Wu (2019): "Contests: Theory and Topics," in *Oxford Research Encyclopedia of Economics and Finance*, Oxford University Press.

[13] ——— (2020): "On the optimal design of biased contests," *Theoretical Economics*, 15, 1435–1470.

[14] Gallice, A. (2017): "An approximate solution to rent-seeking contests with private information," *European Journal of Operational Research*, 256, 673–684.

[15] Ghosh, A. and R. Kleinberg (2014): "Optimal contest design for simple agents," in *Proceedings of the fifteenth ACM conference on Economics and computation*, Palo Alto California USA: ACM, 913–930.

[16] Glazer, A. and R. Hassin (1988): "OPTIMAL CONTESTS," *Economic Inquiry*, 26, 133–143. 3

[17] Hinnosaar, T. (2021): "Optimal Sequential Contests," SSRN Scholarly Paper ID 3795251, Social Science Research Network, Rochester, NY.

[18] Immorlica, N., G. Stoddard, and V. Syrgkanis (2015): "Social Status and Badge Design," in *Proceedings of the 24th International Conference on World Wide Web*, Florence Italy: International World Wide Web Conferences Steering Committee, 473–483.

[19] Konrad, K. A. et al. (2009): "Strategy and dynamics in contests," *OUP Catalogue*. 3

[20] Krishna, V. and J. Morgan (1997): "An Analysis of the War of Attrition and the All-Pay Auction," *Journal of Economic Theory*, 72, 343–362. 3

[21] Körpeoğlu, E. and S.-H. Cho (2018): "Incentives in Contests with Heterogeneous Solvers," *Management Science*, 64, 2709–2715.

[22] Letina, I., S. Liu, and N. Netzer (2020): "Optimal Contest Design: A General Approach," SSRN Scholarly Paper ID 3628174, Social Science Research Network, Rochester, NY.

[23] Liu, B. and J. Lu (2019): "The optimal allocation of prizes in contests with costly entry," *International Journal of Industrial Organization*, 66, 137–161.

[24] Liu, X. and J. Lu (2014): "The effort-maximizing contest with heterogeneous prizes," *Economics Letters*, 125, 422–425. 3

[25] ——— (2017): "Optimal prize-rationing strategy in all-pay contests with incomplete information," *International Journal of Industrial Organization*, 50, 57–90. 2

[26] Moldovanu, B. and A. Sela (2001): "The Optimal Allocation of Prizes in Contests," *The American Economic Review*, 91, 542–558, publisher: American Economic Association. 3

[27] ——— (2006): "Contest architecture," *Journal of Economic Theory*, 126, 70–96. 3

[28] Myerson, R. B. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, 6, 58–73.

[29] Nitzan, S. (1994): "Modelling rent-seeking contests," *European Journal of Political Economy*, 10, 41–60.

[30] Olszewski, W. and R. Siegel (2020): "Performance-maximizing large contests," *Theoretical Economics*, 15, 57–88.

[31] Segev, E. (2020): "Crowdsourcing contests," *European Journal of Operational Research*, 281, 241–255. 3

[32] Sisak, D. (2009): "MULTIPLE-PRIZE CONTESTS - THE OPTIMAL ALLOCATION OF PRIZES," *Journal of Economic Surveys*, 23, 82–114. 3

[33] Vojnovic, M. (2015): *Contest Theory: Incentive Mechanisms and Ranking Methods*, Cambridge: Cambridge University Press. 3

[34] Zhang, M. (2019): "Optimal Contests with Incomplete Information and Convex Effort Costs," SSRN Scholarly Paper ID 3512155, Social Science Research Network, Rochester, NY.

A Proofs for Section 3 (Equilibrium)

Theorem 1. *In an incomplete information contest with* n *agents,* k *prizes and marginal costs* θ_i *drawn according to* $F(.)$ *on* $[0,1]$*, the symmetric Bayes-Nash equilibrium effort function is given by* [2]

$$g(\theta) = \binom{n-1}{k-1}(n-k)\int_{F(\theta)}^{1} \frac{t^{k-1}(1-t)^{n-k-1}}{F^{-1}(t)}dt$$

Proof. Note that for $k=0$ and $k=n$, all students put in 0 effort. So we assume $1 \leq k \leq n-1$. Suppose $n-1$ students are playing the effort function $g(\theta)$ which is decreasing in θ. Then, the utility of a student of type $\theta \in [0,1]$ from playing x is given by:

$$u(\theta, x) = \sum_{i=0}^{k-1} \binom{n-1}{i} F(g^{-1}(x))^{i}(1-F(g^{-1}(x)))^{n-1-i} - \theta x$$

Taking the first order condition, we get

$$\sum_{i=0}^{k-1} \binom{n-1}{i} \frac{f(g^{-1}(x))}{g'(g^{-1}(x))}$$
$$\left[i * (1-F(g^{-1}(x))^{n-1-i}F(g^{-1}(x))^{i-1} - (n-1-i) * F(g^{-1}(x))^{i}(1-F(g^{-1}(x)))^{n-2-i}\right] = \theta$$

The negative term associated with i cancels out the positive term with $i+1$ and thus, the condition simplifies to

$$-\binom{n-1}{k-1}\frac{f(g^{-1}(x))}{g'(g^{-1}(x))}(n-k)F(g^{-1}(x))^{k-1}(1-F(g^{-1}(x)))^{n-k-1} = \theta$$

Now we plug in $g^{-1}(x) = \theta$ to get the condition for g to be the symmetric Bayes-Nash equilibrium. This gives the differential equation:

$$g'(\theta) = -\binom{n-1}{k-1}\frac{f(\theta)}{\theta}(n-k)F(\theta)^{k-1}(1-F(\theta))^{n-k-1}$$

Now using the fact that $g(1) = 0$, we get the symmetric Bayes-Nash equilibrium strategy

$$g(\theta) = \binom{n-1}{k-1}(n-k)\int_{\theta}^{1}\frac{f(\theta)}{\theta}F(\theta)^{k-1}(1-F(\theta))^{n-k-1}d\theta$$

Now replacing $t = F(\theta)$ gives:

$$g(\theta) = \binom{n-1}{k-1}(n-k)\int_{F(\theta)}^{1} \frac{t^{k-1}(1-t)^{n-k-1}}{F^{-1}(t)}dt$$

as required.

[2] More generally, if the payoff from being in top k is v more than the payoff from being in the bottom $n-k$, then the symmetric Bayes-Nash equilibrium function is simply $vg(\theta)$.

Now we check that the soc is satisfied. To simplify calculations, let $g^{-1}(x) = t$ so the agent of type θ is imitating an agent of type t. Then, the foc can be written as:

$$-\binom{n-1}{k-1}(n-k)f(t)F(t)^{k-1}(1-F(t))^{n-k-1} - \theta g'(t) = 0$$

Then the derivative of lhs wrt t gives:

$$\begin{aligned}
&-\binom{n-1}{k-1}(n-k)\Big[F(t)^{k-1}(1-F(t))^{n-k-1}f'(t) + (k-1)f^2(t)(1-F(t))^{n-k-1}F(t)^{k-2} \\
&-(n-k-1)f^2(t)F(t)^{k-1}(1-F(t))^{n-k-2}\Big] - \theta g''(t) \\
=&-\binom{n-1}{k-1}(n-k)F(t)^{k-2}(1-F(t))^{n-k-2}\Big[F(t)(1-F(t))f'(t) + f^2(t)\left(k-1+(2-n)F(t)\right)\Big] - \theta g''(t) \\
=&S(t) - \theta g''(t)
\end{aligned}$$

Observe from the foc that $tg''(t) + g'(t) = S(t)$ and so $g''(t) = \dfrac{S(t) - g'(t)}{t}$

Therefore, the second derivative becomes $S(t) - \theta\dfrac{(S(t) - g'(t))}{t}$ At $t = \theta$, this equals $g'(\theta) < 0$ and therefore, the second order condition for $t = \theta$ to be the optima is satisfied. □

Lemma 2. *For any n, k and F such that $\lim_{\theta\to 0}\theta g(\theta) = 0$, the equilibrium effort function $g(\theta)$ is a density function on $[0,1]$. That is,*

$$\int_0^1 g(\theta)d\theta = 1$$

Proof. We know that

$$\begin{aligned}
g'(\theta) &= -\binom{n-1}{k-1}\frac{f(\theta)}{\theta}(n-k)F(\theta)^{k-1}(1-F(\theta))^{n-k-1} \\
\implies \theta g'(\theta) &= -\binom{n-1}{k-1}f(\theta)(n-k)F(\theta)^{k-1}(1-F(\theta))^{n-k-1} \\
\implies \int_x^1 \theta g'(\theta)d\theta &= -\binom{n-1}{k-1}(n-k)\int_x^1 F(\theta)^{k-1}(1-F(\theta))^{n-k-1}f(\theta)d\theta
\end{aligned}$$

First let us simplify the right hand side:

$$\begin{aligned}
\int_x^1 F(\theta)^{k-1}(1-F(\theta))^{n-k-1}f(\theta)d\theta &= \int_{F(x)}^1 t^{k-1}(1-t)^{n-k-1}dt \\
&= \int_0^1 t^{k-1}(1-t)^{n-k-1}dt - \int_0^{F(x)} t^{k-1}(1-t)^{n-k-1}dt
\end{aligned}$$

$$= \frac{(k-1)!(n-k-1)!}{(n-1)!} - \beta_{F(x)}(k, n-k)$$

Plugging this into the right hand side, we get

$$\int_x^1 \theta g'(\theta) d\theta = \binom{n-1}{k-1}(n-k)\beta_{F(x)}(k, n-k) - 1$$

In particular, for $x = 0$, we get:

$$\int_0^1 \theta g'(\theta) d\theta = -1$$

Simplifying the left hand side using integration by parts and the fact that $g(1) = 0$, we get

$$\int_0^1 \theta g'(\theta) d\theta = \left[\theta g(\theta) - \int g(\theta) d\theta\right]_0^1$$
$$= -\int_0^1 g(\theta) d\theta$$

And thus, the integral of the equilibrium effort function

$$\int_0^1 g(\theta) d\theta = 1$$

irrespective of F on $[0, 1]$, n and k. In other words, the symmetric equilibrium effort function is always a density function.

□

B Proofs for Section 4 (Maximizing effort)

Theorem 2. *Suppose n, $k \in \{1, 2, \ldots, n-1\}$ and F on $[0, 1]$ are such that $\lim_{\theta \to 0} F(\theta)g(\theta) = 0$. Let f, f_l, f_m denote the density functions of θ, $\min_i \theta_i, \max_i \theta_i$ respectively. Then we have that*

- *the expected effort equals*

$$\int_0^1 g(\theta) f(\theta) d\theta = \binom{n-1}{k-1}(n-k) \int_0^1 \frac{t^k (1-t)^{n-k-1}}{F^{-1}(t)} dt$$

- *the expected maximum effort equals*

$$\int_0^1 g(\theta) f_l(\theta) d\theta = \binom{n-1}{k-1}(n-k) \left[\int_0^1 \frac{t^{k-1}(1-t)^{n-k-1}}{F^{-1}(t)} dt - \int_0^1 \frac{t^{k-1}(1-t)^{2n-k-1}}{F^{-1}(t)} dt\right]$$

- *the expected minimum effort equals*

$$\int_0^1 g(\theta) f_m(\theta) d\theta = \binom{n-1}{k-1}(n-k)\int_0^1 \frac{t^{n+k-1}(1-t)^{n-k-1}}{F^{-1}(t)} dt$$

Proof. We prove each in order.

- First, we consider expected effort. Given n, k and F on $[0,1]$, we know from Theorem 1 that the expected effort is given by:

$$\int_0^1 g(\theta) f(\theta) d\theta = \binom{n-1}{k-1}(n-k)\int_0^1 \left[\int_{F(\theta)}^1 \frac{t^{k-1}(1-t)^{n-k-1}}{F^{-1}(t)} dt\right] f(\theta) d\theta$$

  Using integration by parts to evaluate the integral, we get

$$\begin{aligned}\int_0^1 \left[\int_{F(\theta)}^1 \frac{t^{k-1}(1-t)^{n-k-1}}{F^{-1}(t)} dt\right] f(\theta) d\theta &= \int_0^1 \frac{F(\theta)^k (1-F(\theta))^{n-k-1}}{\theta} f(\theta) d\theta \quad (\lim_{\theta\to 0} F(\theta) g(\theta) = 0) \\ &= \int_0^1 \frac{t^k(1-t)^{n-k-1}}{F^{-1}(t)} dt\end{aligned}$$

  Thus, the expected effort is given by

$$\int_0^1 g(\theta) f(\theta) d\theta = \binom{n-1}{k-1}(n-k)\int_0^1 \frac{t^k(1-t)^{n-k-1}}{F^{-1}(t)} dt$$

- Now let's consider expected maximum effort. Given n, k and F on $[0,1]$, we know that the maximum effort will be put in by the student with the lowest marginal cost.

  Let $\theta_l = \min\{\theta_1, \ldots, \theta_n\}$ and let F_l denote the cdf of θ_l. Then, we know that $F_l(t) = \mathbb{P}[\theta_l \le t] = 1 - (1-F(t))^n$ and so the density is $f_l(t) = n(1-F(t))^{n-1} f(t)$

  Thus, from Theorem 1, the expected maximum effort is given by:

$$\begin{aligned}\int_0^1 g(\theta) f_l(\theta) d\theta &= \binom{n-1}{k-1}(n-k)\int_0^1 \left[\int_{F(\theta)}^1 \frac{t^{k-1}(1-t)^{n-k-1}}{F^{-1}(t)} dt\right] f_l(\theta) d\theta \\ &= n\binom{n-1}{k-1}(n-k)\int_0^1 \left[\int_{F(\theta)}^1 \frac{t^{k-1}(1-t)^{n-k-1}}{F^{-1}(t)} dt\right] (1-F(\theta))^{n-1} f(\theta) d\theta \\ &= \binom{n-1}{k-1}(n-k)\int_0^1 \frac{F(\theta)^{k-1}(1-F(\theta))^{n-k-1}}{\theta} f(\theta)\left(1-(1-F(\theta))^n\right) d\theta \quad (\lim_{\theta\to 0} F(\theta) g(\theta) =\end{aligned}$$

$$= \binom{n-1}{k-1}(n-k)\int_0^1 \frac{t^{k-1}(1-t)^{n-k-1}}{F^{-1}(t)}\left(1-(1-t)^n\right)dt$$
$$= \binom{n-1}{k-1}(n-k)\left[\int_0^1 \frac{t^{k-1}(1-t)^{n-k-1}}{F^{-1}(t)}dt - \int_0^1 \frac{t^{k-1}(1-t)^{2n-k-1}}{F^{-1}(t)}dt\right]$$

- Finally, let's consider expected minimum effort.

  Given n, k and F on $[0,1]$, we know that the minimum effort will be put in by the student with the highest marginal cost.

  Let $\theta_m = \max\{\theta_1, \dots, \theta_n\}$ and let F_m denote the cdf of θ_m. Then, we know that $F_m(t) = \mathbb{P}[\theta_m \le t] = F(t)^n$ and so the density is $f_m(t) = nF(t)^{n-1}f(t)$

  Thus, from Theorem 1, the expected minimum effort is given by:

$$\begin{aligned}
\int_0^1 g(\theta)f_m(\theta)d\theta &= \binom{n-1}{k-1}(n-k)\int_0^1 \left[\int_{F(\theta)}^1 \frac{t^{k-1}(1-t)^{n-k-1}}{F^{-1}(t)}dt\right] f_m(\theta)d\theta \\
&= n\binom{n-1}{k-1}(n-k)\int_0^1 \left[\int_{F(\theta)}^1 \frac{t^{k-1}(1-t)^{n-k-1}}{F^{-1}(t)}dt\right] F(\theta)^{n-1}f(\theta)d\theta \\
&= \binom{n-1}{k-1}(n-k)\int_0^1 \frac{F(\theta)^{k-1}(1-F(\theta))^{n-k-1}}{\theta}f(\theta)\left(F(\theta)^n\right)d\theta \qquad (\lim_{\theta\to 0} F(\theta)g(\theta) = 0) \\
&= \binom{n-1}{k-1}(n-k)\int_0^1 \frac{t^{n+k-1}(1-t)^{n-k-1}}{F^{-1}(t)}dt
\end{aligned}$$

□

Theorem 3. *Suppose $F(\theta) = \theta^p$ for $\theta \in [0,1]$ with $p > 1$. Then,*

- *the expected effort equals*

$$\binom{n-1}{k-1}(n-k)\beta\left(k+1-\frac{1}{p}, n-k\right)$$

  and it is maximized at $k = n-1$.

- *the expected maximum effort equals*

$$\binom{n-1}{k-1}(n-k)\left[\beta(k-\frac{1}{p}, n-k) - \beta(k-\frac{1}{p}, 2n-k)\right]$$

  and it is maximized at some interior k

- *the expected minimum effort equals*

$$\binom{n-1}{k-1}(n-k)\left[\beta(n+k-\frac{1}{p}, n-k)\right]$$

and it is maximized at $k = n-1$.

Proof. We know that in the general case, the expected effort is given by

$$\int_0^1 g(\theta)f(\theta)d\theta = \binom{n-1}{k-1}(n-k)\int_0^1 \frac{t^k(1-t)^{n-k-1}}{F^{-1}(t)}dt$$

In particular, when $F(\theta) = \theta^p$ with $p > 1$, we get that the expected effort is

$$\binom{n-1}{k-1}(n-k)\int_0^1 t^{k-\frac{1}{p}}(1-t)^{n-k-1}dt = \binom{n-1}{k-1}(n-k)\beta\left(k+1-\frac{1}{p}, n-k\right)$$

The argument for the definition in other two parts follows exactly in the same way by plugging in $F(\theta) = \theta^p$ in the expressions from Theorem 2.

Now to find the optimal k, we find the ratio of value of objective function at $k+1$ to the value at k by using the identities $\beta(x,y) = \dfrac{\gamma(x)\gamma(y)}{\gamma(x+y)}$ and $\gamma(x+1) = x\gamma(x)$.

- We find the ratio of expected effort for $k+1$ and k:

  $$\frac{\binom{n-1}{k}(n-k-1)\beta(k+2-\frac{1}{p}, n-k-1)}{\binom{n-1}{k-1}(n-k)\beta(k+1-\frac{1}{p}, n-k)} = 1 + \frac{1}{k}\left(1-\frac{1}{p}\right)$$

  For $p > 1$, this ratio is always > 1 and thus, the optimal k is $n-1$.

- We can compare the objective at $k = 1, 2$ and $k = n-1, n-2$ to get that the optimal number of prizes must be interior.

- We find the ratio of expected minimum effort for $k+1$ and k:

  $$\frac{\binom{n-1}{k}(n-k-1)\left[\beta(n+k+1-\frac{1}{p}, n-k-1)\right]}{\binom{n-1}{k-1}(n-k)\left[\beta(n+k-\frac{1}{p}, n-k)\right]} = 1 + \frac{1}{k}\left(n-\frac{1}{p}\right)$$

  Since this ratio is always > 1, we can conclude that the expected minimum effort is maximized at $k = n-1$.

□

C Proofs for Section 5 (Optimal screening contest)

Theorem 4. *Suppose $F(\theta) = \theta$ for $\theta \in [0,1]$. Then, the optimal screening contest with n agents awards $k = \lfloor \frac{n}{2} \rfloor$ prizes.*

Proof. We will do the calculations assuming $F(\theta) = \theta^p$. We'll plug in $p = 1$ towards the end to get the result for the uniform case. For the case where $F(\theta) = \theta^p$ (with $p \geq 1$), we have that the posteriors are

$$f_1(t) = \frac{n}{k}\Pr[X_{(k)}^{n-1} > t^p]pt^{p-1}$$

and

$$f_0(t) = \frac{n}{n-k}\Pr[X_{(k)}^{n-1} \leq t^p]pt^{p-1}$$

We know that if $X_{(k)}^n$ is the kth order statistic in n uniform random samples then its distribution is $\beta(k, n+1-k)$. Thus,

$$\Pr(X_{(k)}^n \leq t) = I_t(k, n+1-k) = \frac{\beta_t(k, n+1-k)}{\beta(k, n+1-k)} = \frac{n!}{(k-1)!(n-k)!}\int_0^t x^{k-1}(1-x)^{n-k}dx$$

A useful object in studying this case will be

$$\begin{aligned}
\int_0^1 t^{\frac{1}{p}} I_t(k, n+1-k)dt &= \frac{1}{\beta(k, n+1-k)}\int_0^1 t^{\frac{1}{p}}\left[\int_0^t x^{k-1}(1-x)^{n-k}dx\right]dt \\
&= \frac{1}{\beta(k, n+1-k)}\left[\frac{p\beta(k, n-k+1)}{1+p} - \frac{p\beta(k+1+\frac{1}{p}, n-k+1)}{1+p}\right] \\
&= \frac{p}{1+p}\left(1 - \frac{\beta(k+1+\frac{1}{p}, n-k+1)}{\beta(k, n+1-k)}\right) \\
&= \frac{p}{1+p}\left(1 - \frac{\Gamma(k+1+\frac{1}{p})\Gamma(n+1)}{\Gamma(n+2+\frac{1}{p})\Gamma(k)}\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
\mu_1 &= \int_0^1 tf_1(t)dt \\
&= \frac{n}{k}\int_0^1 \Pr[X_{(k)}^{n-1} > t^p]pt^p dt \\
&= \frac{n}{k}\int_0^1 \left(1 - \Pr[X_{(k)}^{n-1} \leq t^p]\right)pt^p dt \\
&= \frac{n}{k}\frac{p}{p+1} - \frac{np}{k}\int_0^1 \Pr[X_{(k)}^{n-1} \leq t^p]t^p dt
\end{aligned}$$

$$= \frac{n}{k}\frac{p}{p+1} - \frac{n}{k}\int_0^1 \Pr[X_{(k)}^{n-1} \le z] z^{\frac{1}{p}} dz$$

$$= \frac{n}{k}\mu z \qquad \left(z = \frac{\beta(k+1+\frac{1}{p}, n-k)}{\beta(k, n-k)} = \frac{(kp+1)\Gamma(k+\frac{1}{p})\Gamma(n)}{(np+1)\Gamma(n+\frac{1}{p})\Gamma(k)} \right)$$

Similarly, we get

$$\mu_0 = \int_0^1 t f_0(t) dt = \frac{np}{(n-k)(1+p)}\left(1 - \left(\frac{k}{n}\right)^{\frac{p+1}{p}}\right) = \frac{n}{n-k}\mu(1-z)$$

Plugging the values into our measure for information revealed, we get

$$\begin{aligned}
\frac{k}{n}\left(\mu_1 - \mu\right)^2 + \frac{n-k}{n}\left(\mu_0 - \mu\right)^2 &= \frac{1}{kn}\mu^2\left(nz-k\right)^2 + \frac{1}{(n-k)n}\mu^2\left(k-nz\right)^2 \\
&= \mu^2\frac{(nz-k)^2}{n}\left(\frac{1}{k} + \frac{1}{n-k}\right) \\
&= \mu^2\frac{(nz-k)^2}{k(n-k)}
\end{aligned}$$

Using the expression for expected effort from Theorem 3, we get that the designer's objective in this case takes the form:

$$\max_{k \in \{1,2,\ldots,n-1\}} \frac{\frac{\mu^2}{k(n-k)}\left(k - n\frac{\beta(k+1+\frac{1}{p}, n-k)}{\beta(k, n-k)}\right)^2}{\binom{n-1}{k-1}(n-k)\beta\left(k+1-\frac{1}{p}, n-k\right)}$$

For the uniform case with $p = 1$, this simplifies to

$$\mu^2\frac{k(n-k)}{(n+1)^2}$$

This is clearly maximized at $k = \frac{n}{2}$ and so, we get that awarding $\frac{n}{2}$ prizes is optimal when the contest has n participants with uniform marginal costs.

$\square$