An efficiency ordering of k-price auctions under complete information

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Abstract

We study k-price auctions in a complete information environment and characterize all pure-strategy Nash equilibrium outcomes. In a setting with n agents having ordered valuations, we show that any agent, except those with the lowest $k-2$ valuations, can win in equilibrium. As a consequence, worst-case welfare increases monotonically as we go from $k = 2$ (second-price auction) to $k = n$ (lowest-price auction), with the first-price auction achieving the highest worst-case welfare.

1 Introduction

We study k-price sealed-bid auctions in a complete information environment with n agents having strictly ordered valuations. In a k-price auction, the highest bidder wins the object (with ties broken in favor of agent with highest valuation) and pays the kth highest bid. We fully characterize the set of pure-strategy Nash equilibrium outcomes for all k-price auctions.

Our characterization reveals an ordering of k-price auctions in terms of their worst-case allocative efficiency. Specifically, we show that for each $k \in \{2, \ldots, n\}$, any of the top $n-(k-2)$ agents can win the k-price auction, while the bottom $(k-2)$ agents can never win in equilibrium. In other words, all n agents can win the second-price auction, the top $n-1$ agents can win the third-price auction, and so on, until only the top two agents can win the nth-price auction (where the winner pays the lowest bid). We further show that only the top agent can win the first-price auction ($k = n + 1$). Thus, the worst-case equilibrium allocation under the k-price auction becomes strictly more efficient as k increases from 2 to $n+1$.

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In closely related work, Tauman (2002) and Mathews and Schwartz (2017) also study k-price auctions in complete information environments. Under the restriction to pure strategies that are not weakly dominated, Tauman (2002) shows that for any k-price auction, only the top agent can win in equilibrium. Subsequently, Mathews and Schwartz (2017) constructs an equilibrium in mixed-strategies where the top agent does not win. In comparison, we characterize all pure-strategy Nash equilibria for all k-price auctions and obtain an ordering of these auctions based on their worst-case allocative efficiency.[1]

2 Model

A seller is selling an indivisible object to a set $N = \{1, \ldots, n\}$ of agents. Each agent $i \in N$ has a valuation $v_i > 0$ for the object, and we assume that

$$v_1 > v_2 > \cdots > v_n.$$

We further define $v_{n+1} = 0$. The valuations are assumed to be common knowledge.

The object is sold using a sealed-bid k-price auction. Each agent $i \in N$ simultaneously submits a non-negative bid $b_i \in \mathbb{R}_+$. The object is awarded to the agent who submits the highest bid, with ties resolved in favor of the agent with the highest valuation. The winner pays the kth highest bid, denoted $b_{(k)}$, and all other agents pay zero. The utility of agent $i \in N$ at bid profile $b = (b_1, \ldots, b_n)$ is given by:

$$u_i(b) = \begin{cases} v_i - b_{(k)} & \text{if } i = \min\{j \in N : b_j = b_{(1)}\}, \\ 0 & \text{otherwise.} \end{cases}$$

We characterize pure-strategy Nash equilibrium outcomes for $k \in \{2, \ldots, n+1\}$, where $k = 2$ corresponds to the second-price auction, $k = 3$ the third-price auction, and so on, with $k = n+1$ used (for notational convenience) to represent the first-price auction.[2]

3 Results

We first characterize equilibrium outcomes from the seller's perspective.

[1] Other work on k-price auctions has focused on incomplete information settings (Kagel and Levin (1993), Monderer and Tennenholtz (2000, 2004), Mezzetti and Tsetlin (2009), Azrieli and Levin (2012), Mihelich and Shu (2020), Skitmore (2014)).

[2] In a more general framework, we can define the outcome set for each agent as $X = \mathbb{R}_+ \cup \{-1\}$, where an outcome $x \in \mathbb{R}_+$ represents the payment made by the agent when winning the object, and $x = -1$ denotes the outcome in which the agent does not win. An agent $i \in N$ with valuation $v_i > 0$ is then represented by a preference relation $\succ_i$ over X, such that $x \succ_i y$ for all $x < y \in \mathbb{R}_+$, and $-1 \sim_i v_i$. While we represent these preferences using the utility function u_i, our characterization results, since they focus on pure-strategy Nash equilibria, extend to any utility representation (including those that may capture risk-aversion).

Proposition 1. *Consider a k-price auction with $k \in \{2, \ldots, n+1\}$. There exists a pure-strategy Nash equilibrium in which the seller's revenue is p if and only if*

$$p \in [v_{n-(k-3)}, v_1].$$

Proof. We first show that in any equilibrium, the seller's revenue $p \in [v_{n-(k-3)}, v_1]$. Suppose towards a contradiction that b is an equilibrium profile and $p = b_{(k)} \notin [v_{n-(k-3)}, v_1]$. There are two possibilities:

1. $p > v_1$: In this case, the winner pays more than their valuation, thus receiving a negative utility. It can deviate to bidding 0 instead, and receive a utility ≥ 0. This contradicts the fact that b is a Nash equilibrium.

2. $p < v_{n-(k-3)}$: First, notice that for the second-price auction ($k = 2$), this means $p < 0$, which is not possible. So consider $k \geq 3$. In this case, observe that

   (a) At least $(n - (k-3))$ agents have valuation $> p$ (as $v_1 > \cdots > v_{n-(k-3)} > p$).

   (b) At least k agents bid $\geq p$ at profile b.

   Together, this implies that there are at least three distinct agents with valuations strictly greater than p, each bidding at least p. At least two of these agents are receiving utility 0. At least one of these two agents can deviate by bidding $> b_{(1)}$, and receive strictly positive utility. This contradicts the fact that b is a Nash equilibrium.

Thus, it must be that in any equilibrium, $p \in [v_{n-(k-3)}, v_1]$.

Now we show that for any $p \in [v_{n-(k-3)}, v_1]$, there exists a pure-strategy equilibrium where the seller's revenue is p. Consider the bid profile

$$b = (v_1, \underbrace{p, \ldots, p}_{n-(k-1) \text{ agents}}, \underbrace{v_1, \ldots, v_1}_{(k-2) \text{ agents}}).$$

At this profile, agent 1 wins the good (as ties are broken in favor of agent with highest valuation), and pays a price $b_{(k)} = p$ for the good.[3] Agent 1's utility is $v_1 - p \geq 0$ and for $j \neq 1$, agent j's utility is 0. We now verify that b is indeed a Nash equilibrium. Consider agent $j \in N$. There are three cases:

1. $j = 1$: If $b'_1 \geq v_1$, agent 1's utility does not change. If $b'_1 < v_1$, agent 1's utility is 0. Thus, agent 1 does not have any profitable deviation.

2. $j \in \{2, \ldots, n-(k-2)\}$: If $b'_j > v_1$, agent j's utility will be $v_j - v_1 < 0$. If $b'_j \leq v_1$, agent j's utility remains 0. Thus, agent j does not have any profitable deviation.

[3] Note that we could alternatively construct a profile b with $b_1 > v_1$, and show that it is a Nash equilibrium under any arbitrary tie-breaking rule. Thus, the choice of tie-breaking rule is not important for this result.

3. $j \in \{n-(k-3), \ldots, n\}$: If $b'_j > v_1$, agent j's utility will be $v_j - p \leq 0$. If $b'_j \leq v_1$, agent j's utility remains 0. Thus, agent j does not have any profitable deviation.

Thus, for any $p \in [v_{n-(k-3)}, v_1]$, there exists an equilibrium in which the revenue is p. $\square$

We now characterize equilibrium outcomes from the buyers perspective.

Proposition 2. *Consider a k-price auction with $k \in \{2, \ldots, n+1\}$. There exists a pure-strategy Nash equilibrium in which agent $i \in N$ wins and pays p if and only if*

$$v_i > v_{n-(k-3)} \text{ and } p \in [v_{n-(k-3)}, v_i].$$

Proof. Suppose towards a contradiction that b is an equilibrium profile where agent $i \in N$ wins and $v_i \leq v_{n-(k-3)}$. From Proposition 1, it must pay $p \geq v_{n-(k-3)}$. Thus, the only possibility is that $v_i = p = v_{n-(k-3)}$. In this case, observe that

1. At least $(n-(k-2))$ agents have valuation $> p$ (as $v_1 > \cdots > v_{n-(k-2)} > p$).

2. At least k agents bid $\geq p$ at profile b.

Together, this implies that there are at least two distinct agents with valuations strictly greater than p, each bidding at least p. Further, both these agents are receiving utility 0. At least one of these two agents can deviate by bidding $> b_{(1)}$, and receive strictly positive utility. This contradicts the fact that b is a Nash equilibrium. Thus, it must be that $v_i > v_{n-(k-3)}$. Further, if i wins and pays p, it must be that $p \leq v_i$, and from Proposition 1, $p \geq v_{n-(k-3)}$. Thus, in any equilibrium where i wins and pays p, it must be that $v_i > v_{n-(k-3)}$ and $p \in [v_{n-(k-3)}, v_i]$.

Now we show that for any $i \in N$ such that $v_i > v_{n-(k-3)}$ and $p \in [v_{n-(k-3)}, v_i]$, there exists a Nash equilibrium where agent i wins and pays p. Consider the bid profile b where

$$b_i = v_1 \text{ and } b_{-i} = (\underbrace{p, \ldots, p}_{n-(k-1) \text{ agents}}, \underbrace{v_1, \ldots, v_1}_{(k-2) \text{ agents}}).$$

At this profile, agent i wins the good, and pays a price $b_{(k)} = p$ for the good. Agent i's utility is $v_i - p \geq 0$ and for $j \neq i$, agent j's utility is 0. We now verify that b is indeed a Nash equilibrium. Consider agent $j \in N$. There are three cases:

1. $j = i$: If $b'_i \geq v_1$, agent i's utility does not change. If $b'_i < v_1$, agent i's utility is 0. Thus, agent i does not have any profitable deviation.

2. $j \in \{1, \ldots, n-(k-2)\} \setminus \{i\}$: If $b'_j > v_1$, agent j's utility will be $v_j - v_1 \leq 0$. If $b'_j \leq v_1$, agent j's utility is either 0 or < 0. Thus, agent j does not have any profitable deviation.

3. $j \in \{n-(k-3), \ldots, n\}$: If $b'_j > v_1$, agent j's utility will be $v_j - p \leq 0$. If $b'_j \leq v_1$, agent j's utility remains 0. Thus, agent j does not have any profitable deviation.

Thus, for any $i \in N$ such that $v_i > v_{n-(k-3)}$ and $p \in [v_{n-(k-3)}, v_i]$, there exists a Nash equilibrium where agent i wins and pays p.

$\square$

The characterization in Proposition 2 yields a ranking of k-price auctions in terms of their worst-case efficiency. Formally, we let

$$\underline{W}^k = \min\{v_i : \exists b \text{ such that } b \text{ is an equilibrium under } k\text{-price auction where agent } i \text{ wins}\},$$

denote the worst-case equilibrium welfare under the k-price auction.

Corollary 1. *For $k \in \{2, \ldots, n+1\}$, the worst-case equilibrium welfare of the k-price auction is*

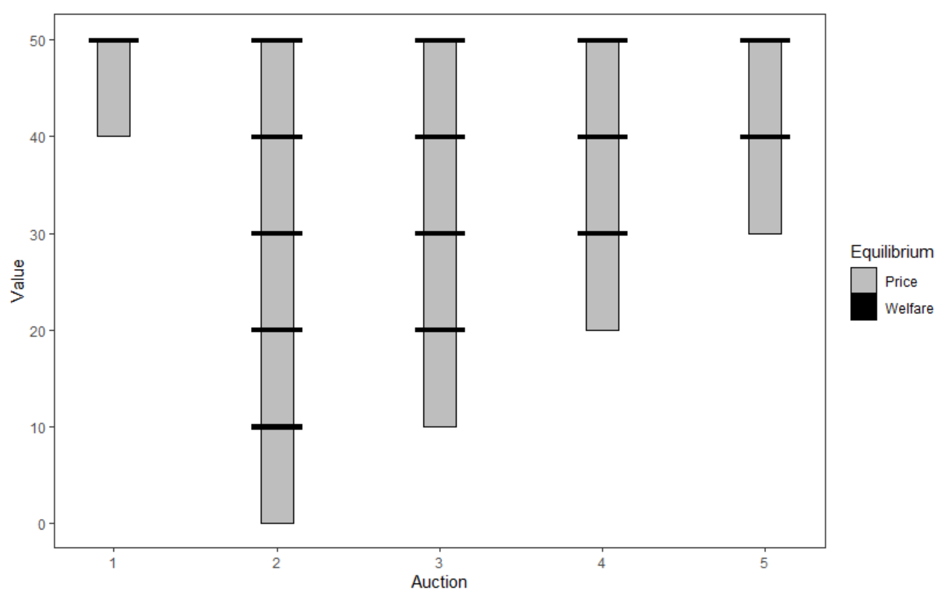
$$\underline{W}^k = v_{n-(k-2)}.$$

Hence,

$$\underline{W}^2 < \underline{W}^3 < \cdots < \underline{W}^n < \underline{W}^{n+1}.$$

Lastly, we illustrate our results through an example with $n = 5$ agents whose valuations are $v_1 = 50, v_2 = 40, v_3 = 30, v_4 = 20$ and $v_5 = 10$. Figure 1 identifies the equilibrium welfare and price (or revenue) possibilities under all k-price auctions. The black bars show the possible equilibrium welfare, while the grey vertical bars show the interval of possible equilibrium prices. For the first-price auction, agent 1 always wins and pays a price between 40 and 50 in equilibrium. In the second-price auction, any agent i can win and pays a price between 0 and v_i in equilibrium, and so on.

Figure 1: k-Price Auctions With 5 Agents



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