# The effect of competition in contests: A unifying approach*

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**Abstract**

We study rank-order contests with finite type spaces and establish the winner-takes-all contest as robustly optimal: it maximizes the total effort of the top $q$ agents, for any $q$, under linear, concave, and even moderately convex cost functions—thereby resolving an open question in contest design. At the same time, the effect of competition is nuanced. We uncover an *interior discouragement effect*: shifting value toward better-ranked prizes may reduce effort when inefficient types are relatively likely. Methodologically, our analysis develops a novel approach based on characterizing symmetric equilibria through the probability of outperforming an arbitrary agent. The representation is broadly applicable and provides a unifying lens that reconciles contrasting results across complete-information and continuum type-space environments, for which we also establish an equilibrium convergence result.

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# 1 Introduction

A central question in contest theory is how prize structures shape agents' incentives to exert effort, and in particular, which structures maximize effort. Under complete information, optimal contests typically feature multiple prizes, allocated in minimally competitive ways (Barut and Kovenock (1998); Fang, Noe, and Strack (2020)). By contrast, in incomplete information settings with a continuum of types, the most competitive winner-takes-all structure is frequently optimal (Moldovanu and Sela (2001); Zhang (2024)). Yet, what drives this divergence remains unclear, and it is an open question which (if any) of these findings extend to the intermediate and fundamental case of a finite type-space. This gap was also noted in a survey article by Sisak (2009), who conjectured that multiple prizes might be optimal:

> *"The case of asymmetric individuals, where types are private information but drawn from discrete, identical or maybe even different distributions, has not been addressed so far. From the results ... on asymmetric types with full information, one could conjecture that multiple prizes might be optimal even with linear costs."*

In this paper, we address this question by studying rank-order contests where ex-ante symmetric agents have private abilities drawn from a finite type-space. The finite type-space framework embeds the complete information as a special case and can approximate any continuum type-space. Thus, our analysis not only bridges a gap in the literature, but provides a unifying approach offering insights into the contrasting results in these extreme environments. Beyond its theoretical appeal, this framework is practically relevant, can accommodate richer non-parametric type-spaces, and enables experimental investigation.

We begin by characterizing the unique symmetric equilibrium of the Bayesian contest game. We show that equilibrium is in mixed strategies and exhibits a monotonic structure: different types randomize over disjoint but contiguous effort intervals, with more efficient types always outperforming less efficient ones. To overcome the analytical complexity of this mixed equilibrium, we introduce a novel representation that characterizes equilibrium effort through the ex-ante probability of outperforming an arbitrary agent. This representation is broadly applicable and may prove valuable in other environments where mixed equilibria hinder analysis. We use it to study how increasing competition by shifting value from lower-ranked prizes to better-ranked prizes affects equilibrium effort. We first fully characterize these effects under linear costs and then identify conditions under which they extend qualitatively to equilibrium under more general cost functions.

Under linear costs, we show that shifting value to the best prize always encourages effort. Intuitively, such a transformation creates strong incentives for the most efficient types, who are cheapest to incentivize, and this encouragement effect more than compensates for the discouragement it induces among less efficient types. Consequently, the winner-takes-all contest maximizes total effort under linear costs, resolving Sisak (2009)'s conjecture in the negative. Moreover, when the designer instead cares about the total effort of the top $q$ agents, the objective effectively places greater weight on the effort of more efficient types. Since winner-takes-all disproportionately strengthens incentives precisely for these types, its optimality persists under this broader objective.

This conclusion extends to concave cost functions as well. Under concavity, marginal effort costs are decreasing, making it even cheaper to induce higher effort levels from efficient types. As a result, shifting value to the best prize further amplifies effort where it is most cost-effective, reinforcing the optimality of the winner-takes-all contest. We therefore establish that winner-takes-all maximizes the total effort of the top $q$ agents, for any $q$, under both linear and concave costs. These results extend the optimality of winner-takes-all established in continuum type-space models (Moldovanu and Sela (2001)) to finite type spaces, while strengthening it to more general objectives that place greater weight on the effort of higher-performing agents.

This intuition, however, does not extend directly to convex costs. When costs are convex, marginal effort costs are increasing, so inducing additional effort at high levels becomes progressively more expensive. As a result, shifting value to the best prize need not encourage effort, since stronger incentives for the efficient types must now be weighed against rising marginal costs. Nevertheless, we show that winner-takes-all optimality persists under moderate convexity. As long as costs are not too convex, the encouragement effect of shifting value to the top prize outweighs the increase in marginal costs. Our analysis further suggests that the degree of convexity required to overturn winner-takes-all optimality shrinks as the environment approaches complete information, where, in contrast, the winner-takes-all contest actually minimizes total effort. In this sense, our analysis provides a unified perspective that reconciles contrasting results in the complete information and incomplete information environments.

Despite the winner-takes-all contest being robustly optimal, it is not generally the case that shifting value toward better-ranked prizes increases effort. We identify an *interior discouragement effect*: shifting value from lower-ranked prizes to better-ranked intermediate (but not top) prizes may reduce aggregate effort when inefficient types are sufficiently likely. Intuitively, such transformations discourage inefficient types—much like shifting value to the top prize—but the encouraging effect on efficient types is muted, relative to the top prize. This is because, as a result of this transformation, efficient types have weaker incentives to exert effort in pursuit of the best prize. Consequently, the encouragement effect on the most efficient types is dampened, and their effort may even decline. When inefficient types are sufficiently prevalent, the discouragement effect dominates and total effort falls.

## Related literature

The existing game-theoretic literature in contests has predominantly focused on the design problem in environments where the type-space is either a continuum, or a singleton (the complete information case), and the results highlight how the structure of the optimal contest can vary significantly depending on the environment. For the continuum type-space, the most competitive winner-takes-all contest has been shown to be optimal under linear or concave costs (Moldovanu and Sela (2001)), in some cases under convex costs (Zhang (2024)), with negative prizes (Liu, Lu, Wang, and Zhang (2018)), and with general architectures (Moldovanu and Sela (2006); Liu and Lu (2014)). In comparison, in the complete information environments, the minimally competitive budget distribution (all agents but one receive an equal positive prize) has been shown to be a feature of the optimal contest quite generally (Barut and Kovenock (1998); Letina, Liu, and Netzer (2023, 2020); Xiao (2018)). In a general framework with many agents, Olszewski and Siegel (2016, 2020) show that awarding multiple prizes of descending sizes is optimal under convex costs. Other related work has examined the effect of competition in complete information setting (Fang, Noe, and Strack (2020)), and continuum type-space setting (Goel (2025); Krishna, Lychagin, Olszewski, Siegel, and Tergiman (2025)).[1]

[1] In early work, Glazer and Hassin (1988) highlight the distinction between the two environments by solving the problem in some special cases. Other related studies include Schweinzer and Segev (2012); Drugov and Ryvkin (2020) who examine the budget allocation problem under different contest success functions. For general surveys of the literature in contest theory, see Corchón (2007); Sisak (2009); Konrad (2009); Vojnović (2015); Fu and Wu (2019); Chowdhury, Esteve-González, and Mukherjee (2023); Beviá and Corchón (2024).

There is a related literature on contests with a finite type-space, much of which assumes binary type-spaces or a small number of agents and focuses on characterizing equilibrium properties under correlated or asymmetric types. Siegel (2014) establishes the existence of a unique equilibrium under general distributional assumptions. With correlated types, Liu and Chen (2016) show that the symmetric equilibrium may be non-monotonic when the degree of absolute correlation is high, Rentschler and Turocy (2016) highlight the possibility of allocative inefficiency in equilibrium, while Tang, Fu, and Wu (2023) and Kuang, Zhao, and Zheng (2024) explore the impact of reservation prices and information disclosure policies, respectively. With asymmetric type distributions, Szech (2011) shows that agents may benefit from revealing partial information about their private types, while Chen (2021) characterizes equilibrium outcomes for varying levels of signal informativeness.[2]

# 2 Model

## Game

A *contest environment* is a tuple $(N+1, \Theta, p)$, where

- $N+1$ is the number of agents,
- $\Theta = \{\theta_1, \dots, \theta_K\}$ is a finite set of types, with $\theta_1 > \cdots > \theta_K$, and
- $p = (p_1, \dots, p_K)$ is a probability distribution over $\Theta$.

Each of the $N+1$ agents has a private type $\theta \in \Theta$, which represents the agent's marginal cost of exerting effort. The types are drawn independently according to $p$. We let $P_k = p_1 + \cdots + p_k$.

A *contest* $v = (v_0, \dots, v_N)$ assigns a prize value to each rank, with $v_0 \leq \cdots \leq v_N$ and $v_0 < v_N$.

[2]Other related work has studied imperfectly discriminating contests (Ewerhart and Quartieri (2020)), contests with altruistic or envious types (Konrad (2004)), and common value all-pay auctions with private asymmetric information (Einy, Goswami, Haimanko, Orzach, and Sela (2017)). There is also some work in mechanism design and auction design with finite type-spaces (Maskin and Riley (1985); Jeong and Pycia (2023); Vohra (2012); Lovejoy (2006); Doni and Menicucci (2013); Elkind (2007)).

Given a contest environment $(N+1, \Theta, p)$, contest $v$, and their private types, the agents simultaneously choose their effort. They are ranked according to their efforts, with ties broken uniformly at random, and awarded the corresponding prizes. An agent who outperforms exactly $m \in \{0, \ldots, N\}$ out of the $N$ other agents is awarded the prize $v_m$. If an agent of type $\theta \in \Theta$ wins prize $v_m$ after exerting effort $x \geq 0$, their vNM utility is

$$v_m - \theta x.$$

The Bayesian game induced by $v$ is strategically equivalent to the game induced by the contest $w$ where $w_m = v_m - v_0$ for all $m \in \{0, \ldots, N\}$. We normalize $v_0 = 0$, and focus on contests in the set

$$\mathcal{V} = \{v \in \mathbb{R}^{N+1} : v_0 \leq v_1 \leq \cdots \leq v_N \text{ where } 0 = v_0 < v_N\}.$$

## Equilibrium

We focus on symmetric Bayes-Nash equilibria. A symmetric Bayes-Nash equilibrium is a strategy profile in which all agents use the same strategy—a mapping from types to (possibly random) effort—such that, for every agent and every type $\theta \in \Theta$, the prescribed effort maximizes expected utility given that all other agents follow the same strategy. We denote a symmetric Bayes-Nash equilibrium by $(X_1, X_2, \ldots, X_K)$, where $X_k \sim F_k$ is the effort exerted by an agent of type $\theta_k$. We further denote by $X \sim F$ the ex-ante equilibrium effort of an arbitrary agent, so that for any $x \in \mathbb{R}$, $F(x) = \sum_{k=1}^{K} p_k F_k(x)$. Accordingly, the expected equilibrium effort of an arbitrary agent is

$$\mathbb{E}[X] = \sum_{k=1}^{K} p_k \mathbb{E}[X_k].$$

## Competition

We are interested in examining how increasing competitiveness of a contest influences the expected equilibrium effort. As is standard in the literature (Fang, Noe, and Strack (2020); Goel (2025)), we define a contest $v \in \mathcal{V}$ as being *more competitive* than $w \in \mathcal{V}$ if the prizes are more unequal, measured using the Lorenz order, i.e.,

$$\sum_{i=0}^{m} v_i \leq \sum_{i=0}^{m} w_i \text{ for all } m \in \{0, 1, \ldots, N\},$$

with equality for $m = N$.

Importantly, if $v$ is more competitive than $w$, $v$ can be obtained from $w$ through a sequence of transfers from lower-ranked prizes to higher-ranked prizes. The marginal effect of such a transfer—from prize $m'$ to prize $m$, with $m > m'$—on expected equilibrium effort is captured by

$$\frac{\partial \mathbb{E}[X]}{\partial v_m} - \frac{\partial \mathbb{E}[X]}{\partial v_{m'}}.$$

Our objective is to understand how this effect may depend on the pair of prizes, the structure of the contest, and the underlying contest environment.

We further discuss implications to the classical design problem of allocating a fixed budget $V \in \mathbb{R}_+$ across prizes to maximize effort. Notice that among all contests $w \in \mathcal{V}$ that distribute the entire budget, the winner-takes-all contest $v = (0, 0, \ldots, 0, V)$ is the most competitive, while the contest that allocates the budget equally among all but the worst-performing agent $v = (0, \frac{V}{N}, \ldots, \frac{V}{N})$ is the least competitive.

## Notation

Suppose an agent outperforms each of the other $N$ agents independently with probability $t \in [0, 1]$. Let

$$H_m^N(t) = \binom{N}{m} t^m (1-t)^{N-m}$$

denote the probability that the agent outperforms exactly $m$ out of $N$ agents. Define

$$H_{\leq m}^N(t) = \sum_{i=0}^{m} H_i^N(t) \text{ and } H_{\geq m}^N(t) = \sum_{i=m}^{N} H_i^N(t),$$

as the probabilities that the agent outperforms at most $m$ and at least $m$ agents, respectively. Finally, given a contest $v \in \mathcal{V}$, define

$$\pi_v(t) = \sum_{m=0}^{N} v_m H_m^N(t),$$

which is the expected prize value obtained by the agent under $v$.

# 3 Equilibrium

In this section, we characterize symmetric Bayes-Nash equilibria of the Bayesian game.

To begin, we establish a robust structural property: different agent types mix over contiguous intervals, with more efficient types choosing higher effort than less efficient ones.[3]

**Lemma 1.** *If $(X_1, \ldots, X_K)$ is an equilibrium, then there exist boundary points*

$$0 = b_0 < b_1 < \cdots < b_K$$

*such that, for each $k$, $X_k$ is continuously distributed on $[b_{k-1}, b_k]$.*

In words, agents of the least efficient type $\theta_1$ mix over the interval $[0, b_1]$, agents of type $\theta_2$ mix over $[b_1, b_2]$, and so on, up to agents of the most efficient type $\theta_K$, who mix over $[b_{K-1}, b_K]$. The supports are disjoint across types because moving from one effort level to another generates the same change in expected prize for all types, while the change in cost is type-dependent. Consequently, two distinct types cannot both be indifferent between the same pair of effort levels, implying that their supports can intersect only at boundary points. As a result, more efficient agents outperform less efficient agents. Mixing arises solely because agents may compete against others of the same type.

The mixing distributions must satisfy the indifference conditions. Notice that if a type-$\theta_k$ agent chooses effort $x_k \in [b_{k-1}, b_k]$, it outperforms an arbitrary agent with probability

$$P_{k-1} + p_k F_k(x_k).$$

Since the agent must be indifferent across all such effort levels, the equilibrium distribution $F_k$ must satisfy, for all $x_k \in [b_{k-1}, b_k]$,

$$\pi_v\big(P_{k-1} + p_k F_k(x_k)\big) - \theta_k x_k = u_k, \tag{1}$$

where $u_k$ denotes the equilibrium utility of type $\theta_k$. While Equation (1) uniquely pins down $F_k$—and one can verify that the resulting distributions constitute an equilibrium—its implicit form makes the equilibrium analytically difficult to study.

[3] Wang (1991) studies common-value auctions with finite signals and finds a similar equilibrium structure.

We develop an alternative representation of the equilibrium that allows us to circumvent these analytical difficulties. Specifically, given a symmetric equilibrium $(X_1, X_2, \dots, X_K)$, define a mapping

$$X_v : [0,1] \to \mathbb{R}_+$$

that assigns to each probability $t \in [0,1]$ of outperforming an arbitrary agent the corresponding equilibrium effort. From Lemma 1, probability $t = 0$ corresponds to zero effort, $t = 1$ corresponds to maximal effort $b_K$, and probability $t = P_k$ maps to the effort $b_k$. This is illustrated in Figure 1.

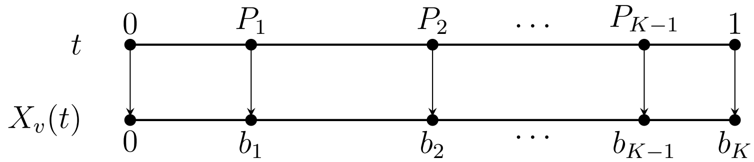



Figure 1: Symmetric equilibrium in terms of probability of outperforming an arbitrary agent.

In general, the indifference condition in Equation (1) can be restated directly in terms of this representation. Specifically, the equilibrium effort schedule must satisfy, for all $t \in [P_{k-1}, P_k]$,

$$\pi_v(t) - \theta_k X_v(t) = u_k. \tag{2}$$

To complete the characterization, it remains to determine the equilibrium utilities. The following iterative argument, initiated by $b_0 = 0$, uniquely pins down utilities for all types:

$$\begin{aligned}
b_0 = 0 \Rightarrow u_1 &= 0 && \text{(type-}\theta_1\text{ agent's utility at } b_0\text{)} \\
\Rightarrow\ b_1 &= \frac{\pi_v(P_1) - u_1}{\theta_1} && (X_v(P_1) \text{ from Equation (2) using } u_1) \\
\Rightarrow\ u_2 &= \pi_v(P_1) - \theta_2 \cdot b_1 && \text{(type-}\theta_2\text{ agent's utility at } b_1\text{)} \\
\Rightarrow\ b_2 &= \frac{\pi_v(P_2) - u_2}{\theta_2} && (X_v(P_2) \text{ from Equation (2) using } u_2) \\
\Rightarrow\ u_3 &= \dots
\end{aligned}$$

**Theorem 1** (Equilibrium Characterization). *For any contest environment $(N+1, \Theta, p)$ and contest $v \in \mathcal{V}$, there exists a unique symmetric equilibrium $(X_1, \dots, X_K)$. It is such that, for each $t \in [0,1]$,*

$$X_v(t) = \frac{\pi_v(t) - u_{k(t)}}{\theta_{k(t)}}, \tag{3}$$

*where $k(t) = \max\{k : P_{k-1} \leq t\}$, and the equilibrium utilities are given by*

$$u_k = \theta_k \left[ \sum_{j=1}^{k-1} \pi_v(P_j) \left( \frac{1}{\theta_{j+1}} - \frac{1}{\theta_j} \right) \right]. \tag{4}$$

**Remark 1.** *The equilibrium under the finite-type space framework exhibits both the mixed structure characteristic of complete information environments (Barut and Kovenock (1998)) and the monotonic structure observed in environments with a continuum of types (Moldovanu and Sela (2001)). The complete information environment is clearly a special case of our model. We also establish an equilibrium convergence result for the continuum type-space environment (Theorem 6 in Appendix D), which implies that the (pure-strategy) equilibrium in any continuum type-space can be well-approximated by the equilibrium of a sufficiently large and appropriately chosen finite type-space. Intuitively, as the finite type-space becomes large, the interval over which an agent of a certain type mixes shrinks, and essentially converges to the effort level prescribed by the pure-strategy equilibrium under the continuum type-space.*

**Remark 2.** *The equilibrium characterization in Theorem 1 extends to environments in which the type space $\mathcal{C} = \{c_1, \ldots, c_K\}$ consists of cost functions. Assume that for each $k \in [K]$, $c_k(0) = 0$, $c_k$ is strictly increasing and differentiable on $(0, \infty)$, and $\lim_{x \to \infty} c_k(x) = \infty$. Suppose further that types are ordered by marginal costs: for every $x > 0$,*

$$c_1'(x) > c_2'(x) > \cdots > c_K'(x).$$

*Then there is a unique symmetric equilibrium in which types mix over disjoint intervals (ordered by type). Moreover, for each $t \in [0, 1]$,*

$$X_v(t) = c_{k(t)}^{-1}\big(\pi_v(t) - u_{k(t)}\big),$$

*where the equilibrium utilities are determined by the same iterative argument as above (with $b_0 = 0$).*

This representation of symmetric equilibrium—formulated in terms of the probability of outperforming an arbitrary agent—provides a unifying framework across environments and is central to our subsequent analysis.

# 4 Competition

In this section, we examine how increasing competitiveness of a contest, by shifting value to better prizes, influences the expected equilibrium effort.

To begin, the equilibrium characterization in Theorem 1 yields a tractable expression for expected equilibrium effort. Since $t$ denotes the probability of outperforming an arbitrary opponent, we can formally write $t = F(X)$, where $X \sim F$ is the equilibrium effort of an arbitrary agent. By the probability integral transform, $F(X)$ is uniformly distributed on $[0, 1]$. Hence, $t$ is ex-ante uniformly distributed on $[0, 1]$, and the expected equilibrium effort can be expressed as

$$\mathbb{E}[X] = \int_0^1 X_v(t)dt.$$

This implies that expected effort is linear in prize values. Solving the integral yields the corresponding coefficients.

**Lemma 2.** *The expected equilibrium effort is*

$$\mathbb{E}[X] = \sum_{m=1}^{N} \alpha_m v_m,$$

*where*

$$\alpha_m = \frac{1}{N+1}\left[\frac{1}{\theta_K} - \sum_{k=1}^{K-1}\left[H_{\geq m}^{N+1}(P_k) + (N-m)H_m^{N+1}(P_k)\right]\left(\frac{1}{\theta_{k+1}} - \frac{1}{\theta_k}\right)\right]. \tag{5}$$

From Lemma 2, the effect of increasing competition by shifting value from a lower-ranked prize $m'$ to a better-ranked prize $m$ is

$$\frac{\partial \mathbb{E}[X]}{\partial v_m} - \frac{\partial \mathbb{E}[X]}{\partial v_{m'}} = \alpha_m - \alpha_{m'},$$

which can be evaluated explicitly using Equation (5). Hence, the effect depends only on the contest environment $(N + 1, \Theta, p)$ and is independent of the specific contest $v \in \mathcal{V}$. In particular, if $\alpha_m - \alpha_{m'} > 0$, a budget-constrained designer seeking to maximize effort would continually shift value from prize $m'$ to $m$ subject to feasibility constraints. And if $\alpha_m - \alpha_{m'} < 0$, value would be shifted in the opposite direction.

**Theorem 2** (Winner-takes-all is optimal)**.** *Consider any contest environment* $(N + 1, \Theta, p)$ *with* $|\Theta| > 1$. *For any prize* $m' \in \{1, \dots, N - 1\}$,

$$\alpha_N - \alpha_{m'} > 0.$$

*Consequently, among all contests* $v \in \mathcal{V}$ *satisfying* $\sum_{m=0}^{N} v_m \leq V$, *the winner-takes-all contest* $v = (0, 0, \dots, 0, V)$ *uniquely maximizes expected equilibrium effort.*

This result resolves the conjecture of Sisak (2009), who—based on results under complete information—suggested that allocating the budget across multiple prizes might be optimal in environments with finitely many types. To see the role of uncertainty, note that when $|\Theta| = K = 1$, we have $\alpha_m - \alpha_{m'} = 0$, so any budget allocation across prizes induces the same expected effort (Barut and Kovenock (1998)). When multiple types are possible, however, the most efficient types are the cheapest to incentivize, and shifting value to the top prize has an encouraging effect on efficient types that more than offsets the discouraging effect on less efficient types. This extends the optimality result of Moldovanu and Sela (2001), established for the continuum case, to environments with finitely many types. Thus, even a small amount of uncertainty (i.e., incomplete information), is sufficient to make the (most competitive) winner-takes-all contest strictly optimal.

Interestingly, however, effort is not necessarily monotonic in the level of competition. Specifically, while transferring value to the best-ranked prize always encourages effort, increasing competition by transferring value to a better-ranked intermediate prize may not.

**Proposition 1** (Interior Discouragement Effect)**.** *Consider any contest environment* $(N+1, \Theta, p)$ *with* $|\Theta| = 2$*. For any prize* $m \in \{2, \ldots, N\}$*,*

$$P_1 > \frac{m}{N} \implies \textit{for all } m' < m,\ \alpha_m - \alpha_{m'} < 0.$$

Proposition 1 identifies a sufficient condition under which shifting value to a better-ranked intermediate prize discourages effort. Intuitively, as with shifting value to the best prize, such a transformation discourages inefficient types. However, in contrast to the best prize, the encouraging effect on efficient types is dampened, because the incentive for securing the best prize is also diluted. Consequently, the most efficient types are not as strongly incentivized, and their equilibrium effort may even decline following the transformation (Figure 2). Because this encouraging force is weaker, it may fail to offset the discouragement among inefficient types. In particular, when the inefficient type is sufficiently likely, the overall effect of shifting value to better-ranked intermediate prizes is negative.

In the extreme case where $P_1 \geq \frac{N-1}{N}$, it follows that any shift in value from a lower-ranked prize to a better-ranked prize discourages effort, except when the better prize is the best prize. Consequently, for the design problem, if the value of the best prize is capped, the optimal contest entails allocating the remaining budget evenly across all intermediate prizes.

# 5 General cost

So far, we have focused on contests with linear effort costs. In this section, we extend the analysis to general cost functions. Our approach is to identify conditions under which the effect of competition under linear costs remains informative under general costs.

## Model

Formally, we enrich the model as follows. Fix a contest environment $(N+1, \Theta, p)$ and contest $v \in \mathcal{V}$. If a type-$\theta_k$ agent chooses effort $x$ and wins prize $v_m$, their vNM utility is

$$v_m - \theta_k c(x),$$

where $c : \mathbb{R}_+ \to \mathbb{R}_+$ is a cost function satisfying $c(0) = 0$, $c'(x) > 0$ for $x > 0$, and $\lim_{x \to \infty} c(x) = \infty$.

We denote a symmetric equilibrium by $(X_1^*, X_2^*, \ldots, X_K^*)$, and let $X^*$ denote the ex-ante equilibrium effort of an arbitrary agent. As before, we define a mapping

$$X_v^* : [0, 1] \to \mathbb{R}_+$$

that assigns to each probability $t \in [0, 1]$ of outperforming an arbitrary agent the corresponding equilibrium effort.

## Equilibrium

For this Bayesian game, we can equivalently describe agents' choices in terms of the cost of effort $c(x)$ rather than effort itself. In other words, we can reinterpret the game as one in which agents directly choose effort costs, are ranked based on their effort costs (which coincides with their ranking based on effort), and are awarded the corresponding prizes. From this change-of-variable interpretation, it follows that the equilibrium effort under linear costs (Theorem 1) more generally characterizes equilibrium effort costs in the present environment. This leads immediately to the following characterization of equilibrium effort in this more general environment.

**Theorem 3** (Equilibrium Characterization)**.** *For any contest environment $(N + 1, \Theta, p)$ with cost function $c(\cdot)$ and contest $v \in \mathcal{V}$, there exists a unique symmetric equilibrium*

$(X_1^*, \dots, X_K^*)$. *It is such that, for each* $t \in [0,1]$,

$$X_v^*(t) = c^{-1}(X_v(t)),$$

*where* $X_v(t)$ *denotes the equilibrium effort with linear costs* $c(x) = x$ *(Theorem 1).*

## Competition

From the equilibrium characterization, the expected equilibrium effort can be expressed as

$$\mathbb{E}[X^*] = \int_0^1 g(X_v(t))\,dt, \quad \text{where } g = c^{-1}.$$

This representation again follows from the probability integral transform: the probability $t = F(X^*)$ of outperforming an arbitrary agent is ex-ante uniformly distributed on $[0,1]$.

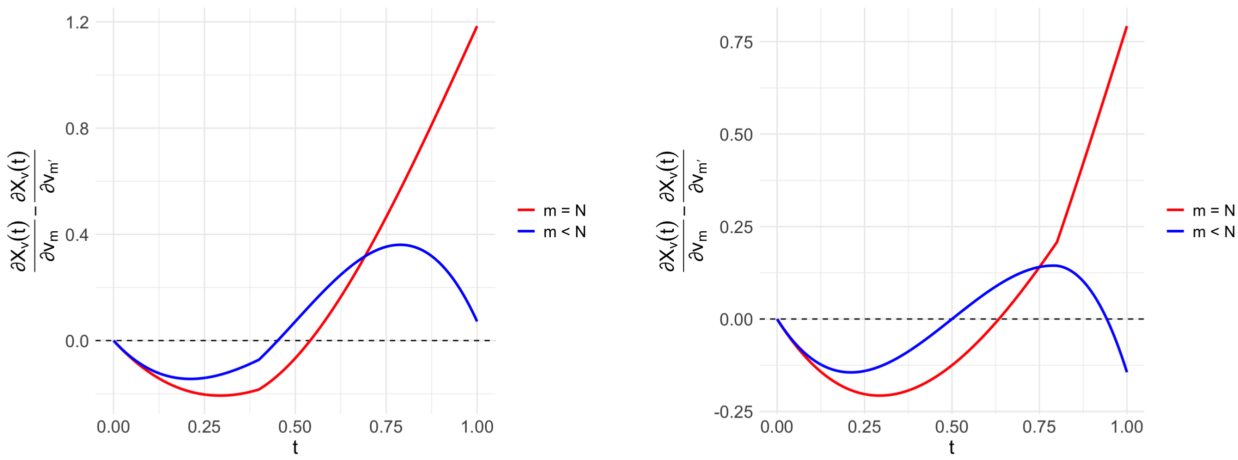
The effect of shifting value from a lower-ranked prize $m'$ to a better-ranked prize $m$ is then

$$\frac{\partial \mathbb{E}[X^*]}{\partial v_m} - \frac{\partial \mathbb{E}[X^*]}{\partial v_{m'}} = \int_0^1 g'(X_v(t)) \underbrace{\left(\frac{\partial X_v(t)}{\partial v_m} - \frac{\partial X_v(t)}{\partial v_{m'}}\right)}_{\substack{\text{i) effect on effort if } c(x)=x \\ \text{ii) effect on effort cost } c(X^*)}} dt, \tag{6}$$

where, from Equations (3) and (4), we obtain that

$$\frac{\partial X_v(t)}{\partial v_m} = \frac{H_m^N(t)}{\theta_{k(t)}} - \left[\sum_{j=1}^{k(t)-1} H_m^N(P_j)\left(\frac{1}{\theta_{j+1}} - \frac{1}{\theta_j}\right)\right]. \tag{7}$$

Equation (6) provides an interpretable and analytically convenient representation for evaluating the effect of shifting value from prize $m'$ to prize $m$ on expected effort. Specifically, $\frac{\partial X_v(t)}{\partial v_m} - \frac{\partial X_v(t)}{\partial v_{m'}}$ represents the effect of the transformation on the effort of a (pseudo)-type $t$ agent under linear cost. From Equation (7), it depends only on the contest environment $(N+1, \Theta, p)$ and the prizes $m, m'$. As discussed earlier, shifting value to better prizes discourages effort among inefficient types and encourages effort among efficient types. However, this encouraging effect is dampened when the better-ranked prize is interior ($m < N$), and may actually even discourage effort among the most efficient types, as illustrated in Figure 2. The term $g'(X_v(t))$ captures any influence that the cost function $c(\cdot)$ or the contest $v \in \mathcal{V}$ may have on how the transformation affects effort, and will be interpreted as simply




(a) Single-crossing ($p = (0.4, 0.6)$) (b) Double-crossing for $m < N$ ($p = (0.8, 0.2)$)

Figure 2: Effect of increasing competition on effort under linear cost ($N = 3, \Theta = \{2, 1\}$ and $m' = 1$)

assigning different weights to different pseudo-types.

We use the representation in Equation (6) to identify conditions under which the effect of the transformation on effort can be inferred from its effect under linear costs ($\alpha_m - \alpha_{m'}$). Specifically, we show that if, under linear costs, the effect on effort across types is single-crossing, then positive effects persist under concave costs, while negative effects persist under convex costs. To illustrate the idea, suppose that under linear costs the effect on effort across types is single-crossing and that the aggregate effect is non-negative ($\alpha_m - \alpha_{m'} \geq 0$). If the cost function $c(\cdot)$ is concave, then $g(\cdot)$ is convex, so that $g'(X_v(t))$ is increasing in $t$. Consequently, this term assigns relatively lower weights to inefficient pseudo-types and larger weights to efficient pseudo-types. Since the aggregate effect on effort under linear costs is assumed to be non-negative, it follows that $\frac{\partial \mathbb{E}[X^*]}{\partial v_m} - \frac{\partial \mathbb{E}[X^*]}{\partial v_{m'}} \geq 0$.

We are now ready to state our main result. The theorem identifies necessary and sufficient conditions for the effect on effort across types under linear costs to be single-crossing, and shows that, under these conditions, positive effects under linear costs persist under concave costs, while negative effects persist under convex costs.

**Theorem 4** (Linear to General)**.** *Consider any contest environment* $(N+1, \Theta, p)$*. Let* $m, m'$

*with $m > m'$ be such that either*

- $m = N$ *or*
- $\sum_{j=1}^{K-1} \left[H^N_{m'}(P_j) - H^N_m(P_j)\right] \left(\frac{1}{\theta_{j+1}} - \frac{1}{\theta_j}\right) \geq 0.$

*Then, the following hold:*

1. *If $\alpha_m - \alpha_{m'} \geq 0$ and $c$ is concave, then for any $v \in \mathcal{V}$, $\dfrac{\partial \mathbb{E}[X^*]}{\partial v_m} - \dfrac{\partial \mathbb{E}[X^*]}{\partial v_{m'}} \geq 0$.*
2. *If $\alpha_m - \alpha_{m'} \leq 0$ and $c$ is convex, then for any $v \in \mathcal{V}$, $\dfrac{\partial \mathbb{E}[X^*]}{\partial v_m} - \dfrac{\partial \mathbb{E}[X^*]}{\partial v_{m'}} \leq 0$.*

**Remark 3.** *Theorem 4 nests and extends the result of Fang, Noe, and Strack (2020) from complete information to general contest environments with finite types. To see this, consider the special case of a complete information environment ($|\Theta| = K = 1$). In this setting, for any $m, m'$ with $m > m'$, the conditions of Theorem 4 are satisfied. Moreover, $\alpha_m - \alpha_{m'} = 0$. It therefore follows that increasing competition always encourages effort under concave costs and always discourages effort under convex costs, recovering the complete information result of Fang, Noe, and Strack (2020).*

## Other objectives

While we have focused on expected effort, Theorem 4 can be readily generalized to other quantities of interest. To illustrate, consider the expected maximum effort, another commonly studied objective in contest design (Archak and Sundararajan (2009); Wasser and Zhang (2023)). From the equilibrium characterization in Theorem 3, the expected maximum effort can be expressed as

$$\mathbb{E}[X^*_{max}] = (N+1) \int_0^1 g(X_v(t)) t^N dt.$$

Thus, the equilibrium effort of a pseudo-type $t$ agent is weighted by the probability that their effort is the maximum, which is $t^N$.

The effect of shifting value from a lower-ranked prize $m'$ to a better-ranked prize $m$ is then

$$\frac{\partial \mathbb{E}[X^*_{max}]}{\partial v_m} - \frac{\partial \mathbb{E}[X^*_{max}]}{\partial v_{m'}} = (N+1) \int_0^1 \underbrace{g'(X_v(t))\, t^N}_{\text{weights}} \underbrace{\left(\frac{\partial X_v(t)}{\partial v_m} - \frac{\partial X_v(t)}{\partial v_{m'}}\right)}_{\text{single-crossing?}} dt. \tag{8}$$

As in Theorem 4, under monotonic weights and single-crossing, the sign of $\alpha_m - \alpha_{m'}$ may be informative about how the transformation affects expected maximum effort. Thus, Theorem 4 provides a simple and general recipe for assessing whether shifting value towards better-ranked prizes encourages or discourages expected effort and related objectives.

## Design problem

The result has implications for the design problem of allocating a fixed prize budget. Suppose the cost function $c$ is concave. It follows from Theorem 4 that, for any contest $v \in \mathcal{V}$, shifting value from any lower-ranked prize $m'$ to the top prize $m = N$ has a positive effect on expected effort. This is because for $m = N$, the single-crossing property holds, and Theorem 2 establishes that $\alpha_N - \alpha_{m'} \geq 0$. A similar implication holds for expected maximum effort. Under concave costs, the weights in Equation (8) are increasing in $t$, implying that the same transformation also increases expected maximum effort. More generally, the following result establishes that, under both linear and concave costs, the winner-takes-all contest is optimal not only for maximizing total effort and maximum effort, but also for maximizing the total effort of the top $q$ agents, for any $q$.

**Theorem 5** (Winner-takes-all is optimal)**.** *Consider any contest environment* $(N + 1, \Theta, p)$ *with cost function* $c(\cdot)$*. If* $c$ *is (weakly) concave, among all contests* $v \in \mathcal{V}$ *satisfying* $\sum_{m=0}^{N} v_m \leq V$*, the winner-takes-all contest* $v = (0, 0, \dots, 0, V)$ *maximizes expected total effort of top* $q$ *agents for any* $q \in [N + 1]$.

Intuitively, as discussed earlier under linear costs, the winner-takes-all contest creates strong incentives for the most efficient pseudo-types—those who are cheapest to incentivize—and this more than compensates for the discouragement it induces among less efficient pseudo-types. Consequently, aggregate effort is maximized (Theorem 2). When the designer instead cares about the total effort of the top $q$ agents, the objective effectively places greater weight on the effort of the more efficient pseudo-types. Since winner-takes-all disproportionately strengthens incentives precisely for these types, its optimality persists under this more selective objective. This conclusion extends to concave cost functions as well. Under concavity, marginal effort costs are decreasing, making it even cheaper to induce higher effort levels from efficient pseudo-types. Thus, shifting value to the best prize amplifies effort where it is most cost-effective, reinforcing the optimality of the winner-takes-all contest.

This intuition, however, does not extend directly to the case of convex costs. When costs are convex, marginal effort costs are increasing, so inducing additional effort at higher effort levels becomes progressively more expensive. As a result, concentrating prize value at the top no longer has an unambiguously dominant effect, since the designer must now weigh stronger incentives for efficient pseudo-types against the rising marginal costs of effort. Nevertheless, we show that the optimality of the winner-takes-all contest persists under moderate convexity. In other words, as long as costs are not too convex, the incentive gains from concentrating prize value at the top continue to outweigh the associated marginal cost increases. The following proposition formalizes this for a parametric class of cost functions.

**Proposition 2.** *Consider any contest environment* $(N+1, \Theta, p)$ *with* $|\Theta| > 1$ *and cost* $c(x) = x^{\alpha}$. *For any* $q \in [N+1]$, *there exists* $\alpha^* > 1$ *such that for all* $\alpha \in [1, \alpha^*)$, *among all contests* $v \in \mathcal{V}$ *satisfying* $\sum_{m=0}^{N} v_m \leq V$, *the winner-takes-all contest* $v = (0, 0, \ldots, 0, V)$ *maximizes the expected total effort of the top* $q$ *agents.*

Proposition 2 reveals a sharp contrast with the complete-information environment, where the (most competitive) winner-takes-all contest instead minimizes total effort under convex costs (Remark 3). Our analysis suggests a reconciliation of these contrasting findings in that the supremacy of the winner-takes-all contest over alternative prize structures in incomplete-information environments diminishes as the environment approaches complete information. This pattern is illustrated in Figure 3, which shows that in a binary-type environment, the effect of shifting value to the top prize—while always positive—shrinks as the probability of the inefficient type, $p_1$, approaches 0 or 1. Based on these findings, we conjecture that the degree of convexity required to overturn the optimality of the winner-takes-all contest likewise decreases as the environment approaches complete information.

**Remark 4.** *Our results (Theorem 5 and Proposition 2) establish that the winner-takes-all contest is robustly optimal in the sense that it maximizes the total effort of the top* $q$ *agents—an objective that places relatively greater weight on the effort of better-ranked agents—provided effort costs are not too convex. More generally, we can identify conditions under which equi-split contests—contests that divide the prize budget equally across a subset of prizes—are optimal. Such contests, together with the null contest* $v = (0, 0, \ldots, 0)$, *constitute the extreme points of the feasible set, allowing us to invoke Bauer's maximum principle to identify conditions when they are optimal. As an application, under linear effort costs and any linear objective of the designer—assigning arbitrary weights to the efforts of differently ranked agents—we can show that there always exists an optimal equi-split contest.*

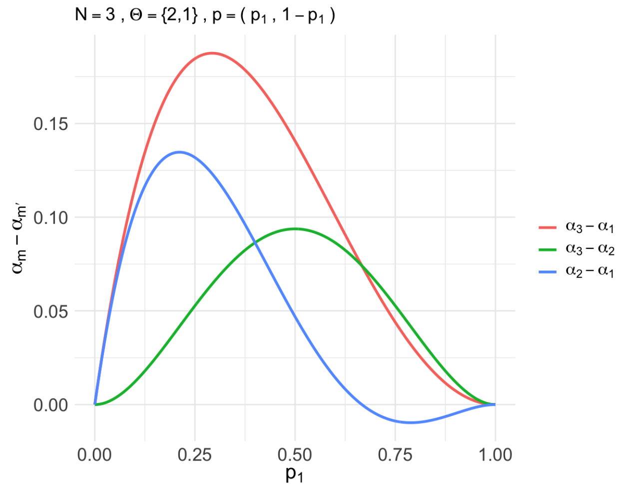



Figure 3: Effect of increasing competition: Complete and Incomplete information.

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# A Proofs for Section 3 (Equilibrium)

**Lemma 1.** *If* $(X_1, \ldots, X_K)$ *is an equilibrium, then there exist boundary points*

$$0 = b_0 < b_1 < \cdots < b_K$$

*such that, for each* $k$*,* $X_k$ *is continuously distributed on* $[b_{k-1}, b_k]$*.*

*Proof.* Suppose $(X_1, \ldots, X_K)$ is an equilibrium, and $X \sim F$ is the ex-ante equilibrium effort. We establish the result in three steps.

Step 1: For each $k$, $X_k$ is a continuous random variable.

Suppose, for contradiction, that $\Pr[X_k = x] > 0$ for some $x$. Under the given profile, there is a positive probability that all $N + 1$ agents choose effort $x$, in which case ties are broken uniformly at random. If a type-$\theta_k$ agent deviates to $x + \epsilon$, it obtains a discontinuous jump in expected prize (since $v_N > v_0$) while the additional cost $\theta_k(x+\epsilon) - \theta_k x$ can be made arbitrarily small. Hence the deviation is profitable, a contradiction.

Without loss of generality, we assume that the support of $X_k$ is closed.

Step 2: There exists $b_K > 0$ such that $X$ is continuously distributed on $[0, b_K]$.

Suppose there exists an interval $(d_1, d_2)$ that is not contained in the support of $X$. Consider a type whose support contains $d_2$. By deviating to $d_1$, the agent obtains the same expected prize while incurring strictly lower cost, yielding a profitable deviation. Hence the support of $X$ cannot contain gaps. It follows that the support is a convex subset of $\mathbb{R}_+$.

Step 3: There exist $b_1 < b_2 < \cdots < b_K$ such that the support of $X_k$ is $[b_{k-1}, b_k]$.

Let $x < y$ lie in the support of $X_k$. Since a type-$\theta_k$ agent must be indifferent between the two,

$$\pi_v(F(y)) - \theta_k y = \pi_v(F(x)) - \theta_k x.$$

For a type-$\theta_j$ agent, the payoff difference between choosing $y$ and $x$ is

$$\pi_v(F(y)) - \theta_j y - \big(\pi_v(F(x)) - \theta_j x\big) = (\theta_k - \theta_j)(y - x).$$

This expression is strictly positive if $\theta_j < \theta_k$ and strictly negative if $\theta_j > \theta_k$. Hence no other type is indifferent between $x$ and $y$, implying that the supports of $X_j$ and $X_k$ intersect in at most boundary points. The ordering follows from the observation that if $\theta_j < \theta_k$, a type-$\theta_j$ agent obtains a higher payoff from $y$ then $x$. Therefore there exist boundary points $0 = b_0 < b_1 < \cdots < b_K$ such that the support of $X_k$ is $[b_{k-1}, b_k]$. $\square$

**Theorem 1** (Equilibrium Characterization)**.** *For any contest environment* $(N+1, \Theta, p)$ *and contest* $v \in \mathcal{V}$*, there exists a unique symmetric equilibrium* $(X_1, \ldots, X_K)$*. It is such that, for each* $t \in [0, 1]$*,*

$$X_v(t) = \frac{\pi_v(t) - u_{k(t)}}{\theta_{k(t)}}, \tag{3}$$

*where* $k(t) = \max\{k : P_{k-1} \leq t\}$*, and the equilibrium utilities are given by*

$$u_k = \theta_k \left[ \sum_{j=1}^{k-1} \pi_v(P_j) \left( \frac{1}{\theta_{j+1}} - \frac{1}{\theta_j} \right) \right]. \tag{4}$$

*Proof.* It remains only to establish the equilibrium utilities, which we do by induction.

Base case. Since zero effort lies in the support of the type-$\theta_1$ strategy and yields payoff 0, we have $u_1 = 0$.

Induction step. Suppose

$$u_k = \theta_k \left[ \sum_{j=1}^{k-1} \pi_v(P_j) \left( \frac{1}{\theta_{j+1}} - \frac{1}{\theta_j} \right) \right].$$

Evaluating the equilibrium effort at $t = P_k$ yields

$$\begin{aligned} X_v(P_k) &= \frac{\pi_v(P_k) - u_k}{\theta_k} \\ &= \frac{\pi_v(P_k)}{\theta_k} - \sum_{j=1}^{k-1} \pi_v(P_j) \left( \frac{1}{\theta_{j+1}} - \frac{1}{\theta_j} \right) \\ &= b_k. \end{aligned}$$

Since $b_k$ lies in the support of the type-$\theta_{k+1}$ strategy, their equilibrium utility is

$$\begin{aligned} u_{k+1} &= \pi_v(P_k) - \theta_{k+1} b_k \\ &= \pi_v(P_k) - \theta_{k+1} \left[ \frac{\pi_v(P_k)}{\theta_k} - \sum_{j=1}^{k-1} \pi_v(P_j) \left( \frac{1}{\theta_{j+1}} - \frac{1}{\theta_j} \right) \right] \\ &= \theta_{k+1} \left[ \sum_{j=1}^{k} \pi_v(P_j) \left( \frac{1}{\theta_{j+1}} - \frac{1}{\theta_j} \right) \right], \end{aligned}$$

which establishes the inductive claim. $\square$

# B Proofs for Section 4 (Competition)

**Lemma 2.** *The expected equilibrium effort is*

$$\mathbb{E}[X] = \sum_{m=1}^{N} \alpha_m v_m,$$

*where*

$$\alpha_m = \frac{1}{N+1}\left[\frac{1}{\theta_K} - \sum_{k=1}^{K-1}\left[H_{\geq m}^{N+1}(P_k) + (N-m)H_m^{N+1}(P_k)\right]\left(\frac{1}{\theta_{k+1}} - \frac{1}{\theta_k}\right)\right]. \tag{5}$$

*Proof.* Since $t = F(X)$ is uniformly distributed on $[0,1]$,

$$\begin{aligned}
\mathbb{E}[X] &= \int_0^1 X_v(t)dt \\
&= \int_0^1 \frac{\pi_v(t) - u_{k(t)}}{\theta_{k(t)}}dt && \text{(Theorem 1)} \\
&= \sum_{k=1}^{K} p_k \cdot \frac{1}{\theta_k} \cdot \left[\int_{P_{k-1}}^{P_k} \frac{\pi_v(t)}{p_k}dt - u_k\right].
\end{aligned}$$

Notice that for any $k \in [K]$, $\int_{P_{k-1}}^{P_k} \frac{\pi_v(t)}{p_k}dt$ is the expected prize awarded to a type-$\theta_k$ agent. To compute this, we instead compute the ex-ante expected total prize awarded to type-$\theta_k$ agents. Notice that for any prize $m \in \{0, \ldots, N\}$, the ex-ante probability that this prize is awarded to a type-$\theta_k$ agent is simply

$$\left[H_{\geq m+1}^{N+1}(P_k) - H_{\geq m+1}^{N+1}(P_{k-1})\right].$$

Thus, the ex-ante expected total prize awarded to type-$\theta_k$ agents is

$$\sum_{m=1}^{N} v_m \left[H_{\geq m+1}^{N+1}(P_k) - H_{\geq m+1}^{N+1}(P_{k-1})\right].$$

By an alternative calculation, which entails adding up over the $N+1$ agents, this expectation should equal

$$(N+1) \cdot p_k \cdot \int_{P_{k-1}}^{P_k} \frac{\pi_v(t)}{p_k}dt.$$

Equating these two, we get that

$$\int_{P_{k-1}}^{P_k} \pi_v(t)dt = \frac{\sum_{m=1}^{N} v_m \left[H_{\geq m+1}^{N+1}(P_k) - H_{\geq m+1}^{N+1}(P_{k-1})\right]}{N+1}.$$

Alternatively, we can also directly use the following fact to compute this integral:

$$\frac{\partial H_{\geq m+1}^{N+1}(t)}{\partial t} = (N+1)H_m^N(t)$$

Substituting this in the above representation, we get

$$\mathbb{E}[X] = \sum_{k=1}^{K} \frac{1}{(N+1)\theta_k} \sum_{m=1}^{N} v_m \left[H_{\geq m+1}^{N+1}(P_k) - H_{\geq m+1}^{N+1}(P_{k-1})\right] - \sum_{k=1}^{K} \frac{p_k u_k}{\theta_k}.$$

From here, we write

$$\mathbb{E}[X] = \sum_{m=1}^{N} \alpha_m v_m$$

where

$$\begin{aligned}
\alpha_m &= \sum_{k=1}^{K} \frac{\left[H_{\geq m+1}^{N+1}(P_k) - H_{\geq m+1}^{N+1}(P_{k-1})\right]}{(N+1)\theta_k} - \sum_{k=1}^{K} p_k \sum_{j=1}^{k-1} H_m^N(P_j)\left(\frac{1}{\theta_{j+1}} - \frac{1}{\theta_j}\right) \\
&= \sum_{k=1}^{K} \frac{\left[H_{\geq m+1}^{N+1}(P_k) - H_{\geq m+1}^{N+1}(P_{k-1})\right]}{(N+1)\theta_k} - \sum_{k=1}^{K-1} (1-P_k) H_m^N(P_k)\left(\frac{1}{\theta_{k+1}} - \frac{1}{\theta_k}\right) \\
&= \frac{1}{N+1}\left[\frac{1}{\theta_K} - \sum_{k=1}^{K-1} H_{\geq m+1}^{N+1}(P_k)\left(\frac{1}{\theta_{k+1}} - \frac{1}{\theta_k}\right)\right] - \frac{(N+1-m)}{N+1} \sum_{k=1}^{K-1}\left[H_m^{N+1}(P_k)\left(\frac{1}{\theta_{k+1}} - \frac{1}{\theta_k}\right)\right] \\
&= \frac{1}{N+1}\left[\frac{1}{\theta_K} - \sum_{k=1}^{K-1}\left[H_{\geq m}^{N+1}(P_k) + (N-m)H_m^{N+1}(P_k)\right]\left(\frac{1}{\theta_{k+1}} - \frac{1}{\theta_k}\right)\right].
\end{aligned}$$

□

**Theorem 2** (Winner-takes-all is optimal)**.** *Consider any contest environment* $(N+1, \Theta, p)$ *with* $|\Theta| > 1$*. For any prize* $m' \in \{1, \ldots, N-1\}$*,*

$$\alpha_N - \alpha_{m'} > 0.$$

*Consequently, among all contests* $v \in \mathcal{V}$ *satisfying* $\sum_{m=0}^{N} v_m \leq V$*, the winner-takes-all contest* $v = (0, 0, \ldots, 0, V)$ *uniquely maximizes expected equilibrium effort.*

*Proof.* From Equation (5), we have that for any prize $m' \in \{1, \ldots, N-1\}$,

$$\alpha_N - \alpha_{m'} = \frac{1}{N+1}\left[\sum_{k=1}^{K-1}\left[H_{\geq m'}^{N+1}(P_k) - H_{\geq N}^{N+1}(P_k) + (N-m')H_{m'}^{N+1}(P_k)\right]\left(\frac{1}{\theta_{k+1}} - \frac{1}{\theta_k}\right)\right].$$

With $|\Theta| = K > 1$, it is straightforward to verify that $\alpha_N - \alpha_{m'} > 0$. It follows that for any contest $v \in \mathcal{V}$, transferring value from any lower-ranked prize $m'$ to the top-prize $N$ leads to an increase in expected effort. □

**Proposition 1** (Interior Discouragement Effect)**.** *Consider any contest environment* $(N+1, \Theta, p)$ *with* $|\Theta| = 2$*. For any prize* $m \in \{2, \ldots, N\}$,

$$P_1 > \frac{m}{N} \implies \textit{for all } m' < m,\ \alpha_m - \alpha_{m'} < 0.$$

*Proof.* From Equation (5), we have that for any prize $m \in \{2, \ldots, N\}$,

$$\begin{aligned}
\alpha_m - \alpha_{m-1} &= \frac{1}{N+1}\left[\left[H_{m-1}^{N+1}(P_1) + (N-(m-1))H_{m-1}^{N+1}(P_1) - (N-m)H_m^{N+1}(P_1)\right]\left(\frac{1}{\theta_2} - \frac{1}{\theta_1}\right)\right] \\
&= \frac{H_{m-1}^{N+1}(P_1)}{N+1}\left(\frac{1}{\theta_2} - \frac{1}{\theta_1}\right)\left[N-m+2-(N-m)\frac{N-m+2}{m}\frac{P_1}{1-P_1}\right] \\
&= \frac{(N-m+2)H_{m-1}^{N+1}(P_1)}{N+1}\left(\frac{1}{\theta_2} - \frac{1}{\theta_1}\right)\left[1 - \frac{(N-m)}{m}\frac{P_1}{1-P_1}\right].
\end{aligned}$$

It is straightforward to verify that if $P_1 > \frac{m}{N}$, then $\alpha_m - \alpha_{m-1} < 0$.

Now for any $m' < m$,

$$\alpha_m - \alpha_{m'} = (\alpha_m - \alpha_{m-1}) + (\alpha_{m-1} - \alpha_{m-2}) + \cdots + (\alpha_{m'+1} - \alpha_{m'}),$$

and the result follows.

□

# C Proofs for Section 5 (General cost)

**Lemma 3.** *Suppose* $a_2 : [0,1] \to \mathbb{R}$ *is such that there exists* $t^* \in [0,1]$ *so that* $a_2(t) \leq 0$ *for* $t \leq t^*$ *and* $a_2(t) \geq 0$ *for* $t \geq t^*$*. Then, for any increasing function* $a_1 : [0,1] \to \mathbb{R}$,

$$\int_0^1 a_1(t)a_2(t)dt \geq a_1(t^*)\int_0^1 a_2(t)dt.$$

*Proof.* Observe that

$$\begin{aligned}
\int_0^1 a_1(t)a_2(t)dt &= \int_0^{t^*} a_1(t)a_2(t)dt + \int_{t^*}^1 a_1(t)a_2(t)dt \\
&\geq \int_0^{t^*} a_1(t^*)a_2(t)dt + \int_{t^*}^1 a_1(t^*)a_2(t)dt \\
&= a_1(t^*)\int_0^1 a_2(t)dt.
\end{aligned}$$

□

**Theorem 4** (Linear to General)**.** *Consider any contest environment* $(N+1, \Theta, p)$*. Let* $m, m'$ *with* $m > m'$ *be such that either*

- $m = N$ *or*
- $\sum_{j=1}^{K-1} \left[H_{m'}^N(P_j) - H_m^N(P_j)\right] \left(\frac{1}{\theta_{j+1}} - \frac{1}{\theta_j}\right) \geq 0.$

*Then, the following hold:*

1. *If* $\alpha_m - \alpha_{m'} \geq 0$ *and* $c$ *is concave, then for any* $v \in \mathcal{V}$*,* $\dfrac{\partial \mathbb{E}[X^*]}{\partial v_m} - \dfrac{\partial \mathbb{E}[X^*]}{\partial v_{m'}} \geq 0.$
2. *If* $\alpha_m - \alpha_{m'} \leq 0$ *and* $c$ *is convex, then for any* $v \in \mathcal{V}$*,* $\dfrac{\partial \mathbb{E}[X^*]}{\partial v_m} - \dfrac{\partial \mathbb{E}[X^*]}{\partial v_{m'}} \leq 0.$

*Proof.* From Equation (6), we have that for any contest $v \in \mathcal{V}$ and prizes $m, m'$ with $m > m'$,

$$\frac{\partial \mathbb{E}[X^*]}{\partial v_m} - \frac{\partial \mathbb{E}[X^*]}{\partial v_{m'}} = \int_0^1 g'\left(X_v(t)\right) \left(\frac{\partial X_v(t)}{\partial v_m} - \frac{\partial X_v(t)}{\partial v_{m'}}\right) dt$$

where, from Equation (7), we obtain that

$$\frac{\partial X_v(t)}{\partial v_m} - \frac{\partial X_v(t)}{\partial v_{m'}} = \left(\frac{H_m^N(t) - H_{m'}^N(t)}{\theta_{k(t)}}\right) - \left[\sum_{j=1}^{k(t)-1} \left(H_m^N(P_j) - H_{m'}^N(P_j)\right) \left(\frac{1}{\theta_{j+1}} - \frac{1}{\theta_j}\right)\right].$$

From here, one can verify that

1. $\left.\frac{\partial X_v(t)}{\partial v_m} - \frac{\partial X_v(t)}{\partial v_{m'}}\right|_{t=0} = 0$
2. $\left.\frac{\partial X_v(t)}{\partial v_m} - \frac{\partial X_v(t)}{\partial v_{m'}}\right|_{t=1} = \begin{cases} \frac{1}{\theta_K} - \left[\sum_{j=1}^{K-1} \left(H_m^N(P_j) - H_{m'}^N(P_j)\right)\left(\frac{1}{\theta_{j+1}} - \frac{1}{\theta_j}\right)\right] & \text{if } m = N \\ \left[\sum_{j=1}^{K-1} \left(H_{m'}^N(P_j) - H_m^N(P_j)\right)\left(\frac{1}{\theta_{j+1}} - \frac{1}{\theta_j}\right)\right] & \text{otherwise} \end{cases}$
3. $\frac{\partial X_v(t)}{\partial v_m} - \frac{\partial X_v(t)}{\partial v_{m'}}$ is continuous in $t$
4. $\frac{\partial X_v(t)}{\partial v_m} - \frac{\partial X_v(t)}{\partial v_{m'}}$ is differentiable at all $t$, except when $t = P_k$. At any $t \in (0,1)$ such that $t \neq P_k$, the derivative has the same sign as the derivative of $H_m^N(t) - H_{m'}^N(t)$ with respect to $t$.

The conditions on $m$ and $m'$ ensure that $\left.\frac{\partial X_v(t)}{\partial v_m} - \frac{\partial X_v(t)}{\partial v_{m'}}\right|_{t=1} \geq 0$. Together with the above properties, this implies that there is some $t^* \in [0,1]$ such that $\frac{\partial X_v(t)}{\partial v_m} - \frac{\partial X_v(t)}{\partial v_{m'}} \leq 0$ for $t \in [0, t^*]$, and $\frac{\partial X_v(t)}{\partial v_m} - \frac{\partial X_v(t)}{\partial v_{m'}} \geq 0$ for $t \in [t^*, 1]$.

If $c$ is concave, $g = c^{-1}$ is convex, and thus, $g'(X_v(t))$ is increasing in $t$. Applying Lemma 3 with $a_1(t) = g'(X_v(t))$ and $a_2(t) = \frac{\partial X_v(t)}{\partial v_m} - \frac{\partial X_v(t)}{\partial v_{m'}}$ gives

$$\begin{aligned}\frac{\partial \mathbb{E}[X^*]}{\partial v_m} - \frac{\partial \mathbb{E}[X^*]}{\partial v_{m'}} &\geq g'(X_v(t^*)) \int_0^1 \left( \frac{\partial X_v(t)}{\partial v_m} - \frac{\partial X_v(t)}{\partial v_{m'}} \right) dt \\ &= g'(X_v(t^*)) (\alpha_m - \alpha_{m'})\end{aligned}$$

and the result follows. An analogous argument applies for the case where $c$ is convex.

□

**Theorem 5** (Winner-takes-all is optimal). *Consider any contest environment $(N+1, \Theta, p)$ with cost function $c(\cdot)$. If $c$ is (weakly) concave, among all contests $v \in \mathcal{V}$ satisfying $\sum_{m=0}^{N} v_m \leq V$, the winner-takes-all contest $v = (0, 0, \ldots, 0, V)$ maximizes expected total effort of top $q$ agents for any $q \in [N+1]$.*

*Proof.* We first establish the result for $q = 1$. For any contest $v \in \mathcal{V}$, the effect of shifting value from any prize $m' \in [N-1]$ to the best prize $m = N$ on expected maximum effort is

$$\frac{\partial \mathbb{E}[X^*_{max}]}{\partial v_N} - \frac{\partial \mathbb{E}[X^*_{max}]}{\partial v_{m'}} = (N+1) \int_0^1 \underbrace{g'(X_v(t))\, t^N}_{\text{increasing weights}} \underbrace{\left( \frac{\partial X_v(t)}{\partial v_N} - \frac{\partial X_v(t)}{\partial v_{m'}} \right)}_{\text{single-crossing}} dt.$$

Since $\alpha_N - \alpha_{m'} > 0$ (Theorem 2), applying Lemma 3 with $a_1(t) = g'(X_v(t))\, t^N$ and $a_2(t) = \frac{\partial X_v(t)}{\partial v_N} - \frac{\partial X_v(t)}{\partial v_{m'}}$ yields

$$\frac{\partial \mathbb{E}[X^*_{max}]}{\partial v_N} - \frac{\partial \mathbb{E}[X^*_{max}]}{\partial v_{m'}} \geq 0.$$

Thus, shifting value toward the best prize increases expected maximum effort, implying that the winner-takes-all contest is optimal for $q = 1$.

We now extend the argument to arbitrary $q \in [N+1]$. The same structure applies. The key observation is that the probability that a pseudo-type $t$ agent ranks among the top $q$ agents is increasing in $t$. Consequently, under concave costs, the induced weights remain monotone increasing in $t$. The single-crossing property holds as before, and Lemma 3 delivers the same sign result. Hence, shifting value toward the best prize increases the expected total effort of the top $q$ agents, implying that the winner-takes-all contest is optimal for any $q$. □

**Proposition 2.** *Consider any contest environment $(N+1, \Theta, p)$ with $|\Theta| > 1$ and cost $c(x) = x^\alpha$. For any $q \in [N+1]$, there exists $\alpha^* > 1$ such that for all $\alpha \in [1, \alpha^*)$, among* *all contests $v \in \mathcal{V}$ satisfying $\sum_{m=0}^{N} v_m \leq V$, the winner-takes-all contest $v = (0, 0, \ldots, 0, V)$ maximizes the expected total effort of the top $q$ agents.*

*Proof.* We first establish the result for $q = N + 1$, i.e., for total expected effort. For any contest $v \in \mathcal{V}$, the effect of shifting value from any prize $m' \in [N-1]$ to the top prize $m = N$ on expected effort is

$$\frac{\partial \mathbb{E}[X^*]}{\partial v_N} - \frac{\partial \mathbb{E}[X^*]}{\partial v_{m'}} = \int_0^1 g'(X_v(t)) \left( \frac{\partial X_v(t)}{\partial v_N} - \frac{\partial X_v(t)}{\partial v_{m'}} \right) dt,$$

where

$$g'(X_v(t)) = \frac{1}{\alpha} X_v(t)^{\frac{1}{\alpha}-1}.$$

From Theorem 2, we know that at $\alpha = 1$ this expression is strictly positive. By continuity of the expression in both prize values $v$ and the curvature parameter $\alpha$, there exist $\epsilon_v > 0$ and $\delta_v > 0$ such that for all contests $v'$ in an $\epsilon_v$-neighborhood of $v$, and all $\alpha < 1 + \delta_v$, the effect of the transformation remains strictly positive.

The collection of such neighborhoods of contests forms an open cover of the feasible set of contests, which is compact. We may therefore extract a finite subcover. Let $\delta_{m'}$ denote the minimum of the corresponding $\delta_v$ across this finite subcover. It follows that for all feasible contests $v$ and all $\alpha < 1 + \delta_{m'}$, shifting value from prize $m'$ to the top prize increases expected effort.

Repeating this argument for each interior prize $m' \in [N - 1]$ and taking the minimum across these interior prizes yields $\alpha^* > 1$ such that, for all $\alpha \in [1, \alpha^*)$, starting from any feasible contest $v \in \mathcal{V}$, shifting value from any lower-ranked prize to the best prize increases expected effort. This establishes the result for $q = N + 1$.

The argument for general $q \in [N + 1]$ is identical. The effect of shifting value to the best prize is strictly positive under linear costs (Theorem 5), continuous in both $v$ and $\alpha$, and the feasible set remains compact. The same continuity and covering argument therefore delivers the result. □

# D Convergence

In this section, we establish an equilibrium convergence result for the continuum type-space. Specifically, we show that if a sequence of (parametric) finite type-space distributions converges to a differentiable distribution over a continuum type-space, then the corresponding sequence of mixed-strategy equilibria converges to the pure-strategy equilibrium in the continuum model. Intuitively, as the finite type-space becomes large, the interval over which a given type mixes shrinks, and essentially converges to the effort level prescribed by the pure-strategy equilibrium under the continuum type-space. Thus, the equilibrium in an appropriate and sufficiently large finite-type space provides a reasonable approximation to the equilibrium strategy under the continuum type-space, and vice versa.

We begin by recalling the symmetric equilibrium under a (parametric) continuum type-space (Moldovanu and Sela (2001)). For this section, we focus on the linear cost case ($c(x) = x$), which is without loss of generality due to the equivalence between convergence in effort cost and in effort.

**Lemma 4.** *Suppose there are $N+1$ agents, each with a private type (marginal cost of effort) drawn from $\Theta = [\underline{\theta}, \overline{\theta}]$ according to a differentiable CDF $G : [\underline{\theta}, \overline{\theta}] \to [0,1]$. For any contest $v \in \mathcal{V}$, there is a unique symmetric Bayes-Nash equilibrium and it is such that for any $\theta \in \Theta$,*

$$X(\theta) = \int_{\theta}^{\overline{\theta}} \frac{\pi_v'(1 - G(t))g(t)}{t} dt.$$

*Proof.* Suppose $N$ agents are playing a strategy $X : [\underline{\theta}, \overline{\theta}] \to \mathbb{R}_+$ so that if an agent's type is $\theta$, it exerts effort $X(\theta)$. Further suppose that $X(\theta)$ is decreasing in $\theta$. Now we want to find the remaining agent's best response to this strategy of the other agents. If the agent's type is $\theta$ and it pretends to be an agent of type $t \in [\underline{\theta}, \overline{\theta}]$, its payoff is

$$\pi_v(1 - G(t)) - \theta X(t).$$

Taking the first order condition, we get

$$\pi_v'(1 - G(t))(-g(t)) - \theta X'(t) = 0.$$

Now we can plug in $t = \theta$ to get the condition for $X(\theta)$ to be a symmetric Bayes-Nash equilibrium. Doing so, we get

$$\pi_v'(1 - G(\theta))(-g(\theta)) - \theta X'(\theta) = 0$$

so that

$$X(\theta) = \int_{\theta}^{\overline{\theta}} \frac{\pi_v'(1 - G(t))g(t)}{t} dt.$$

□

We now state and prove the convergence result.

**Theorem 6.** *Suppose there are $N+1$ agents and fix any contest $v \in \mathcal{V}$. Let $G : [\underline{\theta}, \overline{\theta}] \to [0,1]$ be a differentiable CDF and let $G^1, G^2, \ldots,$ be any sequence of CDF's, each with a finite support, such that for all $\theta \in [\underline{\theta}, \overline{\theta}]$,*

$$\lim_{n\to\infty} G^n(\theta) = G(\theta).$$

*Let $F^n : \mathbb{R} \to [0,1]$ denote CDF of the equilibrium effort under $G^n$, and let $F : \mathbb{R} \to [0,1]$ denote CDF of the equilibrium effort under $G$. Then, the sequence of CDF's $F^1, F^2, \ldots,$ converges to the CDF $F$, i.e., for all $x \in \mathbb{R}$,*

$$\lim_{n\to\infty} F^n(x) = F(x).$$

*Proof.* For the finite support CDF $G^n$, let $\Theta^n = (\theta_1^n, \theta_2^n, \ldots, \theta_{K(n)}^n)$ denote the support and $p^n = (p_1^n, p_2^n, \ldots, p_{K(n)}^n)$ denote the probability mass function. From Theorem 1, let $b^n = (b_1^n, b_2^n, \ldots, b_{K(n)}^n)$ denote the boundary points, $u^n = (u_1^n, u_2^n, \ldots, u_{K(n)}^n)$ denote the equilibrium utilities, and $F_k^n$ denote the equilibrium CDF of agent of type $\theta_k^n$ on support $[b_{k-1}^n, b_k^n]$. Then, the CDF of the equilibrium effort, $F^n : \mathbb{R} \to [0,1]$, is such that for any $x \in \mathbb{R}$,

$$F^n(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ P_{k-1}^n + p_k^n F_k^n(x) & \text{if } x \in [b_{k-1}^n, b_k^n] \\ 1 & \text{if } x \geq b_{K(n)}^n \end{cases}. \tag{9}$$

For the continuum CDF $G : [\underline{\theta}, \overline{\theta}] \to [0,1]$, the CDF of the equilibrium effort, $F : \mathbb{R} \to [0,1]$, is such that for any $x \in \mathbb{R}$,

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 - G(\theta(x)) & \text{if } x \in [0, B] \\ 1 & \text{if } x \geq B \end{cases}. \tag{10}$$

where $\theta(x)$ is the inverse of $X(\theta)$ (from Lemma 4) and $B = X(\underline{\theta})$. The following Lemma will be the key to showing that $F^n(x)$ converges to $F(x)$ for all $x \in \mathbb{R}$.

**Lemma 5.** *Consider any $\theta \in (\underline{\theta}, \overline{\theta})$ and for any $n \in \mathbb{N}$, let $k(n) \in \{0, 1, 2, \ldots, K(n)\}$ be such that $\theta_{k(n)}^n > \theta \geq \theta_{k(n)+1}^n$ (where $\theta_0^n = \infty$ and $\theta_{K(n)+1}^n = 0$). Then,*

$$\lim_{n\to\infty} b_{k(n)}^n = X(\theta) \text{ and } \lim_{n\to\infty} F^n(b_{k(n)}^n) = 1 - G(\theta).$$

*Proof.* From Lemma 4 and Theorem 1, we have

$$X(\theta) = \int_{\theta}^{\overline{\theta}} \frac{\pi_v'(1-G(t))g(t)}{t} dt \text{ and } b^n_{k(n)} = \sum_{j=1}^{k(n)} \frac{\pi_v(P^n_j) - \pi_v(P^n_{j-1})}{\theta^n_j}.$$

Observe that

$$\begin{aligned}
b^n_{k(n)} &= \left[ \frac{\pi_v(P^n_{k(n)})}{\theta^n_{k(n)}} - \sum_{j=1}^{k(n)-1} \pi_v(P^n_j) \left[ \frac{1}{\theta^n_{j+1}} - \frac{1}{\theta^n_j} \right] \right] \\
&= \int_0^{1/\theta^n_{k(n)}} \left[ \pi_v(P^n_{k(n)}) - \pi_v(1 - G^n(1/x)) \right] dx \\
&\xrightarrow{n\to\infty} \int_0^{\frac{1}{\theta}} \left[ \pi_v(1-G(\theta)) - \pi_v(1-G(1/x)) \right] dx \qquad \text{(dominated convergence)} \\
&= \underbrace{\left[ x(\pi_v(1-G(\theta)) - \pi_v(1-G(1/x))) \right]_0^{\frac{1}{\theta}}}_{\text{this is 0}} + \int_0^{\frac{1}{\theta}} \frac{\pi_v'(1-G(1/x))g(1/x)}{x} dx \\
&= \int_{\theta}^{\infty} \frac{\pi_v'(1-G(t))g(t)}{t} \,\mathrm{d}t \qquad \text{(substitute } t = 1/x\text{)} \\
&= X(\theta)
\end{aligned}$$

By definition, we have

$$\begin{aligned}
\lim_{n\to\infty} F^n(b^n_{k(n)}) &= \lim_{n\to\infty} P^n_{k(n)} \\
&= \lim_{n\to\infty} \left[ 1 - G^n(\theta) \right] \\
&= 1 - G(\theta)
\end{aligned}$$

□

Returning to the proof of Theorem 6, fix any $x \in (0, B)$ and let $\theta \in (\underline{\theta}, \overline{\theta})$ be such that $X(\theta) = x$. Then, we know that

$$F(x) = 1 - G(\theta).$$

We want to show that

$$\lim_{n\to\infty} F^n(x) = 1 - G(\theta).$$

Take $\epsilon > 0$. Find $\theta' < \theta$ and $\theta'' > \theta$ such that

$$0 < G(\theta) - G(\theta') = G(\theta'') - G(\theta) < \frac{\epsilon}{4}.$$

Let $x' = X(\theta')$, $x'' = X(\theta'')$, so that $x' > x > x''$. Let $\delta = \min\{x' - x, x - x''\}$. From Lemma 5, let $N(\epsilon)$ be such that for all $n > N(\epsilon)$,

$$\max\{|b^n_{k(n)} - x|, |b^n_{k'(n)} - x'|, |b^n_{k''(n)} - x''|\} < \frac{\delta}{2}$$

and

$$\max\{|F^n(b^n_{k'(n)}) - (1 - G(\theta'))|, |F^n(b^n_{k''(n)}) - (1 - G(\theta''))|\} < \frac{\epsilon}{4},$$

where $k(n), k'(n), k''(n)$ are sequences as defined in Lemma 5 for $\theta$, $\theta'$ and $\theta''$ respectively. Then, for all $n > N(\epsilon)$,

$$\begin{aligned} F^n(x) &> F^n(b^n_{k''(n)}) \\ &> 1 - G(\theta'') - \frac{\epsilon}{4} \\ &> 1 - G(\theta) - \frac{\epsilon}{2} \end{aligned}$$

and

$$\begin{aligned} F^n(x) &< F^n(b^n_{k'(n)}) \\ &< 1 - G(\theta') + \frac{\epsilon}{4} \\ &< 1 - G(\theta) + \frac{\epsilon}{2} \end{aligned}$$

so that $|F^n(x) - (1 - G(\theta))| < \epsilon$. Thus, $\lim_{n\to\infty} F^n(x) = 1 - G(\theta) = F(x)$ for all $x \in \mathbb{R}$. $\square$