Exchange economy (with indivisible goods)

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- Each agent $i \in N$ is endowed with its own house $\omega_i \in X$, where $i \neq j$ implies $\omega_i \neq \omega_j$.
- We will assume, WLOG, that $\omega_i = h_i$.

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  2. efficient (make best use of available resources)
- We will now formalize these ideas.

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- In words, y Pareto dominates x if all agents weakly prefer their house under y than under x, and at least some agent strictly prefers its house under y than x.
- An allocation $x = (x_1, x_2, \ldots, x_n)$ is *Pareto efficient* if there does not exist another allocation y such that y Pareto dominates x.

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- An allocation $y = (y_1, y_2, \dots, y_n)$ *blocks* allocation $x = (x_1, x_2, \dots, x_n)$ if there is some coalition $S \subset N$ such that
  1. for all $i \in S$,
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- An allocation $x = (x_1, x_2, \ldots, x_n)$ is in the *core* if there does not exist another allocation y that blocks x.

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1. Consider a housing economy with four agents in which each agent i is endowed with a house h_i and has the following preferences:

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2. Allocations and their properties:

Allocation	IR	PE	Core
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3. Market outcome? $p = (5, 2, 5, 3)$?

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- A market equilibrium occurs when market prices are such that the demand for houses equals supply of houses.
- A *market equilibrium* is a pair (p, x) where $p : H \to \mathbb{R}$ is a price function and $x : N \to H$ is a function such that for all $i \in N$,
  1. $x_i = x_i(p)$,
  2. x is an allocation

Questions

1. Does a market equilibrium always exist?
2. Can there be multiple market equilibrium allocations?
3. Which of the three properties do equilibrium allocations satisfy?

Some special cases

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- Suppose all agents have the same preference: $h_1 \succ_i h_2 \succ_i \cdots \succ_i h_n$ for all $i \in N$. Then, (p, x) is a market equilibrium if and only if $p_1 > p_2 > \cdots > p_n$ and $x_i = h_i$.

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- In these two very special cases,
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  3. and the equilibrium allocation is IR and PE.
- But what about more general cases, when agents may have other preferences?

First welfare theorem: Markets lead to efficient allocations

Theorem 1.

For any housing economy (with strict preferences),

1. *market equilibrium exists,*
2. *the equilibrium allocation is unique,*
3. *and the equilibrium allocation is Pareto efficient.*

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- But then, it must be that $p(y_i) \geq p_i \geq p(x_i)$ for all i. And for some j, $p(y_j) > p_j \geq p(x_j)$.

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- But then, it must be that $p(y_i) \geq p_i \geq p(x_i)$ for all i. And for some j, $p(y_j) > p_j \geq p(x_j)$.
- But this implies $\sum_i p(y_i) > P$ which is a contradiction.

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  4. If agents and houses remain, go back to step 1. Otherwise, terminate.
- The algorithm terminates and always results in a unique allocation x^{TTC} for any arbitrary preferences.

Example 4: Illustration of TTC

- Consider a housing economy with five agents in which each agent i is endowed with a house h_i and has the following preferences:

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h_3	h_4	h_3	h_3	h_4
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h_5	h_5	h_3	h_5	h_5

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- Step 1: Agents 1, 2, 4 trade

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h_5	h_5	h_3	h_5	h_5

- Step 1: Agents 1, 2, 4 trade
- Step 2: Agent 3 trades

Example 4: Illustration of TTC

- Consider a housing economy with five agents in which each agent i is endowed with a house h_i and has the following preferences:

Agent 1	Agent 2	Agent 3	Agent 4	Agent 5
h_4	h_1	h_1	h_2	h_2
h_3	h_4	h_3	h_3	h_4
h_1	h_3	h_5	h_4	h_3
h_2	h_2	h_2	h_1	h_1
h_5	h_5	h_3	h_5	h_5

- Step 1: Agents 1, 2, 4 trade
- Step 2: Agent 3 trades
- Step 3: Agent 5 trades

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- Then agent $i \in C_j$ can sell its house for p_j. It cannot afford a house owned by agents in $C_1 \cup C_2 \cup \cdots \cup C_{j-1}$. Thus, its utility is maximized from buying the house of its cyclic successor in C_j, which costs exactly p_j.

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- Thus, the allocation x^{TTC} with the above prices constitute a market equilibrium of the housing economy.
- The allocation x^{TTC} is not just Pareto efficient, it is also in the core.

Second welfare theorem: Any efficient allocation can be supported in a market equilibrium

Theorem 2.

Consider a housing economy in which the endowment ω is Pareto optimal. Then, (p, x) is a market equilibrium only if $x = \omega$.

Example 3.

Suppose there are three agents endowed with a house and a parking spot $\omega_i = (h_i, p_i)$ and have the following preferences:

Agent 1	Agent 2	Agent 3
$(h_1, p_2)^1$	$(h_2, p_3)^2$	$(h_1, p_3)^3$
$(h_3, p_1)^3$	$(h_2, p_1)^1$	$(h_3, p_2)^2$
$(h_1, p_1)^2$	$(h_2, p_2)^3$	$(h_3, p_3)^1$
$\dots$	$\dots$	$\dots$

Multiple indivisible goods?

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$(h_1, p_1)^2$	$(h_2, p_2)^3$	$(h_3, p_3)^1$
$\ldots$	$\ldots$	$\ldots$

- The economy in this example has an empty core.

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$(h_1, p_1)^2$	$(h_2, p_2)^3$	$(h_3, p_3)^1$
$\dots$	$\dots$	$\dots$

- The economy in this example has an empty core.
- Significant research in studying existence under natural preference domains.

Summary

- In economies with indivisible goods (one per agent), markets lead to efficient allocations.
- The TTC is a useful algorithm for finding desirable allocation in such economies.
- With multiple indivisible goods, challenges arise.