Top-two condition: Characterizing TTC on restricted domains*

Sumit Goel[†] Yuki Tamura[‡]

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Abstract

We study the object reallocation problem under strict preferences. On the unrestricted domain, Ekici (2024) showed that the Top Trading Cycles (TTC) mechanism is the unique mechanism that is individually rational, pair efficient, and strategyproof. We provide an alternative proof of this result, assuming only minimal richness of the unrestricted domain. This allows us to identify a broad class of restricted domains, those satisfying our *top-two condition*, on which the characterization continues to hold. The condition requires that, within any subset of objects, if two objects can each be most-preferred, they can also be the top two most-preferred objects (in both possible orders). We show that this condition is also necessary in the special case of three objects. These results unify and strengthen prior findings on specific domains such as single-peaked and single-dipped domain, and more broadly, offer a useful criterion for analyzing restricted preference domains.

1 Introduction

This paper studies the object reallocation problem, first introduced by Shapley and Scarf (1974), from a mechanism design perspective. There is a group of agents, each of whom owns an indivisible object. Each agent has a strict preference over the objects, which is their private information. A mechanism specifies how the objects are reallocated based on agents' reported preferences.

The Top Trading Cycles (TTC) mechanism has been shown to be fundamental to object reallocation problem from multiple perspectives. Shapley and Scarf (1974) proposed the

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[†]Division of Social Science, NYU Abu Dhabi; sumitgoel58@gmail.com; 0000-0003-3266-9035

[‡]Department of Economics, Ecole Polytechnique, CREST, IP Paris; yuki.tamura@polytechnique.edu

TTC algorithm (credited to David Gale) as a method for finding an allocation in the core of the exchange economy. Roth and Postlewaite (1977) later showed that the TTC allocation is, in fact, the unique such allocation. Focusing on incentives, Roth (1982) showed that the TTC mechanism is strategyproof. In a seminal contribution, Ma (1994) characterized TTC as the unique mechanism satisfying individual rationality, Pareto efficiency, and strategyproofness.[1] Alternative proofs of this characterization have been proposed by Svensson (1999), Anno (2015), Sethuraman (2016), and Bade (2019). More recently, Ekici (2024) strengthened this characterization by showing that Pareto efficiency can be replaced with pair efficiency, a substantially weaker axiom that only rules out welfare-improving trades between pairs of agents. Ekici and Sethuraman (2024) offer a short proof of this result.[2]

In this paper, we provide a new proof of Ekici (2024)'s characterization result, making minimal use of the richness offered by the unrestricted domain compared to existing proofs. A key ingredient in the existing proofs of Ekici (2024)'s characterization, and even those of Ma (1994)'s characterization, is the construction of preference profiles in which agents in a trading cycle report their endowments to be their second-most preferred objects. At such profiles, while Pareto efficiency directly implies that trading cycles must be executed, Ekici (2024) and Ekici and Sethuraman (2024) exploit the richness of the unrestricted domain to establish the same conclusion under the weaker assumption of pair efficiency. In comparison, we show that trading cycles must be executed with essentially no additional richness requirements. Importantly, our approach enables us to also identify a broad class of restricted domains on which TTC is the unique mechanism that is individually rational, pair efficient, and strategyproof. We refer to such domains as *TTC domains*.

We introduce our richness requirement, the *top-two condition*, and show that it is sufficient for a domain to qualify as a TTC domain. A preference domain satisfies the top-two condition if, within any subset of objects, any two objects that can each be most-preferred can also be the top-two most preferred objects (in both possible orders) within the subset. This condition ensures that, as discussed above, agents involved in a trading cycle can report their endowments to be their second-most preferred objects. Crucially, it also ensures that an agent can report any other object in the cycle as their second-most preferred object, which we show is all the additional richness that is needed for pair efficiency (together with individual rationality and strategyproofness) to imply that the trading cycle be executed at the profile where agents rank their endowment second. From here, individual rationality and strategyproofness suffice to show that trading cycles must be executed at all profiles.

[1]The TTC mechanism has also been characterized using axioms such as group strategyproofness (Bird (1984), Takamiya (2001)), independence of irrelevant rankings (Morrill (2013)), non-bossiness (Miyagawa (2002), Ehlers (2014)), and endowments-swapping-proofness (Fujinaka and Wakayama (2018)).

[2]Some related strands of literature focus on reallocation problems tailored to specific environments (Abdulkadiroğlu and Sönmez (1999), Roth, Sönmez, and Ünver (2004), Schummer and Vohra (2013)) and object allocation problems (Carroll (2014), Hylland and Zeckhauser (1979), Pápai (2000), Pycia and Ünver (2017)). Morrill and Roth (2024) survey this literature, highlighting the relevance of TTC in these environments.

The sufficiency of the top-two condition allows us to strengthen or recover existing characterization results for specific restricted domains, such as the single-dipped domain (Tamura (2023), Hu and Zhang (2024)) and the single-peaked domain with two adjacent peaks (Tamura (2022)). It also enables the classification of certain important but previously unexplored domains as TTC domains. To illustrate, we introduce the *partial agreement domain*, containing preferences that respect a predefined partial order over the objects, and show that it is a TTC domain.[3]

We further show that the top-two condition is necessary in the case of three objects, suggesting that it is not only a minimal richness requirement among existing proofs, but may in fact represent the minimal richness required for a domain to qualify as a TTC domain. To the best of our knowledge, no restricted domain failing the top-two condition is known to be a TTC domain. In fact, non-TTC mechanisms have been identified on some such domains, for example, the single-peaked domain (Bade (2019), Tamura and Hosseini (2022)). These findings, together with our results and existing characterizations across various domains, lead us to conjecture that the top-two condition may be necessary in general, thereby offering a full characterization of TTC domains for any number of objects. At the very least, our results introduce and highlight the top-two condition as a useful, even if potentially incomplete, criterion for analyzing restricted domains.[4]

2 Model

Preliminaries

Let $N = \{1, \dots, n\}$ be a finite set of agents. Let $O = \{o_1, \dots, o_n\}$ be a finite set of indivisible objects such that o_i denotes agent i's endowment. Agents have strict preferences over objects. We denote by $\mathcal{P}$ the set of all strict linear orders over O, and we let $\mathcal{D} \subset \mathcal{P}$ denote the preference domain. Let $P = (P_i)_{i \in N} \in \mathcal{D}^N$ denote a preference profile where $P_i \in \mathcal{D}$ denotes agent i's preference over O. Following standard convention, for $S \subset N$, we let $P_S = (P_i)_{i \in S}$, $P_{-S} = (P_i)_{i \in N \setminus S}$. For each $P_0 \in \mathcal{D}$, we denote by R_0 the "at least as desirable as" relation associated with P_0, i.e., for each pair $o, o' \in O$, $o \; R_0 \; o'$ if and only if either $o \; P_0 \; o'$ or $o = o'$. We refer to a set of agents, their endowments, and their preferences over these objects, as an *economy*.

An *allocation* $x : N \to O$ is a bijection that assigns to each agent an object. Let $\mathcal{X}$ be the set of allocations. For each $x \in \mathcal{X}$ and each $i \in N$, we denote by $x_i \in O$ the assignment

[3] Nicolo and Rodriguez-Alvarez (2017), Fujinaka and Wakayama (2024) investigate a related domain, which they refer to as the common ranking domain. A recent survey by Elkind, Lackner, and Peters (2022) reviews a broad array of domain restrictions in social choice theory, some of which may also prove relevant in the context of object reallocation.

[4] In a similar spirit, Alcalde and Barbera (1994) propose the top dominance criterion for existence of strategyproof and stable mechanisms on restricted domains for the two-sided matching problem.

of agent i under the allocation x.

A *mechanism* $\varphi : \mathcal{D}^N \to \mathcal{X}$ associates with each preference profile $P \in \mathcal{D}^N$ an allocation $x \in \mathcal{X}$.

Axioms

We now introduce some standard properties of allocations and mechanisms.

Given $P \in \mathcal{D}^N$, an allocation $x \in \mathcal{X}$ is individually rational at P if for each $i \in N$, $x_i \ R_i \ o_i$. A mechanism $\varphi : \mathcal{D}^N \to \mathcal{X}$ is *individually rational (IR)* if for each $P \in \mathcal{D}^N$, the allocation $\varphi(P)$ is individually rational at P.

Given $P \in \mathcal{D}^N$, an allocation $x \in \mathcal{X}$ is Pareto efficient at P if there is no other allocation $y \in \mathcal{X}$ such that for each $i \in N$, $y_i \ R_i \ x_i$ and for some $j \in N$, $y_j \ P_j \ x_j$. A mechanism $\varphi : \mathcal{D}^N \to \mathcal{X}$ is *Pareto efficient* if for each $P \in \mathcal{D}^N$, the allocation $\varphi(P)$ is Pareto efficient at P. Given $P \in \mathcal{D}^N$, an allocation $x \in \mathcal{X}$ is pair efficient at P if there is no $i, j \in N$ such that $x_j P_i x_i$ and $x_i P_j x_j$. A mechanism $\varphi : \mathcal{D}^N \to \mathcal{X}$ is *pair efficient* if for each $P \in \mathcal{D}^N$, the allocation $\varphi(P)$ is pair efficient at P.

A mechanism $\varphi : \mathcal{D}^N \to \mathcal{X}$ is *strategyproof (SP)* if for any $P \in \mathcal{D}^N$, there is no $i \in N$ and $P_i' \in \mathcal{D}$ such that $\varphi_i(P_i', P_{-i}) \ P_i \ \varphi_i(P)$. A mechanism $\varphi : \mathcal{D}^N \to \mathcal{X}$ is *group strategyproof (GSP)* if for any $P \in \mathcal{D}^N$, there is no $S \subset N$ and $P_S' \in \mathcal{D}^S$ such that for each $i \in S$, $\varphi_i(P_S', P_{-S}) \ R_i \ \varphi_i(P)$ and for some $j \in S$, $\varphi_j(P_S', P_{-S}) \ P_j \ \varphi_j(P)$.

Top Trading Cycles

We now describe the TTC mechanism. For any strict profile $P \in \mathcal{P}^N$, the TTC algorithm finds an allocation as follows:

1. Each agent points to the agent who owns their most-preferred object.
2. In the ensuing directed graph between the agents, there is at least one cycle. All agents in a cycle are assigned their most-preferred objects and leave the economy.
3. The algorithm repeats with the remaining agents and their endowments.

We let $TTC(P) \in \mathcal{X}$ denote the allocation that results from running this algorithm at profile $P \in \mathcal{P}^N$. For any domain $\mathcal{D} \subset \mathcal{P}$, we define the *TTC mechanism* $\varphi : \mathcal{D}^N \to \mathcal{X}$ as the mechanism that selects for any preference profile $P \in \mathcal{D}^N$ the allocation $\varphi(P) = TTC(P)$. From previous results on the unrestricted domain, we know that this TTC mechanism satisfies all the properties defined above.

Corollary 1. *For any $\mathcal{D} \subset \mathcal{P}$, the TTC mechanism is:*

1. *individually rational,*

2. *Pareto efficient (and hence, pair efficient),*

3. *group strategyproof (and hence, strategyproof).*

For the unrestricted domain $\mathcal{D} = \mathcal{P}$, there is, in fact, no other mechanism that satisfies even the weak combination of individual rationality, pair efficiency, and strategyproofness (Ekici (2024)). In general, we say a domain $\mathcal{D} \subset \mathcal{P}$ is a *TTC domain* if there is no mechanism, other than TTC, that satisfies individual rationality, pair efficiency, and strategyproofness on $\mathcal{D}$. Our goal is to provide a general characterization of such domains.

Notation

We will sometimes describe a preference $P_0 \in \mathcal{D}$ by listing the objects in O in the order specified by P_0. For example, a preference $P_0 \in \mathcal{D}$ such that $o_k \ P_0 \ o_{k+1}$ for all k can be succinctly represented as $o_1 o_2 \dots o_n$.

For any preference $P_0 \in \mathcal{D}$ and subset of objects $O' \subset O$, we let $r_k\left(P_0, O'\right)$ denote the object ranked k-th under the preference P_0, restricted to the subset of objects O'. In other words, $r_k\left(P_0, O'\right)$ is the k-th object in the ordered list of O' according to P_0. For example, $r_1(P_0, O')$ represents the object that is most-preferred according to P_0 among the objects in O'.

Similarly, for any subdomain of preferences $\mathcal{D}' \subset \mathcal{D}$ and subset of objects $O' \subset O$, we let $r_k(\mathcal{D}', O')$ denote the set of objects that can be ranked k-th according to some preference $P_0 \in \mathcal{D}'$ restricted to the objects in O'. Formally,

$$r_k(\mathcal{D}', O') = \{o' \in O' : \text{there exists } P_0 \in \mathcal{D}' \text{ such that } r_k(P_0, O') = o'\}.$$

For example, $r_1(\mathcal{D}, O')$ represents the set of objects that can be most-preferred according to preferences in $\mathcal{D}$ among the objects in O'.

3 Results

In this section, we present our results. We begin by introducing our main richness condition on preference domains, and then show how this condition offers a useful criterion for classifying domains as TTC domains or not TTC domains.

3.1 Top-two condition

Our key richness condition on preference domains requires that within any subset of objects, any two objects that can be most-preferred can also be the top-two most-preferred objects (in both possible orders).

Definition 1. A domain $\mathcal{D} \subset \mathcal{P}$ satisfies the *top-two condition* if for any $O' \subset O$ and any distinct $a, b \in r_1(\mathcal{D}, O')$, there exists a $P_0 \in \mathcal{D}$ such that

1. $a = r_1(P_0, O')$,
2. $b = r_2(P_0, O')$.

In other words, if a and b can each be most-preferred within the objects in O', there must be a preference where a is most-preferred and b is second most-preferred, and also a preference where b is most-preferred and a is second most-preferred.

We first present some simple examples to illustrate this richness condition.[5] Let $n = 3$ and consider the domains $\mathcal{D}_1 = \{o_1o_2o_3, o_2o_3o_1, o_2o_1o_3\}$ and $\mathcal{D}_2 = \{o_1o_2o_3, o_2o_3o_1, o_1o_3o_2\}$. In both cases, only o_1 and o_2 can be most-preferred among O. In $\mathcal{D}_1$, they can also be the top-two most preferred (in both orders), but in $\mathcal{D}_2$, there is no preference where o_2 is most-preferred while o_1 is second-most preferred among O. Thus, $\mathcal{D}_1$ satisfies the top-two condition, while $\mathcal{D}_2$ does not. For another example, consider $n = 4$ and $\mathcal{D}_3 = \{o_1o_2o_3o_4, o_1o_3o_2o_4, o_2o_1o_4o_3, o_2o_4o_3o_1\}$. In this case, observe that within $O' = \{o_1, o_3, o_4\}$, the objects o_1 and o_4 can be most-preferred, but there is no preference in $\mathcal{D}_3$ such that o_4 is most-preferred, and o_1 is second most-preferred, among the objects in O'. Thus, $\mathcal{D}_3$ does not satisfy the top-two condition.

Next, we present some important domains that have been studied in various contexts:

1. Suppose $n \geq 3$ and $\mathcal{D}$ is a *single-peaked domain* (Bade (2019)): This domain contains preferences that are single-peaked with respect to some underlying ordering of the objects. WLOG, say $\mathcal{D} = \mathcal{D}^{SP}$, where $\mathcal{D}^{SP}$ is single-peaked with respect to the ordering $o_1 \to o_2 \to \cdots \to o_n$ so that

   $$\mathcal{D}^{SP} = \{P_0 \in \mathcal{P} : o_p = r_1(P_0, O) \implies o_{k+1}\ P_0\ o_k \text{ for } k < p \text{ and } o_k\ P_0\ o_{k+1} \text{ for } k \geq p\}.$$

   Observe that within any adjacent triple, the two extreme objects can be most-preferred, but there is no preference in which they can be the top-two most-preferred objects. Thus, the single-peaked domain does not satisfy the top-two condition.

2. Suppose $n \geq 3$ and $\mathcal{D}$ is a *single-peaked domain with two adjacent peaks* (Tamura (2022)): This domain further restricts the single-peaked domain, so that only two adjacent objects in the underlying ordering can be most-preferred. WLOG, say $\mathcal{D} = \mathcal{D}^{SP-2}(p)$ for some $p \in \{1, \ldots, n-1\}$, where $\mathcal{D}^{SP-2}(p)$ is defined as

   $$\mathcal{D}^{SP-2}(p) = \{P_0 \in \mathcal{D}^{SP} : r_1(P_0, O) \in \{o_p, o_{p+1}\}\}.$$

   It can be verified that within any restricted subset of objects $O' \subset O$, only two objects can be most-preferred, and since these objects must be adjacent within the subset,

[5] Notice that if $n \leq 2$, any $\mathcal{D} \subset \mathcal{P}$ satisfies the top-two condition.

there exist preferences where they are the top-two most-preferred objects (in both possible orders) within the subset. Thus, the single-peaked domain with two adjacent peaks satisfies the top-two condition.

3. Suppose $n \geq 3$ and $\mathcal{D}$ is a *single-dipped domain* (Tamura (2023)): This domain contains preferences that are single-dipped with respect to some underlying ordering of the objects. WLOG, say $\mathcal{D} = \mathcal{D}^{SD}$, where $\mathcal{D}^{SD}$ is single-dipped with respect to the ordering $o_1 \to o_2 \to \cdots \to o_n$ so that

$$\mathcal{D}^{SD} = \{P_0 \in \mathcal{P} : o_d = r_n(P_0, O) \implies o_k \; P_0 \; o_{k+1} \text{ for } k < d \text{ and } o_{k+1} \; P_0 \; o_k \text{ for } k \geq d\}.$$

It can be verified that within any restricted subset of objects $O' \subset O$, only two objects can be most-preferred, and since these objects must be the extreme objects within the subset, there exist preferences where they are the top-two most-preferred objects (in both possible orders) within the subset. Thus, the single-dipped domain satisfies the top-two condition.

4. Suppose n is arbitrary and $\mathcal{D}$ is a *partial agreement domain*: This domain contains preferences that are consistent with some fixed partial dominance relation over the objects. Formally, say $\mathcal{D} = \mathcal{D}^{PA}(\succ)$ for some partial order $\succ$ on O, where $\mathcal{D}^{PA}(\succ)$ is defined as

$$\mathcal{D}^{PA}(\succ) = \{P_0 \in \mathcal{P} : \text{ for all } a, b \in O, a \succ b \implies a \; P_0 \; b\}.$$

Observe that for any restricted subset of objects $O' \subset O$, if a and b can be most-preferred within O', then there must not be any $c \in O'$ such that $c \succ a$ or $c \succ b$. It follows that there exist preferences where a and b are the top-two most preferred objects (with both possible orders) within O'. Thus, the partial agreement domain satisfies the top-two condition.

3.2 Top-two is sufficient

Our main result establishes that the top-two condition is sufficient for a preference domain to qualify as a TTC domain.

Theorem 1. *If $\mathcal{D} \subset \mathcal{P}$ satisfies the top-two condition, then TTC is the unique mechanism that is individually rational, pair efficient, and strategyproof on $\mathcal{D}$.*

To prove our result, we show that all trading cycles must be executed at any preference profile. The key idea is that if the domain satisfies the top-two condition, then agents in a trading cycle can report their endowment as their second-most preferred object among the remaining objects. At such a profile, individual rationality implies that either all these agents must be assigned their endowment or the cycle must be executed. While Pareto efficiency immediately rules out the former, the core of our argument lies in showing that the same conclusion follows even under the weaker requirement of pair efficiency. Specifically, we show in Lemma 1 that if these agents are instead assigned their endowments, then an agent in the cycle can obtain any object in the cycle (except their most-preferred object) by reporting it as their second-most preferred, ultimately violating pair efficiency. From this point, we use individual rationality and strategyproofness repeatedly to conclude that the trading cycle must also be executed at the original profile, thereby completing the argument. We now present the full proof of Theorem 1.

Proof. Suppose $\mathcal{D} \subset \mathcal{P}$ satisfies the top-two condition, and $\varphi : \mathcal{D}^N \to \mathcal{X}$ is a mechanism that is IR, pair efficient, SP on $\mathcal{D}$. Consider any arbitrary preference profile $P \in \mathcal{D}^N$ and let $x = TTC(P)$. We will show that $\varphi(P) = x$.

Suppose $S \subset N$ denotes a subset of agents who would form a cycle and trade endowments in the first round of TTC at P. Notice that for any $i \in S$, it must be that both $x_i, o_i \in r_1(\mathcal{D}, O)$, and since $\mathcal{D}$ satisfies the top-two condition, we can find a $P'_i \in \mathcal{D}$ such that

1. $x_i = r_1(P'_i, O)$,
2. $o_i = r_2(P'_i, O)$.

We now focus on the profile (P'_S, P_{-S}). By IR, it must be that either

1. $\varphi_i(P'_S, P_{-S}) = x_i$ for all $i \in S$ or
2. $\varphi_i(P'_S, P_{-S}) = o_i$ for all $i \in S$.

From here, the key is to show that the second case cannot hold, and so it must be that $\varphi_i(P'_S, P_{-S}) = x_i$ for all $i \in S$. If φ is Pareto efficient, this follows immediately from the definition. We show that even if φ is pair efficient, under IR and SP, this must be the case.

Lemma 1. *$\varphi_i(P'_S, P_{-S}) = x_i$ for all $i \in S$.*

Proof. Label the agents $S = \{i_1, i_2, \ldots, i_{|S|}\}$ so that agent i_s most prefers the endowment of agent i_{s+1} under P'_{i_s}, and agent $i_{|S|}$ most prefers the endowment of agent i_1:

P'_{i_1}	P'_{i_2}	P'_{i_3}	.	$P'_{i_{\|S\|-1}}$	$P'_{i_{\|S\|}}$
o_{i_2}	o_{i_3}	o_{i_4}	.	$o_{i_{\|S\|}}$	o_{i_1}
o_{i_1}	o_{i_2}	o_{i_3}	.	$o_{i_{\|S\|-1}}$	$o_{i_{\|S\|}}$
.	.	.	.	.	.
.	.	.	.	.	.

Suppose towards a contradiction that $\varphi_i(P'_S, P_{-S}) = o_i$ for all $i \in S$. We first construct a sequence of preference profiles, which differ in only agent i_1's preference. Specifically, for each $k \in \{3, 4, \ldots, |S|\}$, find a preference $P^k_{i_1} \in \mathcal{D}$ such that

$$r_1(P^k_{i_1}, O) = o_{i_2} \text{ and } r_2(P^k_{i_1}, O) = o_{i_k},$$

and define the preference profile

$$P^k = (P^k_{i_1}, P'_{S\setminus\{i_1\}}, P_{-S}).$$

These preferences are illustrated here:

$P^k_{i_1}$	P'_{i_2}	P'_{i_3}	.	$P'_{i_{\|S\|-1}}$	$P'_{i_{\|S\|}}$
o_{i_2}	o_{i_3}	o_{i_4}	.	$o_{i_{\|S\|}}$	o_{i_1}
o_{i_k}	o_{i_2}	o_{i_3}	.	$o_{i_{\|S\|-1}}$	$o_{i_{\|S\|}}$
.	.	.	.	.	.
.	.	.	.	.	.

We now show that for any $k \in \{3, 4, \ldots, |S|\}$,

$$\varphi_{i_1}(P^k) = o_{i_k}.$$

To begin, consider $k = |S|$. Suppose $\varphi_{i_1}(P^{|S|}) \notin \{o_{i_2}, o_{i_{|S|}}\}$. Then agent i_1 can misreport its preference to be such that it most prefers $o_{i_{|S|}}$, followed by o_{i_1}. By IR and pair efficiency, agent i_1 must get $o_{i_{|S|}}$ with this misreport, and SP would be violated. Thus, $\varphi_{i_1}(P^{|S|}) \in \{o_{i_2}, o_{i_{|S|}}\}$. Further, by SP, $\varphi_{i_1}(P^{|S|}) \neq o_{i_2}$, and thus, $\varphi_{i_1}(P^{|S|}) = o_{i_{|S|}}$.

Now consider any $k \in \{3, \ldots, |S| - 1\}$ and assume $\varphi_{i_1}(P^{k+1}) = o_{i_{k+1}}$. Suppose $\varphi_{i_1}(P^k) \notin \{o_{i_2}, o_{i_k}\}$. Then agent i_1 can misreport its preference to be such that it most prefers o_{i_k}, followed by $o_{i_{k+1}}$. Since i_1 can ensure $o_{i_{k+1}}$ (by reporting $P^{k+1}_{i_1}$), agent i_1's assignment with this misreport should be in $\{o_{i_k}, o_{i_{k+1}}\}$. But if i_1 gets $o_{i_{k+1}}$, by IR, i_k should get o_{i_k}, which violates pair efficiency. Therefore, agent i_1 must get o_{i_k} with this misreport, and SP would be violated. Thus, $\varphi_{i_1}(P^k) \in \{o_{i_2}, o_{i_k}\}$. Further, by SP, $\varphi_{i_1}(P^k) \neq o_{i_2}$, and thus, $\varphi_{i_1}(P^k) = o_{i_k}$.

Thus, for any $k \in \{3, 4, \ldots, |S|\}$, $\varphi_{i_1}(P^k) = o_{i_k}$. And in particular, $\varphi_{i_1}(P^3) = o_{i_3}$. By IR, $\varphi_{i_2}(P^3) = o_{i_2}$. But then φ violates pair efficiency, which is a contradiction. $\square$

Going back to the proof of the Theorem, we now have that $\varphi_i(P'_S, P_{-S}) = x_i$ for all $i \in S$. Fix any $j \in S$. By SP, $\varphi_j(P_j, P'_{S\setminus\{j\}}, P_{-S}) = x_j$, and by IR, for each $i \in S$,

$$\varphi_i(P_j, P'_{S\setminus\{j\}}, P_{-S}) = x_i.$$

Now suppose for any $T \subset S$ where $|T| \leq k < |S|$, we have that for each $i \in S$,

$$\varphi_i(P_T, P'_{S\setminus T}, P_{-S}) = x_i.$$

Fix any T of size $k+1$ and any $j \in T$. By SP, $\varphi_j(P_T, P'_{S\setminus T}, P_{-S}) = x_j$, and in fact, for any $i \in T$, $\varphi_i(P_T, P'_{S\setminus T}, P_{-S}) = x_i$. Now there must be some $i \in T$ such that $x_i = o_r$ where $r \in S \setminus T$. By IR, it must be that for each $i \in S$, $\varphi_i(P_T, P'_{S\setminus T}, P_{-S}) = x_i$.

It follows by induction that for each $i \in S$,

$$\varphi_i(P_S, P'_{S\setminus S}, P_{-S}) = \varphi_i(P) = x_i.$$

Now we can iteratively apply this argument to agents in the second cycle, third cycle, and so on, to get that for each $i \in N$, $\varphi_i(P) = x_i$.

It follows that $\varphi(P) = TTC(P)$ for all $P \in \mathcal{P}^N$, and thus, φ must be the TTC mechanism. $\square$

Our proof of Theorem 1 provides an alternative to existing proofs of Ekici (2024)'s characterization (Ekici (2024), Ekici and Sethuraman (2024)), relying only on minimal richness of the unrestricted domain. In our context, the proof in Ekici (2024) requires the domain to satisfy a *top-three condition* (any three objects that can each be most-preferred can also be the top-tree most-preferred objects in all possible orders). The shorter proof by Ekici and Sethuraman (2024) relies on an even stronger *top-k condition*, which requires this property for every k. Notably, existing proofs of Ma (1994)'s characterization with Pareto efficiency also require the domain to satisfy at least the top-two condition. In comparison, our proof establishes a stronger characterization with pair efficiency assuming only the top-two condition.

Theorem 1 enables the classification of several important restricted domains as TTC domains. Specifically, the single-peaked domain with two adjacent peaks, the single-dipped domain, and the partial agreement domain all satisfy the top-two condition. Consequently, by Theorem 1, there is no mechanism, other than TTC, that satisfies individual rationality, pair efficiency, and strategyproofness on these domains. This result not only recovers or strengthens existing characterizations of previously studied domains but also identifies significant, previously unexplored domains where the TTC mechanism is similarly justified.

3.3 Top-two is necessary?

In this subsection, we discuss if the top-two condition is necessary for a domain to qualify as a TTC domain. In the special case where there are only $n = 3$ agents, we are able to show that this is indeed the case.

Proposition 1. *Suppose $n = 3$. If $\mathcal{D} \subset \mathcal{P}$ does not satisfy the top-two condition, there exists a non-TTC mechanism that is individually rational, Pareto efficient (hence, pair efficient), and strategyproof on $\mathcal{D}$.*

Proof. The proof is via construction. Since $n = 3$ and $\mathcal{D} \subset \mathcal{P}$ does not satisfy the top-two condition, there must be a pair of objects that can be most-preferred but cannot simultaneously be the top-two most-preferred objects (in some order). WLOG, suppose $\mathcal{D} \subset \mathcal{P}$ is such that $o_1, o_2 \in r_1(\mathcal{D}, O)$, and for any $P_0 \in \mathcal{D}$,

$$o_1 = r_1(P_0, O) \implies o_3 = r_2(P_0, O).$$

We now construct a mechanism on $\mathcal{D}$ that deviates from TTC on certain select profiles. This mechanism essentially penalizes agent 2 (relative to TTC) for being unable to report its endowment o_2 as the second-most preferred object when it most prefers o_1.

To ease notation, we simply use $r_k(P_0)$ to denote $r_k(P_0, O)$ going forward. Define a subset of preference profiles

$$Diff = \{(P_1, P_2, P_3) \in \mathcal{D}^N : r_1(P_1) = o_2, r_1(P_2) = o_1, \text{ and } o_1 \ P_3 \ o_3\},$$

and consider the mechanism $\varphi : \mathcal{D}^N \to \mathcal{X}$ so that for any $P \in \mathcal{D}^N$,

$$\varphi(P) = \begin{cases} TTC(P) & \text{if } P \notin Diff \\ (o_2, o_3, o_1) & \text{if } P \in Diff \end{cases}.$$

Observe that the mechanism φ is different from TTC on the non-empty subset of preference profiles in $Diff \subset \mathcal{D}^N$, as for any $P \in Diff$, $TTC(P) = (o_2, o_1, o_3)$. Further, it is straightforward to verify that for any $P \in \mathcal{D}^N$, $\varphi(P)$ is individually rational and Pareto efficient, and thus, the mechanism φ is IR and Pareto efficient. We will now show that φ is SP.

Let $P \in \mathcal{D}^N$ be any arbitrary preference profile.

1. Suppose $P \notin Diff$. By definition, $\varphi(P) = TTC(P)$. Since the TTC mechanism is SP, the only cases to consider are those when there is an $i \in N$ and $P_i' \in \mathcal{D}$ such that $(P_i', P_{-i}) \in Diff$.
   (a) Suppose $i = 1$. Since agent 1 is always assigned its TTC outcome, it has no incentive to misreport.
   (b) Suppose $i = 2$. By definition of $Diff$, it must be that $r_1(P_1) = o_2$, $r_1(P_2) \in \{o_2, o_3\}$ and $r_1(P_3) \in \{o_1, o_2\}$. It is easy to verify that in any of these cases, $\varphi_2(P) = r_1(P_2)$. Thus, there is no incentive for agent 2 to misreport.
   (c) Suppose $i = 3$. By definition of $Diff$, it must be that $r_1(P_1) = o_2$, $r_1(P_2) = o_1$ and $o_3 \ P_3 \ o_1$. Since $\varphi_3(P) = o_3$, there is no incentive for agent 3 to misreport.
2. Suppose $P \in Diff$. By definition, $\varphi(P) = (o_2, o_3, o_1)$. It follows that the only cases to consider are those when there is an $i \in N$ and $P_i' \in \mathcal{D}$ such that $(P_i', P_{-i}) \notin Diff$.
   (a) Suppose $i = 1$. Since agent 1 is always assigned its TTC outcome, it has no incentive to misreport.
   (b) Suppose $i = 2$. By definition of $Diff$, it must be that $r_1(P_2) = o_1$, and more precisely, it must be that $o_1 \ P_2 \ o_3 \ P_2 \ o_2$. Note that $\varphi_2(P) = o_3$. The only potential misreport worth considering for agent 2 (if it exists in $\mathcal{D}$) is $o_3 \ P_2' \ o_1 \ P_2' \ o_2$. But since $r_1(P_1) = o_2$, and $r_1(P_3) \in \{o_1, o_2\}$, it follows that in any of these cases, $\varphi_2(P_2', P_{-2}) = o_3$. Thus, there is no incentive for agent 2 to misreport.
   (c) Suppose $i = 3$. By definition of $Diff$, it must be that $r_1(P_1) = o_2$, $r_1(P_2) = o_1$ and $o_1 \ P_3 \ o_3$. Since $\varphi_3(P) = o_1$, there is no incentive for agent 3 to misreport.

Thus, we have constructed a non-TTC mechanism that is IR, Pareto efficient, and SP on $\mathcal{D}$.

□

Together, Theorem 1 and Proposition 1 provide a complete characterization of TTC domains for the case of $n = 3$. Moreover, since our construction in Proposition 1 is actually Pareto efficient, and Theorem 1 implies that TTC remains the unique mechanism satisfying Pareto efficiency, individual rationality, and strategyproofness on domains satisfying the top-two condition, we obtain the following corollary:

Corollary 2. *Suppose $n = 3$ and $\mathcal{D} \subset \mathcal{P}$ is the domain. The following are equivalent:*

1. *$\mathcal{D}$ satisfies the top-two condition.*
2. *There is a unique individually rational, Pareto efficient, and strategyproof mechanism on $\mathcal{D}$.*
3. *There is a unique individually rational, pair efficient, and strategyproof mechanism on $\mathcal{D}$.*

For general n, we know from Theorem 1 that the top-two condition is sufficient for a domain to be a TTC domain. Even for domains that do not satisfy the top-two condition, our proof of Theorem 1 shows that if the top-two property is satisfied for O, the first trading cycle must be executed. More generally, any mechanism that satisfies individual rationality, pair efficiency, and strategyproofness must execute trading cycles until it reaches a sub-economy where the top-two property fails. It is only in such sub-economies that a mechanism can potentially deviate from TTC by not executing the trading cycle while still satisfying the three axioms. We have not yet been able to come up with a precise construction for $n > 3$, but we conjecture that such a construction is possible so that the top-two condition is necessary in general.

For some domains where the top-two property fails for sub-economies with three objects, we are able to leverage the construction from Proposition 1 (developed for $n = 3$) to construct non-TTC mechanisms that satisfy the three axioms. To illustrate the idea, consider again the domain $\mathcal{D}_3 = \{o_1o_2o_3o_4, o_1o_3o_2o_4, o_2o_1o_4o_3, o_2o_4o_3o_1\}$, which fails the top-two property for the triple $O' = \{o_1, o_3, o_4\}$. Define a mechanism which implements TTC except at profiles where agent 2 most prefers its endowment o_2, in which case it invokes the three-agent mechanism from Proposition 1—with appropriate relabeling—on the sub-economy consisting of agents $\{1, 3, 4\}$ and objects $\{o_1, o_3, o_4\}$. It is straightforward to verify that this is a non-TTC mechanism that satisfies individual rationality, Pareto efficiency, and strategyproofness on $\mathcal{D}_3$. We can inductively define such mechanisms for a broad sub-class of domains where the top-two property fails for triples.

The primary challenge lies in extending our construction to domains where the top-two property is violated only for subsets of sizes greater than three. A key aspect of our construction for triples is that an agent unable to report their endowment as their second most-preferred object is penalized, while the interfering agent is rewarded. However, if the top-two property is violated only for subsets of sizes greater than three, there might be multiple agents who can interfere, which complicates the construction and the analysis. A natural domain which exhibits this structure is the circular domain.

Example 1. *Suppose $n \geq 4$ and $\mathcal{D}$ is a circular domain (Kim and Roush (1980), Sato (2010)): This domain contains preferences described by the choice of a most-preferred object, and a clockwise or counterclockwise traversal along some cyclic order on the set of objects. WLOG, say $\mathcal{D} = \mathcal{D}^C$ where $\mathcal{D}^C$ is circular with respect to the cyclic order $o_1 \to o_2 \to \cdots \to o_n \to o_1$ so that*

$$D^C = \{P_0 \in \mathcal{P} : o_p = r_1(P_0, O) \implies P_0 \in \{o_p \dots o_n o_1 \dots o_{p-1}, o_p \dots o_1 o_n \dots o_{p+1}\}\}.$$

Observe that o_1 and o_3 can be most-preferred, and with $n \geq 4$, there is no preference in $\mathcal{D}$ where these two objects can simultaneously be the top-two most-preferred objects. Thus, the circular domain does not satisfy the top-two condition. Note however that within any triple $O' \subset O$, all six linear orders over O' are possible and so there is no triple for which the top-two property is violated.

For $n = 4$, the circular domain $\mathcal{D} = \mathcal{D}^C$ contains the following eight preferences:

Rank								
1	o_1	o_1	o_2	o_2	o_3	o_3	o_4	o_4
2	o_2	o_4	o_3	o_1	o_4	o_2	o_1	o_3
3	o_3	o_3	o_4	o_4	o_1	o_1	o_2	o_2
4	o_4	o_2	o_1	o_3	o_2	o_4	o_3	o_1

Notice that a non-TTC mechanism satisfying the three axioms on $\mathcal{D}^C$ can only potentially deviate from TTC on profiles where none of the four agents most prefer their endowment. This is because if any agent most prefers their endowment, they must be assigned this endowment, and the top-two property is satisfied for the remaining sub-economy. We can try to adopt the idea behind our construction for the case of $n = 3$, and try to penalize agent 3 (relative to TTC) in profiles where it most-prefers o_1, but now there are multiple agents (agent 2 and agent 4) that interfere and can be rewarded instead. This complicates the construction, and the analysis of the designed mechanism.

4 Conclusion

We identify a broad family of preference domains, those satisfying the top-two condition, on which TTC is the unique mechanism satisfying the desirable properties of individual rationality, pair efficiency, and strategyproofness. The search for non-TTC mechanisms satisfying the three properties should thus focus on domains that do not satisfy the top-two condition, and we establish the existence of such mechanisms for the case of three objects. Our findings provide a unifying perspective on previously studied domain restrictions, such as single-peaked and single-dipped domains, while also allowing for the classification of some important and previously unexplored domains as TTC domains or not TTC domains. Our results and analysis suggest several directions for future research. An immediate open question is to determine whether the top-two condition is necessary for TTC to be the unique mechanism satisfying the three properties. Another natural direction would be to explore analogous richness conditions for other characterizations of the TTC mechanism, including those based on axioms such as group strategyproofness.

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