Prizes and effort in contests with private information*

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Abstract

We consider contests where agents have private information about their ability and study the effect of different prizes and competition on the effort exerted by the agents. We characterize the symmetric Bayes-Nash equilibrium strategy function and find that the effect of prizes and competition depend qualitatively on the distribution of abilities among the agents. In particular, if there is an increasing density of inefficient agents, increasing the value of prizes or making the contest more competitive encourages effort. In contrast, if the density is decreasing, these interventions discourage effort. We discuss applications of these results to the design of optimal contests in environments that impose natural constraints on feasible contests including grading contests, contests where agents have concave utilities for prizes, and contests where the designer can only award homogeneous prizes of a fixed value.

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1 Introduction

Contests are situations in which agents compete with one another by investing effort or resources to win prizes. Such competitive situations are common in many social and economic contexts, including college admissions, classroom settings, labor markets, R&D races, sporting events, politics, etc. While some of these situations arise naturally, there are many others where the contest designer can carefully design the rules of the contest so as to satisfy their objectives. The designer's objective, and the structural elements of the contest that it can and cannot manipulate may vary depending upon the situation.

In this paper, we focus on situations where the contest participants have private information about their abilities and the designer can manipulate the values of the different prizes $v_1, \ldots, v_n$ to influence the effort exerted by the participants. For such environments, we study how the prizes and their inequality influence the effort exerted by the agents and also discuss applications to the design of optimal contests in various settings. We note here that while there is a significant literature that studies optimal contest design in incomplete information environments, it typically assumes that the designer's objective is to maximize effort given a budget that it can divide arbitrarily among the n prizes. But there are settings where effort does not carry any intrinsic value for the designer, and also the set feasible contests are constrained or different. Our objective in this paper is to obtain a more general comparison of how different contests compare in terms of the effort they induce and then identify optimal contests in some natural environments where the designer's objective and the set of feasible contests are different from what is assumed in the classical case.

We discuss three different environments. First, we consider the case of a professor deciding a rank-based grading scheme for a class. For these grading contests, we assume that the value of a grade is determined by the information it reveals about the type of the student, and in particular, equals its expected productivity. Under this assumption, we characterize the set of feasible contests for the designer. Since the effort exerted by the students in these grading contests may be desirable (if it adds to their productivity as under the human capital theory of education), or a social waste (as under the signalling theory of education), we identify both effort-maximizing and effort-minimizing grading contests. Our second application considers the case of a designer who can costlessly award an arbitrary number of agents with a prize of homogeneous value. This might be the case when the prize takes the form of access to digital content, or a free trial to services. In comparison to grading contests, the value of the prize in these cases does not change based on the number of agents or ranks of the agents that it is awarded to. Our last application is to the case where contest participants have concave utilities for prizes. We make parametric assumptions and discuss how the effort maximizing contest changes as the agents values for the prizes become more concave. We will now discuss our results.

We first study the effect of different prizes on the equilibrium function. We find that increasing the value of first prize v_1 encourages effort while increasing the value of the last prize v_n discourages effort for all agent types, and so these effects persist in expectation as well. On the other hand, the effect of increasing any intermediate prize $v_2, \ldots, v_{n-1}$ is mixed in that it encourages effort from the less efficient agents while discouraging effort from the more efficient agents. Moreover, this transfer of effort from the more efficient to less efficient is balanced in the sense that the total area under the equilibrium effort function remains the same. This property of the equilibrium function relies on the existence of agents with almost zero marginal costs of effort, and to the best of our knowledge, ours is the first paper to study contests in this domain. Importantly, the property leads to some interesting implications for the overall effects of these prizes.

The overall effect on effort of increasing any intermediate prize v_i depends qualitatively on the distribution of abilities in the population and we obtain natural sufficient conditions on the distribution under which these prizes encourage or discourage effort. If the distribution of abilities is such that there is an increasing density of less efficient agents, then increasing the values of intermediate prizes encourages effort. In contrast, if there is an increasing density of more efficient agents, these prizes discourage effort. Intuitively, this is because any intermediate prize leads to a balanced transfer of effort from more efficient to less efficient agents. And so if there is a greater density of less efficient agents, these prizes have a positive effect in expectation. And if there is a greater density of more efficient agents, these prizes have a negative effect in expectation. For the case of uniform distribution of abilities, the intermediate prizes do not effect the expected effort. More generally, in the uniform case, decreasing the value of any intermediate prize leads to an equilibrium function that is a mean preserving spread of the original effort function. So if the designer has an increasing and concave objective, it would prefer to increase the value of the intermediate prizes, and if its objective is increasing and convex, it would prefer to reduce these values.

We also study how the competitiveness of a contest influences effort and as with the effect of prizes, we find that the effect of competition also depends qualitatively on the prior distribution of abilities. To study the effects of competition, we focus on a parametric subclass of distributions and examine how the expected effort changes as we increase competition by increasing the inequality between the top and bottom prizes. When the density of less efficient agents is increasing, more competitive contests induce higher expected effort. In contrast, when the density of more efficient agents is increasing, making the contest more competitive reduces expected effort. This is because under the parametric class of distributions, absolute values of the expected effects of prizes decreases as we go down the ranks of the prizes. The effect of increasing inequality then follows from the fact the expected effects of the intermediate prizes are positive when there is an increasing density of inefficient agents and negative when this density is decreasing.

Lastly, we discuss applications to the design of optimal contests in three different environments. For grading contests, under the assumption that the value of a grade equals the expected productivity of the agent who gets it, we show that more informative grading schemes lead to more competitive prize vectors. We then use our results on the effect of competition to derive optimal grading contests. When the density of inefficient agents is increasing, the effort-maximizing contest awards a unique grade to each agent while the effort-minimizing contest awards only two different grades, say A and B, in some distribution. When the density of efficient agents is increasing, the effort-maximizing contest awards a unique grade to the best agent while pooling the remaining agents with a common grade, and the effort-minimizing contest pools few of the top agents while awarding a unique grade to the remaining agents at the bottom. For our second application, we consider a designer who has a budget that it can split arbitrarily across n prizes and the agents have a common parametric concave utility function for prizes and linear effort costs. We derive the effort-maximizing contest and show that as the utility becomes more concave, the optimal prize vector becomes less competitive. Our last application is to settings where the designer can only choose the number of winners to award with a costless homogeneous prize. In this case, we show that if the density is monotone, the optimal contest awards either a single prize or $n-1$ prizes.

1.1 Literature review

There is a vast literature in contest theory studying the effect of manipulating various features of a contest on the equilibrium effort of the agents and consequently designing these features so as to maximize effort. This work closely relates to the optimal contest design problem in which the designer wants to *distribute a given budget among prizes* $v_1, v_2, \ldots, v_n$ *so as to maximize the effort of the* n *competing agents.* This problem was first posed by Galton in 1902 and has since been studied in various different environments. The environments generally differ in the assumptions they make about the *contest success function (csf)*, the *abilities or valuations of the agents* which may be symmetric or asymmetric, and the *information* agents have about the abilities of other agents which may be complete or incomplete. We briefly discuss the literature on this problem for the perfectly discriminatory csf, which ranks agents according to the effort they put in and awards them the corresponding prizes.

In the complete information setting with symmetric agents, the results generally suggest that increasing competition by increasing prize inequality discourages effort and so distributing the budget equally among the top $n-1$ agents maximizes total effort. This is for instance the case in Glazer and Hassin [30] who consider a model with concave utility and linear costs. In the special case when utility is also linear, Barut and Kovenock [2] show that any equilibrium induces the same expected aggregate effort and the only restriction posed by optimality is that the last prize v_n should be zero. More recently, Fang et al. [23] generalize these results and find that with linear utility and convex costs, increasing prize inequality reduces each agent's effort in the sense of second order stochastic dominance. In contrast, our work in the incomplete information environment identifies a sufficient condition on the distribution of abilities under which competition actually encourages effort.

In an incomplete environment with ex-ante symmetric agents, which is the focus of this paper, a wide range of models have been studied in the literature. In many of these settings, a winner-take-all prize structure has been shown to be optimal for maximizing effort. Glazer and Hassin [30] show that if the agents have concave utility and linear costs, the optimal prize vector involves setting all the later half of the prizes to 0. Zhang [61] considers a model with convex effort costs and identifies a necessary and sufficient condition under which the winner-takes-all prize structure is optimal among a general class of mechanisms. In a setting with linear utility and linear or concave costs, Moldovanu and Sela [44] find that awarding only a single prize is optimal. For convex costs, they identify conditions under which awarding more than one prize might be optimal. In comparison, our paper obtains a more complete ordering of contests in terms of the effort they induce. The analysis allows us to study and solve the optimal contest design problem in other natural environments where the set of feasible prize vectors may be constrained or different.

There has been relatively little work on the optimal prize distribution problem with asymmetric agents. In a complete information setting with linear utility and linear costs, Clark and Riis [14] provide examples where splitting the budget into more than one prize might be optimal. We are not aware of any work on the problem in an incomplete information environment with ex-ante asymmetric agents. There has also been work on this problem for the ratio-form contest success function, which is the most popular alternative to the perfectly discriminatory csf. Clark and Riis [15] consider a complete information setting with symmetric agents having linear utility and costs and find that under the Tullock csfs, which represents a parametric subclass of ratio-form csfs, increasing prize inequality leads to an increase in total effort. Szymanski and Valletti [59] show that with asymmetric agents, a second prize might be optimal. Other related work that looks at the design of optimal contests under some different assumptions include Krishna and Morgan [37], Liu and Lu [42], Cohen and Sela [16]. Sisak [57] provides a more detailed survey of the literature on this problem.

In addition to manipulating the values of prizes to influence the effort, there are other structural elements of a contest that have also been considered in the literature. These include winner selection mechanisms, introduction of dynamics with sequential decision making, introduction of asymmetry via head starts, information disclosure at intermediate stages, entry constraints, introducing multiple contests, group contests, etc. Fu and Wu [26] provides a survey of the theoretical literature on optimal contest design from these different perspectives. More general surveys of the theoretical literature in contest theory can be found in Corchón [18], Vojnović [60], Konrad et al. [34], Segev [52].

The settings we focus on for our applications have also been studied in the literature. In particular, there has been significant work on the design of optimal grading schemes (Moldovanu et al. [46], Rayo [50], Popov and Bernhardt [49], Chan et al. [8], Dubey and Geanakoplos [20], Zubrickas [62]). Moldovanu et al. [46] consider a setting where the designer can associate grades with arbitrary monetary prizes subject to budget and individual rationality constraints and find that the optimal grading scheme awards the top grade to a unique agent and a single grade to all the remaining agents. Dubey and Geanakoplos [20] consider a complete information environment where agents care about relative ranks and find that absolute grading is generally better than relative grading and that it's better to clump scores into coarse categories. Other related papers look at the signalling value of grades under different models or assumptions (Costrell [19], Betts [5], Zubrickas [62], Boleslavsky and Cotton [6]). The setting of our last application where the designer can only choose the number of winners to receive a fixed homogeneous prize was also considered in Liu and Lu [42] under different distributional assumptions. They find that the expected effort is single peaked in the number of prizes. In comparison, we identify natural conditions on the prior distribution under which awarding just a single prize or $n-1$ prizes is optimal.

The paper proceeds as follows. In section 2, we present the model of a contest in an incomplete-information environment. Section 3 characterizes the symmetric Bayes-Nash equilibrium of this game and discusses some important properties of the equilibrium function. In section 4, we discuss applications of our results to the design of optimal contest in three different settings. Section 5 concludes. All proofs are relegated to the appendix.

2 Model

There is a single contest designer and n agents. The designer chooses a vector of prizes $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ such that $v_i \geq v_{i+1}$ for all i. The agents compete for these prizes by exerting costly effort. Each agent i is privately informed about its marginal cost of effort $\theta_i \in [0,1]$ which is drawn independently from $[0,1]$ according to a distribution $F : [0,1] \to [0,1]$. The distribution F is common knowledge. Given a vector of prizes $\mathbf{v}$, marginal cost of effort θ_i, and belief F about the marginal costs of effort of other agents, each agent i simultaneously chooses an effort level e_i. The designer ranks the agents in order of the efforts they put in and awards them the corresponding prizes. The agent who puts in the maximum effort is awarded prize v_1. Agent with the second highest effort is awarded prize v_2 and so on. If agent i puts in effort e_i and wins prize v_i, its final payoff is

$$v_i - \theta_i e_i$$

Given a prize structure $\mathbf{v}$ and belief F, an agent's strategy $\sigma_i : [0,1] \to \mathbb{R}_+$ maps its marginal cost of effort to the level of effort it puts in. A strategy profile $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a Bayes-Nash equilibrium of the game if for all agents i and type $\theta_i \in [0,1]$, agent i's expected payoff from playing $\sigma_i(\theta_i)$ is at least as high as its payoff from playing anything else given that all other agents are playing σ_{-i}. We'll focus on the symmetric Bayes-Nash equilibrium of this contest game. This is a Bayes-Nash equilibrium where all agents are playing the same strategy $g_{\mathbf{v}} : [0,1] \to \mathbb{R}_+$.

Given a prior distribution F, we'll assume the designer's preferences over the different contests or prize vectors $\mathbf{v}$ is defined by a monotone utility function $U : \mathbb{R}_+ \to \mathbb{R}$ so that the designer prefer $\mathbf{v}$ over $\mathbf{v}'$ if and only if $\mathbb{E}[U(g_{\mathbf{v}}(\theta))] \geq \mathbb{E}[U(g_{\mathbf{v}'}(\theta))]$ where $g_{\mathbf{v}}$ represents the symmetric Bayes-Nash equilibrium function under prize vector $\mathbf{v}$. We'll impose conditions on U as required to illustrate our results.

3 Equilibrium

In this section, we first characterize the symmetric Bayes-Nash equilibrium of the game for arbitrary prize vectors and then discuss how the equilibrium changes as we vary different prizes. Note that our model is similar to the model studied in Moldovanu and Sela [44]. The difference is that Moldovanu and Sela [44] assume that the agents marginal costs of effort are in an interval $[m, 1]$ with $m > 0$, and the agents in our model have marginal costs in $[0, 1]$. While the symmetric Bayes-Nash equilibrium strategy function takes the same form as in Moldovanu and Sela [44], it also satisfies an interesting property due to the presence of agents with negligible marginal costs of effort which we will discuss later. The following result displays the symmetric Bayes-Nash equilibrium strategy of the contest game

(Moldovanu and Sela [44]).

Lemma 1. *In a contest with n agents, prizes $\mathbf{v} = (v_1, v_2, \ldots, v_{n-1}, v_n)$ and prior cdf F, the symmetric Bayes-Nash equilibrium strategy function is given by*

$$g_{\mathbf{v}}(\theta) = \sum_{i=1}^{n} m_i(\theta) v_i$$

where,

$$m_i(\theta) = \binom{n-1}{i-1} \int_{F(\theta)}^{1} \frac{\left[(1-t)^{n-i-1} t^{i-2}\right]}{F^{-1}(t)} \left((n-1)t - (i-1)\right) dt$$

for all $i \in \{1, 2, \ldots, n\}$.

The proof uses the standard approach of assuming that $n-1$ agents are playing the same strategy and then getting conditions under which that strategy is the best response for the remaining agent:

$$-f(\theta) \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} \left[(1-F(\theta))^{n-i-1} F(\theta)^{i-1}\right] = \theta g_v'(\theta)$$

Using the boundary condition $g_v(1) = 0$ pins down the form of the function

$$g_v(\theta) = \sum_{i=1}^{n-1} (v_i - v_{i+1}) \frac{(n-1)!}{(i-1)!(n-i-1)!} \int_{F(\theta)}^{1} \frac{\left[(1-t)^{n-i-1} t^{i-1}\right]}{F^{-1}(t)} dt$$

We can then rewrite the expression as in the lemma by combining the two terms with coefficient v_i. Lastly, we check that the second order condition is satisfied. The full proof is in the appendix.

In case an agent's value for prize v is given by some utility function $u(v)$ and all agents share the same utility function u for prizes, the equilibrium in Theorem 1 is simply as if prize i was $u(v_i)$ instead of v_i. The following corollary states this formally.

Corollary 1. *In a contest with n agents, each with utility function u for prizes, prizes $\mathbf{v} = (v_1, v_2, \ldots, v_{n-1}, v_n)$ and prior cdf F, the symmetric Bayes-Nash equilibrium strategy function is given by*

$$g_v(\theta) = \sum_{i=1}^{n} m_i(\theta) u(v_i)$$

where,

$$m_i(\theta) = \binom{n-1}{i-1} \int_{F(\theta)}^{1} \frac{[(1-t)^{n-i-1} t^{i-2}]}{F^{-1}(t)} \left((n-1)t - (i-1)\right) dt$$

for $i \in [n-1]$ and $m_n(\theta) = -\sum_{i=1}^{n-1} m_i(\theta)$.

We will use this when we discuss the effort-maximizing contest for agents with concave utility for prizes.

Now that we know what the equilibrium function looks like, we focus our attention on studying how the equilibrium changes as we vary the values of different prizes.

Theorem 1. *Consider a setting with n agents and prior distribution of abilities F. Suppose $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ are two prize vectors such that $v_i > w_i$ and $v_j = w_j$ for all $j \neq i$.*

1. *If $i = 1$, then $\mathbb{E}[U(g_{\mathbf{v}}(\theta))] \geq \mathbb{E}[U(g_{\mathbf{w}}(\theta))]$ for any increasing function U.*

2. *If $i = n$, then $\mathbb{E}[U(g_{\mathbf{w}}(\theta))] \geq \mathbb{E}[U(g_{\mathbf{v}}(\theta))]$ for any increasing function U.*

3. *If $i \in \{2, \ldots, n-1\}$ and the density f is increasing, then $\mathbb{E}[U(g_{\mathbf{v}}(\theta))] \geq \mathbb{E}[U(g_{\mathbf{w}}(\theta))]$ for any increasing and concave function U.*

4. *If $i \in \{2, \ldots, n-1\}$ and the density f is decreasing, then $\mathbb{E}[U(g_{\mathbf{w}}(\theta))] \geq \mathbb{E}[U(g_{\mathbf{v}}(\theta))]$ for any increasing and convex function U.*

In words, the first prize always encourages effort and the last prize discourages effort irrespective of the prior distribution of abilities. The effect of the intermediate prizes depends on the prior distribution of abilities. If the prior distribution is such that the density of less efficient agents is increasing, then higher values of intermediate prizes encourage effort. But if this density is decreasing, then these prizes discourage effort in expectation. This is partly because any intermediate prize i has a different effect on different types of agents. The more efficient agents who are fighting for the top prizes put in lesser effort as the gain from winning these better prizes has gone down. In contrast, the less efficient agents who generally get lower prizes now put in more effort to get the increased prize i. Importantly, the decrease in effort of the more efficient agents and the increase for the less efficient agents cancel out so that there is basically a *transfer* of effort from the more efficient agents to the less efficient agents as any intermediate prize is increased. Note that the existence of agents with negligible marginal costs of effort is important for the equilibrium to have this property. This property is formally stated in the following lemma which is the key to proving Theorem 1.

Lemma 2. *For any number of agents n and prior F, the following hold:*

1. $\theta m_i(\theta) \leq 1$ *for all* $i \in \{1, \ldots, n-1\}, \theta \in [0, 1]$
2. $\lim_{\theta \to 0} \theta m_i(\theta) = 0$ *for all* $i \in \{1, \ldots, n-1\}$
3. $\int_0^1 m_1(\theta) = 1$ *and* $\int_0^1 m_i(\theta) = 0$ *for* $i \in \{2, n-1\}$
4. $m_1(\theta) > 0$ *for all* θ *and monotone decreasing*
5. *For* $i \in \{2, n-1\}$, *there exist* $t_i^1 < t_i^2$ *such that*

$$m_i(\theta) = \begin{cases} <= 0 & \textit{if } \theta \leq t_i^1 \\ > 0 & \textit{otherwise} \end{cases}$$

$$m_i'(\theta) = \begin{cases} >= 0 & \textit{if } \theta \leq t_i^2 \\ < 0 & \textit{otherwise} \end{cases}$$

The first property provides an upper bound on the marginal effect of prize i on the effort of agent of type θ and is fairly straightforward. The second property says that something stronger is true for the most efficient agents. It says that the effort cost of an agent goes to zero as its marginal cost of effort goes to 0. The third property says that the overall effect of any prize i other than the first prize is zero. So increasing any prize i just transfers the effort from some set of agents to another set of agents. The first prize is special in that its overall effect is positive and equals 1 irrespective of the prior distribution. To prove this, we apply integration by parts to integrate $\int_0^1 m_i(\theta)d\theta$ and using the second property, we get that it equals $-\int_0^1 \theta m_i'(\theta)d\theta$. By Leibniz rule, we know

$$m_i'(\theta) = -\binom{n-1}{i-1}\frac{[(1-F(\theta))^{n-i-1}F(\theta)^{i-2}]}{\theta}\left((n-1)F(\theta)-(i-1)\right)f(\theta)$$

and so we are able to integrate $\theta m_i'(\theta)$ to get the result. Moving on, the fourth property says the marginal effect of the first prize is positive for all types of agents and also decreasing in their type. And finally, the last property describes exactly how the effort transfers from one set of agents to another as we increase the value of some prize. In particular, for any prize i that is not the first prize, increasing its value leads to a transfer of effort from more efficient agents ($\theta \leq t_i^1$) to less efficient agents ($\theta > t_i^1$). These properties are illustrated in Figure 1 for the case of $n = 5$ agents and prior cdf $F(\theta) = \theta^3$.

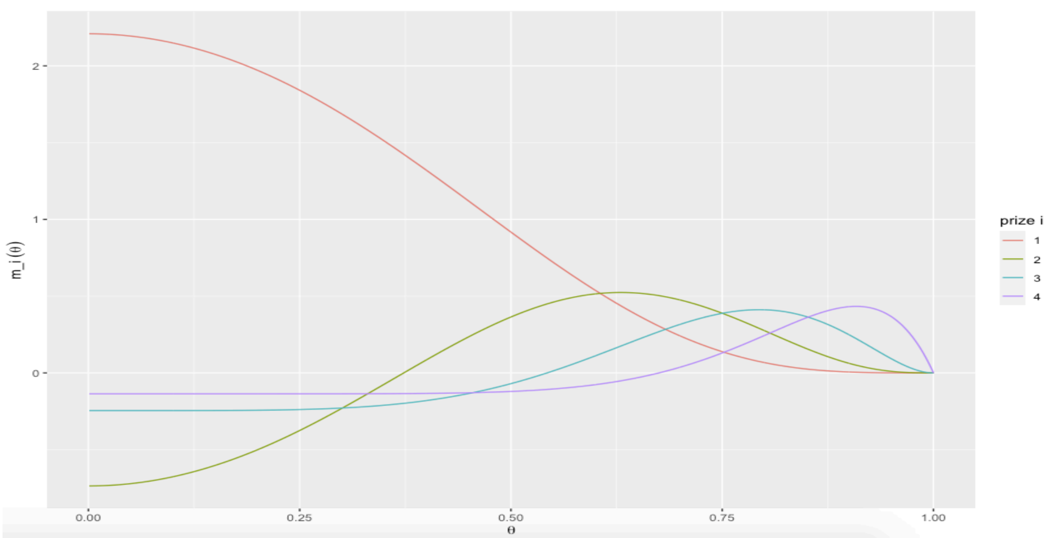

Figure 1: The marginal effect of prizes on effort for $n = 5$ and $F(\theta) = \theta^3$.

The overall expected effect of each prize i as described in Theorem 1 depends on the prior distribution. If there is a higher proportion of less efficient agents who are positively influenced by intermediate prizes, the overall effect of these intermediate prizes is positive. On the other hand, if there is a higher proportion of more efficient agents who are discouraged by these intermediate prizes, the overall effect is negative. The full proofs of Lemma 2 and Theorem 1 are in the appendix.

Note that both the second and third parts of Theorem 1 apply to the case where F is uniform. Thus, we have the following corollary:

Corollary 2. *Consider a setting with n agents and uniform prior $F(\theta) = \theta$. Suppose $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ are two prize vectors such that $v_j > w_j$ for some $j \in \{2, \ldots, n-1\}$ and $v_{-j} = w_{-j}$.*

1. *if U is any increasing and concave function, then $\mathbb{E}[U(g_{\mathbf{v}}(\theta))] \geq \mathbb{E}[U(g_{\mathbf{w}}(\theta))]$*

2. *if U is any increasing and convex function, then $\mathbb{E}[U(g_{\mathbf{w}}(\theta))] \geq \mathbb{E}[U(g_{\mathbf{v}}(\theta))]$*

In other words, for the uniform case, $g_{\mathbf{w}}(\theta)$ is a mean-preserving spread of $g_{\mathbf{v}}(\theta)$ which implies the comparisons in the corollary.

Theorem 1 identifies natural conditions on the prior distributions under which the expected marginal effects are positive or negative. Next, we consider a natural parametric class of the priors for which we can compute and compare the expected marginal effects of prizes. We then discuss what this comparison implies for the effects of competition in contests on the effort exerted by the agents under these priors. First, let us formally define what it means for a prize vector to be more competitive than another.

Definition 3.1. A prize vector $\mathbf{v} = (v_1, v_2, \ldots, v_{n-1}, v_n)$ is more competitive than $\mathbf{w} = (w_1, w_2, \ldots, w_{n-1}, w_n)$ if $\mathbf{v}$ majorizes $\mathbf{w}$ (i.e. $\sum_{i=1}^{k} v_i \geq \sum_{i=1}^{k} w_i$ for all $k \in [n]$ and $\sum_{i=1}^{n} v_i = \sum_{i=1}^{n} w_i$).

This is the definition that was also considered in Fang et al. [23]. The next result describes the effect of competition on expected effort and expected minimum effort and how it depends on the prior distribution.

Theorem 2. *Consider a setting with n agents and prior $F(\theta) = \theta^p$. Suppose $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ are two prize vectors such that $\mathbf{v}$ is more competitive than $\mathbf{w}$. Then, we have the following:*

1. *if* $p \geq 1$, *then*

$$\mathbb{E}[g_{\mathbf{v}}(\theta)] \geq \mathbb{E}[g_{\mathbf{w}}(\theta)]$$

2. *if* $\frac{1}{2} \leq p \leq 1$, $v_1 = w_1, v_n = w_n$, *then*

$$\mathbb{E}[g_{\mathbf{v}}(\theta)] \leq \mathbb{E}[g_{\mathbf{w}}(\theta)]$$

3. *if* $p \geq \frac{1}{2}$ *and* $v_n = w_n$, *then*

$$\mathbb{E}[g_{\mathbf{v}}(\theta_{max})] \leq \mathbb{E}[g_{\mathbf{w}}(\theta_{max})]$$

To prove this, we first show that for $F(\theta) = \theta^p$,

$$\mathbb{E}[m_i(\theta)] = \binom{n-1}{i-1}\beta(i - \frac{1}{p}, n-i)\frac{(n-i)(p-1)}{np-1}$$

which implies that

$$\frac{\mathbb{E}[m_{i+1}(\theta)]}{\mathbb{E}[m_i(\theta)]} = \frac{n-i}{i}\frac{i-\frac{1}{p}}{n-i-1}\frac{n-i-1}{n-i} = \frac{i-\frac{1}{p}}{i} < 1$$

It follows then that for $p \geq 1$, the expected marginal effects $\mathbb{E}[m_i(\theta)]$ are positive and decreasing in i. Since $\mathbf{w}$ can be obtained from $\mathbf{v}$ by a sequence of Robinhood transfers which involve a transfer of value from a top prize to a bottom prize, each of which reduces expected effort, we get that the expected effort goes down as prize vector becomes less competitive. An analagous argument holds for $\frac{1}{2} \leq p \leq 1$.

For the case of expected minimum effort, we again find that

$$\mathbb{E}[m_i(\theta_{max})] = \binom{n-1}{i-1}\beta\left(n+i-1-\frac{1}{p}, n-i\right)\frac{(n-i)(np-1)}{2np-p-1}$$

which implies

$$\frac{\mathbb{E}[m_{i+1}(\theta_{max})]}{\mathbb{E}[m_i(\theta_{max})]} = \frac{n+i-1-\frac{1}{p}}{i} > 1$$

It follows then that less competitive prize vectors lead to higher expected minimum effort.

The fact that the expected marginal effect of prize v_i is decreasing in i for $p \geq 1$ is perhaps a bit surprising since these are distributions where a large proportion of the agents are inefficient (the density is increasing in θ). Since these inefficient agents are generally competing for the intermediate prizes, one might expect that as their proportion increases (p increases), the expected marginal effect of the later prizes would increase relative to the earlier prizes. While we do get that the ratio $\frac{\mathbb{E}[m_{i+1}(\theta)]}{\mathbb{E}[m_i(\theta)]} = \frac{i-\frac{1}{p}}{i}$ increases as p increases, it only goes to 1 as $p \to \infty$. Thus, the effect is not big enough to make later prizes more valuable than the earlier prizes in terms of the effort they induce. In contrast, for the case of expected minimum effort, the designer is putting even more weight on the effort of the least efficient agents. And it turns out that the marginal effect of prize v_i on expected minimum effort do increase as we increase i.

4 Applications

In this section, we discuss some applications to the design of optimal contests in environments where the set of feasible prize vectors available to the designer may be constrained. In particular, we will consider three different environments. First, we'll consider settings where the designer can commit to a grading scheme and the value of these grades is determined by the information they reveal about the type of the agents. Second, we'll consider settings where agents have concave utilities for prizes and we'll derive the effort-maximizing prize structure under the standard constraint that the designer has a budget that it must allocate across prizes. We'll also discuss how the optimal prize structure changes as the degree of concavity increases in the population. At last, we'll consider settings where the contest designer is constrained to award homogeneous prizes of a fixed value and only needs to decide how many prizes it must award to maximize effort.

4.1 Optimal grading contests

First, we focus on the design of grading schemes where the contest designer doesn't have an explicit budget that it can distribute across prizes but instead, can choose a distribution of grades that it can award based on the rank of the agents. This is generally the case in classroom settings where the professor awards grades to students based on their performance in exams. For instance, the professor may commit to giving grades A and B to the top 50% and bottom 50% respectively, or it may give $A+, A-, B+$, and $B-$ with distribution $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Formally, we define a grading contest as follows:

Definition 4.1. A grading contest with n agents is defined by a strictly increasing sequence of natural numbers $s = (s_1, s_2, \ldots, s_k)$ such that $s_k = n$.

The interpretation of grading contest s is that the top s_1 agents get grade g_1, next $s_2 - s_1$ get grade g_2 and generally, $s_k - s_{k-1}$ get grade g_k. There is a natural partial order over these grading contests in terms of how much information they reveal about the quality of the agents. From above, the grading contest that awards the grades $A+, A-, B+$, and $B-$ in equal proportion is more informative about the agents type then the one that awards just A and B in equal proportion. More generally, we can say the following:

Definition 4.2. A grading contest s is more informative than s' if s' is a subsequence of s.

Clearly, the rank revealing contest $s^* = (1, 2, \ldots, n)$ is more informative than any other grading contest.

To discuss how these grading contests compare in terms of the effort they induce, we need to define how the agents assign value to these grades. We assume that the value of a grade is determined by the information it reveals about the type of the agent. More precisely, we suppose that there is a publicly known wage function $w : \Theta \to \mathbb{R}_+$ which maps an agent's marginal cost to its productivity and is monotone decreasing. So if the market could observe the type of the agent to be θ, the agent would be offered a wage of $w(\theta)$. Given this wage function, we assume that a grading contest s induces the prize vector which is the expected productivity or wage of the agent given its grade. That is, if the market has a posterior belief f over the type of the agent, then the agent will get a wage equal to $\int_0^1 w(\theta) f(\theta) d\theta$.

Under this assumption, we get that the rank revealing contest $s^* = (1, 2, \ldots, n)$ induces the prize vector

$$v_i = \mathbb{E}[w(\theta) | \theta = \theta_{(i)}^n]$$

where $\theta^n_{(i)}$ is the ith order statistic in a random sample of n observations. This is because the rank revealing contest reveals the exact rank of the agent in a random sample of n observations. Note here that since $\theta^n_{(i)}$ is stochastically dominated by $\theta^n_{(j)}$ for all $i < j$ and w is monotone decreasing, the prize vector induced by s^* is monotone decreasing $v_1 > v_2 \cdots > v_n$.

Now we can define the prize vectors induced by arbitrary grading contests s in terms of the v_i's as defined above. An arbitrary grading contest $s = (s_1, s_2, \ldots, s_k)$ induces the prize vector $v(s)$ where

$$v(s)_i = \frac{v_{s_{j-1}+1} + v_{s_{j-1}+2} + \cdots + v_{s_j}}{s_j - s_{j-1}}$$

and j is such that $s_{j-1} < i \leq s_j$

This is because if an agent gets grade g_j in the grading contest $s = (s_1, s_2, \ldots, s_k)$, then the market learns that the agent's type θ must be ranked at one of $\{s_{j-1}+1, \ldots, s_j\}$ and further, it is equally likely to be ranked at any of these positions. The form of the prize vector above then follows from the assumption that the value of grade equals its expected productivity under the posterior induced by the grade.

Given this framework, we can now ask how the different grading schemes compare in terms of the effort they induce.

Theorem 3. *Consider a setting with n agents and prior cdf $F(\theta) = \theta^p$. Suppose grading scheme s is more informative than s'. Then, we have the following:*

- *if $p \geq 1$, then $\mathbb{E}[g_{v(s)}(\theta)] \geq \mathbb{E}[g_{v(s')}(\theta)]$*
- *if $\frac{1}{2} \leq p \leq 1$, $v(s)_1 = v(s')_1$, and $v(s)_n = v(s')_n$, then $\mathbb{E}[g_{v(s)}(\theta)] \leq \mathbb{E}[g_{v(s')}(\theta)]$*
- *if $p \geq \frac{1}{2}$ and $v(s)_n = v(s')_n$, then $\mathbb{E}[g_{v(s)}(\theta_{max})] \geq \mathbb{E}[g_{v(s')}(\theta_{max})]$*

Proof. Observe that if s is more informative than s', then it induces a prize vector $v(s)$ that is more competitive than the prize vector $v(s')$. The result then follows directly from Theorem 2.

□

With this comparison, we can now find the optimal grading contest in these settings.

Corollary 3. *Consider a setting with n agents and prior cdf $F(\theta) = \theta^p$. Then, we have the following:*

- *if $p \geq 1$, the rank revealing contest $s = (1, 2, \ldots, n)$ maximizes expected effort among all grading contests.*
- *if $\frac{1}{2} \leq p \leq 1$, the contest $s = (1, n)$ in which the best agent gets a unique grade and all other agents get a common grade maximizes expected effort among all grading contests.*
- *if $p \geq \frac{1}{2}$, the contest $s = (n-1, n)$ in which the worst agent gets a unique grade and all other agents get a common grade maximizes expected minimum effort among all grading contests.*

Note that when the designer has a budget that it can distribute across prizes, the expected effort maximizing prize vector allocates the entire budget to the first prize. But as we see in the corollary, the optimal grading contest depends on the prior distribution of abilities. If the density of agents is increasing in θ so that there is a greater proportion of inefficient agents, the effort maximizing grading contest awards a unique grade to each agent. But when the density is decreasing, the optimal grading contest awards a unique grade to the best agent and pools the rest of the agents by awarding them a common grade. And for the case where the designer wants to maximize expected minimum effort, which is perhaps a reasonable objective in a classroom environment, the optimal grading contest awards a common grade to everyone except the least efficient agent.

4.2 Optimal contests where agents have concave utility

Now we consider the contest design problem in a typical environment where the designer has a budget of B that it can allocate across prizes $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ such that $v_i \geq v_{i+1}$. It is known that when agent's utility $u(v) = v$, the effort maximizing contest awards the entire prize budget B to the first prize ([44]). In this section, we consider the contest design problem where agents have concave utilities. More precisely, we assume that under a prize vector $\mathbf{v}$, if agent i of type θ_i puts in effort e_i and wins prize j, its payoff equals $u(v_j) - \theta_i e_i$ where $u(v_j) = v_j^r$. The next result characterizes the expected effort maximizing contest in this environment.

Theorem 4. *Suppose there are n agents with utility $u(v) = v^r$ and the prior cdf on marginal costs is $F(\theta) = \theta^p$ with $p \geq 1$. The prize structure that maximizes expected effort is*

$$v(r) = (v_1(r), c_2^{\frac{1}{1-r}} v_1(r), \ldots c_{n-1}^{\frac{1}{1-r}} v_1(r), 0)$$

where

$$c_i = \frac{\mathbb{E}[m_i(\theta)]}{\mathbb{E}[m_1(\theta)]} < 1$$

and $v_1(r)$ is such that

$$\sum v_i(r) = B$$

Further, if $1 \geq r > r' > 0$, then $v(r)$ is more competitive than $v(r')$.

The proof proceeds by using corollary 1 to identify the Bayes-Nash equilibrium function so that the problem becomes $\max_{\mathbf{v}} \sum_{i=1}^{n-1} v^r \mathbb{E}[m_i(\theta)]$ such that $\sum v_i = B$. Solving this constrained optimization problem characterizes the optimal contest. To show that the optimal contest become more competitive as r increases, we define $f_k(r)$ as the sum of the first k prizes in the optimal contest under r and show that this sum is increasing in r. Thus, as the agents utility for prizes becomes more concave, the effort maximizing prize vector becomes less competitive.

4.3 Optimal contests with costless homogeneous prizes

For our last application, we consider a setting where the contest designer can award arbitrarily many prizes of a fixed value a. More precisely, the set of prize vectors available to the designer is given by

$$B = \{\mathbf{v} \in \mathbb{R}^n : \exists k \text{ such that } v_i = a \text{ if } i \leq k \text{ and } v_i = 0 \text{ if } i > k\}$$

This might be the case in online contests run on platforms like Leetcode and Kaggle where the platforms essentially have an unlimited supply of digitial certificates or medals that they can award to participants in the contest. The designer wants to chose the number of prizes so as to maximize the expected effort. Note that this problem was also considered in [42] but under different distributional assumptions. In their setting, the authors found that the expected effort was single-peaked in k. For our setting, we have the following as a corollary of Theorem 1.

Corollary 4. *Consider a setting with n agents and prior F. Then, we have the following:*

1. *if F is such that the density f is increasing, then the contest that awards $k = n - 1$ prizes maximizes expected effort among the contests in B.*
2. *if F is such that the density f is decreasing, then the contest that awards $k = 1$ prize maximizes the expected effort among the contests in B.*

In fact, in the first case where the density is increasing, we actually have something stronger. In this case, awarding $k = n - 1$ second order stochastically dominates awarding fewer number of prizes. In other words, for any concave increasing utility function U, the contest designer's optimal decision would be to award $k = n - 1$ prizes.

5 Conclusion

In contests where agents have private information about their abilities, we study the effect different prizes have on the effort exerted by the agents in equilibrium. While increasing the value of first prize encourages effort for all agents, increasing any intermediate prize leads to a balanced transfer of effort from the more efficient agents to less efficient agents. The overall expected effect then depends on the prior distribution of abilities. If the density of agents is increasing in inefficiency, the expected effects are positive. If this density is decreasing, the expected effects are negative. For a parametric subclass of priors with monotone density functions, we also study the effects of competition on expected effort and expected minimum effort.

We also discuss the application of these results to the design of effort-maximizing contests in settings where the set of feasible prize vectors may be constrained due to various reasons. First, we consider the design of grading contests under the assumption that the value of a grade is determined by the information it reveals about the type of the agent. We find that when the density function is increasing, more informative contests lead to higher expected effort and therefore, the rank revealing contest maximizes expected effort among all grading contests. Second, we consider the setting where the designer has a budget that it must allocate across different prizes and the agents have concave utilities for prizes. We find the optimal contest in this setting and show that it becomes more competitive as the degree of concavity decreases. Lastly, we consider settings where the designer can only choose the number of homogeneous prizes to award and show that when the prior density is monotone, it is optimal to award either 1 or $n-1$ prizes depending on whether the density is decreasing or increasing.

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A Proofs for Section 3 (Equilibrium)

Lemma 1. *In a contest with n agents, prizes $\mathbf{v} = (v_1, v_2, \ldots, v_{n-1}, v_n)$ and prior cdf F, the symmetric Bayes-Nash equilibrium strategy function is given by*

$$g_{\mathbf{v}}(\theta) = \sum_{i=1}^{n} m_i(\theta) v_i$$

where,

$$m_i(\theta) = \binom{n-1}{i-1} \int_{F(\theta)}^{1} \frac{[(1-t)^{n-i-1} t^{i-2}]}{F^{-1}(t)} \left((n-1)t - (i-1)\right) dt$$

for all $i \in \{1, 2, \ldots, n\}$.

Proof. Suppose $n-1$ agents are playing a monotone decreasing strategy $g(\theta)$. Let $\theta^n_{(j)}$ denote the jth order statistic from n random draws with $\theta^n_{(0)} = 0$ and $\theta^n_{n+1} = 1$. Then, an agent of type θ's utility from putting in x units of effort is given by:

$$\begin{aligned} u(\theta, x) &= \sum_{i=1}^{n} v_i \Pr[\theta^{n-1}_{(i-1)} \leq g^{-1}(x) \leq \theta^{n-1}_{(i)}] - \theta x \\ &= \sum_{i=1}^{n} v_i \binom{n-1}{i-1} F(g^{-1}(x))^{i-1} (1 - F(g^{-1}(x)))^{n-i} - \theta x \end{aligned}$$

Now, differentiating with respect to x gives:

$$\frac{\partial u(\theta, x)}{\partial x} = \frac{f(g^{-1}(x))}{g'(g^{-1}(x))} \sum_{i=1}^{n} v_i \binom{n-1}{i-1}$$
$$\left[(1 - F(g^{-1}(x)))^{n-i} (i-1) F(g^{-1}(x))^{i-2} - F(g^{-1}(x))^{i-1} (n-i)(1 - F(g^{-1}(x)))^{n-i-1}\right] - \theta$$

. Setting it to 0 and plugging in $g(\theta) = x$ gives the condition for $g(\theta)$ to be a symmetric Bayes-Nash equilibrium:

$$f(\theta) \sum_{i=1}^{n} v_i \binom{n-1}{i-1} \left[(1 - F(\theta))^{n-i} (i-1) F(\theta)^{i-2} - F(\theta)^{i-1} (n-i)(1 - F(\theta))^{n-i-1}\right] = \theta g'(\theta)$$

An alternate way to write this condition is:

$$-f(\theta)\sum_{i=1}^{n-1}(v_i - v_{i+1})\,\frac{(n-1)!}{(i-1)!(n-i-1)!}\left[(1-F(\theta))^{n-i-1}F(\theta)^{i-1}\right] = \theta g'(\theta)$$

Using the boundary condition $g(1) = 0$, we get that the symmetric Bayes-Nash equilibrium function is given by

$$\int_\theta^1 \frac{f(\theta)}{\theta}\sum_{i=1}^{n-1}(v_i - v_{i+1})\,\frac{(n-1)!}{(i-1)!(n-i-1)!}\left[(1-F(\theta))^{n-i-1}F(\theta)^{i-1}\right]d\theta$$

Replacing $F(\theta) = t$, we get

$$g(\theta) = \int_{F(\theta)}^1 \frac{1}{F^{-1}(t)}\sum_{i=1}^{n-1}(v_i - v_{i+1})\,\frac{(n-1)!}{(i-1)!(n-i-1)!}\left[(1-t)^{n-i-1}t^{i-1}\right]dt$$

Bringing the summation outside:

$$g_v(\theta) = \sum_{i=1}^{n-1}(v_i - v_{i+1})\,\frac{(n-1)!}{(i-1)!(n-i-1)!}\int_{F(\theta)}^1 \frac{\left[(1-t)^{n-i-1}t^{i-1}\right]}{F^{-1}(t)}dt$$

We can also write the equilibrium function as $g_v(\theta) = \sum_{i=1}^n m_i(\theta)v_i$ where for $i \geq 2$,

$$\begin{aligned}
m_i(\theta) &= \frac{(n-1)!}{(i-1)!(n-i-1)!}\int_{F(\theta)}^1 \frac{\left[(1-t)^{n-i-1}t^{i-1}\right]}{F^{-1}(t)}dt - \frac{(n-1)!}{(i-2)!(n-i)!}\int_{F(\theta)}^1 \frac{\left[(1-t)^{n-i}t^{i-2}\right]}{F^{-1}(t)}dt \\
&= \frac{(n-1)!}{(i-2)!(n-i-1)!}\int_{F(\theta)}^1 \left(\frac{\left[(1-t)^{n-i-1}t^{i-1}\right]}{(i-1)F^{-1}(t)} - \frac{\left[(1-t)^{n-i}t^{i-2}\right]}{(n-i)F^{-1}(t)}\right)dt \\
&= \frac{(n-1)!}{(i-2)!(n-i-1)!}\int_{F(\theta)}^1 \frac{\left[(1-t)^{n-i-1}t^{i-2}\right]}{F^{-1}(t)}\left(\frac{t}{(i-1)} - \frac{1-t}{n-i}\right)dt \\
&= \binom{n-1}{i-1}\int_{F(\theta)}^1 \frac{\left[(1-t)^{n-i-1}t^{i-2}\right]}{F^{-1}(t)}\left((n-1)t - (i-1)\right)dt
\end{aligned}$$

For $i = 1$, we have that

$$m_1(\theta) = (n-1)\int_{F(\theta)}^1 \frac{\left[(1-t)^{n-2}\right]}{F^{-1}(t)}dt$$

Now we check that the second order condition is satisfied. To simplify calculations, let $g^{-1}(x) = t$ so the agent of type θ is imitating an agent of type t. Then, the foc can be written as:

$$f(t)\sum_{i=1}^{n} v_i \binom{n-1}{i-1}\left[(1-F(t))^{n-i}(i-1)F(t)^{i-2} - F(t)^{i-1}(n-i)(1-F(t))^{n-i-1}\right] - \theta g'(t) = 0$$

or alternatively

$$-\sum_{i=1}^{n-1} (v_i - v_{i+1})\binom{n-1}{i-1}(n-i)f(t)\left[(1-F(t))^{n-i-1}F(t)^{i-1}\right] - \theta g'(t) = 0$$

Let $V(t) = -\sum_{i=1}^{n-1} (v_i - v_{i+1})\binom{n-1}{i-1}(n-i)f(t)\left[(1-F(t))^{n-i-1}F(t)^{i-1}\right]$

Then, the foc is that $V(t) = tg'(t)$ and so $V'(t) = tg''(t) + g'(t)$. Taking the derivative of lhs of the foc wrt t gives

$$V'(t) - \theta g''(t) = V'(t) - \theta \frac{(V'(t) - g'(t))}{t}$$

At $t = \theta$, we get that this equals $g'(\theta)$ which we know is < 0. Thus, the second order condition is satisfied.

$\square$

Theorem 1. *Consider a setting with n agents and prior distribution of abilities F. Suppose $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ are two prize vectors such that $v_i > w_i$ and $v_j = w_j$ for all $j \neq i$.*

1. *If $i = 1$, then $\mathbb{E}[U(g_{\mathbf{v}}(\theta))] \geq \mathbb{E}[U(g_{\mathbf{w}}(\theta))]$ for any increasing function U.*
2. *If $i = n$, then $\mathbb{E}[U(g_{\mathbf{w}}(\theta))] \geq \mathbb{E}[U(g_{\mathbf{v}}(\theta))]$ for any increasing function U.*
3. *If $i \in \{2, \ldots, n-1\}$ and the density f is increasing, then $\mathbb{E}[U(g_{\mathbf{v}}(\theta))] \geq \mathbb{E}[U(g_{\mathbf{w}}(\theta))]$ for any increasing and concave function U.*
4. *If $i \in \{2, \ldots, n-1\}$ and the density f is decreasing, then $\mathbb{E}[U(g_{\mathbf{w}}(\theta))] \geq \mathbb{E}[U(g_{\mathbf{v}}(\theta))]$ for any increasing and convex function U.*

Proof. Let's prove each of the claims in order.

For the first claim, we know from lemma 2 that $g_v(\theta) > g_w(\theta)$ for all $\theta \in [0,1]$ since $m_1(\theta) > 0$ for all $\theta \in [0,1]$. As a result,

$$\mathbb{P}[g_w(\theta) \leq x] = \mathbb{P}[\theta \geq g_w^{-1}(x)] \geq \mathbb{P}[\theta \geq g_v^{-1}(x)] = \mathbb{P}[g_v(\theta) \leq x]$$

Note that the result also follows from Theorem 1.A.17 in [55].

The second claim again follows from the fact that $m_n(\theta) < 0$ for all θ.

Now let's prove the third claim. First we'll show that $\mathbb{E}[m_j(\theta)] \geq 0$. Note that $\mathbb{E}[m_j(\theta)] = \int_0^1 m_j(\theta) f(\theta) d\theta$. For any $j \in \{2, \ldots, n-1\}$, we know from Lemma 2 that there exists t_j^1 such that $m_j(\theta) = \begin{cases} <= 0 & \text{if } \theta \leq t_j^1 \\ > 0 & \text{otherwise} \end{cases}$

Using this, we have that

$$\begin{aligned}
\mathbb{E}[m_j(\theta)] &= \int_0^1 m_j(\theta) f(\theta) d\theta \\
&= \int_0^{t_j^1} m_j(\theta) f(\theta) d\theta + \int_{t_j^1}^1 m_j(\theta) f(\theta) d\theta \\
&\geq \int_0^{t_j^1} m_j(\theta) f(t_j^1) d\theta + \int_{t_j^1}^1 m_j(\theta) f(t_j^1) d\theta \\
&= f(t_j^1) \int_0^1 m_j(\theta) d\theta \\
&= 0
\end{aligned}$$

. It follows then that $\mathbb{E}[g_v(\theta)] \geq \mathbb{E}[g_w(\theta)]$. In addition, we know from lemma 2 that there exists t_j^1 such that $g_v(\theta) - g_w(\theta) = \begin{cases} < 0 & \text{if } \theta < t_j^1 \\ = 0 & \text{if } \theta = t_j^1 \\ > 0 & \text{otherwise} \end{cases}$

Let $G_v(x) = \mathbb{P}[g_v(\theta) \leq x]$ denote the cdf of effort under prize vector $\mathbf{v}$. Then, from above, we have that

$$G_v(x) - G_w(x) = \begin{cases} < 0 & \text{if } x < g_v(t_j^1) \\ = 0 & \text{if } x = g_v(t_j^1) \\ > 0 & \text{otherwise} \end{cases}$$

Thus, we have that $\mathbb{E}[g_v(\theta)] \geq \mathbb{E}[g_w(\theta)]$ and also the sign of $G_v(x) - G_w(x)$ changes exactly once from $-$ to $+$ as x increases. It follows then from Theorem 4.A.22 in [55] that $g_v(\theta)$ second order stochastically dominates $g_w(\theta)$.

The argument for the case of decreasing density is analagous. $\square$

Lemma 2. *For any number of agents n and prior F, the following hold:*

1. $\theta m_i(\theta) \leq 1$ *for all* $i \in \{1, \dots, n-1\}, \theta \in [0, 1]$

2. $\lim_{\theta \to 0} \theta m_i(\theta) = 0$ *for all* $i \in \{1, \dots, n-1\}$

3. $\int_0^1 m_1(\theta) = 1$ *and* $\int_0^1 m_i(\theta) = 0$ *for* $i \in \{2, n-1\}$

4. $m_1(\theta) > 0$ *for all* θ *and monotone decreasing*

5. *For* $i \in \{2, n-1\}$, *there exist* $t_i^1 < t_i^2$ *such that*

$$m_i(\theta) = \begin{cases} <= 0 & \text{if } \theta \leq t_i^1 \\ > 0 & \text{otherwise} \end{cases}$$

$$m_i'(\theta) = \begin{cases} >= 0 & \text{if } \theta \leq t_i^2 \\ < 0 & \text{otherwise} \end{cases}$$

Proof. We'll prove the properties one by one.

The first property provides an upper bound on the marginal effect of any prize i on the effort of agent of type θ: $m_i(\theta) \leq \frac{1}{\theta}$. To prove this, consider a case where prize i increases by Δ. Then from the characterization in Theorem 1, it follows that an agent of type θ will increase its effort by $\Delta m_i(\theta)$. This would correspond to an increase in cost of $\Delta \theta m_i(\theta)$. But the overall gain from the increased prize is $\leq \Delta$. Since the change in cost must be less than the gain in prize, we get a simple bound of $\theta m_i(\theta) \leq 1$ for all θ.

The second property essentially says that the cost of the most efficient agent goes to zero. We'll first prove this property for $i \in \{2, n-1\}$ by squeeze theorem. First let's obtain an upper bound:

$$
\begin{aligned}
\theta m_i(\theta) &= \binom{n-1}{i-1}\theta\int_{F(\theta)}^{1}\frac{\left[(1-t)^{n-i-1}t^{i-2}\right]}{F^{-1}(t)}\left((n-1)t-(i-1)\right)dt\\
&\le \binom{n-1}{i-1}\theta\int_{F(\theta)}^{1}\frac{\left[(1-t)^{n-i-1}t^{i-2}\right]}{\theta}\left((n-1)t-(i-1)\right)dt\\
&= \binom{n-1}{i-1}\int_{F(\theta)}^{1}(1-t)^{n-i-1}t^{i-2}\left((n-1)t-(i-1)\right)dt
\end{aligned}
$$

At $\theta = 0$, the integral equals $(n-1)\beta(i, n-i) - (i-1)\beta(i-1, n-i) = 0$

For the lower bound, we have

$$
\theta m_i(\theta) \ge \binom{n-1}{i-1}\theta\int_{F(\theta)}^{1}(1-t)^{n-i-1}t^{i-2}\left((n-1)t-(i-1)\right)dt
$$

which goes to 0 as $\theta \to 0$. Thus, $\lim_{\theta\to 0}\theta m_i(\theta) = 0$ for $i \in \{2, 3, \dots, n-1\}$

Now we'll prove the limit is 0 for $i = 1$. We have

$$
\theta m_1(\theta) = \theta(n-1)\int_{F(\theta)}^{1}\frac{\left[(1-t)^{n-2}\right]}{F^{-1}(t)}dt
$$

. If the integral is finite, we are done. If it is infinite, we can apply L-Hospital's rule to get

$$
\lim_{\theta\to 0}\frac{m_1(\theta)}{\frac{1}{\theta}} = \lim_{\theta\to 0}-\theta^2 m_1'(\theta) = \lim_{\theta\to 0}\theta^2(n-1)f(\theta)\frac{(1-F(\theta))^{n-2}}{\theta} = 0
$$

Now we prove the third property. By Leibniz rule, for $i \ge 2$, we have

$$
m_i'(\theta) = -\binom{n-1}{i-1}\frac{\left[(1-F(\theta))^{n-i-1}F(\theta)^{i-2}\right]}{\theta}\left((n-1)F(\theta)-(i-1)\right)f(\theta)
$$

Since $\lim_{\theta\to 0}\theta m_i(\theta) = 0$, we have that $\int_0^1 \theta m_i'(\theta) = -\int_0^1 m_i(\theta)d\theta$

From above, we have that

$$
\begin{aligned}
\int_0^1 \theta m_i'(\theta)d\theta &= -\int_0^1 \theta\binom{n-1}{i-1}\frac{\left[(1-F(\theta))^{n-i-1}F(\theta)^{i-2}\right]}{\theta}\left((n-1)F(\theta)-(i-1)\right)f(\theta)d\theta\\
&= -\binom{n-1}{i-1}\int_0^1\left[(1-t)^{n-i-1}t^{i-2}\right]\left((n-1)t-(i-1)\right)dt\\
&= 0
\end{aligned}
$$

Thus, we get that $\int_0^1 m_i(\theta)d\theta = 0$ for $i \ge 2$. For $i = 1$, we have that

$$
m_1(\theta) = (n-1)\int_{F(\theta)}^{1}\frac{\left[(1-t)^{n-2}\right]}{F^{-1}(t)}dt
$$

so that $m_1'(\theta) = -(n-1)\dfrac{(1-F(\theta))^{n-2}}{\theta}f(\theta)$ and thus, $\int_0^1 \theta m_1'(\theta)d\theta = -1$. This gives that $\int_0^1 m_1(\theta)d\theta = 1$.

The fourth property follows from the fact that $m_1'(\theta) < 0$ and $m_1(1) = 0$.

For the last property, we can use the expression for $m_i'(\theta)$ to get that $t_i^2 = F^{-1}(\frac{i-1}{n-1})$. The claim on existence of $t_i^1 < t_i^2$ then follows from the fact that $\int_0^1 m_i(\theta)d\theta = 0$ for $i \in \{2, \ldots, n-1\}$.

$\square$

Theorem 2. *Consider a setting with* n *agents and prior* $F(\theta) = \theta^p$. *Suppose* $\mathbf{v} = (v_1, \ldots, v_n)$ *and* $\mathbf{w} = (w_1, \ldots, w_n)$ *are two prize vectors such that* $\mathbf{v}$ *is more competitive than* $\mathbf{w}$. *Then, we have the following:*

1. *if* $p \geq 1$, *then*
$$\mathbb{E}[g_{\mathbf{v}}(\theta)] \geq \mathbb{E}[g_{\mathbf{w}}(\theta)]$$

2. *if* $\frac{1}{2} \leq p \leq 1$, $v_1 = w_1, v_n = w_n$, *then*
$$\mathbb{E}[g_{\mathbf{v}}(\theta)] \leq \mathbb{E}[g_{\mathbf{w}}(\theta)]$$

3. *if* $p \geq \frac{1}{2}$ *and* $v_n = w_n$, *then*
$$\mathbb{E}[g_{\mathbf{v}}(\theta_{max})] \leq \mathbb{E}[g_{\mathbf{w}}(\theta_{max})]$$

Proof. First, we show that the expected marginal effect of prize v_i is

$$\mathbb{E}[m_i(\theta)] = \binom{n-1}{i-1}\beta(i - \frac{1}{p}, n-i)\frac{(n-i)(p-1)}{np-1}$$

This follows from the following calculations:

$$\begin{aligned}
\mathbb{E}[m_i(\theta)] &= -\int_0^1 F(\theta)m_i'(\theta)d\theta \\
&= \binom{n-1}{i-1}\int_0^1 \frac{[(1-F(\theta))^{n-i-1}F(\theta)^{i-1}]}{\theta}\left((n-1)F(\theta) - (i-1)\right)f(\theta)d\theta \\
&= \binom{n-1}{i-1}\int_0^1 \frac{[(1-t)^{n-i-1}t^{i-1}]}{F^{-1}(t)}\left((n-1)t - (i-1)\right)dt
\end{aligned}$$

For $F(\theta) = \theta^p$,

$$
\begin{aligned}
\mathbb{E}[m_i(\theta)] &= \binom{n-1}{i-1}\left((n-1)\beta(i+1-\frac{1}{p}, n-i) - (i-1)\beta(i-\frac{1}{p}, n-i)\right) \\
&= \binom{n-1}{i-1}\left((n-1)\beta(i-\frac{1}{p}, n-i)\frac{i-\frac{1}{p}}{n-\frac{1}{p}} - (i-1)\beta(i-\frac{1}{p}, n-i)\right) \\
&= \binom{n-1}{i-1}\beta(i-\frac{1}{p}, n-i)\left((n-1)\frac{i-\frac{1}{p}}{n-\frac{1}{p}} - (i-1)\right) \\
&= \binom{n-1}{i-1}\beta(i-\frac{1}{p}, n-i)\frac{(n-i)(p-1)}{np-1}
\end{aligned}
$$

Now observe that

$$\frac{\mathbb{E}[m_{i+1}(\theta)]}{\mathbb{E}[m_i(\theta)]} = \frac{n-i}{i}\frac{i-\frac{1}{p}}{n-i-1}\frac{n-i-1}{n-i} = \frac{i-\frac{1}{p}}{i} < 1$$

Note that for $p \geq 1$, these marginal effects are positive and thus, the marginal effect of prize i is decreasing in i. This implies that the change in expected effort from increasing any prize $i \in [n-1]$ is positive and the change from increasing v_i is greater than that from increasing v_j for any $i < j$. Since w can be obtained from v via a sequence of Robinhood operations which involve replacing v_i by $v_i - \epsilon$ and v_j by $v_j + \epsilon$, each of which reduces expected effort, we get that the expected effort under w will be lesser than the expected effort under v when $p \geq 1$. So if v is more competitive than w, then $\mathbb{E}[g_v(\theta)] \geq \mathbb{E}[g_w(\theta)]$. For $\frac{1}{2} \leq p \leq 1$, the expected marginal effects are < 0 but the ratio is still < 1. Thus, the expected marginal effects are actually increasing in i and so we get the inequality in the second item.

For the inequality in the third item, we first show that

$$\mathbb{E}[m_i(\theta_{max})] = \binom{n-1}{i-1}\beta\left(n+i-1-\frac{1}{p}, n-i\right)\frac{(n-i)(np-1)}{2np-p-1}$$

for $i \in \{1, 2, \ldots, n-1\}$.

$$\mathbb{E}[m_i(\theta_{max})] = \int_0^1 m_i(\theta) nF(\theta)^{n-1} f(\theta) d\theta$$

$$= \binom{n-1}{i-1} \int_0^1 \frac{[(1-F(\theta))^{n-i-1}F(\theta)^{i-2}]}{\theta} \left((n-1)F(\theta)-(i-1)\right) F(\theta)^n f(\theta) d\theta$$
$$= \binom{n-1}{i-1} \int_0^1 \frac{[(1-t)^{n-i-1}t^{n+i-2}]}{F^{-1}(t)} \left((n-1)t-(i-1)\right) dt$$

For the case of $F(\theta) = \theta^p$, we get that

$$\mathbb{E}[m_i(\theta_{max})] = \binom{n-1}{i-1} \left((n-1)\beta\left(n+i-\frac{1}{p}, n-i\right) - (i-1)\beta\left(n+i-1-\frac{1}{p}, n-i\right)\right)$$
$$= \binom{n-1}{i-1} \beta\left(n+i-1-\frac{1}{p}, n-i\right)\left((n-1)\frac{n+i-1-\frac{1}{p}}{2n-1-\frac{1}{p}} - (i-1)\right)$$
$$= \binom{n-1}{i-1} \beta\left(n+i-1-\frac{1}{p}, n-i\right)\frac{(n-i)(np-1)}{2np-p-1}$$

Now observe that
$$\frac{\mathbb{E}[m_{i+1}(\theta_{max})]}{\mathbb{E}[m_i(\theta_{max})]} = \frac{n+i-1-\frac{1}{p}}{i} > 1$$
Thus, the marginal effect of prize i is increasing in i. Again, since w can be obtained from v via a sequence of Robinhood operations which involve replacing v_i by $v_i - \epsilon$ and v_j by $v_j + \epsilon$, each of which increases expected minimum effort, we get that the expected minimum effort under w will be greater than that under v. It follows that if v is more competitive than w and both have the same last prize, then $\mathbb{E}[g_v(\theta_{max})] \leq \mathbb{E}[g_w(\theta_{max})]$ $\square$

B Proofs for Section 4 (Applications)

Theorem 4. *Suppose there are n agents with utility $u(v) = v^r$ and the prior cdf on marginal costs is $F(\theta) = \theta^p$ with $p \geq 1$. The prize structure that maximizes expected effort is*

$$v(r) = (v_1(r), c_2^{\frac{1}{1-r}} v_1(r), \ldots c_{n-1}^{\frac{1}{1-r}} v_1(r), 0)$$

where

$$c_i = \frac{\mathbb{E}[m_i(\theta)]}{\mathbb{E}[m_1(\theta)]} < 1$$

and $v_1(r)$ is such that

$$\sum v_i(r) = B$$

Further, if $1 \geq r > r' > 0$, then $v(r)$ is more competitive than $v(r')$.

Proof. From corollary 1, we know that the Bayes-Nash equilibrium function takes the form

$$g_v(\theta) = \sum_{i=1}^{n} m_i(\theta) u(v_i)$$

where,

$$m_i(\theta) = \binom{n-1}{i-1} \int_{F(\theta)}^{1} \frac{[(1-t)^{n-i-1} t^{i-2}]}{F^{-1}(t)} \left((n-1)t - (i-1)\right) dt$$

for $i \in [n-1]$ and $m_n(\theta) = -\sum_{i=1}^{n-1} m_i(\theta)$. Given this form of the equilibrium function, the problem is

$$\max_{\mathbf{v}} \sum_{i=1}^{n-1} u(v_i) \mathbb{E}[m_i(\theta)]$$

such that $\sum_{i=1}^{n-1} v_i = B$.

Check that the solution will satisfy the equation

$$V_1(r) \left[1 + \sum_{i=2}^{n-1} c_i^{\frac{1}{1-r}}\right] = B$$

where $c_i = \frac{\mathbb{E}[m_i(\theta)]}{\mathbb{E}[m_1(\theta)]} < 1$ and $c_i > c_{i+1}$ for all i. Note that c_i does not depend on r.

Let $f_k(r) = V_1(r) \left[1 + \sum_{i=2}^{k} c_i^{\frac{1}{1-r}}\right]$.

I want to show that $f_k'(r) > 0$ for all k.

If I can show $f_k'(r)$ is single peaked in k, that would imply the result since $f_n(r) = 0$.

Check that $V_1'(r) = \dfrac{-\left[\sum_{i=2}^{n-1} c_i^{\frac{1}{1-r}} \log(c_i)\right] V_1^2(r)}{(1-r)^2 B}$ Plugging it in, we get

$$f_k'(r) = V_1(r) \left[\frac{1}{(1-r)^2} \sum_{i=2}^{k} c_i^{\frac{1}{1-r}} \log(c_i)\right] + V_1'(r) \left[1 + \sum_{i=2}^{k} c_i^{\frac{1}{1-r}}\right]$$

$$= V_1(r) \left[\frac{1}{(1-r)^2} \sum_{i=2}^{k} c_i^{\frac{1}{1-r}} \log(c_i)\right] - \frac{\left[\sum_{i=2}^{n-1} c_i^{\frac{1}{1-r}} \log(c_i)\right] V_1^2(r)}{(1-r)^2 B} \left[1 + \sum_{i=2}^{k} c_i^{\frac{1}{1-r}}\right]$$

$$
\begin{aligned}
&= \frac{V_1(r)}{(1-r)^2}\sum_{i=2}^{k} c_i^{\frac{1}{1-r}}\log(c_i)\left[1-\frac{V_1(r)}{B}\left(1+\sum_{i=2}^{k} c_i^{\frac{1}{1-r}}\right)\right] - \frac{V_1^2(r)}{B(1-r)^2}\sum_{i=k+1}^{n-1} c_i^{\frac{1}{1-r}}\log(c_i)\left[1+\sum_{i=2}^{k} c_i^{\frac{1}{1-r}}\right] \\
&= \frac{V_1(r)}{B(1-r)^2}\sum_{i=2}^{k} c_i^{\frac{1}{1-r}}\log(c_i)\left[B-f_k(r)\right] - \frac{V_1(r)f_k(r)}{B(1-r)^2}\sum_{i=k+1}^{n-1} c_i^{\frac{1}{1-r}}\log(c_i) \\
&= \frac{V_1(r)}{B(1-r)^2}\left(B\sum_{i=2}^{k} c_i^{\frac{1}{1-r}}\log(c_i) - f_k(r)\sum_{i=2}^{n-1} c_i^{\frac{1}{1-r}}\log(c_i)\right)
\end{aligned}
$$

To show that the term inside the bracket is positive, we basically need to show that for any decreasing sequence $1 \geq d_1 > d_2 > \ldots d_n > 0$, we have that

$$
h(k) = \sum_{i=1}^{n} d_i \sum_{i=1}^{k} d_i \log(d_i) - \sum_{i=1}^{k} d_i \sum_{i=1}^{n} d_i \log(d_i) \geq 0
$$

for any $k \in [n]$

Observe that

$$
\begin{aligned}
\Delta(k) &= h(k+1) - h(k) \\
&= d_{k+1}\log(d_{k+1})\sum_{i=1}^{n} d_i - d_{k+1}\sum_{i=1}^{n} d_i \log(d_i) \\
&= d_{k+1}\left(\log(d_{k+1})\sum_{i=1}^{n} d_i - \sum_{i=1}^{n} d_i \log(d_i)\right)
\end{aligned}
$$

Since d_k is a decreasing sequence, it follows that if $\Delta(k) < 0$, then $\Delta(j) < 0$ for all $j > k$. But observe that $h(n) = 0$. So we just need to show that $h(1) > 0$ which is obvious. $\square$