Chapter-4

Digital Image Transforms

Review Questions

1. What is the complex conjugate and the Hermitian matrix of T?

$$T = \begin{pmatrix} 2+2j & 3 \\ -2 & 7-j \end{pmatrix}$$

Solution

Complex conjugate of $T = \begin{pmatrix} \mathbf{2-2j} & \mathbf{3} \\ \mathbf{-2} & \mathbf{7+j} \end{pmatrix}$

Hermitian matrix of $T = \begin{pmatrix} \mathbf{2-2j} & \mathbf{-2} \\ \mathbf{3} & \mathbf{7+j} \end{pmatrix}$

2.Check whether the matrix A = $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is orthogonal or not.

Solution

If $A^{-1} = A^T$, then A is said to be orthogonal.

$A^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$(\text{Since } \cos^2\theta + \sin^2\theta = 1)$

$A^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$\therefore A^{-1} = A^T$

So, matrix **A is orthogonal**.

3. Prove that the product of two orthogonal matrices is another orthogonal matrix.

Solution

Let A and B be two orthogonal matrices. So, $A^{-1} = A^T$ and $B^{-1} = B^T$.

$\therefore (AB)^{-1} = A^{-1} \cdot B^{-1} = A^T \cdot B^T = (AB)^T$

If $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ then,

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(\mathrm{AB})^{-1} = -1 \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(\mathrm{AB})^{T} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\therefore (AB)^{-1} = (AB)^{T}$

Thus, the product of two orthogonal matrices is another orthogonal matrix.

Numerical Problems

1. Check whether the matrix $A = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$ is orthogonal or not.

<u>Solution</u>

If $A^{-1} = A^{T}$, then A is said to be orthogonal.

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0.5 & -1 \\ 0 & 1 \end{pmatrix}$$

$$A^{T} = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$$

$\therefore A^{-1} \neq A^{T}$

So, matrix **A is not orthogonal**.

2. What is the complex conjugate and Hermitian matrix of F.

$$F = \begin{pmatrix} 1+j & 1-j \\ -2 & 7+j \end{pmatrix}$$

Solution:

$$F = \begin{pmatrix} 1-j & 1+j \\ -2 & 7-j \end{pmatrix}$$

3. Consider the basis images in given in text book, show that the original image can be reconstructed for the given image.

$$F = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$$

What is the reconstructed image if only two largest factors are retained?

Solution:

Let the basis images are given as

$$H_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}; H_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix};$$
$$H_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}; H_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix};$$

Show that for the image $f = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$, the image can be constructed using basis images.

The transformed coefficient can be calculated by performed element wise product of the original image and the basis images.

$$t_1 = \langle H_1, f \rangle = \frac{1}{2}[1*1+2*1+3*1+8*1] = \frac{1}{2}[14] = 7$$

$$t_2 = \langle H_2, f \rangle = \frac{1}{2}[1*1+2*-1+3*1+8*-1] = \frac{1}{2}[-6] = -3$$

$$t_3 = \langle H_3, f \rangle = \frac{1}{2}[1*1+2*1-3*1-8*1] = \frac{1}{2}[-8] = -4$$

$$t_4 = \langle H_4, f \rangle = \frac{1}{2}[1*1-2*1-3*1+8*1] = \frac{1}{2}[4] = 2$$

The reconstructed image is the product of the transformed coefficients and the basis image.

$$= 7*\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} - \frac{4}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + 2*\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$$

4. Apply DFT to the following sequences:

    a. { 2, 4, 6, 1 }

    b. { 1, 2, 8, 9 }

    and prove that the inverse transform works. What is the need for Fourier transforms?

    <u>Solution</u>

$$W_{4x4} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$$

a)Applying DFT,

$$F(u) = W \cdot f(x)$$

$$F(u) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{13} \\ \mathbf{-4-3j} \\ \mathbf{3} \\ \mathbf{-4+3j} \end{pmatrix}$$

Applying IDFT,

$$f(x) = \frac{1}{N} \cdot W^* \cdot F(u)$$

$$f(x) = \frac{1}{4} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \cdot \begin{pmatrix} 13 \\ -4-3j \\ 3 \\ -4+3j \end{pmatrix} = \frac{1}{4} \cdot \begin{pmatrix} 8 \\ 16 \\ 24 \\ 4 \end{pmatrix} = \begin{pmatrix} \mathbf{2} \\ \mathbf{4} \\ \mathbf{6} \\ \mathbf{1} \end{pmatrix}$$

Thus, DFT works for the given sequence.

b)Applying DFT,

$$F(u) = W \cdot f(x)$$

$$F(u) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} \mathbf{20} \\ \mathbf{-7-7j} \\ \mathbf{-2} \\ \mathbf{-7+7j} \end{pmatrix}$$

Applying IDFT,

$$f(x) = \frac{1}{N} \cdot W^* \cdot F(u)$$

$$f(x) = \frac{1}{4} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \cdot \begin{pmatrix} 20 \\ -7-7j \\ -2 \\ -7+7j \end{pmatrix} = \frac{1}{4} \cdot \begin{pmatrix} 4 \\ 8 \\ 32 \\ 36 \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{8} \\ \mathbf{9} \end{pmatrix}$$

Thus, DFT works for the given sequence.

5. Apply DFT to the following matrices:

a. $\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$

b. $\begin{pmatrix} 7 & 0 \\ 3 & 1 \end{pmatrix}$

<u>Solution</u>

a) Applying DFT,

$$y = W \cdot F \cdot W^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$y = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathbf{5} & \mathbf{-1} \\ \mathbf{3} & \mathbf{1} \end{pmatrix}$$

Applying IDFT,

$$F = W^* \cdot y \cdot W^{*T} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$F = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 4 & 6 \\ 4 & 2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 8 & 8 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} \mathbf{2} & \mathbf{2} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$$

Thus, DFT works for the given image.

b) Applying DFT,

$$y = W \cdot F \cdot W^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 7 & 0 \\ 3 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$y = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 7 & 7 \\ 4 & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathbf{11} & \mathbf{9} \\ \mathbf{3} & \mathbf{5} \end{pmatrix}$$

Applying IDFT,

$$F = W^* \cdot y \cdot W^{*T} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 11 & 9 \\ 3 & 5 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$F = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 20 & 2 \\ 8 & -2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 28 & 0 \\ 12 & 4 \end{pmatrix} = \begin{pmatrix} \mathbf{7} & \mathbf{0} \\ \mathbf{3} & \mathbf{1} \end{pmatrix}$$

Thus, DFT works for the given image.

6. Apply DCT to the following sequences:

a. { 2, 4, 6, 1 }

b. { 1, 2, 8, 9 }

<u>Solution</u>

The 4x4 DCT matrix, $C = \begin{pmatrix} 0.50 & 0.50 & 0.50 & 0.50 \\ 0.65 & 0.27 & -0.27 & -0.65 \\ 0.50 & -0.50 & -0.50 & 0.50 \\ 0.27 & -0.65 & 0.65 & -0.27 \end{pmatrix}$

a)Applying DCT,

$$y = C \cdot F$$

$$y = \begin{pmatrix} 0.50 & 0.50 & 0.50 & 0.50 \\ 0.65 & 0.27 & -0.27 & -0.65 \\ 0.50 & -0.50 & -0.50 & 0.50 \\ 0.27 & -0.65 & 0.65 & -0.27 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 6 \\ 1 \end{pmatrix}$$

$$\boldsymbol{y} = \begin{pmatrix} \mathbf{6.50} \\ \mathbf{0.11} \\ \mathbf{-3.50} \\ \mathbf{1.57} \end{pmatrix}$$

Applying IDCT,

$$F = C^T \cdot y$$

$$F = \begin{pmatrix} 0.50 & 0.50 & 0.50 & 0.50 \\ 0.65 & 0.27 & -0.27 & -0.65 \\ 0.50 & -0.50 & -0.50 & 0.50 \\ 0.27 & -0.65 & 0.65 & -0.27 \end{pmatrix} \cdot \begin{pmatrix} 6.50 \\ 0.11 \\ -3.50 \\ 1.57 \end{pmatrix}$$

$$\boldsymbol{F} = \begin{pmatrix} \mathbf{2} \\ \mathbf{4} \\ \mathbf{6} \\ \mathbf{1} \end{pmatrix}$$

Thus, DCT works for the given image.

b)Applying DCT,

$$y = C \cdot F$$

$$y = \begin{pmatrix} 0.50 & 0.50 & 0.50 & 0.50 \\ 0.65 & 0.27 & -0.27 & -0.65 \\ 0.50 & -0.50 & -0.50 & 0.50 \\ 0.27 & -0.65 & 0.65 & -0.27 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 8 \\ 9 \end{pmatrix}$$

$$\boldsymbol{y} = \begin{pmatrix} \mathbf{10.00} \\ \mathbf{-6.85} \\ \mathbf{0} \\ \mathbf{1.74} \end{pmatrix}$$

Applying IDCT,

$$F = C^T \cdot y$$

$$F = \begin{pmatrix} 0.50 & 0.50 & 0.50 & 0.50 \\ 0.65 & 0.27 & -0.27 & -0.65 \\ 0.50 & -0.50 & -0.50 & 0.50 \\ 0.27 & -0.65 & 0.65 & -0.27 \end{pmatrix} \cdot \begin{pmatrix} 10.00 \\ -6.85 \\ 0 \\ 1.74 \end{pmatrix}$$

$$\boldsymbol{F} = \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{8} \\ \mathbf{9} \end{pmatrix}$$

Thus, DCT works for the given image.

5. Show that the Walsh transform works for the following image

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

Solution:

The forward transform is given as

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$$
$$=\frac{1}{2}\begin{pmatrix}6 & 0\\0 & -2\end{pmatrix}$$
$$=\begin{pmatrix}3 & 0\\0 & -1\end{pmatrix}$$

The inverse transform is as shown below

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}3 & 0\\0 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$$
$$=\frac{1}{2}\begin{pmatrix}3 & -1\\3 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$$
$$=\frac{1}{2}\begin{pmatrix}2 & 4\\4 & 2\end{pmatrix}$$
$$=\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}$$

6. Prove that the Hadamard transform works for the following images:

a. $\begin{pmatrix}2 & 2\\2 & 1\end{pmatrix}$

Solution:

The forward transform is given as

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$
$$=\frac{1}{2}\begin{pmatrix} 7 & 1 \\ 1 & -1 \end{pmatrix}$$
$$=\begin{pmatrix} 3.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}$$

The inverse transform is given as

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$
$$=\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$$

b. $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

Solution: Same as the previous problem

7. Prove that the Slant transform holds for the image $\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$.

Solution:

2 X 2 Slant transform kernel is same as Hadamard kernel.

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}3 & 2\\ 1 & 2\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$
$$=\begin{pmatrix}4 & 0\\ 1 & 1\end{pmatrix}$$

The inverse transform is given as

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}4 & 0\\ 1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$
$$=\begin{pmatrix}3 & 2\\ 1 & 2\end{pmatrix}$$

8. Find the Eigen values and the Eigen vector for the following images:

(a) $\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$

Solution:

$$\begin{pmatrix}1-\lambda & 2\\ 2 & 1-\lambda\end{pmatrix}=0$$
$$=(1-\lambda)(1-\lambda)-4=0$$

The characteristic equation is $\lambda^2-2\lambda-3=0$

So λ is 3 or -1.

$when\ \lambda = -1$

$$\begin{pmatrix} 1-(-1) & 2 \\ 2 & 1-(-1) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The equations obtained are

$2x_1 + 2x_2 = 0$ and

$2x_1 + 2x_2 = 0$

Many solutions are possible, but simplest is $x_1 = -1, hence\ \ x_2 = 1$

So the first eigen vector is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$when\ \lambda = 3$

$$\begin{pmatrix} 1-3 & 2 \\ 2 & 1-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The equations obtained are

$-2x_1 + 2x_2 = 0$ and

$2x_1 - 2x_2 = 0$

Many solutions are possible, but simplest is $x_1 = 1, hence\ \ x_2 = 1$

So the first eigen vector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The normalized eigen vectors are given as

-0.7071	0.7071
0.7071	0.7071

(b)

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$$

Solution: The eigen values of this matrix is given 0, 3 and 15.

The normalized eigen vectors are given as

0.3333	-0.6667	0.6667
0.6667	-0.3333	-0.6667
0.6667	0.6667	0.3333

9. Let two of three Eigen values of a 3 × 3 matrix are −1 and 2 and if the determinant value equals 4. What is the third Eigen value?

Solution:

The product of the eigen values is equal to the determinant value.

Therefore -1 * 2 * x = 4

x = -4/2 = -2.