DATA AND
ARTIFICIAL INTELLIGENCE

Eigenvalues, Eigenvectors, and Eigendecomposition

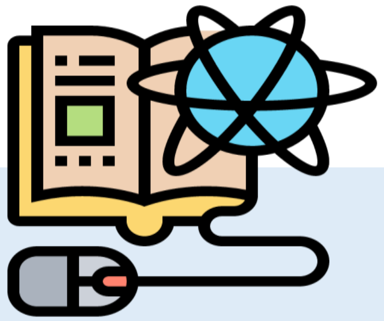
Learning Objectives

By the end of this lesson, you will be able to:

- Define eigenvalues and eigenvectors
- Describe eigenvalues of a square matrix
- List the properties of eigenvalues
- Illustrate the eigendecomposition

Eigenvalues

Eigenvalues Definition



- Eigenvalues are a concept in linear algebra.
- It is mostly used in matrix equations.
- The term **Eigen** in German implies **appropriate** or **characteristic**.
- Other names for the concept of an eigenvalue include characteristic value, characteristic root, appropriate value, and latent root.

Eigenvalues Definition

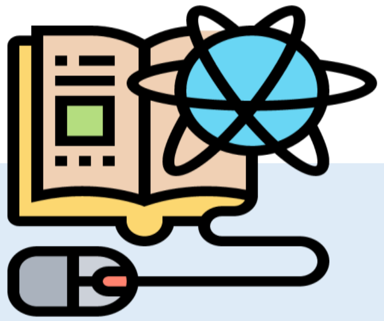
The eigenvalue is a scalar that converts the eigenvector.

Formula

$$Av = \lambda v$$

The number or scalar value λ is an eigenvalue of A.

Eigenvalues Definition



- An eigenvector in mathematics relates to the real non-zero eigenvalues that point in the direction that the transformation stretches.
- The eigenvalue is regarded as a factor by which the eigenvector is stretched.
- There is an eigenvalue for every real matrix.
- The presence of the eigenvalue for any complex matrices is equivalent to the fundamental theorem of algebra.

Eigenvectors

What Are Eigenvectors?

- Eigenvectors are non-zero special vectors that do not change direction when a linear transformation is performed.
- Only a scalar component may affect it.
- If A is a linear transformation from a vector space V and x is a non-zero vector in V, then v is an eigenvector of A if A(X) is a scalar multiple of vector x.
- An eigenspace of vector x is made up of all eigenvectors that have the same eigenvalue as the zero vector.
- The vector zero is not an eigenvector.

What Are Eigenvectors?

If A is a **nxn** matrix and λ is an eigenvalue of matrix A, then x, a non-zero vector, is an eigenvector if it satisfies the expression provided below:

Equation

$$Ax = \lambda x$$

Where:
x indicates an eigenvector of **A** referring to eigenvalue, **λ**.

- There could be many eigenvectors associated with a single eigenvalue.
- The eigenvectors are directly dependent on separate eigenvalues.

Eigenvalues of a Square Matrix

If **$A_{n \times n}$** is a square matrix, then **$[A- \lambda I]$** is referred to as an Eigen or characteristic matrix, and it is an indeterminate scalar.

The eigen matrix determinant and eigen equation can be expressed as:

Determinant

$|A- \lambda I|$

Equation

$|A- \lambda I| = 0$

Where:
I is the identity matrix

Eigenvalues of a Square Matrix

The roots of an Eigen matrix are called Eigen roots.

Eigenvalues of a diagonal matrix and a triangular matrix are equivalent to primary diagonal elements.

The eigenvalues of a scalar matrix are scalars.

Properties of Eigenvalues

Eigenvectors with different Eigenvalues are linearly Independent.

The Eigenvalues of singular matrices are zero.

If A is a square matrix, then $\boldsymbol{\lambda = 0}$ is not one of its eigenvalues.

For a matrix's scalar multiple, if A is a square matrix and λ is one of its eigenvalues, then $a\lambda$ is an eigenvalue of aA.

Properties of Eigenvalues

If A is a square matrix, λ is one of its eigenvalues, and **$n \geq 0$** is an integer, then λ^n is one of An's eigenvalues.

If A is a square matrix, and p(x) is a polynomial in variable x, then p(λ) is the eigenvalue of matrix p(A).

The eigenvalue of A^{-1} is λ^{-1} if the eigenvalue of the square matrix A is λ.

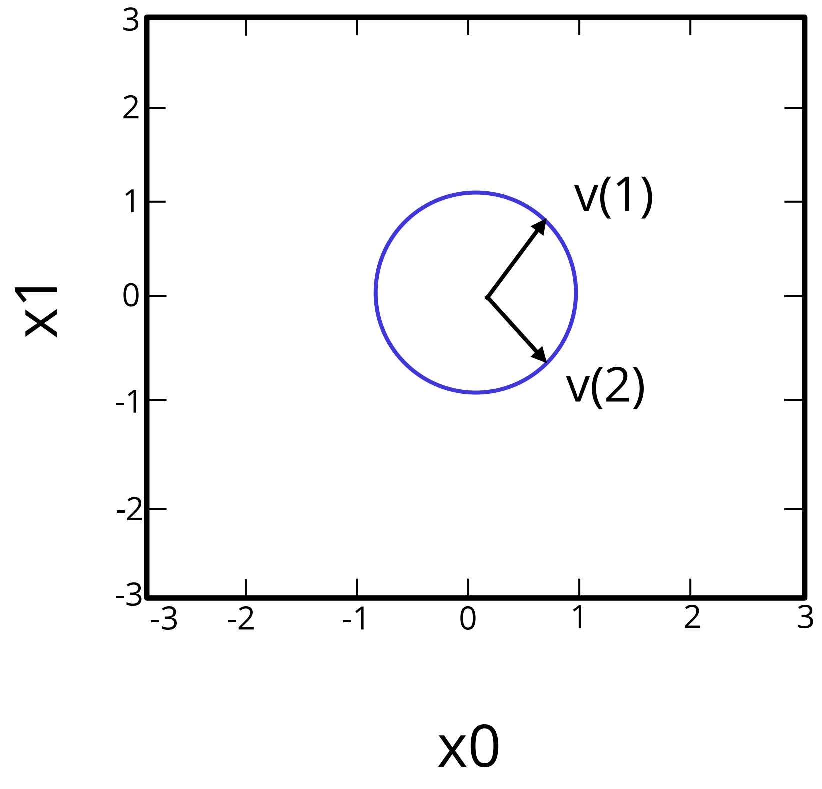
If the square matrix A has the eigenvalue λ, then the eigenvalue of A^T is λ.

Effect of Eigenvectors and Eigenvalues

The matrix A has two orthonormal eigenvectors, v(1) and v(2), which have eigenvalues of λ1 and λ2, respectively.

Set of all unit vectors u € R2 as a unit circle

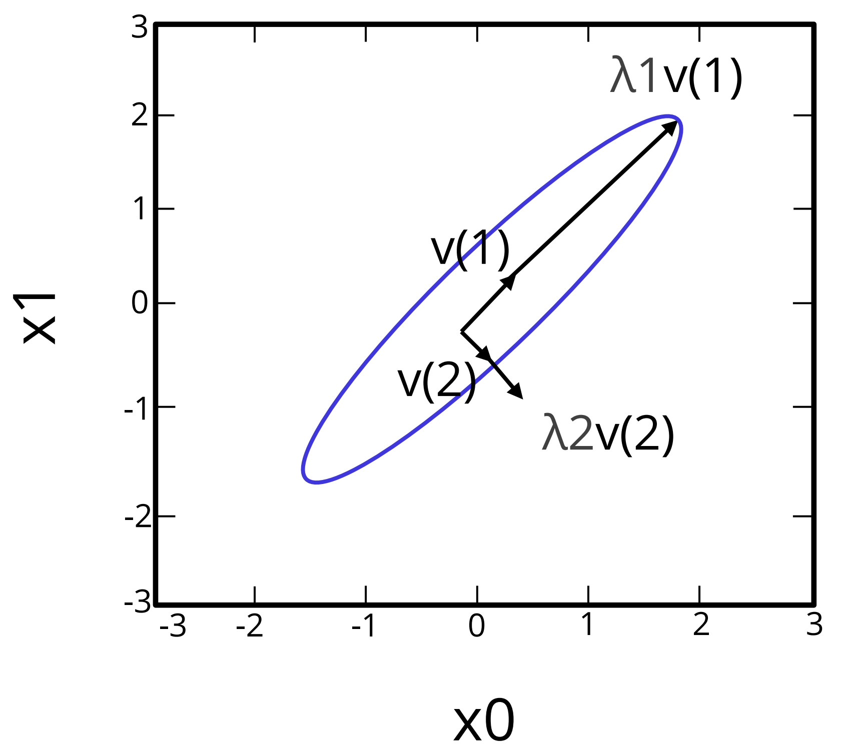
Before multiplication



Effect of Eigenvectors and Eigenvalues

Matrix A distorts the unit circle by scaling the space in the direction v(i) by λi.

After multiplication



Eigendecomposition

Eigendecomposition

Eigendecomposition

Eigendecomposition is the factorization of a matrix into its canonical form, which represents the matrix in terms of its eigenvalues and eigenvectors.

Matrix decompositions are handy for breaking down a matrix into its constituent parts to simplify a variety of more complex operations.

This decomposition is also utilized in machine learning approaches, such as Principal Component Analysis, or PCA.

Eigendecomposition

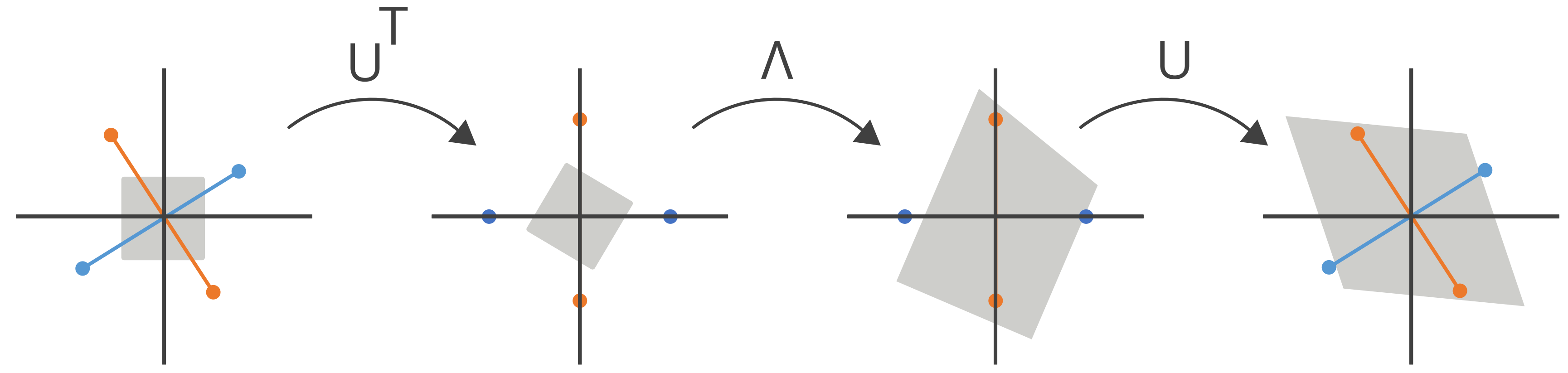
If A contains a set of eigenvectors v1, v2,... represented by matrix V and associated eigenvalues λ1, λ2,... represented by a vector, then the following is the formula for A's eigendecomposition:

Formula

$$A = V \, \mathrm{diag}(\lambda) \, v^{-1}$$

Eigendecomposition

Consider the following image where, U is an orthogonal matrix, U^T is the rotation of matrix U and Λ is the diagonal matrix



The unit vector U is rotated and transformed to U^T.

U^T is scaled-up by Λ.

Finally, the scaled-up U^T is rotated back to U.

Eigendecomposition

A = QAQ-1 explains how A may be decomposed into a similarity transformation.

$$
\begin{array}{ccccccc}
\mathbf{A} & & \mathbf{Q} & & \boldsymbol{\lambda} & & \mathbf{Q^{-1}} \\
\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} & = & \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} & & \begin{bmatrix} \lambda 1 & 0 & 0 \\ 0 & \lambda 2 & 0 \\ 0 & 0 & \lambda 3 \end{bmatrix} & & \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix}^{-1} \\
 & & \text{Eigenvectors of A} & & \text{Eigenvalues of A} & & \text{Eigenvectors of A}
\end{array}
$$

Key Takeaways

- Eigenvalues are the special set of scalar vectors that are associated with the system of linear equations.
- The existence of the eigenvalue for any complex matrices is equal to the fundamental theorem of algebra.
- Eigenvectors are the special vectors (non-zero) that don't change the direction when any linear transformation is applied.
- Eigenvectors with distinct eigenvalues are linearly independent
- Eigenvalues of a triangular matrix and diagonal matrix are identical to the components on the principal diagonals.

Knowledge Check

Knowledge Check 1

What is the determinant of the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix}$?

A. 2

B. 4

C. 6

D. None of above

simplilearn

Knowledge Check 1

What is the determinant of the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix}$?

A. 2

B. 4

C. 6

D. None of above

The correct answer is **C**

This can be computed by cofactor expansion or row reduction.

Knowledge Check 2	Suppose the determinant of a 2 × 2 matrix A is equal to 5. What is the determinant of 2A?

A. 5

B. 10

C. 20

D. 25

Knowledge Check 1

Suppose the determinant of a 2 × 2 matrix A is equal to 5. What is the determinant of 2A?

A. 5

B. 10

C. 20

D. 25

The correct answer is **C**

Two rows get multiplied by 2, so the determinant is multiplied by 2 twice.