Derivatives-II

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1 Implicit Differentiation

Differentiating a function of type $y = f(x)$ is easy. Now, we will try to tackle the harder case where a function is an equation (linear or non-linear) $g(x, y) = c$.

An example: Consider a function $xy = k$, where k is a constant.

Let $y = f(x)$. Therefore, we can write $xf(x) = k$.

Differentiate both sides w.r.t. x, we get:

$$1 \times (f(x)) + x \times (f'(x)) = 0$$

We know that, if $y = f(x)$, then

$$\frac{dy}{dx} = f'(x)$$

Plugging this back into the above equation, we have:

$$\begin{aligned} f(x) + x \times \frac{dy}{dx} &= 0 \\ x \times \frac{dy}{dx} &= -f(x) \\ x \times \frac{dy}{dx} &= -y \\ \frac{dy}{dx} &= -\frac{y}{x} \end{aligned}$$

Consider another example: $y^2 + x = 4$. What is $\frac{dy}{dx}$?

Let $t = y^2 = (f(x))^2$.

Therefore, $\frac{dt}{dy} = 2y$ and we already know that $\frac{dy}{dx} = f'(x)$.

By chain rule:

$$\begin{aligned} \frac{dt}{dx} &= \frac{dt}{dy} \times \frac{dy}{dx} \\ \frac{dt}{dx} &= 2y \times \frac{dy}{dx} \end{aligned}$$

Now, we are all set to calculate the derivative of $f(x, y)$.

$$
\begin{aligned}
2y \times \frac{dy}{dx} + 1 &= 0 && \text{(taking derivative on both sides)} \\
2y \times \frac{dy}{dx} &= -1 \\
\frac{dy}{dx} &= \frac{-1}{2y}
\end{aligned}
$$

One last example and that's it. Consider the function: $xy^2 + \frac{1}{3}x^3 - xy = 5$.

Let $t = y^2 = (f(x))^2$. This implies $\frac{dt}{dy} = 2y$ and $\frac{dt}{dx} = 2y\frac{dy}{dx}$ (by the chain rule).

Let's now compute $\frac{dy}{dx}$.

$$
\begin{aligned}
\frac{d}{dx}(xy^2) + \frac{d}{dx}(\frac{1}{3}x^3) - \frac{d}{dx}(xy) &= 0 \\
\underbrace{y^2\frac{d}{dx}(x) + x\frac{d}{dx}(y^2)}_{\text{applying the product rule}} + x^2 - \underbrace{y\frac{d}{dx}(x) - x\frac{d}{dx}(y)}_{\text{applying the product rule}} &= 0 \\
y^2 + 2xy\frac{dy}{dx} + x^2 - y - x\frac{dy}{dx} &= 0 \\
(2xy - x)\frac{dy}{dx} + x^2 + y^2 - y &= 0 \\
(2xy - x)\frac{dy}{dx} &= (y - x^2 - y^2) \\
\frac{dy}{dx} &= \frac{y - x^2 - y^2}{2xy - x}
\end{aligned}
$$

The recipe:

- Apply $\frac{d}{dx}$ to both sides, reducing the RHS to zero.
- Apply the chain rule rigorously.
- Solve for $\frac{dy}{dx}$ by carefully applying the product rule.

2 Elasticity of Demand

Definition: Let the demand function be $q = f(p)$ (continuous and differentiable). Then, the price elasticity of demand, ϵ_p is defined as

$$
\boxed{\epsilon_p = -\frac{pf'(p)}{f(p)}}
$$

If the previous definitions seems too complicated, you can also use the following definition:

$$\epsilon_{p,q} = -\frac{p}{q} \times \frac{dq}{dp}$$

Interpretation: The price elasticity of demand measures the *percentage change in demand* due to *percentage change in the unit price*.

An example: The unit price p and the quantity demanded (in slices) q of pizza is given by the following equation: $p = -0.05q + 400 \quad (0 \leq q \leq 24000)$. What is the price elasticity of demand for pizza? Calculate elasticities at $p = 100$, $p = 300$.

First, rewrite the demand in its standard form (q as a function of p). $q = 8000 - 20p$

Now, compute $\frac{dq}{dp}$.

$\frac{dq}{dp} = -20$

The elasticity of demand is:

$$\begin{aligned}
&\epsilon = \frac{p}{q} \times \frac{dq}{dp} \\
\implies &\epsilon = \frac{p}{q} \times -20 \\
\implies &\epsilon = \frac{p}{8000 - 20p} \times -20 \\
\implies &\epsilon = \frac{-20p}{8000 - 20p} \\
\implies &\epsilon = \frac{-p}{400 - p}
\end{aligned}$$

It is now easy to compute the elasticities at given prices.

$$\begin{aligned}
&\text{When } p = 100, \quad \epsilon = \frac{-100}{400 - 100} = -\frac{1}{3} \\
&\text{When } p = 300, \quad \epsilon = \frac{-300}{400 - 300} = -3
\end{aligned}$$

Types of Elasticity

1. When $|\epsilon(p)| = 0$, we say that the demand is perfectly inelastic at price p.
2. When $0 < |\epsilon(p)| < 1$, we say that the demand is inelastic at price p.
3. When $|\epsilon(p)| = 1$, the demand is unitary elastic at price p.
4. When $1 < |\epsilon(p)| \leq \infty$, the demand is elastic at price p.
5. When $|\epsilon(p)| \to \infty$, the demand becomes perfectly elastic at price p.

In the previous example, the elasticity of demand is *inelastic* when $p = 100$ and becomes *elastic* as $p = 300$. Guess what happens when $p = 400$.