Introduction to Matrix Algebra-III

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1 Transpose of a Matrix

Let A be an $m \times n$. We define a new matrix denoted by A^T which contains all the elements from A except that the rows and the columns are interchanged. Therefore, the dimension of A^T is $n \times m$.

An example:

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$dim(A) = 1 \times 3$

$$A^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$dim(A^T) = 3 \times 1$

1.1 Properties of Transpose

1 $(A^T)^T = A$ (the transpose of the transpose of a matrix is the original matrix).

2 $(A^T + B^T) = (A + B)^T$.

3 For any scalar k, $(kA)^T = kA^T$.

4 For any two matrices A and B such that matrix multiplication is possible, $(AB)^T = B^T \times A^T$.

2 Determinant of a Matrix

For any $n \times n$ (square) matrix A, there exists a unique number called the determinant of a matrix. It usually denoted by $detA$ or $|A|$. Please note that determinants are defined only for square matrices.

2.1 Determinant of a 1×1 Matrix

If A is a 1×1 matrix with its element a, the determinant of the matrix is a.

2.2 Determinant of a 2×2 Matrix

Let there be a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then, the determinant of A is $ad - bc$.
An example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$$

$|A| = 1 \times 1 - (2 \times 3) = -7$.

2.3 Determinant of a 3×3 Matrix

While we did not cover this case in the lecture, it is important to know how to compute the determinant of a 3×3 matrix.

Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$|A| = a \times \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \times \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \times \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

This is just one of the ways to calculate the determinant of a 3×3 matrix. You can actually pick any particular row or column and construct second order determinants by deleting the elements along the row and the column.

3 Inverse of a Matrix

We will stick to 2×2 case for the inverse of the matrix. We will begin with some rules.

1. A square matrix A is said to be singular if $|A| = 0$ and non-singular if $|A| \neq 0$.
2. For two matrices A and B, the determinant of the product of the matrices will be equal to the product of the determinants of the two matrices. $|AB| = |A| \times |B|$.

Definition: A square matrix A is *invertible* or *non-singular* if there exists another matrix B such that

$$AB = BA = I$$

We call this matrix B the inverse of the matrix A.

The inverse of a non-singular matrix A, denoted by A^{-1}, is defined as:

$$A^{-1} = \frac{adj(A)}{|A|}$$

where $adj(A)$ is known as the adjoint matrix of A.

A square matrix A is invertible if and only if A is a non-singular matrix.

We already know $|A|$. Let's understand how to construct the adjoint matrix. As promised, we only consider a 2×2 matrix.

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then,

$$adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So, the adjoint matrix, $adj(A)$, of a 2×2 matrix, A, can be generated by switching the diagonal elements and changing the signs of the non-diagonal elements.

An example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\implies adj(A) = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

Let's extend this example to compute A^{-1}.

$$\begin{aligned}
A^{-1} &= \frac{adj(A)}{|A|} \\
|A| &= 1 \times 4 - 2 \times 3 = -2 \\
A^{-1} &= \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} && \text{(since we already know } adj(A).\text{)} \\
A^{-1} &= \begin{bmatrix} \frac{-1}{2} \times 4 & \frac{-1}{2} \times (-2) \\ \frac{-1}{2} \times (-3) & \frac{-1}{2} \times 1 \end{bmatrix} && \text{(this step follows the rule of scalar multiplication)} \\
A^{-1} &= \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}
\end{aligned}$$

4 Linear Equations & Matrices

4.1 Basics

Let's take a step back and think of a system of linear equations. Consider the following set of equations:

$$\begin{aligned}
2x + y &= 4 \\
x - 2y &= -3
\end{aligned}$$

We can represent this equation (or any other set of equations) in the matrix form. Let's define a few matrices to do this. Let A be the matrix containing the coefficients on the variables, X be the column vector containing the unknowns, and B be the column vector containing the constants.

We can write $AX = B$. In our example,

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$$
$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$
$$B = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

A general structure, given n variables, would be as follows: A will be an $n \times n$ matrix, X and B two column vectors of dimension $n \times 1$. We can solve these equations using matrix algebra. There are two ways to do this. Before we get into these methods, it is important to understand the different types of solutions. A linear system of equations can have exactly one solution (a unique solution) or infinite solutions or no solution.

Given a matrix equation $A \times X = B$,

- When $|A| \neq 0$, there exists a unique solution.
- When $|A| = 0$, it can either mean infinitely many solutions or no solutions.
  - If $adj(A) \times B$ is not a zero matrix, then we have no solution.
  - If $adj(A) \times B$ is a zero matrix, we may have infinitely many solutions.

4.2 Matrix Inverse Method

Let's consider a two-variables two-equation system $AX = B$, where A, X and B have already been defined above. We can show that $X = (A^{-1}) \times B$.

$$\begin{aligned}
AX &= B \\
(A^{-1})AX &= (A^{-1}) \times B && \text{(multiplying } A^{-1} \text{ to both sides)} \\
(A^{-1}A)X &= A^{-1} \times B && \text{(since we know that matrix multiplication is associative)} \\
IX &= A^{-1} \times B && \text{(since } A^{-1} \times A = I\text{)} \\
X &= A^{-1} \times B && \text{(multiplying identity matrix to a matrix yields the matrix itself)}
\end{aligned}$$

An example: We have a set of equations in x and y.

$$2x + y = 4$$
$$x - 2y = -3$$

Let's write down all the matrices.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$$
$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$
$$B = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

We are all set to apply the matrix inverse method to solve for x and y. We will first need A^{-1}.

$$
\begin{aligned}
A &= \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \\
|A| &= 2 \times -2 - (1 \times 1) = -5 \\
adj(A) &= \begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix} \\
A^{-1} &= \frac{-1}{5} \begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix} \\
A^{-1} &= \begin{bmatrix} \frac{-1}{5} \times -2 & \frac{-1}{5} \times -1 \\ \frac{-1}{5} \times -1 & \frac{-1}{5} \times 2 \end{bmatrix} \\
A^{-1} &= \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{-2}{5} \end{bmatrix}
\end{aligned}
$$

Now that we have A^{-1}, all we need to do is to multiply this matrix to B.

$$
\begin{aligned}
X &= A^{-1} \times B \\
X &= \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{-2}{5} \end{bmatrix} \times \begin{bmatrix} 4 \\ -3 \end{bmatrix} \\
X &= \begin{bmatrix} \frac{2}{5} \times 4 + \frac{1}{5} \times -3 \\ \frac{1}{5} \times 4 + \frac{-2}{5} \times -3 \end{bmatrix} \\
X &= \begin{bmatrix} \frac{8}{5} + \frac{-3}{5} \\ \frac{4}{5} + \frac{6}{5} \end{bmatrix} \\
X &= \begin{bmatrix} \frac{8-3}{5} \\ \frac{4+6}{5} \end{bmatrix} \\
X &= \begin{bmatrix} \frac{5}{5} \\ \frac{10}{5} \end{bmatrix} \\
X &= \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\end{aligned}
$$

Therefore, $x = 1$ and $y = 2$. I will encourage you to verify this using standard methods of solving two equations in two variables.

4.3 Cramer's Rule

Let $AX = B$ be a system of equations such that A is the coefficients matrix, X the matrix of unknowns, and B the matrix of constants. For the sake of simplicity, we consider a 2×2 linear equation system.

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

Therefore, $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, and $B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

Cramer's rule says that the solution to this system is as follows:

$$\begin{aligned} x &= \frac{D_x}{|A|} \\ y &= \frac{D_y}{|A|} \end{aligned}$$

where $D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ and $D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$.

An example:

$$\begin{aligned} x + y &= 5 \\ 2x - y &= 1 \end{aligned}$$

We need to write down the three determinants that we need to compute x and y. It will help if you can list down all the elements you need for this exercise. $a_1 = 1$, $b_1 = 1$, $c_1 = 5$, $a_2 = 2$, $b_2 = -1$, and $c_2 = 1$.

- $|A| = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3$
- $D_x = \begin{vmatrix} 5 & 1 \\ 1 & -1 \end{vmatrix} = -6$
- $D_y = \begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix} = -9$

Now that we have all the terms, we can calculate x and y.

$$\begin{aligned} x &= \frac{D_x}{|A|} = \frac{-6}{-3} = 2 \\ y &= \frac{D_y}{|A|} = \frac{-9}{-3} = 3 \end{aligned}$$