

Integral Calculus

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1 Indefinite Integral

Suppose that we have the first derivative with us and we wish to recover the function. Integrals come handy in that task.

Definition: A function F is the antiderivative of function f on an interval I if

$$F'(x) = f(x) \quad \forall x \in I$$

Let's first recover the function manually using the rule. Let's say $f(x) = 2x$. What is $F(x)$? We know that the derivative of x^2 is $2x$. Therefore, the antiderivative of $2x$ is $x^2 + C$ (where C is an arbitrary constant).

Let's consider another example. Take a firm whose marginal cost is $2x + 3x^2$. You're also told that the firm's fixed cost is ₹100. What is the cost function? The antiderivative of $2x$ is x^2 and that of $3x^2$ is x^3 (check by differentiating x^3 w.r.t. x). Therefore, the cost function must be $C(x) = x^2 + 3x + C$. We also know that the fixed cost is ₹100.

$$C(0) = 100 = (0)^2 + (0)^3 + C \implies C = 100.$$

Hence, the cost function must be

$$C(x) = x^2 + x^3 + 100$$

Definition: Given that $f(x) = F'(x)$:

$$\int f(x)dx = F(x) + C$$

Some terminology- \int is the symbol of the integral, $f(x)$ is known as the integrand, and dx indicates the variable of integration. We have already seen C in action. It is the arbitrary constant. Why do we need this arbitrary constant? Let's take an example: consider $F(x) = x^2$ and $G(x) = x^2 + 1$. Also, think about the following function $f(x) = 2x$. Whether you differentiate $F(x)$ or $G(x)$, you arrive at $f(x)$. Now that we know that an integrand may be associated with many different functions, we should also agree that we need to add a constant to make the original function 'generic'.

1.1 Integral Rules

- **Constant Rule:** $\int k dx = kx + C$.
- **Constant Multiple Rule:** $\int k \cdot f(x) dx = k \int f(x) dx$.
- **Sum/Diff Rule:** $\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$.
- **Power Rule:** $\int x^n dx = \frac{x^{n+1}}{n+1} + C$.
- **Log Rule:** $\int \frac{1}{x} dx = \log |x| + C \quad x \neq 0$.
- **Exponent Rule I:** $\int a^{kx} dx = \frac{a^{kx}}{k \log(a)} + C \quad x \neq 0$.
- **Exponent Rule II:** $\int e^{kx} dx = \frac{1}{k} e^{kx} + C \quad k \neq 0$.

Some examples:

1. What is $\int x^2 dx$?

Answer: Using the power rule, we get $\int x^2 dx = \frac{1}{3}x^3 + C$.

2. What is $\int (x^2 + 2x) dx$?

Answer: Using the sum rule, we get $\int (x^2 + 2x) dx = \int x^2 dx + \int 2x dx = \frac{1}{3}x^3 + x^2 + C$

3. What is $\int e^{2x} dx$?

Answer: Using the exponent rule, we get: $\int e^{2x} dx = \frac{1}{2}e^{2x} + C$

2 Definite Integral

2.1 Area and Definite Integral

The concept of the integral is closely tied to finding the area of a curve. Consider Figure 1. As the function $y = f(x)$ moves from $x = a$ to $x = b$, what is the area under the shaded region?

Let $F(x)$ be an arbitrary indefinite integral of $f(x)$. We also define the area under the curve as $A(x)$. Then, $A(x) = F(x) + C$. At any given point on this curve, the area will be zero. Therefore, $A(a) = F(a) + C \implies C = -F(a)$.

Hence,

$$A(x) = F(x) - F(a) \quad \text{where } F(x) = \int f(x) dx$$

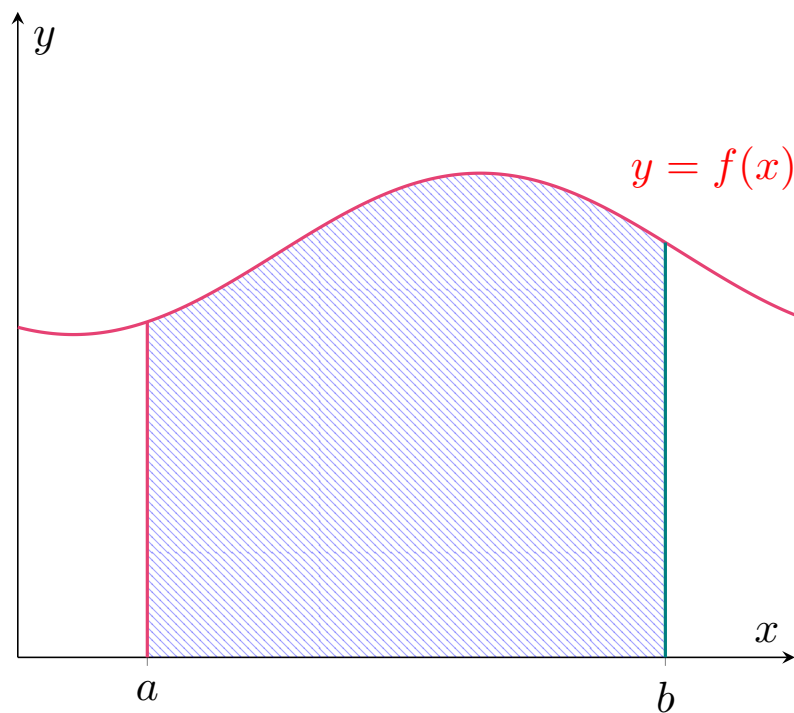


Figure 1

An example: Compute the area under the line $f(x) = x$ over the interval $[1, 3]$.

Answer: Let's first compute the indefinite integral $\int f(x)dx$.

$$F(x) = \int f(x)dx = \int xdx = \frac{1}{2}x^2 + C.$$

The area under $f(x)$ will be $A = F(3) - F(1) = \frac{9}{2} - \frac{1}{2} = 4$. See Figure 2 for the details.

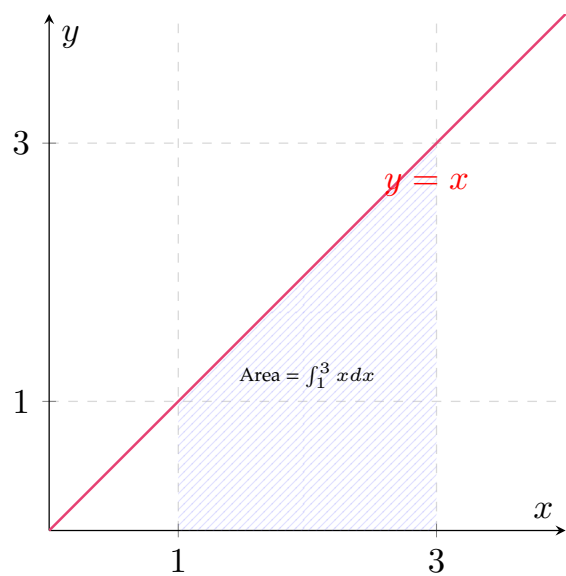


Figure 2

2.2 Definite Integral: Definition

$$\int_a^b f(x)dx = \Big|_a^b F(x) = F(b) - F(a)$$

where F is any arbitrary indefinite integral of f over an interval containing both a and b .

Some examples.

1. Evaluate $\int_1^2 \frac{x}{2} dx$

Answer: $\int_1^2 \frac{x}{2} dx = \Big|_1^2 x = 2 - 1 = 1.$

2. Evaluate $\int_2^4 4x^3 dx$.

Answer: $\int_2^4 4x^3 dx = \Big|_2^4 x^4 = 4^4 - 2^4 = 240.$

3. Evaluate: $\int_1^3 \frac{1}{x} dx$.

Answer: $\int_1^3 \frac{1}{x} dx = \Big|_1^3 3 \log |x| = \log |3| - \log |1| \approx 1.1.$

4. Evaluate: $\int_0^3 (e^x - 1) dx$.

Answer: $\int_0^3 (e^x - 1) dx = \int_0^3 e^x dx - \int_0^3 1 \cdot dx = \Big|_0^3 e^x - \Big|_0^3 x = (e^3 - 1) - (3 - 0) = e^3 - 4.$

2.3 Properties of Definite Integrals

$$(1) \quad \int_a^b f(x)dx = - \int_b^a f(x)dx$$

$$(2) \quad \int_a^a f(x)dx = 0$$

$$(3) \quad \int_a^b kf(x)dx = k \int_a^b f(x)dx \quad (k \text{ is a constant})$$

$$(4) \quad \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad (\text{where } a < b < c)$$

3 Applications in Economics

3.1 Consumer Surplus

It is the difference between the willingness to pay for a good and the actual price that the consumer pays for the good. Imagine that the demand for Pretentious Coffee Shop's *matcha latte* is given by $q = 10 - p$. The supply of *matcha latte* is given by $q = -2 + p$. The equilibrium price, $p^* = 6$ and the equilibrium quantity, $q^* = 4$. Figure 3 shows the equilibrium. Let's try to understand the concept

of willingness to pay (WTP). Imagine a world without matcha lattes. The WTP is $p = 10 - 0 = 10$. What if we were all fighting for a solitary, elusive pitcher of matcha latte? What will be the WTP? The WTP is $p = 10 - 1 = 9$. As more and more match lattes are produced, the WTP falls. The reservation price is 10 and the actual price (based on market equilibrium) is ₹6. The difference is ₹4.

Now, we are all set to compute the consumer surplus (CS). Let p^* and q^* be the market equilibrium price and quantity respectively. The area under the demand curve ($f(q)$) will represent the consumer surplus. We can use the following integral:

$$CS = \int_0^{q^*} f(q) dq - p^* \cdot q^* \quad \text{where } f(q) \text{ is the inverse demand}$$

$$CS = \int_0^4 (10 - q) dq - 6 \cdot 4$$

$$CS = \int_0^4 10 \cdot dq - \int_0^4 q dq - 24 \quad \text{(using the sum rule)}$$

$$CS = \left|_0^4 10q - \left|_0^4 \frac{q^2}{2} - 24 \quad \text{(using the power rule)}$$

$$CS = (40 - 0) - (8 - 0) - 24$$

$$CS = 8$$

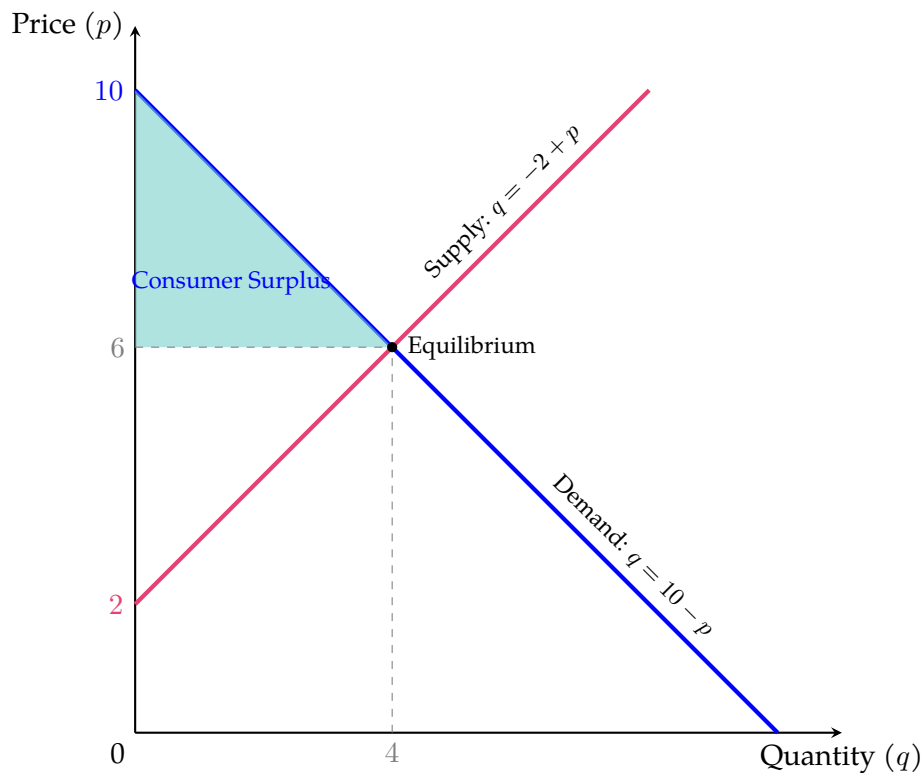


Figure 3

3.2 Producer Surplus

It refers to the difference between the price that the seller receives and their marginal cost. Please note that the supply curve also represents the marginal cost curve for a firm. We will focus on the Pretentious Coffee Shop. By now, we know that the equilibrium price of their matcha latte is $p^* = 6$. When $q = 1$, the marginal cost is 3 and the producer surplus is ₹3. When $q = 4$, the marginal cost is 6. Therefore, the producer surplus at the equilibrium is zero. What happens when $q = 5$? The marginal cost now exceeds the price. Hence, the Pretentious Coffee Shop may stop selling the latte at this stage. Figure 4 depicts the equilibrium.

We can now formally define the producer surplus. Let p^* and q^* be the market equilibrium price and quantity respectively. The producer surplus is the area above the supply curve ($g(q)$). We can use the following integral:

$$PS = p^* \cdot q^* - \int_0^{q^*} g(q) dq \quad \text{where } g(q) \text{ is the inverse supply curve}$$

$$PS = p^* q^* - \int_0^4 (2 + q) dq$$

$$PS = 6 \cdot 4 - \int_0^4 2 dq - \int_0^4 q dq \quad (\text{using the sum rule})$$

$$PS = 24 - \left|_0^4 2q - \left|_0^4 \frac{q^2}{2} \right.\right.$$

$$PS = 24 - (8 - 0) - (8 - 0)$$

$$PS = 24 - 16$$

$$PS = 8$$

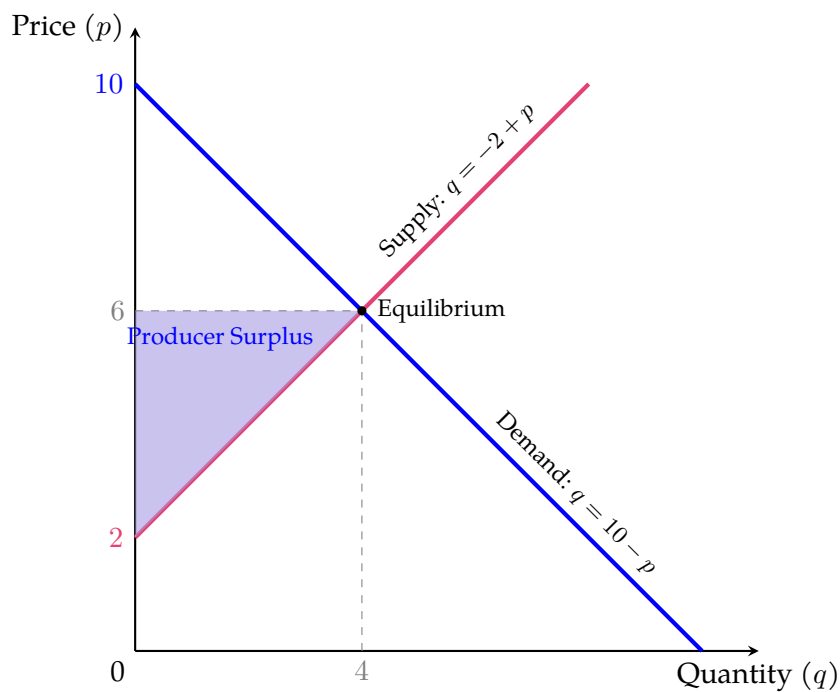


Figure 4