Endterm (Set C)

SIAS, Krea University (AY 2025-26)

Mathematical Methods for Economics (Course Code: **ECON211**)

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Short Answer Questions-I

1. (1 point) Let $f(x) = \sqrt{x} + 4$ and $g(x) = f^{-1}(x)$. Find $g'(6)$.

Solution: We know that: $\boxed{g'(a) = \frac{1}{f'(g(a))}}$

$\therefore g'(6) = \frac{1}{f'(g(6))}$

Let $g(6) = k \implies f(k) = 6$.

$$\begin{aligned} \sqrt{k} + 4 &= 6 \\ \implies \sqrt{k} &= 6 - 4 \\ \implies \sqrt{k} &= 2 \\ \implies k &= 4 \\ f'(x) &= \frac{1}{2\sqrt{x}} \\ \implies f'(4) &= \frac{1}{2\sqrt{2}} \\ \implies f'(4) &= \frac{1}{4} \\ \implies g'(6) &= 4 \end{aligned}$$

Answer: $\boxed{g'(6) = 4}$

2. (1 point) Suppose that f and g are continuous on $[0, 4]$ and that $\int_0^4 (f(x) - g(x))dx = 2$ and $\int_0^4 (3f(x) - 4g(x))dx = 3$. Find $\int_0^4 (f(x) + g(x))dx$.

Solution: Let $\int_0^4 f(x)dx = a$ and $\int_0^4 g(x)dx = b$. Then,

$$\begin{aligned} a - b &= 2 \\ 3a - 4b &= 3 \\ 3a - 3b &= 6 && \text{(multiplying the first equation by 3)} \\ \implies b &= 3 && \text{(differencing the previous two equations to eliminate } a\text{)} \\ \implies a &= 5 \\ \int_0^4 (f(x) + g(x))dx &= \int_0^4 f(x)dx + \int_0^4 g(x)dx && \text{(applying the sum rule)} \\ \implies \int_0^4 (f(x) + g(x))dx &= 8 \end{aligned}$$

Answer : $\boxed{\int_0^4 (f(x) + g(x))dx = 8}$

3. (1 point) Compute: $\int (9x^2 + \frac{5}{x} + e^{6x})dx$

Solution:

$$\int (9x^2 + \frac{5}{x} + e^{6x})dx = \int 9x^2 dx + \int \frac{5}{x} dx + \int e^{6x} dx \qquad \text{(applying the sum rule)}$$

$$\implies \int (9x^2 + \frac{5}{x} + e^{6x})dx = 3x^3 + 5\ln|x| + \frac{e^{6x}}{6} + C$$

Answer : $\boxed{\int (9x^2 + \frac{5}{x} + e^{6x})dx = 3x^3 + 5\ln|x| + \frac{e^{6x}}{6} + C}$

4. (1 point) Is $\lim_{x\to 0} |x-1| = \lim_{x\to 0} |x| - 1$? Explain briefly.

Solution: Consider $f(x) = |x-1|$.

$$f(x) = \begin{cases} 1-x & x < 1 \\ x-1 & x > 1 \end{cases}$$

$$\begin{aligned} \text{LHL: } & \lim_{x\to 0^-} |x-1| = 1 && \text{(since } |x-1| = 1-x \text{ when } x < 1) \\ \text{RHL: } & \lim_{x\to 0^+} |x-1| = 1 && \text{(since } |x-1| = 1-x \text{ when } x < 1) \\ \therefore & \lim_{x\to 0} |x-1| = 1 \end{aligned}$$

Let $g(x) = |x| - 1$.

$$g(x) = \begin{cases} -x-1 & x < 0 \\ x-1 & x > 0 \end{cases}$$

$$\begin{aligned} \text{LHL: } & \lim_{x\to 0^-} |x| - 1 = -1 && \text{(since } |x| = -x \text{ when } x < 0) \\ \text{RHL: } & \lim_{x\to 0^+} |x| - 1 = -1 && \text{(since } |x| = x \text{ when } x > 0) \\ \therefore & \lim_{x\to 0} |x| - 1 = -1 \end{aligned}$$

Answer: $\boxed{\lim_{x\to 0} |x-1| \neq \lim_{x\to 0} |x| - 1}$

5. (1 point) Let $f(x) = \frac{12}{4+x}$. Find $f^{-1}(x)$.

Solution: Let $y = f(x)$.

$$\begin{aligned} y &= \frac{12}{x+4} \\ \implies y(x+4) &= 12 \\ \implies xy + 4y &= 12 \\ \implies xy &= 12 - 4y \\ \implies x &= \frac{12-4y}{y} \\ \implies f^{-1}(x) &= \frac{12-4x}{x} \end{aligned}$$

Answer: $\boxed{f^{-1}(x) = \frac{12-4x}{x}}$

6. (1 point) Let $f(x) = x^{x+2}$. Find $f'(x)$.

Solution: Let $y = x^{x+2}$.

$$\begin{aligned} \ln y &= (x+2)\ln x && \text{(using the property of log } a = b^c \implies \ln a = c \ln b) \\ \implies \frac{1}{y}\frac{dy}{dx} &= \frac{d}{dx}((x+2)\ln x) && \text{(differentiating both sides w.r.t. } x) \\ \implies \frac{1}{y}\frac{dy}{dx} &= \ln x \frac{d}{dx}(x+2) + (x+2)\frac{d}{dx}(\ln x) && \text{(applying the product rule to the RHS)} \\ \implies \frac{1}{y}\frac{dy}{dx} &= \ln x + \frac{x+2}{x} \\ \implies \frac{dy}{dx} &= y\left(\ln x + \frac{x+2}{x}\right) \\ \implies \frac{dy}{dx} &= x^{x+2}\left(\ln x + \frac{x+2}{x}\right) \end{aligned}$$

Answer: $\boxed{\frac{dy}{dx} = x^{x+2}\left(\ln x + \frac{x+2}{x}\right)}$

7. (1 point) Determine if the function $f(x) = x^2 - 6x + 5$ is increasing or decreasing in $[1, 5]$.

Solution: First derivative tells us whether (or where) a function is increasing or decreasing.

$$\begin{aligned} f'(x) &= 2x - 6 \\ \text{Condition for increasing function} : 2x - 6 &\geq 0 \\ \therefore \text{ the function is increasing when } x &\geq 3 \\ \text{Condition for decreasing function} : 2x - 6 &\leq 0 \\ \therefore \text{ the function is decreasing when } x &\leq 3 \end{aligned}$$

Answer: $f(x)$ is decreasing in $[1, 3]$ and increasing in $[3, 5]$.

8. (1 point) Let $x^2y^3 + x^3y^2 = 9$. Find $\frac{dy}{dx}$.

Solution: Let $u = x^2y^3$, $v = x^3y^2$, and $c = 9$.

Applying the product rule, we get:

$$\begin{aligned}
u' &= 2xy^3 + 3x^2y^2\frac{dy}{dx} \\
v' &= 2x^3y\frac{dy}{dx} + 3x^2y^2 \\
c' &= 0 \\
u' + v' &= c' \\
\therefore \left[2xy^3 + 3x^2y^2\frac{dy}{dx}\right] + \left[2x^3y\frac{dy}{dx} + 3x^2y^2\right] &= 0 \\
\implies (2xy^3 + 3x^2y^2) + (3x^2y^2 + 2x^3y)\frac{dy}{dx} &= 0 \\
\implies (3x^2y^2 + 2x^3y)\frac{dy}{dx} &= -(2xy^3 + 3x^2y^2) \\
\implies \frac{dy}{dx} &= -\frac{(2xy^3 + 3x^2y^2)}{(3x^2y^2 + 2x^3y)} \\
\implies \frac{dy}{dx} &= -\frac{y}{x}\frac{(3x + 2y)}{(2x + 3y)}
\end{aligned}$$

Answer: $\boxed{\frac{dy}{dx} = -\frac{y}{x}\frac{(3x + 2y)}{(2x + 3y)}}$

Short Answer Questions-II

9. (2+1 points) The total cost of producing *Phantom cigarettes* is $C(q) = 3q^2 + 10q + 48$. Find the value of q which minimizes the average cost. Show that the marginal cost is equal to the average cost at this point (where the average cost is being minimized).

Solution: We know that the average cost is:

$$AC(q) = \frac{C(q)}{q}$$

The average cost of producing *Phantom cigarettes* is:

$$AC(q) = 3q + 10 + \frac{48}{q}$$

We need to find the first derivative and set it to zero.

$$\begin{aligned}
\frac{d(AC(q))}{dq} &= \frac{d}{dq}\left(3q + 10 + \frac{48}{q}\right) \\
&= 3 - \frac{48}{q^2} \\
\text{FOC:} \quad \frac{d(AC(q))}{dq} &= 0 \\
\implies 3 - \frac{48}{q^2} &= 0 \\
\implies 3q^2 &= 48 \\
\implies q^2 &= 16 \\
\implies q^* &= 4 \\
\text{SOC:} \quad \frac{d^2(AC(q))}{dq^2} &> 0 \qquad \text{(for minimum)} \\
\frac{d^2(AC(q))}{dq^2} &= \frac{96}{q^3} \\
\frac{96}{q^3} &> 0 \text{ when } q = 4
\end{aligned}$$

We also need to compute the marginal cost.

$$MC(q) = 6q + 10$$

When $q = 4$,

$$\begin{aligned}
\text{Average cost:} \quad AC(q=4) &= 3(4) + 10 + \frac{48}{4} \\
&= 12 + 10 + 12 \\
&= 34 \\
\text{Marginal cost:} \quad MC(q=4) &= 6(4) + 10 \\
&= 34
\end{aligned}$$

Answer: The quantity that minimizes the average cost is $\boxed{q^* = 4}$. When $q = 4$, $\boxed{AC = MC = 34}$.

10. (3 points) Let $f(x, y) = 9x^{1/3}y^{2/3}$.

(a) (1 point) Determine the degree of homogeneity.

Solution: The degree of homogeneity can be calculated using:

$$f(tx, ty) = t^k f(x, y)$$

$$\begin{aligned}
f(tx, ty) &= 9(tx)^{1/3}(ty)^{2/3} \\
&= 9t^{1/3}x^{1/3}t^{2/3}y^{2/3} \\
&= (t^{1/3+2/3})9(x^{1/3}y^{2/3}) \\
&= t^1 f(x, y) \\
\implies k &= 1
\end{aligned}$$

Answer: $f(x, y)$ is homogeneous of degree 1.

(b) (2 points) Compute all first and second order partial derivatives.

Solution:

f_x	$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(9x^{1/3}y^{2/3}\right)\\ &= 9\cdot\frac{1}{3}x^{-2/3}y^{2/3}\\ &= 3x^{-2/3}y^{2/3}\\ &= 3\left(\frac{y}{x}\right)^{2/3}\end{aligned}$
f_y	$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(9x^{1/3}y^{2/3}\right)\\ &= 9x^{1/3}\cdot\frac{2}{3}y^{-1/3}\\ &= 6x^{1/3}y^{-1/3}\\ &= 6\left(\frac{x}{y}\right)^{1/3}\end{aligned}$
f_{xx}	$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(3x^{-2/3}y^{2/3}\right)\\ &= 3y^{2/3}\cdot\left(-\frac{2}{3}\right)x^{-5/3}\\ &= -2x^{-5/3}y^{2/3}\\ &= -2\left(\frac{y^2}{x^5}\right)^{1/3}\end{aligned}$
f_{yy}	$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(6x^{1/3}y^{-1/3}\right)\\ &= 6x^{1/3}\cdot\left(-\frac{1}{3}\right)y^{-4/3}\\ &= -2x^{1/3}y^{-4/3}\\ &= -2\left(\frac{x}{y^4}\right)^{1/3}\end{aligned}$
f_{xy}	$\begin{aligned}\frac{\partial^2 f}{\partial x\partial y} &= \frac{\partial}{\partial y}\left(3x^{-2/3}y^{2/3}\right)\\ &= 3x^{-2/3}\cdot\frac{2}{3}y^{-1/3}\\ &= 2x^{-2/3}y^{-1/3}\\ &= 2\left(\frac{1}{x^2y}\right)^{1/3}\end{aligned}$
f_{yx}	$\begin{aligned}\frac{\partial^2 f}{\partial y\partial x} &= \frac{\partial}{\partial x}\left(6x^{1/3}y^{-1/3}\right)\\ &= 6y^{-1/3}\cdot\frac{1}{3}x^{-2/3}\\ &= 2x^{-2/3}y^{-1/3}\\ &= 2\left(\frac{1}{x^2y}\right)^{1/3}\end{aligned}$

11. (1+1+1 points) Let $Y = 2K + 2\sqrt{L}$. Compute the marginal products and the marginal rate of technical substitution.

Solution:

$$MP_K = 2$$
$$MP_L = \frac{1}{\sqrt{L}}$$
$$MRTS_{K,L} = \frac{MP_K}{MP_L}$$
$$MRTS_{K,L} = 2\sqrt{L}$$
$$MRTS_{L,K} = \frac{1}{2\sqrt{L}}$$

12. (3 points) Given the demand function for comedy shows on *Ruinmyshow*: $p = \dfrac{36}{q+5} - 5$,

(a) ($\frac{1}{2}$ points) Compute the total revenue.

Solution: Total revenue, $TR = p \cdot q$.

$$p = \frac{36}{q+5} - 5$$
$$\implies TR = \frac{36q}{q+5} - 5q$$

(b) ($\frac{1}{2}$ points) Compute the marginal revenue.

Solution: We know that $MR = \dfrac{d}{dq}(TR)$.

$$TR = \frac{36q}{q+5} - 5q$$
$$\text{Let } u = 36q, \quad v = q+5$$
$$\implies u' = 36, \quad v' = 1$$
$$MR = \frac{vu' - uv'}{v^2} - 5$$
$$\implies MR = \frac{36(q+5) - 36q}{(q+5)^2} - 5$$
$$\implies MR = \frac{180}{(q+5)^2} - 5$$
$$\implies MR = \frac{180 - 5(q+5)^2}{(q+5)^2}$$

Answer: $\boxed{MR = \dfrac{180 - 5(q+5)^2}{(q+5)^2}}$

(c) (2 points) Compute the revenue-maximizing price and quantity.

Solution: We know that the revenue is maximized when $MR = 0$.

$$\begin{aligned}
180 - 5(q+5)^2 &= 0 \quad \text{(note that the denominator can't be zero.)} \\
\implies 5(q+5)^2 &= 180 \\
\implies (q+5)^2 &= 36 \\
\implies (q+5) &= \pm 6 \\
\implies q+5 &= 6 \qquad\qquad \text{(discarding the negative value.)} \\
\implies q &= 1
\end{aligned}$$

Plugging the value into the demand equation, we get $p = \dfrac{36}{1+5} - 5$

$$\begin{aligned}
\implies p &= \frac{36}{6} - 5 \\
\implies p &= 6 - 5 \\
\implies p &= 1
\end{aligned}$$

Answer: The revenue-maximizing price is $\boxed{p = 1}$ and the quantity is $\boxed{q = 1}$.

Long Answer Questions

13. (5 points) Consider $f(x, y) = \dfrac{x^3}{3} + \dfrac{y^3}{3} + x^2 + \dfrac{y^2}{2} - 3x - 6y + 2$. Find and classify all stationary points.

Solution: To find the stationary points, we first compute the first-order partial derivatives and set them equal to zero.

$$\begin{aligned}
f_x &= \frac{\partial f}{\partial x} = (x^2 + 2x - 3) = 0 \\
f_y &= \frac{\partial f}{\partial y} = (y^2 + y - 6) = 0
\end{aligned}$$

We have two quadratic equations to be solved.

$$\begin{aligned}
(x+3)(x-1) &= 0 \\
(y+3)(y-2) &= 0 \\
x &= 1, -3 \\
y &= 2, -3
\end{aligned}$$

The stationary points are the combinations of these x and y values:

$$\begin{matrix} (-3,-3) & (-3,2) \\ (1,-3) & (1,2) \end{matrix}$$

To classify the stationary points, we use the second derivative test, which requires the second-order partial derivatives.

$$\begin{aligned}
f_{xx} &= \frac{\partial^2 f}{\partial x^2} = 2x + 2 \\
f_{yy} &= \frac{\partial^2 f}{\partial y^2} = 2y + 1 \\
f_{xy} &= \frac{\partial^2 f}{\partial x \partial y} = 0
\end{aligned}$$

The determinant of the Hessian matrix is $D = f_{xx} f_{yy} - (f_{xy})^2 = (2x+2)(2y+1)$.

We evaluate the Hessian determinant $D = f_{xx} f_{yy} - (f_{xy})^2$ at each stationary point:

- If $D > 0$ and $f_{xx} > 0$: **Local Minimum**
- If $D > 0$ and $f_{xx} < 0$: **Local Maximum**
- If $D < 0$: **Saddle Point**
- If $D = 0$: **Test Inconclusive**

Stationary Point	f_{xx}	f_{yy}	D	Classification
$(-3, -3)$	-4	-5	$20 \quad (> 0)$	Local Maximum
$(-3, 2)$	-4	5	$-20 \quad (< 0)$	Saddle Point
$(1, -3)$	4	-5	$-20 \quad (< 0)$	Saddle Point
$(1, 2)$	4	5	$20 \quad (> 0)$	Local Minimum

14. (5 points) The demand for robots in *Tatooine* is given by $p = 19 - 2q$ and the supply of robots is given by $p = 3 + 2q$.

(a) (1 point) Compute the equilibrium price and quantity.

Solution: The equilibrium can be found out by setting demand = supply.

$$
\begin{aligned}
19 - 2q &= 3 + 2q \\
\implies 4q &= 16 \\
\implies q^* &= 4 \\
\implies p^* &= 11
\end{aligned}
$$

Answer: The equilibrium quantity is $\boxed{q^* = 4}$ and the equilibrium price is $\boxed{p^* = 11}$.

(b) (1+1 points) Compute the consumer surplus and producer surplus.

Solution: Given inverse demand ($D(q)$) and inverse supply ($S(q)$), we know that:

$$CS = \int_{q=0}^{q=q^*} D(q))dq - p^*q*$$

$$PS = p^*q* - \int_{q=0}^{q=q^*} (S(q))dq$$

We also know from the previous calculation that $p^* = 11$ and $q^* = 4$.

$$
\begin{aligned}
CS &= \int_0^4 (19 - 2q)dq - 44 \\
&= \Big|_0^4 (19q - q^2) - 44 \\
&= 76 - 16 - 44 \\
&= 16
\end{aligned}
$$

$$
\begin{aligned}
PS &= 44 - \int_0^4 (3 + 2q)dq \\
&= 44 - \Big|_0^4 (3q + q^2) \\
&= 44 - (12 + 16) \\
&= 16
\end{aligned}
$$

Answer: $\boxed{CS = 16}$, $\boxed{PS = 16}$

(c) (1+1 points) Now, suppose that the Damiyo (the ruler of Tatooine), sensing that the robots are valuable, announces a price floor of 13. Compute the new consumer surplus and producer surplus.

Solution: When $p = 13$, we should compute the quantity demanded and the quantity supplied.

$$\begin{aligned} 19 - 2q &= 13 \\ 2q &= 6 \\ \text{Demand:} \quad q &= 3 \\ 3 + 2q &= 13 \\ 2q &= 10 \\ \text{Supply:} \quad q &= 5 \end{aligned}$$

At this price, three robots will be sold in *Tatooine*.

$$\begin{aligned} CS &= \int_0^3 (19 - 2q) - 13 \times 3 \\ &= \Big|_0^3 (19q - q^2) - 39 \\ &= (57 - 9) - 39 \\ &= 9 \\ PS &= 39 - \int_0^3 (3 + 2q)dq \\ &= 39 - \Big|_0^3 (3q + q^2) \\ &= 39 - (9 + 9) \\ &= 21 \end{aligned}$$

Answer: $\boxed{CS = 9}$, $\boxed{PS = 21}$