Introduction to Matrix Algebra-II

Sumit

July 19, 2025

1 Matrix Operations

1.1 Scalar Multiplication

We need to define the term 'scalar' first. Any real number will be labelled as a scalar. Multiplying a scalar to a matrix requires us to multiply **each element** of the matrix by the scalar. Formally, if we have a matrix $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ and a scalar k, the product $k \times A$ is

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}$$

An example: let $k = 2$ and $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$$kA = 2 \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$kA = \begin{bmatrix} 2 \times 1 & 2 \times 2 \\ 2 \times 3 & 3 \times 4 \end{bmatrix}$$

$$kA = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

1.2 Matrix Addition

Two matrices A and B can only be added if they are of the same order (dimension). Let the dimension of the two matrices be $m \times n$. The sum of two matrices will be defined as:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$
$$B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}$$
$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{pmatrix}$$

Example 1: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ -2 & -3 \end{bmatrix}$

- Step 1: Check the dimensions of the two matrices. It turns out that $dim(A) = 2 \times 2$ and $dim(B) = 2 \times 2$. They are of the same dimension.
- Step 2: Add each corresponding element from the two matrices. Call each element of the sum of matrices c_{ij}.
  - $c_{11} = a_{11} + b_{11} = 1 + (-1) = 0$
  - $c_{12} = a_{12} + b_{12} = 2 + 0 = 0$
  - $c_{21} = a_{21} + b_{21} = 3 + (-2) = 1$
  - $c_{22} = a_{22} + b_{22} = 4 + (-3) = 1$
- Step 3: Write the matrix (call it C).

$$C = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

1.2.1 Properties of Matrix Addition

1. Associative: matrix addition is associative

$$A + B = B + A$$

2. Commutative: matrix addition is commutatitve

$$A + (B + C) = (A + B) + C$$

3. Additive zero: adding a zero matrix (of the same order) should keep the matrix unchanged.

$$A + O = A$$

4. Additive inverse: adding the additive inverse of a matrix to the matrix should yield a zero matrix.

$$A + (-A) = O$$

1.3 Matrix Equality

Two matrices A and B are said to be equal if and only if each element of the two matrices are identical.

An example:

$$A = \begin{pmatrix} x & 1 \\ 2 & 2y \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix}$$

If $A = B$, then

$$\begin{aligned} x &= -1 \quad \text{since } a_{11} = b_{11} \\ 2y &= 2 \quad \text{since } a_{22} = b_{22} \end{aligned}$$

Therefore, $x = -1$ and $y = 1$.

1.4 Matrix Multiplication

Given two matrices A and B whose dimensions are $m_1 \times n_1$ and $m_2 \times n_2$ respectively, $A \times B$ is possible if and only if

$$n_1 = m_2$$

In plain language, for matrix multiplication to happen, you must have the number of columns in matrix A to match the number of rows in matrix B.

An example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$
$$B = \begin{pmatrix} -1 & -2 & -3 \end{pmatrix}$$

We must first note the dimensions of the two matrices.

$$dim(A) = 2 \times 3$$
$$dim(B) = 1 \times 3$$

$n_1 = 3$, but $m_2 = 1$. Therefore, $A \times B$ is not defined in this case.

If matrix multiplication is indeed possible, the resultant matrix $A \times B$ will be of order(dimension) $m_1 \times n_2$. What will be process of this multiplication? You will need to take an entire **row** from A and multiply each element of that row with all the **columns** from B.

An example:

Consider two matrices

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$
$$B = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$$

$dim(A) = 2 \times 2$ and $dim(B) = 2 \times 2$. Since, $n_1 = m_2 = 2$, matrix multiplication is possible here. We also know that $A \times B$ will be of dimension 2×2. Let us call this matrix C.

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

Each of the elements of C will come from the multiplication of rows (of A) and columns (of B).

Step 1- Let's now dismantle matrix A and create two different row matrices.

$$A_{R1} = \begin{bmatrix} 1 & 2 \end{bmatrix} \qquad A_{R2} = \begin{bmatrix} 3 & 4 \end{bmatrix}$$

Step 2- Let's create two different column matrices from matrix B.

$$B_{C1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad B_{C2} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

Step 3- Multiply A_{R1} and B_{C1}, A_{R1} and B_{C2}, A_{R2} and B_{C1}, and A_{R2} and B_{C2}.

$$\begin{aligned} c_{11} &= A_{R1} \times B_{C1} = 1 \times -1 + 2 \times 0 = -1 \\ c_{12} &= A_{R1} \times B_{C2} = 1 \times 0 + 2 \times -2 = -4 \\ c_{21} &= A_{R2} \times B_{C1} = 3 \times -1 + 4 \times 0 = -3 \\ c_{22} &= A_{R2} \times B_{C2} = 3 \times 0 + 4 \times -2 = -8 \end{aligned}$$

Step 4- Write the matrix C.

$$C = \begin{bmatrix} -1 & -4 \\ -3 & -8 \end{bmatrix}$$

1.4.1 Properties of Matrix Multiplication

1. Non-commutative: $A \times B \neq B \times A$.
2. Associative: $A(BC) = (AB)C$.
3. Distributive: $A(B + C) = AB + AC$.
4. Multiplicative identity: $IA = AI = A$.
5. Multiplicative zero: $OA = AO = O$.