Functions-I

Sumit

August 10, 2025

1 What are functions?

In order to understand sets, we will return to sets. Consider two sets A and B.

1.1 Ordered Pairs

Think of points on the xy plane. We typically represent them as a pair. Consider $x = 0$ and $y = 1$. We can write this point as $(0, 1)$. Any such construction is known as an ordered pair, where each element of a pair comes from two different sets. These are called ordered pairs because the order in which elements appear matters. Let $\langle x_1, y_1 \rangle$ be an ordered pair and $\langle x_2, y_2 \rangle$ be another ordered pair. These two pairs can only be identical if and only if $x_1 = x_2$ and $y_1 = y_2$.

1.2 Cartesian Product

The Cartesian product of these two sets, $A \times B$ is defined as the ordered pairs of elements from each of the sets. Let $A = \{1, 2\}$ and $B = \{a, b\}$. The Cartesian product of these two sets is

$$A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$$

1.3 Definition of function

A function, $f : A \to B$ between two sets A and B is a subset of the Cartesian product $A \times B$ such that for every $a \in A$, there exists a unique $b \in B$.

An example:

Let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$. Then,

$$f = \{(1, x), (2, y), (3, z)\}$$

$f(1) = x$, $f(2) = y$, and $f(3) = z$.

More generally, set A is known as the domain of the function and set B the codomain of the function.

Another example:

Let $f : A \to B$ where $A = \{1, 2\}$ and $B = \{x, y\}$ such that $f(1) = x$ and $f(2) = y$. In this case, the domain of the function f is $\{1, 2\}$ and the codomain happens to be $\{x, y\}$.

Before we proceed, we must also be able to understand functions more intuitively. To put it in simple terms, a function is a mapping from an input to an output.

$$\boxed{\underbrace{\text{INPUT}}_{x} \quad \underbrace{\rightarrow}_{\text{FUNCTION}} \quad \underbrace{\text{OUTPUT}}_{y}}$$

2 Domain and Range

2.1 Domain

The domain of a function is the set of all possible values that the input can take such that the function is defined.

An example: Consider $f(x) = \dfrac{1}{x-2}$. What is the domain of this function?

When $x = 2$, the function is not defined. All other values are possible. Therefore, we can say that the domain of the function $f(x)$ is: $D_f = x \in (-\infty, 2) \cup (2, \infty)$. You can also write the domain as: $D_f = x \in \mathbb{R} \setminus \{2\}$.

2.2 Range

The range of a function is defined as the set of all possible values of output that the function can generate.

An example: Consider $f(x) = x^2 + 1$. What is the range of this function?

Please note that we have a square on the RHS whose minimum value, as long we stick to real numbers, will be zero. Therefore, the minimum possible value of the output can be 1. Hence, the range of the function $f(x)$ is $R_f = f(x) \in [1, \infty)$.

You can verify this in a more structured manner. Consider:

$$\begin{aligned} y &= x^2 + 1 \\ \implies x^2 &= y - 1 \\ \implies x &= \sqrt{y-1} \end{aligned}$$

Looking at the RHS, we know that $y \geq 1$ for x to be defined. Therefore, the range that we calculated using a verbal argument is correct.