Endterm (Set B)

SIAS, Krea University (AY 2025-26)
Mathematical Methods for Economics (Course Code: **ECON211**)
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Short Answer Questions-I

1. (1 point) Let $f(x) = x^{x-2}$. Find $f'(x)$.

Solution: Let $y = x^{x-2}$.

$$
\begin{aligned}
\ln y &= (x-2)\ln x && \text{(using the property of log } a = b^c \implies \ln a = c \ln b\text{)}\\
\implies \frac{1}{y}\frac{dy}{dx} &= \frac{d}{dx}((x-2)\ln x) && \text{(differentiating both sides w.r.t. } x\text{)}\\
\implies \frac{1}{y}\frac{dy}{dx} &= \ln x \frac{d}{dx}(x-2) + (x-2)\frac{d}{dx}(\ln x) && \text{(applying the product rule to the RHS)}\\
\implies \frac{1}{y}\frac{dy}{dx} &= \ln x + \frac{x-2}{x}\\
\implies \frac{dy}{dx} &= y\left(\ln x + \frac{x-2}{x}\right)\\
\implies \frac{dy}{dx} &= x^{x-2}\left(\ln x + \frac{x-2}{x}\right)
\end{aligned}
$$

Answer: $\boxed{\frac{dy}{dx} = x^{x-2}\left(\ln x + \frac{x-2}{x}\right)}$

2. (1 point) Determine if the function $f(x) = x^2 - 8x + 15$ is increasing or decreasing in $[3, 5]$.

Solution: First derivative tells us whether (or where) a function is increasing or decreasing.

$$
\begin{aligned}
f'(x) &= 2x - 8\\
\text{Condition for increasing function} : 2x - 8 &\geq 0\\
\therefore \text{ the function is increasing when } x &\geq 4\\
\text{Condition for decreasing function} : 2x - 8 &\leq 0\\
\therefore \text{ the function is decreasing when } x &\leq 4
\end{aligned}
$$

Answer: $f(x)$ is decreasing in $[3, 4]$ and increasing in $[4, 5]$.

3. (1 point) Let $x^2y^3 + x^3y^2 = 7$. Find $\frac{dy}{dx}$.

Solution: Let $u = x^2y^3$, $v = x^3y^2$, and $c = 7$.

Applying the product rule, we get:

$$\begin{aligned} u' &= 2xy^3 + 3x^2y^2\frac{dy}{dx} \\ v' &= 2x^3y\frac{dy}{dx} + 3x^2y^2 \\ c' &= 0 \\ u' + v' &= c' \\ \therefore \left[2xy^3 + 3x^2y^2\frac{dy}{dx}\right] + \left[2x^3y\frac{dy}{dx} + 3x^2y^2\right] &= 0 \\ \implies (2xy^3 + 3x^2y^2) + (3x^2y^2 + 2x^3y)\frac{dy}{dx} &= 0 \\ \implies (3x^2y^2 + 2x^3y)\frac{dy}{dx} &= -(2xy^3 + 3x^2y^2) \\ \implies \frac{dy}{dx} &= -\frac{(2xy^3 + 3x^2y^2)}{(3x^2y^2 + 2x^3y)} \\ \implies \frac{dy}{dx} &= -\frac{y}{x}\frac{(3x + 2y)}{(2x + 3y)} \end{aligned}$$

Answer: $\boxed{\frac{dy}{dx} = -\frac{y}{x}\frac{(3x + 2y)}{(2x + 3y)}}$

4. (1 point) Let $f(x) = \sqrt{x} + 5$ and $g(x) = f^{-1}(x)$. Find $g'(7)$.

Solution: We know that: $\boxed{g'(a) = \frac{1}{f'(g(a))}}$

$\therefore g'(7) = \frac{1}{f'(g(7))}$

Let $g(7) = k \implies f(k) = 7$.

$$\begin{aligned} \sqrt{k} + 5 &= 7 \\ \implies \sqrt{k} &= 7 - 5 \\ \implies \sqrt{k} &= 2 \\ \implies k &= 4 \\ f'(x) &= \frac{1}{2\sqrt{x}} \\ \implies f'(4) &= \frac{1}{2\sqrt{2}} \\ \implies f'(4) &= \frac{1}{4} \\ \implies g'(7) &= 4 \end{aligned}$$

Answer: $\boxed{g'(7) = 4}$

5. (1 point) Suppose that f and g are continuous on $[0, 4]$ and that $\int_0^4 (f(x) - g(x))dx = 4$ and $\int_0^4 (3f(x) - 4g(x))dx = 11$. Find $\int_0^4 (f(x) + g(x))dx$.

Solution: Let $\int_0^4 f(x)dx = a$ and $\int_0^4 g(x)dx = b$. Then,

$$\begin{aligned}
a - b &= 4 \\
3a - 4b &= 11 \\
3a - 3b &= 12 && \text{(multiplying the first equation by 3)} \\
\implies b &= 1 && \text{(differencing the previous two equations to eliminate } a\text{)} \\
\implies a &= 5 \\
\int_0^4 (f(x) + g(x))dx &= \int_0^4 f(x)dx + \int_0^4 g(x)dx && \text{(applying the sum rule)} \\
\implies \int_0^4 (f(x) + g(x))dx &= 6
\end{aligned}$$

Answer : $\boxed{\int_0^4 (f(x) + g(x))dx = 6}$

6. (1 point) Compute: $\int (6x^2 + \frac{3}{x} + e^{4x})dx$

Solution:

$$\begin{aligned}
\int (6x^2 + \frac{3}{x} + e^{4x})dx &= \int 6x^2dx + \int \frac{3}{x}dx + \int e^{4x}dx && \text{(applying the sum rule)} \\
\implies \int (6x^2 + \frac{3}{x} + e^{4x})dx &= 2x^3 + 3\ln|x| + \frac{e^{4x}}{4} + C
\end{aligned}$$

Answer : $\boxed{\int (6x^2 + \frac{3}{x} + e^{4x})dx = 2x^3 + 3\ln|x| + \frac{e^{4x}}{4} + C}$

7. (1 point) Is $\lim_{x\to 0} |x - 3| = \lim_{x\to 0} |x| - 3$? Explain briefly.

Solution: Consider $f(x) = |x - 3|$.

$$f(x) = \begin{cases} 3 - x & x < 3 \\ x - 3 & x > 3 \end{cases}$$

$$\begin{aligned}
\text{LHL: } \lim_{x\to 0^-} |x - 3| &= 3 && \text{(since } |x-3| = 3 - x \text{ when } x < 3\text{)} \\
\text{RHL: } \lim_{x\to 0^+} |x - 3| &= 3 && \text{(since } |x-3| = 3 - x \text{ when } x < 3\text{)} \\
\therefore \lim_{x\to 0} |x - 3| &= 3
\end{aligned}$$

Let $g(x) = |x| - 3$.

$$g(x) = \begin{cases} -x - 3 & x < 0 \\ x - 3 & x > 0 \end{cases}$$

$$\begin{aligned}
\text{LHL: } \lim_{x\to 0^-} |x| - 3 &= -3 && \text{(since } |x| = -x \text{ when } x < 0\text{)} \\
\text{RHL: } \lim_{x\to 0^+} |x| - 3 &= -3 && \text{(since } |x| = x \text{ when } x > 0\text{)} \\
\therefore \lim_{x\to 0} |x| - 3 &= -3
\end{aligned}$$

Answer: $\boxed{\lim_{x\to 0} |x-3| \neq \lim_{x\to 0} |x| - 3}$

8. (1 point) Let $f(x) = \dfrac{9}{3+x}$. Find $f^{-1}(x)$.

Solution: Let $y = f(x)$.

$$\begin{aligned} y &= \frac{9}{x+3} \\ \implies y(x+3) &= 9 \\ \implies xy + 3y &= 9 \\ \implies xy &= 9 - 3y \\ \implies x &= \frac{9-3y}{y} \\ \implies f^{-1}(x) &= \frac{9-3x}{x} \end{aligned}$$

Answer: $\boxed{f^{-1}(x) = \dfrac{9-3x}{x}}$

Short Answer Questions-II

9. (3 points) Let $f(x, y) = 6x^{2/3}y^{1/3}$.

(a) (1 point) Determine the degree of homogeneity.

Solution: The degree of homogeneity can be calculated using:

$$f(tx, ty) = t^k f(x, y)$$

$$\begin{aligned} f(tx, ty) &= 6(tx)^{2/3}(ty)^{1/3} \\ &= 6t^{2/3}x^{2/3}t^{1/3}y^{1/3} \\ &= (t^{1/3+2/3})6(x^{2/3}y^{1/3}) \\ &= t^1 f(x, y) \\ \implies k &= 1 \end{aligned}$$

Answer: $f(x, y)$ is homogeneous of degree 1.

(b) (2 points) Compute all first and second order partial derivatives.

Solution:

f_x	$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(6x^{2/3}y^{1/3}\right) \\ &= 6 \cdot \frac{2}{3}x^{-1/3}y^{1/3} \\ &= 4x^{-1/3}y^{1/3} \\ &= 4\left(\frac{y}{x}\right)^{1/3} \end{aligned}$
f_y	$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(6x^{2/3}y^{1/3}\right) \\ &= 6x^{2/3} \cdot \frac{1}{3}y^{-2/3} \\ &= 2x^{2/3}y^{-2/3} \\ &= 2\left(\frac{x}{y}\right)^{2/3} \end{aligned}$
f_{xx}	$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left[4\left(\frac{y}{x}\right)^{1/3}\right] \\ &= 4y^{1/3} \cdot \left(-\frac{1}{3}\right)x^{-4/3} \\ &= -\frac{4}{3}x^{-4/3}y^{1/3} \\ &= -\frac{4}{3}\left(\frac{y}{x^4}\right)^{1/3} \end{aligned}$
f_{yy}	$\begin{aligned} \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left[2\left(\frac{x}{y}\right)^{2/3}\right] \\ &= 2x^{2/3} \cdot \left(-\frac{2}{3}\right)y^{-5/3} \\ &= -\frac{4}{3}x^{2/3}y^{-5/3} \\ &= -\frac{4}{3}\left(\frac{x^2}{y^5}\right)^{1/3} \end{aligned}$
f_{xy}	$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial y}\left[4\left(\frac{y}{x}\right)^{1/3}\right] \\ &= 4x^{-1/3} \cdot \frac{1}{3}y^{-2/3} \\ &= \frac{4}{3}x^{-1/3}y^{-2/3} \\ &= \frac{4}{3}\left(\frac{1}{xy^2}\right)^{1/3} \end{aligned}$
f_{yx}	$\begin{aligned} \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial x}\left[2\left(\frac{x}{y}\right)^{2/3}\right] \\ &= 2y^{-2/3} \cdot \frac{2}{3}x^{-1/3} \\ &= \frac{4}{3}x^{-1/3}y^{-2/3} \\ &= \frac{4}{3}\left(\frac{1}{xy^2}\right)^{1/3} \end{aligned}$

10. (1+1+1 points) Let $U = x + 2\sqrt{y}$. Compute the marginal utilities and the marginal rate of substitution.

Solution:

$$MU_x = 1$$
$$MU_y = \frac{1}{\sqrt{y}}$$
$$MRS_{x,y} = \frac{MU_x}{MU_y}$$
$$MRS_{x,y} = \sqrt{y}$$

11. (3 points) Given the demand function for comedy shows on *Ruinmyshow*: $p = \dfrac{25}{q+4} - 4$,

 (a) ($\frac{1}{2}$ points) Compute the total revenue.

Solution: Total revenue, $TR = p \cdot q$.

$$p = \frac{25}{q+4} - 4$$
$$\implies TR = \frac{25q}{q+4} - 4q$$

 (b) ($\frac{1}{2}$ points) Compute the marginal revenue.

Solution: We know that $MR = \dfrac{d}{dq}(TR)$.

$$TR = \frac{25q}{q+4} - 4q$$
$$\text{Let } u = 25q, \quad v = q+4$$
$$\implies u' = 25, \quad v' = 1$$
$$MR = \frac{vu' - uv'}{v^2} - 4$$
$$\implies MR = \frac{25(q+4) - 25q}{(q+4)^2} - 4$$
$$\implies MR = \frac{100}{(q+4)^2} - 4$$
$$\implies MR = \frac{100 - 4(q+4)^2}{(q+4)^2}$$

Answer: $\boxed{MR = \dfrac{100 - 4(q+4)^2}{(q+4)^2}}$

 (c) (2 points) Compute the revenue-maximizing price and quantity.

Solution: We know that the revenue is maximized when $MR = 0$.

$$\begin{aligned} 100 - 4(q+4)^2 &= 0 \quad \text{(note that the denominator can't be zero.)} \\ \implies 4(q+4)^2 &= 100 \\ \implies (q+4)^2 &= 25 \\ \implies (q+4) &= \pm 5 \\ \implies q+4 &= 5 \qquad \text{(discarding the negative value.)} \\ \implies q &= 1 \end{aligned}$$

Plugging the value into the demand equation, we get $p = \dfrac{25}{1+4} - 4$

$$\begin{aligned} \implies p &= \frac{25}{5} - 4 \\ \implies p &= 5 - 4 \\ \implies p &= 1 \end{aligned}$$

Answer: The revenue-maximizing price is $\boxed{p = 1}$ and the quantity is $\boxed{q = 1}$.

12. (2+1 points) The total cost of producing *Phantom cigarettes* is $C(q) = 2q^2 + 10q + 50$. Find the value of q which minimizes the average cost. Show that the marginal cost is equal to the average cost at this point (where the average cost is being minimized).

Solution: We know that the average cost is:

$$AC(q) = \frac{C(q)}{q}$$

The average cost of producing *Phantom cigarettes* is:

$$AC(q) = 2q + 10 + \frac{50}{q}$$

We need to find the first derivative and set it to zero.

$$\begin{aligned} \frac{d(AC(q))}{dq} &= \frac{d}{dq}\left(2q + 10 + \frac{50}{q}\right) \\ &= 2 - \frac{50}{q^2} \\ \text{FOC:} \quad \frac{d(AC(q))}{dq} &= 0 \\ \implies 2 - \frac{50}{q^2} &= 0 \\ \implies 2q^2 &= 50 \\ \implies q^2 &= 25 \\ \implies q^* &= 5 \\ \text{SOC:} \quad \frac{d^2(AC(q))}{dq^2} &> 0 \qquad \text{(for minimum)} \\ \frac{d^2(AC(q))}{dq^2} &= \frac{100}{q^3} \\ \frac{100}{q^3} &> 0 \text{ when } q = 5 \end{aligned}$$

We also need to compute the marginal cost.

$$MC(q) = 4q + 10$$

When $q = 5$,

$$\begin{aligned} \text{Average cost:} \quad AC(q=5) &= 2q + 10 + \frac{50}{5} \\ &= 10 + 10 + 10 \\ &= 30 \\ \text{Marginal cost:} \quad MC(q=5) &= 4(5) + 10 \\ &= 30 \end{aligned}$$

Answer: The quantity that minimizes the average cost is $\boxed{q^* = 5}$. When $q = 5$, $\boxed{AC = MC = 30}$.

Long Answer Questions

13. (5 points) The demand for robots in *Tatooine* is given by $p = 18 - 2q$ and the supply of robots is given by $p = 2 + 2q$.

(a) (1 point) Compute the equilibrium price and quantity.

Solution: The equilibrium can be found out by setting demand = supply.

$$\begin{aligned} 18 - 2q &= 2 + 2q \\ \implies 4q &= 16 \\ \implies q^* &= 4 \\ \implies p^* &= 10 \end{aligned}$$

Answer: The equilibrium quantity is $\boxed{q^* = 4}$ and the equilibrium price is $\boxed{p^* = 10}$.

(b) (1+1 points) Compute the consumer surplus and producer surplus.

Solution: Given inverse demand ($D(q)$) and inverse supply ($S(q)$), we know that:

$$CS = \int_{q=0}^{q=q^*} D(q))dq - p^*q*$$

$$PS = p^*q* - \int_{q=0}^{q=q^*} (S(q))dq$$

We also know from the previous calculation that $p^* = 10$ and $q^* = 4$.

$$\begin{aligned} CS &= \int_0^4 (18 - 2q)dq - 40 \\ &= \Big|_0^4 (18q - q^2) - 40 \\ &= 72 - 16 - 40 \\ &= 16 \end{aligned}$$

$$\begin{aligned} PS &= 40 - \int_0^4 (2 + 2q)dq \\ &= 40 - \Big|_0^4 (2q + q^2) \\ &= 40 - (8 + 16) \\ &= 16 \end{aligned}$$

Answer: $\boxed{CS = 16}$, $\boxed{PS = 16}$

(c) (1+1 points) Now, suppose that the Damiyo (the ruler of Tatooine), sensing that the robots are valuable, announces a price floor of 12. Compute the new consumer surplus and producer surplus.

Solution: When $p = 14$, we should compute the quantity demanded and the quantity supplied.

$$\begin{aligned} 18 - 2q &= 12 \\ 2q &= 6 \\ \text{Demand:} \quad q &= 3 \\ 2 + 2q &= 12 \\ 2q &= 10 \\ \text{Supply:} \quad q &= 5 \end{aligned}$$

At this price, three robots will be sold in *Tatooine*.

$$\begin{aligned} CS &= \int_0^3 (18 - 2q) - 12 \times 3 \\ &= \Big|_0^3 (18q - q^2) - 36 \\ &= (54 - 9) - 36 \\ &= 9 \\ PS &= 36 - \int_0^3 (2 + 2q)dq \\ &= 36 - \Big|_0^3 (2q + q^2) \\ &= 36 - (6 + 9) \\ &= 21 \end{aligned}$$

Answer: $\boxed{CS = 9}$, $\boxed{PS = 21}$

14. (5 points) Consider $f(x, y) = \frac{x^3}{3} + \frac{y^3}{3} + x^2 + \frac{y^2}{2} - 3x - 6y + 3$. Find and classify all stationary points.

Solution: To find the stationary points, we first compute the first-order partial derivatives and set them equal to zero.

$$\begin{aligned} f_x &= \frac{\partial f}{\partial x} = (x^2 + 2x - 3) = 0 \\ f_y &= \frac{\partial f}{\partial y} = (y^2 + y - 6) = 0 \end{aligned}$$

We have two quadratic equations to be solved.

$$\begin{aligned} (x + 3)(x - 1) &= 0 \\ (y + 3)(y - 2) &= 0 \\ x &= 1, -3 \\ y &= 2, -3 \end{aligned}$$

The stationary points are the combinations of these x and y values:

$$\begin{matrix} (-3, -3) & (-3, 2) \\ (1, -3) & (1, 2) \end{matrix}$$

To classify the stationary points, we use the second derivative test, which requires the second-order partial derivatives.

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 2x + 2$$
$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = 2y + 1$$
$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 0$$

The determinant of the Hessian matrix is $D = f_{xx}f_{yy} - (f_{xy})^2 = (2x+2)(2y+1)$.
We evaluate the Hessian determinant $D = f_{xx}f_{yy} - (f_{xy})^2$ at each stationary point:

- If $D > 0$ and $f_{xx} > 0$: **Local Minimum**
- If $D > 0$ and $f_{xx} < 0$: **Local Maximum**
- If $D < 0$: **Saddle Point**
- If $D = 0$: **Test Inconclusive**

Stationary Point	f_{xx}	f_{yy}	D	**Classification**
$(-3,-3)$	-4	-5	$20 \quad (> 0)$	Local Maximum
$(-3,2)$	-4	5	$-20 \quad (< 0)$	Saddle Point
$(1,-3)$	4	-5	$-20 \quad (< 0)$	Saddle Point
$(1,2)$	4	5	$20 \quad (> 0)$	Local Minimum