Limits

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1 Limits

Let there be a function $f(x)$. Then, $f(x)$ has a number L as the limit as x approaches a (by which we mean that x is very close to a, but not equal to a). We write:

$$\lim_{x \to a} f(x) = L$$

An example:

Consider the function $f(x) = \dfrac{x^2 - 4}{x - 2}$. What is the limit of $f(x)$ when x approaches 2?

Please note that $x = 2$ is not in the domain of the function. Let's use some values around $x = 2$. Table 1 shows these values. Figure 1 plots the function. We can see that the limit approaches 4 as we move closer to $x = 2$, but of course, the value of the function itself is not defined at $x = 2$.

Table 1: Values of $f(x)$ as $x \to 2$

x	1.9	1.99	1.999	1.9999	2.0001	2.001	2.01	2.1
$f(x)$	3.9	3.99	3.999	3.9999	4.0001	4.001	4.01	4.1

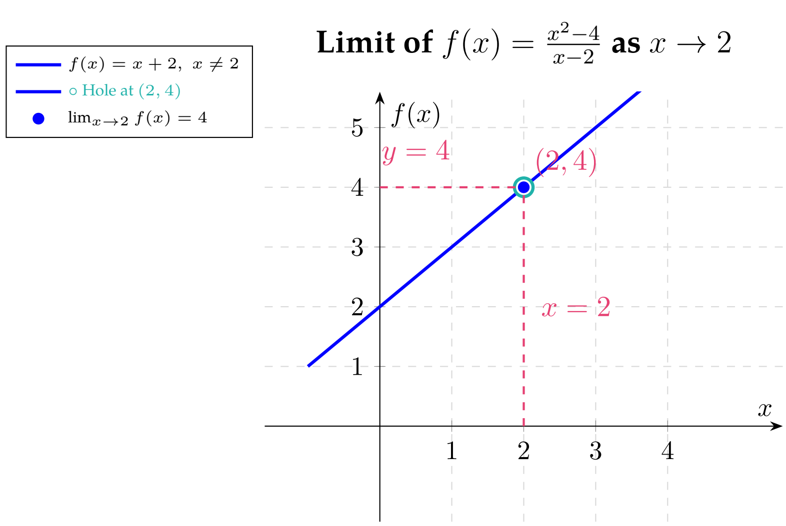

Figure 1

1.1 One-sided Limits

1.1.1 Left-hand limits (LHL)

We write:

$$\lim_{x \to a^-} f(x) = L$$

when we choose a value of x very close to a, but less than a.

1.1.2 Right-hand limits (RHL)

We write:

$$\lim_{x \to a^+} f(x) = L$$

when we select a value of x very close to a, but more than a.

In general, we say that the actual limit of the function to be defined if and only if LHL = RHL.

An example:

Consider the following piece-wise function.

$$f(x) = \begin{cases} x - 1 & \text{if } x \leq 1 \\ x + 1 & \text{if } x > 1 \end{cases}$$

What is $\lim_{x \to 1} f(x)$?

We will first calculate the LHL. In order to do so, we need to be clear about the form of the function. When $x \leq 1$, $f(x) = x - 1$.

Therefore, LHL = $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x - 1 = 0$.

Similarly, when $x > 1$, $f(x) = x + 1$.

Therefore, RHL = $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x + 1 = 2$.

We can see that LHL $\neq$ RHL. Therefore, the limit is not defined.

1.2 Properties of Limits

Let $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ and let c be some constant. Then

1. $\lim_{x \to a} cf(x) = cL$
2. $\lim_{x \to a} (f(x) + g(x)) = L + M$
3. $\lim_{x \to a} (f(x) - g(x)) = L - M$
4. $\lim_{x \to a} (f(x)\dot{g}(x)) = LM$
5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{L}{M}$ if $M \neq 0$
6. $\lim_{x \to a} f(x)^p = A^p$ (if A^p is defined and p is a real number)
7. $\lim_{x \to a} (x)^{1/n} = a^{1/n}$

1.3 Some Simple Tricks

- If you are given a rational expression, try to factorize the numerator or the denominator to eliminate the common factor.

Example 1: $g(x) = x^2 - 6x + 8$ and $h(x) = x - 2$. What is $\lim_{x\to 2} f(x)$ where $f(x) = \dfrac{g(x)}{h(x)}$?

$$\begin{aligned}
&\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2 - 6x + 8}{x - 2} \\
\implies &\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{(x-2)(x-4)}{x-2} && \text{(factorizing the numerator)} \\
\implies &\lim_{x\to 2} f(x) = \lim_{x\to 2} x + 2 \\
\implies &\lim_{x\to 2} f(x) = 4
\end{aligned}$$

- If you have an irrational expression, try to rationalize it to compute the limit.

Example 2: $f(x) = \dfrac{\sqrt{x+1} - 1}{x}$. What is $\lim_{x\to 0} f(x)$?

$$\begin{aligned}
&\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sqrt{x+1} - 1}{x} \\
&\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{(\sqrt{x+1} - 1) \times (\sqrt{x+1} + 1)}{x \times (\sqrt{x+1} + 1)} \\
&\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{(x+1) - 1}{x \times (\sqrt{x+1} + 1)} && \text{(since } (a+b)(a-b) = a^2 - b^2\text{)} \\
&\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x}{x \times (\sqrt{x+1} + 1)} \\
&\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{\sqrt{x+1} + 1} \\
&\lim_{x\to 0} f(x) = \frac{1}{2}
\end{aligned}$$

- If you want to compute the limit at infinity for a rational expression, start with dividing the whole thing by the highest power of x that occurs in the denominator.

Example 3: $f(x) = \dfrac{2x^2 + 3}{5x^2 + x}$. What is $\lim_{x\to\infty} f(x)$?

$$\begin{aligned}
&\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^2 + 3}{5x^2 + x} \\
\implies &\lim_{x\to\infty} = \lim_{x\to\infty} \frac{2 + 3/x^2}{5 + 1/x^2} \\
\implies &\lim_{x\to\infty} = \frac{\lim_{x\to\infty} 2 + \lim_{x\to\infty} 3/x^2}{\lim_{x\to\infty} 5 + \lim_{x\to\infty} 1/x^2} \\
\implies &\lim_{x\to\infty} = \frac{2 + 3(0)}{5 + 1(0} \\
\implies &\lim_{x\to\infty} = \frac{2}{5}
\end{aligned}$$

2 Continuity

A function is continuous at $x = a$ if

$$LHL = RHL = \lim_{x \to a} f(x) = f(a)$$

A simple checklist-

- Check if $f(a)$ is **defined**.
- Calculate the LHL and the RHL. Verify they match.
- Verify if $\lim_{x \to a} = f(a)$.

We will take a couple of examples to check whether a function is continuous (or not).
Example 1: Is $f(x) = x^2$ continuous at $x = 0$?

$$\text{LHL: } \lim_{x \to 0^-} x^2 = 0$$

$$\text{RHL: } \lim_{x \to 0^+} x^2 = 0$$

LHL = RHL

$$f(0) = (0)^2 = 0$$

LHL = RHL = $f(0)$.
Therefore, the function is continuous at $x = 0$.

Example 2: Is $f(x) = |x|$ continuous at $x = 0$?
Note that $f(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x >= 0 \end{cases}$.

$$\text{LHL: } \lim_{x \to 0^-} -x = -(0) = 0$$

$$\text{RHL: } \lim_{x \to 0^+} x = 0$$

LHL = RHL

$$f(0) = 0$$

Therefore, the function is continuous at $x = 0$.

3 Derivative

3.1 Slope of a line

We know that if we have a line passing through (x_1, y_1) and (x_2, y_2), the slope of the line $y = mx + c$ is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

We will extend this idea to a real-valued function.

3.2 The Rate of Change of a Function

Let there be a function $f(x)$. Let x_1 and x_2 be two values in the domain of the function. The corresponding values of the function at these points are $f(x_1)$ and $f(x_2)$ respectively. Then, the slope (or the rate of change) of the function is

$$\text{Slope} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

An example: Consider $f(x) = x^2$. What is the average rate of change of the function when x moves from 2 to 4?

Using the formula, we get:

$$\text{Slope} = \frac{f(4) - f(2)}{4 - 2)}$$
$$\text{Slope} = \frac{16 - 4}{2}$$
$$\text{Slope} = 6$$

We can say that the average rate of change of x^2 between $x = 2$ and $x = 4$ is 6.

3.3 Derivative

Definition: The derivative of the function $f(x)$ at any point a is given by:

$$f'(0) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

At first sight, this formula may seem intimidating. Let's unpack it using the idea of slope. When $x = a$, $f(x) = f(a)$. When $x = a + h$, $f(x) = f(a + h)$.

What will be the average change of the function from $x = a$ to $x = a + h$? It will be:

$$\frac{f(a+h) - f(a)}{(a+h) - a}$$
$$= \frac{f(a+h) - f(a)}{h}$$

Alright, then why do we insert limits here? The answer lies in the fact that h tends to be very small and we would like to compute the slope of any function at a given point (not just between two points).

An example: Let $f(x) = x^2$. What is $f'(2)$?

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$
$$\implies f'(2) = \lim_{h \to 0} \frac{(2+h)^2 - 4}{h} \qquad \text{(since } f(x) = x^2\text{)}$$
$$\implies f'(2) = \lim_{h \to 0} \frac{4 + h^2 + 2ah - 4}{h} \qquad \text{(since } (a+b)^2 = a^2 + b^2 + 2ab\text{)}$$
$$\implies f'(2) = \lim_{h \to 0} \frac{h^2 + 2ah}{h}$$
$$\implies f'(2) = \lim_{h \to 0} (h + 2a)$$
$$\implies f'(2) = 2a$$