Derivatives-I

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1 Derivatives

We know, by now, how to compute the derivative at a given point in the domain of a function. Let's generalize that idea. The derivative (denoted by $f'(x)$ and also by $\frac{dy}{dx}$) of a function $f(x)$ is given by:

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

When the above limit exists, we say that the function is **differentiable**. Sometimes, a function may not have a derivative at a given point in its domain.

An example: Let $f(x) = |x|$. What is $f'(x)$?

When $x > 0$, $f(x) = x$. Therefore,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$\implies f'(x) = \lim_{h \to 0} \frac{(x+h) - x}{h}$$
$$\implies f'(x) = \lim_{h \to 0} \frac{h}{h}$$
$$\implies f'(x) = 1$$

When $x < 0$, $f(x) = -x$. Therefore,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$\implies f'(x) = \lim_{h \to 0} \frac{-(x+h) - (-x)}{h}$$
$$\implies f'(x) = \lim_{h \to 0} \frac{-h}{h}$$
$$\implies f'(x) = -1$$

What happens when $x = 0$? Since the derivative right below zero is -1 and right above zero is +1, we can conclude that the function is not differentiable at $x = 0$.

2 Derivative Rules

2.1 Derivative of a Constant

Let c be a constant. Then,

$$f'(c) = 0$$

Proof: We know that:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$\implies f'(x) = \lim_{h\to 0} \frac{c-c}{h} \quad \text{since } f(x) = f(x+h) = c$$
$$\implies f'(x) = 0$$

2.2 Power Rule

Let $f(x) = x^n$. Then,

$$f'(x) = nx^{n-1}$$

An example: What is the derivative of $f(x) = x^6$?
Using the power rule and $n = 6$, we get: $f'(x) = 6x^5$.

2.3 Constant Multiple Rule

For any real-valued function $f(x)$ and a real number c:

$$f'(cx) = cf'(x)$$

An example: Let $f(x) = 10x^2$. We know that $\frac{d}{dx}(x^2) = 2x$ (by power rule). We can now apply the constant multiple rule to get: $f'(x) = 10(2x) = 20x$.

2.4 Sum (and difference) Rule

Let there be two real-valued and differentiable functions $f(x)$ and $g(x)$. Then,

$$\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$$

An example: Let $f(x) = x^2 + 1$ and $g(x) = x^3 - 5$. Using power and constant rules, we get $f'(x) = 2x$ and $g'(x) = 3x^2$.

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$
$$\implies \frac{d}{dx}(f(x) + g(x)) = 2x + 3x^2$$

2.5 Product Rule

If there are two real-valued and differentiable functions $f(x)$ and $g(x)$. Then,

$$\boxed{\frac{d}{dx}(f(x)g(x)) = f'(x) \cdot g(x) + g'(x) \cdot f(x)}$$

An example: Let $f(x) = x^2 + 1$ and $g(x) = x^3 - 5$. We already know that $f'(x) = 2x$ and $g'(x) = 3x^2$. We will use these values while applying the rule.

$$\begin{aligned}
&\frac{d}{dx}(f(x)g(x)) = 2x \times (x^3 - 5) + 3x^2 \times (x^2 + 1) \\
\implies &\frac{d}{dx}(f(x)g(x)) = 2x^4 - 10x + 3x^4 + 3x^2 \\
\implies &\frac{d}{dx}(f(x)g(x)) = 5x^4 + 3x^2 - 10x
\end{aligned}$$

2.6 Quotient Rule

If there are two real-valued and differentiable functions $f(x)$ and $g(x)$. Then,

$$\boxed{\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}}$$

An example: Let $f(x) = x - 1$ and $g(x) = x + 1$. $f'(x) = 1$ and $g'(x) = 1$.

$$\begin{aligned}
&\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{(x+1)\cdot(1) - (x-1)\cdot(1)}{(x+1)^2} \\
\implies &\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{2}{(x+1)^2}
\end{aligned}$$

2.7 Chain Rule

Let $f(x)$ and $g(x)$ be two real-valued functions such that f is differentiable at $g(x)$. Then, the composite function, say $h(x)$, is differentiable at x and is given by:

$$\boxed{h'(x) = f'(g(x)) \cdot g'(x)}$$

Sounds too complicated. An example will help us understand this rule. Let $f(x) = x + 1$ and $g(x) = x^2$. The composite function $h(x) = f(g(x)) = f(x^2) = x^2 + 1$. Let's verify the chain rule.

$$\begin{aligned}
h'(x) &= 2x \\
f'(g(x)) &= 1 && \text{(since } f'(x) = 1\text{)} \\
g'(x) &= 2x \\
\implies f'(g(x)) \times g'(x) &= 2x
\end{aligned}$$

The much easier way to learn the chain rule is to use the Leibniz's notation. Let $y = f(u)$ and $u = f(x)$. Then,

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}}$$

An example: $y = u^2$, $u = x - 1$.

$$\begin{aligned} \frac{dy}{du} &= 2u \\ \frac{du}{dx} &= 1 \\ \implies \frac{dy}{dx} &= 2u \times 1 \\ \implies \frac{dy}{dx} &= 2(x-1) \end{aligned}$$

3 Applications

In economics, the derivative is widely applied to compute 'marginal' quantities. For instance, if you have the cost function of a firm, you can compute the derivative of the cost function in order to get the marginal cost function.

Type	Function	Marginal Function
Total cost	$C(q)$	$C'(q)$
Total revenue	$R(q)$	$R'(q)$
Profit	$\pi(q)$	$\pi'(q)$

Let's take an example.

If the cost function is $C(q) = 10 + 2q + q^2$, determine the marginal cost. Calculate the marginal cost when $q = 10$, $q = 100$, and $q = 1000$.

$$\begin{aligned} MC(q) &= 2 + 2q && (MC(q) = C'(q)) \\ MC(10) &= 2 + 2(10) = 22 \\ MC(100) &= 2 + 2(100) = 202 \\ MC(1000) &= 2 + 2(1000) = 2002 \end{aligned}$$

The **marginal cost** to produce x units represents the actual cost of $(x+1)^{th}$ unit. So, the approximate cost of producing the 11th unit (in the above example) is ₹22.