Constrained Optimization: A Very Short Introduction

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September 20, 2025

1 Introduction

Consider a two-variable function $f(x, y)$. Consider also an equality constraint defined by $g(x, y) = c$. We will assume that both f and g are differentiable. Our goal is to:

$$\text{max(min)} f(x, y) \quad \text{subject to } g(x, y) = c$$

2 The Lagrange Multiplier Method

This is the method developed by the eighteenth century mathematician named Joseph-Louis Lagrange.
Let f and g be two continuously differentiable functions defined on a set S and suppose that (x_0, y_0) is an interior point of S that solves the problem we stated above. That is, (x_0, y_0) represents the local maximizer (minimizer) of $f(x, y)$ subject to $g(x, y) = c$. Suppose also that $g_x(x_0, y_0) \neq 0$ and $g_y(x_0, y_0) \neq 0$. Then, there exists a unique number λ (the Lagrange multiplier) such that (x_0, y_0) is a stationary point of the Lagrangian

$$\mathcal{L}(x, y) = f(x, y) + \lambda(c - g(x, y))$$

(x_0, y_0) satisfies the following first-order conditions:

$$\begin{aligned}
\mathcal{L}_x(x_0, y_0) &= f_x(x_0, y_0) - \lambda g_x(x_0, y_0) = 0 \\
\mathcal{L}_y(x_0, y_0) &= f_y(x_0, y_0) - \lambda g_y(x_0, y_0) = 0 \\
g(x_0, y_0) &= c
\end{aligned}$$

An example:

$$\max_{x,y} \quad xy \quad \text{subject to } x + y = 4 \text{ and } x \geq 0, y \geq 0$$

The Lagrangian is

$$\mathcal{L}(x, y) = xy + \lambda(4 - x - y)$$

The first-order derivatives are:

$$
\begin{aligned}
\mathcal{L}_x &= y - \lambda = 0 \\
\mathcal{L}_y &= x - \lambda = 0 \\
x + y &= 4 \\
\implies 2\lambda &= 4 \\
\implies \lambda &= 2 \\
\implies x &= y = \lambda = 2
\end{aligned}
$$

Please note that $g_x = 1$ and $g_y = 1$. So, both derivatives of the constraint do not vanish implying we are on the right track.
Hence, the solution is $(2, 2, 4)$.

Another example:

$$\max_{x,y} \quad x^\alpha y^\beta \quad \text{subject to } px + y = m$$

where $p > 0$, $\alpha > 0$, $\beta > 0$, and $m > 0$.
The Lagrangian is:

$$\mathcal{L}_{x,y} = x^\alpha y^\beta + \lambda(m - px - y)$$

The first-order conditions are:

$$
\begin{aligned}
\alpha x^{\alpha-1} y^\beta - p\lambda &= 0 \\
\beta x^\alpha y^{\beta-1} - \lambda &= 0 \\
px + y &= m \\
\implies \alpha y &= p\beta x && \text{(from the first two FOCs)} \\
\implies y &= p\frac{\beta}{\alpha}x \\
\implies px + p\frac{\beta}{\alpha}x &= m \\
\implies p\Big(1 + \frac{\beta}{\alpha}\Big)x &= m \\
\implies x &= \frac{\alpha m}{(\alpha+\beta)p} \\
\implies y &= \frac{\beta m}{(\alpha+\beta)}
\end{aligned}
$$

Hence, the solution is $\Big(\dfrac{\alpha m}{(\alpha+\beta)p}, \dfrac{\beta m}{(\alpha+\beta)}\Big)$.
Please also note that if this were a utility function, the condition for maximization will be: MRS equals the ratio of prices. In this case, $MRS = p$.

2.1 Interpreting the Lagrange Multiplier

The Lagrange multiplier represents the rate at which the optimal value of the function changes w.r.t. changes in the constant c. Therefore,

$$\frac{df^*(c)}{dc} = \lambda(c)$$

An example:

$$\max \quad xy \quad \text{subject to } x + y = k$$

The first-order conditions will yield: $y = x = \lambda$. Therefore, $x + x = k \implies x(k) = \frac{k}{2}$.

At this point, the function becomes: $f^*(k) = \left(\frac{k}{2}\right)^2 = \frac{k^2}{4}$.

Compute the derivative w.r.t. k: $\frac{df^*(k)}{dk} = \frac{k}{2} = \lambda$.

2.2 Utility Maximization: Lagrangian Approach

Consider a utility function $u(x, y)$ and a budget constraint $px + qy = m$ where p is the unit price of good x, q the unit price of good y, and m is the total budget allocated to x and y. We want to optimize the following function:

$$\max \quad u(x, y) \quad \text{subject to } px + qy = m$$

The Lagrangian is:

$$\mathcal{L} = u(x, y) + \lambda(m - px - qy)$$

The first order conditions are:

$$\begin{aligned} MU_x &= \lambda p \\ MU_y &= \lambda q \\ px + qy &= m \end{aligned}$$

We can eliminate λ and write:

$$\frac{MU_x}{MU_y} = \frac{p}{q}$$

We already know that the MRS is the ratio of the marginal utilities. Therefore, $MRS = \frac{p}{q}$.

We will end the discussion with some standard utility functions.

2.2.1 Cobb Douglas Utility Function

Example: maximize $u(x, y) = 3x^{1/3}y^{2/3}$ subject to $px + qy = m$.

The Lagrangian is:

$$\mathcal{L} = 3x^{1/3}y^{2/3} + \lambda(m - px - qy)$$

The FOCs are:

$$\frac{1}{3}\left(\frac{y}{x}\right)^{2/3} = \lambda p \tag{1}$$

$$\frac{2}{3}\left(\frac{x}{y}\right)^{1/3} = \lambda q \tag{2}$$

$$px + qy = m \tag{3}$$

Dividing the first equation by the second one, we get: $\frac{y}{2x} = \frac{p}{q}$. Plug this back into the third equation.

$$\begin{aligned} px + 2px &= m \\ \implies x^* &= \frac{m}{3p} \\ \implies y^* &= \frac{2m}{3q} \end{aligned}$$

2.3 Utility Maximization: Other Approaches

While we can optimize well-behaved utility functions using the Lagrange method, we will need to look elsewhere when utility functions that generate **corner solutions**.

2.3.1 Perfect Substitutes

Consider $u = 2x + y$ and the constraint is $x + y = 10$.
Let's try to optimize this function using the Lagrange method.
The Lagrangian is: $\mathcal{L} = 2x + y + \lambda(10 - x - y)$.
FOCs are:

$$2 - \lambda = 0$$
$$1 - \lambda = 0$$
$$x + y = 10$$

It turns out that $\lambda = 2$ and $\lambda = 1$ (not possible!).
What should we do now? We should rely on the ratio of the marginal utility to the price. Let p be the price of x and q be the price of y.
$\frac{MU_x}{p} = 2 > \frac{MU_y}{q} = 1$
Since these are perfect substitutes, an individual is better off discarding the good that provides lesser marginal utility per rupee. Therefore, the solution is: $(10, 0)$.

2.3.2 Perfect Complements

Consider $u = min(x, y)$ and the constraint is $x + y = 10$.
This function is not even differentiable. However, the utility is maximized when $x = y$. Therefore, the solution is: $(5, 5)$.

2.3.3 Quasi-linear Utility

Let $u = \sqrt{x} + 2y$. The constraint is $x + 8y = 4$.
The Lagrangian is: $\sqrt{x} + 2y + \lambda(4 - x - 8y)$.
The FOCs are:

$$\frac{1}{2\sqrt{x}} = \lambda \tag{4}$$
$$2 = 8\lambda \tag{5}$$
$$x + 8y = 4 \tag{6}$$

Using the first two equations, we get $\frac{1}{2\sqrt{x}} = \frac{1}{4} \implies 4x = 16 \implies x = 4$.
Plugging the value of x into the constraint, we have $y = 0$. Solution: $(4, 0)$ (a corner solution)