Unconstrained Optimization

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1 Introduction

While optimization is a broad topic, our goal in this lecture will be focused on a narrow issue. We will dwell in the realm of nicely-behaved (continuous and differentiable) functions. We will also restrict ourselves to think about local optima. We will also use compact notations for the partial derivatives.

$$f_x = \frac{\partial f}{\partial x} \quad f_y = \frac{\partial f}{\partial y} \quad f_{xx} = \frac{\partial^2 f}{\partial x^2} \quad f_{yy} = \frac{\partial^2 f}{\partial y^2} \quad f_{xy} = \frac{\partial^2 f}{\partial y \partial x} \quad f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$$

2 Local Optima

Let (x_0, y_0) be a **stationary point** of the function f of two variables. Then, we should have (at this point):
Step 1: Find and set the first-order partial derivatives to zero to find (x_0, y_0). Therefore, the necessary condition is:

$$\begin{aligned} f_x &= 0 \\ f_y &= 0 \end{aligned}$$

Step 2: In order to classify (or label) the stationary, we need all the second-order derivatives. We also need the *Hessian* of the function.

Let f be a twice-differentiable function of two variables. The **Hessian** of f at $\mathbf{x}$ is

$$H(\mathbf{x}) = \begin{pmatrix} f_{xx}(\mathbf{x}) & f_{xy}(\mathbf{x}) \\ f_{yx}(\mathbf{x}) & f_{yy}(\mathbf{x}) \end{pmatrix}$$

We define the determinant of the Hessian as: $D = (f_{xx}f_{yy}) - (f_{xy})^2$.

Case	Second Order Conditions	Conclusion	Concavity
1.	$D > 0$ **and** $f_{xx} < 0$	Local Maximum	Concave.
2.	$D > 0$ **and** $f_{xx} > 0$	Local Minimum	Convex.
3.	$D < 0$	Saddle Point	Convex (one side), Concave (other side).
4.	$D = 0$	**Inconclusive**	-

3 Examples

1- $f(x, y) = (x-1)^2 + (y-1)^2$
To find the stationary points, we first compute the first-order partial derivatives and set them equal to zero.

$$\begin{aligned} f_x &= \frac{\partial}{\partial x}((x-1)^2 + (y-1)^2) \\ &= 2(x-1) \\ f_y &= \frac{\partial}{\partial y}((x-1)^2 + (y-1)^2) \\ &= 2(y-1) \end{aligned}$$

We look for potential stationary points by setting both partial derivatives to zero.

$$\begin{aligned} f_x &= 0 \\ \implies 2(x-1) &= 0 \\ \implies x &= 1 \\ f_y &= 0 \\ \implies 2(y-1) &= 0 \\ \implies y &= 1 \end{aligned}$$

We got only one candidate point: $(1, 1)$.
Let's now test for the second order condition for which we need the second order partial derivatives.

$$\begin{aligned} f_{xx} &= \frac{\partial}{\partial x}(2(x-1)) \\ &= 2 \end{aligned} \quad \begin{aligned} f_{yy} &= \frac{\partial}{\partial y}(2(y-1)) \\ &= 2 \end{aligned} \quad \begin{aligned} f_{xy} &= \frac{\partial}{\partial y}(2(x-1)) \\ &= 0 \end{aligned} \quad \begin{aligned} f_{yx} &= \frac{\partial}{\partial x}(2(y-1)) \\ &= 0 \end{aligned}$$

We can see that $f_{xx} > 0$ and $f_{yy} > 0$. Therefore, it seems like $(1, 1)$ represents the local minimum, but we must also check the value of the determinant of the Hessian. $D > 0$ implies that this is indeed the local minimum.

2- $f(x,y) = 60x + 34y - 6x^2 - 3y^2 - 4xy + 5$.

First Order Partial Derivatives
$f_x = 60 - 12x - 4y$
$f_y = 34 - 6y - 4x$
Set these two partial derivatives to zero.

$$\begin{aligned} 12x + 4y &= 60 \\ 4x + 6y &= 34 \end{aligned}$$

Solving these equations, we get: $x = 3$, $y = 4$.
We have only candidate stationary point: $(3,4)$.
Second Order Partial Derivatives

$f_{xx} = \frac{\partial}{\partial x}(60 - 12x - 4y) = -12$	$f_{yy} = \frac{\partial}{\partial y}(34 - 6y - 4x) = -6$	$f_{xy} = \frac{\partial}{\partial y}(60 - 12x - 4y) = -4$	$f_{yx} = \frac{\partial}{\partial x}(34 - 6y - 4x) = -6$

$f_{xx} < 0$, $f_{yy} < 0$, and $D > 0 \implies$ Local maximum

3- $f(x,y) = 4x^2 - xy + y^2 - x^3$.
First Order Partial Derivatives
$f_x = 8x - y - 3x^2$
$f_y = -x + 2y$
Set these two partial derivatives to zero.

$$\begin{aligned} 8x^2 - y - 3x^2 &= 0 \\ x &= 2y \\ \implies 16y - y - 3(2y)^2 &= 0 \\ \implies 15y - 12y^2 &= 0 \\ \implies 3y(5 - 4y) &= 0 \\ \implies y &= 0, \frac{5}{4} \\ \implies x &= 0, \frac{5}{2} \end{aligned}$$

We have two stationary points: $(0,0)$ and $(\frac{5}{2}, \frac{5}{4})$.
Second Order Partial Derivatives

$f_{xx} = \frac{\partial}{\partial x}(8x - y - 3x^2) = 8 - 6x$	$f_{yy} = \frac{\partial}{\partial y}(-x + 2y) = 2$	$f_{xy} = \frac{\partial}{\partial y}(8x - y - 3x^2) = -1$	$f_{yx} = \frac{\partial}{\partial x}(-x + 2y) = -1$

Based on the values, we have a local minimum at $(0,0)$ and a saddle point at $(\frac{5}{2}, \frac{5}{4})$.