# Single Variable Optimization-II

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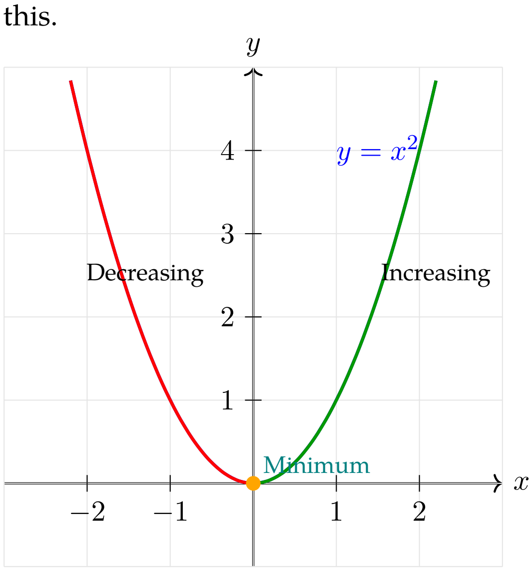
September 1, 2025

## 1 Increasing and Decreasing Functions

Consider a function $f(x)$ defined in an interval $I$.

- $f(x)$ is an increasing function if $x_1 > x_2 \implies f(x_1) \geq f(x_2)$.
- $f(x)$ is a decreasing function if $x_1 > x_2 \implies f(x_1) \leq f(x_2)$.
- $f(x)$ is a strictly increasing function if $x_1 > x_2 \implies f(x_1) > f(x_2)$.
- $f(x)$ is a strictly decreasing function if $x_1 < x_2 \implies f(x_1) < f(x_2)$.

We can also use the derivative to classify a function as increasing or decreasing. A function is increasing if $f'(x) \geq 0$ and decreasing if $f'(x) \leq 0$.
An example: $f(x) = x^2$.
Compute the first derivative. $f'(x) = 2x$.
Applying the derivative condition, we get: $2x \geq 0$ for the function to be increasing. Therefore, the function $f(x) = x^2$ is increasing whenever $x \geq 0$. So, do we conclude that this function is increasing? The answer is no, because we should also look for the values of $x$ where $f'(x) \leq 0$ or $2x \leq 0$. Hence, the function $f(x)$ is decreasing when $x \leq 0$. You should plot the function to verify this.



## 2 Concave and Convex Functions

The first order derivative is useful when we want to figure out whether a function has a stationary point or is increasing (decreasing). Similarly, the second order derivative tells us whether a function is convex or concave. It goes without saying that we must have a twice-differentiable function defined in an interval. Luckily, a lot of functions that you are going to encounter in economics will be well-behaved functions (cost, revenue, profit, utility, production functions). Let $I$ be the interval.

- $f''(x) \leq 0 \implies$ the function is concave in interval $I$.
- $f''(x) \geq 0 \implies$ the function is convex in interval $I$.

An example: $f(x) = x^3 - x$.
Differentiate this function twice to get $f''(x) = 6x$. We can see that for all $x < 0$, the second order derivative will be negative and for all $x > 0$, the second order derivative will be positive. Therefore, $f(x)$ is concave in $(-\infty, 0]$ and convex in $[0, \infty)$.

## 3 First Order Derivative Test

Let $x = c$ be a stationary point. Then,

- If $f'(x) \geq 0 \quad \forall x < c$ and $f'(x) \leq 0 \quad \forall x > c$, then $c$ must be a maximum point.
- If $f'(x) \leq 0 \quad \forall x < c$ and $f'(x) \geq 0 \quad \forall x > c$, then $c$ must be a minimum point.

An example: Classify the stationary points of $x^3 - 12x$ using the first order derivative test.

$$
\begin{aligned}
& f'(x) = 3x^2 - 12 \\
\implies & 3x^2 - 12 = 0 \qquad \text{(using the necessary condition for stationary points)} \\
\implies & x^2 - 4 = 0 \\
\implies & x^2 = 4 \\
\implies & x = \pm 2
\end{aligned}
$$

When $c = -2$,

$$
\begin{aligned}
f'(-3) &= 3(-3)^2 - 12 > 0 \\
f'(-1) &= 3(-1)^2 - 12 < 0
\end{aligned}
$$

When $c = 2$,

$$
\begin{aligned}
f'(1) &= 3(1)^2 - 12 < 0 \\
f'(3) &= 3(3)^2 - 12 > 0
\end{aligned}
$$

The first order derivative test suggests that $x = -2$ is the maximum point and $x = 2$ is the minimum point for the given function.

## 4 Absolute and Local Extrema

**Definition: Absolute Extrema**

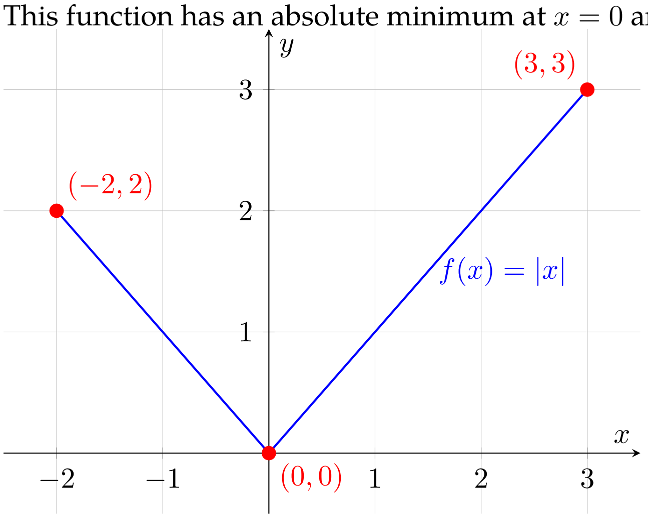
- If $c \in D_f$, then $f(c)$ is the absolute maximum value of $f(x)$ if $f(c) \geq f(x) \quad \forall x \in D_f$.
- If $c \in D_f$, then $f(c)$ is the absolute minimum value of $f(x)$ if $f(c) \leq f(x) \quad \forall x \in D_f$.

**Definition: Local Extrema**

- If $c \in D_f$, then $f(c)$ is a local maximum value of $f(x)$ if $f(c) > f(x)$ for all $x$ in some open interval containing $c$.
- If $c \in D_f$, then $f(c)$ is a local minimum value of $f(x)$ if $f(c) < f(x)$ for all $x$ in some open interval containing $c$.

An example: Consider $f(x) = x^3 - 12x$. The domain of the function is $(-\infty, \infty)$. We can easily see that this function does not have an absolute maximum or minimum.
Let's consider an interval $I = [-3, 3]$ (a part of the domain). Can we identify the absolute minimum and maximum points for $I$? The rule tells us to check for the stationary points as well as the end points. $f(-3) = 9$, $f(3) = -9$, $f(-2) = 16$, and $f(2) = -16$.
Within interval $I$, $x = -2$ and $x = 2$ represent absolute minimum and absolute maximum points respectively. $x = -2$ and $x = -1$ are also the local maximum and local minimum points respectively. Please take a couple of values close to $\pm 2$ to verify that this is indeed true.

Let's take another example. $f(x) = |x|$ and $I = [-2, 3]$.
$f(-2) = 2$, $f(0) = 0$, and $f(3) = 3$.
This function has an absolute minimum at $x = 0$ and an absolute maximum at $x = 3$.



# 5 Extreme Value Theorem

If a function $f$ is continuous in a closed interval $[a, b]$, then $f$ is guaranteed to have an absolute minimum point $c$ and an absolute maximum point $d$ on $[a, b]$ such that:

$$f(c) \leq f(x) \leq f(d) \quad \forall x \in [a, b]$$

## 5.1 Finding Extreme Values: A Cookbook

Step 1 Determine whether the function is continuous on $[a, b]$.

Step 2 Using the necessary condition, find all stationary points that are in $(a, b)$.

Step 3 Evaluate the value of the function at the stationary points **and** the endpoints $(a, b)$.

Step 4 Classify the largest and the smallest value of the function as the absolute maximum and the absolute minimum respectively.

An example: $f(x) = \dfrac{2}{x+2}$ and interval $I = [1, 3]$.
Note that this function does not have any critical point. So, now we are only left to evaluate the endpoints. When $x = 1$, $f(1) = \dfrac{2}{3}$. When $x = 3$, $f(3) = \dfrac{2}{5}$.
Therefore, the function has an absolute minimum at $x = 3$ and an absolute maximum at $x = 1$.

# 6 Mean Value Theorem

Suppose that $f(x)$ is continuous on $[a, b]$ and differentiable on $(a, b)$.
Then, there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

An example: Consider $f(x) = x^3 - x$. Test the MVT on $f(x)$ in $[0, 1]$.

$$f'(c) = \frac{f(2) - f(0)}{2 - 0}$$
$$f'(c) = \frac{(2)^3 - 2 - (0^3 - 0)}{2}$$
$$f'(c) = \frac{6}{2}$$
$$f'(c) = 3$$

Using the given function, we can write:

$$
\begin{aligned}
f'(x) &= 3x^2 - 1 \\
\implies f'(c) &= 3c^2 - 1 \\
\implies 3c^2 - 1 &= 3 \\
\implies 3c^2 &= 4 \\
\implies c^2 &= \frac{4}{3} \\
\implies c &= \frac{2}{\sqrt{3}} && \text{(why?)}
\end{aligned}
$$

## 7 Second Derivative Test and Inflection Points

We have already seen the second derivative test in the previous lecture. We will expand the test to fully characterize the stationary points.
Let $c$ be a stationary point of a function $f(x)$. Then,

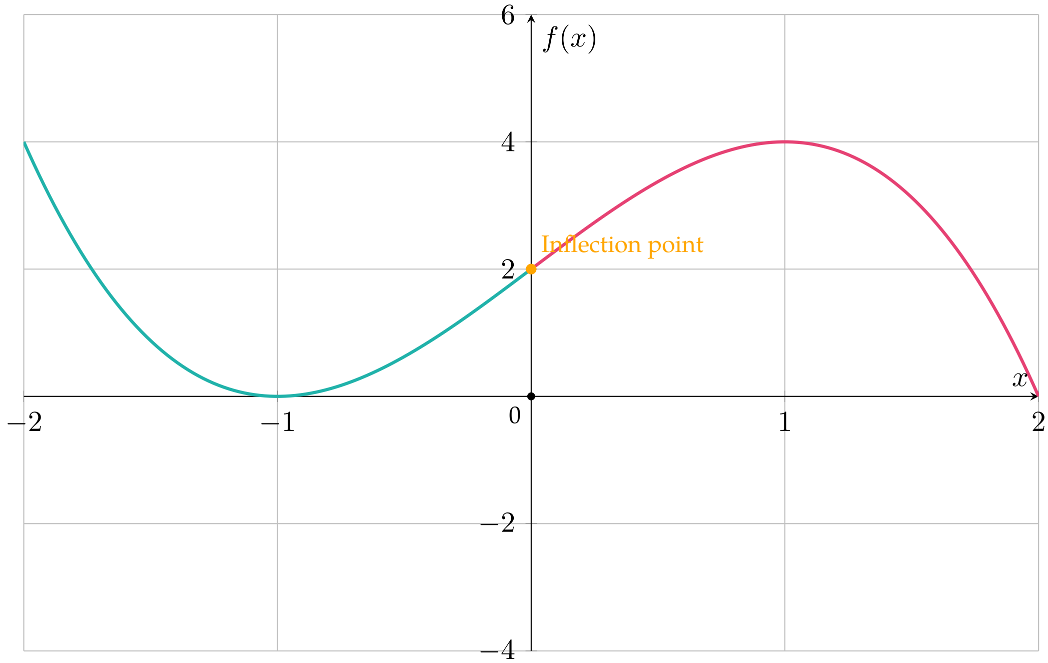
$f'(c) = 0 \quad \& \quad f''(c) \leq 0 \Rightarrow$ MAXIMUM
$f'(c) = 0 \quad \& \quad f''(c) \geq 0 \Rightarrow$ MINIMUM
$f'(c) = 0 \quad \& \quad f''(c) = 0 \Rightarrow$ ????

We will zoom in on the second part of the last condition. We can label such a point where the second derivate vanishes as the **inflection point** if

$f''(x) \geq 0$ when $x > c$ and $f''(x) \leq 0$ when $x < c$
**OR**
$f''(x) \leq 0$ when $x > c$ and $f''(x) \geq 0$ when $x < c$

`tl;dr version`: $c$ is an inflection point if the second derivative changes sign around $x = c$.
An example: $f(x) = 2 + 3x - x^3$.
$f'(x) = 3 - 3x^2$ and $f''(x) = -6x$.
Since the necessary condition for an inflection point is $f''(x) = 0$, therefore, the inflection point for this function is $x = 0$.
Let's see what happens when we calculate the value of the second derivative below and above $x = 0$.
We will use $x = -1$ and $x = 1$.
When $x = -1$, $f''(-1) = 6 > 0$. When $x = 1$, $f''(1) = -6 < 0$.
We can see that the function changes signs at $x = 0$.
Therefore, $x = 0$ represents the inflection point for $f(x) = 2 + 3x - x^3$.

| Interval | Second Derivative | Curvature |
|---|---|---|
| $(-\infty, 0)$ | **Positive** | Convex |
| $(0, \infty)$ | **Negative** | Concave |




Consider another example: $f(x) = \frac{1}{2}x^4 + x^3 - 6x^2$.
The second derivative is: $f''(x) = 6x^2 + 6x - 12$.
The condition for inflection point is $f''(x) = 0$.

$$\begin{aligned} 6x^2 + 6x - 12 &= 0 \\ 6(x^2 + x - 2) &= 0 \\ 6(x+2)(x-1) &= 0 \end{aligned}$$

The two possible candidates for the inflection point are $x = -2$ and $x = 1$.
Let's pick $x = -3$, $x = 0$, and $x = 2$ for the sufficient condition test.
When $x = -3$, $f'' = 6(-3)^2 + 6(-3) - 12 = 24 > 0$.
When $x = 0$, $f'' = 6(0) + 6(0) - 12 = -12 < 0$.
When $x = 2$, $f'' = 6(2)^2 + 6(2) - 12 = 24 > 0$.
Therefore, $x = -2$ and $x = 1$ are the inflection points.