

### Short Answer Questions-I

1. (1 point) Let  $f(x) = \sqrt{x} + 4$  and  $g(x) = f^{-1}(x)$ . Find  $g'(6)$ .

**Solution:** We know that:  $g'(a) = \frac{1}{f'(g(a))}$

$$\therefore g'(6) = \frac{1}{f'(g(6))}$$

$$\text{Let } g(6) = k \implies f(k) = 6.$$

$$\sqrt{k} + 4 = 6$$

$$\implies \sqrt{k} = 6 - 4$$

$$\implies \sqrt{k} = 2$$

$$\implies k = 4$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$\implies f'(4) = \frac{1}{2\sqrt{2}}$$

$$\implies f'(4) = \frac{1}{4}$$

$$\implies g'(6) = 4$$

**Answer:**  $g'(6) = 4$

2. (1 point) Suppose that  $f$  and  $g$  are continuous on  $[0, 4]$  and that  $\int_0^4 (f(x) - g(x))dx = 2$  and  $\int_0^4 (3f(x) - 4g(x))dx = 3$ . Find  $\int_0^4 (f(x) + g(x))dx$ .

**Solution:** Let  $\int_0^4 f(x)dx = a$  and  $\int_0^4 g(x)dx = b$ . Then,

$$a - b = 2$$

$$3a - 4b = 3$$

$$3a - 3b = 6$$

$$\implies b = 3$$

$$\implies a = 5$$

(multiplying the first equation by 3)

(differencing the previous two equations to eliminate  $a$ )

$$\int_0^4 (f(x) + g(x))dx = \int_0^4 f(x)dx + \int_0^4 g(x)dx$$

(applying the sum rule)

$$\implies \int_0^4 (f(x) + g(x))dx = 8$$

**Answer :**  $\int_0^4 (f(x) + g(x))dx = 8$

3. (1 point) Compute:  $\int(9x^2 + \frac{5}{x} + e^{6x})dx$

**Solution:**

$$\begin{aligned}\int(9x^2 + \frac{5}{x} + e^{6x})dx &= \int 9x^2 dx + \int \frac{5}{x} dx + \int e^{6x} dx && \text{(applying the sum rule)} \\ \Rightarrow \int(9x^2 + \frac{5}{x} + e^{6x})dx &= 3x^3 + 5 \ln|x| + \frac{e^{6x}}{6} + C\end{aligned}$$

**Answer :**  $\int(9x^2 + \frac{5}{x} + e^{6x})dx = 3x^3 + 5 \ln|x| + \frac{e^{6x}}{6} + C$

4. (1 point) Is  $\lim_{x \rightarrow 0} |x - 1| = \lim_{x \rightarrow 0} |x| - 1$ ? Explain briefly.

**Solution:** Consider  $f(x) = |x - 1|$ .

$$f(x) = \begin{cases} 1 - x & x < 1 \\ x - 1 & x > 1 \end{cases}$$

$$\text{LHL: } \lim_{x \rightarrow 0^-} |x - 1| = 1 \quad \text{(since } |x - 1| = 1 - x \text{ when } x < 1)$$

$$\text{RHL: } \lim_{x \rightarrow 0^+} |x - 1| = 1 \quad \text{(since } |x - 1| = 1 - x \text{ when } x < 1)$$

$$\therefore \lim_{x \rightarrow 0} |x - 1| = 1$$

Let  $g(x) = |x| - 1$ .

$$g(x) = \begin{cases} -x - 1 & x < 0 \\ x - 1 & x > 0 \end{cases}$$

$$\text{LHL: } \lim_{x \rightarrow 0^-} |x| - 1 = -1 \quad \text{(since } |x| = -x \text{ when } x < 0)$$

$$\text{RHL: } \lim_{x \rightarrow 0^+} |x| - 1 = -1 \quad \text{(since } |x| = x \text{ when } x > 0)$$

$$\therefore \lim_{x \rightarrow 0} |x| - 1 = -1$$

**Answer:**  $\lim_{x \rightarrow 0} |x - 1| \neq \lim_{x \rightarrow 0} |x| - 1$

5. (1 point) Let  $f(x) = \frac{12}{4 + x}$ . Find  $f^{-1}(x)$ .

**Solution:** Let  $y = f(x)$ .

$$\begin{aligned}y &= \frac{12}{x+4} \\ \Rightarrow y(x+4) &= 12 \\ \Rightarrow xy + 4y &= 12 \\ \Rightarrow xy &= 12 - 4y \\ \Rightarrow x &= \frac{12 - 4y}{y} \\ \Rightarrow f^{-1}(x) &= \frac{12 - 4x}{x}\end{aligned}$$

**Answer:**  $f^{-1}(x) = \frac{12 - 4x}{x}$

6. (1 point) Let  $f(x) = x^{x+2}$ . Find  $f'(x)$ .

**Solution:** Let  $y = x^{x+2}$ .

$$\begin{aligned}\ln y &= (x+2) \ln x && \text{(using the property of } \log a = b^c \Rightarrow \ln a = c \ln b) \\ \Rightarrow \frac{1}{y} \frac{dy}{dx} &= \frac{d}{dx}((x+2) \ln x) && \text{(differentiating both sides w.r.t. } x) \\ \Rightarrow \frac{1}{y} \frac{dy}{dx} &= \ln x \frac{d}{dx}(x+2) + (x+2) \frac{d}{dx}(\ln x) && \text{(applying the product rule to the RHS)} \\ \Rightarrow \frac{1}{y} \frac{dy}{dx} &= \ln x + \frac{x+2}{x} \\ \Rightarrow \frac{dy}{dx} &= y \left( \ln x + \frac{x+2}{x} \right) \\ \Rightarrow \frac{dy}{dx} &= x^{x+2} \left( \ln x + \frac{x+2}{x} \right)\end{aligned}$$

**Answer:**  $\frac{dy}{dx} = x^{x+2} \left( \ln x + \frac{x+2}{x} \right)$

7. (1 point) Determine if the function  $f(x) = x^2 - 6x + 5$  is increasing or decreasing in  $[1, 5]$ .

**Solution:** First derivative tells us whether (or where) a function is increasing or decreasing.

$$\begin{aligned}f'(x) &= 2x - 6 \\ \text{Condition for increasing function : } 2x - 6 &\geq 0 \\ \therefore \text{ the function is increasing when } x &\geq 3 \\ \text{Condition for decreasing function : } 2x - 6 &\leq 0 \\ \therefore \text{ the function is increasing when } x &\leq 3\end{aligned}$$

**Answer:**  $f(x)$  is decreasing in  $[1, 3]$  and increasing in  $[3, 5]$ .

8. (1 point) Let  $x^2y^3 + x^3y^2 = 9$ . Find  $\frac{dy}{dx}$ .

**Solution:** Let  $u = x^2y^3$ ,  $v = x^3y^2$ , and  $c = 9$ .

Applying the product rule, we get:

$$u' = 2xy^3 + 3x^2y^2 \frac{dy}{dx}$$

$$v' = 2x^3y \frac{dy}{dx} + 3x^2y^2$$

$$c' = 0$$

$$u' + v' = c'$$

$$\therefore \left[ 2xy^3 + 3x^2y^2 \frac{dy}{dx} \right] + \left[ 2x^3y \frac{dy}{dx} + 3x^2y^2 \right] = 0$$

$$\Rightarrow (2xy^3 + 3x^2y^2) + (3x^2y^2 + 2x^3y) \frac{dy}{dx} = 0$$

$$\Rightarrow (3x^2y^2 + 2x^3y) \frac{dy}{dx} = -(2xy^3 + 3x^2y^2)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(2xy^3 + 3x^2y^2)}{(3x^2y^2 + 2x^3y)}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y(3x + 2y)}{x(2x + 3y)}$$

<b>Answer:</b> $\frac{dy}{dx} = -\frac{y(3x + 2y)}{x(2x + 3y)}$
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## Short Answer Questions-II

9. (2+1 points) The total cost of producing *Phantom cigarettes* is  $C(q) = 3q^2 + 10q + 48$ . Find the value of  $q$  which minimizes the average cost. Show that the marginal cost is equal to the average cost at this point (where the average cost is being minimized).

**Solution:** We know that the average cost is:

$$AC(q) = \frac{C(q)}{q}$$

The average cost of producing *Phantom cigarettes* is:

$$AC(q) = 3q + 10 + \frac{48}{q}$$

We need to find the first derivative and set it to zero.

$$\begin{aligned}\frac{d(AC(q))}{dq} &= \frac{d}{dq} \left( 3q + 10 + \frac{48}{q} \right) \\ &= 3 - \frac{48}{q^2} \\ \text{FOC: } \frac{d(AC(q))}{dq} &= 0 \\ \Rightarrow 3 - \frac{48}{q^2} &= 0 \\ \Rightarrow 3q^2 &= 48 \\ \Rightarrow q^2 &= 16 \\ \Rightarrow q^* &= 4 \\ \text{SOC: } \frac{d^2(AC(q))}{dq^2} &> 0 \quad (\text{for minimum}) \\ \frac{d^2(AC(q))}{dq^2} &= \frac{96}{q^3} \\ \frac{96}{q^3} &> 0 \text{ when } q = 4\end{aligned}$$

We also need to compute the marginal cost.

$$MC(q) = 6q + 10$$

When  $q = 4$ ,

$$\begin{aligned}\text{Average cost: } AC(q = 4) &= 3(4) + 10 + \frac{48}{4} \\ &= 12 + 10 + 12 \\ &= 34 \\ \text{Marginal cost: } MC(q = 4) &= 6(4) + 10 \\ &= 34\end{aligned}$$

**Answer:** The quantity that minimizes the average cost is  $q^* = 4$ . When  $q = 4$ ,  $AC = MC = 34$ .

10. (3 points)  $f(x, y) = 9x^{1/3}y^{2/3}$

(a) (1 point) Determine the degree of homogeneity.

**Solution:** The degree of homogeneity can be calculated using:

$$\begin{aligned}f(tx, ty) &= t^k f(x, y) \\ f(tx, ty) &= 9(tx)^{1/3}(ty)^{2/3} \\ &= 9t^{1/3}x^{1/3}t^{2/3}y^{2/3} \\ &= (t^{1/3+2/3})9(x^{1/3}y^{2/3}) \\ &= t^1 f(x, y) \\ \Rightarrow k &= 1\end{aligned}$$

**Answer:**  $f(x, y)$  is homogeneous of degree 1.

(b) (2 points) Compute all first and second order partial derivatives.

**Solution:**

$$\begin{aligned}f_x \quad \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x} (9x^{1/3}y^{2/3}) \\&= 9 \cdot \frac{1}{3}x^{-2/3}y^{2/3} \\&= 3x^{-2/3}y^{2/3} \\&= 3\left(\frac{y}{x}\right)^{2/3}\end{aligned}$$

$$\begin{aligned}f_y \quad \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y} (9x^{1/3}y^{2/3}) \\&= 9x^{1/3} \cdot \frac{2}{3}y^{-1/3} \\&= 6x^{1/3}y^{-1/3} \\&= 6\left(\frac{x}{y}\right)^{1/3}\end{aligned}$$

$$\begin{aligned}f_{xx} \quad \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} (3x^{-2/3}y^{2/3}) \\&= 3y^{2/3} \cdot \left(-\frac{2}{3}\right)x^{-5/3} \\&= -2x^{-5/3}y^{2/3} \\&= -2\left(\frac{y^2}{x^5}\right)^{1/3}\end{aligned}$$

$$\begin{aligned}f_{yy} \quad \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} (6x^{1/3}y^{-1/3}) \\&= 6x^{1/3} \cdot \left(-\frac{1}{3}\right)y^{-4/3} \\&= -2x^{1/3}y^{-4/3} \\&= -2\left(\frac{x}{y^4}\right)^{1/3}\end{aligned}$$

$$\begin{aligned}f_{xy} \quad \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial y} (3x^{-2/3}y^{2/3}) \\&= 3x^{-2/3} \cdot \frac{2}{3}y^{-1/3} \\&= 2x^{-2/3}y^{-1/3} \\&= 2\left(\frac{1}{x^2y}\right)^{1/3}\end{aligned}$$

$$\begin{aligned}f_{yx} \quad \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial x} (6x^{1/3}y^{-1/3}) \\&= 6y^{-1/3} \cdot \frac{1}{3}x^{-2/3} \\&= 2x^{-2/3}y^{-1/3} \\&= 2\left(\frac{1}{x^2y}\right)^{1/3}\end{aligned}$$

11. (1+1+1 points) Let  $Y = 2K + 2\sqrt{L}$ . Compute the marginal products and the marginal rate of technical substitution.

**Solution:**

$$\begin{aligned}MP_K &= 2 \\MP_L &= \frac{1}{\sqrt{L}} \\MRTS_{K,L} &= \frac{MP_K}{MP_L} \\MRTS_{K,L} &= 2\sqrt{L} \\MRTS_{L,K} &= \frac{1}{2\sqrt{L}}\end{aligned}$$

12. (3 points) Given the demand function for comedy shows on *Ruinmyshow*:  $p = \frac{36}{q+5} - 5$ ,

(a) ( $\frac{1}{2}$  points) Compute the total revenue.

**Solution:** Total revenue,  $TR = p \cdot q$ .

$$\begin{aligned}p &= \frac{36}{q+5} - 5 \\ \Rightarrow TR &= \frac{36q}{q+5} - 5q\end{aligned}$$

(b) ( $\frac{1}{2}$  points) Compute the marginal revenue.

**Solution:** We know that  $MR = \frac{d}{dq}(TR)$ .

$$\begin{aligned}TR &= \frac{36q}{q+5} - 5q \\ \text{Let } u &= 36q, \quad v = q+5 \\ \Rightarrow u' &= 36, \quad v' = 1 \\ MR &= \frac{vu' - uv'}{v^2} - 5 \\ \Rightarrow MR &= \frac{36(q+5) - 36q}{(q+5)^2} - 5 \\ \Rightarrow MR &= \frac{180}{(q+5)^2} - 5 \\ \Rightarrow MR &= \frac{180 - 5(q+5)^2}{(q+5)^2}\end{aligned}$$

<b>Answer:</b> $MR = \frac{180 - 5(q+5)^2}{(q+5)^2}$
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(c) (2 points) Compute the revenue-maximizing price and quantity.

**Solution:** We know that the revenue is maximized when  $MR = 0$ .

$$\begin{aligned}180 - 5(q + 5)^2 &= 0 \quad (\text{note that the denominator can't be zero.}) \\ \Rightarrow 5(q + 5)^2 &= 180 \\ \Rightarrow (q + 5)^2 &= 36 \\ \Rightarrow (q + 5) &= \pm 6 \\ \Rightarrow q + 5 &= 6 && (\text{discarding the negative value.}) \\ \Rightarrow q &= 1\end{aligned}$$

Plugging the value into the demand equation, we get  $p = \frac{36}{1 + 5} - 5$

$$\begin{aligned}\Rightarrow p &= \frac{36}{6} - 5 \\ \Rightarrow p &= 6 - 5 \\ \Rightarrow p &= 1\end{aligned}$$

**Answer:** The revenue-maximizing price is  $p = 1$  and the quantity is  $q = 1$ .

## Long Answer Questions

13. (5 points) Consider  $f(x, y) = \frac{x^3}{3} + \frac{y^3}{3} + x^2 + \frac{y^2}{2} - 3x - 6y + 2$ . Find and classify all stationary points.

**Solution:** To find the stationary points, we first compute the first-order partial derivatives and set them equal to zero.

$$\begin{aligned}f_x &= \frac{\partial f}{\partial x} = (x^2 + 2x - 3) = 0 \\ f_y &= \frac{\partial f}{\partial y} = (y^2 + y - 6) = 0\end{aligned}$$

We have two quadratic equations to be solved.

$$\begin{aligned}(x + 3)(x - 1) &= 0 \\ (y + 3)(y - 2) &= 0 \\ x &= 1, -3 \\ y &= 2, -3\end{aligned}$$

The stationary points are the combinations of these  $x$  and  $y$  values:

$$\begin{array}{cc}(-3, -3) & (-3, 2) \\ (1, -3) & (1, 2)\end{array}$$

To classify the stationary points, we use the second derivative test, which requires the second-order partial derivatives.

$$\begin{aligned}f_{xx} &= \frac{\partial^2 f}{\partial x^2} = 2x + 2 \\ f_{yy} &= \frac{\partial^2 f}{\partial y^2} = 2y + 1 \\ f_{xy} &= \frac{\partial^2 f}{\partial x \partial y} = 0\end{aligned}$$

The determinant of the Hessian matrix is  $D = f_{xx}f_{yy} - (f_{xy})^2 = (2x + 2)(2y + 1)$ .

We evaluate the Hessian determinant  $D = f_{xx}f_{yy} - (f_{xy})^2$  at each stationary point:



- If  $D > 0$  and  $f_{xx} > 0$ : **Local Minimum**
- If  $D > 0$  and  $f_{xx} < 0$ : **Local Maximum**
- If  $D < 0$ : **Saddle Point**
- If  $D = 0$ : **Test Inconclusive**

Stationary Point	$f_{xx}$	$f_{yy}$	$D$	Classification
$(-3, -3)$	-4	-5	20 ( $> 0$ )	Local Maximum
$(-3, 2)$	-4	5	-20 ( $< 0$ )	Saddle Point
$(1, -3)$	4	-5	-20 ( $< 0$ )	Saddle Point
$(1, 2)$	4	5	20 ( $> 0$ )	Local Minimum

14. (5 points) The demand for robots in *Tatooine* is given by  $p = 19 - 2q$  and the supply of robots is given by  $p = 3 + 2q$ .

(a) (1 point) Compute the equilibrium price and quantity.

**Solution:** The equilibrium can be found out by setting demand = supply.

$$19 - 2q = 3 + 2q$$

$$\Rightarrow 4q = 16$$

$$\Rightarrow q^* = 4$$

$$\Rightarrow p^* = 11$$

**Answer:** The equilibrium quantity is  $q^* = 4$  and the equilibrium price is  $p^* = 11$ .

(b) (1+1 points) Compute the consumer surplus and producer surplus.

**Solution:** Given inverse demand ( $D(q)$ ) and inverse supply ( $S(q)$ ), we know that:

$$CS = \int_{q=0}^{q=q^*} D(q) dq - p^* q^*$$

$$PS = p^* q^* - \int_{q=0}^{q=q^*} S(q) dq$$

We also know from the previous calculation that  $p^* = 11$  and  $q^* = 4$ .

$$\begin{aligned} CS &= \int_0^4 (19 - 2q) dq - 44 \\ &= \left| 19q - q^2 \right|_0^4 - 44 \\ &= 76 - 16 - 44 \\ &= 16 \end{aligned}$$

$$\begin{aligned} PS &= 44 - \int_0^4 (3 + 2q) dq \\ &= 44 - \left| 3q + q^2 \right|_0^4 \\ &= 44 - (12 + 16) \\ &= 16 \end{aligned}$$

**Answer:**  $CS = 16$ ,  $PS = 16$

- (c) (1+1 points) Now, suppose that the Damiyo (the ruler of Tatooine), sensing that the robots are valuable, announces a price floor of 13. Compute the new consumer surplus and producer surplus.

**Solution:** When  $p = 13$ , we should compute the quantity demanded and the quantity supplied.

$$19 - 2q = 13$$

$$2q = 6$$

$$\text{Demand: } q = 3$$

$$3 + 2q = 13$$

$$2q = 10$$

$$\text{Supply: } q = 5$$

At this price, three robots will be sold in *Tatooine*.

$$\begin{aligned} CS &= \int_0^3 (19 - 2q) - 13 \times 3 \\ &= \left| 19q - q^2 \right|_0^3 - 39 \\ &= (57 - 9) - 39 \\ &= 9 \end{aligned}$$

$$\begin{aligned} PS &= 39 - \int_0^3 (3 + 2q) dq \\ &= 39 - \left| 3q + q^2 \right|_0^3 \\ &= 39 - (9 + 9) \\ &= 21 \end{aligned}$$

**Answer:**  $CS = 9$ ,  $PS = 21$