Endterm (Set A)

SIAS, Krea University (AY 2025-26)

Mathematical Methods for Economics (Course Code: **ECON211**)

25 September 2025

Maximum Points: 30 **Duration**: 100 minutes

Short Answer Questions-I

1. (1 point) Is $\lim_{x\to 0} |x-2| = \lim_{x\to 0} |x| - 2$? Explain briefly.

Solution: Consider $f(x) = |x-2|$.

$$f(x) = \begin{cases} 2-x & x < 2 \\ x-2 & x > 2 \end{cases}$$

$$\begin{aligned} \text{LHL: } \lim_{x\to 0^-} |x-2| &= 2 && \text{(since } |x-2| = 2-x \text{ when } x<2) \\ \text{RHL: } \lim_{x\to 0^+} |x-2| &= 2 && \text{(since } |x-2| = 2-x \text{ when } x<2) \\ \therefore \lim_{x\to 0} |x-2| &= 2 \end{aligned}$$

Let $g(x) = |x| - 2$.

$$g(x) = \begin{cases} -x-2 & x < 0 \\ x-2 & x > 0 \end{cases}$$

$$\begin{aligned} \text{LHL: } \lim_{x\to 0^-} |x|-2 &= -2 && \text{(since } |x| = -x \text{ when } x<0) \\ \text{RHL: } \lim_{x\to 0^+} |x|-2 &= -2 && \text{(since } |x| = x \text{ when } x>0) \\ \therefore \lim_{x\to 0} |x|-2 &= -2 \end{aligned}$$

Answer: $\boxed{\lim_{x\to 0} |x-2| \neq \lim_{x\to 0} |x| - 2}$

2. (1 point) Let $f(x) = \dfrac{6}{2+x}$. Find $f^{-1}(x)$.

Solution: Let $y = f(x)$.

$$\begin{aligned} y &= \frac{6}{x+2} \\ \implies y(x+2) &= 6 \\ \implies xy + 2y &= 6 \\ \implies xy &= 6 - 2y \\ \implies x &= \frac{6-2y}{y} \\ \implies f^{-1}(x) &= \frac{6-2x}{x} \end{aligned}$$

Answer: $\boxed{f^{-1}(x) = \frac{6-2x}{x}}$

3. (1 point) Let $f(x) = x^{x-1}$. Find $f'(x)$.

Solution: Let $y = x^{x-1}$.

$$\begin{aligned}
\ln y &= (x-1)\ln x && \text{(using the property of log } a = b^c \implies \ln a = c\ln b\text{)}\\
\implies \frac{1}{y}\frac{dy}{dx} &= \frac{d}{dx}((x-1)\ln x) && \text{(differentiating both sides w.r.t. } x\text{)}\\
\implies \frac{1}{y}\frac{dy}{dx} &= \ln x\frac{d}{dx}(x-1) + (x-1)\frac{d}{dx}(\ln x) && \text{(applying the product rule to the RHS)}\\
\implies \frac{1}{y}\frac{dy}{dx} &= \ln x + \frac{x-1}{x}\\
\implies \frac{dy}{dx} &= y\left(\ln x + \frac{x-1}{x}\right)\\
\implies \frac{dy}{dx} &= x^{x-1}\left(\ln x + \frac{x-1}{x}\right)
\end{aligned}$$

Answer: $\boxed{\frac{dy}{dx} = x^{x-1}\left(\ln x + \frac{x-1}{x}\right)}$

4. (1 point) Determine if the function $f(x) = x^2 - 4x + 3$ is increasing or decreasing in $[1, 3]$.

Solution: First derivative tells us whether (or where) a function is increasing or decreasing.

$$\begin{aligned}
f'(x) &= 2x - 4\\
\text{Condition for increasing function} : 2x - 4 &\geq 0\\
\therefore \text{ the function is increasing when } x &\geq 2\\
\text{Condition for decreasing function} : 2x - 4 &\leq 0\\
\therefore \text{ the function is decreasing when } x &\leq 2
\end{aligned}$$

Answer: $f(x)$ is decreasing in $[1, 2]$ and increasing in $[2, 3]$.

5. (1 point) Let $x^2y^3 + x^3y^2 = 5$. Find $\frac{dy}{dx}$.

Solution: Let $u = x^2y^3$, $v = x^3y^2$, and $c = 5$.

Applying the product rule, we get:

$$\begin{aligned}
u' &= 2xy^3 + 3x^2y^2\frac{dy}{dx} \\
v' &= 2x^3y\frac{dy}{dx} + 3x^2y^2 \\
c' &= 0 \\
u' + v' &= c' \\
\therefore \left[2xy^3 + 3x^2y^2\frac{dy}{dx}\right] + \left[2x^3y\frac{dy}{dx} + 3x^2y^2\right] &= 0 \\
\implies (2xy^3 + 3x^2y^2) + (3x^2y^2 + 2x^3y)\frac{dy}{dx} &= 0 \\
\implies (3x^2y^2 + 2x^3y)\frac{dy}{dx} &= -(2xy^3 + 3x^2y^2) \\
\implies \frac{dy}{dx} &= -\frac{(2xy^3 + 3x^2y^2)}{(3x^2y^2 + 2x^3y)} \\
\implies \frac{dy}{dx} &= -\frac{y}{x}\frac{(3x + 2y)}{(2x + 3y)}
\end{aligned}$$

Answer: $\boxed{\frac{dy}{dx} = -\frac{y}{x}\frac{(3x + 2y)}{(2x + 3y)}}$

6. (1 point) Let $f(x) = \sqrt{x} + 3$ and $g(x) = f^{-1}(x)$. Find $g'(5)$.

Solution: We know that: $\boxed{g'(a) = \frac{1}{f'(g(a))}}$

$\therefore g'(5) = \frac{1}{f'(g(5))}$

Let $g(5) = k \implies f(k) = 5.$

$$\begin{aligned}
\sqrt{k} + 3 &= 5 \\
\implies \sqrt{k} &= 5 - 3 \\
\implies \sqrt{k} &= 2 \\
\implies k &= 4 \\
f'(x) &= \frac{1}{2\sqrt{x}} \\
\implies f'(4) &= \frac{1}{2\sqrt{2}} \\
\implies f'(4) &= \frac{1}{4} \\
\implies g'(5) &= 4
\end{aligned}$$

Answer: $\boxed{g'(5) = 4}$

7. (1 point) Suppose that f and g are continuous on $[0, 4]$ and that $\int_0^4 (f(x) - g(x))dx = 2$ and $\int_0^4 (3f(x) - 4g(x))dx = 4$. Find $\int_0^4 (f(x) + g(x))dx$.

Solution: Let $\int_0^4 f(x)dx = a$ and $\int_0^4 g(x)dx = b$. Then,

$$\begin{aligned} a - b &= 2 \\ 3a - 4b &= 4 \\ 3a - 3b &= 6 && \text{(multiplying the first equation by 3)} \\ \implies b &= 2 && \text{(differencing the previous two equations to eliminate } a\text{)} \\ \implies a &= 4 \\ \int_0^4 (f(x) + g(x))dx &= \int_0^4 f(x)dx + \int_0^4 g(x)dx && \text{(applying the sum rule)} \\ \implies \int_0^4 (f(x) + g(x))dx &= 6 \end{aligned}$$

Answer : $\boxed{\int_0^4 (f(x) + g(x))dx = 6}$

8. (1 point) Compute: $\int (3x^2 + \frac{2}{x} + e^{3x})dx$

Solution:

$$\begin{aligned} \int (3x^2 + \frac{2}{x} + e^{3x})dx &= \int 3x^2 dx + \int \frac{2}{x} dx + \int e^{3x} dx && \text{(applying the sum rule)} \\ \implies \int (3x^2 + \frac{2}{x} + e^{3x})dx &= x^3 + 2\ln|x| + \frac{e^{3x}}{3} + C \end{aligned}$$

Answer : $\boxed{\int (3x^2 + \frac{2}{x} + e^{3x})dx = x^3 + 2\ln|x| + \frac{e^{3x}}{3} + C}$

Short Answer Questions-II

9. (3 points) Given the demand function for comedy shows on *Ruinmyshow*: $p = \dfrac{16}{q+3} - 3$,

   (a) ($\frac{1}{2}$ points) Compute the total revenue.

   Solution: Total revenue, $TR = p \cdot q$.

   $$\begin{aligned} p &= \frac{16}{q+3} - 3 \\ \implies TR &= \frac{16q}{q+3} - 3q \end{aligned}$$

   (b) ($\frac{1}{2}$ points) Compute the marginal revenue.

Solution: We know that $MR = \frac{d}{dq}(TR)$.

$$\begin{aligned}
TR &= \frac{16q}{q+3} - 3q \\
\text{Let } u &= 16q, \quad v = q+3 \\
\implies u' &= 16, \quad v' = 1 \\
MR &= \frac{vu' - uv'}{v^2} - 3 \\
\implies MR &= \frac{16(q+3) - 16q}{(q+3)^2} - 3 \\
\implies MR &= \frac{48}{(q+3)^2} - 3 \\
\implies MR &= \frac{48 - 3(q+3)^2}{(q+3)^2}
\end{aligned}$$

Answer: $\boxed{MR = \frac{48 - 3(q+3)^2}{(q+3)^2}}$

(c) (2 points) Compute the revenue-maximizing price and quantity.

Solution: We know that the revenue is maximized when $MR = 0$.

$$\begin{aligned}
48 - 3(q+3)^2 &= 0 \quad \text{(note that the denominator can't be zero.)} \\
\implies 3(q+3)^2 &= 48 \\
\implies (q+3)^2 &= 16 \\
\implies (q+3) &= \pm 4 \\
\implies q+3 &= 4 \quad \text{(discarding the negative value.)} \\
\implies q &= 1
\end{aligned}$$

Plugging the value into the demand equation, we get $p = \frac{16}{1+3} - 3$

$$\begin{aligned}
\implies p &= \frac{16}{4} - 3 \\
\implies p &= 4 - 3 \\
\implies p &= 1
\end{aligned}$$

Answer: The revenue-maximizing price is $\boxed{p = 1}$ and the quantity is $\boxed{q = 1}$.

10. (2+1 points) The total cost of producing *Phantom cigarettes* is $C(q) = 2q^2 + 10q + 32$. Find the value of q which minimizes the average cost. Show that the marginal cost is equal to the average cost at this point (where the average cost is being minimized).

Solution: We know that the average cost is:

$$AC(q) = \frac{C(q)}{q}$$

The average cost of producing *Phantom cigarettes* is:

$$AC(q) = 2q + 10 + \frac{32}{q}$$

We need to find the first derivative and set it to zero.

$$\begin{aligned}
\frac{d(AC(q))}{dq} &= \frac{d}{dq}\left(2q + 10 + \frac{32}{q}\right) \\
&= 2 - \frac{32}{q^2} \\
\text{FOC:} \quad \frac{d(AC(q))}{dq} &= 0 \\
\implies 2 - \frac{32}{q^2} &= 0 \\
\implies 2q^2 &= 32 \\
\implies q^2 &= 16 \\
\implies q^* &= 4 \\
\text{SOC:} \quad \frac{d^2(AC(q))}{dq^2} &> 0 && \text{(for minimum)} \\
\frac{d^2(AC(q))}{dq^2} &= \frac{64}{q^3} \\
\frac{64}{q^3} &> 0 \text{ when } q = 4
\end{aligned}$$

We also need to compute the marginal cost.

$$MC(q) = 4q + 10$$

When $q = 4$,

$$\begin{aligned}
\text{Average cost:} \quad AC(q = 4) &= 2q + 10 + \frac{32}{4} \\
&= 8 + 10 + 8 \\
&= 26 \\
\text{Marginal cost:} \quad MC(q = 4) &= 4(4) + 10 \\
&= 26
\end{aligned}$$

Answer: The quantity that minimizes the average cost is $\boxed{q^* = 4}$. When $q = 4$, $\boxed{AC = MC = 26}$.

11. (3 points) Let $f(x, y) = 6x^{1/3}y^{2/3}$.

(a) (1 point) Determine the degree of homogeneity.

Solution: The degree of homogeneity can be calculated using:

$$f(tx, ty) = t^k f(x, y)$$

$$\begin{aligned}
f(tx, ty) &= 6(tx)^{1/3}(ty)^{2/3} \\
&= 6t^{1/3}x^{1/3}t^{2/3}y^{2/3} \\
&= (t^{1/3+2/3})6(x^{1/3}y^{2/3}) \\
&= t^1 f(x, y) \\
\implies k &= 1
\end{aligned}$$

Answer: $f(x, y)$ is homogeneous of degree 1.

(b) (2 points) Compute all first and second order partial derivatives.

Solution:

f_x	$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(6x^{1/3}y^{2/3}\right)\\ &= 6\cdot\frac{1}{3}x^{-2/3}y^{2/3}\\ &= 2x^{-2/3}y^{2/3}\\ &= 2\left(\frac{y}{x}\right)^{2/3}\end{aligned}$
f_y	$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(6x^{1/3}y^{2/3}\right)\\ &= 6x^{1/3}\cdot\frac{2}{3}y^{-1/3}\\ &= 4x^{1/3}y^{-1/3}\\ &= 4\left(\frac{x}{y}\right)^{1/3}\end{aligned}$
f_{xx}	$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(2x^{-2/3}y^{2/3}\right)\\ &= 2y^{2/3}\cdot\left(-\frac{2}{3}\right)x^{-5/3}\\ &= -\frac{4}{3}x^{-5/3}y^{2/3}\\ &= -\frac{4}{3}\left(\frac{y^2}{x^5}\right)^{1/3}\end{aligned}$
f_{yy}	$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(4x^{1/3}y^{-1/3}\right)\\ &= 4x^{1/3}\cdot\left(-\frac{1}{3}\right)y^{-4/3}\\ &= -\frac{4}{3}x^{1/3}y^{-4/3}\\ &= -\frac{4}{3}\left(\frac{x}{y^4}\right)^{1/3}\end{aligned}$
f_{xy}	$\begin{aligned}\frac{\partial^2 f}{\partial x\partial y} &= \frac{\partial}{\partial y}\left(2x^{-2/3}y^{2/3}\right)\\ &= 2x^{-2/3}\cdot\frac{2}{3}y^{-1/3}\\ &= \frac{4}{3}x^{-2/3}y^{-1/3}\\ &= \frac{4}{3}\left(\frac{1}{x^2y}\right)^{1/3}\end{aligned}$
f_{yx}	$\begin{aligned}\frac{\partial^2 f}{\partial y\partial x} &= \frac{\partial}{\partial x}\left(4x^{1/3}y^{-1/3}\right)\\ &= 4y^{-1/3}\cdot\frac{1}{3}x^{-2/3}\\ &= \frac{4}{3}x^{-2/3}y^{-1/3}\\ &= \frac{4}{3}\left(\frac{1}{x^2y}\right)^{1/3}\end{aligned}$

12. (1+1+1 points) Let $U = 2\sqrt{x} + y$. Compute the marginal utilities and the marginal rate of substitution.

Solution:

$$MU_x = \frac{1}{\sqrt{x}}$$
$$MU_y = 1$$
$$MRS_{x,y} = \frac{MU_x}{MU_y}$$
$$MRS_{x,y} = \frac{1}{\sqrt{x}}$$

Long Answer Questions

13. (5 points) Consider $f(x,y) = \frac{x^3}{3} + \frac{y^3}{3} + x^2 + \frac{y^2}{2} - 3x - 6y + 5$. Find and classify all stationary points.

Solution: To find the stationary points, we first compute the first-order partial derivatives and set them equal to zero.

$$f_x = \frac{\partial f}{\partial x} = (x^2 + 2x - 3) = 0$$
$$f_y = \frac{\partial f}{\partial y} = (y^2 + y - 6) = 0$$

We have two quadratic equations to be solved.

$$(x+3)(x-1) = 0$$
$$(y+3)(y-2) = 0$$
$$x = 1, -3$$
$$y = 2, -3$$

The stationary points are the combinations of these x and y values:

$$\begin{array}{cc} (-3,-3) & (-3,2) \\ (1,-3) & (1,2) \end{array}$$

To classify the stationary points, we use the second derivative test, which requires the second-order partial derivatives.

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 2x + 2$$
$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = 2y + 1$$
$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 0$$

The determinant of the Hessian matrix is $D = f_{xx}f_{yy} - (f_{xy})^2 = (2x+2)(2y+1)$.

We evaluate the Hessian determinant $D = f_{xx}f_{yy} - (f_{xy})^2$ at each stationary point:

- If $D > 0$ and $f_{xx} > 0$: **Local Minimum**
- If $D > 0$ and $f_{xx} < 0$: **Local Maximum**
- If $D < 0$: **Saddle Point**
- If $D = 0$: **Test Inconclusive**

Stationary Point	f_{xx}	f_{yy}	D	Classification
$(-3,-3)$	-4	-5	$20 \quad (>0)$	Local Maximum
$(-3,2)$	-4	5	$-20 \quad (<0)$	Saddle Point
$(1,-3)$	4	-5	$-20 \quad (<0)$	Saddle Point
$(1,2)$	4	5	$20 \quad (>0)$	Local Minimum

14. (5 points) The demand for robots in *Tatooine* is given by $p = 20 - 2q$ and the supply of robots is given by $p = 4 + 2q$.

(a) (1 point) Compute the equilibrium price and quantity.

Solution: The equilibrium can be found out by setting demand = supply.

$$\begin{aligned} 20 - 2q &= 4 + 2q \\ \implies 4q &= 16 \\ \implies q^* &= 4 \\ \implies p^* &= 12 \end{aligned}$$

Answer: The equilibrium quantity is $\boxed{q^* = 4}$ and the equilibrium price is $\boxed{p^* = 12}$.

(b) (1+1 points) Compute the consumer surplus and producer surplus.

Solution: Given inverse demand ($D(q)$) and inverse supply ($S(q)$), we know that:

$$CS = \int_{q=0}^{q=q^*} D(q))dq - p^*q*$$
$$PS = p^*q* - \int_{q=0}^{q=q^*} (S(q))dq$$

We also know from the previous calculation that $p^* = 12$ and $q^* = 4$.

$$\begin{aligned} CS &= \int_0^4 (20 - 2q)dq - 48 \\ &= \Big|_0^4 (20q - q^2) - 48 \\ &= 80 - 16 - 48 \\ &= 16 \end{aligned}$$

$$\begin{aligned} PS &= 48 - \int_0^4 (4 + 2q)dq \\ &= 48 - \Big|_0^4 (4q + q^2) \\ &= 48 - (16 + 16) \\ &= 16 \end{aligned}$$

Answer: $\boxed{CS = 16}$, $\boxed{PS = 16}$

(c) (1+1 points) Now, suppose that the Damiyo (the ruler of Tatooine), sensing that the robots are valuable, announces a price floor of 14. Compute the new consumer surplus and producer surplus.

Solution: When $p = 14$, we should compute the quantity demanded and the quantity supplied.

$$\begin{aligned} 20 - 2q &= 14 \\ 2q &= 6 \\ \text{Demand:} \quad q &= 3 \\ 4 + 2q &= 14 \\ 2q &= 10 \\ \text{Supply:} \quad q &= 5 \end{aligned}$$

At this price, three robots will be sold in *Tatooine*.

$$\begin{aligned} CS &= \int_0^3 (20 - 2q) - 14 \times 3 \\ &= \Big|_0^3 (20q - q^2) - 42 \\ &= (60 - 9) - 42 \\ &= 9 \\ PS &= 42 - \int_0^3 (4 + 2q) dq \\ &= 42 - \Big|_0^3 (4q + q^2) \\ &= 42 - (12 + 9) \\ &= 21 \end{aligned}$$

Answer: $\boxed{CS = 9}$, $\boxed{PS = 21}$