

Mathematics of Finance-II

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1 Simple and compound interest

Let's say you borrow ₹1000 from a bank which offers loans at an interest rate of 5% per year. Then, after a year, you will have to return the one thousand rupees you originally borrowed plus an interest. ₹1000 you borrowed is known as the **principal**. You will also have to add ₹50 as the interest amount.

- **Simple interest rate:** If you deposit x into your bank account and the bank offers an interest rate of $r\%$ per year, your bank balance after a year is $x + rx = x(1 + r)$. Your bank balance after n years will be $x(1 + r)n$.
 - If I deposit ₹100 with a bank that offers a simple interest rate of 6%, how much will be my bank balance at the end of year five?
- **Compound interest rate:** If you deposit x into your bank account and the bank offers an interest rate that compounds annually at $r\%$, your bank balance after n years will be $x(1 + r)^n$.
 - If I deposit ₹100 with a bank that offers a compound interest rate of 6%, how much will be my bank balance at the end of year five?

2 Interest Periods and Effective Rates

In the examples above, we assumed that interest is accrued only annually to an account. However, banks across the world offer other interest schemes. Sometimes, interest is added quarterly or daily. So, we need to distinguish between the annual rate and the periodic rate (quarterly or daily). Let the principal be x that yields interest at the rate r per year, but the interest is credited every p periods per year. We assume that this interest is compounded. Then, what will be the amount after, let's say, t periods?

$$x\left(1 + \frac{r}{p}\right)^t$$

The periodic interest rate is given by:

$$\frac{r}{p}$$

An example: If you invest ₹1000 at 12% annual interest compounded monthly ($p = 12$), then how much will be your balance after 24 months? What is the periodic interest rate?

$$\text{Balance} = 1000\left(1 + \frac{0.12}{12}\right)^{24} = 1000 \times (1.01)^{24}$$

$$\text{Monthly interest rate} = \frac{r}{p} = \frac{0.12}{12} = 0.01 = 1\%$$

2.1 Effective Interest Rates

This is a useful measure to compare different investments (with varying interest rates and differing time periods). When interest is added p times during the year at the rate $\frac{r}{p}$ per period, then the effective rate, EAR , is defined as

$$EAR = \left(1 + \frac{r}{p}\right)^p - 1$$

We return to the previous example and compute the effective annual interest rate (a.k.a. EAR).

$$EAR = \left(1 + \frac{0.12}{12}\right)^{12} - 1 = (1 + 0.01)^{12} - 1 = 1.126825 - 1 = 0.126825 \approx 12.7\%$$

Now, let's think about a particular situation. You buy a financial product from a bank that offers you an interest rate of r_1 in the first year, r_2 in the second year, and so on. What will be the balance in your account after n years? What will be the EAR ?

The balance, after n years, will be:

$$\text{Balance} = P(1 + r_1)(1 + r_2) \cdots (1 + r_n)$$

The EAR will be:

$$EAR = [(1 + r_1)(1 + r_2) \cdots (1 + r_n)]^{\frac{1}{n}} - 1$$

What if we muddy the waters even further? You buy a financial product that offers you an interest rate of r_1 for p_1 periods (these periods could be daily, monthly, etc.) , r_2 for p_2 periods, and so on. There are, in total, P periods. What will be the balance after n years? What is the effective interest rate?

$$\text{Balance} = P(1 + r_1)^{p_1}(1 + r_2)^{p_2} \cdots (1 + r_m)^{p_m}$$

The EAR will be:

$$EAR = [(1 + r_1)^{p_1} \times (1 + r_2)^{p_2} \cdots (1 + r_m)^{p_m}]^{1/n} - 1$$

3 Continuous Compounding

So far, we have seen that interest is compounded every month or quarter or year. This means that the investment grows by a small amount during every period. However, with **continuous compounding**, we visualize interest being accrued to your account every moment. So, the number of periods becomes infinite. If the annual interest rate is r and there is continuous compounding of interest, then after t years, a principal of x_0 will become

$$x(t) = x_0 e^{rt}$$

With continuous compounding of interest at the rate r , the principal increases annually by e^r .

An example: If you invest ₹1000 at 12% annual rate compounded continuously for a year, what will be the balance after a year:

$$\text{Balance} = 1000 \times e^{0.12} = 1127.5$$

What is the EAR ? It is now 12.75%, slightly higher than the case with monthly compounding.

4 Present Value

4.1 Present Discounted Value

Let one year nominal interest rate be i_t . So, if you invest a rupee this year, you expect to get $\text{₹}(1 + i_t)$ next year. We can turn this argument around and instead ask the following question: how much a rupee next year is worth this year? If the interest rate or the discount rate is $p\%$ per year and $r = \frac{p}{100}$, and amount K that is due in t years has the *present discounted value* or PDV:

- with annual interest payments, $K(1 + r)^{-t}$
- with continuous compounding of interest, Ke^{-rt}

4.2 Present Value

4.2.1 Present Value of an Annuity

The present value of an annuity a per payment period for n periods, at the rate of r per period, where each payment is credited at the end of the period, is given by

$$P_n = \frac{a}{1 + r} + \cdots + \frac{a}{(1 + r)^n} = \frac{a}{r} \left[1 - \frac{1}{(1 + r)^n} \right]$$

where $r = \frac{p}{100}$.

4.2.2 Present Value: General Formula

In the above formulation, we assume that the interest rate as well as the future payments remain the same. Let's generalize this relationship. Denote today's payment as z_t , the payment next year by z_{t+1} , the payment two years later as z_{t+2} . The present value of this sequence of payments will be:

$$V_t = z_t + \frac{1}{(1 + i_t)} z_{t+1} + \frac{1}{(1 + i_t)(1 + i_{t+1})} z_{t+2} + \cdots$$

Two things are worth noting here.

- Each payment is multiplied by its respective discount factor.
- The further away you move in time, the smaller will be today's value of that payment.

4.2.3 Present Value of a Perpetuity

Let's now return to a special case where the interest rates are constant and the payments are also constant but they go on forever. We assume that, when you invest in this type of financial product, you receive your first payment next year. What will be the present value?

$$\begin{aligned}
V_t &= \frac{1}{(1+i)}z + \frac{1}{(1+i)(1+i)}z + \dots \\
&= \frac{1}{(1+i)} \left[1 + \frac{1}{(1+i)} + \dots \right] z && \text{(factoring out } \frac{1}{(1+i)} \text{)} \\
\Rightarrow V_t &= \frac{1}{(1+i)} \times \frac{1}{(1 - (\frac{1}{(1+i)}))} z && \text{(using the formula for an infinite GP)} \\
\Rightarrow V_t &= \frac{z}{i}
\end{aligned}$$

4.3 Future Value of an Annuity

If an amount a is credited to an account each period for n periods, earning r per period, the future value of the account is

$$F_n = \frac{a}{r} [(1+r)^n - 1]$$

4.4 Bonds, Yields, Yada Yada

We will discuss one special case¹ before concluding this section: bonds, G-Secs, and T-bills. Each of these are financial instruments of varying maturities² A bond, for instance, is a "debt instrument which an investor loans money to an entity (typically government or corporate) which borrows the funds for a definite period of time at a variable or a fixed interest rate"³. With each bond, we associate a **coupon payment** and a **face value**. The coupon payment is the amount of interest you, as an investor, receive at a fixed interval. The face value is the amount that you will receive at maturity. Bonds of different maturities have an associated interest rate known as the **yield to maturity** or the **yield**⁴.

- The ratio of coupon payments to the face value is called the **coupon rate**.
- The **current yield** is the ratio of coupon payment to the price of the bond.

An example: consider a bond that has a face value of ₹10,000. It is priced at ₹12,000, and you receive coupon payments of ₹2,000 each year for two years. What is the current yield of this bond and the yield to maturity associated with this bond?

$$\text{Current Yield} = \frac{\text{Coupon Payment}}{\text{Price of the bond}}$$

In this case, the current yield = $\frac{2000}{12000} \approx 16.7\%$.

To compute the YTM, you need to know the present value of the bond is given. It is the price of the bond! We have to compute the annualized interest rate.

$$12000 = \frac{2000}{(1+i)} + \frac{2000 + 10000}{(1+i)^2}$$

¹I have (rather shamelessly) used Blanchard's macroeconomics textbook to write this up.

²The maturity of a bond is the length of time which the bond promises to make payments to the holder of the bond.

³Source: RBI FAQs

⁴Bonds with shorter maturity (less than a year) have yields that are termed as **short-term interest rates** whereas bonds with longer maturity (more than a year) have yields that are called **long-term interest rates**. You can observe the health of the economy by plotting these yields against maturity!

After some math torture (or using a calculator), you will arrive at the following number: $YTM \approx 6.3\%$.

Appendix A: Basic Rules of Logarithms

Let $a > 0$, $a \neq 1$, and also let x and y be positive real numbers. Then we have the following rules:

- 1 **Product Rule:** $\log_a(xy) = \log_a x + \log_a y$
- 2 **Quotient Rule:** $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
- 3 **Power Rule:** $\log_a(x^b) = b \times \log_a x$
- 4 **Logarithm of one:** $\log_a 1 = 0$
- 5 **Logarithm of the base:** $\log_a a = 1$

Appendix B: Rules for Exponential Equations

- 1 **Inverse Property:** The natural logarithm (\ln) and exponential (e^x) are inverses. $\ln(e^x) = x$ and $e^{\ln(x)} = x$.
- 2 If you have an equation of the form: $\ln(x) = c$, the solution is $x = e^c$.
- 3 If you have an equation of the form $ae^{bx} = c$, the solution is $x = \frac{\ln(c/a)}{b}$.