# Mathematics of Finance-I

Sumit

July 30, 2025

## 1 Summation Operator

Consider a department with seven teams headed by seven managers. Each manager handles a team of 10. You'd want to measure the number of hours logged by each team. You can write:

$$T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7$$

Now imagine the tediousness of the task had there been 100 teams. We can use a compact way to write out such additions. Let $T_i$ be any team. $i$ runs from 1 to 7. We make use of the greek letter sigma ($\Sigma$) to represent the summation.

$$\sum_{i=1}^{i=7} T_i$$

Please note some stylistic elements of this operation.

- **Index** ($i$)- a placeholder required for the summation.
- **Lower limit** ($i = a$)- the starting value that the index takes. In the example above, $a = 1$.
- **Upper limit** ($i = b$)- the ending value that the index takes. In the example we have seen, $b = 7$.
- **Expression**- the expression that is being summed up. In the example above, it is $T_i$.

An example: Consider the following summation: $\sum_{i=1}^{i=3}(2i - 3)$. Write the expansion of the summation.

$$\begin{aligned}
&\sum_{i=1}^{i=3}(2i - 3) \\
&\qquad = \underbrace{[2(1) - 3]}_{i=1} + \underbrace{[2(2) - 3]}_{i=2} + \underbrace{[2(3) - 3]}_{i=3} \\
&\qquad = -1 + 1 + 3 \\
&\qquad = 3
\end{aligned}$$

## 1.1 Rules for Sums

- For two expressions $a$ and $b$, $\sum_{i=1}^{i=n}(a_i + b_i) = \sum_{i=1}^{i=n} a_i + \sum_{i=1}^{i=n} b_i$
- Given any expression $a$ and a constant $k$, $\sum_{i=1}^{i=n} ka_i = k\sum_{i=1}^{i=n} a_i$.
- For any constant $k$, $\sum_{i=1}^{i=n} k = kn$.

Let's use these rules to establish a useful relationship in statistics. Let there be $n$ numbers $x_1, x_2, \cdots, x_n$ and $m_x$ be their average. Therefore,

$$m_x = \frac{1}{n}\sum_{i=1}^{i=n} x_i$$

. We can show that:

$$\sum_{i=1}^{i=n}(x_i - m_x) = 0$$

and

$$\sum_{i=1}^{i=n}(x_i - m_x)^2 = \sum_{i=1}^{i=n}(x_i^2) - nm_x^2$$

We begin with the first term:

$$\begin{aligned}
&\sum_{i=1}^{i=n}(x_i - m_x) = \sum_{i=1}^{i=n}(x_i) - \sum_{i=1}^{i=n}(m_x) && \text{(using the first rule)}\\
\implies &\sum_{i=1}^{i=n}(x_i - m_x) = n \times m_x - \sum_{i=1}^{i=n}(m_x) && \text{(since } m_x = \frac{1}{n}\sum_{i=1}^{i=n} x_i\text{)}\\
\implies &\sum_{i=1}^{i=n}(x_i - m_x) = n \times m_x - \sum_{i=1}^{i=n}(m_x \times 1) \\
& && \text{(multiplying } m_x \text{ by 1 should keep the expression intact)}\\
\implies &\sum_{i=1}^{i=n}(x_i - m_x) = n \times m_x - m_x\sum_{i=1}^{i=n} 1 && \text{(since } m_x \text{ is a constant)}\\
\implies &\sum_{i=1}^{i=n}(x_i - m_x) = n \times m_x - m_x \times n && \text{(since for any constant } k\text{, the sum is } kn\text{)}\\
\implies &\sum_{i=1}^{i=n}(x_i - m_x) = 0
\end{aligned}$$

We can now proceed to the harder proof.

$$\sum_{i=1}^{i=n}(x_i - m_x)^2 = \sum_{i=1}^{i=n}(x_i^2 - 2x_i \times m_x + (m_x^2))$$

(expanding the expression using $(a-b)^2 = a^2 - 2ab + b^2$)

$$\implies \sum_{i=1}^{i=n}(x_i - m_x)^2 = \sum_{i=1}^{i=n} x_i^2 + \sum_{i=1}^{i=n} -2x_i \times m_x + \sum_{i=1}^{i=n} m_x^2 \qquad \text{(using the first rule)}$$

$$\implies \sum_{i=1}^{i=n}(x_i - m_x)^2 = \sum_{i=1}^{i=n} x_i^2 - 2m_x \sum_{i=1}^{i=n} x_i + \sum_{i=1}^{i=n} m_x^2 \qquad \text{(using the second rule)}$$

$$\implies \sum_{i=1}^{i=n}(x_i - m_x)^2 = \sum_{i=1}^{i=n} x_i^2 - 2m_x(n \times m_x) + \sum_{i=1}^{i=n} m_x^2 \qquad \text{(since we know that } m_x = \sum_{i=1}^{i=n}\text{)}$$

$$\implies \sum_{i=1}^{i=n}(x_i - m_x)^2 = \sum_{i=1}^{i=n} x_i^2 - 2nm_x^2 + \sum_{i=1}^{i=n} m_x^2 \times 1$$

(trivially multiplying the last term by 1 should keep the expression unchanged)

$$\implies \sum_{i=1}^{i=n}(x_i - m_x)^2 = \sum_{i=1}^{i=n} x_i^2 - 2nm_x^2 + m_x^2 \sum_{i=1}^{i=n} 1 \qquad \text{(following the second rule)}$$

$$\implies \sum_{i=1}^{i=n}(x_i - m_x)^2 = \sum_{i=1}^{i=n} x_i^2 - 2nm_x^2 + nm_x^2 \qquad \text{(following the third rule)}$$

$$\implies \sum_{i=1}^{i=n}(x_i - m_x)^2 = \sum_{i=1}^{i=n}(x_i^2) - nm_x^2$$

# 2 Series and Sequences

## 2.1 Arithmetic Series

An arithmetic series is a series of numbers in which you move from one number to the next one by *adding* a constant.

Some examples: $1, 2, 3$ (constant = 1); $1, 3, 5$ (constant = 2); $2, 5, 8$ (constant = 3).

Any general arithmetic progression (AP) will be of the following form:

$$a, a+d, a+2d, a+3d, \cdots$$

where $a$ is the first term of the AP and $d$ is the constant term, a.k.a, common difference.

Some features of an AP:

- The $n^{th}$ term of an AP is given by: $a + (n-1)d$.
- The sum of the first $n$ terms of an AP is given by: $\sum_{i=1}^{i=n} = \frac{n}{2}[2a + (n-1)d]$

## 2.2 Geometric Series

A geometric series or a geometric progression (GP) is a series of numbers in which you move from one number to the next one by *multiplying* a constant.

Some examples: $2, 4, 8$ (constant = 2); $5, 20, 80$ (constant = 4); $100, 20, 4$ (constant = $\frac{1}{5}$).
Any general geometric progression will be of the following form:

$$a, a \times r, a \times r^2, a \times r^3, \cdots$$

where $a$ is the first term, and $r$ the multiplicative constant, a.k.a., common ratio.
Some features of a GP:

- The $n^{th}$ term of a GP is given by: $a \times r^{n-1}$.
- The sum of the first $n$ terms of a GP is given by: $\sum_{i=1}^{i=n} = \frac{a \times (r^n - 1)}{r - 1}$.
  - **Special Case:** When $0 < r < 1$ and $n \to \infty$, the sum is given by:

$$\sum_{i=1}^{\infty} = \frac{a}{1 - r}$$