Functions-II

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September 24, 2025

1 Types of Functions

1.1 Linear Functions

A simple linear function is defined as

$$y = mx + c$$

where m is the slope of the line and c the intercept.
An example:

$$y = 2x + 1$$

This function represents a line with a slope = 2 and intercept = 1.
The equation of a line that passses through two points (x_1, y_1) and (x_2, y_2) can be written as:

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

An example:
What is the equation of the line that passes through $(0, 3)$, $(-1, 1)$?
Step 1: Compute the slope of the line.

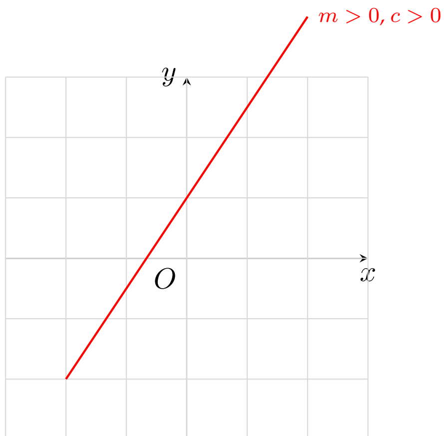
$$\begin{aligned}
m &= \frac{(1-3)}{(-1-0)} \\
m &= \frac{-2}{-1} \\
m &= 2
\end{aligned}$$

Step 2: Write the equation of the line using *any* given point.

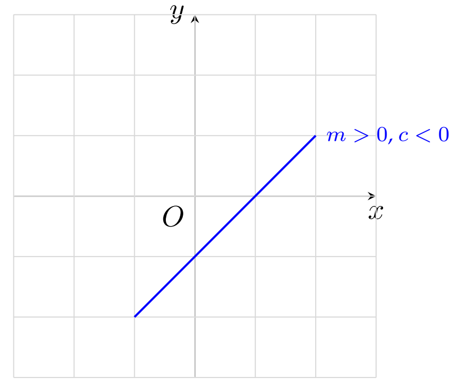
$$\begin{aligned}
(y - 3) &= 2 \times (x - 0) && \text{(using given values)} \\
y - 3 &= 2x \\
y &= 2x + 3
\end{aligned}$$

1.1.1 Graph

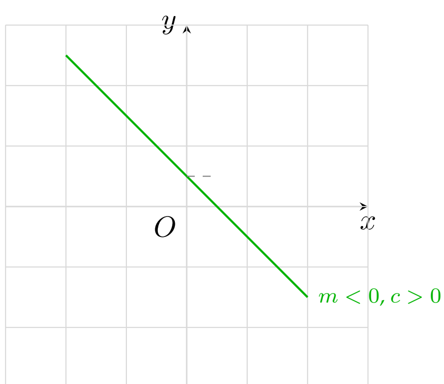
Figure 1 shows a graph with many possible scenarios of slope and intercept.



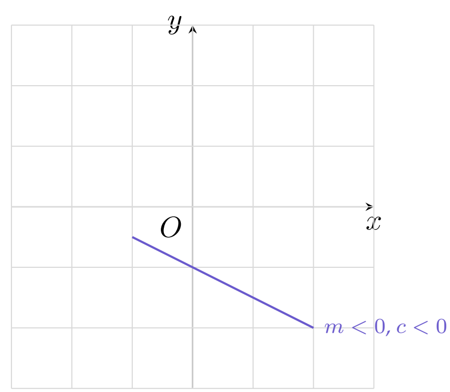
(a) $m > 0, c > 0$



(b) $m > 0, c < 0$



(c) $m < 0, c > 0$



(d) $m < 0, c < 0$

Figure 1: $y = mx + c$ with varying slope (m) and intercept (c).

1.1.2 Supply and Demand

Consider a generic market with the following demand and supply setup.

$$\begin{aligned} \text{Supply} &: P = \alpha + \beta \times Q_S \\ \text{Demand} &: P = \gamma - \delta \times Q_D \end{aligned}$$

where $\alpha, \beta, \gamma, \delta > 0$.
What are the equilibrium price and quantity?
At equilibrium, $\boxed{\text{Demand} = \text{Supply}}$. Therefore,

$$\begin{aligned} \alpha + \beta Q^* &= \gamma - \delta Q^* \\ \implies (\beta + \delta) Q^* &= \gamma - \alpha \\ \implies Q^* &= \frac{\gamma - \alpha}{\beta + \delta} \\ P^* &= \alpha + \beta Q^* \\ \implies P^* &= \alpha + \beta \frac{\gamma - \alpha}{\beta + \delta} \\ \implies P^* &= \frac{\alpha\delta + \beta\gamma}{\beta + \delta} \end{aligned}$$

1.1.3 Linear Inequalities

A general form of linear inequality is given by:

$$\begin{aligned} \alpha x + \beta y &\leq b \\ \alpha x + \beta y &\geq b \end{aligned}$$

1. Budget Sets

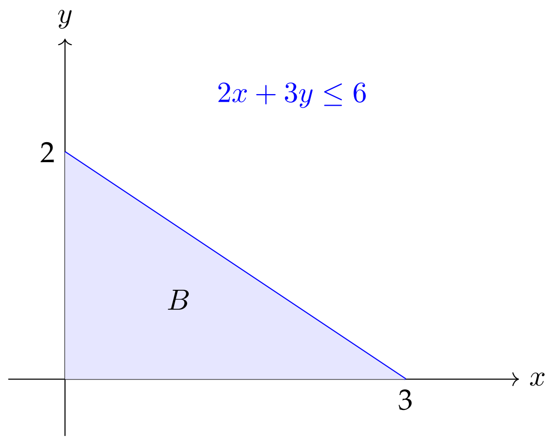
   A person's income is M and they wish to spend it on espressos (x) and cakes (y). The price of a cup of espresso is p_x and the price of a slice of a cake is p_y. Suppose that this person (who wishes not to be named) wants to consume non-zero quantities of coffee and cake $(x \geq 0, y \geq 0)$. What is their budget set?

   $$B = \{(x, y) : p_x x + p_y y \leq M, x \geq 0, y \geq 0\}$$

   An example: Consider an individual with income, $M = 6$, who wishes to spend it on two goods x and y. The price of good x is 2 and the price of good y is 3. What is the budget set?

   The budget set is defined as:

   $$B = \{(x, y) \in \mathbb{R}^2 : 2x + 3y \leq 6, x \geq 0, y \geq 0\}$$



1.2 Quadratic Functions

A general quadratic function is defined as:

$$f(x) = ax^2 + bx + c$$

1.2.1 Graph of a Quadratic Function

The graph of the function $f(x) = ax^2 + bx + c$ is called a **parabola**. Informally, when $a < 0$, we get an inverted-U and when $a > 0$, we have a U-shaped parabola. Figure 2 represents all possible scenarios.

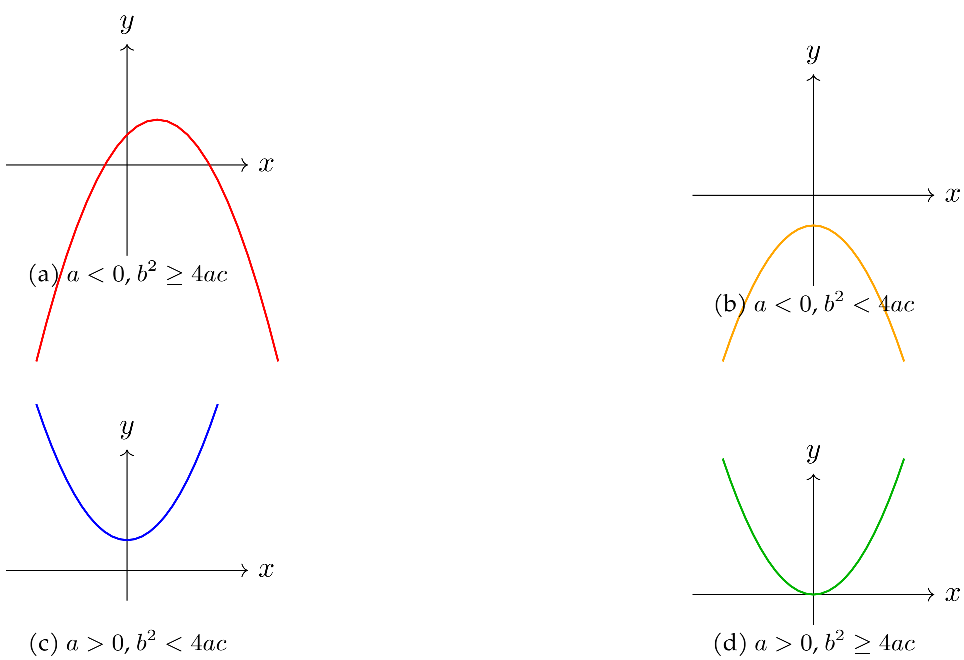


Figure 2: The graph of the parabola $y = ax^2 + bx + c$ for different values of a, b, and c.
(a) Inverted-U shaped with real roots, (b) Inverted-U shaped with no real roots,
(c) U-shaped with no real roots, (d) U-shaped with (at least) one real root.

1.2.2 Maximum and Minimum Values of a Quadratic Function

While we haven't yet formally started discussing maxima and minima, we will discuss max-min for quadratic functions (informally).

- If $a > 0$, then $f(x) = ax^2 + bx + c$ has its **minimum** value at $x = \dfrac{-b}{2a}$.
- if $a < 0$, the $f(x) = ax^2 + bx + c$ has its **maximum** value at $x = \dfrac{-b}{2a}$

An example:
Consider two functions: $f(x) = -3x^2 + 30x - 60$ and $g(x) = 2x^2 + 6x + 24$.
For $f(x)$, $a < 0$. Therefore, the function $f(x)$ achieves its maximum value. $a = -3$, $b = 30$.

$$\begin{aligned} x^* &= \frac{-30}{2 \times -1} \\ \implies x^* &= 15 \end{aligned}$$

For $g(x)$, $a > 0$. Therefore, the function $g(x)$ achieves its minimum value. $a = 2$, $b = 6$.

$$\begin{aligned} x^* &= \frac{6}{2 \times 2} \\ \implies x^* &= \frac{3}{2} \end{aligned}$$

1.3 Polynomials

A general polynomial of degree n is defined as:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \tag{1}$$

According to the fundamental theorem of algebra, every polynomial of degree n can be written as a product of polynomials of degree 1 and 2.

Suppose that $a_0, a_1, \ldots, a_n$ are all integers. Then all possible integer roots of Equation 1 must be factors of the constant term a_0.

1.3.1 Factoring Polynomials

Let $P(x)$ and $Q(x)$ be two polynomials such that the degree of $P(x)$ is at least as large as that of $Q(x)$. Then, there always exist unique polynomials $q(x)$ and $r(x)$ such that

$$P(x) = Q(x) \times q(x) + r(x)$$

Therefore,

$$\frac{P(x)}{Q(x)} = q(x) + \frac{r(x)}{Q(x)}$$

An example:
Let $P(x) = x^3 - 4x^2 + 8x + 16$ and $Q(x) = x - 2$.

$$
\begin{array}{r}
x^2 - 2x \quad + 4 \\
x-2 \overline{\smash{\big)}\; x^3 - 4x^2 + 8x + 16} \\
\underline{-x^3 + 2x^2} \qquad\qquad\quad \\
-2x^2 + 8x \qquad\; \\
\underline{2x^2 - 4x} \qquad\; \\
4x + 16 \\
\underline{-4x \quad + 8} \\
24
\end{array}
$$

Therefore,

$$
\begin{aligned}
x^3 - 4x^2 + 8x + 16 &= (x-2)(x^2 - 2x + 4) + 24 \\
\frac{x^3 - 4x^2 + 8x + 16}{x-2} &= x^2 - 2x + 4 + \frac{24}{x-2}
\end{aligned}
$$

If a polynomial $P(x)$ has a factor $x - a$, then:

$$P(a) = 0$$

An example:

Consider a polynomial $P(x) = x^2 - kx + 4$. What should be the value of k such that $P(x)$ is divisible by $x - 2$.

We will use the rule we wrote above. If $P(x)$ is divisble by $x - 2$, then

$$
\begin{aligned}
P(2) &= 0 \\
\implies (2)^2 - k(2) + 4 &= 0 \\
\implies 4 - 2k + 4 &= 0 \\
\implies 2k &= 8 \\
\implies k &= 4
\end{aligned}
$$