Course Name- Engineering Physics
Course Code- BSC 102
UNIT-1 Mathematical Physics
Syllabus:- Concepts of Del Operator; Gradient of scalar, divergence and Curl of vector, Gauss divergence theorem and Stokes theorem.

Scalar and vector quantities

Scalar quantities:- The scalar quantities have only magnitude and no direction. In order to specify scalar quantities, we need
(a). A standard of measurement of physical quantities i.e. a unit.
(b). A number which states how many time the given unit is contained in that physical quantity i.e. numerical values.

For example, mass of a stone is 10 kilogram is a compete statement, there is no mention of direction, hence mass is a scalar quantity. Kg is a unit of the physical quantity mass and 10 is the numerical value i.e. kilogram is contained ten times in the given stone. Other examples of scalar quantities are length, time, temperature, speed, etc

Vector quantities:- Vector quantities have both magnitude and direction. In order to specify vector quantities, we need.

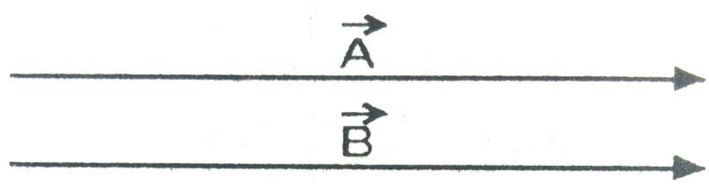
a). A standard of measurement of physical quantity i.e. a unit.
b). A number which states how many times the given unit is contained in that physical quantity i.e. numerical value.
c). The statement of direction.

For example, velocity of a body is 10m/sec due east is complete statement. Other examples of vector quantities are acceleration, force, momentum, electric field intensity, etc.

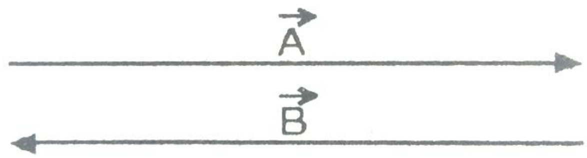
Representation of vectors:- A vector can be represented graphically by a line ab pointed in the direction from a to b and is denoted by $\overrightarrow{ab}$ or $\vec{A}$ or a bold letter $\boldsymbol{A}$. The magnitude of the vector quantity is given by the length of the line and is denoted by $|\overrightarrow{ab}|$ or $|\vec{A}|$.

Types of vectors:-

1. Equal vectors:- Two vectors $\vec{A}$ and $\vec{B}$ are said to be equal if they have equal magnitude and the same direction. Vectors $\vec{A}$ and $\vec{B}$ are equal vectors implies $\vec{A} = \vec{B}$.



2. Negative vectors:- The negative vector of a given vector is that vector whose magnitude is equal to that of the given vector but direction is opposite. Vector $\vec{B}$ is equal in magnitude but opposite in direction to that of vector $\vec{A}$. Thus $\vec{A} = -\vec{B}$



3. Null vector:- A vector whose magnitude is zero is called null or zero vector and it may have any direction. Null vector is pulling a rope from both the end with equal forces at opposite direction.

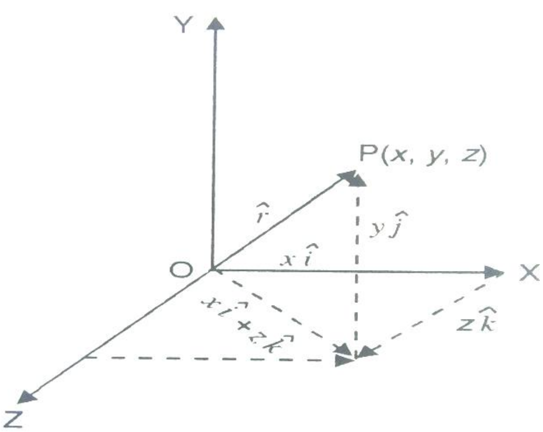
4. Unit vector:- A vector whose magnitude is unity and direction is same as that of the given vector is called a unit vector.

   As vector $\vec{A}$ can be written as $\vec{A} = |A|\hat{A}$ where $|A|$ represent the magnitude of the vector $\vec{A}$ and $\hat{A}$ gives the direction of $\vec{A}$ and is called a unit vector. It is defined as
   $\hat{A} = \frac{\vec{A}}{|A|}$, $\hat{A}$ is read as A hat or A caret.

Unit vectors in Cartesian coordinates system:- The most common unit vector of right handed Cartesian coordinate system are shown in figure. $\hat{\imath}$ is a unit vector along X-axis, $\hat{\jmath}$ and $\hat{k}$ are unit vectors along Y and Z axes. In other words, $\hat{\imath}\,,\hat{\jmath}$ and $\hat{k}$ unit vectors give the direction of a vector along X, Y and Z axes, respectively. Since $\hat{\imath}\,,\hat{\jmath}$ and $\hat{k}$ are perpendicular to each other, therefore, they are called orthogonal unit vectors.

Position vector:- A vector drawn from the origin of the coordinate system to any point in the space is called the position vector of the point with respect to the origin.

If the coordinates of a point P in the space be (x,y,z) with respect to the origin of the coordinate system, then the vector drawn from the origin O to the point P is called a position vector. It is represented by $\overrightarrow{OP} = \vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}$z, and $|\vec{r}| = (x^2 + y^2 + z^2)^{1/2}$ is the magnitude of the position vector $\vec{r}$

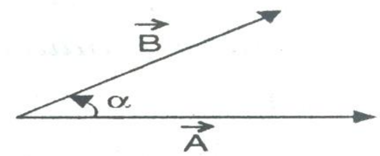


Product of vectors:- When two vectors are multiplied, we get either a scalar quantity or a vector quantity. The product of vectors which gives the scalar quantity is called scalar product or dot product, while the product of vectors which gives the vector quantity is called vector product or cross product.

Scalar product or dot product:- The scalar product or dot product of two vectors $\vec{A}\ and\ \vec{B}$ is defined as the product of the magnitudes of the vectors and the cosine of the minimum angle between their direction.

Let the angle between the direction of the vectors $\vec{A}$ and $\vec{B}$ be α, then the scalar product or dot product of vectors $\vec{A}$ and $\vec{B}$ is given by

$\vec{A}.\vec{B} = |\vec{A}||\vec{B}| Cos\ \alpha$



Examples

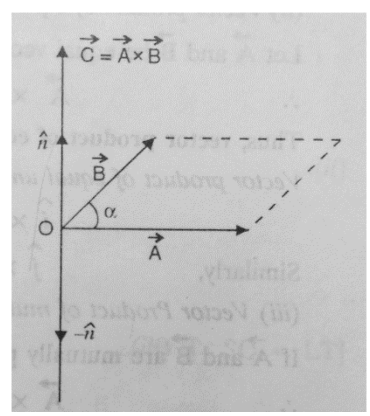
1. Work is the scalar quantity, which is the dot product of force ($\vec{F}$) and displacement ($\vec{S}$) thus, W= $\vec{F}.\vec{S}$
2. Electric flux is a scalar quantity, which is the dot product of electric field intensity ($\vec{E}$) and the area vector (d$\vec{S}$)
$$\therefore d\phi_E = \vec{E}.d\vec{S}$$
3. Magnetic flux is a scalar quantity, which is the dot product of magnetic flux density ($\vec{B}$) and the area vector (d$\vec{S}$)
$$\therefore d\phi_B = \vec{B}.d\vec{S}$$

Properties of scalar product

1. Scalar product is commutative. i.e. $\vec{A}.\vec{B} = \vec{B}.\vec{A}$
2. Scalar product or dot product of mutually perpendicular unit vectors is $\hat{\imath}.\hat{\jmath} = |1||1|Cos\ 90 = 0$, Similarly $\hat{\jmath}.\hat{\imath} = \hat{\jmath}.\hat{k} = \hat{k}.\hat{\imath} = 0$
3. Scalar product of equal unit vectors is $\hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = |1||1|Cos0 = 1$

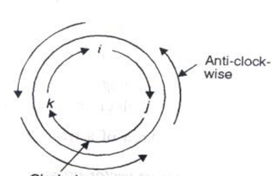
Vector product or cross product of two vectors:- The vector product of two vectors $\vec{A}$ and $\vec{B}$ is defined as the vector having a magnitude equal to the product of the magnitudes of the vectors $\vec{A}$ and $\vec{B}$ and the sine of the angle between their directions and the direction perpendicular to the plane containing vectors $\vec{A}$ and $\vec{B}$.

Let $\vec{A}$ and $\vec{B}$ be the two vectors in a plane inclined to an angle α with each other, the vector product of $\vec{A}$ and $\vec{B}$ is given by $\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|Sin\ \alpha\hat{n} = \vec{C}$ where $\hat{n}$ is the unit vector indicating the direction of the resultant vector $\vec{C}$ of vectors $\vec{A}$ and $\vec{B}$. The magnitude of vector $\vec{C}$ is given by $|\vec{C}| = |\vec{A}||\vec{B}|Sin\ \alpha$



Properties of vector product

1. Vector product of vectors does not obey the commutative law, i.e.
$$\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$$



2. Vector product of equal unit vectors is zero. $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$
3. Vector product of mutually perpendicular unit vectors when taken clockwise is positive, however when taken anticlockwise is negative i.e.
   $\hat{\imath} \times \hat{\jmath} = \hat{k},\ \hat{\jmath} \times \hat{k} = \hat{\imath},\ \ \hat{k} \times \hat{\imath} = \hat{\jmath}$
   $\hat{\jmath} \times \hat{\imath} = -\hat{k}, \hat{k} \times \hat{\jmath} = -\hat{\imath},\ \ \hat{\imath} \times \hat{k} = -\hat{\jmath}$

The Scalar triple product:- Let $\vec{A}$, $\vec{B}$ and $\vec{C}$ be the three vectors, then the scalar product or dot product of vector $\vec{A}$ with the vector product of the vectors $\vec{B}$ and $\vec{C}$ is called the scalar triple product of vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$. It is denoted by $\vec{A} .(\vec{B} \times \vec{C})$.

The vector triple product:- Let $\vec{A}$, $\vec{B}$ and $\vec{C}$ be the three vectors, then the vector product of vector $\vec{A}$ with the vector product of the vectors $\vec{B}$ and $\vec{C}$ is called the vector triple product of vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$. It is denoted by $\vec{A} \times (\vec{B} \times \vec{C})$. The vector triple product gives the vector quantity. The $\vec{A} \times (\vec{B} \times \vec{C})$ is normal to the plane containing $\vec{A}$ and $(\vec{B} \times \vec{C})$.

Note:- The scalar triple product $\vec{A} . (\vec{B} \times \vec{C})$ can be written as $\vec{A} .\vec{B} \times \vec{C}$ but the vector triple product $\vec{A} \times (\vec{B} \times \vec{C})$ cannot be written as $\vec{A} \times \vec{B} \times \vec{C}$

Identity:- $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} . \vec{C}) - \vec{C}(\vec{A} . \vec{B})$

Scalar and vector fields:-

If a physical quantity (scalar or vector) varies from point to point in a space, it can be expressed as a continuous function of the position of a point in the region of space, such a continuous function is called the function of position. The region in which this function specifies the physical quantity is called the field. Types of fields

1. Scalar field corresponding to scalar quantities
2. Vector field corresponding to vector quantities

Scalar field:- The region of space in which a scalar quantity has unique value at every point is called the scalar field. A scalar field can be expressed as $\phi = \phi(\vec{r}) = \phi(x, y, z)$, where ϕ is a scalar quantity. Examples:- distribution of temperature in space. In scalar field, surfaces corresponding to equal values of the scalar quantities can be drawn. These surfaces are known as level surfaces and each level surface has a definite constant value. Two level surfaces cannot cross each other and there is a constant difference between the values of the scalar quantity for any two successive level surfaces.

Vector field:- The region of space in which a vector quantity has a unique value (both magnitude and direction) at every point is called the vector field. Vector field is represented by $\vec{A} = \vec{A}(r) = \vec{A}(x, y, z)$, where $\vec{A}$ is a vector quantity. Example:- Electric field intensity, gravitational force on a body.

Derivatives of products

1. Derivatives of a product of a scalar and a vector both being the function of scalar. Let $k\vec{A}$ be the product of scalar k and the vector $\vec{A}$, where k and $\vec{A}$ both are the function of time 't' then
$\frac{d}{dt}(k\vec{A}) = \frac{dk}{dt}\vec{A} + k\frac{d\vec{A}}{dt}$, if $\vec{A} = \hat{\imath}Ax + \hat{\jmath}Ay + \hat{k}Az$

$\frac{d}{dt}(k\vec{A}) = \frac{dk}{dt}(\hat{\imath}Ax + \hat{\jmath}Ay + \hat{k}Az) + k\frac{d}{dt}(\hat{\imath}Ax + \hat{\jmath}Ay + \hat{k}Az)$
2. Derivatives of a scalar or dot product of two vectors, both the vectors are the function of a scalar time t
$\frac{d}{dt}(\vec{A}.\vec{B}) = \frac{d\vec{A}}{dt}.\vec{B} + \vec{A}.\frac{d\vec{B}}{dt}$
3. $\frac{d}{dt}(\vec{A} \times \vec{B}) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$
4. Derivative of a scalar triple product of vectors
$\frac{d}{dt}(\vec{A}.(\vec{B} \times \vec{C})) = \frac{d\vec{A}.}{dt}.(\vec{B} \times \vec{C}) + \vec{A}.\ (\frac{d\vec{B}}{dt} \times \vec{C}) + \vec{A}.\ (\vec{B} \times \frac{d\vec{C}}{dt})$

5. Derivative of a vector triple product of vectors
$\frac{d}{dt}(\vec{A} \times (\vec{B} \times \vec{C})) = \frac{d\vec{A}.}{dt} \times (\vec{B} \times \vec{C}) + \vec{A} \times (\frac{d\vec{B}}{dt} \times \vec{C}) + \vec{A} \times (\vec{B} \times \frac{d\vec{C}}{dt})$

Partial differentiation of scalar and vector fields:- When a given physical quantity (scalar or vector) is the function of more than one variable, then its total differential may be written in terms of its partial differential coefficients.

$$\vec{E} = \vec{E}(x, y, z)$$
$$\overrightarrow{dE} = \frac{\partial\vec{E}}{\partial x}dx + \frac{\partial\vec{E}}{\partial y}dy + \frac{\partial\vec{E}}{\partial z}dz$$

Where, $\frac{\partial\vec{E}}{\partial x}$, $\frac{\partial\vec{E}}{\partial y}$ and $\frac{\partial\vec{E}}{\partial z}$ represent the rate of change of $\vec{E}$ with respect to x, y and z respectively.

Now, consider a scalar quantity, $V = V(x, y, z)$

Therefore, $dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$, where $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$ represent the rate of change of V with respect to x, y and z respectively.

The Del operator:- The del operator $\vec{\nabla}$ is called the vector differential operator. This operator was introduced by Sir Hamilton and is read as nabla or del. Del $\vec{\nabla}$ is defined as $\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$. When a scalar field ϕ is operated upon by $\vec{\nabla}$ operator we get a vector field.

Laplacian operator:- The operator $\vec{\nabla}.\vec{\nabla} = \nabla^2$ is called the laplacian operator.

$\vec{\nabla}.\vec{\nabla} = (\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z})\ .\ (\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z})$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Gradient of a scalar field:- When a scalar field ϕ is operated upon by a $\vec{\nabla}$ operator, we get a vector function which is called the gradient of a scalar field (ϕ). The gradient of a scalar field ϕ is written as

grad $\phi = \vec{\nabla}\phi = (\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z})\phi$

$= \hat{\imath}\frac{\partial \phi}{\partial x} + \hat{\jmath}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$

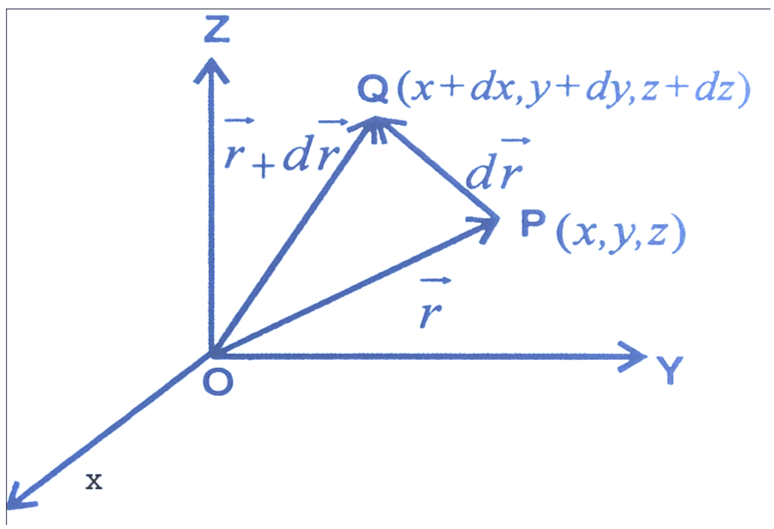
Gradient of ϕ is a vector quantity.

Physical meaning of gradient of the scalar function:- Consider a scalar function ϕ associated with every point of a certain region in space.

Let P and Q be two points in this region having coordinates (x, y, z) and $(x + dx, y + dy, z + dz)$, respectively. The position vectors of P and Q are

$\vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z$ ...(1)

And $\vec{r} + d\vec{r} = \hat{\imath}(x + dx) + \hat{\jmath}(y + dy) + \hat{k}(z + dz)$ ... (2)



Subtracting equation (1) from (2), we get

$d\vec{r} = \hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz$ ... (3)

Let change of ϕ in going from P to Q is $d\phi$, then

$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$

$$d\phi = \left[\hat{\imath}\frac{\partial \phi}{\partial x} + \hat{\jmath}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right].(\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz)$$

$d\phi = \vec{\nabla}\phi . d\vec{r}$, *where* $\vec{\nabla}\phi = \hat{\imath}\frac{\partial \phi}{\partial x} + \hat{\jmath}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$ and $d\vec{r} = (\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz)$

The quantity $\vec{\nabla}\phi$ is called gradient of ϕ or grad ϕ. It is a vector function of (x, y, z).

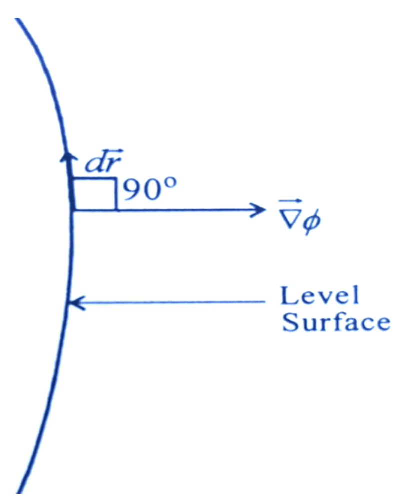
Physical interpretation of grad ϕ:- Since $d\phi = \vec{\nabla}\phi . d\vec{r}$, if θ is the angle between $\vec{\nabla}\phi$ *and* $d\vec{r}$, then

$d\phi = \vec{\nabla}\phi . d\vec{r}$

$|d\phi| = |\nabla\phi||dr| \cos\theta,$

$\left|\frac{d\phi}{dr}\right| = |\nabla\phi| \cos\theta,$

Now, $\left|\frac{d\phi}{dr}\right|$ now will be maximum if $\theta = 0$ i.e. $\cos 0 = 1$ it implies that



$$\left|\frac{d\phi}{dr}\right|_{Max} = |\nabla\phi|$$

Or

$|\nabla\phi| = \left|\frac{d\phi}{dr}\right|_{Max}$

Thus magnitude of grad $\phi$is equal to the maximum rate of change of ϕ

Generally, gradient means change in the value of a quantity (as temperature, pressure, or concentration) with change in a given variable and especially per unit on a linear scale.

Thus the angle between vector $\vec{\nabla}\phi$ and displacement $d\vec{r}$ is zero. Thus, the direction of $\vec{\nabla}\phi$ is the same as the direction of displacement along which the rate of change of ϕ is maximum.

If the displacement $d\vec{r}$ is taken along the level surface i.e. a surface on which ϕ is constant, the $d\phi = 0$

Since, $d\phi = \vec{\nabla}\phi . d\vec{r}$, for $d\phi = 0$

$\vec{\nabla}\phi . d\vec{r} = 0$. When dot product is zero, it implies that $\vec{\nabla}\phi$ is normal to the level surface.

Numerical

Que1. If $\vec{r} = \hat{\imath}x + \hat{\jmath}\hat{y}$ + $\hat{k}$z is position vector at any point, calculate the gradient $\left[\frac{1}{|r|}\right]$.

Sol. If $\vec{r} = \hat{\imath}x + \hat{\jmath}\hat{y}$ + $\hat{k}$z, then $|r| = (x^2 + y^2 + z^2)^{1/2}$

Or $\left[\frac{1}{|r|}\right] = \frac{1}{(x^2+y^2+z^2)^{1/2}} = (x^2 + y^2 + z^2)^{-1/2}$

Now, grad $\left[\frac{1}{|r|}\right] = \vec{\nabla}\left[\frac{1}{|r|}\right] = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{-1/2}$

$= \hat{\imath}\frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{-1/2} + \hat{\jmath}\frac{\partial}{\partial y}(x^2 + y^2 + z^2)^{-1/2} + \hat{k}\frac{\partial}{\partial z}(x^2 + y^2 + z^2)^{-1/2}$

$=-\frac{1}{2}\hat{\imath}(x^2 + y^2 + z^2)^{\frac{1}{2}-1}(2x) - \frac{1}{2}\hat{\jmath}(x^2 + y^2 + z^2)^{\frac{1}{2}-1}(2y) \quad -\frac{1}{2}\hat{k}(x^2 + y^2 + z^2)^{\frac{1}{2}-1}(2z)$

$= \frac{-\hat{\imath}x}{(x^2+y^2+z^2)^{3/2}} + \frac{-\hat{\jmath}y}{(x^2+y^2+z^2)^{3/2}} + \frac{-\hat{k}z}{(x^2+y^2+z^2)^{3/2}}$

$= \frac{-(\hat{\imath}x + \hat{\jmath}y + \hat{k}z)}{(x^2+y^2+z^2)^{3/2}} = \frac{-\vec{r}}{|r|^3}$

grad $\left[\frac{1}{|r|}\right] = \frac{-\vec{r}}{|r|^3}$

Que2. If $\vec{r} = \hat{\imath}x + \hat{\jmath}\hat{y}$ + $\hat{k}$z is position vector, calculate the gradient(r^n)

Sol. Given $\vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z$ then
$|r| = (x^2 + y^2 + z^2)^{1/2}$ and

$$|r|^n = (x^2 + y^2 + z^2)^{n/2}$$

$\therefore$ gradient(r^n) = $\vec{\nabla}|r|^n$ = $\left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{n/2}$

$=\hat{\imath}\frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{n/2} + \hat{\jmath}\frac{\partial}{\partial y}(x^2 + y^2 + z^2)^{n/2} + \hat{k}\frac{\partial}{\partial z}(x^2 + y^2 + z^2)^{n/2}$

$= \hat{\imath}\frac{n}{2}(x^2 + y^2 + z^2)^{\frac{n}{2}-1}(2x) + \hat{\jmath}\frac{n}{2}(x^2 + y^2 + z^2)^{\frac{n}{2}-1}(2y) + \hat{k}\frac{n}{2}(x^2 + y^2 + z^2)^{\frac{n}{2}-1}(2z)$

$= n(\hat{\imath}x + \hat{\jmath}y + \hat{k}z)\,(x^2 + y^2 + z^2)^{\frac{n-2}{2}}$

Thus gradient(r^n)= $n|r|^{n-2}\vec{r}$

Que3. If $\vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z$ and $\vec{A}$ is a constant vector show that $\vec{\nabla}(\vec{A}.\vec{r}) = \vec{A}$
Sol. Here $\vec{r} = \hat{\imath}x + \hat{\jmath}\hat{y} + \hat{k}z$ and $\vec{A} = \hat{\imath}Ax + \hat{\jmath}Ay + \hat{k}Az$
Now first we calculate $\vec{A}.\vec{r} = (\hat{\imath}x + \hat{\jmath}\hat{y} + \hat{k}z).(\hat{\imath}Ax + \hat{\jmath}Ay + \hat{k}Az)$

$$\vec{A}.\vec{r} = xAx + yAy + zAz$$

Now, $\vec{\nabla}(\vec{A}.\vec{r}) = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xAx + yAy + zAz)$
$= (\hat{\imath}Ax + \hat{\jmath}Ay + \hat{k}Az) = \vec{A}$

Que4. Prove that $\vec{\nabla}(uv) = u\vec{\nabla}v + v\vec{\nabla}u$, where u *and* v both are scalars.
Sol. $\vec{\nabla}(uv) = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(uv)$
$= \hat{\imath}\frac{\partial}{\partial x}(uv) + \hat{\jmath}\frac{\partial}{\partial y}(uv) + \hat{k}\frac{\partial}{\partial z}(uv)$

$u\left(\hat{\imath}\frac{\partial v}{\partial x} + \hat{\jmath}\frac{\partial v}{\partial y} + \hat{k}\frac{\partial v}{\partial z}\right) + v\left(\hat{\imath}\frac{\partial u}{\partial x} + \hat{\jmath}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z}\right) = u\vec{\nabla}v + v\vec{\nabla}u$

Hence $\vec{\nabla}(uv) = u\vec{\nabla}v + v\vec{\nabla}u$

Que5. If $\phi(x, y, z) = 3x^2y - y^3z^2$, Calculate the gradient of ϕ at the point (1,-2,-1)
Sol. Given $\phi(x, y, z) = 3x^2y - y^3z^2$
Now grad ϕ = $\vec{\nabla}\phi = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(3x^2y - y^3z^2)$
$= \hat{\imath}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{\jmath}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$
$= \hat{\imath}(6xy) + \hat{\jmath}(3x^2 - 3y^2z^2) + \hat{k}\,(-2y^3z)$
At point (1,-2,-1)

$$\vec{\nabla}\phi(1, -2, -1) = -12\hat{\imath} - 9\hat{\jmath} - 16\hat{k}$$

Que6. Find $\vec{\nabla}\phi$ if $\phi = x^{\frac{3}{2}} + y^{\frac{3}{2}} + z^{\frac{3}{2}}$
Sol. $\vec{\nabla}\phi = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^{\frac{3}{2}} + y^{\frac{3}{2}} + z^{\frac{3}{2}}\right)$

$= \hat{\imath}\frac{\partial}{\partial x}\left(x^{\frac{3}{2}} + y^{\frac{3}{2}} + z^{\frac{3}{2}}\right) + \hat{\jmath}\frac{\partial}{\partial y}\left(x^{\frac{3}{2}} + y^{\frac{3}{2}} + z^{\frac{3}{2}}\right) + \hat{k}\frac{\partial}{\partial z}\left(x^{\frac{3}{2}} + y^{\frac{3}{2}} + z^{\frac{3}{2}}\right)$

$= \hat{\imath}\frac{3}{2}x^{\frac{3}{2}-1} + \hat{\jmath}\frac{3}{2}y^{\frac{3}{2}-1} + \hat{k}\frac{3}{2}z^{\frac{3}{2}-1}$

$= \frac{3}{2}\left(\hat{\imath}x^{\frac{1}{2}} + \hat{\jmath}y^{\frac{1}{2}} + \hat{k}z^{\frac{1}{2}}\right)$

Que7. Find $\vec{\nabla}\phi$ if $\phi = \log|r|$

Sol. Since $\phi = \log|r| = \log(x^2 + y^2 + z^2)^{\frac{1}{2}} = \frac{1}{2}\log(x^2 + y^2 + z^2)$

$\vec{\nabla}\phi = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{2}\log(x^2 + y^2 + z^2)\right)$

$= \frac{1}{2}\left(\hat{\imath}\frac{\partial}{\partial x}\log(x^2 + y^2 + z^2) + \hat{\jmath}\frac{\partial}{\partial y}\log(x^2 + y^2 + z^2) + \hat{k}\frac{\partial}{\partial z}\log(x^2 + y^2 + z^2)\right)$

$= \frac{1}{2}\left(\hat{\imath}\frac{2x}{(x^2+y^2+z^2)}) + \hat{\jmath}\frac{2y}{(x^2+y^2+z^2)} + \hat{k}\frac{2z}{(x^2+y^2+z^2)}\right) = \frac{(\hat{\imath}x+\hat{\jmath}y+\hat{k}z)}{(x^2+y^2+z^2)} = \frac{\vec{r}}{|r|^2}$

Que8. If $\vec{r} = \hat{\imath}x + \hat{\jmath}\hat{y} + \hat{k}$z is position vector, calculate $\vec{\nabla}r$.

Sol. Since $\vec{r} = \hat{\imath}x + \hat{\jmath}\hat{y} + \hat{k}$z, then $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$

$$\vec{\nabla}r = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left((x^2 + y^2 + z^2)^{\frac{1}{2}}\right)$$

$=\left(\hat{\imath}\frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{\frac{1}{2}} + \hat{\jmath}\frac{\partial}{\partial y}(x^2 + y^2 + z^2)^{\frac{1}{2}} + \hat{k}\frac{\partial}{\partial z}(x^2 + y^2 + z^2)^{\frac{1}{2}}\right)$

$=\Big(\hat{\imath}\frac{1}{2}(x^2 + y^2 + z^2)^{\frac{1}{2}-1}(2x) + \hat{\jmath}\frac{1}{2}(x^2 + y^2 + z^2)^{\frac{1}{2}-1}(2y) +$

$\hat{k}\frac{1}{2}(x^2 + y^2 + z^2)^{\frac{1}{2}-1}(2z)\Big)$

$= \frac{(\hat{\imath}x+\hat{\jmath}y+\hat{k}z)}{(x^2+y^2+z^2)^{\frac{1}{2}}} = \frac{\vec{r}}{|r|}$

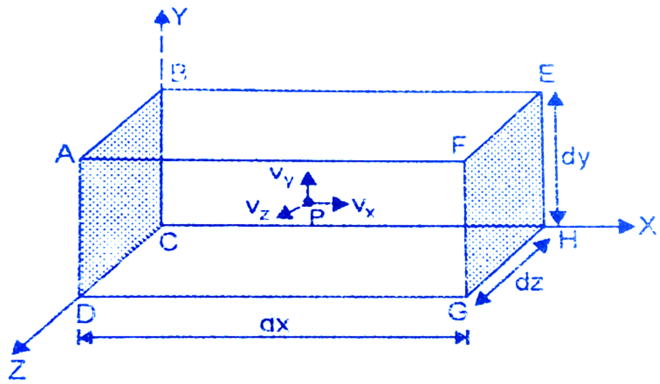
Que9. Evaluate $\vec{\nabla}\psi$, where $\psi = (x^2 + y^2 + z^2)e^{-\sqrt{x^2+y^2+z^2}}$

Divergence of a vector field:- If a vector $\vec{A}$ is operated with $\vec{\nabla}$ scalar, it is called divergence of a vector $\vec{A}$, thus $\vec{A} = \hat{\imath}Ax + \hat{\jmath}Ay + \hat{k}A$z, thus

$\vec{\nabla}.\vec{A} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(\hat{\imath}Ax + \hat{\jmath}Ay + \hat{k}A$z)

$=\left(\frac{\partial Ax}{\partial x} + \frac{\partial Ay}{\partial y} + \frac{\partial Az}{\partial z}\right)$ = a scalar, $\vec{\nabla}.\vec{A}$ is called divergence of a vector field, $\vec{A}$

Physical significance of divergence or expression for divergence in terms of Cartesian coordinates:- Consider a volume element with sides dx, dy and dz along x, y and z direction enclosing the point P at its center, where the expression for divergence of vector field $\vec{V}$ is to be obtained.



Consider the flow of some fluid through this volume element($dxdydz$), this fluid flows in or out of the volume element through all its six faces. Volume of fluid entering the face ABCD per second = $Vx\ (Area\ of\ surface\ ABCD)$

= $Vxdydz$

Where Vx is the velocity of fluid along x-axis.

Suppose if there is source of fluid inside this volume element, then velocity of fluid and volume of the fluid coming out also increases.

Thus, Outward flow of fluid through face $EFGH$ per second = $(Vx + dVx)dydz$

Net outward flow of fluid per second along x-axis

=$(Vx + dVx)dydz - Vxdydz = dVxdydz$

Since $dVx = \frac{\partial Vx}{\partial x} dx$+0+0

(Since fluid is flowing along x-axis hence the y and z components are constant)

Net outward flow of fluid per second along x-axis = $\frac{\partial Vx}{\partial x} dxdydz = \frac{\partial Vx}{\partial x} dV$

Similarly, Net outward flow of fluid per second along y-axis = $\frac{\partial Vy}{\partial y} dxdydz = \frac{\partial Vy}{\partial y} dV$

And Net outward flow of fluid per second along z-axis = $\frac{\partial Vz}{\partial z} dxdydz = \frac{\partial Vz}{\partial z} dV$

Net outward flow of fluid through volume element per second = $\frac{\partial Vx}{\partial x} dV + \frac{\partial Vy}{\partial y} dV +$ $\frac{\partial Vz}{\partial z} dV$

$$= \left(\frac{\partial Vx}{\partial x} + \frac{\partial Vy}{\partial y} + \frac{\partial Vz}{\partial z}\right) dV$$

Net outward flow of fluid through the volume element per second per unit volume = $\left(\frac{\partial Vx}{\partial x} + \frac{\partial Vy}{\partial y} + \frac{\partial Vz}{\partial z}\right) = \vec{\nabla}.\vec{V}$

Thus, divergence of a function at a point is the net outward flow per unit volume per second at that point. This gives physical meaning of divergence.

1. If Div$\vec{V}$ at a point is positive, it means the point is a source of fluid or fluid is expanding.
2. If Div$\vec{V}$ at a point is negative, it means the point is a sink of fluid or the fluid is contracting.

3. If $\text{Div}\vec{V} = 0$, it means the point is neither source or a sink of fluid. In other words the volume of the fluid entering the volume element is equal to the volume of fluid leaving the volume element.

If the divergence of a fluid is zero than the field is said to be solenoidal field e.g. magnet field is solenoidal because flux lines for the magnetic field leaving from the north pole and entering from the south pole and then go from south pole to north pole from the interior body of the magnet. In other words, they form closed curves, there is no starting or end point of these lines. If magnetic field is denoted by $\vec{B}$, then divergence of $\vec{B}$ is zero, i.e. $\vec{\nabla}.\vec{B}= 0$

Que1. Calculate $\text{Div}\vec{r}$, where $\vec{r}$ is a position vector and $\vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z$.

Sol. Given $\vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z$

Now $\text{Div}\vec{r} = \vec{\nabla}.\vec{r} = \left(\frac{\hat{\imath}\partial}{\partial x} + \frac{\hat{\jmath}\partial}{\partial y} + \frac{\hat{k}\partial}{\partial z}\right)(\hat{\imath}x + \hat{\jmath}y + \hat{k}z) = \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = 1+1+1= 3$

Que2. Show that $\vec{\nabla}.(r^n\vec{r}) = (n+3)r^n, where\ \vec{r}$ is a position vector and $\vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z$.

Sol. Given $\vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z$ then

$|r| = (x^2+y^2+z^2)^{1/2}$ and

$$|r|^n = (x^2+y^2+z^2)^{n/2}$$

$\therefore \vec{\nabla}.(r^n\vec{r}) = \left(\frac{\hat{\imath}\partial}{\partial x} + \frac{\hat{\jmath}\partial}{\partial y} + \frac{\hat{k}\partial}{\partial z}\right)(x^2+y^2+z^2)^{n/2}(\hat{\imath}x + \hat{\jmath}y + \hat{k}z)$

$= \frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2}x + \frac{\partial}{\partial y}(x^2+y^2+z^2)^{n/2}y + \frac{\partial}{\partial z}(x^2+y^2+z^2)^{n/2}z$

$=\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2x)(x) + (x^2+y^2+z^2)^{\frac{n}{2}} + \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2y)(y) + (x^2+y^2+z^2)^{\frac{n}{2}} + \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2z)(z) + (x^2+y^2+z^2)^{\frac{n}{2}}$

$=n(x^2+y^2+z^2)^{\frac{n}{2}-1}(x^2+y^2+z^2)^1 + 3(x^2+y^2+z^2)^{\frac{n}{2}}$

$=n(x^2+y^2+z^2)^{\frac{n}{2}-1+1} + 3(x^2+y^2+z^2)^{\frac{n}{2}}$

$=n(x^2+y^2+z^2)^{\frac{n}{2}} + 3(x^2+y^2+z^2)^{\frac{n}{2}}$

$=(n+3)(x^2+y^2+z^2)^{\frac{n}{2}}$

$=(n+3)r^n$

Que4. Show that $\vec{\nabla}.\left(\frac{\vec{r}}{|r|^3}\right) = 0$, where $\vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z$

$$|r| = (x^2+y^2+z^2)^{1/2}$$
$$|r|^3 = (x^2+y^2+z^2)^{3/2}$$

$$\vec{\nabla}.\left(\frac{\vec{r}}{|r|^3}\right) = \left(\frac{\hat{\imath}\partial}{\partial x}+\frac{\hat{\jmath}\partial}{\partial y}+\frac{\hat{k}\partial}{\partial z}\right)\left(\frac{(\hat{\imath}x+\hat{\jmath}y+\hat{k}z)}{(x^2+y^2+z^2)^{3/2}}\right)$$

$=\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right)+\frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right)+\frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right)$

$=\frac{\partial}{\partial x}(x)(x^2+y^2+z^2)^{-3/2}+\frac{\partial}{\partial y}(y)(x^2+y^2+z^2)^{-3/2}+\frac{\partial}{\partial z}(z)(x^2+y^2+z^2)^{-3/2}$

$=(x)\left(\frac{-3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2x)+1(x^2+y^2+z^2)^{-3/2}+(y)\left(\frac{-3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2y)+1(x^2+y^2+z^2)^{-3/2}+(z)\left(\frac{-3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2z)+1(x^2+y^2+z^2)^{-3/2}$

$=\frac{-3\,(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}}+3(x^2+y^2+z^2)^{-3/2}$

$= -3(x^2+y^2+z^2)^{1-5/2}+3(x^2+y^2+z^2)^{-3/2}$
$= -3(x^2+y^2+z^2)^{-3/2}+3(x^2+y^2+z^2)^{-3/2}=0$

Que5. Show that $\vec{\nabla}.(\vec{A}+\vec{B}) = \vec{\nabla}.\vec{A}+\vec{\nabla}.\vec{B}$
Sol. Let $\vec{A} = \hat{\imath}Ax+\hat{\jmath}Ay+\hat{k}Az$ and
$\vec{B} = \hat{\imath}Bx+\hat{\jmath}By+\hat{k}Bz$

$\therefore \vec{\nabla}.(\vec{A}+\vec{B})= \left(\frac{\hat{\imath}\partial}{\partial x}+\frac{\hat{\jmath}\partial}{\partial y}+\frac{\hat{k}\partial}{\partial z}\right).\left((Ax+Bx)\hat{\imath}+(Ay+By)\hat{\jmath}+(Az+Bz)\hat{k}\right)$

$=\left(\frac{\partial}{\partial x}(Ax+Bx)+\frac{\partial}{\partial y}(Ay+By)+\frac{\partial}{\partial z}(Az+Bz)\right)$

$=\left(\frac{\partial}{\partial x}(Ax)+\frac{\partial}{\partial y}(Ay)+\frac{\partial}{\partial z}(Az)\right)+\left(\frac{\partial}{\partial x}(Bx)+\frac{\partial}{\partial y}(By)+\frac{\partial}{\partial z}(Bz)\right)$

$\left(\frac{\hat{\imath}\partial}{\partial x}+\frac{\hat{\jmath}\partial}{\partial y}+\frac{\hat{k}\partial}{\partial z}\right).\left(Ax\hat{\imath}+Ay\hat{\jmath}+Az\hat{k}\right)+$

$$\left(\frac{\hat{\imath}\partial}{\partial x}+\frac{\hat{\jmath}\partial}{\partial y}+\frac{\hat{k}\partial}{\partial z}\right).\left(Bx\hat{\imath}+By\hat{\jmath}+Bz\hat{k}\right)$$

$= \vec{\nabla}.\vec{A}+\vec{\nabla}.\vec{B}$

Que6. If $\vec{V} = x^2z\hat{\imath}-2y^3z^2\hat{\jmath}+xy^2z\hat{k}$, then find $\vec{\nabla}.\vec{V}$ at a point $(1,-1,1)$.
Sol. Given $\vec{V} = x^2z\hat{\imath}-2y^3z^2\hat{\jmath}+xy^2z\hat{k}$

$\vec{\nabla}.\vec{V} = \left(\frac{\hat{\imath}\partial}{\partial x}+\frac{\hat{\jmath}\partial}{\partial y}+\frac{\hat{k}\partial}{\partial z}\right).\left(x^2z\hat{\imath}-2y^3z^2\hat{\jmath}+xy^2z\hat{k}\right)$

$=\frac{\partial}{\partial x}(x^2z)+\frac{\partial}{\partial y}(-2y^3z^2)+\frac{\partial}{\partial z}(xy^2z)$

$\vec{\nabla}.\vec{V} = 2xz-6y^2z^2+xy^2$

$\vec{\nabla}.\vec{V}$ at point $(1,-1,1)$ $\;i.e., x=1, y=-1$ and $z=1$

$\vec{\nabla}.\vec{V} = 2(1)(1) - 6(-1)^2 + (-1)^2 = 2 - 6 + 1 = -3$

Que7. If $\phi = 2x^3yz^2$ then find $Div(grad\phi)$

Sol. $grad\phi = \vec{\nabla}\phi = \left(\frac{\hat{\imath}\partial}{\partial x} + \frac{\hat{\jmath}\partial}{\partial y} + \frac{\hat{k}\partial}{\partial z}\right)(2x^3yz^2)$

$= \left(\frac{\hat{\imath}\partial}{\partial x}(2x^3yz^2) + \frac{\hat{\jmath}\partial}{\partial y}(2x^3yz^2) + \frac{\hat{k}\partial}{\partial z}(2x^3yz^2)\right) = \hat{\imath}6x^2yz^2 + \hat{\jmath}2x^3z^2 + \hat{k}4x^3yz$

Now, $Div(grad\phi) = \vec{\nabla}.(\vec{\nabla}\phi) = \left(\frac{\hat{\imath}\partial}{\partial x} + \frac{\hat{\jmath}\partial}{\partial y} + \frac{\hat{k}\partial}{\partial z}\right)(\hat{\imath}6x^2yz^2 + \hat{\jmath}2x^3z^2 + \hat{k}4x^3yz)$

$= \frac{\partial}{\partial x}(6x^2yz^2) + \frac{\partial}{\partial y}(2x^3z^2) + \frac{\partial}{\partial z}(4x^3yz)$

$= 12xyz^2 + 0 + 4x^3y$

Thus $Div(grad\phi) = 12xyz^2 + 0 + 4x^3y$

Que8. Show that $\vec{\nabla}.(\vec{\nabla}\phi) = \nabla^2\phi$

Sol. $\vec{\nabla}.(\vec{\nabla}\phi) = \vec{\nabla}.\left(\frac{\hat{\imath}\partial}{\partial x} + \frac{\hat{\jmath}\partial}{\partial y} + \frac{\hat{k}\partial}{\partial z}\right)\phi$

$= \left(\frac{\hat{\imath}\partial}{\partial x} + \frac{\hat{\jmath}\partial}{\partial y} + \frac{\hat{k}\partial}{\partial z}\right).\left(\frac{\hat{\imath}\partial\phi}{\partial x} + \frac{\hat{\jmath}\partial\phi}{\partial y} + \frac{\hat{k}\partial\phi}{\partial z}\right)$

$= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)$

$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$

$= \nabla^2\phi$,

where $\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$ is called Laplacian operator

Hence $\vec{\nabla}.(\vec{\nabla}\phi) = \nabla^2\phi$

Que9. Show that If $\vec{A} = 3y^2z^2\hat{\imath} + 4x^3z^2\hat{\jmath} - 3x^2y^2\hat{k}$ is a solenoidal vector.

Sol. The vector $\vec{A}$ will be solenoidal vector if div$\vec{A}$= 0 i.e. $\vec{\nabla}.\vec{A}=0$

$= \left(\frac{\hat{\imath}\partial}{\partial x} + \frac{\hat{\jmath}\partial}{\partial y} + \frac{\hat{k}\partial}{\partial z}\right).\left(3y^2z^2\hat{\imath} + 4x^3z^2\hat{\jmath} - 3x^2y^2\hat{k}\right)$

$\frac{\partial}{\partial x}(3y^2z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(-3x^2y^2)$

$=0+0-0=0$

Thus $\vec{A}$ is a solenoidal vector.

Que10. Evaluate Divergence of the given field, $\vec{F} = 2x^2\hat{\imath} + (4y^2 + z^2)\hat{\jmath} - 3yz\hat{k}$

Sol. Div$\vec{F} = \vec{\nabla}.\vec{F}$

$= \left(\frac{\hat{\imath}\partial}{\partial x} + \frac{\hat{\jmath}\partial}{\partial y} + \frac{\hat{k}\partial}{\partial z}\right).\left(2x^2\hat{\imath} + (4y^2 + z^2)\hat{\jmath} - 3yz\hat{k}\right)$

$=\frac{\partial}{\partial x}(2x^2)+\frac{\partial}{\partial y}(4y^2+z^2)+\frac{\partial}{\partial z}(-3yz)=4x+8y-3y=4x+5y$

Que11. Find the value of a constant 'b' so that the vector field $\vec{A}=x^2\hat{\imath}+(y-2xy)\hat{\jmath}+(x+bz)\hat{k}$ is a solenoidal.

Sol. The vector $\vec{A}$ will be solenoidal vector if div$\vec{A}$= 0 i.e. $\vec{\nabla}.\vec{A}$=0

Thus $\left(\frac{\hat{\imath}\partial}{\partial x}+\frac{\hat{\jmath}\partial}{\partial y}+\frac{\hat{k}\partial}{\partial z}\right).\left(x^2\hat{\imath}+(y-2xy)\hat{\jmath}+(x+bz)\hat{k}\right)=0$

$\frac{\partial}{\partial x}(x^2)+\frac{\partial}{\partial y}(y-2xy)+\frac{\partial}{\partial z}(x+bz)=0$

$$2x+(1-2x)+b=0$$
$$or\ b=-1$$

Que11. Evaluate Div$\hat{r}$

Sol. We have $\vec{r}=|r|\hat{r}$ or $\hat{r}=\frac{\vec{r}}{|r|}$

Now, Div $\hat{r}=\left(\frac{\hat{\imath}\partial}{\partial x}+\frac{\hat{\jmath}\partial}{\partial y}+\frac{\hat{k}\partial}{\partial z}\right).\left(\frac{(\hat{\imath}x+\hat{\jmath}y+\hat{k}z)}{(x^2+y^2+z^2)^{1/2}}\right)$

$=\left(\frac{\hat{\imath}\partial}{\partial x}+\frac{\hat{\jmath}\partial}{\partial y}+\frac{\hat{k}\partial}{\partial z}\right).\left((\hat{\imath}x+\hat{\jmath}y+\hat{k}z)(x^2+y^2+z^2)^{-1/2}\right)$

$=\frac{\partial}{\partial x}(x)(x^2+y^2+z^2)^{-1/2}+\frac{\partial}{\partial y}(y)(x^2+y^2+z^2)^{-1/2}+\frac{\partial}{\partial z}(z)(x^2+y^2+z^2)^{-1/2}$

$=(x)\left(\frac{-1}{2}\right)(x^2+y^2+z^2)^{-3/2}(2x)+1(x^2+y^2+z^2)^{-1/2}+(y)\left(\frac{-1}{2}\right)(x^2+y^2+z^2)^{-3/2}(2y)+1(x^2+y^2+z^2)^{-1/2}+(z)\left(\frac{-1}{2}\right)(x^2+y^2+z^2)^{-3/2}(2z)+1(x^2+y^2+z^2)^{-1/2}$

$=-(x^2+y^2+z^2)^{-3/2}(x^2+y^2+z^2)+3(x^2+y^2+z^2)^{-1/2}$

$=-(x^2+y^2+z^2)^{-3/2+1}+3(x^2+y^2+z^2)^{-1/2}$

$=2(x^2+y^2+z^2)^{-1/2}$

$$\left(\frac{2}{(x^2+y^2+z^2)^{1/2}}\right)=\frac{2}{|r|}$$

Evaluate 12 If $\vec{r}$ is a position vector and $\vec{r}=\hat{\imath}x+\hat{\jmath}y+\hat{k}z$ evaluate grad$\frac{1}{|r|^2}$

Sol. We have $|r|=(x^2+y^2+z^2)^{1/2}$

Thus, $|r|^2=(x^2+y^2+z^2)^{2/2}=(x^2+y^2+z^2)^1$

Now grad$\frac{1}{|r|^2}=\left(\frac{\hat{\imath}\partial}{\partial x}+\frac{\hat{\jmath}\partial}{\partial y}+\frac{\hat{k}\partial}{\partial z}\right).(x^2+y^2+z^2)^{-1}$

$=\left(\frac{\hat{\imath}\partial}{\partial x}+\frac{\hat{\jmath}\partial}{\partial y}+\frac{\hat{k}\partial}{\partial z}\right).((x^2+y^2+z^2)^{-1})$

$=\hat{\imath}(-1)(x^2+y^2+z^2)^{-2}(2x)+\hat{\jmath}(-1)(x^2+y^2+z^2)^{-2}(2y)+\hat{k}(-1)(x^2+y^2+z^2)^{-2}(2z)$

$=-(2x\hat{\imath}+2y\hat{\jmath}+2z\hat{k})(x^2+y^2+z^2)^{-2}$

$=\frac{-2(\hat{\imath}x+\hat{\jmath}y+\hat{k}z)}{(x^2+y^2+z^2)^2}=\frac{-2\vec{r}}{|r|^4}$

As $|r|=(x^2+y^2+z^2)^{1/2}$
$|r|^4=\ (x^2+y^2+z^2)^{4/2}=(x^2+y^2+z^2)^2$

Curl of a vector:- If a vector field $\vec{A}$ is operated with $\vec{\nabla}$ vectorially, it is called the curl of the vector field $\vec{A}$ and is written as $\vec{\nabla}\times\vec{A}$. Let a vector field $\vec{A}$ is written as $\vec{A}=\hat{\imath}Ax+\hat{\jmath}Ay+\hat{k}Az$, then

$\vec{\nabla}\times\vec{A}=\left(\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times(\hat{\imath}Ax+\hat{\jmath}Ay+\hat{k}Az)$

$=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ Ax & Ay & Az\end{vmatrix}=\hat{\imath}\left(\frac{\partial Az}{\partial y}-\frac{\partial Ay}{\partial z}\right)-\hat{\jmath}\left(\frac{\partial Az}{\partial x}-\frac{\partial Ax}{\partial z}\right)+\hat{k}\left(\frac{\partial Ay}{\partial y}-\frac{\partial Ax}{\partial y}\right)$

$$=\hat{\imath}\left(\frac{\partial Az}{\partial y}-\frac{\partial Ay}{\partial z}\right)+\hat{\jmath}\left(\frac{\partial Ax}{\partial z}-\frac{\partial Az}{\partial x}\right)+\hat{k}\left(\frac{\partial Ay}{\partial x}-\frac{\partial Ax}{\partial y}\right)$$

Thus, $x, y\ and\ z$ components of Curl $\vec{A}$ are given by

$\left|(\vec{\nabla}\times\vec{A})_x\right|=\left(\frac{\partial Az}{\partial y}-\frac{\partial Ay}{\partial z}\right)$

$\left|(\vec{\nabla}\times\vec{A})_y\right|=\left(\frac{\partial Ax}{\partial z}-\frac{\partial Az}{\partial x}\right)$

$\left|(\vec{\nabla}\times\vec{A})_z\right|=\left(\frac{\partial Ay}{\partial x}-\frac{\partial Ax}{\partial y}\right)$

Curl of a vector field is a vector quantity. Since Curl is associated in hydrodynamics with rotation of liquid, so it is also termed as rot. If curl of vector field is zero such a vector field is called irrotational vector field.

Numerical

Que1. Show that Curl $\vec{r}=0$ i.e. $\vec{\nabla}\times\vec{r}=0$ where $\vec{r}$ is a position vector and $\vec{r}=\hat{\imath}x+\hat{\jmath}y+\hat{k}z$.

Sol. Given $\vec{r}=\hat{\imath}x+\hat{\jmath}y+\hat{k}z$

Now curl$\vec{r}=\vec{\nabla}\times\vec{r}=\left(\frac{\hat{\imath}\partial}{\partial x}+\frac{\hat{\jmath}\partial}{\partial y}+\frac{\hat{k}\partial}{\partial z}\right)\times(\hat{\imath}x+\hat{\jmath}y+\hat{k}z)$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{\imath}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = 0$$

Que2. If $\vec{A} = \hat{\imath}xz^2 + \hat{\jmath}2xyz + \hat{k}2xyz^3$, find the Curl $\vec{A}$ at the point $(1, -1, -1)$

Sol. Given $\vec{A} = \hat{\imath}xz^2 + \hat{\jmath}2xyz + \hat{k}2xyz^3$

Now Curl $\vec{A} = \vec{\nabla} \times \vec{A} = \left(\frac{\hat{\imath}\partial}{\partial x} + \frac{\hat{\jmath}\partial}{\partial y} + \frac{\hat{k}\partial}{\partial z}\right) \times (\hat{\imath}xz^2 + \hat{\jmath}2xyz + \hat{k}2xyz^3)$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^2 & 2xyz & 2xyz^3 \end{vmatrix}$$

$$= \hat{\imath}\left(\frac{\partial}{\partial y}(2xyz^3) - \frac{\partial}{\partial z}(2xyz)\right) + \hat{\jmath}\left(\frac{\partial}{\partial z}(xz^2) - \frac{\partial}{\partial x}(2xyz^3)\right) + \hat{k}\left(\frac{\partial}{\partial x}(2xyz) - \frac{\partial}{\partial y}(xz^2)\right)$$

$$= \hat{\imath}(2xz^3 - 2xy) + \hat{\jmath}(2xz) - 2yz^3) + \hat{k}(2yz - 0)$$

Now Curl $\vec{A}$ at the point $(1, -1, -1)$

$$= \hat{\imath}(2(1)(-1)^3 - 2(1)(-1)) + \hat{\jmath}(2(1)(-1) - 2(-1)(-1)^3) + \hat{k}(2(-1)(-1))$$

$$= \hat{\imath}(0) + \hat{\jmath}(-4) + \hat{k}(2)$$

$$= -4\hat{\jmath} + 2\hat{k}$$

Que3. Show that $\vec{\nabla} \times |r|^n\vec{r} = 0$, *where* $\vec{r}$ is a position vector and $\vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z$

Sol. Given $\vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z$ then

$|r| = (x^2 + y^2 + z^2)^{1/2}$ and

$$|r|^n = (x^2 + y^2 + z^2)^{n/2}$$

$\therefore \vec{\nabla} \times |r|^n\vec{r} = \left(\frac{\hat{\imath}\partial}{\partial x} + \frac{\hat{\jmath}\partial}{\partial y} + \frac{\hat{k}\partial}{\partial z}\right) \times |r|^n(\hat{\imath}x + \hat{\jmath}y + \hat{k}z)$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ |r|^n x & |r|^n y & |r|^n z \end{vmatrix}$$

$$= \hat{\imath}\left[\frac{\partial}{\partial y}(x^2 + y^2 + z^2)^{\frac{n}{2}})(z) - \frac{\partial}{\partial z}(x^2 + y^2 + z^2)^{\frac{n}{2}}(y)\right] + \hat{\jmath}\left[\frac{\partial}{\partial z}(x^2 + y^2 + z^2)^{\frac{n}{2}}(x) - \frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{\frac{n}{2}}(z)\right] + \hat{k}\left[\frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{\frac{n}{2}}(y) - \frac{\partial}{\partial y}(x^2 + y^2 + z^2)^{\frac{n}{2}}(x)\right]$$

$$= \hat{\imath}\left[\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2yz) - \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2zy)\right]$$
$$+\hat{\jmath}\left[\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2zx) - \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2xz)\right]$$
$$+\hat{k}\left[\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2xy) - \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2yx)\right]$$

$=\hat{\imath}(0)+\hat{\jmath}(0)+\hat{k}(0)=0$

Que4. If $\vec{A}$ is a constant vector, show that Curl $(\vec{A}\times\vec{r}) = 2\vec{A}$

Sol. Curl $(\vec{A}\times\vec{r}) = \left(\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\left[(\hat{\imath}Ax+\hat{\jmath}Ay+\hat{k}Az)\times(\hat{\imath}x+\hat{\jmath}y+\hat{k}z)\right]$

$$= \left(\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ Ax & Ay & Az\\ x & y & z\end{vmatrix}$$

$$= \left(\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\left[\hat{\imath}(zAy-yAz)+\hat{\jmath}(xAz-zAx)+\hat{k}(yAx-xAy)\right]$$

$$= \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ (zAy-yAz) & (xAz-zAx) & (yAx-xAy)\end{vmatrix}$$

$$= \hat{\imath}\left(\frac{\partial}{\partial y}(yAx-xAy)-\frac{\partial}{\partial z}(xAz-zAx)\right)+\hat{\jmath}\left(\frac{\partial}{\partial z}(zAy-yAz)-\frac{\partial}{\partial x}(yAx-xAy)\right)$$
$$+\hat{k}\left(\frac{\partial}{\partial x}(xAz-zAx)-\frac{\partial}{\partial y}(zAy-yAz)\right)$$

$$= \hat{\imath}(Ax+Ax)+\hat{\jmath}(Ay+Ay))+\hat{k}(Az+Az)$$

$$= \hat{\imath}(2Ax)+\hat{\jmath}(2Ay)+\hat{k}(2Az)$$

$= 2(Ax\hat{\imath}+Ay\hat{\jmath}+Az\hat{k}) = 2\vec{A}$

Hence Curl $(\vec{A}\times\vec{r}) = 2\vec{A}$

Que5. Show that $|r|^n\vec{r}$ is an irrotational for any value of n, but it is solenoidal for n= -3, $\vec{r} = \hat{\imath}x+\hat{\jmath}y+\hat{k}z$

Sol. The condition for irrotational vector is $\vec{\nabla}\times|r|^n\vec{r} = 0$,

Since $\vec{r} = \hat{\imath}x+\hat{\jmath}y+\hat{k}z$ then

$|r| = (x^2+y^2+z^2)^{1/2}$ and

$$|r|^n = (x^2+y^2+z^2)^{n/2}$$

$\therefore \vec{\nabla}\times|r|^n\vec{r} = \left(\frac{\hat{\imath}\partial}{\partial x}+\frac{\hat{\jmath}\partial}{\partial y}+\frac{\hat{k}\partial}{\partial z}\right)\times|r|^n(\hat{\imath}x+\hat{\jmath}y+\hat{k}z)$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ |r|^n x & |r|^n y & |r|^n z \end{vmatrix}$$

$$= \hat{\imath}\left[\frac{\partial}{\partial y}(x^2+y^2+z^2)^{\frac{n}{2}})(z) - \frac{\partial}{\partial z}(x^2+y^2+z^2)^{\frac{n}{2}}(y)\right] + \hat{\jmath}\left[\frac{\partial}{\partial z}(x^2+y^2+z^2)^{\frac{n}{2}}(x) - \frac{\partial}{\partial x}(x^2+y^2+z^2)^{\frac{n}{2}}(z)\right] + \hat{k}\left[\frac{\partial}{\partial x}(x^2+y^2+z^2)^{\frac{n}{2}}(y) - \frac{\partial}{\partial y}(x^2+y^2+z^2)^{\frac{n}{2}}(x)\right]$$

$$= \hat{\imath}\left[\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2yz) - \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2zy)\right] + \hat{\jmath}\left[\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2zx) - \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2xz)\right] + \hat{k}\left[\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2xy) - \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2yx)\right]$$

$=\hat{\imath}(0) + \hat{\jmath}(0) + \hat{k}(0) = 0$

For the second part, the vector field is solenoidal if $\vec{\nabla}.(r^n\vec{r}) = 0$
$\vec{\nabla}.(r^n\vec{r}) = (n+3)r^n, where\ \vec{r}$ is a position vector
$|r| = (x^2+y^2+z^2)^{1/2}$ and

$$|r|^n = (x^2+y^2+z^2)^{n/2}$$

$\therefore \vec{\nabla}.(r^n\vec{r}) = \left(\frac{\hat{\imath}\partial}{\partial x} + \frac{\hat{\jmath}\partial}{\partial y} + \frac{\hat{k}\partial}{\partial z}\right)(x^2+y^2+z^2)^{n/2}(\hat{\imath}x + \hat{\jmath}y + \hat{k}z)$

$= \frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2}x + \frac{\partial}{\partial y}(x^2+y^2+z^2)^{n/2}y + \frac{\partial}{\partial z}(x^2+y^2+z^2)^{n/2}z$

$= \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2x)(x) + (x^2+y^2+z^2)^{\frac{n}{2}} + \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2y)(y) + (x^2+y^2+z^2)^{\frac{n}{2}} + \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2z)(z) + (x^2+y^2+z^2)^{\frac{n}{2}}$

$=n(x^2+y^2+z^2)^{\frac{n}{2}-1}(x^2+y^2+z^2)^1 + 3(x^2+y^2+z^2)^{\frac{n}{2}}$

$=n(x^2+y^2+z^2)^{\frac{n}{2}-1+1} + 3(x^2+y^2+z^2)^{\frac{n}{2}}$

$=n(x^2+y^2+z^2)^{\frac{n}{2}} + 3(x^2+y^2+z^2)^{\frac{n}{2}}$

$=(n+3)(x^2+y^2+z^2)^{\frac{n}{2}}$

$=(n+3)r^n$
Now if n= -3, then
$\vec{\nabla}.(r^n\vec{r})=(-3+3)r^n = 0,$
Hence, solenoidal for n= -3.

Que6. If $\vec{A} = \hat{\imath}x^2y - \hat{\jmath}2xz + \hat{k}2yz$, find the Curl Curl $\vec{A}$

Sol. Given $\vec{A} = \hat{\imath}x^2y - \hat{\jmath}2xz + \hat{k}2yz$

Now Curl $\vec{A} = \vec{\nabla} \times \vec{A} = \left(\frac{\hat{\imath}\partial}{\partial x} + \frac{\hat{\jmath}\partial}{\partial y} + \frac{\hat{k}\partial}{\partial z}\right) \times (\hat{\imath}x^2y - \hat{\jmath}2xz + \hat{k}2yz)$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix}$$

$$= \hat{\imath}\left(\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(-2xz)\right) + \hat{\jmath}\left(\frac{\partial}{\partial z}(x^2y) - \frac{\partial}{\partial x}(2yz)\right) + \hat{k}\left(\frac{\partial}{\partial x}(-2xz) - \frac{\partial}{\partial y}(x^2y)\right)$$

$$= \hat{\imath}(2z + 2x) + \hat{\jmath}(0) - \hat{k}(x^2 + 2z)$$

Now Curl Curl $\vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2z + 2x) & 0 & -(x^2 + 2z) \end{vmatrix} = (2x + 2)\hat{\jmath}$

Que7. Show that Curl $k\vec{A} = k$Curl $\vec{A} + grad\ k \times \vec{A}$

Sol. Curl $k\vec{A} = \vec{\nabla} \times k\overrightarrow{A} = \left(\frac{\hat{\imath}\partial}{\partial x} + \frac{\hat{\jmath}\partial}{\partial y} + \frac{\hat{k}\partial}{\partial z}\right) \times (\hat{\imath}kAx + \hat{\jmath}kAy + \hat{k}A\mathrm{z})$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ kAx & kAy & kAz \end{vmatrix}$$

$$= \hat{\imath}\left(\frac{\partial}{\partial y}(kAz) - \frac{\partial}{\partial z}(kAy)\right) + \hat{\jmath}\left(\frac{\partial}{\partial z}(kAx) - \frac{\partial}{\partial x}(kAz)\right) + \hat{k}\left(\frac{\partial}{\partial x}(kAy) - \frac{\partial}{\partial y}(kAx)\right)$$

$$= \hat{\imath}\left(k\frac{\partial Az}{\partial y} + Az\frac{\partial k}{\partial y} - k\frac{\partial Ay}{\partial z} - Ay\frac{\partial k}{\partial z}\right) + \hat{\jmath}\left(k\frac{\partial Ax}{\partial z} + Ax\frac{\partial k}{\partial z} - k\frac{\partial Az}{\partial x} - Az\frac{\partial k}{\partial x}\right) + \hat{k}\left(k\frac{\partial Ay}{\partial x} + Ay\frac{\partial k}{\partial x} - k\frac{\partial Ax}{\partial y} - Ax\frac{\partial k}{\partial y}\right)$$

$$=k\left[\hat{\imath}\left(\frac{\partial Az}{\partial y} - \frac{\partial Ay}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial Ax}{\partial z} - k\frac{\partial Az}{\partial x}\right) + \hat{k}\left(\frac{\partial Ay}{\partial x} - k\frac{\partial Ax}{\partial y}\right)\right] + \hat{\imath}\left(Az\frac{\partial k}{\partial y} - Ay\frac{\partial k}{\partial z}\right) + \hat{\jmath}\left(Ax\frac{\partial k}{\partial z} - Az\frac{\partial k}{\partial x}\right) + \hat{k}\left(Ay\frac{\partial k}{\partial x} - Ax\frac{\partial k}{\partial y}\right)$$

$$k\vec{\nabla} \times \vec{A} + \vec{\nabla}k \times \vec{A}$$

Hence, Curl $k\vec{A} = k$Curl $\vec{A} + grad\ k \times \vec{A}$

Que8. Show that $\vec{\nabla} \times \vec{\nabla}\mathrm{U} = 0$

Sol. $\vec{\nabla}\times\vec{\nabla}\text{U} = \left(\frac{\hat{\imath}\partial}{\partial x}+\frac{\hat{\jmath}\partial}{\partial y}+\frac{\hat{k}\partial}{\partial z}\right)\times\left(\frac{\hat{\imath}\partial u}{\partial x}+\frac{\hat{\jmath}\partial u}{\partial y}+\frac{\hat{k}\partial u}{\partial z}\right)$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \end{vmatrix}$$

$$= \hat{\imath}\left(\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right)-\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\right)\right)+\ \hat{\jmath}\left(\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right)\right)+\hat{k}\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)-\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\right)$$

$$= \hat{\imath}(0)+\ \hat{\jmath}(0)+\hat{k}(0) = 0$$

Que9. Show that $\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{\nabla}.\vec{E})-\nabla^2\vec{E}$ or $CurlCurl\vec{E} = GradDiv\vec{E}-\nabla^2\vec{E}$

Sol. $CurlCurl\vec{E}$=$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})$

Now first consider

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Ex & Ey & Ez \end{vmatrix}$$

$$= \hat{\imath}\left(\frac{\partial Ez}{\partial y}-\frac{\partial Ey}{\partial z}\right)+\ \hat{\jmath}\left(\frac{\partial Ex}{\partial z}-\frac{\partial Ez}{\partial x}\right)+\hat{k}\left(\frac{\partial Ey}{\partial x}-\frac{\partial Ex}{\partial y}\right)$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial Ez}{\partial y}-\frac{\partial Ey}{\partial z}\right) & \left(\frac{\partial Ex}{\partial z}-\frac{\partial Ez}{\partial x}\right) & \left(\frac{\partial Ey}{\partial x}-\frac{\partial Ex}{\partial y}\right) \end{vmatrix}$$

Taking only the x- component, we have

$$= \hat{\imath}\left(\frac{\partial}{\partial y}\left(\frac{\partial Ey}{\partial x}-\frac{\partial Ex}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial Ex}{\partial z}-\frac{\partial Ez}{\partial x}\right)\right)$$

$$= \hat{\imath}\left(\frac{\partial^2 Ey}{\partial x\partial y}-\frac{\partial^2 Ex}{\partial y^2}-\frac{\partial^2 Ex}{\partial z^2}+\frac{\partial^2 Ez}{\partial x\partial z}\right)$$

Adding & subtracting $\frac{\partial^2 Ex}{\partial x^2}, we\ get$

$$= \hat{\imath}\left(\frac{\partial^2 Ex}{\partial x^2} + \frac{\partial^2 Ey}{\partial x \partial y} + \frac{\partial^2 Ez}{\partial x \partial z} - \frac{\partial^2 Ex}{\partial x^2} - \frac{\partial^2 Ex}{\partial y^2} - \frac{\partial^2 Ex}{\partial z^2}\right)$$

$$= \hat{\imath}\left(\frac{\partial}{\partial x}\left(\frac{\partial Ex}{\partial x} + \frac{\partial Ey}{\partial y} + \frac{\partial Ez}{\partial z}\right)\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\hat{\imath}Ex$$

$$(\vec{\nabla} Div \vec{E})_{x-comp} - (\nabla^2 E)_{x-comp}$$

Similarly, we can evaluate for y and z components.
Hence, $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla}.\vec{E}) - \nabla^2 \vec{E}$

Que.10. Prove that Div $(\vec{A} \times \vec{B})$ = $\vec{B}.Curl\vec{A} - \vec{A}.Curl\,\vec{B}$

Sol. We have $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ Ax & Ay & Az \\ Bx & By & Bz \end{vmatrix}$

$= \hat{\imath}(AyBz - ByAz) + \hat{\jmath}(AzBx - AxBz) + \hat{k}(AxBy - AyBx)$

$$\vec{\nabla}.\left(\vec{A} \times \vec{B}\right) = \left(\frac{\hat{\imath}\partial}{\partial x} + \frac{\hat{\jmath}\partial}{\partial y} + \frac{\hat{k}\partial}{\partial z}\right).\left(\hat{\imath}(AyBz - ByAz) + \hat{\jmath}(AzBx - AxBz) + \hat{k}(AxBy - AyBx)\right)$$

$$=\left(\frac{\partial}{\partial x}(AyBz - ByAz) + \frac{\partial}{\partial y}(AzBx - AxBz) + \frac{\partial}{\partial z}(AxBy - AyBx)\right)$$

$$= Ay\frac{\partial Bz}{\partial x} + Bz\frac{\partial Ay}{\partial x} - By\frac{\partial Az}{\partial x} - Az\frac{\partial By}{\partial x} + Az\frac{\partial Bx}{\partial y} + Bx\frac{\partial Az}{\partial y} - Ax\frac{\partial Bz}{\partial y} - Bz\frac{\partial Ax}{\partial y} + Ax\frac{\partial By}{\partial z} + By\frac{\partial Ax}{\partial z} - Ay\frac{\partial Bx}{\partial z} - Bx\frac{\partial Ay}{\partial z}$$

$$= Bx\left(\frac{\partial Az}{\partial y} - \frac{\partial Ay}{\partial z}\right) + By\left(\frac{\partial Ax}{\partial z} - \frac{\partial Az}{\partial x}\right) + Bz\left(\frac{\partial Ay}{\partial x} - \frac{\partial Ax}{\partial y}\right) - Ax\left(\frac{\partial Bz}{\partial y} - \frac{\partial By}{\partial z}\right) - Ay\left(\frac{\partial Bx}{\partial z} - \frac{\partial Bz}{\partial x}\right) - Az\left(\frac{\partial By}{\partial x} - \frac{\partial Bx}{\partial y}\right)$$

$$=\left(\hat{\imath}Bx + \hat{\jmath}By + \hat{k}Bz\right).\left(\hat{\imath}\left(\frac{\partial Az}{\partial y} - \frac{\partial Ay}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial Ax}{\partial z} - \frac{\partial Az}{\partial x}\right) + \hat{k}\left(\frac{\partial Ay}{\partial x} - \frac{\partial Ax}{\partial y}\right)\right)$$
$$-\left(\hat{\imath}Ax + \hat{\jmath}Ay + \hat{k}Az\right).\left(\hat{\imath}\left(\frac{\partial Bz}{\partial y} - \frac{\partial By}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial Bx}{\partial z} - \frac{\partial Bz}{\partial x}\right) + \hat{k}\left(\frac{\partial By}{\partial x} - \frac{\partial Bx}{\partial y}\right)\right)$$

$=\vec{B}.Curl\vec{A} - \vec{A}.Curl\,\vec{B}$

Hence Div $(\vec{A} \times \vec{B})$ = $\vec{B}.Curl\vec{A} - \vec{A}.Curl\,\vec{B}$ proved

Que.11. If $\vec{A}$ and $\vec{B}$ are irrotational, Prove that $\vec{A} \times \vec{B}$ is solenoidal.

Sol. Given that $\vec{A}$ and $\vec{B}$ are irrotational, implies that $Curl\vec{A} = 0 \; and \; Curl\,\vec{B} = 0$

We have to show Div($\vec{A} \times \vec{B}$)= 0

Since Div ($\vec{A} \times \vec{B}$) = $\vec{B}.Curl\vec{A} - \vec{A}.Curl\,\vec{B}$

= $\vec{B}.(0) - \vec{A}.(0)$ = 0

Que12. A rigid body is rotating with a uniform angular velocity $\vec{\omega}$, about its axis passing through it. Show that Curl $\vec{V} = 2\vec{\omega}$, where $\vec{V}$ is the linear velocity.

Sol. Since we know that $\vec{V} = \vec{\omega} \times \vec{r}$, where $\vec{r}$ is the position vector.

$$\vec{V} = \hat{\imath}Vx + \hat{\jmath}Vy + \hat{k}Vz$$

And

$$\vec{\omega} = \hat{\imath}\omega_x + \hat{\jmath}\omega_y + \hat{k}\omega_z$$

Now Curl $\vec{V} = Curl(\vec{\omega} \times \vec{r}) = \vec{\nabla} \times (\vec{\omega} \times \vec{r})$

$$=\vec{\nabla} \times \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix} = \vec{\nabla} \times \left[\hat{\imath}\left(\omega_y z - \omega_z y\right) + \hat{\jmath}(\omega_z x - \omega_x z) + \hat{k}(\omega_x y - \omega_y x)\right]$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \left(\omega_y z - \omega_z y\right) & (\omega_z x - \omega_x z) & (\omega_x y - \omega_y x) \end{vmatrix}$$

$$= \hat{\imath}\left(\frac{\partial}{\partial y}\left(\omega_x y - \omega_y x\right) - \frac{\partial}{\partial z}(\omega_z x - \omega_x z)\right) + \hat{\jmath}\left(\frac{\partial}{\partial z}\left(\omega_y z - \omega_z y\right) - \frac{\partial}{\partial x}\left(\omega_x y - \omega_y x\right)\right)$$
$$+ \hat{k}\left(\frac{\partial}{\partial x}(\omega_z x - \omega_x z) - \frac{\partial}{\partial y}\left(\omega_y z - \omega_z y\right)\right)$$

$$=\hat{\imath}(\omega_x + \omega_x) + \hat{\jmath}\left(\omega_y + \omega_y\right) + \hat{k}(\omega_z + \omega_z) = \hat{\imath}(2\omega_x) + \hat{\jmath}\left(2\omega_y\right) + \hat{k}(2\omega_z) = 2\vec{\omega}$$

Hence Curl $\vec{V} = 2\vec{\omega}$

Que13. Show that $\vec{E} = 6xy\hat{\imath} + \hat{\jmath}(3x^2 - 3y^2)$ is irrotational field.

Sol. $\vec{E}$ will be irrotational field if $\vec{\nabla} \times \vec{E}$ = 0

Now,

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy & (3x^2 - 3y^2) & 0 \end{vmatrix}$$

$$= \hat{\imath}\left(0 - \frac{\partial}{\partial z}(3x^2 - 3y^2)\right) + \hat{\jmath}\left(\frac{\partial}{\partial z}(6xy) - 0\right) + \hat{k}\left(\frac{\partial}{\partial x}(3x^2 - 3y^2) - \frac{\partial}{\partial y}(6xy)\right)$$

$$= \hat{\imath}(0) + \hat{\jmath}(0) + \hat{k}(6x - 6x) = 0$$

$\vec{\nabla} \times \vec{E}$=0
Thus, $\vec{E}$ is irrotational field.

Que14. Find the curl of the vector field $\vec{A}$ *such that* $A_x = x^2 - z^2, A_y = 2$ *and* $A_z = 9xz$.
Sol.

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - z^2 & 2 & 9xz \end{vmatrix}$$

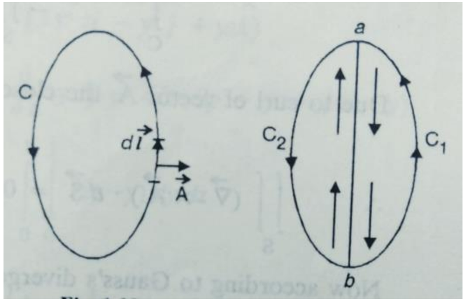
$$= \hat{\imath}(0 - 0) + \hat{\jmath}(-2z - 9z) + \hat{k}(0 - 0) = -11z\hat{\jmath}$$

Stokes theorem:-

According to this theorem, the line integral of a vector field $\vec{A}$ around any closed curve is equal to the surface integral of the curl of $\vec{A}$ taken over any surface S of which the curve is a bounding edge.

$$\oint \vec{A}\,.\,d\vec{l} = \oint_s \left(\vec{\nabla} \times \vec{A}\right).\,d\vec{s}$$

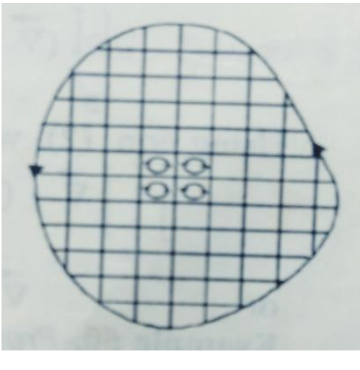
Proof:- Let $\vec{A}$ be the vector field acting on the surface enclosed by closed curve C. Then the line integral of vector $\vec{A}$ along a closed curve is given by $\oint \vec{A}\,.\,d\vec{l}$, where $d\vec{l}$ is the length of a small element of the path.



Divide the area enclosed by the closed curve C into two equal parts by drawing a line ab as shown in figure. We have now two closed curves C_1 and C_2. Therefore, the line integral of vector $\vec{A}$ along a closed curve C can be written as

$$\oint \vec{A}\,.\,d\vec{l} = \oint_{c_1} \vec{A}.\,d\vec{l} \;+\; \oint_{c_2} \vec{A}.\,d\vec{l} \qquad \dots (1)$$

If the area enclosed by the curve C is divided into large number of small areas such as $ds_1, ds_2\,, ds_3 \dots\dots\dots ds_n$ bounded by the curves $C_1, C_2 \dots\dots C_n$ as shown in figure, then



$$\oint_c \vec{A}.\vec{dl} = \sum \oint_{c_n} \vec{A}.\vec{dl} \quad \ldots (2)$$

According to the definition of curl, we have

$$\oint_c \vec{A}.\vec{dl} = Curl\,\vec{A}.\overrightarrow{ds_n} \quad \ldots. (3)$$

Putting this value in (2), we get

$$\oint_c \vec{A}.\vec{dl} = \sum Curl\,\vec{A}.\overrightarrow{ds_n} \quad \ldots (4)$$

As $ds \to 0$ $then$ $\sum Curl\,\vec{A}.\overrightarrow{ds_n} = \oint_s Curl\,\vec{A}.d\vec{s}$

Hence equation (4) can be written as

$$\oint_c \vec{A}.\vec{dl} = \oint_s Curl\,\vec{A}.d\vec{s}$$

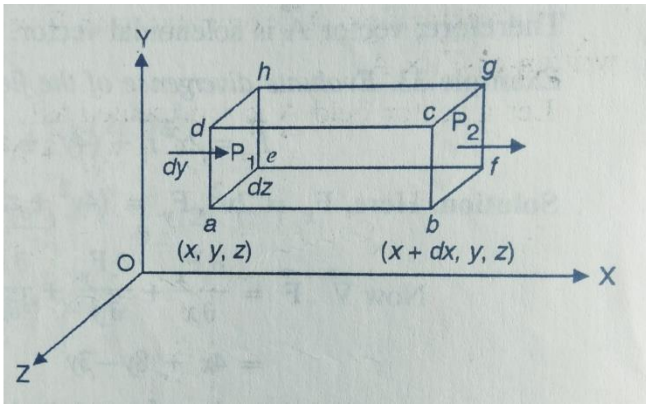
Or $\oint \vec{A}\,.\,d\vec{l} = \oint_s (\vec{\nabla} \times \vec{A}).\,d\vec{s}$, which is the Stoke's theorem

Gauss's divergence theorem:-

According to this theorem, the surface integral of a vector field $\vec{A}$ over the closed surface is equal to the volume integral of the divergence of vector field $\vec{A}$ over the volume V enclosed by the surface.

$$\oint_s \overrightarrow{A.ds} = \oint_v Div\,\vec{A}\,dv = \oint_v (\vec{\nabla}.\vec{A})dv$$

Proof:- Consider a surface S which encloses a volume V. Let this volume be divided into number of elementary volumes in the form of parallelopiped having volume dv with its sides $dx, dy\ and\ dz$ along the three axes. Let A_x, A_y and A_z be the components of $\vec{A}$ along $x, y\ and\ z$ axis respectively. If the value of A_x at P_1(midpoint of the face adhe) is A_x (P_1), then the flux through face adhe of the cube is $-A_x\,(P_1)dydz$, where $dydz$ is the area of the face adhe.



If dA_x is the change in A_x with x, then

$$dA_x = \frac{\partial Ax}{\partial x}dx$$

Flux through the face bcgf = $\left[A_x\,(P_1) + \frac{\partial Ax}{\partial x}dx\right]dydz$

Hence, flux through the faces adhe and bcgf = $\left[A_x\,(P_1) + \frac{\partial Ax}{\partial x}dx\right]dydz - A_x\,(P_1)dydz$

$$= \frac{\partial Ax}{\partial x}dxdydz$$

Similarly, flux through the faces abfe and dcgh = $\frac{\partial Ay}{\partial y} dxdydz$

Also, flux through the faces adcb and efgh = $\frac{\partial Az}{\partial z} dxdydz$

Total flux through the whole surfaces of cube is

$$\vec{A}.d\vec{s} = \left(\frac{\partial Ax}{\partial x} + \frac{\partial Ay}{\partial y} + \frac{\partial Az}{\partial z}\right) dxdydz$$

$$= \left(\vec{\nabla}.\vec{A}\right) dxdydz = \left(\vec{\nabla}.\vec{A}\right) dV$$

Where $dV = dxdydz$ is the volume of the cube.

Taking the sum of fluxes of all the elementary cubes constituting the surface S

$$\oint_s \overrightarrow{A.ds} = \oint_v \left(\vec{\nabla}.\vec{A}\right) dv$$

The converse of this theorem is also true, hence the Gauss's divergence theorem enables us to transform a volume integral into surface integral and vice-versa.

Numerical

Que1. Evaluate $\oint_s \overrightarrow{A.ds}$, where $\vec{A} = 2x^2z\hat{\imath} - y^2\hat{\jmath} + yz\hat{k}$, S is the surface bounded by $x = 0, x = 1, y = 1, y = 1$ and $z = 0, z = 1.$

Sol. According to Gauss's divergence theorem,

$$\oint_s \overrightarrow{A.ds} = \oint_v \left(\vec{\nabla}.\vec{A}\right) dv = \oint_v \left(\vec{\nabla}.\vec{A}\right) dxdydz$$

$$\left(\vec{\nabla}.\vec{A}\right) = \left(\frac{\hat{\imath}\partial}{\partial x} + \frac{\hat{\jmath}\partial}{\partial y} + \frac{\hat{k}\partial}{\partial z}\right).\left(2x^2z\hat{\imath} - y^2\hat{\jmath} + yz\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(2x^2z) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz) = 4xz - 2y + y = (4xz - y)$$

$$\oint_s \overrightarrow{A.ds} = \iiint_{0,0,0}^{1,1,1} (4xz - y)\, dxdydz = \iint_{0,0}^{1,1} (2z - y) dydz = \int_0^1 \left((2z - \frac{1}{2}\right) dz = \left(1 - \frac{1}{2}\right)$$

$$= \frac{1}{2}$$

Que2. If $\vec{r}$ is a position vector and $\vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z$. $Show$ $\oint_s \overrightarrow{r.ds} = 3V$, Where V is the volume enclosed by the surface S.

Sol. According to the Gauss's Divergence theorem

$$\oint_s \overrightarrow{A.ds} = \oint_v \left(\vec{\nabla}.\vec{A}\right) dv$$

Replacing $\vec{A}$ with $\vec{r}$ in the above equation.

$$\oint_s \vec{r}.\overrightarrow{ds} = \oint_v (\vec{\nabla}.\vec{r})dv$$

Since $\vec{\nabla}.\vec{r} = 3$

$$\oint_s \vec{r}.\overrightarrow{ds} = \oint_v 3dv = 3V$$

$$Hence \oint_s \vec{r}.\overrightarrow{ds} = 3V$$

Que3. Evaluate $\oint_s \overrightarrow{A.ds}$, Where $\vec{A} = x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k}$ and S is the surface of the cube bounded by x=0, x=1, y=0, y=1 and z=0 and z=1.

Sol. According to the Gauss's Divergence theorem

$$\oint_s \overrightarrow{A.ds} = \oint_v (\vec{\nabla}.\vec{A})dv = \int_0^1\int_0^1\int_0^1 (\vec{\nabla}.\vec{A})dxdydz$$

Now $(\vec{\nabla}.\vec{A}) = \left(\frac{\hat{\imath}\partial}{\partial x} + \frac{\hat{\jmath}\partial}{\partial y} + \frac{\hat{k}\partial}{\partial z}\right).\left(x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k}\right) = 2x + 2y + 2z = 2(x + y + z)$

$$\oint_s \overrightarrow{A.ds} = \int_0^1\int_0^1\int_0^1 2(x + y + z)dxdydz$$

$$= 2\int_0^1\int_0^1 \left(\frac{1}{2} + y + z\right)dydz$$

$$2\int_0^1 \left(\frac{1}{2} + \frac{1}{2} + z\right)dz = 2\left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right) = 2\left(\frac{3}{2}\right) = 3$$

Que4. Evaluate $\oint_s \overrightarrow{A.ds}$, Where $\vec{A} = 2x^2z\hat{\imath} - y^2\hat{\jmath} + yz\hat{k}$ and S is the surface of the cube bounded by x=0, x=1, y=0, y=1 and z=0 and z=1.

Sol. According to the Gauss's Divergence theorem

$$\oint_s \overrightarrow{A.ds} = \oint_v (\vec{\nabla}.\vec{A})dv = \int_0^1\int_0^1\int_0^1 (\vec{\nabla}.\vec{A})dxdydz$$

Now $(\vec{\nabla}.\vec{A}) = \left(\frac{\hat{\imath}\partial}{\partial x} + \frac{\hat{\jmath}\partial}{\partial y} + \frac{\hat{k}\partial}{\partial z}\right).\left(2x^2z\hat{\imath} - y^2\hat{\jmath} + yz\hat{k}\right) = 2xz - 2y + y = (4xz - y)$

$$\oint_s \overrightarrow{A.ds} = \int_0^1\int_0^1\int_0^1 (4xz - y)dxdydz$$

$$= \int_0^1\int_0^1 (2z - y)dydz$$

$$\int_0^1 \left(2z - \frac{1}{2}\right)dz = \left(1 - \frac{1}{2}\right) = \frac{1}{2}$$

Que5. If $\vec{r}$ is a position vector and $\vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z$. *Show* $\oint \vec{r}.\overrightarrow{dr} = 0$.

Sol. According to the Stokes theorem

$$\oint \vec{A}\,.\,\overrightarrow{dr} = \oint_s (\vec{\nabla} \times \vec{A}).\,d\vec{s}$$

Replacing $\vec{A}$ with $\vec{r}$ in the above equation.

$$\oint \vec{A}\,.\,\overrightarrow{dr} = \oint_s (\vec{\nabla} \times \vec{r}).\,d\vec{s}$$

Since $\vec{\nabla} \times \vec{r} = 0$ Hence, $\oint \vec{A}\,.\,\overrightarrow{dr} = 0$

Que6. If $\vec{E} = -y\hat{\imath} + x\hat{\jmath}$ then calculate the line integral $\oint \vec{E}.\overrightarrow{dl}$ for a closed curve. $x^2 + y^2 = r^2,\ z = 0$ Hence verify the Stokes theorem.

Sol. Given $\vec{E} = -y\hat{\imath} + x\hat{\jmath}$

Now $\oint \vec{E}.\overrightarrow{dl} = \int(-y\hat{\imath} + x\hat{\jmath}).(dx\hat{\imath} + dy\hat{\jmath}) = -\int ydx + \int xdy$

Given $x^2 + y^2 = r^2$ (It is equation of circle with radius r)

Put $x = rcos\theta \quad \therefore dx = -rsin\theta d\theta$

And $y = rsin\theta \quad \therefore dy = rcos\theta d\theta$

$$\int \vec{E}.\overrightarrow{dl} = -\int_0^{2\pi} rsin\,\theta(-rsin\theta d\theta) + \int_0^{2\pi} rcos\,\theta(rcos\theta d\theta)$$

$$\int_0^{2\pi} r^2(Sin^2\theta + cos^2\theta)d\theta = \int_0^{2\pi} r^2\, d\theta = r^2 2\pi = 2\pi r^2$$

According to the Stokes theorem

$$\oint \vec{E}\,.\,\overrightarrow{dl} = \oint_s (\vec{\nabla} \times \vec{E}).\,d\vec{s}$$

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix}$$

$$= \hat{\imath}(0 - 0) + \hat{\jmath}(0 - 0) + \hat{k}(1 + 1) = 2\hat{k}$$

Now, $\oint_s (\vec{\nabla} \times \vec{E}).d\vec{s} = \oint_s 2\hat{k}.d\vec{s}$

$$= \oint_s 2\hat{k}.\hat{k}ds = \oint_s 2ds = 2\,(Area\ of\ a\ circle)$$

$$= 2\pi r^2$$

Hence LHS = RHS.