Chapter 9

Topic III: Triconnectivity

In DEFINITION 6.95 we introduced the notion of a biconnected or "2-connected" graph. From EXERCISE 6.96(2) it is evident that a connected graph G is not biconnected if and only if it has an articulation point. The intuitive idea of an articulation point is shown in FIGURE 9.1(a). There, the graph $G = (V,E)$ has its edge set E partitioned into two subsets, F and $F^c = E - F$. For any subset $S \subseteq E$, let V(S) denote the set of vertices of edges in S. For the vertex x of FIGURE 9.1(a) to be an articulation point, we must have $|F| \geqslant 1$, $|F^c| \geqslant 1$, and $|V(F) \cap V(F^c)| = 1$. If G is connected and the articulation point x together with all edges of I_x (the set of edges incident on x, see DEFINITION 6.6) are removed from the graph then G becomes disconnected. Thus an articulation point is a vertex of G whose deletion separates G into two or more components. This idea is generalized in DEFINITION 9.2.

9.1 PICTORIAL REPRESENTATION OF 1- AND 2-SEPARATION SETS.

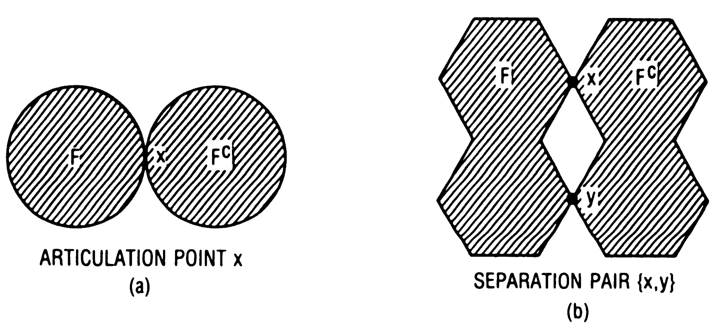


Figure 9.1

9.2 DEFINITION.

Let $G = (V,E)$ be a connected graph and let k be any positive integer. Let $\mathcal{S}_k = \{F: F \subseteq E, |F| \geqslant k, |F^c| \geqslant k, |V(F) \cap V(F^c)| = k\}$. If $\mathcal{S}_k$ is nonempty then G is called a *k-separated* graph. If F is in $\mathcal{S}_k$ then the set of vertices $V(F) \cap$

$V(F^c)$ is called a k-separation set for G. The 1-separation sets are the *articulation* points and the 2-separation sets are called *separation pairs*.

In a certain sense, the k-separation sets provide a measure of the degree of connectedness of a graph. There are many possible variations on this idea and a general study of connectivity will not concern us here. One natural definition of connectivity is given in DEFINITION 9.3.

9.3 DEFINITION.

Let $G = (V,E)$ be a graph. We denote by $k(G)$ a non-negative integer called the *connectivity* of G. If G is disconnected then $k(G) = 0$. For connected G, if $|E| = 0$ then $k(G) = 1$, if $|E| = 1$ then $k(G) = 2$, and if G is a cycle of length 3 then $k(G) = 3$. For all other connected G, $k(G)$ is the smallest integer k such that $\mathcal{S}_k$ of DEFINITION 9.2 is nonempty. We say that G is *t-connected* for any integer $0 \leqslant t \leqslant k(G)$.

We shall not use DEFINITION 9.3 except for the cases $k(G) = 0,1,2,3$. We want to say that the graph consisting of one vertex has connectivity 1, the graph consisting of a single edge has connectivity 2, and a cycle of length 3 has connectivity 3 for technical reasons. For these graphs $\mathcal{S}_k$ is empty for all $k = 1,2,\ldots$. sometimes these graphs are said to have ''infinite connectivity'' for this reason. Our graphs of the form $G = (V,E)$ do not have loops or multiple edges (such as the edge ''a'' of FIGURE 6.1 or the pair of edges ''e'' and ''f'' of FIGURE 6.1). If these more general graphs are considered then the graphs consisting of a single loop, a pair of vertices joined by two edges, and a pair of vertices joined by three edges would also have ''infinite connectivity.'' All other graphs have finite connectivity $k(G)$. In fact, it is easy to see that $k(G)$ is always less than or equal to the minimal degree of a vertex of G. From DEFINITION 9.3, we see that a graph is 3-connected or ''triconnected'' if its connectivity is 3 or more. Equivalently, we may say that *a graph* $G = (V,E)$ *is triconnected if it is biconnected with at least three edges and has no separation pair* (DEFINITION 9.2). Some ideas associated with connectivity are explored in EXERCISE 9.38(1).

At this point the reader should recall the idea of a bridge graph of a cycle in a graph (DEFINITION 6.110) and related ideas (FIGURES 6.98, 6.99, DEFINITIONS 6.100, 6.101, 6.102, EXERCISE 6.109(3)). In BASIC NOTATION 8.3 we introduced some terminology for segment graphs. This terminology has obvious extensions to the slightly more general setting of bridge graphs which we now describe. Let $\mathcal{C} = (x_1,x_2,\ldots,x_k,x_1)$ be a cycle of a biconnected graph G as shown in FIGURE 9.4. Let $\mathcal{C}'$ be the corresponding broken cycle obtained by removing the edge $\{x_k,x_1\}$ from $\mathcal{C}$. We associate $\mathcal{C}'$ with a directed path $(x_1,x_2,\ldots,x_k)$ as shown in FIGURE 9.4. FIGURE 9.4 shows a bridge A relative to the cycle $\mathcal{C}$. As in BASIC NOTATION 8.3, let RANGE(A) be the vertices common to A and $\mathcal{C}$ (the ''vertices of attachment'' of A to $\mathscr{C}$). RANGE(A) is linearly ordered by the order of the vertices in $\mathscr{C}'$. We call the first vertex in RANGE(A) LOW1(A) and the last vertex TAIL(A). The directed path in $\mathscr{C}'$ from LOW1(A) to TAIL(A) is called SPAN(A). If $v_1 < v_2 < \ldots < v_r$ is RANGE(A) then the directed paths from v_i to v_{i+1}, $i = 1, \ldots, r-1$, are called the proper gaps of A. The directed path in $\mathscr{C}$ from v_r to v_1 is called the cospan of A. In FIGURE 9.4, SPAN(B) is contained in a proper gap of A. When this happens, we say that A *arches* B or B *is arched by* A.

9.4 BRIDGES OF A BROKEN CYCLE.

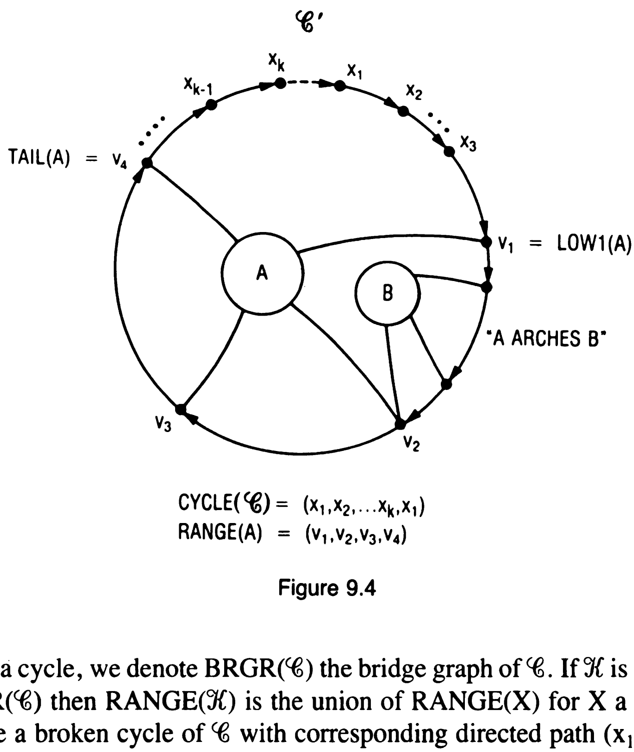


Figure 9.4

If $\mathscr{C}$ is a cycle, we denote BRGR($\mathscr{C}$) the bridge graph of $\mathscr{C}$. If $\mathscr{H}$ is a component of BRGR($\mathscr{C}$) then RANGE($\mathscr{H}$) is the union of RANGE(X) for X a vertex of $\mathscr{H}$. Let $\mathscr{C}'$ be a broken cycle of $\mathscr{C}$ with corresponding directed path $(x_1, x_2, \ldots, x_k)$. We define LOW1($\mathscr{H}$) to be the first vertex of RANGE($\mathscr{H}$) on this path and TAIL($\mathscr{H}$) to be the last vertex. As the graph G is biconnected, we always have LOW1($\mathscr{H}$) < TAIL($\mathscr{H}$) in the order on $\mathscr{C}'$. If $v_1 < v_2 < \ldots < v_r$ is RANGE($\mathscr{H}$) then the proper gaps and cospan of $\mathscr{H}$ are defined as in the previous paragraph for a single bridge. If $\mathscr{H}_1$ and $\mathscr{H}_2$ are components and RANGE($\mathscr{H}_2$) is contained in a proper gap of $\mathscr{H}_1$ then we say $\mathscr{H}_1$ arches $\mathscr{H}_2$ and we write $\mathscr{H}_1 \geqslant \mathscr{H}_2$. These ideas are illustrated in FIGURE 9.5. In FIGURE 9.5, $\mathscr{H}_1$ arches $\mathscr{H}_2$. It is possible to have components $\mathscr{H}_1$ and $\mathscr{H}_2$ such that $\mathscr{H}_1$ arches $\mathscr{H}_2$ and $\mathscr{H}_2$ arches $\mathscr{H}_1$ but $\mathscr{H}_1 \neq \mathscr{H}_2$. This can only happen if $\mathscr{H}_1$ and $\mathscr{H}_2$ have only one bridge. This situation is shown in FIGURE 9.6. Note that in FIGURE 9.6, RANGE($\mathscr{H}_1$) = RANGE ($\mathscr{H}_2$) and there are only two vertices in each set. If there are no such mutually arching bridges then the arching relation on the set of components of BRGR($\mathscr{C}$) is an order relation (define $\mathscr{H} \geq \mathscr{H}$ in this case). In EXERCISE 9.38 the reader is asked to prove THEOREM 9.7 (recall DEFINITION 1.2 of CHAPTER 1 which defines "order relation").

9.5 ARCHING RELATION ON COMPONENTS OF BRIDGE GRAPH.

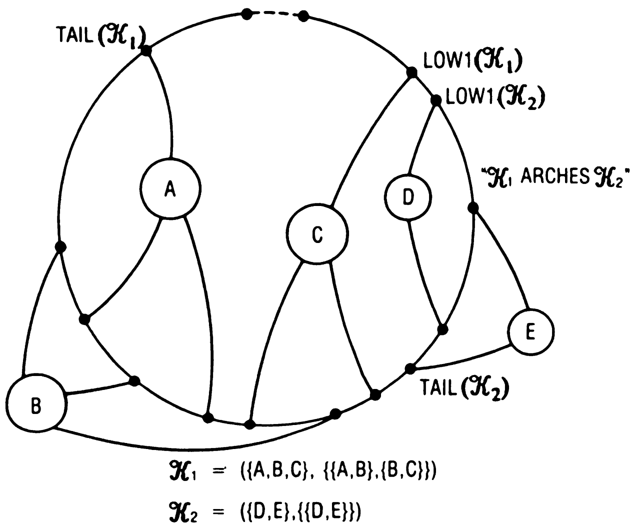


Figure 9.5

9.6 MUTUALLY ARCHING COMPONENTS OF BRIDGE GRAPH.

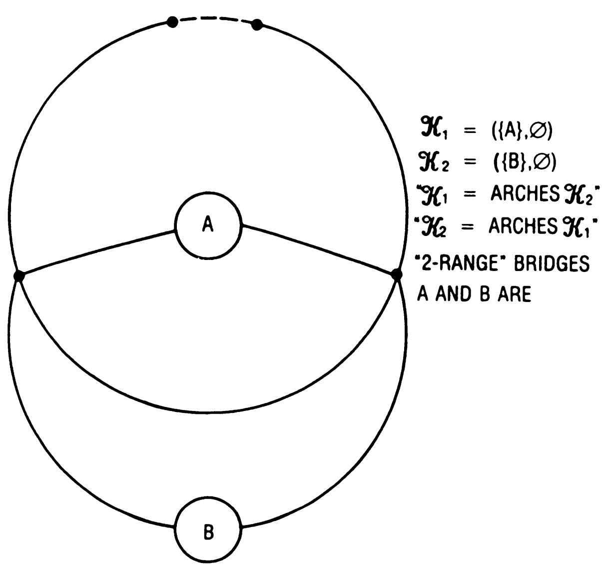


Figure 9.6

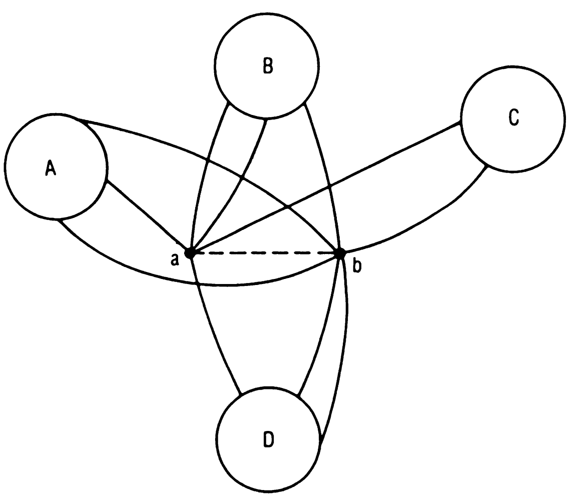
9.7 THEOREM. (Restate to deal with mutually arching bridges!)

Let G = (V,E) be a biconnected graph and let $\mathscr{C}'$ be a broken cycle of G. The arching relation on components of BRGR($\mathscr{C}$) is an order relation.

Proof. EXERCISE 9.38(2).

We now look more closely at the notion of a "separation pair." In DEFINITION 6.100 and EXERCISE 6.109(7) we defined the concept of a bridge relative to a subgraph. Using the definition of EXERCISE 6.109(7), for example, let H = ({a,b},ϕ) be the subgraph consisting of the two vertices a and b of G and no edges. The bridges relative to this subgraph are the equivalence classes of the equivalence relation defined in EXERCISE 6.109(7). That is, two edges e and f are equivalent if there is a path joining e to f without a or b as internal vertices. These equivalance classes are as shown in FIGURE 9.8. We call such equivalence classes *separation classes* of a,b.

9.8 SEPARATION CLASSES OF A PAIR OF VERTICES.



A,B,C,D ARE "SEPARATION CLASSES" OF THE PAIR {a,b}. THE EDGE JOINING a AND b MAY BE A "TRIVIAL" SEPARATION CLASS.

Figure 9.8

We call a separation class *nontrivial* if it has at least two edges. Thus, a pair of vertices of a biconnected graph G = (V,E) is a separation pair if and only if it has at least two nontrivial separation classes.

We now develop the relationship between bridges of a cycle and separation classes of a pair of vertices {a,b}. As we have just noted, if the pair {a,b} is a separation pair in the sense of DEFINITION 9.2 then there must exist at least two nontrivial separation classes of {a,b}. Let $\mathscr{C}$ be a cycle containing the vertices a and b of the separation pair {a,b}. There are three basic cases of interest to us, shown in FIGURES 9.10, 9.11, and 9.12. The circled regions A, B, X, etc., in these FIGURES are subgraphs, the detailed structure of which is not shown explicitly. In FIGURE 9.10, the cycle $\mathscr{C} = (v_1, v_2, \ldots, v_8, v_1)$ lies entirely within one separation class A. As remarked above, there must exist another nontrivial separation class B as shown in FIGURE 9.10. Clearly, the separation class B is also a bridge of the cycle $\mathscr{C}$. Note that the RANGE(B) = {a,b}.

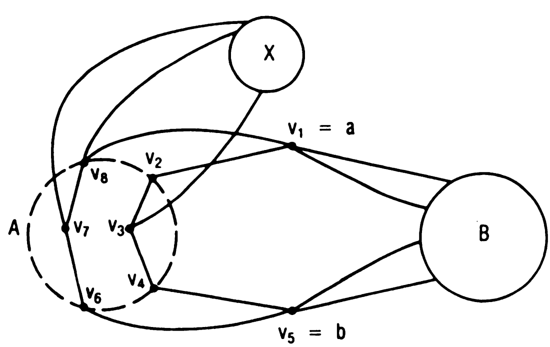
9.9 DEFINITION.

Let $\mathscr{C}$ be a cycle and B a bridge of $\mathscr{C}$. If RANGE(B) has exactly two elements then we call B a *2-range* bridge of $\mathscr{C}$. In other words, B has two vertices of attachment to $\mathscr{C}$.

Thus, FIGURE 9.10 shows that if $\mathscr{C}$ is contained in a single separation class then $\mathscr{C}$ must have a 2-range bridge B with RANGE(B) = {a,b}. If one considers the bridge X in FIGURE 9.10 it is apparent that X and B are joined in BRGR($\mathscr{C}$). Thus, in the case of FIGURE 9.10, one can have many bridges of $\mathscr{C}$ and have BRGR($\mathscr{C}$) either connected or disconnected.

Another basic case is shown in FIGURE 9.11. In this case, the cycle has all of its edges in separation class A except for the one edge {a,b} which is a trivial separation class. Again, there must be a 2-range bridge B and BRGR($\mathscr{C}$) may be connected or disconnected.

9.10 CASE I OF THEOREM 9.13.



$\mathscr{C} = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_1)$
A (WHICH INCLUDES X) AND B ARE SEPARATION CLASSES.

Figure 9.10

9.11 CASE II OF THEOREM 9.13.

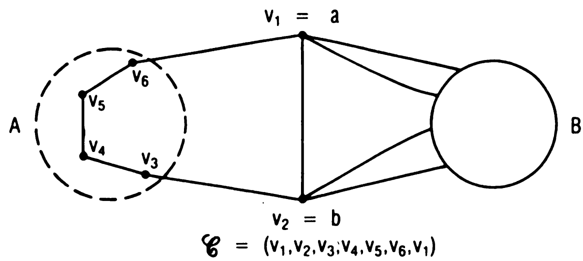


Figure 9.11

9.12 CASE III OF THEOREM 9.13.

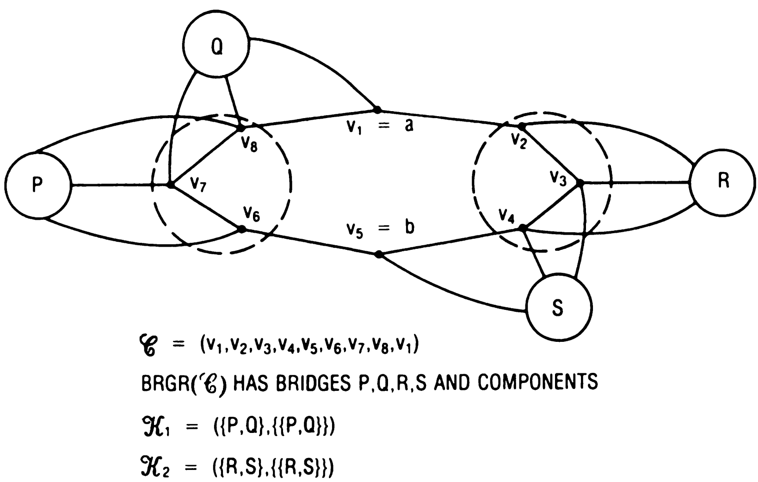


Figure 9.12

The third basic case is shown in FIGURE 9.12. In this case the cycle is contained in two separation classes. Obviously, a cycle cannot be contained in three separation classes. Case III is broken down into two subcases, where $\mathscr{C}$ has a vertex of degree 2 in G and where $\mathscr{C}$ does not have a vertex of degree 2. It is the latter case that is shown in FIGURE 9.12. Note that the fact that $\mathscr{C}$ does not have a vertex of degree 2 forces the bridge graph, BRGR($\mathscr{C}$) to have at least two components, as shown in FIGURE 9.12. One or both of these components may have only one vertex (a bridge of $\mathscr{C}$). These observations form the basis for the proof of the "only if" part of THEOREM 9.13.

9.13 THEOREM.

Let $G = (V,E)$ be a biconnected graph with $|E| > 3$, and let $\mathcal{C}$ be a cycle of G. $\mathcal{C}$ contains a separation pair $\{a,b\}$ of G if and only if either

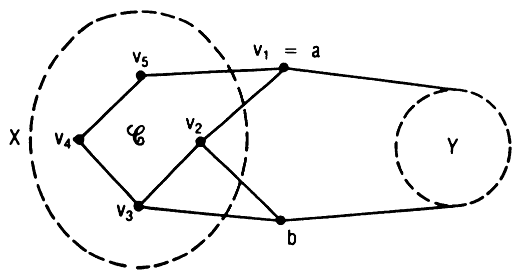
(1) $\mathcal{C}$ has a vertex of degree two in G
(2) $\mathcal{C}$ has a nontrivial 2-range bridge
or
(3) BRGR($\mathcal{C}$) is disconnected

Proof. The "only if" part follows from FIGURES 9.10, 9.11, and 9.12 and the related discussion. To prove the converse, suppose that $\mathcal{C}$ has a vertex v of degree 2 in G. If $\mathcal{C}$ has only three edges then condition (2) holds. If $\mathcal{C}$ has more than three edges then the vertices of $\mathcal{C}$ adjacent to v form a separation pair. If $\mathcal{C}$ has a nontrivial 2-range bridge B and if RANGE(B) = $\{a,b\}$ then a and b again are a separation pair. Finally, assume that $\mathcal{C}$ has no 2-range bridge and BRGR ($\mathcal{C}$) is disconnected. From FIGURE 9.6, we see that if BRGR($\mathcal{C}$) contains two mutually arching components (relative to some broken cycle $\mathcal{C}'$) then at least one of them is a nontrivial 2-range bridge. Thus BRGR($\mathcal{C}$) does not contain any mutually arching components, and by THEOREM 9.7, the arching relation on components is an order relation. Let $\mathcal{K}$ be a minimal component with respect to the arching relation. It is easily seen that $\{a,b\}$ is a separation pair where a = LOW1($\mathcal{K}$) and b = TAIL($\mathcal{K}$).

THEOREM 9.13 suggests a way of deciding whether or not a graph is triconnected. One could examine every cycle $\mathcal{C}$ and check whether or not any of the conditions (1), (2), or (3) of THEOREM 9.13 are satisfied. The obvious problem is that for all but the most trivial graphs there are too many cycles. It is an interesting and important fact, however, that it is only necessary to examine the cycles in a tree of cycles for the graph (see DEFINITION 6.107). We now explain why this is the case.

Let $G = (V,E)$ be a biconnected graph and let $\{a,b\}$ be a separation pair. If $\mathcal{C}$ is a cycle in G, then we have seen that $\mathcal{C}$ either has all edges entirely in one separation class or has edges in two classes. These two possibilities are shown in FIGURES 9.10, 9.11, and 9.12. If the cycle does not contain both a and b then the cycle cannot have edges in two different separation classes as shown in FIGURES 9.11 or 9.12 (this forces a and b to both be on $\mathcal{C}$). If $\mathcal{C}$ has all edges in one separation class it may have both a and b as vertices as in FIGURE 9.10, but it also may contain just one or neither of the vertices a and b as shown in FIGURE 9.14. In any case, we have the trivial LEMMA 9.15.

9.14 THE CYCLE IS CONTAINED IN A SEPARATION CLASS.



$\mathscr{C} = (v_1, v_2, v_3, v_4, v_5, v_1)$

Figure 9.14

9.15 LEMMA.

Let G be a biconnected graph with separation pair {a,b}. Let $\mathscr{C}$ be a cycle such that not both a and b are vertices of $\mathscr{C}$. Then the edges of $\mathscr{C}$ are all contained in one separation class of G.

A second fact we need concerns the bridges of a cycle $\mathscr{C}$ such as that of LEMMA 9.15. Again, let G be biconnected with cycle $\mathscr{C}$ and assume that {a,b} is a separation pair with a on $\mathscr{C}$ and b not on $\mathscr{C}$. The fact that b is not on $\mathscr{C}$ implies by LEMMA 9.15 that all edges of $\mathscr{C}$ belong to the same separation class of {a,b}. FIGURE 9.16(a) shows a bridge B of $\mathscr{C}$. Note, a is not a vertex of RANGE(B), and hence, whether or not b is in B, the following is true: given any edge e of B, there is a path that does not contain a or b as interior vertices joining e to at least one edge of $\mathscr{C}$. Thus, all edges of B belong to the same separation class of {a,b} as the edges of $\mathscr{C}$. It is also easy to see that if a is in RANGE(B) but b is not in B, then all edges of B lie in the same separation class as the edges of $\mathscr{C}$. FIGURE 9.16(b) shows the situation with a in RANGE(B) and b in B. Note in this case that the edges s,t,u,v, and w of B do not belong to the same separation class of {a,b} as the edges of $\mathscr{C}$. These observations lead to LEMMA 9.17.

9.16 THE INTUITIVE IDEA OF LEMMA 9.17.

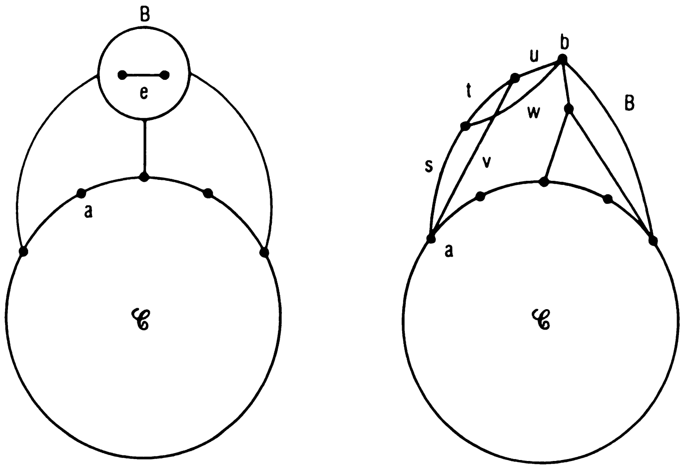


Figure 9.16

9.17 LEMMA.

Let G be a biconnected graph and let 𝒞 be a cycle of G. Let {a,b} be a separation pair of G with a and b not both on 𝒞. Then a and b belong to the same bridge B of 𝒞.

Proof. Let X be a bridge of 𝒞 that does not contain both a and b as vertices. As noted above, every edge of X belongs to the same separation class as the edges of 𝒞 relative to the separation pair {a,b}. Thus, if there does not exist a bridge B with both a and b as vertices then there is only one separation class of {a,b}, contradicting the choice of a and b. Thus, the bridge B must exist.

Let G, 𝒞, {a,b}, and B be as in LEMMA 9.17. Let 𝒞′ be any broken cycle associated wtih 𝒞 and let B′ denote the carrier of B relative to 𝒞′. The reader should review DEFINITION 6.102, FIGURE 6.103, and EXERCISE 6.109(3). It is shown in EXERCISE 6.109(3) that the carrier B′ is just B together with the edges of the SPAN(B) in 𝒞′. The carrier B′ is a biconnected graph and, as a consequence of LEMMA 9.17, B′ has both a and b as vertices (in fact, they are vertices of B). We now assert that under the hypothesis of LEMMA 9.17 the pair {a,b} is a separation pair of B′. Using FIGURE 9.14 as an example, note that the cycle 𝒞 is contained in a separation class X (a consequence of LEMMA 9.15). In FIGURE 9.14, for example, suppose that 𝒞′ is 𝒞 − $\{v_4,v_5\}$. Then a and b are contained in the bridge B = Y ∪ $\{v_3,b\}$ ∪ $\{v_2,b\}$ of 𝒞, and B′ is B together with the two edges $\{v_1,v_2\}$ and $\{v_2,v_3\}$, which are the edges of SPAN(B) relative to 𝒞′. Generally, all separation classes of {a,b} in G other than X belong to the bridge B. Thus these *remain* separation classes of {a,b} in the biconnected graph B′ and at least one of them must be nontrivial. We must show that there is another nontrivial separation class of {a,b} in B′. In general, given any vertex c of RANGE(B) and any vertex d of B there is a path *in* B from d to c. Assume b is not on $\mathscr{C}$. Choose $c \neq a$. Consider the path P in B from b to c. By interchanging the roles of a and b if necessary, we may assume that a is not on this path. The vertex c is on some edge e of $\mathscr{C}'$. Thus, the edges of P together with the edge e (totalling at least two edges) belong to a nontrivial separation class of {a,b} in B′. These edges are also in X since e is in X and hence this class is distinct from the one mentioned above. Thus {a,b} is a separation pair for B′. These ideas are summarized in LEMMA 9.18.

9.18 LEMMA.

Let B be the bridge of LEMMA 9.17 that contains the separation pair {a,b} and let B′ be the carrier of B relative to the broken cycle $\mathscr{C}'$. The separation pair {a,b} of G is also a separation pair of B′.

Recalling the definition of the cycle tree of a biconnected graph G (DEFINITION 6.107), we can now prove THEOREM 9.19 by induction (EXERCISE 9.38(3)).

9.19 THEOREM.

Let G be a biconnected graph and let CYCTR(G) be any cycle tree of G in the sense of DEFINITION 6.107. If {a,b} is a separation pair of G then a and b must both lie on some cycle of CYCTR(G).

Using THEOREM 9.19, we can now check the conditions of THEOREM 9.13 on a much smaller class of cycles of G.

9.20 REASONABLE TRICONNECTIVITY ALGORITHM.

Given a biconnected graph G, construct a cycle tree CYCTR(G) for G and then check each of the conditions of THEOREM 9.13 for each cycle that occurs as a vertex of CYCTR(G). Only cycles corresponding to nontrivial bridges need be checked.

The reader should "execute" ALGORITHM 9.20 on the cycle tree of FIGURE 6.108.

The reader has probably noticed that, although REASONABLE TRICONNECTIVITY ALGORITHM 9.20 reduces the number of cycles of G that one need inspect for separation pairs, at each cycle it still is necessary to consider all of the graph G. One can improve this situation by reducing the size of the graph associated with each cycle as we now describe.

FIGURE 9.22(a) shows a biconnected graph G with two bridges A and B.

The carriers of the two bridges are labeled A′ and B′ and are shown in FIGURE 9.22(b) and (c). The cycle $\mathscr{C}$ in this figure is (1,2,3,4,5,1) and $\mathscr{C}'$ is (1,2,3,4,5). Among the separation pairs of B′ we find {5,9} {7,8}, and {2,4}. The first two pairs are separation pairs of G but the pair {2,4} is not a separation pair of G. To avoid introducing new separation pairs, such as {2,4} in this example, we introduce the idea of a *closed carrier* in DEFINITION 9.21 and *reduced carrier*, DEFINITION 9.25.

9.21 DEFINITION.

Let G be a biconnected graph with broken cycle $\mathscr{C}'$. Let X be a nontrivial bridge of $\mathscr{C}$ with SPAN(X) = $(v_1,\ldots,v_p)$, $p \geqslant 3$. The bridge X together with the cycle $(v_1,\ldots,v_p,v_1)$ will be called the *closed carrier* of X. If $p = 2$ (X has just two vertices of attachment to $\mathscr{C}$) then the *closed carrier* of X is defined to be simply the carrier of X.

The closed carriers of the bridges A and B of FIGURE 9.22 are shown in FIGURE 9.23.

9.22 THE CARRIER HAS SEPARATION PAIRS NOT IN G.

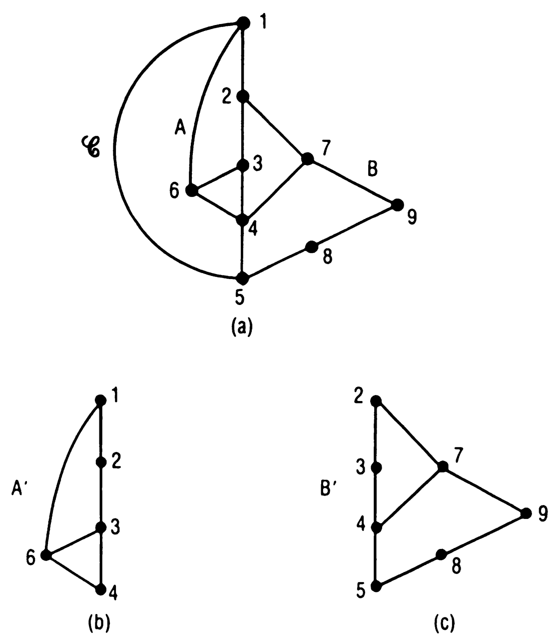


Figure 9.22

9.23 THE CLOSED CARRIERS OF BRIDGES A AND B OF FIGURE 9.22.

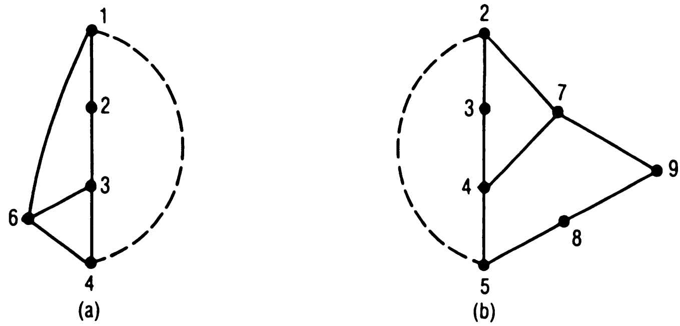


Figure 9.23

We require one more step in connection with the closed carriers of FIGURE 9.23. In both FIGURE 9.23(a) and (b) there are vertices of degree two on the span of the carrier. In FIGURE 9.23(a), for example, the vertex 2 is of degree two. Vertices of degree two can be eliminated by considering any two edges incident on a vertex of degree two as part of the same edge. Equivalently, we replace all proper gaps of the bridge by a single edge. The result for FIGURE 9.23 is shown in FIGURE 9.24. We call the resulting graphs the *reduced carriers* of $\mathscr{C}'$ as stated in DEFINITION 9.25.

9.24 THE REDUCED CARRIERS OF FIGURE 9.23.

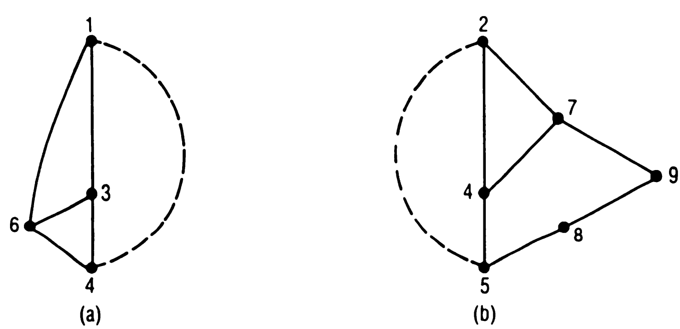


Figure 9.24

9.25 DEFINITION.

Let G be a biconnected graph and let $\mathscr{C}'$ be a broken cycle of G. Let B be a nontrivial bridge and B' its closed carrier with respect to $\mathscr{C}$. If each proper gap of B in B' is replaced by a single edge then the resulting graph will be called a *reduced carrier* of $\mathscr{C}'$. From DEFINITION 9.25 we see that for each broken cycle $\mathscr{C}'$ and each nontrivial bridge B of $\mathscr{C}'$ there is a unique (up to isomorphism) reduced carrier. Thus we may speak of the "set of reduced carriers" of $\mathscr{C}'$. The reader should note that a reduced carrier is actually a reduced *closed* carrier, but we avoid this more cumbersome terminology. One approach to improving REASONABLE TRICONNECTIVITY ALGORITHM 9.20 is provided by THEOREM 9.26.

9.26 THEOREM.

Let G be a biconnected graph and let $\mathscr{C}'$ be a broken cycle of G. Let B' be a reduced carrier of $\mathscr{C}'$ and let {a,b} be vertices of B' not both of which are vertices of $\mathscr{C}'$. All separation pairs of B' are of this type, and such a pair {a,b} is a separation pair of B' if and only if it is a separation pair of G.

Proof. The basic idea of the proof is to consider the pair of vertices {a,b} in B' and compare the separation classes of {a,b} in B' and in G. Assume that not both a and b are vertices of $\mathscr{C}$ and $|\text{RANGE(B)}| \geq 3$. A little thought reveals that the separation classes of {a,b} in B' and in G are exactly the same except for the class associated with the edges of SPAN(B) $= (v_1, \ldots, v_p)$. In G, this class includes all edges of $\mathscr{C}$ and also all edges belonging to bridges of $\mathscr{C}$ other than the bridge B. In B', this class includes the edges obtained from $(v_1, \ldots, v_p)$ by eliminating vertices of proper gaps of B, as described in DEFINITION 9.25, and the edge $\{v_p, v_1\}$ ($|\text{RANGE(B)}| \geq 3$). By our assumptions, this class still contains at least three edges. Thus, if {a,b} is a separation pair for B' it is obviously also a separation pair for G and conversely. That all separation pairs of B' are of this type follows by noting (for $|\text{RANGE(B)}| \geq 3$) that B is a bridge of the cycle of B' obtained from $(v_1, \ldots, v_p)$ as described in DEFINITION 9.25. Apply THEOREM 9.13 to B and this cycle in B'. The case where $|\text{RANGE(B)}| = 2$ is trivial.

It is immediate from THEOREM 9.13 that if $\mathscr{C}'$ does not contain any separation pairs of G then any bridge of $\mathscr{C}'$ is either trivial (one edge) or has at least three "vertices of attachment" to $\mathscr{C}'$ (i.e., three vertices in its range). Thus, given a broken cycle $\mathscr{C}'$ of G, we can use the conditions of THEOREM 9.13 to check for separation pairs of G on $\mathscr{C}'$. If none are found then recursively we check the reduced carriers of $\mathscr{C}'$. Thus THEOREM 9.26 seems to provide the basic recursive structure for finding a separation pair of a biconnected graph G. But there remains one small difficulty! Consider the graph of FIGURE 9.27 with cycle $\mathscr{C}$ and bridge B. Let $\mathscr{C}' = (1,2,3)$. The reduced carrier B' is shown in FIGURE 9.27(b) and is isomorphic to the graph G in this case. This is the same problem confronted by the reader in EXERCISE 6.118(3). If this case occurs then one can choose a new cycle $\tilde{\mathscr{C}}$ in the carrier B′ that includes the edges of SPAN(B). This new cycle has at least one more edge than $\mathscr{C}$ and has the edge e, deleted in going from $\mathscr{C}$ to $\mathscr{C}'$, as a trivial bridge. Thus any closed carrier (and hence reduced carrier) of $\bar{\mathscr{C}}$ has fewer edges than G and allows the recursion to proceed. The reader is asked to develop these ideas in EXERCISE 9.38(4). The new cycle and its bridges (both trivial) for the graph of FIGURE 9.27(a) are shown in FIGURE 9.27(c).

9.27 A GRAPH WITH ISOMORPHIC REDUCED CARRIER.

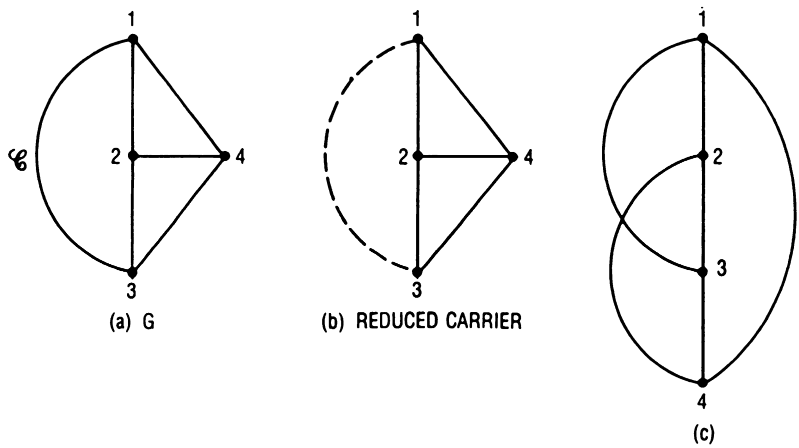


Figure 9.27

We now are in a position to improve on REASONABLE TRICONNECTIVITY ALGORITHM 9.20. G = (V,E) is a biconnected graph with at least three edges ($|E| \geqslant 3$).

9.28 MORE REASONABLE TRICONNECTIVITY ALGORITHM.

(1) Choose a cycle $\mathscr{C}$ in G = (V,E). Let $\mathscr{C}'$ be a corresponding broken cycle.
(2) If $|E| = 3$ then STOP. G has no separation pairs. Otherwise, check the conditions of THEOREM 9.13 and see if $\mathscr{C}$ has a pair of vertices {a,b} that are a separation pair of G. If there is a separation pair STOP, otherwise go to (3).
(3) If $\mathscr{C}$ has only trivial bridges STOP, G has no separation pairs. Otherwise, if $\mathscr{C}$ has only one bridge B, replace $\mathscr{C}$ by a cycle in the carrier B′, choosing the cycle such that it contains the edges of SPAN(B), and go to (2). Otherwise go to (4).
(4) Construct the reduced carriers of the nontrivial bridges of $\mathscr{C}'$ and recursively apply the algorithm to each reduced carrier.

The reader should construct some examples to illustrate the various situations that arise in ALGORITHM 9.28. The purpose of ALGORITHM 9.28 is to find a separation pair of G if one exists. We have actually characterized the set of all separation pairs in the process of proving THEOREM 9.13 and THEOREM 9.26. This result is stated in THEOREM 9.30. We need one more definition to state THEOREM 9.30.

9.29 DEFINITION.

Let G be a biconnected graph and let $\mathscr{C} = (v_1, \ldots, v_p, v_1)$ be a cycle of G with $\mathscr{C}' = (v_1, \ldots, v_p)$. Let {a,b} be a pair of vertices of $\mathscr{C}$. Let P be the path joining a to b in $\mathscr{C}'$ and let Q be the complementary path joining a to b in $\mathscr{C}$. The pair {a,b} will be called *bridge complete* if every bridge B (trivial and nontrivial) of $\mathscr{C}$ either has all of its vertices of attachment in P or all in Q.

A pair {a,b} is thus bridge complete if no bridge of $\mathscr{C}$ has part of its range interior to P and part interior to Q. If a and b are not adjacent vertices of $\mathscr{C}$ and if either P or Q has all its internal vertices of degree 2 in G then {a,b} is bridge complete. In particular, if a and b are not adjacent and are themselves the adjacent vertices to a vertex of degree 2 of G on $\mathscr{C}$ then {a,b} is bridge complete. We now state a "cycle based" necessary and sufficient condition for a pair of vertices of G to be a separation pair.

9.30 THEOREM.

Let G be a biconnected graph and let $\mathscr{C} = (v_1, \ldots, v_p, v_1)$ be a cycle of G with $\mathscr{C}' = (v_1, \ldots, v_p)$. Then a pair {a,b} is a separation pair of G if and only if one of the following conditions holds:

(1) The vertices a and b are on $\mathscr{C}$, are not adjacent, and the pair {a,b} is bridge complete.
(2) The vertices a and b are on $\mathscr{C}$ and they are the vertices of attachment of a nontrivial 2-range bridge of $\mathscr{C}$.
(3) The pair {a,b} is a separation pair for a reduced carrier of $\mathscr{C}'$.

We now indicate the tree structures analogous to the bicomponent tree (DEFINITION 6.104, FIGURE 6.105) and the tree of cycles (DEFINITION 6.107, FIGURE 6.108) that are associated with the recursion of THEOREM 9.30 and ALGORITHM 9.28. Consider the graph G of FIGURE 9.31. Relative to the broken cycle (1,2,3,4), there are two reduced carriers, shown as the sons of G in FIGURE 9.31. The bridge that produces the first carrier has the separation pair {1,3} as vertices of attachment. The pair {1,3} satisfies both conditions (1) and (2) of THEOREM 9.30. There are no other separation pairs on (1,2,3,4). The first son of G is triconnected as may be seen by considering the broken cycle (1,3,5,6). There are no separation pairs on this cycle and all bridges of this cycle are trivial. The second son of G in FIGURE 9.31 has two reduced carriers relative to the cycle (1,3,4,7,8). They are both "3-cycles" as shown in FIGURE 9.31.

9.31 THE REDUCED CARRIER TREE OF A GRAPH.

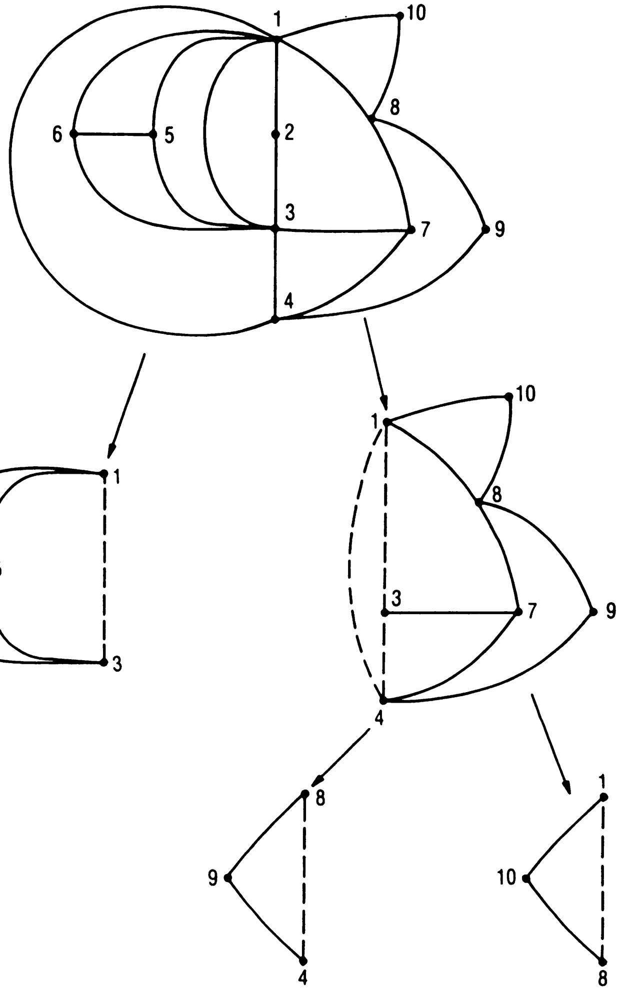


Figure 9.31

Just as in the case of FIGURE 6.105, where we observed that we had too much information labeling the vertices of the tree, we may greatly reduce the structures associated with the vertices of the reduced carrier tree (FIGURE 9.31). At each level of the recursion, we select a broken cycle and extract its reduced carriers. When the carriers are removed, all that is left is a cycle with perhaps some trivial bridges. If this process is repeated the whole graph is eventually decomposed into a tree with vertices labeled by such structures. This process is carried out for the tree of FIGURE 9.31 in FIGURE 9.32. Motivated by DEFINITION 9.33, we call trees such as that of FIGURE 9.32 *Hamiltonian structure trees*.

9.32 HAMILTONIAN STRUCTURE TREE OF FIGURE 9.31.

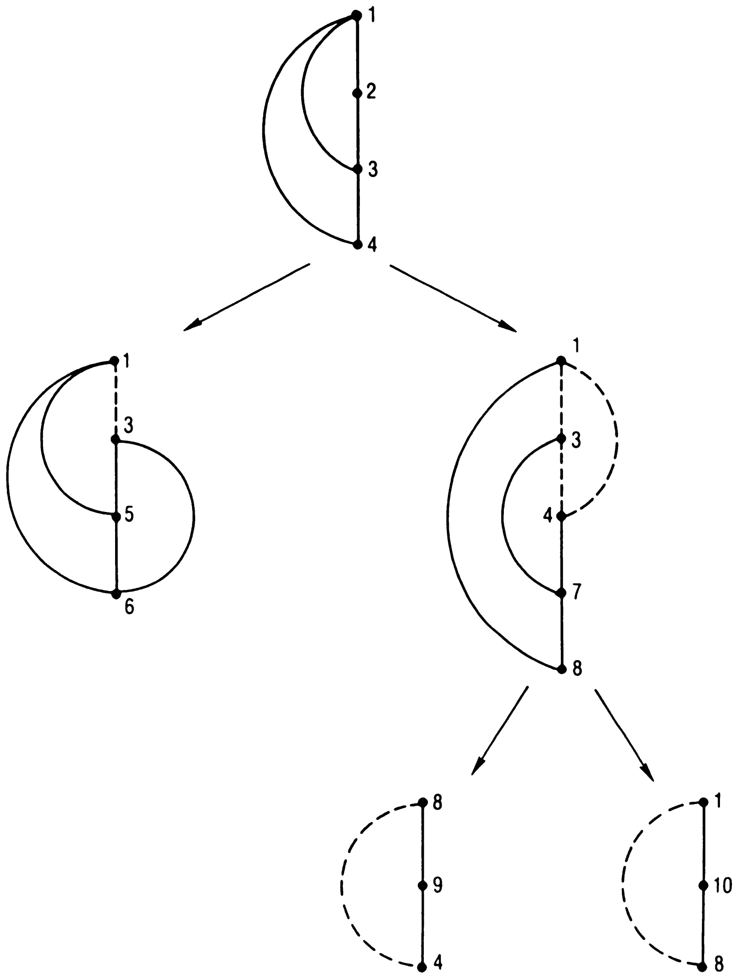


Figure 9.32

9.33 DEFINITION.

A graph G = (V,E) will be called *Hamiltonian* if there is a cycle $\mathscr{C}$ that has all of V as a vertex set. The cycle $\mathscr{C}$ is called a *Hamiltonian cycle for* G.

Note that all vertices of the Hamiltonian structure tree of FIGURE 9.32 are Hamiltonian graphs with Hamiltonian cycles as indicated by the vertical sequence of vertices. There are a number of interesting questions about Hamiltonian structure trees that we leave for the reader to explore in EXERCISE 9.38(5). The

Hamiltonian structure tree points out quite strikingly the importance of the test for bridge completeness, THEOREM 9.30(1). One may develop a depth first version of the search for separation pairs along the lines suggested by ALGORITHM 9.34.

9.34 DEPTH FIRST SEARCH FOR SEPARATION PAIRS.

(1) Given a biconnected graph G, construct a Hamiltonian structure tree such as that shown in FIGURE 9.32.
(2) Inspect the vertices of the tree constructed in step (1) in postorder, reconstructing the reduced carrier tree (see FIGURE 9.31 for an example). When a vertex of the Hamiltonian structure tree is arrived at in postorder, the carriers of its sons have been constructed and all associated separation pairs have been found. Find all separation pairs on the cycle associated with that vertex and construct the carrier associated with that vertex.

There remains two basic questions. First, how do we construct the Hamiltonian structure tree in the first place, and second, how do we perform the test for bridge completeness? Both of these procedures can be carried out using lineal spanning trees and the ideas associated with TOPICS I and II. We refer the reader to the references at the end of the chapter for details. One can construct the segment forest of the segment graph of each cycle as in TOPIC I. If the segment forest is not a tree then we know the graph is not triconnected. This approach leads to a good but not the best possible test for triconnectivity. The best test for triconnectivity of biconnected graphs $G = (V,E)$ has worst-case complexity $0(|E|)$. It turns out that the question of the connectivity of the segment forest can be settled without constructing all of the edges of the forest. This idea leads to the linear in edges algorithm. One can, in the process of constructing the Hamiltonian structure tree of graph G, stop decomposing the reduced carriers as soon as a triconnected reduced carrier is found. One would then have a tree consisting of Hamiltonian structures and more complex triconnected graphs for some (perhaps none) of the terminal nodes. There is a natural way of splitting the Hamiltonian structures further into smaller Hamiltonian structures until the whole graph is broken up into "triconnected components." There are different approaches to the problem of decomposing a graph into triconnected components that we leave to the reader to explore in EXERCISE 9.38(6). We conclude our discussion of triconnectivity by deriving one of the most important facts about triconnected graphs, namely, that they have only one embedding (in a very natural sense) in the sphere or plane.

Two different embeddings of the same graph are shown in FIGURE 9.36. We may regard these embeddings as being in the plane or on the surface of a sphere. Intuitively, an "embedding" of a graph $G = (V,E)$ is a representation or drawing of the graph where the vertices V are represented by distinct points on the surface and the edges E are represented by smooth arcs joining the respective vertices.

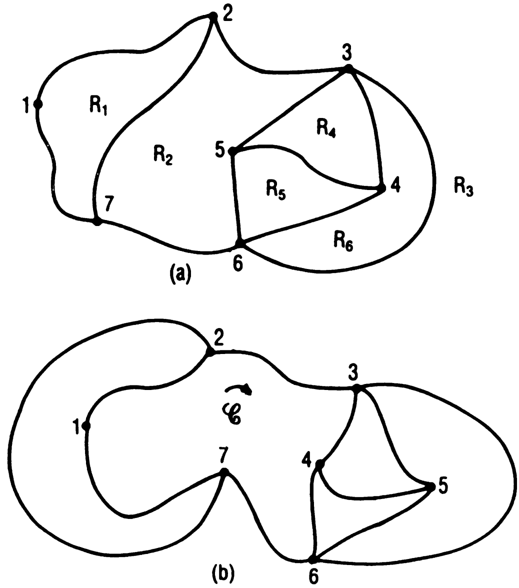
An embedding is planar if the different arcs representing edges have, at most, endpoints in common. The embeddings of FIGURE 9.36 are planar. A graph G is *planar* if it has a planar embedding. The embedding of FIGURE 9.36(a) has six *regions* or *domains* labeled $R_1, R_2, \ldots, R_6$. If we regard the embedding as being on the surface of a sphere then all domains $R_1, \ldots, R_6$ are finite. For this reason it is slightly better from an intuitive point of view to think of spherical rather than planar embeddings of graphs. Regarding the embedding of FIGURE 9.36(a) as being on the surface of a sphere, we may identify with each region its bounding cycle. With R_1 we associate the cycle (1,2,7,1) where the vertices are listed in clockwise order viewed from the interior of the region. Thus with R_2 we associate the cycle (2,3,5,6,7,2), etc. These bounding cycles are called the "domain boundaries" of the embedding. If the vertices and smooth arcs of a spherical embedding are allowed to drift around on the surface of the sphere without crossing over each other, we get the intuitive idea of "isotopically equivalent" embeddings. Note that the domain boundaries of two embeddings are the same if and only if the two embeddings are isotopically equivalent. We take this as intuitively obvious for our purposes. We express this fact in DEFINITION 9.35.

9.35 DEFINITION.

Two planar or spherical embeddings of a graph G are regarded as equivalent (isotopically) if they have the same domain boundaries.

Referring to FIGURE 9.36, the reader will note that the cycle $\mathscr{C} = (1,2,3,4,6,7,1)$ is a domain boundary in FIGURE 9.36(b) but not in 9.36(a). Thus these two embeddings are not equivalent. The basic result we require is that if the graph G is triconnected there can be only one embedding.

9.36 TWO DIFFERENT EMBEDDINGS OF THE SAME GRAPH.



$\mathscr{C}$ is a domain boundary of (b) but not of (a).

Figure 9.36

9.37 THEOREM.

A planar triconnected graph has only one embedding in the sphere or plane up to isotopic equivalence.

Proof. We illustrate the idea of the proof using FIGURE 9.36. Suppose there are two inequivalent embeddings. Consider the cycle $\mathscr{C} = (1,2,3,4,6,7,1)$ which is a domain boundary in FIGURE 9.36(b) but not in FIGURE 9.36(a). Because $\mathscr{C}$ is not a domain boundary in FIGURE 9.36(a) it must have at least two bridges. Because $\mathscr{C}$ is a domain boundary in FIGURE 9.36(b), these bridges can all be embedded outside of $\mathscr{C}$ and hence none of them are joined by an edge in the bridge graph of $\mathscr{C}$ (DEFINITION 6.110). Thus the bridge graph is "discrete" or without edges. Hence it is disconnected and, by THEOREM 9.13, $\mathscr{C}$ would contain a separation pair, contradicting triconnectivity of G.

9.38 EXERCISE.

(1) Characterize all graphs $G = (V,E)$ for which the set $\mathscr{S}_k$ of DEFINITION 9.2 is empty for all k. Consider also the case of more general graphs (see

DEFINITION 6.3). Prove that for all other graphs the connectivity is less than or equal to the minimal degree of a vertex.

(2) Prove that the only mutually arching components of a bridge graph are as shown in FIGURE 9.6. Use this result to prove THEOREM 9.7.

(3) Give a careful inductive proof of THEOREM 9.19.

(4) Let G = (V,E) be a biconnected graph and let B be a bridge of a broken cycle $\mathscr{C}'$ of G. Let B′ be the carrier of B. Prove that there always is a cycle $\mathscr{C}'$ of B′ that contains all of the edges of SPAN(B). Prove that the idea of FIGURE 9.27 used to continue the recursion of ALGORITHM 9.28 (step (3)) always works.

(5) Describe a good algorithm (analogous to the planarity test of TOPIC I for example) for constructing Hamiltonian structure trees and carrying out ALGORITHM 9.34, DEPTH-FIRST SEARCH FOR SEPARATION PAIRS. *Discuss data structures.*

(6) Define a notion of ''triconnected components'' and construct them algorithmically. Are your tricomponents unique in some sense? (See the references at the end of PART II, in particular the papers by Hopcroft and Tarjan, MacLane, and Vo.)