ON A CONTROL PROBLEM FOR A HEAVY CHAIN WITH LOADS

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A control problem for a mixed hyperbolic problem is considered. The dependence between the control and the solution is given explicitly.

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In [1] some control problems for heavy chain systems, i.e. trolleys carrying a fixed length chain that may carry a load, were addressed in the partial derivatives equations framework.

The problem that follows was formulated as a result of the discussion on this theme.

Consider a crane carrying a mass $m > 0$ at the end of a heavy chain of length $l > 0$. The crane is put onboard the ship, and the load is traveling under the sea for civil engineering applications. It is desired to find a control algorithm to stabilize the load while system is subjected to disturbances consisting of a subsea stream occurring at a certain depth: over the subinterval of $(0, l)$.

Mathematically, the system is governed by the following mixed problem (assuming a homogeneous mass distribution):

$$\frac{\partial}{\partial x}\Big((x+m)g\frac{\partial u(x,t)}{\partial x}\Big) - \frac{\partial^2 u(x,t)}{\partial t^2} + f(x,t) = 0, \tag{1}$$

in the domain

$$(x,t): \quad 0 < x < l, \quad t > 0,$$

with homogeneous initial conditions

$$u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = 0, \tag{2}$$

and boundary value conditions

$$\frac{\partial^2 u(0,t)}{\partial t^2} = g\frac{\partial u(0,t)}{\partial x}, \quad u(l,t) = v(t), \tag{3}$$

with g the gravity constant, and $v(t)$ is freely chosen control variable. The function f represents the stream forces. The question is: how $u(0,t)$ depends on f and v?

In the present paper we give an answer to this question.

By substituting

$$y = 2\sqrt{\frac{x+m}{g}}, \quad y_1 = 2\sqrt{\frac{m}{g}}, \quad y_2 = 2\sqrt{\frac{l+m}{g}},$$

$$u_1(y,t) = u(x,t), \quad f_1(y,t) = f(x,t)$$

we reduce it to a problem for the equation

$$\frac{\partial^2 u_1(y,t)}{\partial y^2} + \frac{1}{y}\frac{\partial u_1(y,t)}{\partial y} - \frac{\partial^2 u_1(y,t)}{\partial t^2} + f_1(y,t) = 0, \tag{4}$$

in the domain

$$(y,t): \quad y_1 < y < y_2, \quad t > 0,$$

with homogeneous initial conditions

$$u_1(y,0) = 0, \quad \frac{\partial u_1(y,0)}{\partial t} = 0, \tag{5}$$

and boundary value conditions

$$\frac{\partial^2 u_1(y_1,t)}{\partial t^2} = \frac{2}{y_1}\frac{\partial u_1(y_1,t)}{\partial y}, \quad u_1(y_2,t) = v(t). \tag{6}$$

To solve (4)-(6) we use the Laplace transform:

$$X(y,s) = \int_0^{+\infty} e^{-st} u_1(y,t)dt,$$

$$F(y,s) = \int\limits_0^{+\infty} e^{-st} f_1(y,t)dt, \quad V(s) = \int\limits_0^{+\infty} e^{-st} v(t)dt.$$

As a result we obtain a problem for the ordinary differential equation with a parameter:

$$\frac{\partial^2 X(y,s)}{\partial y^2} + \frac{1}{y}\frac{\partial X(y,s)}{\partial y} - s^2 X(y,t) + F(y,t) = 0, \qquad (7)$$

on the interval

$$y_1 < y < y_2$$

with condition

$$\lim_{s \to +\infty} s^2 X(y,s) = 0 \qquad (8)$$

and boundary value conditions

$$s^2 X(y_1,s) = \frac{2}{y_1}\frac{\partial X(y_1,s)}{\partial y}, \quad X(y_2,s) = V(s). \qquad (9)$$

By substituting

$$z = isy, \quad X(y,s) = Z(z,s), \quad H(z,s) = F(y,s),$$

we transform (7) to the equation

$$z\frac{\partial^2 Z(z,s)}{\partial z^2} + \frac{\partial Z(z,s)}{\partial z} + zZ(z,t) + H(z,t) = 0. \qquad (10)$$

The general solution for (10) may be expressed as follows:

$$Z(z,s) = A(s)J_0(z) + B(s)Y_0(z) + \frac{\pi}{2}\int\limits_0^z (Y_0(z)J_0(\xi) - J_0(z)Y_0(\xi))\frac{\xi}{s^2}H(\xi,s)d\xi,$$

where $J_0(z)$ and $Y_0(z)$ are the Bessel functions. Hence:

$$X(y,s) = A(s)J_0(isy) + B(s)Y_0(isy) -$$

$$-\frac{\pi}{2}\int\limits_0^y (Y_0(isy)J_0(is\tau) - J_0(isy)Y_0(is\tau))\tau F(\tau, s)d\tau.$$

It is more convenient to use the modified Bessel functions, using the equality [2, p. 95]: $I_0(z)K^{0\prime(z)-I_0'(z)K_0(z)=-1/z}$,

$$X(y, s) = A(s)I_0(sy) + B(s)K_0(sy) + G(y, s),$$

where

$$G(y, s) = -\int\limits_0^y (K_0(sy)I_0(s\tau) - I_0(sy)K_0(s\tau))\tau F(\tau, s)d\tau.$$

We find $A(s)$ and $B(s)$ by the boundary value conditions, using the formulae $I_0'(z) = I_1(z), K_0'(z) = -K_1(z)$, $zI_2(z) = zI_0(z) - 2I_1(z) = zI_0(z) - 2I_0'(z), zK_2(z) = zK_0(z) + 2K_1(z) = zK_0(z) - 2K_0'(z)$, [2, p. 93]:

$$sy_1(I_2(sy_1)A(s) + K_2(sy_1)B(s)) = \frac{2}{s}\frac{\partial G}{\partial y}(y_1, s) - sy_1G(y_1, s),$$

$$I_0(sy_2)A(s) + K_0(sy_2)B(s) = V(s) - G(y_2, s).$$

The determinant for this system is

$$\Delta(s) = (sy_1I_0(sy_1) - 2I_1(sy_1))K_0(sy_2) - (sy_1K_0(sy_1) + 2K_1(sy_1))I_0(sy_2) =$$

$$= sy_1(I_2(sy_1)K_0(sy_2) - K_2(sy_1)I_0(sy_2)).$$

The system's solution is

$$A(s) = \Big(\Big(\frac{2}{s}\frac{\partial G}{\partial y}(y_1, s) - sy_1G(y_1, s)\Big)K_0(sy_2) +$$

$$+ sy_1K_2(sy_1)(G(y_2, s) - V(s))\Big)/\Delta(s),$$

$$B(s) = \Big(sy_1I_2(sy_1)(V(s) - G(y_2, s)) -$$

$$-\Big(\frac{2}{s}\frac{\partial G}{\partial y}(y_1,s)-sy_1G(y_1,s)\Big)I_0(sy_2)\Big)/\Delta(s).$$

Theorem 1. *All roots of the determinant $\Delta(s)$ lie on the imaginary axis.*

Proof. Since $\Delta(-iy)=\overline{\Delta(iy)}$, we may explore only the roots with positive imaginary part. We consider the corresponding spectral problem

$$X''(y)+\frac{X'(y)}{y}+\lambda X(y)=0,\quad 0<y_1<y_2,$$

$$2X'(y_1)+\lambda y_1X(y_1)=0,\quad X(y_2)=0.$$

By substituting $z=\sqrt{\lambda}y$, we obtain the equation

$$X''(z)+\frac{X'(z)}{z}+X(z)=0,\quad 0<z_1<z_2, \tag{11}$$

and boundary value conditions

$$2X'(z_1)+z_1X(z_1)=0,\quad X(z_2)=0. \tag{12}$$

We substitute the general solution for (11),

$$X(z)=AJ_0(z)+BY_0(z),$$

into boundary value conditions (12). We obtain the following system:

$$2AJ_0'(z_1)+2BY_0'(z_1)+Az_1J_0(z_1)+Bz_1Y_0(z_1)=0,$$

$$AJ_0(z_2)+BY_0(z_2)=0,$$

where we exclude constants A and B. As a result, we write the characteristic equation for the spectral problem:

$$\Delta=(2J_0'(z_1)+z_1J_0(z_1))Y_0(z_2)-$$

$$-(2Y_0'(z_1) + z_1 Y_0(z_1))J_0(z_2) = 0,$$

or

$$\Delta = (2J_0'(\sqrt{\lambda}y_1) + \sqrt{\lambda}y_1 J_0(\sqrt{\lambda}y_1))Y_0(\sqrt{\lambda}y_2)-$$
$$-(2Y_0'(\sqrt{\lambda}y_1) + \sqrt{\lambda}y_1 Y_0(\sqrt{\lambda}y_1))J_0(\sqrt{\lambda}y_2) = 0. \tag{13}$$

The roots of characteristic equation (13) are positive numbers. If λ_n and λ_m are distinct eigenvalues, $X_n(y)$ and $X_m(y)$ are their corresponding eigenfunctions, then the following relation holds:

$$(\lambda_m - \lambda_n)\int\limits_{y_1}^{y_2} yX_n(y)X_m(y)dy =$$

$$= \int\limits_{y_1}^{y_2} [(yX_n''(y) + X_n'(y))X_m(y) - (yX_m''(y) + X_m'(y))X_n(y)]dy =$$

$$= \int\limits_{y_1}^{y_2} [(yX_n'(y))'X_m(y) - (yX_m'(y))'X_n(y)]dy =$$

$$= y_1 X_m'(y_1)X_n(y_1) - y_1 X_n'(y_1)X_m(y_1) = (\lambda_n - \lambda_m)\frac{y_1^2}{2}X_n(y_1)X_m(y_1).$$

It delivers the equality

$$\int\limits_{y_1}^{y_2} yX_n(y)X_m(y)dy + \frac{y_1^2}{2}X_n(y_1)X_m(y_1) = 0.$$

This equality allows us [3, 4] to introduce in a Hilbert space H, which is the direct product of spaces $L_2(y_1, y_2)$ and C, a scalar product and a corresponding positive self-adjoint compact operator with the same eigenvalues as the spectral problem. In fact, the considered spectral problem may be rewritten in the following integral form:

$$X(y) = \lambda\int\limits_{y_1}^{y_2} \ln\Big(\frac{2y_2}{|\xi - y| + (\xi + y)}\Big)\xi X(\xi)d\xi-$$

$$-\frac{\lambda y_1^2}{2}\ln\Big(\frac{y}{y_2}\Big)X(y_1).$$

If $U=(X(y),X_1)$ and $W=(Z(y),Z_1)$ are two elements of H with the scalar product

$$(U,W)=\int\limits_{y_1}^{y_2} yX(y)\overline{Z(y)}dy+\frac{y_1^2}{2}X_1\overline{Z_1},$$

then such operator L is delivered by the formula

$$LU=\Big(\int\limits_{y_1}^{y_2}\ln\Big(\frac{2y_2}{|\xi-y|+(\xi+y)}\Big)\xi X(\xi)d\xi-$$

$$-\frac{\lambda y_1^2}{2}\ln\Big(\frac{y}{y_2}\Big)X_1,\int\limits_{y_1}^{y_2}\ln\Big(\frac{y_2}{\xi}\Big)\xi X(\xi)d\xi-$$

$$-\frac{\lambda y_1^2}{2}\ln\Big(\frac{y_1}{y_2}\Big)X_1\Big).$$

The spectral problem takes the form $LU=\mu U$, where $\mu=1/\lambda$.

Theorem 2. *Asymptotical behavior as Re $s\to+\infty$*

$$\Delta(s)\sim-\frac{\sqrt{y_1/y_2}}{2}e^{(y_2-y_1)s}.$$

Proof. According to asymptotics of the modified Bessel functions as *Re* $z\to+\infty$,

$$I_\nu(z)\sim\frac{1}{\sqrt{2\pi z}}e^z,\quad K_\nu(z)\sim\frac{\sqrt{\pi}}{\sqrt{2z}}e^{-z}:$$

$$\Delta(s) \sim \frac{sy_1 - 2}{2s\sqrt{y_1 y_2}} e^{(y_1-y_2)s} - \frac{sy_1 + 2}{2s\sqrt{y_1 y_2}} e^{(y_2-y_1)s} \sim$$

$$\sim -\frac{sy_1 + 2}{2s\sqrt{y_1 y_2}} e^{(y_2-y_1)s} \sim -\frac{\sqrt{y_1/y_2}}{2} e^{(y_2-y_1)s}.$$

Corollary. $X(y,s)$-image (i.e. origin of that function) equals to zero at $t<0$.

The roots of the determinant are simple poles (see below) located on the imaginary axes. To discard the corresponding singularities, we have to discard the residues in the roots. Let $\Delta(\lambda_k) = 0, Im\lambda_k > 0, k = 1,2,3,...$, then

$$res_{s=\lambda_k} X(y,s) = I_0(\lambda_k y) res_{s=\lambda_k} A(s) + K_0(\lambda_k y) res_{s=\lambda_k} B(s),$$

using the equalities $I_1'(z) = -\frac{1}{z} I_1(z) - I_0(z)$ and $K_1'(z) = -\frac{1}{z} K_1(z) + K_0(z)$,

$$(sy_1 I_0(sy_1) - 2I_1(sy_1))K_0(sy_2) = (sy_1 K_0(sy_1) + 2K_1(sy_1))I_0(sy_2).$$

$$\Delta'(\lambda_k) res_{s=\lambda_k} X(y_1,s) = \Delta'(\lambda_k)\Big(I_0(\lambda_k y_1) res_{s=\lambda_k} A(s) + K_0(\lambda_k y_1) res_{s=\lambda_k} B(s)\Big) =$$

$$= I_0(\lambda_k y_1)\Big(\Big(\frac{2}{\lambda_k}\frac{\partial G}{\partial y}(y_1,\lambda_k) - \lambda_k y_1 G(y_1,\lambda_k)\Big)K_0(\lambda_k y_2) +$$

$$+(\lambda_k y_1 K_0(\lambda_k y_1) + 2K_1(\lambda_k y_1))(G(y_2,\lambda_k) - V(\lambda_k))\Big) +$$

$$+K_0(\lambda_k y_1)\Big((\lambda_k y_1 I_0(\lambda_k y_1) - 2I_1(\lambda_k y_1))(V(\lambda_k) - G(y_2,\lambda_k)) -$$

$$-\Big(\frac{2}{\lambda_k}\frac{\partial G}{\partial y}(y_1,\lambda_k) - \lambda_k y_1 G(y_1,\lambda_k)\Big)I_0(\lambda_k y_2)\Big) =$$

$$= I_0(\lambda_k y_1)\Big(\Big(\frac{2}{\lambda_k}\frac{\partial G}{\partial y}(y_1,\lambda_k) - \lambda_k y_1 G(y_1,\lambda_k)\Big)K_0(\lambda_k y_2) +$$

$$+(\lambda_k y_1 I_0(\lambda_k y_1) - 2I_1(\lambda_k y_1))\frac{K_0(\lambda_k y_2)}{I_0(\lambda_k y_2)}(G(y_2,\lambda_k) - V(\lambda_k))\Big) +$$

$$+K_0(\lambda_k y_1)\Big((\lambda_k y_1 I_0(\lambda_k y_1) - 2I_1(\lambda_k y_1))(V(\lambda_k) - G(y_2,\lambda_k)) -$$

$$-\Big(\frac{2}{\lambda_k}\frac{\partial G}{\partial y}(y_1,\lambda_k) - \lambda_k y_1 G(y_1,\lambda_k)\Big)I_0(\lambda_k y_2)\Big) =$$

$$= \Big(\Big(\frac{2}{\lambda_k}\frac{\partial G}{\partial y}(y_1,\lambda_k) - \lambda_k y_1 G(y_1,\lambda_k)\Big) +$$

$$+\frac{\lambda_k y_1 I_0(\lambda_k y_1) - 2I_1(\lambda_k y_1)}{I_0(\lambda_k y_2)}(G(y_2,\lambda_k) - V(\lambda_k))\Big) \times$$

$$\times (I_0(\lambda_k y_1)(K_0(\lambda_k y_2) - K_0(\lambda_k y_1)(I_0(\lambda_k y_2)).$$

By discarding the residues, we obtain the condition:

$$V(\lambda_k) = G(y_2,\lambda_k) - \frac{\lambda_k^2 y_1 G(y_1,\lambda_k) - 2G'_y(y_1,\lambda_k)}{\lambda_k^2 y_1 I_2(\lambda_k y_1)} I_0(\lambda_k y_2).$$

We also note that if f and v may be continued analytically as entire or meromorphic functions, then so does X.

Theorem 3. *All zeroes of the determinant $\Delta(s)$ are simple.*

Proof. We set $k = y_2/y_1 > 0, x = z_1$, then $z_2 = kx$, and the determinant may be rewritten as follows:

$$\Delta(-ix) = (2J'_0(x) + xJ_0(x))Y_0(kx) - (2Y'_0(x) + xY_0(x))J_0(kx).$$

We consider a function

$$P(x) = \frac{\Delta(-ix)}{(2J'_0(x) + xJ_0(x))Y_0(kx)} = \frac{2Y'_0(x) + xY_0(x)}{2J'_0(x) + xJ_0(x)} - \frac{J_0(kx)}{Y_0(kx)}.$$

We prove that $P(x)$ has no multiple roots at $x>0$. On the contrary, if $P(x)=P'(x)=0$, then we set $v(f,g)=f'g-g'f$, so:

$$v(xJ_0(x)+2J_0'(x),xY_0(x)+2Y_0'(x))=$$

$$=(xJ_0(x)+2J_0'(x))'(xY_0(x)+2Y_0'(x))-$$
$$-(xJ_0(x)+2J_0'(x))(xY_0(x)+2Y_0'(x))'=$$
$$=(J_0+xJ_0'(x)+2J_0''(x))(xY_0(x)+2Y_0'(x))-$$
$$-(J_0+xJ_0'(x)+2J_0''(x))(xY_0(x)+2Y_0'(x))'=$$
$$=(x^2-2)v(J_0(x),Y_0(x))+2x(J_0''(x)Y_0(x)-Y_0''(x)J_0(x))+$$
$$+4(J_0''(x)Y_0'(x)-Y_0''(x)J_0'(x))=$$
$$=(x^2-2)v(J_0(x),Y_0(x))+2((J_0'(x)+xJ_0(x))Y_0(x)-(Y_0'(x)+xY_0(x))J_0(x))-$$
$$-(4/x)((J_0'(x)+xJ_0(x))Y_0'(x)-(Y_0'(x)+xY_0(x))J_0'(x))=$$
$$=x^2v(J_0(x),Y_0(x))=-2x,$$
$$v(J_0(kx),Y_0(kx))=k(J_0'(kx)Y_0(kx)-J_0'(kx)Y_0(kx))=-2k/x.$$

Hence, the equalities $P(x)=P'(x)=0$ take the form:

$$\frac{2Y_0'(x)+xY_0(x)}{2J_0'(x)+xJ_0(x)}=\frac{J_0(kx)}{Y_0(kx)},$$

$$\Big(Y_0(x)+2\frac{Y_0'(x)}{x}\Big)^2=\Big(\frac{Y_0(kx)}{\sqrt{k}}\Big)^2,\quad \Big(J_0(x)+2\frac{J_0'(x)}{x}\Big)^2=\Big(\frac{J_0(kx)}{\sqrt{k}}\Big)^2.$$

Or, taking into account the equality

$$\frac{d}{dx}H_0^{(1)}(x)=-H_1^{(1)}(x)$$

($H_\nu^{(1)}(x)=J_\nu(x)+iY_\nu(x)$ is the Hankel function of the first kind):

$$H_0^{(1)}(x)-2H_1^{(1)}(x)/x=\pm(J_0(kx)\pm iY_0(kx))\sqrt{k},x>0$$

(any two of four signs are possible).

For certainty we consider the first case (the other cases may be done in a quite same manner, as it follows from the proof given below):

$$H_0^{(1)}(x) - 2H_1^{(1)}(x)/x = H_0^{(1)}(kx)\sqrt{k}.$$

We use the known formula [2]:

$$H_\nu^{(1)}(x) = \Big(\frac{2}{\pi x}\Big)^{1/2} \frac{e^{i(x-\pi\nu/2-\pi/4)}}{\Gamma(\nu+1/2)} \int\limits_0^\infty e^{-u}u^{\nu-1/2}\Big(1+\frac{iu}{2x}\Big)^{\nu-1/2} du.$$

The main equality takes the form:

$$\int\limits_0^\infty e^{-u}u^{-1/2}\Big(1+\frac{iu}{2x}\Big)^{-1/2} du + \frac{4}{ix}\int\limits_0^\infty e^{-u}u^{1/2}\Big(1+\frac{iu}{2x}\Big)^{1/2} du =$$

$$= \frac{e^{i(k-1)x}}{\sqrt{k}} \int\limits_0^\infty e^{-u}u^{-1/2}\Big(1+\frac{iu}{2kx}\Big)^{-1/2} du.$$

Integrating the second integral by parts, we obtain:

$$\int\limits_0^\infty e^{-u}u^{-1/2}\Big(1+\frac{iu}{2x}\Big)^{-1/2}\Big(1+\frac{2i}{x}-\frac{2u}{x^2}\Big) du =$$

$$= \frac{e^{i(k-1)x}}{\sqrt{k}} \int\limits_0^\infty e^{-u}u^{-1/2}\Big(1+\frac{iu}{2kx}\Big)^{-1/2} du.$$

Finally, using the formula

$$\int\limits_0^\infty e^{-u}u^{-1/2}\Big(1+\frac{iu}{2x}\Big)^{-1/2}\Big(-\frac{2u}{x^2}\Big) du =$$

$$= \Big(-\frac{1}{x^2}\Big)\int\limits_0^\infty e^{-u}u^{-1/2}\Big(1+\frac{iu}{2x}\Big)^{-3/2} du,$$

we obtain the equality

$$\int\limits_0^\infty e^{-u}u^{-1/2}\Big(1+\frac{iu}{2x}\Big)^{-1/2}\Big(1+\frac{2i}{x}-\frac{2}{x(2x+iu)}\Big)du=$$

$$=\frac{e^{i(k-1)x}}{\sqrt{k}}\int\limits_0^\infty e^{-u}u^{-1/2}\Big(1+\frac{iu}{2kx}\Big)^{-1/2}du,$$

which we rewrite as follows:

$$\int\limits_0^\infty e^{it}t^{-1/2}\Big(1+\frac{t}{2x}\Big)^{-1/2}\Big(1+\frac{2i}{x}-\frac{2}{x(2x+t)}\Big)dt=$$

$$=\frac{e^{i(k-1)x}}{\sqrt{k}}\int\limits_0^\infty e^{it}t^{-1/2}\Big(1+\frac{t}{2kx}\Big)^{-1/2}dt.$$

We multiply both sides of this equality by $e^{i\alpha}$. Let ϕ is a value which delivers the maximum to the function

$$f(\alpha)=Im\int\limits_0^\infty e^{i(t+\alpha)}t^{-1/2}\Big(1+\frac{t}{2x}\Big)^{-1/2}\Big(1+\frac{2i}{x}-\frac{2}{x(2x+t)}\Big)dt=$$

$$=\int\limits_0^\infty \cos(t+\alpha)t^{-1/2}\Big(1+\frac{t}{2x}\Big)^{-1/2}\frac{2}{x}dt+$$

$$+\int\limits_0^\infty \sin(t+\alpha)t^{-1/2}\Big(1+\frac{t}{2x}\Big)^{-1/2}\Big(1-\frac{2}{x(2x+t)}\Big)dt.$$

The statement of the theorem will be proved if we prove the inequality

$$\max_\alpha Im\int\limits_0^\infty e^{i(t+\alpha)}t^{-1/2}\Big(1+\frac{t}{2kx}\Big)^{-1/2}dt=$$

$$= \max_{\alpha} \int\limits_0^{\infty} \sin(t+\alpha) t^{-1/2} \Big(1 + \frac{t}{2kx}\Big)^{-1/2} dt \leqslant f(\phi),$$

or

$$\max_{\alpha} g(\alpha) \leqslant f(\phi),$$

where

$$g(\alpha) = \int\limits_0^{\infty} \sin(t+\alpha) t^{-1/2} \Big(1 + \frac{t}{2x}\Big)^{-1/2} dt.$$

Let ψ is a value which delivers the maximum to $g(\alpha)$. Then

$$g'(\psi) = \int\limits_0^{\infty} \cos(t+\psi) t^{-1/2} \Big(1 + \frac{t}{2x}\Big)^{-1/2} dt = 0,$$

$$\cos(\psi) \int\limits_0^{\infty} (\cos t) t^{-1/2} \Big(1 + \frac{t}{2x}\Big)^{-1/2} dt =$$

$$= \sin(\psi) \int\limits_0^{\infty} (\sin t) t^{-1/2} \Big(1 + \frac{t}{2x}\Big)^{-1/2} dt.$$

Hence

$$f(\psi - \alpha) = \int\limits_0^{\infty} \cos(t+\psi-\alpha) t^{-1/2} \Big(1 + \frac{t}{2x}\Big)^{-1/2} \frac{2}{x} dt +$$

$$+ \int\limits_0^{\infty} \sin(t+\psi-\alpha) t^{-1/2} \Big(1 + \frac{t}{2x}\Big)^{-1/2} \Big(1 - \frac{1}{x^2(1+t/(2x))}\Big) dt =$$

$$= \Big(\frac{2}{x}\sin\alpha + \cos\alpha\Big) \int\limits_0^{\infty} \sin(t+\psi) t^{-1/2} \Big(1 + \frac{t}{2x}\Big)^{-1/2} dt -$$

$$-\frac{\cos\alpha}{x^2}\int\limits_0^\infty \sin(t+\psi)t^{-1/2}\Big(1+\frac{t}{2x}\Big)^{-3/2}dt+$$

$$+\frac{\sin\alpha}{x^2}\int\limits_0^\infty \cos(t+\psi)t^{-1/2}\Big(1+\frac{t}{2x}\Big)^{-3/2}dt.$$

We set

$$A=\int\limits_0^\infty \sin(t+\psi)t^{-1/2}\Big(1+\frac{t}{2x}\Big)^{-1/2}dt,$$

$$B=\int\limits_0^\infty \sin(t+\psi)t^{-1/2}\Big(1+\frac{t}{2x}\Big)^{-3/2}dt,$$

$$C=\int\limits_0^\infty \cos(t+\psi)t^{-1/2}\Big(1+\frac{t}{2x}\Big)^{-3/2}dt.$$

Then the demanded inequality may be rewritten as follows:

$$\Big(A-\frac{B}{x^2}\Big)\cos\alpha+\Big(\frac{2A}{x}+\frac{C}{x^2}\Big)\sin\alpha\geqslant A.$$

It is solvable if

$$\Big(A-\frac{B}{x^2}\Big)^2+\Big(\frac{2A}{x}+\frac{C}{x^2}\Big)^2\geqslant A^2,$$

or

$$2A(2A-B)x^2+B^2+4xAC+C^2\geqslant 0.$$

It is more convenient to rewrite this inequality as follows: $\psi=\pi/4+\theta$,

$$A_1=\int\limits_0^\infty e^{-2x\nu}\nu^{-1/2}(1+\nu^2)^{-1/4}\cos\Big(\theta-\frac{1}{2}\operatorname{arctg}\nu\Big)d\nu>0,$$

$$B_1=\int\limits_0^\infty e^{-2x\nu}\nu^{-1/2}(1+\nu^2)^{-3/4}\cos\Big(\theta-\frac{3}{2}\operatorname{arctg}\nu\Big)d\nu>0,$$

$$C_1 = -\int_0^\infty e^{-2x\nu}\nu^{-1/2}(1+\nu^2)^{-3/4}\sin\Bigl(\theta - \frac{3}{2}\operatorname{arctg}\nu\Bigr)d\nu.$$

We note that

$$\int_0^\infty e^{-2x\nu}\nu^{-1/2}(1+\nu^2)^{-1/4}\cos\Bigl(\theta + \frac{\pi}{4} - \frac{1}{2}\operatorname{arctg}\nu\Bigr)d\nu = 0,$$

and by the monotonousness of the function before the cosine we obtain: $\dfrac{\pi}{4} < \psi < \dfrac{3\pi}{8}$, i.e. $0 < \theta < \dfrac{\pi}{8}$.

The following inequality holds:

$$\gamma A_1 \geqslant B_1, \tag{14}$$

where $\gamma = (1+\sqrt{2})/2$. In fact, let $\nu = \operatorname{tg}(2t)$, $\quad 0 < t < \pi/4$, then the inequality

$$\sqrt{1+\nu^2}\cos\Bigl(\theta - \frac{1}{2}\operatorname{arctg}\nu\Bigr) \geqslant \cos\Bigl(\theta - \frac{3}{2}\operatorname{arctg}\nu\Bigr),$$

which implies estimate (14), takes the form

$$\cos(\theta - t) \geqslant \cos(2t)\cos(\theta - 3t),$$

or $\cos(\theta - t) \geqslant \cos(\theta - 5t)$. We set

$$f(t) = \cos(\theta - t)/\cos(\theta - 5t), \quad 0 < t < \pi/4,$$

then

$$f(t) = \cos(4t) + \sin(4t)\operatorname{tg}(\theta - t) \leqslant$$
$$\leqslant \sqrt{1+\operatorname{tg}^2(\theta - t)} = 1/\cos(\theta - t) \leqslant \sqrt{2} = 2\gamma - 1.$$

Estimate (14) is proved; it also implies $2A_1 > B_1$.

Also,

$$B_1 > -C_1. \tag{15}$$

In fact,

$$B_1 + C_1 = \int\limits_0^\infty e^{-2x\nu}\nu^{-1/2}(1+\nu^2)^{-3/4}(\cos\Big(\theta - \frac{3}{2}\operatorname{arctg}\nu\Big) - \sin\Big(\theta - \frac{3}{2}\operatorname{arctg}\nu\Big))d\nu =$$

$$= \sqrt{2}\int\limits_0^{\pi/2} e^{-2x\operatorname{tg}t}\sin\Big(\frac{\pi}{4} - \theta + \frac{3t}{2}\Big)\frac{dt}{\sqrt{\sin t}} > 0,$$

since

$$0 < \frac{\pi}{4} - \theta < \frac{\pi}{4} - \theta + \frac{3t}{2} < \pi - \theta < \pi.$$

Corollary of (15): if $C_1 < 0$, then $B_1^2 > C_1^2$.

Now we prove the main inequality ($x \geqslant 0$):

$$2A_1(2A_1 - B_1)x^2 + B_1^2 + 4xA_1C_1 + C_1^2 \geqslant 0.$$

We omit index 1 below. If $C \geqslant 0$, then the inequality is obviously true by the corollary of estimate (14). If $C < 0$, then we transform the expression as follows:

$$2A(2A - B)x^2 + B^2 + 4xAC + C^2 = B(B - 2Ax^2) + (2xA + C)^2.$$

In the case $x < \sqrt{B/(2A)}$ the main inequality is obviously true. If $x \geqslant \sqrt{B/(2A)}$, then we introduce a constant $0 < q \leqslant 1$ and estimate the expression by using inequalities (14), (15) and their corollaries:

$$2A(2A - B)x^2 + B^2 + 4xAC + C^2 =$$

$$= (1 - q^2)(2Ax)^2 - (1 - q^2)C/q^2 - 2ABx^2 + B^2 + (2qxA + C/q)^2 \geqslant$$

$$\geqslant (1 - q^2)(2AB - C^2/q^2) \geqslant$$

$$\geqslant (1 - q^2)(2B^2/\gamma - C^2/q^2) \geqslant (1/q^2 - 1)(q^2 - \gamma/2)2C^2/\gamma.$$

As the last step, we set $q = \sqrt{\gamma/2} < 1$. The theorem is proved.

Now we explore the distribution of the determinant's roots on the upper halfplane, in fact, on the upper imaginary semiaxis. We set

$$\Delta(x) = -x\delta(x), \quad \delta(x) = J_2(x)Y_0(kx) - Y_2(x)J_0(kx).$$

To continue $\delta(x)$ analytically from the positive semiaxis to the right halfplane we will use the following formulae [2, pp. 29, 88, 185]:

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x), \quad H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x);$$

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{n+2m}}{m!(n+m)!}, \quad n = 0, 1, 2, ...;$$

$$H_\nu^{(1)}(x) = \Big(\frac{2}{\pi x}\Big)^{1/2} \frac{e^{i(x-\pi\nu/2-\pi/4)}}{\Gamma(\nu+1/2)} \int_0^\infty e^{-u} u^{\nu-1/2} \Big(1+\frac{iu}{2x}\Big)^{\nu-1/2} du, \quad Re \quad \nu > -1/2;$$

$$H_\nu^{(2)}(x) = \Big(\frac{2}{\pi x}\Big)^{1/2} \frac{e^{i(x-\pi\nu/2-\pi/4)}}{\Gamma(\nu+1/2)} \int_0^\infty e^{-u} u^{\nu-1/2} \Big(1-\frac{iu}{2x}\Big)^{\nu-1/2} du, \quad Re \quad \nu > -1/2.$$

If $x = iy, y > 0$, we use $H_0^{(1)} : \delta(x) = -i(J_2(x)H_0^{(1)}(kx) - H_2^{(1)}(x)J_0(kx))$.
If $x = -iy, y > 0$, we use $H_0^{(2)} : \delta(x) = i(J_2(x)H_0^{(2)}(kx) - H_2^{(2)}(x)J_0(kx))$.
In both cases $Im\delta(x) = 0$. We also note that if $x \to 0$ then $J_n(x) = (x/2)^n/n! + O(x^{n+2})$ and [2, p. 75]

$$Y_n(x) = -\Big(\frac{x}{2}\Big)^{-n} \sum_{m=0}^{n-1} \frac{(n-m-1)!}{m!} \Big(\frac{x}{2}\Big)^{2m} + 2\Big(\gamma + \ln\Big(\frac{x}{2}\Big)\Big) J_n(x) + O(x^n),$$

hence $\delta(x) \sim Cx^{-2}$ when $x \to 0$.

Since $s = 0$ is a regular point for $X(y, s)$, we may consider function

$$F(x) = -x^2\delta(x) = x^2(Y_2(x)J_0(kx) - J_2(x)Y_0(kx)),$$

which is analytic on the right half plane and has only real roots. To count the number of its roots, N, which lie on the interval $(0, R)$ on the real axis, we use the argument principle. We take a contour which consists of the interval $[-iR, iR]$ and the semicircle $\Gamma = \{z : r = R, -\pi/2 \leqslant \phi \leqslant \pi/2\}$. The interval $[-iR, iR]$ gives no contribution to the variation of the argument since $F(z)$ is real there. On the semicircle we use the asymptotic formulae [2, p. 222]

$$J_n(z) \sim \sqrt{\frac{2}{\pi z}}\Big(\cos\Big(z-\frac{\pi n}{2}-\frac{\pi}{4}\Big)(1+...)-\sin\Big(z-\frac{\pi n}{2}-\frac{\pi}{4}\Big)\Big(\frac{4n^2-1}{8z}+...\Big)\Big),$$

$$Y_n(z) \sim \sqrt{\frac{2\pi}{z}}\Big(\sin\Big(z-\frac{\pi n}{2}-\frac{\pi}{4}\Big)(1+...)+\cos\Big(z-\frac{\pi n}{2}-\frac{\pi}{4}\Big)\Big(\frac{4n^2-1}{8z}+...\Big)\Big),$$

Hence, if R is large enough then $F(z) \sim 2k^{-1/2} z \sin((k-1)z)$. Therefore, N exceeds on 1 number of roots of $\sin((k-1)z)$ which lie on the interval $(0, R)$, so $N = 1 + [(k-1)R/\pi]$ (we may always suppose that the right end of the interval is not a root).

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