Categorical Set Theory
An Introduction to the Foundations of Mathematics

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Contents

Introduction

what is the purpose, on meta-

note to the experts in set-, topos- and category theory

1 The Category of Sets

1.1 The Axiom of Sets and Functions

Our long list of axioms starts with a context in which we can express all the other axioms. We would like to introduce two distinct primitive notions, that of *sets* and that of *functions* between sets. We are not concerned with the ontology of these notions, i.e. we do not care what they actually are. Instead we rely on our intuition to guide us to a list of axioms. Everything which satisfies this list of axioms can be considered a theory of sets.

So what *is* our intuition for sets and functions? Cantor, Function

Axiom 1.1.1 (Category Set)

There are two distinct primitive notions of **Sets** (which we denote by capital letters) and **Functions** (which we denote by lowercase letters). Every function comes with a specified set called its **domain** and a specified set called its **codomain**. We denote a function f with domain X and codomain Y by $f : X \to Y$ or $X \xrightarrow{f} Y$.

Functions f and g such that the codomain of f equals the domain of g are called **composable**. For any composable pair of functions $f : X \to Y$ and $g : Y \to Z$ we suppose to have a unique **composite function** $g \circ f : X \to Z$ with the domain of f to the codomain of g.

Given functions $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$ we suppose that composition is **associative** in the sense that the two composite morphism $h \circ (g \circ f) : X \to W$ and $(h \circ g) \circ f : X \to W$ are equal.

Furthermore we declare that every set X has a dedicated **identity function** $\mathrm{id}_X : X \to X$. These functions are subject to the assumption that for every function $f : A \to B$ the composites $f \circ \mathrm{id}_A : A \to B$ and $\mathrm{id}_B \circ f : A \to B$ both are equal to the function f.

⋆ **Remark 1.1.2**

In the light of definition this axiom just states "There exists a category Set, whose objects we call *sets* and whose morphisms we call *functions*".

isomorphisms

1.2 The Axioms of Primitive Sets and Emptyness

Up until now basically everything could count as a category of sets. For example terminal category

rework axiom, $\{0, 1\}$ should have coprod UAE

Axiom 1.2.1 (Primitive Sets)

We declare that the category Set has the following properties:

- There is a dedicated set $*$ called the **point** with the property that for every set X there is a unique function $X \to \{*\}$.
- There is a dedicated set $\emptyset$ called the **empty set** with the property that for every set X there is a unique function $\emptyset \to X$.
- There is a dedicated set $\{0, 1\}$, which has precisely two distinct elements $0 : \{*\} \to \{0, 1\}$ and $1 : \{*\} \to \{0, 1\}$ and the property that for any set X with given elements $x, x' \in X$ there exists a unique function $[x, x'] : \{0, 1\} \to X$ making the following diagram commute

$$\begin{tikzcd}
\{*\} \arrow[r, "0"] \arrow[dr, "x"'] & \{0,1\} \arrow[d, dashed, "{[x,x']}"] & \{*\} \arrow[l, "1"'] \arrow[dl, "x'"] \\
& X &
\end{tikzcd}$$

- The three sets $\{*\}$, $\emptyset$ and $\{0, 1\}$ are pairwise non-isomorphic.

We should remark on why we chose to call the set $\emptyset$ the empty set. To do this we should give a precise definition of what it means to contain an element.

Definition 1.2.2

Given a set X we call a function $x : \{*\} \to X$ an **element** of X. We often abbreviate notation in writing $x \in X$. For a function $f : X \to Y$ we often write $f(x) \in Y$ for the composite function $f \circ x : \{*\} \to Y$. image preimage of element

If there exists an element $x \in X$, the set X is **inhabited**, otherwise we call it **uninhabited**.

Now the fourth part of the axiom of primitive sets implies that the empty set is in fact empty.

Lemma 1.2.3

The empty set $\emptyset$ is uninhabited.

Proof by contradiction

Assume there exists an element $x \in \emptyset$, i.e. a function $x : \{*\} \to \emptyset$.

Then the composite function $\{*\} \xrightarrow{x} \emptyset \to *$ is a function from $*$ to itsself. Since the identity function $\mathrm{id} : \{*\} \to \{*\}$ is another function from the point to itsself, but by assumption there is a unique function from the terminal object to itsself, both functions are equal.

In the same way we can show that the composite function $\emptyset \to \{*\} \xrightarrow{x} \emptyset$ has to coincide with the identity function $\mathrm{id}_\emptyset : \emptyset \to \emptyset$.

This shows that the element $\{*\} \xrightarrow{x} \emptyset$ constitutes an isomorphism with inverse function $\emptyset \to \{*\}$, a contradiction to the fourth part of the axiom of primitive sets.

$\square$

cannot show that uninhabited implies empty COUNTEREXAMPLE! have to put it as axiom

Axiom 1.2.4 (Emptyness †)

We assert that the empty set $\emptyset$ is the unique uninhabited set in the sense that for every uninhabited set X the unique morphism $\emptyset \to X$ is an isomorphism.

♨ **Exercise 1.2.5**

Show that for every uninhabited set X and every set Y there exists a unique function $X \to Y$. Deduce that every morphism between uninhabited sets is an isomorphism.

diagram with two elements commutes if elements equal

constant functions

1.3 The Axioms of Generation and Separation

intuition, explain parallel function

Axiom 1.3.1 (Generation and Separation)
We declare that the category Set has the following properties:

- Any two parallel functions $f, g : X \to Y$ are equal if and only if they agree on each point, i.e. if for every element $x \in X$ the elements $f(x) \in Y$ and $g(x) \in Y$ are equal.
- For any set X any two elements $x, x' \in X$ are equal, if and only if there exists a **separating function** $s : X \to \{0, 1\}$ with the property that the following diagram commutes.

$$\begin{tikzcd}
\{*\} \arrow[r, "x"] \arrow[dr, "0"'] & X \arrow[d, "s"] & \{*\} \arrow[l, "x'"'] \arrow[dl, "1"] \\
& \{0,1\} &
\end{tikzcd}$$

Lemma 1.3.2
Any set X with only one unique element $x \in X$ is isomorphic to the point.

Proof There is a unique function $X \to \{*\}$ and the given element $x : \{*\} \to X$. We claim that these are inverse to each other. By the defining property of the point $\{*\}$, the composite $\{*\} \xrightarrow{x} X \to \{*\}$ equals the identity on $\{*\}$. But then the diagram

$$\begin{tikzcd}
& X \arrow[drr, bend left, "\mathrm{id}_X"] & & \\
\{*\} \arrow[ur, "x"] \arrow[drr, bend left, "\mathrm{id}_{\{*\}}"] \arrow[dr, "x"'] & & & X \\
& X \arrow[r] & \{*\} \arrow[ur, "x"'] &
\end{tikzcd}$$

commutes. Since $x \in X$ is the unique point of X, the generation axiom ensures that the composite $X \to \{*\} \xrightarrow{x} X$ equals the identity function id_X. $\square$

1.4 Injective and Surjective Functions

motivate injection, surjection

Definition 1.4.1
A function $f : X \to Y$ is **surjective**, if for every element $y \in Y$ there exists an (not necessarily unique) element $x \in X$ such that the evaluation $f(x) \in Y$ and the element $y \in Y$ are equal.

$$\begin{tikzcd}
& X \arrow[d, "f"] \\
\{*\} \arrow[ur, dashed, "x"] \arrow[r, "y"'] & Y
\end{tikzcd}$$

We indicate surjections as $f : X \twoheadrightarrow Y$.

Example 1.4.2
Suppose that X is inhabited by some element $x \in X$. Then the unique morphism $X \to \{*\}$ is surjective since the unique element $* \in \{*\}$ has the preimage $x \in X$.

However for an inhabited set X, the unique morphism $\emptyset \to X$ is not surjective. Any preimage of $x \in X$ would make $\emptyset$ inhabited, contradicting lemma 1.2.3. More generally, any morphism out of an uninhabited set is not surjective.

isomorphism is surj

composition, cancellation

motivate epimorphism

Definition 1.4.3
A function $X \to Y$ is a **epimorphism**, if it has the pre-cancellation property, i.e. if any two functions $t, t' : Y \to T$ such that the composites $t \circ f$ and $t' \circ f$ agree are already equal.

example split epi

Lemma 1.4.4
A function $f : X \to Y$ is surjective if and only if it is an epimorphism.

Proof ($\Rightarrow$) Suppose f is surjective and that $t, t' : Y \to T$ are functions such that the composites $t \circ f$ and $t' \circ f$ agree.

Since f is surjective every $y \in Y$ admits a preimage $x \in X$ making the diagram

$$\begin{tikzcd}
& & Y \arrow[dr, "t"] & \\
\{*\} \arrow[r, dashed, "x"] \arrow[urr, bend left, "y"] \arrow[drr, bend right, "y"'] & X \arrow[ur, "f"] \arrow[dr, "f"'] & & T \\
& & Y \arrow[ur, "t'"'] &
\end{tikzcd}$$

commute. This means that t and t' agree on every element $y \in Y$, and thus are equal by the generation axiom. Therefore f is an epimorphism.

(⇐) For the converse assume that $f : X \to Y$ has the pre-cancellation property.

If Y is uninhabited, so is X. Thus by exercise 1.2.5 f is an isomorphism and in particular a surjection by example 1.4.2.

If Y has precisely one element then it is isomorphic to the point by lemma 1.3.2. The set X must be inhabited, since otherwise by exercise 1.2.5 there exists a unique function $X \to \{0,1\}$, so the pre-cancellation property applied onto the two composites $X \to Y \xrightarrow{0} \{0,1\}$ and $X \to Y \xrightarrow{1} \{0,1\}$ would imply that the two elements $0 \in \{0,1\}$ and $1 \in \{0,1\}$ coincide, contradicting the axiom of primitive sets. Thus by example 1.4.2 the function f is surjective.

Finally, if Y has at least two distinct elements $y, y' \in Y$, we assume that it is not surjective and work towards a contradiction. Suppose

finish

□

Exercise 1.4.5

Show that a morphism $f : X \to Y$ is a surjection, if and only if for every commutative square of functions as depicted below the unique function $Y \to \{*\}$ makes the two triangles commute.

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & \{*\} \arrow[d, "1"] \\
Y \arrow[r, "s"'] \arrow[ur, dashed] & \{0,1\}
\end{tikzcd}$$

Definition 1.4.6

A function $f : X \to Y$ is **injective**, if any two elements $x, x' \in X$ such that the values $f(x) \in Y$ and $f(x') \in Y$ agree are already equal.

$$\begin{tikzcd}
\{0,1\} \arrow[r, "{[x,x']}"] \arrow[d] & X \arrow[d, "f"] \\
\{*\} \arrow[r, "y"'] \arrow[ur, dashed] & Y
\end{tikzcd}$$

We indicate injections as $f : X \hookrightarrow Y$.

Example 1.4.7

Any function $\{*\} \to X$ is injective. This is trivially true, since $\{*\}$ has precisely one element, so every two preimages of an element in X have to coincide.

The function $\{0,1\} \to \{*\}$ is not injective. The unique element $* \in \{*\}$ has both $0 \in \{0,1\}$ and $1 \in \{0,1\}$ as preimages, since the diagram

$$\begin{tikzcd}
\{*\} \arrow[r, "0"] \arrow[dr, "*"'] & \{0,1\} \arrow[d] & \{*\} \arrow[l, "1"'] \arrow[dl, "*"] \\
& \{*\} &
\end{tikzcd}$$

commutes by the uniqueness property of functions into $\{0,1\}$. But by assumption on the Boolean set $\{0,1\}$ the elements $0 \in \{0,1\}$ and $1 \in \{0,1\}$ are distinct.

isomorphism is inj

composition, cancellation, motivate monomorphism

Definition 1.4.8
A function $X \to Y$ is a **monomorphism**, if it has the post-cancellation property, i.e. if any two functions $t, t' : T \to X$ such that the composites $f \circ t$ and $f \circ t'$ agree are already equal.

example split mono

Lemma 1.4.9
A function $f : X \to Y$ is injective if and only if it is a monomorphism.

Proof ($\Leftarrow$) The post-cancellation property specializes to elements $x, x' : \{*\} \to X$.

($\Rightarrow$) Suppose that $f : X \to Y$ is an injection and that T is an arbitrary set. Let $t, t' : T \to X$ be two functions so that $f \circ t$ and $f \circ t'$ agree. Every element $w \in T$ gives rise to a diagram

$$
\begin{tikzcd}
 & T \arrow[dr, "t"] \arrow[drr, bend left, "f \circ t"] & & \\
\{*\} \arrow[ur, "w"] \arrow[dr, "w"'] & & X \arrow[r, "f"] & Y \\
 & T \arrow[ur, "t'"'] \arrow[urr, bend right, "f \circ t'"'] & &
\end{tikzcd}
$$

in which everything but the left square commutes by assumption and the left square commutes by f being an injection. But this shows that t and t' agree on every element $w \in T$, hence have to be equal by the generation axiom.

$\square$

Exercise 1.4.10
Show that a function $f : X \to Y$ is an injection, if and only if for every commutative square of functions as depicted below there exists some function $Y \to \{0,1\}$ which makes the two triangles commute.

$$
\begin{tikzcd}
X \arrow[r, "s"] \arrow[d, "f"'] & \{0,1\} \arrow[d] \\
Y \arrow[r] \arrow[ur, dashed, "s'"] & \{*\}
\end{tikzcd}
$$

1.5 The Axioms of Balance and Image Factorizations

discuss bijection

Exercise 1.5.1
Show that a monomorphism, which is split epic, is an isomorphism. Show that an epimorphism, which is split monic, is an isomorphism.

motivate balance

Axiom 1.5.2 (Balance †)
Every bijection is an isomorphism.

Would like to be unique

Exercise 1.5.3
Let $f : X \to Y$ be a function, which is an isomorphism.
Show that the inverse function $f^{-1} : Y \to X$ is unique.

motivate image factorization

Axiom 1.5.4 (Image Factorization †)
Every function $f : X \to Y$ can be written as a composite function $i \circ p$, in which $p : X \to \mathrm{Img}(f)$ is a surjection and $\mathrm{Img}(f) \to Y$ is an injection. We assert that this factorization is unique in the sense that for every other factorization of f into a surjection $p' : X \to I$ and an injection $i' : I \to Y$ there is some function $c : \mathrm{Img}(f) \to I$ making the following diagram commute.

$$\begin{tikzcd}
& \mathrm{Img}(f) \arrow[dr, hook, "i"] \arrow[dd, dashed, "c"] & \\
X \arrow[ur, two heads, "p"] \arrow[dr, two heads, "p'"'] & & Y \\
& I \arrow[ur, hook, "i'"'] &
\end{tikzcd}$$

Exercise 1.5.5
Show that the comparison function $c : \mathrm{Img}(f) \to I$ in the previous axiom is uniquely determined. Show that it is a bijection and thus an isomorphism.

functorial factorizations?

epi-mono WFS?

discuss subsets, image of a subset, introduce existential quantifier

1.6 The Axioms of Products and Sums

Axiom 1.6.1 (Products)

For any two sets X and Y there exists the **product** set $X \times Y$ and two functions $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$ called **projections**, satisfying that for any set T and functions $f : T \to X$ and $g : T \to Y$ there exists a unique function $(f, g) : T \to X \times Y$ making the following diagram commute.

$$\begin{tikzcd} & T \arrow[dl, "f"'] \arrow[d, dashed, "{(f,g)}"] \arrow[dr, "g"] & \\ X & X \times Y \arrow[l, "\mathrm{pr}_X"] \arrow[r, "\mathrm{pr}_Y"'] & Y \end{tikzcd}$$

product with terminal, associativity, commutativity, diagonal

Axiom 1.6.2 (Sums)

For any two sets X and Y there exists their **sum** $X + Y$ and two functions $\mathrm{incl}_X : X \to X + y$ and $\mathrm{incl}_Y : Y \to X + Y$ called **inclusions**, satisfying that for any set T and functions $f : X \to T$ and $g : Y \to T$ there exists a unique function $[f, g] : X + Y \to T$ making the following diagram commute.

$$\begin{tikzcd} X \arrow[r, "\mathrm{incl}_X"] \arrow[dr, "f"'] & X \times Y \arrow[d, dashed, "{[f,g]}"] & Y \arrow[l, "\mathrm{incl}_Y"'] \arrow[dl, "g"] \\ & T & \end{tikzcd}$$

sum with empty, associativity, commutativity, codiagonal

Lemma 1.6.3 (Sums are disjoint)

Let $X + Y$ be the sum of two sets X and Y. Every element $t \in X + Y$ is either the image of an element $x \in X$ under incl_X or the image of an element $y \in Y$ under incl_Y.

Proof WLOG both X and Y inhabited.

Assume $t \in X + Y$ is an element, which is neither the image of some $x \in X$ nor the image of some $y \in Y$. By separation there is a function $s : X + Y \to \{0, 1\}$, which sends the element $t \in X + Y$ to $1 \in \{0, 1\}$ and every other element of $X + Y$ to $0 \in \{0, 1\}$. Furthermore we have the constant zero function $\mathrm{const}_0 : X + Y \to \{0, 1\}$. The two functions are distinct, since they differ in their value at $t \in X + Y$. Meanwhile both functions make the diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{incl}_X"] \arrow[dr, "\mathrm{const}_0"'] & X + Y \arrow[d, dashed] & Y \arrow[l, "\mathrm{incl}_Y"'] \arrow[dl, "\mathrm{const}_0"] \\ & \{0,1\} & \end{tikzcd}$$

commute, since both composites $\mathrm{const}_0 \circ \mathrm{incl}_X$ and $s \circ \mathrm{incl}_X$ evaluate to $0 \in \{0, 1\}$ for every element of X and analogously for Y. This contradicts the uniqueness in the definition of a sum of sets.

Now assume that there exists an element $t \in X + Y$, which is both the image of an element $x \in X$ and an element $y \in Y$. The constant functions $\mathrm{const}_1 : X \to \{0, 1\}$ and

$\mathrm{const}_0 : Y \to \{0,1\}$ induce a unique function $\mathbb{1}_X : X+Y \to \{0,1\}$ making the diagram

$$\begin{tikzcd}
 & \{*\} \arrow[dl, "x"'] \arrow[d, "t"] \arrow[dr, "y"] & \\
X \arrow[r, "\mathrm{incl}_X"] \arrow[dr, "\mathrm{const}_1"'] & X+Y \arrow[d, "\mathbb{1}_X"] & Y \arrow[l, "\mathrm{incl}_Y"'] \arrow[dl, "\mathrm{const}_0"] \\
 & \{0,1\} &
\end{tikzcd}$$

commute. This is impossible, since the two elements $0 \in \{0,1\}$ and $1 \in \{0,1\}$ are distinct.
□

☕ **Exercise 1.6.4**

Show that there is a bijection between $\{0,1\} + \{0,1\}$ and $\{0,1\} \times \{0,1\}$.
Hint: Don't forget the universal property of $\{0,1\}$.

☕☕ **Exercise 1.6.5**

Show that for every set X there is a bijection between $X + X$ and $X \times \{0,1\}$.
Hint: Use the universal property of the product to obtain maps $X \to X \times \{0,1\}$.

1.7 Boolean Logic

formally introduce terms

We have enough tools at hand to give an exhaustive list of all functions $\{0,1\} \to \{0,1\}$. The universal property of $\{0,1\}$ ensures that all of the following functions exist, while the generation axiom implies that they are distinct.

$$\begin{array}{rcl} \{0,1\} & \xrightarrow{\text{false}} & \{0,1\} \\ 0 & \longmapsto & 0 \\ 1 & \longmapsto & 0 \end{array} \quad \begin{array}{rcl} \{0,1\} & \xrightarrow{\text{id}} & \{0,1\} \\ 0 & \longmapsto & 0 \\ 1 & \longmapsto & 1 \end{array} \quad \begin{array}{rcl} \{0,1\} & \xrightarrow{t_\neg} & \{0,1\} \\ 0 & \longmapsto & 1 \\ 1 & \longmapsto & 0 \end{array} \quad \begin{array}{rcl} \{0,1\} & \xrightarrow{\text{true}} & \{0,1\} \\ 0 & \longmapsto & 1 \\ 1 & \longmapsto & 1 \end{array}$$

Similarly exercise 1.6.4 provides us with a method to construct functions out of the set $\{0,1\} \times \{0,1\}$. We could give an exhaustive list of all functions $\{0,1\} \times \{0,1\} \to \{0,1\}$, but we refrain from doing so and focus on just the following examples.

Definition 1.7.1

A **term** in n variables is a function $t : \{0,1\}^{\times n} \to \{0,1\}$ represented by a specific choice of composites of the logical connectives above and diagonal functions. Two terms are **tautologically equivalent**, if they represent the same function. A term is a **tautology**, if it is **tautologically equivalent** to the term $\mathbf{1}$.

Example 1.7.2

The term $x \vee \neg x$ is a tautology. It describes the function

any function is represented by a term, coprod models OR!

Discuss propositions, and, or, xor

1.8 The Axiom of Intersections

motivate intersections

Axiom 1.8.1 (Intersections)
For every pair of functions $f : X \to Z$ and $g : Y \to Z$ with the same codomain there exists a dedicated set called the **pullback** of f and g, which we denote by

$$X \underset{f=g}{\times} Y \qquad \text{or} \qquad \{(x, y) \in X \times Y \mid f(x) = g(y)\}$$

together with functions

$$\mathrm{pr}_X : X \underset{f=g}{\times} Y \to X \qquad \text{and} \qquad \mathrm{pr}_Y : X \underset{f=g}{\times} Y \to Y$$

with the universal property that for every pair of functions $x : T \to X$ and $y : T \to Y$, which make the square

$$\begin{tikzcd} T \arrow[r, "y"] \arrow[d, "x"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

there exists a unique function $t : T \to X \underset{f=g}{\times} Y$ making the following diagram commute.

$$\begin{tikzcd} T \arrow[dr, dashed, "t"] \arrow[drr, bend left, "y"] \arrow[ddr, bend right, "x"'] & & \\ & X \underset{f=g}{\times} Y \arrow[r, "\mathrm{pr}_Y"] \arrow[d, "\mathrm{pr}_X"'] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z \end{tikzcd}$$

Remark 1.8.2
There universal property of the product $X \times Y$ yields a unique function $X \underset{f=g}{\times} Y \to X \times Y$ making the diagram

$$\begin{tikzcd} & X \underset{f=g}{\times} Y \arrow[dl, "\mathrm{pr}_X"'] \arrow[d, "i_{f=g}"] \arrow[dr, "\mathrm{pr}_Y"] & \\ X & X \times Y \arrow[l, "\mathrm{pr}_X"] \arrow[r, "\mathrm{pr}_Y"'] & Y \end{tikzcd}$$

commute. This function is an injection: Take two elements $t, t' \in X \underset{f=g}{\times} Y$ which evaluate under

$i_{f=g} : X \underset{f=g}{\times} Y \to X \times Y$ to the same element $(x, y) \in X \times Y$, or more specifically make the diagram

$$\begin{tikzcd}
& \{*\} \arrow[d, "{t,t'}"] \arrow[ddl, bend right, "x"'] \arrow[ddr, bend left, "y"] & \\
& X \underset{f=g}{\times} Y \arrow[d, "i_{f=g}"] \arrow[dl, "\mathrm{pr}_X"'] \arrow[dr, "\mathrm{pr}_Y"] & \\
{} & X \times Y \arrow[l, "\mathrm{pr}_X"] \arrow[r, "\mathrm{pr}_Y"'] & {}
\end{tikzcd}$$

commute. But then both t and t' make the diagram

$$\begin{tikzcd}
\{*\} \arrow[dr, dashed, "t"] \arrow[drr, bend left, "y"] \arrow[ddr, bend right, "x"'] & & \\
& X \underset{f=g}{\times} Y \arrow[r, "\mathrm{pr}_Y"] \arrow[d, "\mathrm{pr}_X"'] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}$$

commute and hence have to coincide.

Example 1.8.3

Intersection of subsets, disjoint

Example 1.8.4

Preimage

Example 1.8.5

Fiber

Lemma 1.8.6

A function $f : X \to Y$ is injective if and only if for every element $y \in Y$ the fiber $f^{-1}(\{y\})$ is either empty or a singleton.

Proof todo □

Exercise 1.8.7

Show that a function $f : X \to Y$ is surjective, if and only if for every element $y \in Y$ the fiber $f^{-1}(\{y\})$ is inhabited.

Deduce that a function $f : X \to Y$ is a bijection, if and only if for every element $y \in Y$ the fiber $f^{-1}(\{y\})$ is a singleton.

Example 1.8.8
graph of function

Properties of pullbacks

1.9 The Axiom of Characteristic Functions

Subobjects? Complements? Balanced? Implies Separation

Axiom 1.9.1 (Characteristic Functions)
We assert that for every injection $i_A : A \hookrightarrow X$ there exists a uniquely determined **characteristic function** $\chi_A : X \to \{0,1\}$, which makes the square

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i_A"'] & \{*\} \arrow[d, "1"] \\
X \arrow[r, "\chi_A"'] & \{0,1\}
\end{tikzcd}$$

commute and makes i_A the fiber of χ_A at $1 \in \{0,1\}$.

element in A iff $\chi_A(x) = 1$

♨ **Exercise 1.9.2**
Show that the axiom of characteristic functions implies the axiom of separating functions.

Lemma 1.9.3
The axiom of characteristic functions implies the axiom of balance.

Proof Let $f : X \to Y$ be a bijective function. Since it is an injection there exists a characteristic function $\chi_X : Y \to \{0,1\}$ making the square

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & \{*\} \arrow[d, "1"] \\
Y \arrow[r, "\chi_X"'] \arrow[ur, dashed] & \{0,1\}
\end{tikzcd}$$

commute. Since f is surjective by exercise 1.4.5 the unique function $Y \to \{*\}$ makes the whole diagram above commute. This means that the outer square of the diagram

$$\begin{tikzcd}
Y \arrow[dr, dashed, "g"] \arrow[drr, bend left] \arrow[ddr, bend right, equal] & & \\
& X \arrow[r] \arrow[d, "f"] & \{*\} \arrow[d] \\
& Y \arrow[r, "\chi_X"'] & \{0,1\}
\end{tikzcd}$$

commutes and by the universal property of the fiber $X = \chi^{-1}(\{1\})$ there exists a function $g : Y \to X$, which makes the whole diagram commute. This renders f split epic and since it is injective it is an isomorphism by exercise 1.5.1. $\square$

Example 1.9.4

Let $A \subseteq X$ be a subset with inclusion $i_A : A \to X$ classified by $\chi_A : X \to \{0,1\}$. The **complement** of $A \subseteq X$ is the subset $X \setminus A \subseteq X$ classified by $X \xrightarrow{\chi_A} \{0,1\} \xrightarrow{\neg} \{0,1\}$. Equivalently it is the fiber of χ_A at $0 \in \{0,1\}$.

Lemma 1.9.5

Given the inclusion of a subset $i_A : A \to X$ and an inclusion of its complement $i_{X\setminus A} : X \setminus A \to X$ the canonical function $[i_A, i_{X\setminus A}] : A + X \setminus A \to X$ induced by the two inclusions is a bijection.

Proof The inclusions i_A and $i_{X\setminus A}$ are classified by characteristic functions $\chi_A : X \to \{0,1\}$ and $\chi_{X\setminus A} : X \xrightarrow{\chi_A} \{0,1\} \xrightarrow{\neg} \{0,1\}$.

Pick an element $x \in X$, then either $\chi_A(x) = 1$ or $\chi_A(x) = 0$ holds. This means that $x \in X$ can be regarded either as an element of the fiber $A = f^{-1}(\{1\})$ or the fiber $X \setminus A = f^{-1}(\{0\})$. This shows that the induced function $[i_A, i_{X\setminus A}] : X + X \setminus A \to X$ is surjective.

Take two elements $x, x' \in A + X \setminus A$ such that $[i_A, i_{X\setminus A}](x) = [i_A, i_{X\setminus A}](x') \in X$. Post-composing with $\chi_A : X \to \{0,1\}$ we find that finish □

♨ **Exercise 1.9.6**

Let X be an inhabited set. Show that an inclusion $A \to X$ is a split monic if and only if A is inhabited.

remark on AC

Example 1.9.7

Consider two subsets $A, B \subseteq X$ with classifying characteristic functions $\chi_A : X \to \{0,1\}$ and $\chi_B : X \to \{0,1\}$. The subset classified by the composite function $X \xrightarrow{(\chi_A,\chi_B)} \{0,1\} \times \{0,1\} \xrightarrow{\wedge} \{0,1\}$ is the **intersection** $A \cap B \subseteq X$ defined in example 1.8.3.

proof

Example 1.9.8

Consider two subsets $A, B \subseteq X$ with classifying characteristic functions $\chi_A : X \to \{0,1\}$ and $\chi_B : X \to \{0,1\}$. The subset classified by the composite function $X \xrightarrow{(\chi_A,\chi_B)} \{0,1\} \times \{0,1\} \xrightarrow{\vee} \{0,1\}$ is the **union** $A \cup B \subseteq X$ of the subsets.

has inclusions $j_A : A \to A \cup B$ and $j_B : B \to A \cup B$

show that it is image of $A + B \to X$

1.10 The Axiom of Unions

introduce pushouts

Lemma 1.10.1
Consider two subsets $A, B \subseteq X$. The commutative square

$$\begin{tikzcd} A \cap B \arrow[r, "k_B"] \arrow[d, "k_A"'] & B \arrow[d, "j_B"] \\ A \arrow[r, "j_A"'] & A \cup B \end{tikzcd}$$

of example 1.9.8 is both a pullback and a pushout.

Proof pullback, since $A \cup B \to X$ is mono

pushout, since actual pushout provides epi-mono factorization □

coequalizer

⋆ **Proposition 1.10.2**
The axiom of image factorizations is obsolete.

Proof use proof of topos seminar? □

1.11 The Axiom of Sets of Functions

motivate cartesian closure and currying

$\mathrm{Map}(T, X \times Y) \cong \mathrm{Map}(T, X) \times \mathrm{Map}(T, Y)$

$\mathrm{Map}(T, -)$ conservative?

Exercise 1.11.1
A function $f : X \to Y$ is an isomorphism, if and only if for every set T the induced function $f_* : \mathrm{Map}(T, X) \to \mathrm{Map}(T, Y)$ is an isomorphism.

1.12 Relations

Definition 1.12.1
A **relation** on a set X is a subset $R \subseteq X \times X$. It is

- **reflexive**, if for every element $x \in X$ the element $(x, x) \in X \times X$ is contained in R.
- **symmetric**, if for every element $(x, y) \in R$ the element $(y, x) \in X \times X$ is contained in R.
- **antisymmetric**, if for every element $(x, y) \in R$ such that $(y, x) \in R$ already $x = y \in X$ holds.
- **transitive**, if for any two elements $(x, y) \in R$ and $(y, z) \in R$ the element $(x, z) \in X \times X$ is contained in R.

Given a relation R on a set X with subset inclusion $i_R : R \to X \times X$, its **opposite relation** R^{op} on X is the relation defined by the subset inclusion $R \xrightarrow{i_R} X \times X \xrightarrow{(12)} X \times X$.

motivate categorical reformulation: constructing functions is hard

Remark 1.12.2
Consider a set X and a subset $R \subseteq X \times X$ with given inclusion $i_R : R \to X \times X$. Above properties admit a more categorical formulation.

- The relation R is reflexive in the above sense, if and only if the diagonal morphism $\Delta_X : X \to X \times X$ factors via $i_R : R \to X \times X$, i.e. the lifting problem depicted on the right admits a solution. diagram

  This is because the reflexivity condition above is by the generation axiom equivalent to the commutativity of the outer square in the diagram

$$\begin{tikzcd}
X \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right, "\Delta_X"'] & & \\
& R \arrow[r] \arrow[d, "i_R"'] & \{*\} \arrow[d, "1"] \\
& X \times X \arrow[r, "\chi_R"'] & \{0,1\}
\end{tikzcd}$$

  which by the universal property of the pullback R is equivalent to the existence of the dashed arrow.

- symmetry similar, exercise?
- antisymmetry, exercise?

$$\begin{tikzcd}
R \times R^{\mathrm{op}} \arrow[rr] \arrow[dd] \arrow[dr, dashed, "i_R = i_{R^{\mathrm{op}}}"'] & & R^{\mathrm{op}} \arrow[dd, "i_{R^{\mathrm{op}}}"] \\
& X \arrow[dr, "\Delta_X"] & \\
R \arrow[rr, "i_R"'] & & X \times X
\end{tikzcd}$$

- transitivity, not so easy, prove self

$$
\begin{tikzcd}
R \underset{\mathrm{pr}_2=\mathrm{pr}_1}{\times} R \arrow[r] \arrow[d] & R \arrow[d, "\mathrm{pr}_1"] \\
R \arrow[r, "\mathrm{pr}_2"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
 & R \arrow[d, "i_R"] \\
R \underset{\mathrm{pr}_2=\mathrm{pr}_1}{\times} R \arrow[r, "?"] \arrow[ur, dashed] & X \times X
\end{tikzcd}
$$

more categorical formulation? see nlab-correspondence

Definition 1.12.3
An **equivalence relation** on X is a reflexive, transitive and symmetric relation.

quotient by equivalence relation

char of equivalence relations as pb of identity relation along surj, char of surjection

Definition 1.12.4
A **partial order** P on a set X is a reflexive, transitive and antisymmetric relation.

It is a **total order**, if for every element $(x, y) \in X \times X$ we have that $(x, y) \in P$ or $(y, x) \in P$.

Example 1.12.5
The set $\{0, 1\}$ can be equipped with the total order $\leq := \{(0, 0), (0, 1), (1, 1)\} \subseteq \{0, 1\}^{\times 2}$, which we call the **standard order**. depict, opposite order

Example 1.12.6
The set $\{0, 1\}^{\times 2}$ has a partial order grid and lexicographic

♨ **Exercise 1.12.7**
Let $(P, \preceq)$ be a partially ordered set and X an arbitrary set. Show that the characteristic function

$$asd$$

defines a partial order on $\mathrm{Map}(X, P)$.

lattices and Knaster-Tarski FPT?

1.13 Powersets, First Order Logic and the Subobject Lattice

recall characteristic functions

Definition 1.13.1
The **powerset** of a set X is the set $\mathcal{P}(X) := \mathrm{Map}(X, \{0,1\})$.

♨ **Exercise 1.13.2**
How many subsets does $\{0,1\}$ have? How many subsets does $\{0,1,2\} = \{*\}+\{*\}+\{*\}$ have?

Remark 1.13.3
The functions

$$\begin{aligned}\cap_X :&\mathcal{P}(X)\times\mathcal{P}(X) \cong \mathrm{Map}(X, \{0,1\}\times\{0,1\}) \xrightarrow{\wedge_*} \mathcal{P}(X)\\ \cup_X :&\mathcal{P}(X)\times\mathcal{P}(X) \cong \mathrm{Map}(X, \{0,1\}\times\{0,1\}) \xrightarrow{\vee_*} \mathcal{P}(X)\end{aligned}$$

map subsets $A, B \subseteq X$ to their intersection $A \cap_X B \subseteq X$ respectively union $A \cup_X B \subseteq X$. The function

$$\neg_X : \mathcal{P}(X) \xrightarrow{\neg_*} \mathcal{P}(X)$$

sends a subset $A \subseteq X$ to its complement $\neg_X A \subseteq X$.

X and $\emptyset$ as neutral elements, distributivity (follows from boolean tautologies)

Remark 1.13.4
By exercise 1.12.7 the powerset $\mathcal{P}(X) = \mathrm{Map}(X, \{0,1\})$ inherits a partial order from the standard order on $\{0,1\}$ called the **partial order by inclusion**. This is legitimized by the fact that for subsets $A, B \subseteq X$ the following assertions are equivalent:

1. There exists a (necessarily injective) function making the triangle on the right commute. triangle
2. The characteristic functions satisfy $\chi_A \leq \chi_B$ with respect to the partial order by inclusion.

minmal element $\emptyset$, maximal element X, joins $\cup$ and meets $\cap$

compatibility with $\cap$ and $\cup$ (partially ordered monoid)

♨♨ **Exercise 1.13.5**
Every function $f : X \to Y$ induces an order-preserving map $f^{-1}[-] : \mathcal{P}(Y) \to \mathcal{P}(X)$, which maps a subset $B \subseteq Y$ to its preimage $f^{-1}[B] \subseteq X$ under f.

functorial

Proposition 1.13.6
Every function $f : X \to Y$ induces an order-preserving map $f[-] : \mathcal{P}(X) \to \mathcal{P}(Y)$, which maps a subset $A \subseteq X$ to its image $f[A] \subseteq Y$ under f.

functorial

preimage-image adjunction

examples and counterexamples for intersections and stuff

Definition 1.13.7

For any set X there exists the **singleton map** $X \xrightarrow{\{-\}} \mathcal{P}(X)$, which sends an element $x \in X$ to the singleton subset $\{x\} \subseteq X$. It is given by the transpose of the characteristic function of the diagonal. elaborate, natural transformation

classify mono + epi

Powersets + subobj classifier imply cartesian closed

smallest subobject $\emptyset \subseteq X$, largest subobject $X \subseteq X$, arbitrary intersections and unions...

Lemma 1.13.8

Let X be a set and $p : I \to \mathcal{P}(X)$ be a function, regarded as an I-indexed family of propositions on X.

Then the sets $\{x \in X \mid \exists i \in I : p_i(x)\}$ and $\{x \in X \mid \forall i \in I : p_i(x)\}$ exist UMP

Proof Consider the transpose $\widehat{p} : I \times X \to \{0,1\}$ of the map $p : I \to \mathcal{P}(X)$ and note that it suffices to construct the set $\{x \in X \mid \exists i \in I : \widehat{p}(i,x) = 1\}$. This can be realized by first taking the the pullback

$$\begin{tikzcd}
\{(i,x) \in I \times X \mid \widehat{p}(i,x) = 1\} \arrow[r] \arrow[d, "i"'] & \{*\} \arrow[d, "1"] \\
I \times X \arrow[r] & \{0,1\}
\end{tikzcd}$$

and then considering the image factorization of the map $\mathrm{pr}_X \circ i$. elaborate?

Want

$$\begin{tikzcd}
I \times \{x \in X \mid \forall i \in I : p_i(x)\} \arrow[r] \arrow[d, "I \times j"'] & \{*\} \arrow[d, "1"] \\
I \times X \arrow[r, "\widehat{p}"'] & \{0,1\}
\end{tikzcd}$$

consider transpose $\widetilde{p} : X \to \mathcal{P}(I)$ and pullback

$$\begin{tikzcd}
\{x \in X \mid \widetilde{p}(x) = \mathbf{1}\} \arrow[r] \arrow[d, "j"'] & \mathrm{Map}(I, \{*\}) = \{*\} \arrow[d, "\mathbf{1}"] \\
X \arrow[r, "\widetilde{p}"'] & \mathrm{Map}(I, \{0,1\})
\end{tikzcd}$$

$\square$

$\forall i \in I : p_i(x) \Leftrightarrow \neg(\exists i \in I : \neg p_i(x))$...

rules for interchanging forall and exists

set of all equivalence relations, set of all partial orders etc...

Proposition 1.13.9

Let X be a set and $p : I \to \mathcal{P}(X)$ be an I-indexed family of subsets. Then the set $\{A \subseteq X \mid \forall i \in I : A \subseteq_X p(i)\}$ has a $\subseteq$-maximal element $\bigcap_{i \in I} p(i) \subseteq X$.

Proposition 1.13.10
For any set X the powerset $\mathcal{P}(X)$ is a complete lattice.

Knaster-Tarski (see nlab CSB)

1.14 Cardinality

Definition 1.14.1

Two sets X and Y have the same **cardinality**, if there exists a bijection $X \cong Y$. This is denoted as $|X| = |Y|$.

The set X is of **lesser cardinality** than Y, if there exists an injection $X \hookrightarrow Y$. This is denoted as $|X| \leq |Y|$.

Since every identity function is injective and injections compose, cardinality defines a reflexive and transitive *meta-relation* on the *meta-collection* of isomorphism classes of sets. The rest of this section is devoted to showing that it in fact defines a *meta-theoretical partial order*. More specifically, the following theorem shows that it is an anti-symmetric *meta-relation*.

Theorem 1.14.2 (Cantor-Schröder-Bernstein)

Let X and Y be sets, such that $|X| \leq |Y|$ and $|Y| \leq |X|$. Then $|X| = |Y|$.

Proof see nlab

Suppose we have injections $i : X \hookrightarrow Y$ and $j : Y \hookrightarrow X$. We have to construct a bijection $f : X \cong Y$, or equivalently provide $f : X \to Y$ and its inverse $g : Y \to X$. Naively we would expect that there is some subset $A \subseteq X$ such that $f|_A = i|_A$ and that there exists some subset $B \subseteq Y$ such that $g|_B = j|_B$. In an ideal situation we would have that $B = \neg_Y i(A)$ and that $A = \neg_X j(B)$, since then i restricts to an isomorphism $i|_A : A \cong i(A)$ and j restricts to an isomorphism $j|_B : B \cong j(B)$ and the universal property of the disjoint unions $Y = i(A) + B$ and $X = A + j(B)$ provide us with functions

$$\begin{tikzcd} A \arrow[r, "\subseteq"] \arrow[dr, "i|_A"'] & X \arrow[d, dashed, "f"] & j(B) \arrow[l, "\supseteq"'] \arrow[d, "(j|_B)^{-1}"] \\ & Y & B \arrow[l, "\supseteq"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} i(A) \arrow[r, "\subseteq"] \arrow[d, "(i|_A)^{-1}"'] & Y \arrow[d, dashed, "g"'] & B \arrow[l, "\supseteq"'] \arrow[dl, "j|_B"] \\ A \arrow[r, "\subseteq"'] & X & \end{tikzcd}$$

which are inverse to each other by construction. Note that this ideal situation is equivalent to saying that there is a subset $A \subseteq X$, which satisfies $\neg_X j(\neg_X i(A)) = \neg_X j(B) = A$, or in other words that $A \in \mathcal{P}(X)$ is a fixed point of the order preserving function

$$\mathcal{P}(X) \xrightarrow{i[-]} \mathcal{P}(Y) \xrightarrow{\neg_Y} \mathcal{P}(Y)^{\mathrm{op}} \xrightarrow{j[-]} \mathcal{P}(X)^{\mathrm{op}} \xrightarrow{\neg_X} \mathcal{P}(X).$$

By the Knaster-Tarski Fixed-Point Theorem REF such a subset $A \subseteq X$ always exists. □

would like finite sets

1.15 The Axioms of Infinity and Countable Choice

introduce universal recursive object $\mathbb{N}$: natural numbers as smallest nonempty set with infinitely many successors

Axiom 1.15.1 (Infinity)
There exists a **set of natural numbers** $\mathbb{N}$ containing an element $0 : \{*\} \to \mathbb{N}$ together with a **successor function** $\mathrm{succ} : \mathbb{N} \to \mathbb{N}$, which satisfies the following universal property:

For every set X, element $x_0 \in X$ and endomorphism $f : X \to X$, there exists a unique function $x : \mathbb{N} \to \mathbb{N}$ making the diagram

$$\begin{tikzcd} & \mathbb{N} \arrow[r, "\mathrm{succ}"] \arrow[d, dashed, "x"] & \mathbb{N} \arrow[d, dashed, "x"] \\ \{*\} \arrow[ur, "0"] \arrow[dr, "x_0"'] & & \\ & X \arrow[r, "f"'] & X \end{tikzcd}$$

commute. A function $x : \mathbb{N} \to X$ is commonly called a **sequence** in X and for $n \in \mathbb{N}$ one writes $x_n \in X$ instead of $x(n) \in X$.

discuss Peano

(P1) Zero is a natural number.

(P2) Every natural number has a unique succesor.

(P3) No natural number has zero as its successor.

(P4) Distinct natural numbers have different successors.

can do:

Lemma 1.15.2
The successor function $\mathrm{succ} : \mathbb{N} \to \mathbb{N}$ is injective and its image $\mathrm{succ}(\mathbb{N}) \subseteq \mathbb{N}$ decomposes $\mathbb{N}$ into $\mathbb{N} = \{0\} + \mathrm{succ}(\mathbb{N})$. In other words, the diagram

$$\{*\} \xrightarrow{0} \mathbb{N} \xleftarrow{\mathrm{succ}} \mathbb{N}$$

is a coproduct diagram and in particular provides an isomorphism $\mathbb{N} \cong \{*\} + \mathbb{N}$.

Proof Johnstone

We show that the set $\{*\} + \mathbb{N}$ satisfies the universal property of the natural numbers. To this end consider the diagram

$$\begin{array}{ccccc} \{*\} & \xrightarrow{\mathrm{incl}_1} & \{*\} + \mathbb{N} & \xrightarrow{\mathrm{incl}_2 \circ [0,\mathrm{succ}]} & \{*\} + \mathbb{N} \\ * & \longmapsto & * & \longmapsto & 0 \\ & & n & \longmapsto & \mathrm{succ}(n) \end{array}$$

and an arbitrary diagram

$$\{*\} \xrightarrow{x_0} X \xrightarrow{f} X$$

With $y : \mathbb{N} \to X$ being the function induced by the diagram

$$\begin{tikzcd} & \mathbb{N} \arrow[r, "\mathrm{succ}"] \arrow[d, dashed, "y"] & \mathbb{N} \arrow[d, dashed, "y"] \\ \{*\} \arrow[ur, "0"] \arrow[dr, "f(x_0)"'] & & \\ & X \arrow[r, "f"'] & X \end{tikzcd}$$

we obtain a commutative diagram

$$\begin{tikzcd}
& \{*\}+\mathbb{N} \arrow[r, "\mathrm{incl}_2\circ{[0,\mathrm{succ}]}"] \arrow[dd, "{[x_0,y]}"] & \{*\}+\mathbb{N} \arrow[dd, "{[x_0,y]}"] \\
\{*\} \arrow[ur, "\mathrm{incl}_1"] \arrow[dr, "x_0"'] & & \\
& X \arrow[r, "f"'] & X
\end{tikzcd}$$

Conversely, if $[x_0, y'] : \{*\} + \mathbb{N} \to X$ is an arbitrary map making the diagram

$$\begin{tikzcd}
& \{*\}+\mathbb{N} \arrow[r, "\mathrm{incl}_2\circ{[0,\mathrm{succ}]}"] \arrow[dd, "{[x_0,y']}"] & \{*\}+\mathbb{N} \arrow[dd, "{[x_0,y']}"] \\
\{*\} \arrow[ur, "\mathrm{incl}_1"] \arrow[dr, "x_0"'] & & \\
& X \arrow[r, "f"'] & X
\end{tikzcd}$$

commute, then the diagrams

$$\begin{tikzcd}
\mathbb{N} \arrow[rr, "\mathrm{succ}"] \arrow[d, "\mathrm{incl}_2"] \arrow[dd, bend right=60, "y'"'] & & \mathbb{N} \arrow[d, "\mathrm{incl}_2"'] \arrow[dd, bend left=60, "y'"] \\
\{*\}+\mathbb{N} \arrow[urr, "{[0,\mathrm{succ}]}"] \arrow[d, "{[x,y']}"] & & \{*\}+\mathbb{N} \arrow[d, "{[x,y']}"'] \\
X \arrow[rr, "f"'] & & X
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\{*\} \arrow[r, "\mathrm{incl}_1"'] \arrow[rr, bend left, "0"] \arrow[dr, "x"'] & \{*\}+\mathbb{N} \arrow[r, "{[0,\mathrm{succ}]}"'] \arrow[d, "{[x,y']}"] & \mathbb{N} \arrow[d, "y'"] \\
& X \arrow[r, "f"'] & X
\end{tikzcd}$$

commute. Hence the function y' satisfies the same universal property as the function y, showing that y and y' agree.

Specializing this discussion to the diagram

$$\{*\} \xrightarrow{0} \mathbb{N} \xrightarrow{\mathrm{succ}} \mathbb{N}$$

we arrive at the desired isomorphism

$$\begin{tikzcd}
& \{*\}+\mathbb{N} \arrow[r, "\mathrm{incl}_2\circ{[0,\mathrm{succ}]}"] \arrow[dd, "{[0,\mathrm{succ}]}", "\cong"'] & \{*\}+\mathbb{N} \arrow[dd, "{[0,\mathrm{succ}]}", "\cong"'] \\
\{*\} \arrow[ur, "\mathrm{incl}_1"] \arrow[dr, "0"'] & & \\
& \mathbb{N} \arrow[r, "\mathrm{succ}"'] & \mathbb{N}
\end{tikzcd}$$

since the diagram

$$\begin{tikzcd}
& \mathbb{N} \arrow[r, "\mathrm{succ}"] \arrow[d, "\mathrm{succ}"] & \mathbb{N} \arrow[d, "\mathrm{succ}"] \\
\{*\} \arrow[ur, "0"] \arrow[r, "\mathrm{succ}(0)"'] & \mathbb{N} \arrow[r, "\mathrm{succ}"'] & \mathbb{N}
\end{tikzcd}$$

commutes. □

have shown that succ is injective and that zero is not the successor of any natural number! missing Peano-axiom: no loops!

(P5) If a statement holds for zero and if the statement for a natural number implies the statement for its successor, then it holds for all natural numbers.

Lemma 1.15.3 (Peano's Fifth Postulate)
Any subset $A \subseteq \mathbb{N}$, which contains $0 \in X$ and is *closed under successors* in the sense that $\mathrm{succ}(A) \subseteq_{\mathbb{N}} A$ is the whole of $\mathbb{N}$.

Proof adapt (simplify???) Johnstone Thm. 6.14 □

construct arithmetic, refer to later sections for rules, motivate transpose by wanting to construct map out of $\mathbb{N}$

For an element $n \in \mathbb{N}$, want function $\ell_n : \mathbb{N} \to \mathbb{N}, m \mapsto n + m$ and want that $0 + m = m$ and $(\mathrm{succ}\, n) + m = \mathrm{succ}(n + m)$.

Definition 1.15.4
The natural numbers admit an **addition** function $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, whose transpose $L_\bullet : \mathbb{N} \to \mathrm{Map}(\mathbb{N}, \mathbb{N})$ is the unique function induced from the requirement that

$$\begin{tikzcd}
 & \mathbb{N} \arrow[r, "\mathrm{succ}"] \arrow[d, dashed, "\ell_\bullet"] & \mathbb{N} \arrow[d, dashed, "\ell_\bullet"] \\
\{*\} \arrow[ur, "0"] \arrow[r, "\mathrm{id}_{\mathbb{N}}"'] & \mathrm{Map}(\mathbb{N}, \mathbb{N}) \arrow[r, "\mathrm{succ}_*"'] & \mathrm{Map}(\mathbb{N}, \mathbb{N})
\end{tikzcd}$$

commutes.

construct standard order on $\mathbb{N}$

Lemma 1.15.5
The function $\leq : \mathbb{N} \times \mathbb{N} \xrightarrow{(\mathrm{pr}_1, +)} \mathbb{N} \times \mathbb{N}$ is injective and defines a total order on $\mathbb{N}$, called the **standard order**.

Proof cf. Johnstone Prop. 6.17 □

$\mathbb{N}$ wellordered?

Axiom 1.15.6 (Countable Choice)
Every surjective function $f : X \to \mathbb{N}$ admits a section $s : \mathbb{N} \to X$ satisfying $fs = \mathrm{id}_{\mathbb{N}}$.

Lemma 1.15.7
as exercise? The axiom of countable choice is equivalent to the requirement that every lifting problem of the form

$$\begin{tikzcd}
 & X \arrow[d, two heads, "p"] \\
\mathbb{N} \arrow[ur, dashed, "s"] \arrow[r, "f"'] & Y
\end{tikzcd}$$

in which p is a surjection admits a solution.

Proof The existence of lifts for all lifting problems implies the axiom of countable choice by choosing $f = \mathrm{id}_{\mathbb{N}}$. Conversely consider an arbitrary lifting problem as depicted above and form the pullback

$$\begin{tikzcd}
P \arrow[r, "f'"] \arrow[d, two heads, "p'"] & X \arrow[d, two heads, "p"] \\
\mathbb{N} \arrow[r, "f"'] \arrow[u, bend left, "s'"] & Y
\end{tikzcd}$$

Since by REF a pullback of a surjection is a surjection, the function q is surjective, hence by the axiom of countable choice admits a section $s' : \mathbb{N} \to P$. But then $s = f' \circ s'$ defines a lift as required. $\square$

1.16 Finiteness

Lemma 1.16.1 (MO410013)
The following assertions are equivalent

1. There exists $n \in \mathbb{N}$ such that $X \cong [n]$.
2. The set X is isomorphic to a subset $E \subseteq X$ and X is <u>not</u> isomorphic to $\mathbb{N}$.
3. Every nonempty subset of $\mathcal{P}(X)$ has a maximal element. (Tarski)
4. Every nonempty subset of $\mathcal{P}(X)$ has a minimal element (Tarksi)

Lemma 1.16.2 (MO410013)
The following assertions are equivalent.

1. There is <u>no</u> surjection $X \to \mathbb{N}$.
2. The set X is Noetherian.
3. The set X is Artinian.

Lemma 1.16.3 (MO410013)
The following assertions are equivalent.

1. Every injective endomorphism $X \to X$ is a bijection.
2. There is <u>no</u> injection $\mathbb{N} \to X$.

Definition 1.16.4
A set X is **finite**, if it is either the empty set or isomorphic to $Y + \{*\}$ for a finite set Y.

Definition 1.16.5
The set of **Kuratowski-finite subsets** of a set X is defined as the smallest subset of $\mathcal{P}_{\text{fin}}(X) \subseteq \mathcal{P}(X)$, which contains the empty subset $\emptyset \subseteq X$ and each singleton subset $\{x\} \subseteq X$ for $x \in X$ and which is closed under binary unions. explicit construction: smallest sub-$\vee$-lattice

A subset $K \subseteq X$ is **Kuratowski-finite**, if $K \in \mathcal{P}_{\text{fin}}(X)$. A set X is **Kuratowski-finite**, if $X \in \mathcal{P}_{\text{fin}}(X)$.

Definition 1.16.6
A set X is **Kuratowski-finite**, if no NNO! see Ortega

Definition 1.16.7
A set X is **Dedekind-finite**, if every injective endomorphism $X \to X$ is bijective.

Definition 1.16.8
A set X is **Tarski-finite**, if every inhabited subobject of $\mathcal{P}(X)$ has a $\subseteq$-minimal element.

Definition 1.16.9
A set X is **Artin-finite**, if every descending sequence of subsets

$$A_1 \supseteq_X A_2 \supseteq_X A_3 \supseteq_X \dots$$

stabilizes.

Theorem 1.16.10
For a set X the following assertions are equivalent.

(i) The set X is finite.

(ii) The set X is Kuratowski-finite.

(iii) The set X is Dedekind-finite.

(iv) The set X is Tarski-finite.

Proof. (i)$\Rightarrow$(iii) The empty set is Dedekind-finite. Thus let X be isomorphic to some set $Y + \{*\}$, for which we assume that Y is finite and Dedekind-finite. Consider an injective endomorphism $X \to X$. It gives rise to an injective endomorphism $f : Y + \{*\} \to Y + \{*\}$. ???

Suppose X is Kuratowski-finite. Let $x \in X$ and consider $X' = X \setminus \{x\}$. □

finite AC

pigeon-hole principle

remark that up until now finite sets are valid set theory

1.17 Cantor Diagonalization and the Continuum

1.18 The Axiom of Choice and Zorn's Lemma

2 Elementary Number Theory and Elementary Geometry

2.1 The Natural Numbers

arithmetic and order of natural numbers, prime numbers, initial monoid

2.2 The Integers

arithmetic, initial abelian group

2.3 Modular Arithmetic?

modular arithmetic

2.4 The Rational Numbers

2.5 The Real Numbers

2.6 The Euclidean Plane

2.7 The Complex Numbers

complex numbers, algebraically closed, complex exponential function

2.8 The p-adic Numbers?

3 Category Theory

3.1 Categories, Functors, Natural Transformations

3.2 Limits and Colimits

3.3 Adjoint Functors

3.4 Monoidal Structures?

4 Independence Results

4.1 Sheaf Theoretic Forcing

4.2 Forcing CH

4.3 Forcing $\neg\mathsf{CH}$

4.4 Forcing $\neg\mathsf{C}$