常用积分公式及其证明

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目录

前言

本文章所包含的积分公式均摘自《同济大学高等数学第 7 版上册》，并对原书中的一些错误进行修正。为方便读者查看，第一部分仅列出公式，第二部分再给出不完全证明，目录上带有超链接，点击可跳跃查看。本文只考虑如何证明，并不涉及一些细节如被积函数的定义域问题；为了方便，通常不考虑常数因子 C 的变化，C 可以吸收任何常数即 $C + a = C + b$. 本文章严格按照数学符号标准编写：变量、函数如 $f(x)$ 写作斜体，函数符号如 sin、常数如 e、微分符号 d 写作直立体。

在第二部分证明中，为了方便，其中的一些常用的过程我们用 **use** $\cdots$ 的方式替代。其中 **use** 的内容在第 2 第 3 章有证明方法，点击超链接即可查看。

本文仅因学习交流而作，未经作者同意，不允许用作于任何商业用途，侵权必究。因编写仓促，错误难免，如有错误请指正。

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1 积分公式

1.1 含有 $ax + b$ 的积分

$$\int \frac{\mathrm{d}x}{ax+b} = \frac{1}{a} \ln |ax+b| + C \tag{1.1}$$

$$\int (ax+b)^{\mu} \, \mathrm{d}x = \frac{1}{a(\mu+1)} (ax+b)^{\mu+1} + C \tag{1.2}$$

$$\int \frac{x}{ax+b} \, \mathrm{d}x = \frac{1}{a^2} (ax+b-b\ln|ax+b|) + C \tag{1.3}$$

$$\int \frac{x^2}{ax+b} \, \mathrm{d}x = \frac{1}{a^3} \left[\frac{1}{2} (ax+b)^2 - 2b(ax+b) + b^2 \ln |ax+b| \right] + C \tag{1.4}$$

$$\int \frac{\mathrm{d}x}{x(ax+b)} = -\frac{1}{b} \ln \left| \frac{ax+b}{x} \right| + C \tag{1.5}$$

$$\int \frac{\mathrm{d}x}{x^2(ax+b)} = -\frac{1}{bx} + \frac{a}{b^2} \ln \left| \frac{ax+b}{x} \right| + C \tag{1.6}$$

$$\int \frac{x}{(ax+b)^2} \, \mathrm{d}x = \frac{1}{a^2} \left(\ln |ax+b| + \frac{b}{ax+b} \right) + C \tag{1.7}$$

$$\int \frac{x^2}{(ax+b)^2} \, \mathrm{d}x = \frac{1}{a^3} \left(ax+b-2b\ln|ax+b| - \frac{b^2}{ax+b} \right) + C \tag{1.8}$$

$$\int \frac{\mathrm{d}x}{x\left(ax+b\right)^2} = \frac{1}{b\left(ax+b\right)} - \frac{1}{b^2}\ln\left|\frac{ax+b}{x}\right| + C \tag{1.9}$$

1.2 含有 $\sqrt{ax+b}$ 的积分

$$\int \sqrt{ax+b}\,\mathrm{d}x = \frac{2}{3a}\sqrt{\left(ax+b\right)^3} + C \tag{1.10}$$

$$\int x\sqrt{ax+b}\,\mathrm{d}x = \frac{2}{15a^2}\left(3ax-2b\right)\sqrt{\left(ax+b\right)^3} + C \tag{1.11}$$

$$\int x^2\sqrt{ax+b}\,\mathrm{d}x = \frac{2}{105a^3}\left(15a^2x^2-12abx+8b^2\right)\sqrt{\left(ax+b\right)^3} + C \tag{1.12}$$

$$\int \frac{x}{\sqrt{ax+b}}\,\mathrm{d}x = \frac{2}{3a^2}\left(ax-2b\right)\sqrt{ax+b} + C \tag{1.13}$$

$$\int \frac{x^2}{\sqrt{ax+b}}\,\mathrm{d}x = \frac{2}{15a^2}\left(3a^2x^2-4abx+8b^2\right)\sqrt{ax+b} + C \tag{1.14}$$

$$\int \frac{\mathrm{d}x}{x\sqrt{ax+b}} = \begin{cases} \dfrac{1}{\sqrt{b}}\ln\left|\dfrac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + C & \text{if } (b>0) \\ \dfrac{2}{\sqrt{-b}}\arctan\sqrt{\dfrac{ax+b}{-b}} + C & \text{if } (b<0) \end{cases} \tag{1.15}$$

$$\int \frac{\mathrm{d}x}{x^2\sqrt{ax+b}} = -\frac{ax+b}{bx} - \frac{a}{2b}\int \frac{\mathrm{d}x}{x\sqrt{ax+b}} \tag{1.16}$$

$$\int \frac{\sqrt{ax+b}}{x}\,\mathrm{d}x = 2\sqrt{ax+b} + b\int \frac{\mathrm{d}x}{x\sqrt{ax+b}} \tag{1.17}$$

$$\int \frac{\sqrt{ax+b}}{x^2}\,\mathrm{d}x = -\frac{\sqrt{ax+b}}{x} + \frac{a}{2}\int \frac{\mathrm{d}x}{x\sqrt{ax+b}} \tag{1.18}$$

1.3 含有 $x^2 \pm a^2$ 的积分

$$\int \frac{\mathrm{d}x}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C \tag{1.19}$$

$$\begin{aligned} \int \frac{\mathrm{d}x}{\left(x^2+a^2\right)^n} = &\frac{x}{2\left(n-1\right)a^2\left(x^2+a^2\right)^{n-1}} \\ &+ \frac{2n-3}{2\left(n-1\right)a^2}\int \frac{\mathrm{d}x}{\left(x^2+a^2\right)^{n-1}} \end{aligned} \tag{1.20}$$

$$\int \frac{\mathrm{d}x}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C \tag{1.21}$$

1.4 含有 $ax^2+b\ (a>0)$ 的积分

$$\int\frac{\mathrm{d}x}{ax^2+b}=\begin{cases}\dfrac{1}{\sqrt{ab}}\arctan\sqrt{\dfrac{a}{b}}x+C & \text{if } (b>0)\\ \dfrac{1}{2\sqrt{-ab}}\ln\left|\dfrac{\sqrt{a}x-\sqrt{-b}}{\sqrt{a}x+\sqrt{-b}}\right|+C & \text{if } (b<0)\end{cases}\tag{1.22}$$

$$\int\frac{x}{ax^2+b}\,\mathrm{d}x=\frac{1}{2a}\ln\left|ax^2+b\right|+C\tag{1.23}$$

$$\int\frac{x^2}{ax^2+b}\,\mathrm{d}x=\frac{x}{a}-\frac{b}{a}\int\frac{\mathrm{d}x}{ax^2+b}\tag{1.24}$$

$$\int\frac{\mathrm{d}x}{x\left(ax^2+b\right)}=\frac{1}{2b}\ln\frac{x^2}{|ax^2+b|}+C\tag{1.25}$$

$$\int\frac{\mathrm{d}x}{x^2\left(ax^2+b\right)}=-\frac{1}{bx}-\frac{a}{b}\int\frac{\mathrm{d}x}{ax^2+b}\tag{1.26}$$

$$\int\frac{\mathrm{d}x}{x^3\left(ax^2+b\right)}=\frac{a}{2b^2}\ln\frac{|ax^2+b|}{x^2}-\frac{1}{2bx^2}+C\tag{1.27}$$

$$\int\frac{\mathrm{d}x}{\left(ax^2+b\right)^2}=\frac{x}{2b\left(ax^2+b\right)}+\frac{1}{2b}\int\frac{\mathrm{d}x}{ax^2+b}\tag{1.28}$$

1.5 含有 $ax^2+bx+c\,(a>0)$ 的积分

$$\int\frac{\mathrm{d}x}{ax^2+bx+c}=\begin{cases}\dfrac{2}{\sqrt{4ac-b^2}}\arctan\dfrac{2ax+b}{\sqrt{4ac-b^2}}+C & \text{if } \left(b^2<4ac\right)\\ \dfrac{1}{\sqrt{b^2-4ac}}\ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|+C & \text{if } \left(b^2>4ac\right)\end{cases}\tag{1.29}$$

$$\int\frac{x}{ax^2+bx+c}\,\mathrm{d}x=\frac{1}{2a}\ln\left|ax^2+bx+c\right|-\frac{b}{2a}\int\frac{\mathrm{d}x}{ax^2+bx+c}\tag{1.30}$$

1.6 含有 $\sqrt{x^2+a^2}\,(a>0)$ 的积分

$$\int\frac{\mathrm{d}x}{\sqrt{x^2+a^2}}=arsh\frac{x}{a}+C_1=\ln\left(x+\sqrt{x^2+a^2}\right)+C\tag{1.31}$$

$$\int\frac{\mathrm{d}x}{\sqrt{\left(x^2+a^2\right)^3}}=\frac{x}{a^2\sqrt{x^2+a^2}}+C\tag{1.32}$$

$$\int\frac{x}{\sqrt{x^2+a^2}}\,\mathrm{d}x=\sqrt{x^2+a^2}+C\tag{1.33}$$

$$\int\frac{x}{\sqrt{\left(x^2+a^2\right)^3}}\,\mathrm{d}x=-\frac{1}{\sqrt{x^2+a^2}}+C\tag{1.34}$$

$$\int\frac{x^2}{\sqrt{x^2+a^2}}\,\mathrm{d}x=\frac{x}{2}\sqrt{x^2+a^2}-\frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right)+C\tag{1.35}$$

$$\int\frac{x^2}{\sqrt{\left(x^2+a^2\right)^3}}\,\mathrm{d}x=-\frac{x}{\sqrt{x^2+a^2}}+\ln\left(x+\sqrt{x^2+a^2}\right)+C\tag{1.36}$$

$$\int \frac{\mathrm{d}x}{x\sqrt{x^2+a^2}} = \frac{1}{a}\ln\frac{\sqrt{x^2+a^2}-a}{|x|}+C \tag{1.37}$$

$$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{a^2x}+C \tag{1.38}$$

$$\int \sqrt{x^2+a^2}\,\mathrm{d}x = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right)+C \tag{1.39}$$

$$\begin{aligned}\int \sqrt{\left(x^2+a^2\right)^3}\,\mathrm{d}x &= \frac{x}{8}\left(2x^2+5a^2\right)\sqrt{x^2+a^2}\\ &\quad+\frac{3}{8}a^4\ln\left(x+\sqrt{x^2+a^2}\right)+C\end{aligned} \tag{1.40}$$

$$\int x\sqrt{x^2+a^2}\,\mathrm{d}x = \frac{1}{3}\sqrt{\left(x^2+a^2\right)^3}+C \tag{1.41}$$

$$\begin{aligned}\int x^2\sqrt{x^2+a^2}\,\mathrm{d}x &= \frac{x}{8}\left(2x^2+a^2\right)\sqrt{x^2+a^2}\\ &\quad-\frac{a^4}{8}\ln\left(x+\sqrt{x^2+a^2}\right)+C\end{aligned} \tag{1.42}$$

$$\int \frac{\sqrt{x^2+a^2}}{x}\,\mathrm{d}x = \sqrt{x^2+a^2}+a\ln\frac{\sqrt{x^2+a^2}-a}{|x|}+C \tag{1.43}$$

$$\int \frac{\sqrt{x^2+a^2}}{x^2}\,\mathrm{d}x = -\frac{\sqrt{x^2+a^2}}{x}+\ln\left(x+\sqrt{x^2+a^2}\right)+C \tag{1.44}$$

1.7 含有 $\sqrt{x^2-a^2}\,(a>0)$ 的积分

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = \frac{x}{|x|}\operatorname{arccosh}\frac{|x|}{a}+C_1 = \ln\left|x+\sqrt{x^2-a^2}\right|+C \tag{1.45}$$

$$\int \frac{\mathrm{d}x}{\sqrt{\left(x^2-a^2\right)^3}} = -\frac{x}{a^2\sqrt{x^2-a^2}}+C \tag{1.46}$$

$$\int \frac{x}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \sqrt{x^2-a^2}+C \tag{1.47}$$

$$\int \frac{x}{\sqrt{\left(x^2-a^2\right)^3}}\,\mathrm{d}x = -\frac{1}{\sqrt{x^2-a^2}}+C \tag{1.48}$$

$$\int \frac{x^2}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \frac{x}{2}\sqrt{x^2-a^2}+\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+C \tag{1.49}$$

$$\int \frac{x^2}{\sqrt{\left(x^2-a^2\right)^3}}\,\mathrm{d}x = -\frac{x}{\sqrt{x^2-a^2}}+\ln\left|x+\sqrt{x^2-a^2}\right|+C \tag{1.50}$$

$$\int \frac{\mathrm{d}x}{x\sqrt{x^2-a^2}} = \frac{1}{a}\arccos\frac{a}{|x|}+C \tag{1.51}$$

$$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x}+C \tag{1.52}$$

$$\int \sqrt{x^2-a^2}\,\mathrm{d}x = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+C \tag{1.53}$$

$$\int \sqrt{\left(x^2-a^2\right)^3}\,\mathrm{d}x = \frac{x}{8}\left(2x^2-5a^2\right)\sqrt{x^2-a^2} + \frac{3}{8}a^4 \ln\left|x+\sqrt{x^2-a^2}\right| + C \tag{1.54}$$

$$\int x\sqrt{x^2-a^2}\,\mathrm{d}x = \frac{1}{3}\sqrt{\left(x^2-a^2\right)^3} + C \tag{1.55}$$

$$\int x^2\sqrt{x^2-a^2}\,\mathrm{d}x = \frac{x}{8}\left(2x^2-a^2\right)\sqrt{x^2-a^2} - \frac{a^4}{8}\ln\left|x+\sqrt{x^2-a^2}\right| + C \tag{1.56}$$

$$\int \frac{\sqrt{x^2-a^2}}{x}\,\mathrm{d}x = \sqrt{x^2-a^2} - a\arccos\frac{a}{|x|} + C \tag{1.57}$$

$$\int \frac{\sqrt{x^2-a^2}}{x^2}\,\mathrm{d}x = -\frac{\sqrt{x^2-a^2}}{x} + \ln\left|x+\sqrt{x^2-a^2}\right| + C \tag{1.58}$$

1.8 含有 $\sqrt{a^2-x^2}\,(a>0)$ 的积分

$$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C \tag{1.59}$$

$$\int \frac{\mathrm{d}x}{\sqrt{\left(a^2-x^2\right)^3}} = \frac{x}{a^2\sqrt{a^2-x^2}} + C \tag{1.60}$$

$$\int \frac{x}{\sqrt{a^2-x^2}}\,\mathrm{d}x = -\sqrt{a^2-x^2} + C \tag{1.61}$$

$$\int \frac{x}{\sqrt{\left(a^2-x^2\right)^3}}\,\mathrm{d}x = \frac{1}{\sqrt{a^2-x^2}} + C \tag{1.62}$$

$$\int \frac{x^2}{\sqrt{a^2-x^2}}\,\mathrm{d}x = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C \tag{1.63}$$

$$\int \frac{x^2}{\sqrt{\left(a^2-x^2\right)^3}}\,\mathrm{d}x = \frac{x}{\sqrt{a^2-x^2}} - \arcsin\frac{x}{a} + C \tag{1.64}$$

$$\int \frac{\mathrm{d}x}{x\sqrt{a^2-x^2}} = \frac{1}{a}\ln\frac{a-\sqrt{a^2-x^2}}{|x|} + C \tag{1.65}$$

$$\int \frac{\mathrm{d}x}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x} + C \tag{1.66}$$

$$\int \sqrt{a^2-x^2}\,\mathrm{d}x = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C \tag{1.67}$$

$$\int \sqrt{\left(a^2-x^2\right)^3}\,\mathrm{d}x = \frac{x}{8}\left(5a^2-2x^2\right)\sqrt{a^2-x^2} + \frac{3}{8}a^4\arcsin\frac{x}{a} + C \tag{1.68}$$

$$\int x\sqrt{a^2-x^2}\,\mathrm{d}x = -\frac{1}{3}\sqrt{\left(a^2-x^2\right)^3} + C \tag{1.69}$$

$$\int x^2\sqrt{a^2-x^2}\,\mathrm{d}x = \frac{x}{8}\left(2x^2-a^2\right)\sqrt{a^2-x^2} + \frac{a^4}{8}\arcsin\frac{x}{a} + C \tag{1.70}$$

$$\int \frac{\sqrt{a^2-x^2}}{x}\,\mathrm{d}x = \sqrt{a^2-x^2} + a\ln\frac{a-\sqrt{a^2-x^2}}{|x|} + C \tag{1.71}$$

$$\int \frac{\sqrt{a^2-x^2}}{x^2}\,\mathrm{d}x = -\frac{\sqrt{a^2-x^2}}{x} - \arcsin\frac{x}{a} + C \tag{1.72}$$

1.9 含有 $\sqrt{\pm ax^2+bx+c}\,(a>0)$ 的积分

$$\int \frac{\mathrm{d}x}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C \tag{1.73}$$

$$\begin{aligned}\int \sqrt{ax^2+bx+c}\,\mathrm{d}x &= \frac{2ax+b}{4a}\sqrt{ax^2+bx+c} \\ &\quad + \frac{4ac-b^2}{8\sqrt{a^3}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C\end{aligned} \tag{1.74}$$

$$\begin{aligned}\int \frac{x}{\sqrt{ax^2+bx+c}}\,\mathrm{d}x &= \frac{1}{a}\sqrt{ax^2+bx+c} \\ &\quad - \frac{b}{2\sqrt{a^3}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C\end{aligned} \tag{1.75}$$

$$\int \frac{\mathrm{d}x}{\sqrt{c+bx-ax^2}} = \frac{1}{\sqrt{a}}\arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C \tag{1.76}$$

$$\begin{aligned}\int \sqrt{c+bx-ax^2}\,\mathrm{d}x &= \frac{2ax-b}{4a}\sqrt{c+bx-ax^2} \\ &\quad + \frac{b^2+4ac}{8\sqrt{a^3}}\arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C\end{aligned} \tag{1.77}$$

$$\begin{aligned}\int \frac{x}{\sqrt{c+bx-ax^2}}\,\mathrm{d}x &= -\frac{1}{a}\sqrt{c+bx-ax^2} \\ &\quad + \frac{b}{2\sqrt{a^3}}\arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C\end{aligned} \tag{1.78}$$

1.10 含有 $\sqrt{\pm\frac{x-a}{x-b}}$ 或 $\sqrt{(x-a)\,(b-x)}$ 的积分

$$\begin{aligned}\int \sqrt{\frac{x-a}{x-b}}\,\mathrm{d}x &= (x-b)\sqrt{\frac{x-a}{x-b}} \\ &\quad + (b-a)\ln\left(\sqrt{|x-a|}+\sqrt{|x-b|}\right)+C\end{aligned} \tag{1.79}$$

$$\int \sqrt{\frac{x-a}{b-x}}\,\mathrm{d}x = (x-b)\sqrt{\frac{x-a}{b-x}}+(b-a)\arcsin\sqrt{\frac{x-a}{b-a}}+C \tag{1.80}$$

$$\int \frac{\mathrm{d}x}{\sqrt{(x-a)(b-x)}} = 2\arcsin\sqrt{\frac{x-a}{b-a}}+C(a<b) \tag{1.81}$$

$$\begin{aligned}\int \sqrt{(x-a)(b-x)}\,\mathrm{d}x &= \frac{2x-a-b}{4}\sqrt{(x-a)(b-x)} \\ &\quad + \frac{(b-a)^2}{4}\arcsin\sqrt{\frac{x-a}{b-a}}+C(a<b)\end{aligned} \tag{1.82}$$

1.11 含有三角函数的积分

$$\int \sin x\,\mathrm{d}x = -\cos x+C \tag{1.83}$$

$$\int \cos x\,\mathrm{d}x = \sin x+C \tag{1.84}$$

$$\int \tan x \,\mathrm{d}x = -\ln|\cos x| + C \tag{1.85}$$

$$\int \cot x \,\mathrm{d}x = \ln|\sin x| + C \tag{1.86}$$

$$\int \sec x \,\mathrm{d}x = \ln\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + C = \ln|\sec x + \tan x| + C \tag{1.87}$$

$$\int \csc x \,\mathrm{d}x = \ln\left|\tan\frac{x}{2}\right| + C = \ln|\csc x - \cot x| + C \tag{1.88}$$

$$\int \sec^2 x \,\mathrm{d}x = \tan x + C \tag{1.89}$$

$$\int \csc^2 x \,\mathrm{d}x = -\cot x + C \tag{1.90}$$

$$\int \sec x \tan x \,\mathrm{d}x = \sec x + C \tag{1.91}$$

$$\int \csc x \cot x \,\mathrm{d}x = -\csc x + C \tag{1.92}$$

$$\int \sin^2 x \,\mathrm{d}x = \frac{x}{2} - \frac{1}{4}\sin 2x + C \tag{1.93}$$

$$\int \cos^2 x \,\mathrm{d}x = \frac{x}{2} + \frac{1}{4}\sin 2x + C \tag{1.94}$$

$$\int \sin^n x \,\mathrm{d}x = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x \,\mathrm{d}x \tag{1.95}$$

$$\int \cos^n x \,\mathrm{d}x = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x \,\mathrm{d}x \tag{1.96}$$

$$\int \frac{\mathrm{d}x}{\sin^n x} = -\frac{1}{n-1}\frac{\cos x}{\sin^{n-1} x} + \frac{n-2}{n-1}\int \frac{\mathrm{d}x}{\sin^{n-2} x} \tag{1.97}$$

$$\int \frac{\mathrm{d}x}{\cos^n x} = \frac{1}{n-1}\frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1}\int \frac{\mathrm{d}x}{\cos^{n-2} x} \tag{1.98}$$

$$\begin{aligned}\int \cos^m x \sin^n x \,\mathrm{d}x &= \frac{1}{m+n}\cos^{m-1} x \sin^{n+1} x + \frac{m-1}{m+n}\int \cos^{m-2} x \sin^n x \,\mathrm{d}x \\ &= -\frac{1}{m+n}\cos^{m+1} x \sin^{n-1} x + \frac{n-1}{m+n}\int \cos^m x \sin^{n-2} x \,\mathrm{d}x\end{aligned} \tag{1.99}$$

$$\int \sin ax \cos bx \,\mathrm{d}x = -\frac{1}{2(a+b)}\cos(a+b)x - \frac{1}{2(a-b)}\cos(a-b)x + C \tag{1.100}$$

$$\int \sin ax \sin bx \,\mathrm{d}x = -\frac{1}{2(a+b)}\sin(a+b)x + \frac{1}{2(a-b)}\sin(a-b)x + C \tag{1.101}$$

$$\int \cos ax \cos bx \,\mathrm{d}x = \frac{1}{2(a+b)}\sin(a+b)x + \frac{1}{2(a-b)}\sin(a-b)x + C \tag{1.102}$$

$$\int \frac{\mathrm{d}x}{a+b\sin x} = \frac{2}{\sqrt{a^2-b^2}}\arctan\frac{a\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}} + C(a^2>b^2) \tag{1.103}$$

$$\int \frac{\mathrm{d}x}{a+b\sin x} = \frac{1}{\sqrt{b^2-a^2}}\ln\left|\frac{a\tan\frac{x}{2}+b-\sqrt{b^2+a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2+a^2}}\right| + C(a^2<b^2) \tag{1.104}$$

$$\int \frac{\mathrm{d}x}{a+b\cos x} = \frac{2}{a+b}\sqrt{\frac{a+b}{a-b}}\arctan\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right) + C(a^2>b^2) \tag{1.105}$$

$$\int \frac{\mathrm{d}x}{a+b\cos x} = \frac{1}{a+b}\sqrt{\frac{a+b}{b-a}} \ln\left|\frac{\tan\frac{x}{2}+\sqrt{\frac{a+b}{b-a}}}{\tan\frac{x}{2}-\sqrt{\frac{a+b}{b-a}}}\right| + C(a^2<b^2) \tag{1.106}$$

$$\int \frac{\mathrm{d}x}{a^2\cos^2 x + b^2\sin^2 x} = \frac{1}{ab}\arctan\left(\frac{b}{a}\tan x\right) + C \tag{1.107}$$

$$\int \frac{\mathrm{d}x}{a^2\cos^2 x - b^2\sin^2 x} = \frac{1}{2ab}\ln\left|\frac{b\tan x + a}{b\tan x - a}\right| + C \tag{1.108}$$

$$\int x\sin ax \,\mathrm{d}x = \frac{1}{a^2}\sin ax - \frac{1}{a}x\cos ax + C \tag{1.109}$$

$$\int x^2\sin ax \,\mathrm{d}x = -\frac{1}{a}x^2\cos ax + \frac{2}{a^2}x\sin ax + \frac{2}{a^3}\cos ax + C \tag{1.110}$$

$$\int x\cos ax \,\mathrm{d}x = \frac{1}{a^2}\cos ax + \frac{1}{a}x\sin ax + C \tag{1.111}$$

$$\int x^2\cos ax \,\mathrm{d}x = \frac{1}{a}x^2\sin ax + \frac{2}{a^2}x\cos ax - \frac{2}{a^3}\sin ax + C \tag{1.112}$$

1.12 含有反三角函数的积分 $(a>0)$

$$\int \arcsin\frac{x}{a}\,\mathrm{d}x = x\arcsin\frac{x}{a} + \sqrt{a^2-x^2} + C \tag{1.113}$$

$$\int x\arcsin\frac{x}{a}\,\mathrm{d}x = \left(\frac{x^2}{2}-\frac{a^2}{4}\right)\arcsin\frac{x}{a} + \frac{x}{4}\sqrt{a^2-x^2} + C \tag{1.114}$$

$$\int x^2\arcsin\frac{x}{a}\,\mathrm{d}x = \frac{x^3}{3}\arcsin\frac{x}{a} + \frac{1}{9}\left(x^2+2a^2\right)\sqrt{a^2-x^2} + C \tag{1.115}$$

$$\int \arccos\frac{x}{a}\,\mathrm{d}x = x\arccos\frac{x}{a} - \sqrt{a^2-x^2} + C \tag{1.116}$$

$$\int x\arccos\frac{x}{a}\,\mathrm{d}x = \left(\frac{x^2}{2}-\frac{a^2}{4}\right)\arccos\frac{x}{a} - \frac{x}{4}\sqrt{a^2-x^2} + C \tag{1.117}$$

$$\int x^2\arccos\frac{x}{a}\,\mathrm{d}x = \frac{x^3}{3}\arccos\frac{x}{a} - \frac{1}{9}\left(x^2+2a^2\right)\sqrt{a^2-x^2} + C \tag{1.118}$$

$$\int \arctan\frac{x}{a}\,\mathrm{d}x = x\arctan\frac{x}{a} - \frac{a}{2}\ln\left(a^2+x^2\right) + C \tag{1.119}$$

$$\int x\arctan\frac{x}{a}\,\mathrm{d}x = \frac{1}{2}\left(a^2+x^2\right)\arctan\frac{x}{a} - \frac{a}{2}x + C \tag{1.120}$$

$$\int x^2\arctan\frac{x}{a}\,\mathrm{d}x = \frac{x^3}{3}\arctan\frac{x}{a} - \frac{a}{6}x^2 + \frac{a^3}{6}\ln\left(a^2+x^2\right) + C \tag{1.121}$$

1.13 含有指数函数的积分

$$\int a^x \, \mathrm{d}x = \frac{1}{\ln a} a^x + C \tag{1.122}$$

$$\int \mathrm{e}^{ax} \, \mathrm{d}x = \frac{1}{a} \mathrm{e}^{ax} + C \tag{1.123}$$

$$\int x \, \mathrm{e}^{ax} \, \mathrm{d}x = \frac{1}{a^2}(ax-1) \, \mathrm{e}^{ax} + C \tag{1.124}$$

$$\int x^n \, \mathrm{e}^{ax} \, \mathrm{d}x = \frac{1}{a} x^n \, \mathrm{e}^{ax} - \frac{n}{a} \int x^{n-1} \, \mathrm{e}^{ax} \, \mathrm{d}x \tag{1.125}$$

$$\int x a^x \, \mathrm{d}x = \frac{x}{\ln a} a^x - \frac{1}{(\ln a)^2} a^x + C \tag{1.126}$$

$$\int x^n a^x \, \mathrm{d}x = \frac{1}{\ln a} x^n a^x - \frac{n}{\ln a} \int x^{n-1} a^x \, \mathrm{d}x \tag{1.127}$$

$$\int \mathrm{e}^{ax} \sin bx \, \mathrm{d}x = \frac{1}{a^2+b^2} \mathrm{e}^{ax} \left(a \sin bx - b \cos bx\right) + C \tag{1.128}$$

$$\int \mathrm{e}^{ax} \cos bx \, \mathrm{d}x = \frac{1}{a^2+b^2} \mathrm{e}^{ax} \left(b \sin bx + a \cos bx\right) + C \tag{1.129}$$

$$\begin{aligned} \int \mathrm{e}^{ax} \sin^n bx \, \mathrm{d}x = {} & \frac{1}{a^2+b^2n^2} \mathrm{e}^{ax} \sin^{n-1} bx \left(a \sin bx - nb \cos bx\right) \\ & + \frac{n(n-1)b^2}{a^2+b^2n^2} \int \mathrm{e}^{ax} \sin^{n-2} bx \, \mathrm{d}x \end{aligned} \tag{1.130}$$

$$\begin{aligned} \int \mathrm{e}^{ax} \cos^n bx \, \mathrm{d}x = {} & \frac{1}{a^2+b^2n^2} \mathrm{e}^{ax} \cos^{n-1} bx \left(a \cos bx + nb \sin bx\right) \\ & + \frac{n(n-1)b^2}{a^2+b^2n^2} \int \mathrm{e}^{ax} \cos^{n-2} bx \, \mathrm{d}x \end{aligned} \tag{1.131}$$

1.14 含有对数函数的积分

$$\int \ln x \, \mathrm{d}x = x \ln x - x + C \tag{1.132}$$

$$\int \frac{\mathrm{d}x}{x \ln x} = \ln |\ln x| + C \tag{1.133}$$

$$\int x^n \ln x \, \mathrm{d}x = \frac{1}{n+1} x^{n+1} \left(\ln x - \frac{1}{n+1}\right) + C \tag{1.134}$$

$$\int (\ln x)^n \, \mathrm{d}x = x(\ln x)^n - n \int (\ln x)^{n-1} \, \mathrm{d}x \tag{1.135}$$

$$\int x^m (\ln x)^n \, \mathrm{d}x = \frac{1}{m+1} x^{m+1} (\ln x)^n - \frac{n}{m+1} \int x^m (\ln x)^{n-1} \, \mathrm{d}x \tag{1.136}$$

1.15 含有双曲函数的积分

$$\int \sinh x \, \mathrm{d}x = \cosh x + C \tag{1.137}$$

$$\int \cosh x \, \mathrm{d}x = \sinh x + C \tag{1.138}$$

$$\int \tanh x \, \mathrm{d}x = \ln \cosh x + C \tag{1.139}$$

$$\int \sinh^2 x \, \mathrm{d}x = -\frac{x}{2} + \frac{1}{4} \sinh 2x + C \tag{1.140}$$

$$\int \cosh^2 x \, \mathrm{d}x = \frac{x}{2} + \frac{1}{4} \sinh 2x + C \tag{1.141}$$

1.16 定积分 $(m, n \in \mathbb{Z})$

$$\int_{-\pi}^{\pi} \cos nx \, \mathrm{d}x = \int_{-\pi}^{\pi} \sin nx \, \mathrm{d}x = 0 \tag{1.142}$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, \mathrm{d}x = 0 \tag{1.143}$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, \mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases} \tag{1.144}$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, \mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases} \tag{1.145}$$

$$\int_{0}^{\pi} \sin mx \sin nx \, \mathrm{d}x = \int_{0}^{\pi} \cos mx \cos nx \, \mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \\ \frac{\pi}{2} & \text{if } m = n \end{cases} \tag{1.146}$$

$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{2}} \sin^n x \, \mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^n x \, \mathrm{d}x \\ I_n &= \frac{n-1}{n} I_{n-2} = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \ldots \cdot \frac{4}{5} \cdot \frac{2}{3} & \text{if } \{x = 2n+1, n > 0\} I_1 = 1 \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \ldots \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} & \text{if } \{x = 2n, n > 0\} I_0 = \frac{\pi}{2} \end{cases} \end{aligned} \tag{1.147}$$

2 积分公式证明

2.1 含有 $ax + b$ 的积分

$$\int \frac{\mathrm{d}x}{ax + b} = \frac{1}{a} \ln |ax + b| + C \tag{2.1}$$

Proof.

$$\begin{aligned}\int \frac{\mathrm{d}x}{ax+b} &= \frac{1}{a}\int \frac{\mathrm{d}(ax+b)}{ax+b} \\ &= \frac{1}{a}\ln|ax+b|+C\end{aligned}$$

■

$$\int (ax+b)^{\mu}\ \mathrm{d}x = \frac{1}{a(\mu+1)}(ax+b)^{\mu+1}+C \tag{2.2}$$

Proof.

$$\begin{aligned}\int (ax+b)^{\mu}\ \mathrm{d}x &= \frac{1}{a}\int (ax+b)^{\mu}\ \mathrm{d}(ax+b) \\ &= \frac{1}{a(\mu+1)}(ax+b)^{\mu+1}+C\end{aligned}$$

■

$$\int \frac{x}{ax+b}\ \mathrm{d}x = \frac{1}{a^2}(ax+b-b\ln|ax+b|)+C \tag{2.3}$$

Proof.

$$\begin{aligned}\int \frac{x}{ax+b}\ \mathrm{d}x &= \frac{1}{a}\int \left[\frac{ax+b}{ax+b}-\frac{b}{ax+b}\right]\ \mathrm{d}x \\ &= \frac{1}{a}\left[x-\frac{a}{b}\int \frac{\mathrm{d}(ax+b)}{ax+b}\right] \\ &= \frac{1}{a^2}(ax-b\ln|ax+b|)+C\end{aligned}$$

■

It seems that we have lost the $\frac{b}{a^2}$ (as $\frac{b}{a^2}+C=C$),so we have another way:

Proof.

$$\begin{aligned}&\text{let } u = ax+b \\ \int \frac{x}{ax+b}\ \mathrm{d}x &= \frac{1}{a^2}\int \frac{u-b}{u}\ \mathrm{d}u \\ &= \frac{1}{a^2}\int \left[1-\frac{b}{u}\right]\ \mathrm{d}u \\ &= \frac{1}{a^2}(u-b\ln|u|) \\ &= \frac{1}{a^2}(ax+b-b\ln|ax+b|)+C\end{aligned}$$

■

$$\int \frac{x^2}{ax+b}\ \mathrm{d}x = \frac{1}{a^3}\left[\frac{1}{2}(ax+b)^2-2b(ax+b)+b^2\ln|ax+b|\right]+C \tag{2.4}$$

Proof.

$$
\begin{aligned}
&\text{let } u = ax + b \\
\int \frac{x^2}{ax+b}\,\mathrm{d}x &= \frac{1}{a^3}\int \frac{(u-b)^2}{u}\,\mathrm{d}u \\
&= \frac{1}{a^3}\int \left[u - 2b + \frac{b^2}{u}\right]\,\mathrm{d}u \\
&= \frac{1}{a^3}\left(\frac{u^2}{2} - 2bu + b^2\ln|u|\right) + C \\
&= \frac{1}{a^3}\left[\frac{1}{2}(ax+b)^2 - 2b(ax+b) + b^2\ln|ax+b|\right] + C
\end{aligned}
$$

■

$$
\int \frac{\mathrm{d}x}{x(ax+b)} = -\frac{1}{b}\ln\left|\frac{ax+b}{x}\right| + C \tag{2.5}
$$

Proof.

$$
\begin{aligned}
\int \frac{\mathrm{d}x}{x(ax+b)} &= \int \left[\frac{1}{bx} - \frac{a}{b}\frac{1}{ax+b}\right]\,\mathrm{d}x \\
&= \frac{1}{b}\ln|x| - \frac{1}{b}\ln|ax+b| + C \\
&= -\frac{1}{b}\ln\left|\frac{ax+b}{x}\right| + C
\end{aligned}
$$

■

$$
\int \frac{\mathrm{d}x}{x^2(ax+b)} = -\frac{1}{bx} + \frac{a}{b^2}\ln\left|\frac{ax+b}{x}\right| + C \tag{2.6}
$$

Proof.

$$
\begin{aligned}
\int \frac{\mathrm{d}x}{x^2(ax+b)} &= \int \left[\frac{1}{bx^2} - \frac{a}{b^2x} + \frac{a^2}{b^2}\frac{1}{ax+b}\right]\,\mathrm{d}x \\
&= -\frac{1}{bx} - \frac{a}{b^2}\ln|x| + \frac{a}{b^2}\ln|ax+b| + C \\
&= -\frac{1}{bx} + \frac{a}{b^2}\ln\left|\frac{ax+b}{x}\right| + C
\end{aligned}
$$

■

$$
\int \frac{x}{(ax+b)^2}\,\mathrm{d}x = \frac{1}{a^2}\left(\ln|ax+b| + \frac{b}{ax+b}\right) + C \tag{2.7}
$$

Proof.

$$
\begin{aligned}
&\text{let } u = ax+b \\
\int \frac{x}{(ax+b)^2}\,\mathrm{d}x &= \frac{1}{a^2}\int \frac{u-b}{u^2}\,\mathrm{d}u \\
&= \frac{1}{a^2}\int \left[\frac{1}{u}-\frac{b}{u^2}\right]\,\mathrm{d}u \\
&= \frac{1}{a^2}\left(\ln|u|+\frac{b}{u}\right)+C \\
&= \frac{1}{a^2}\left(\ln|ax+b|+\frac{b}{ax+b}\right)+C
\end{aligned}
$$

■

$$
\int \frac{x^2}{(ax+b)^2}\,\mathrm{d}x = \frac{1}{a^3}\left(ax+b-2b\ln|ax+b|-\frac{b^2}{ax+b}\right)+C \tag{2.8}
$$

Proof.

$$
\begin{aligned}
\int \frac{x^2}{(ax+b)^2}\,\mathrm{d}x &= -\frac{x^2}{a(ax+b)}+\int \frac{2x}{a(ax+b)}\,\mathrm{d}x \\
\textbf{use } \int \frac{x}{ax+b}\,\mathrm{d}x &= \frac{1}{a^2}\left(ax+b-b\ln|ax+b|\right)+C \\
&= -\frac{x^2}{a(ax+b)}+\frac{2}{a^3}\left(ax+b-b\ln|ax+b|\right)+C \\
&= \frac{1}{a^3}\left(ax+b-2b\ln|ax+b|+\frac{a^2x^2+2abx+b^2}{ax+b}-\frac{a^2x^2}{ax+b}\right)+C \\
&= \frac{1}{a^3}\left(ax+b-2b\ln|ax+b|+\frac{2b(ax+b)}{ax+b}-\frac{b^2}{ax+b}\right)+C \\
&= \frac{1}{a^3}\left(ax+b-2b\ln|ax+b|-\frac{b^2}{ax+b}\right)+C
\end{aligned}
$$

■

$$
\int \frac{\mathrm{d}x}{x\,(ax+b)^2} = \frac{1}{b\,(ax+b)}-\frac{1}{b^2}\ln\left|\frac{ax+b}{x}\right|+C \tag{2.9}
$$

Proof.

$$
\begin{aligned}
\int \frac{\mathrm{d}x}{x\,(ax+b)^2} &= \int \left[-\frac{1}{b^2x}+\frac{a}{b(ax+b)^2}+\frac{a}{b^2(ax+b)}\right]\,\mathrm{d}x \\
&= -\frac{1}{b^2}\ln|x|+\frac{1}{b(ax+b)}+\frac{1}{b^2}\ln|ax+b|+C \\
&= \frac{1}{b\,(ax+b)}-\frac{1}{b^2}\ln\left|\frac{ax+b}{x}\right|+C
\end{aligned}
$$

■

2.2 含有 $\sqrt{ax+b}$ 的积分

$$\int \sqrt{ax+b}\,\mathrm{d}x = \frac{2}{3a}\sqrt{(ax+b)^3} + C \tag{2.10}$$

Proof.

$$\begin{aligned}
&\text{let } u = \sqrt{ax+b} \text{ ,then } x = \frac{u^2-b}{a} \\
\int \sqrt{ax+b}\,\mathrm{d}x &= \int u\,\mathrm{d}\left(\frac{u^2-b}{a}\right) \\
&= \int \frac{2u^2}{a}\,\mathrm{d}u \\
&= \frac{2u^3}{3a} + C \\
&= \frac{2}{3a}\sqrt{(ax+b)^3} + C
\end{aligned}$$

■

$$\int x\sqrt{ax+b}\,\mathrm{d}x = \frac{2}{15a^2}\left(3ax-2b\right)\sqrt{(ax+b)^3} + C \tag{2.11}$$

Proof.

$$\begin{aligned}
&\text{let } u = \sqrt{ax+b} \text{ ,then } x = \frac{u^2-b}{a} \\
\int x\sqrt{ax+b}\,\mathrm{d}x &= \int \frac{u^2-b}{a} u\,\mathrm{d}\left(\frac{u^2-b}{a}\right) \\
&= \int \frac{2u^4-2bu^2}{a^2}\,\mathrm{d}u \\
&= \frac{2u^5}{5a^2} - \frac{2bu^3}{3a^2} + C \\
&= \frac{6(ax+b)-10b}{15a^2}\sqrt{(ax+b)^3} + C \\
&= \frac{2}{15a^2}\left(3ax-2b\right)\sqrt{(ax+b)^3} + C
\end{aligned}$$

■

$$\int x^2\sqrt{ax+b}\,\mathrm{d}x = \frac{2}{105a^3}\left(15a^2x^2-12abx+8b^2\right)\sqrt{(ax+b)^3} + C \tag{2.12}$$

Proof.

$$
\begin{aligned}
&\text{let } u=\sqrt{ax+b}\ \text{,then } x=\frac{u^2-b}{a} \\
\int x^2\sqrt{ax+b}\,\mathrm{d}x &= \int\left(\frac{u^2-b}{a}\right)^2 u\,\mathrm{d}\left(\frac{u^2-b}{a}\right) \\
&= \int\frac{2u^6-4bu^4+2b^2u^2}{a^3}\,\mathrm{d}u \\
&= \frac{2u^7}{7a^3}-\frac{4bu^5}{5a^3}+\frac{2b^2u^3}{3a^3}+C \\
&= \frac{30(ax+b)^2-84b(ax+b)+70b^2}{105a^2}\sqrt{(ax+b)^3}+C \\
&= \frac{2}{105a^3}\left(15a^2x^2-12abx+8b^2\right)\sqrt{(ax+b)^3}+C
\end{aligned}
$$

■

$$
\int\frac{x}{\sqrt{ax+b}}\,\mathrm{d}x=\frac{2}{3a^2}\,(ax-2b)\sqrt{ax+b}+C \tag{2.13}
$$

Proof.

$$
\begin{aligned}
&\text{let } u=\sqrt{ax+b}\ \text{,then } x=\frac{u^2-b}{a} \\
\int\frac{x}{\sqrt{ax+b}}\,\mathrm{d}x &= \int\frac{\frac{u^2-b}{a}}{u}\,\mathrm{d}\left(\frac{u^2-b}{a}\right) \\
&= \int\frac{2(u^2-b)}{a^2}\,\mathrm{d}u \\
&= \frac{2}{a^3}\left(\frac{u^3}{3}-bu\right)+C \\
&= \frac{2\left[(ax+b)-3b\right]}{3a^2}\sqrt{ax+b}+C \\
&= \frac{2}{3a^2}\,(ax-2b)\sqrt{ax+b}+C
\end{aligned}
$$

■

$$
\int\frac{x^2}{\sqrt{ax+b}}\,\mathrm{d}x=\frac{2}{15a^2}\left(3a^2x^2-4abx+8b^2\right)\sqrt{ax+b}+C \tag{2.14}
$$

Proof.

$$\text{let } u=\sqrt{ax+b}\text{ ,then } x=\frac{u^2-b}{a}$$

$$\begin{aligned}
\int\frac{x^2}{\sqrt{ax+b}}\,\mathrm{d}x &= \int\frac{\left(\frac{u^2-b}{a}\right)^2}{u}\,\mathrm{d}\left(\frac{u^2-b}{a}\right)\\
&= \int\frac{2\left(u^4-2bu^2+b^2\right)}{a^2}\,\mathrm{d}u\\
&= \frac{2}{a^2}\left(\frac{u^5}{5}-\frac{2bu^3}{3}+\frac{b^2u}{1}\right)+C\\
&= \frac{2\left[3(ax+b)^2-10b(ax+b)+15b^2\right]}{15a^2}\sqrt{ax+b}+C\\
&= \frac{2}{15a^2}\left(3a^2x^2-4abx+8b^2\right)\sqrt{ax+b}+C
\end{aligned}$$

■

$$\int\frac{\mathrm{d}x}{x\sqrt{ax+b}}=\begin{cases}\dfrac{1}{\sqrt{b}}\ln\left|\dfrac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right|+C & \text{if } (b>0)\\[2ex] \dfrac{2}{\sqrt{-b}}\arctan\sqrt{\dfrac{ax+b}{-b}}+C & \text{if } (b<0)\end{cases}\tag{2.15}$$

Proof.

$$\text{let } u=\sqrt{ax+b}\text{ ,then } x=\frac{u^2-b}{a}$$

$$\begin{aligned}
\int\frac{\mathrm{d}x}{x\sqrt{ax+b}} &= \int\frac{1}{\frac{u^2-b}{a}u}\,\mathrm{d}\left(\frac{u^2-b}{a}\right)\\
&= \int\frac{2}{u^2-b}\,\mathrm{d}u
\end{aligned}$$

$$\text{if } b>0:$$

$$\begin{aligned}
\int\frac{2}{u^2-b}\,\mathrm{d}u &= \frac{1}{\sqrt{b}}\int\left[\frac{1}{u-\sqrt{b}}-\frac{1}{u+\sqrt{b}}\right]\,\mathrm{d}u\\
&= \frac{1}{\sqrt{b}}\left(\ln\left|u-\sqrt{b}\right|-\ln\left|u+\sqrt{b}\right|\right)+C\\
&= \frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right|+C
\end{aligned}$$

$$\text{if } b<0:$$

$$\text{let } u=\sqrt{-b}\tan v\text{ ,then } v=\arctan\frac{u}{\sqrt{-b}}$$

$$\begin{aligned}
\int\frac{2}{u^2-b}\,\mathrm{d}u &= \int\frac{2}{\left(\sqrt{-b}\right)^2\tan^2 v+\left(\sqrt{-b}\right)^2}\,\mathrm{d}\left(\sqrt{-b}\tan v\right)\\
&= \int\frac{2\sec^2 v}{\sqrt{-b}\sec^2 v}\,\mathrm{d}v\\
&= \frac{2v}{\sqrt{-b}}+C\\
&= \frac{2\arctan\frac{u}{\sqrt{-b}}}{\sqrt{-b}}+C
\end{aligned}$$

$$= \frac{2}{\sqrt{-b}} \arctan \sqrt{\frac{ax+b}{-b}} + C$$

■

$$\int \frac{\mathrm{d}x}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{\mathrm{d}x}{x\sqrt{ax+b}} \tag{2.16}$$

Proof.

$$\text{let } u = \sqrt{ax+b} \text{ ,then } x = \frac{u^2-b}{a}$$

$$\int \frac{\mathrm{d}x}{x^2\sqrt{ax+b}} = \int \frac{1}{\left(\frac{u^2-b}{a}\right)^2 u} \mathrm{d}(\frac{u^2-b}{a})$$

$$= \int \frac{2a}{(u^2-b)^2} \mathrm{d}u$$

$$\text{if } b > 0:$$

$$\int \frac{2a}{(u^2-b)^2} \mathrm{d}u = \int \left[\frac{A}{\left(u+\sqrt{b}\right)^2} + \frac{B}{\left(u-\sqrt{b}\right)^2} + \frac{C}{\left(u+\sqrt{b}\right)} + \frac{D}{\left(u-\sqrt{b}\right)}\right] \mathrm{d}u$$

$$= \int \left[\frac{a}{2b\left(u+\sqrt{b}\right)^2} + \frac{a}{2b\left(u-\sqrt{b}\right)^2} + \frac{a}{2b\sqrt{b}\left(u+\sqrt{b}\right)} - \frac{a}{2b\sqrt{b}\left(u-\sqrt{b}\right)}\right] \mathrm{d}u$$

$$= -\frac{a}{2b\left(u+\sqrt{b}\right)} - \frac{a}{2b\left(u-\sqrt{b}\right)} - \int \frac{a}{b\,(u^2-b)} \mathrm{d}u$$

$$= -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{\mathrm{d}x}{x\sqrt{ax+b}}$$

$$\text{if } b < 0:$$

$$\text{let } u = \sqrt{-b}\tan v \text{ ,then } v = \arctan\frac{u}{\sqrt{-b}}$$

$$= \int \frac{2a}{\left(\sqrt{-b}^2 + \tan^2 v - b\right)} \mathrm{d}(\sqrt{-b}\tan v)$$

$$= \frac{2a}{\sqrt{-b}^3}\int \cos^2 v \,\mathrm{d}v$$

$$\textbf{use } \int \cos^2 x \,\mathrm{d}x = \frac{x}{2} + \frac{1}{4}\sin 2x + C$$

$$= \frac{av}{\sqrt{-b}^3} + \frac{a}{2\sqrt{-b}^3}\sin 2v + C$$

$$= \frac{a}{\sqrt{-b}^3}\arctan\frac{u}{\sqrt{-b}} + \frac{a}{\sqrt{-b}^3}\sin\arctan\frac{u}{\sqrt{-b}}\cos\arctan\frac{u}{\sqrt{-b}} + C$$

$$\textbf{use } \arctan\frac{x}{a} = \arcsin\sqrt{\frac{x^2}{x^2+a^2}}$$

$$\textbf{use } \arctan\frac{x}{a} = \arccos\sqrt{\frac{a^2}{x^2+a^2}}$$

$$= -\frac{a}{2b}\frac{2}{\sqrt{-b}}\arctan\sqrt{\frac{ax+b}{-b}} - \frac{a}{b\sqrt{-b}}\sqrt{\frac{u^2}{u^2-b}}\sqrt{\frac{\sqrt{-b}^2}{u^2-b}} + C$$

$$
\begin{aligned}
&= -\frac{a}{2b}\int\frac{\mathrm{d}x}{x\sqrt{ax+b}} - \frac{a}{b}\frac{u}{u^2-b} + C \\
&= -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int\frac{\mathrm{d}x}{x\sqrt{ax+b}} + C
\end{aligned}
$$

■

$$
\int\frac{\sqrt{ax+b}}{x}\,\mathrm{d}x = 2\sqrt{ax+b} + b\int\frac{\mathrm{d}x}{x\sqrt{ax+b}} \tag{2.17}
$$

Proof.

$$
\begin{aligned}
I = \int\frac{\sqrt{ax+b}}{x}\,\mathrm{d}x = x\frac{\sqrt{ax+b}}{x} - \int x\left(\frac{\sqrt{ax+b}}{x}\right)'\,\mathrm{d}x \\
&= \sqrt{ax+b} + \int\frac{ax+2b}{2x\sqrt{ax+b}}\,\mathrm{d}x \\
&= \sqrt{ax+b} + \frac{1}{2}\int\frac{ax+b+b}{x\sqrt{ax+b}}\,\mathrm{d}x \\
&= \sqrt{ax+b} + \frac{1}{2}\int\frac{\sqrt{ax+b}}{x}\,\mathrm{d}x + \frac{1}{2}\int\frac{b}{x\sqrt{ax+b}}\,\mathrm{d}x \\
I &= \sqrt{ax+b} + \frac{1}{2}I + \frac{1}{2}\int\frac{b}{x\sqrt{ax+b}}\,\mathrm{d}x \\
I &= 2\sqrt{ax+b} + b\int\frac{\mathrm{d}x}{x\sqrt{ax+b}}
\end{aligned}
$$

■

$$
\int\frac{\sqrt{ax+b}}{x^2}\,\mathrm{d}x = -\frac{\sqrt{ax+b}}{x} + \frac{a}{2}\int\frac{\mathrm{d}x}{x\sqrt{ax+b}} \tag{2.18}
$$

Proof.

$$
\begin{aligned}
\int\frac{\sqrt{ax+b}}{x^2}\,\mathrm{d}x &= -\frac{\sqrt{ax+b}}{x} + \int\frac{\left(\sqrt{ax+b}\right)'}{x}\,\mathrm{d}x \\
&= -\frac{\sqrt{ax+b}}{x} + \frac{a}{2}\int\frac{\mathrm{d}x}{x\sqrt{ax+b}}
\end{aligned}
$$

■

2.3 含有 $x^2 \pm a^2$ 的积分

$$
\int\frac{\mathrm{d}x}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C \tag{2.19}
$$

Proof.

$$
\text{let } x = a\tan u
$$

$$
\begin{aligned}
\int\frac{\mathrm{d}x}{x^2+a^2} &= \int\frac{\mathrm{d}(a\tan u)}{a^2\left(\tan^2 u+1\right)} \\
&= \frac{1}{a}\int\,\mathrm{d}u \\
&= \frac{1}{a}\arctan\frac{x}{a} + C
\end{aligned}
$$

∎

$$\int \frac{\mathrm{d}x}{\left(x^2+a^2\right)^n} = \frac{x}{2\left(n-1\right)a^2\left(x^2+a^2\right)^{n-1}} + \frac{2n-3}{2\left(n-1\right)a^2}\int \frac{\mathrm{d}x}{\left(x^2+a^2\right)^{n-1}} \tag{2.20}$$

Proof.

$$\begin{aligned}
&\text{let } x = a\tan u \\
I_n &= \int \frac{\mathrm{d}x}{\left(x^2+a^2\right)^n} \\
I_{n-1} &= \int \frac{\mathrm{d}x}{\left(x^2+a^2\right)^{n-1}} \\
&= \frac{x}{\left(x^2+a^2\right)^{n-1}} - \int \frac{2x^2(1-n)}{\left(x^2+a^2\right)^n}\,\mathrm{d}x \\
&= \frac{x}{\left(x^2+a^2\right)^{n-1}} + 2(n-1)\int \left[\frac{x^2+a^2}{\left(x^2+a^2\right)^n} - \frac{a^2}{\left(x^2+a^2\right)^n}\right]\,\mathrm{d}x \\
&= \frac{x}{\left(x^2+a^2\right)^{n-1}} + 2(n-1)\left[I_{n-1} - a^2 I_n\right] \\
I_n &= \frac{x}{2\left(n-1\right)a^2\left(x^2+a^2\right)^{n-1}} + \frac{2n-3}{2\left(n-1\right)a^2}\int \frac{\mathrm{d}x}{\left(x^2+a^2\right)^{n-1}}
\end{aligned}$$

∎

$$\int \frac{\mathrm{d}x}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C \tag{2.21}$$

Proof.

$$\begin{aligned}
\int \frac{\mathrm{d}x}{x^2-a^2} &= \frac{1}{2a}\int \left[\frac{1}{x-a} - \frac{1}{x+a}\right]\,\mathrm{d}x \\
&= \frac{1}{2a}\left[\ln|x-a| - \ln|x+a|\right] + C \\
&= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C
\end{aligned}$$

∎

2.4 含有 $ax^2+b\ (a>0)$ 的积分

$$\int \frac{\mathrm{d}x}{ax^2+b} = \begin{cases} \dfrac{1}{\sqrt{ab}}\arctan\sqrt{\dfrac{a}{b}}x + C & \text{if } (b>0) \\ \dfrac{1}{2\sqrt{-ab}}\ln\left|\dfrac{\sqrt{a}x-\sqrt{-b}}{\sqrt{a}x+\sqrt{-b}}\right| + C & \text{if } (b<0) \end{cases} \tag{2.22}$$

Proof.

$$
\begin{aligned}
&\text{if } (b>0): \\
&\qquad \text{let } x=\frac{\sqrt{b}}{\sqrt{a}}\tan u \text{ ,then } u=\arctan\left(\frac{\sqrt{a}}{\sqrt{b}}x\right) \\
&\int\frac{\mathrm{d}x}{ax^2+b}=\int\frac{\mathrm{d}\left(\frac{\sqrt{b}}{\sqrt{a}}\tan u\right)}{b\left(\tan^2 u+1\right)} \\
&\qquad\qquad=\int\frac{\mathrm{d}u}{\sqrt{ab}} \\
&\qquad\qquad=\frac{1}{\sqrt{ab}}\arctan\sqrt{\frac{a}{b}}x+C \\
&\text{if } (b<0): \\
&\int\frac{\mathrm{d}x}{ax^2+b}=\int\frac{\mathrm{d}x}{\sqrt{a}^2x^2-\sqrt{-b}^2} \\
&\qquad\qquad=\frac{1}{2\sqrt{-b}}\int\left[\frac{1}{\sqrt{a}x-\sqrt{-b}}-\frac{1}{\sqrt{a}x+\sqrt{-b}}\right]\mathrm{d}x \\
&\qquad\qquad=\frac{1}{2\sqrt{-ab}}\left[\ln\left|\sqrt{a}x-\sqrt{-b}\right|-\ln\left|\sqrt{a}x+\sqrt{-b}\right|\right]+C \\
&\qquad\qquad=\frac{1}{2\sqrt{-ab}}\ln\left|\frac{\sqrt{a}x-\sqrt{-b}}{\sqrt{a}x+\sqrt{-b}}\right|+C
\end{aligned}
$$

■

$$\int\frac{x}{ax^2+b}\,\mathrm{d}x=\frac{1}{2a}\ln\left|ax^2+b\right|+C \tag{2.23}$$

Proof.

$$
\begin{aligned}
\int\frac{x}{ax^2+b}\,\mathrm{d}x&=\frac{1}{2a}\int\frac{\mathrm{d}(ax^2+b)}{ax^2+b} \\
&=\frac{1}{2a}\ln\left|ax^2+b\right|+C
\end{aligned}
$$

■

$$\int\frac{x^2}{ax^2+b}\,\mathrm{d}x=\frac{x}{a}-\frac{b}{a}\int\frac{\mathrm{d}x}{ax^2+b} \tag{2.24}$$

Proof.

$$
\begin{aligned}
\int\frac{x^2}{ax^2+b}\,\mathrm{d}x&=\int\left[\frac{x^2+\frac{b}{a}}{ax^2+b}-\frac{b}{a\left(ax^2+b\right)}\right]\mathrm{d}x \\
&=\frac{x}{a}-\frac{b}{a}\int\frac{\mathrm{d}x}{ax^2+b}
\end{aligned}
$$

■

$$\int\frac{\mathrm{d}x}{x\left(ax^2+b\right)}=\frac{1}{2b}\ln\frac{x^2}{|ax^2+b|}+C \tag{2.25}$$

Proof.

$$\begin{aligned}
\int \frac{\mathrm{d}x}{x\left(ax^2+b\right)} &= \frac{1}{2a}\int \frac{\mathrm{d}x^2}{x^2\left(x^2+\frac{b}{a}\right)} \\
&= \frac{1}{2b}\int \left[\frac{1}{x^2}-\frac{1}{x^2+\frac{b}{a}}\right]\,\mathrm{d}x^2 \\
&= \frac{1}{2b}\left[\ln x^2-\ln\left|x^2+\frac{b}{a}\right|\right] \\
&= \frac{1}{2b}\ln\frac{x^2}{|ax^2+b|}+C
\end{aligned}$$

■

$$\int \frac{\mathrm{d}x}{x^2\left(ax^2+b\right)} = -\frac{1}{bx}-\frac{a}{b}\int \frac{\mathrm{d}x}{ax^2+b} \tag{2.26}$$

Proof.

$$\begin{aligned}
\int \frac{\mathrm{d}x}{x^2\left(ax^2+b\right)} &= \int \left[\frac{1}{bx^2}-\frac{a}{b\left(ax^2+b\right)}\right]\,\mathrm{d}x \\
&= -\frac{1}{bx}-\frac{a}{b}\int \frac{\mathrm{d}x}{ax^2+b}
\end{aligned}$$

■

$$\int \frac{\mathrm{d}x}{x^3\left(ax^2+b\right)} = \frac{a}{2b^2}\ln\frac{|ax^2+b|}{x^2}-\frac{1}{2bx^2}+C \tag{2.27}$$

Proof.

$$\begin{aligned}
\int \frac{\mathrm{d}x}{x^3\left(ax^2+b\right)} &= \frac{1}{2a}\int \frac{\mathrm{d}x^2}{x^4\left(x^2+\frac{b}{a}\right)} \\
&= \frac{1}{2a}\int \left[\frac{a}{bx^4}-\frac{a^2}{b^2x^2}+\frac{\frac{a^2}{b^2}}{x^2+\frac{b}{a}}\right]\,\mathrm{d}x^2 \\
&= \frac{1}{2a}\left[-\frac{a}{bx^2}-\frac{a^2}{b^2}\ln x^2+\frac{a^2}{b^2}\ln\left|x^2+\frac{b}{a}\right|\right]+C \\
&= \frac{a}{2b^2}\ln\frac{|ax^2+b|}{x^2}-\frac{1}{2bx^2}+C
\end{aligned}$$

■

$$\int \frac{\mathrm{d}x}{\left(ax^2+b\right)^2} = \frac{x}{2b\left(ax^2+b\right)}+\frac{1}{2b}\int \frac{\mathrm{d}x}{ax^2+b} \tag{2.28}$$

Proof.

$$
\begin{aligned}
\int \frac{\mathrm{d}x}{\left(ax^2+b\right)^2} &= \int \frac{1}{2ax}\frac{2ax}{\left(ax^2+b\right)^2}\,\mathrm{d}x \\
&= -\frac{1}{2ax}\frac{1}{ax^2+b} - \int \frac{1}{2ax^2}\frac{1}{ax^2+b}\,\mathrm{d}x \\
\text{if } b>0 \\
&= -\frac{1}{2ax}\frac{1}{ax^2+b} - \frac{1}{2a}\int \left[\frac{Ax+B}{ax^2+b} + \frac{C}{x^2} + \frac{D}{x}\right]\mathrm{d}x \\
&= -\frac{1}{2ax}\frac{1}{ax^2+b} - \frac{1}{2a}\int \left[\frac{-\frac{a}{b}}{ax^2+b} + \frac{\frac{1}{b}}{x^2}\right]\mathrm{d}x \\
&= -\frac{1}{2ax}\frac{1}{ax^2+b} + \frac{1}{2b}\int \frac{1}{ax^2+b}\,\mathrm{d}x - \frac{1}{2ab}\int \frac{1}{x^2}\,\mathrm{d}x \\
&= \frac{1}{2abx} - \frac{1}{2ax}\frac{1}{ax^2+b} + \frac{1}{2b}\int \frac{1}{ax^2+b}\,\mathrm{d}x \\
&= \frac{x}{2b\left(ax^2+b\right)} + \frac{1}{2b}\int \frac{\mathrm{d}x}{ax^2+b} \\
\text{if } b<0 \\
&= -\frac{1}{2ax}\frac{1}{ax^2+b} - \frac{1}{2a}\int \left[\frac{A}{\sqrt{a}x-\sqrt{-b}} + \frac{B}{\sqrt{a}x+\sqrt{-b}} + \frac{C}{x^2} + \frac{D}{x}\right]\mathrm{d}x \\
&= -\frac{1}{2ax}\frac{1}{ax^2+b} - \frac{1}{2a}\int \left[\frac{-\frac{a}{2b\sqrt{-b}}}{\sqrt{a}x-\sqrt{-b}} + \frac{\frac{a}{2b\sqrt{-b}}}{\sqrt{a}x+\sqrt{-b}} + \frac{\frac{1}{b}}{x^2}\right]\mathrm{d}x \\
&= -\frac{1}{2ax}\frac{1}{ax^2+b} + \frac{1}{2abx} \\
&+ \frac{1}{4b\sqrt{-ab}}\left[\ln\left|\sqrt{a}x-\sqrt{-b}x\right| - \ln\left|\sqrt{a}x+\sqrt{-b}x\right|\right] + C \\
&= \frac{x}{2b\left(ax^2+b\right)} + \frac{1}{2b}\frac{1}{2\sqrt{-ab}}\ln\left|\frac{\sqrt{a}x-\sqrt{-b}}{\sqrt{a}x+\sqrt{-b}}\right| + C \\
&= \frac{x}{2b\left(ax^2+b\right)} + \frac{1}{2b}\int \frac{\mathrm{d}x}{ax^2+b}
\end{aligned}
$$

■

2.5 含有 $ax^2+bx+c\,(a>0)$ 的积分

$$
\int \frac{\mathrm{d}x}{ax^2+bx+c} = \begin{cases} \dfrac{2}{\sqrt{4ac-b^2}}\arctan\dfrac{2ax+b}{\sqrt{4ac-b^2}} + C & \text{if } \left(b^2<4ac\right) \\ \dfrac{1}{\sqrt{b^2-4ac}}\ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| + C & \text{if } \left(b^2>4ac\right) \end{cases} \tag{2.29}
$$

Proof.

$$
\begin{aligned}
&\text{if } \left(b^2<4ac\right) \\
\int \frac{\mathrm{d}x}{ax^2+bx+c} &= \int \frac{1}{\frac{4ac-b^2}{4a} + \left(\frac{b}{2\sqrt{a}}+\sqrt{a}x\right)^2}\,\mathrm{d}x \\
\text{let } u &= \frac{b}{2\sqrt{a}} + \sqrt{a}x
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{a}} \int \frac{1}{\frac{4ac-b^2}{4a} + u^2} \,\mathrm{d}u \\
&= \frac{1}{\sqrt{a}} \int \frac{4a}{(4ac-b^2)\left(\frac{4au^2}{4ac-b^2}+1\right)} \,\mathrm{d}u \\
&= \frac{4\sqrt{a}}{4ac-b^2} \int \frac{1}{\frac{4au^2}{4ac-b^2}+1} \,\mathrm{d}u \\
\text{let } v &= \frac{2\sqrt{a}u}{\sqrt{4ac-b^2}} \\
&= \frac{2}{\sqrt{4ac-b^2}} \int \frac{1}{1+v^2} \,\mathrm{d}v \\
&= \frac{2}{\sqrt{4ac-b^2}} \arctan v + C \\
&= \frac{2}{\sqrt{4ac-b^2}} \arctan \frac{2ax+b}{\sqrt{4ac-b^2}} + C
\end{aligned}
$$

$$
\text{if } \left(b^2 > 4ac\right)
$$

$$
\begin{aligned}
\int \frac{\mathrm{d}x}{ax^2+bx+c} &= \int \frac{1}{a\left(x - \frac{-b-\sqrt{b^2-4ac}}{2a}\right)\left(x - \frac{-b+\sqrt{b^2-4ac}}{2a}\right)} \,\mathrm{d}x \\
&= \frac{1}{a} \int \left[\frac{A}{x+\frac{b+\sqrt{b^2-4ac}}{2a}} + \frac{B}{x+\frac{b-\sqrt{b^2-4ac}}{2a}}\right] \mathrm{d}x \\
&= \frac{1}{a} \int \left[\frac{-\frac{a}{\sqrt{b^2-4ac}}}{x+\frac{b+\sqrt{b^2-4ac}}{2a}} + \frac{\frac{a}{\sqrt{b^2-4ac}}}{x+\frac{b-\sqrt{b^2-4ac}}{2a}}\right] \mathrm{d}x \\
&= \frac{1}{\sqrt{b^2-4ac}} \left[\ln\left|x + \frac{b-\sqrt{b^2-4ac}}{2a}\right| - \ln\left|x + \frac{b+\sqrt{b^2-4ac}}{2a}\right|\right] + C \\
&= \frac{1}{\sqrt{b^2-4ac}} \ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| + C
\end{aligned}
$$

■

$$
\int \frac{x}{ax^2+bx+c} \,\mathrm{d}x = \frac{1}{2a} \ln\left|ax^2+bx+c\right| - \frac{b}{2a} \int \frac{\mathrm{d}x}{ax^2+bx+c} \tag{2.30}
$$

***Proof*.**

$$
\begin{aligned}
\int \frac{x}{ax^2+bx+c} \,\mathrm{d}x &= \frac{1}{2a} \int \frac{2ax+b-b}{ax^2+bx+c} \,\mathrm{d}x \\
&= \frac{1}{2a} \int \left[\frac{2ax+b}{ax^2+bx+c} - \frac{b}{ax^2+bx+c}\right] \mathrm{d}x \\
&= \frac{1}{2a} \int \frac{\mathrm{d}\left(ax^2+bx+c\right)}{ax^2+bx+c} - \frac{b}{2a} \int \frac{\mathrm{d}x}{ax^2+bx+c} \\
&= \frac{1}{2a} \ln\left|ax^2+bx+c\right| - \frac{b}{2a} \int \frac{\mathrm{d}x}{ax^2+bx+c}
\end{aligned}
$$

■

2.6 含有 $\sqrt{x^2+a^2}\,(a>0)$ 的积分

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \operatorname{arcsinh}\frac{x}{a} + C_1 = \ln\left(x+\sqrt{x^2+a^2}\right) + C \tag{2.31}$$

Proof.

$$\begin{aligned}
&\text{let } x = a\tan u \\
\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} &= \int \frac{1}{a\sqrt{\tan^2 u+1}}\,\mathrm{d}(a\tan u) \\
&= \int \sec u\,\mathrm{d}u \\
\textbf{use } \int \sec x\,\mathrm{d}x &= \ln|\sec x+\tan x| + C \\
&= \ln|\sec u+\tan u| + C \\
&= \ln\left|\sec\arctan\frac{x}{a}+\tan\arctan\frac{x}{a}\right| + C \\
\textbf{use } \arctan\frac{x}{a} &= \operatorname{arcsec}\sqrt{\frac{x^2}{a^2}+1} \\
&= \ln\left|\sec\operatorname{arcsec}\sqrt{\frac{x^2}{a^2}+1}+\frac{x}{a}\right| + C \\
&= \ln\left|\sqrt{\frac{x^2}{a^2}+1}+\frac{x}{a}\right| + C \\
&= \ln\left(x+\sqrt{x^2+a^2}\right) + C
\end{aligned}$$

■

another way

Proof.

$$\begin{aligned}
&\text{let } x = a\sinh u \\
\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} &= \frac{1}{a}\int \frac{\mathrm{d}(a\sinh u)}{\cosh u} \\
&= \int \frac{\cosh u}{\cosh u}\,\mathrm{d}u \\
&= u + C = \operatorname{arcsinh}\frac{x}{a} + C \\
&= \ln\left(x+\sqrt{x^2+a^2}\right) + C
\end{aligned}$$

■

$$\int \frac{\mathrm{d}x}{\sqrt{(x^2+a^2)^3}} = \frac{x}{a^2\sqrt{x^2+a^2}} + C \tag{2.32}$$

Proof.

$$
\begin{aligned}
&\text{let } x = a\tan u \\
\int \frac{\mathrm{d}x}{\sqrt{\left(x^2+a^2\right)^3}} &= \int \frac{\mathrm{d}(a\tan u)}{\left(a^2+a^2\tan^2 u\right)^{3/2}} \\
&= \frac{1}{a^2}\int \cos u \,\mathrm{d}u = \frac{\sin u}{a^2} + C \\
&= \frac{\sin\arctan\frac{x}{a}}{a^2} + C \\
\textbf{use } \arctan\frac{x}{a} &= \arcsin\sqrt{\frac{x^2}{x^2+a^2}} \\
&= \frac{\sin\arcsin\sqrt{\frac{x^2}{x^2+a^2}}}{a^2} + C \\
&= \frac{x}{a^2\sqrt{x^2+a^2}} + C
\end{aligned}
$$

■

$$\int \frac{x}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \sqrt{x^2+a^2} + C \tag{2.33}$$

Proof.

$$
\begin{aligned}
\int \frac{x}{\sqrt{x^2+a^2}}\,\mathrm{d}x &= \frac{1}{2}\int \frac{\mathrm{d}\left(x^2+a^2\right)}{\sqrt{x^2+a^2}} \\
&= \sqrt{x^2+a^2} + C
\end{aligned}
$$

■

$$\int \frac{x}{\sqrt{\left(x^2+a^2\right)^3}}\,\mathrm{d}x = -\frac{1}{\sqrt{x^2+a^2}} + C \tag{2.34}$$

Proof.

$$
\begin{aligned}
\int \frac{x}{\sqrt{\left(x^2+a^2\right)^3}}\,\mathrm{d}x &= \frac{1}{2}\int \frac{\mathrm{d}\left(x^2+a^2\right)}{\sqrt{\left(x^2+a^2\right)^3}} \\
&= -\frac{1}{\sqrt{x^2+a^2}} + C
\end{aligned}
$$

■

$$\int \frac{x^2}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \frac{x}{2}\sqrt{x^2+a^2} - \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right) + C \tag{2.35}$$

Proof.

$$
\begin{aligned}
&\text{let } x = a\tan u \\
\int \frac{x^2}{\sqrt{x^2+a^2}}\,\mathrm{d}x &= \int \frac{a^2\tan^2 u}{\sqrt{a^2+a^2\tan^2 u}}\,\mathrm{d}(a\tan u) \\
&= a^2 \int \tan^2 u \sec u\,\mathrm{d}u \\
&= a^2 \int \sec u\left(\sec^2 u - 1\right)\,\mathrm{d}u \\
&= a^2 \int \sec^3 u\,\mathrm{d}u - a^2 \int \sec u\,\mathrm{d}u \\
\textbf{use } \int \frac{\mathrm{d}x}{\cos^n x} &= \frac{1}{n-1}\frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1}\int \frac{\mathrm{d}x}{\cos^{n-2} x} \\
&= \frac{1}{2}a^2 \tan u \sec u + \frac{1}{2}a^2 \int \sec u\,\mathrm{d}u - a^2 \int \sec u\,\mathrm{d}u \\
\textbf{use } \int \sec x\,\mathrm{d}x &= \ln|\sec x + \tan x| + C \\
&= \frac{1}{2}a^2 \tan u \sec u - \frac{1}{2}a^2 \ln|\tan u + \sec u| + C \\
&= \frac{1}{2}a^2 \tan\arctan\frac{x}{a}\sec\arctan\frac{x}{a} - \frac{1}{2}a^2 \ln\left|\tan\arctan\frac{x}{a} + \sec\arctan\frac{x}{a}\right| + C \\
\textbf{use } \arctan\frac{x}{a} &= \operatorname{arcsec}\sqrt{\frac{x^2}{a^2}+1} \\
&= \frac{a^2}{2}\frac{x}{a}\sqrt{\frac{x^2}{a^2}+1} - \frac{a^2}{2}\ln\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2}+1}\frac{x}{a}\right| + C \\
&= \frac{x}{2}\sqrt{x^2+a^2} - \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right) + C
\end{aligned}
$$

■

$$
\int \frac{x^2}{\sqrt{\left(x^2+a^2\right)^3}}\,\mathrm{d}x = -\frac{x}{\sqrt{x^2+a^2}} + \ln\left(x+\sqrt{x^2+a^2}\right) + C \tag{2.36}
$$

Proof.

$$
\begin{aligned}
&\text{let } x = a\tan u \\
\int \frac{x^2}{\sqrt{\left(x^2+a^2\right)^3}}\,\mathrm{d}x &= \int \frac{a^2\tan^2 u}{\sqrt{\left(a^2\tan^2 u + a^2\right)^3}}\,\mathrm{d}(a\tan u) \\
&= \int \frac{\sin^2 u}{\cos u}\,\mathrm{d}u \\
&= \int \frac{1-\cos^2 u}{\cos u}\,\mathrm{d}u \\
&= \int (\sec u - \cos u)\,\mathrm{d}u \\
\textbf{use } \int \sec x\,\mathrm{d}x &= \ln|\sec x + \tan x| + C \\
&= \ln|\tan u + \sec u| - \sin u + C
\end{aligned}
$$

$$
\begin{aligned}
&= \ln\left|\tan\arctan\frac{x}{a}+\sec\arctan\frac{x}{a}\right| - \sin\arctan\frac{x}{a} + C \\
\textbf{use}\ \ \arctan\frac{x}{a} &=\ \operatorname{arcsec}\sqrt{\frac{x^2}{a^2}+1} \\
\textbf{use}\ \ \arctan\frac{x}{a} &= \arcsin\sqrt{\frac{x^2}{x^2+a^2}} \\
&= -\frac{x}{\sqrt{x^2+a^2}} + \ln\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}+1}\right| + C \\
&= -\frac{x}{\sqrt{x^2+a^2}} + \ln\left(x+\sqrt{x^2+a^2}\right) + C
\end{aligned}
$$

■

$$
\int\frac{\mathrm{d}x}{x\sqrt{x^2+a^2}} = \frac{1}{a}\ln\frac{\sqrt{x^2+a^2}-a}{|x|} + C \tag{2.37}
$$

***Proof*.**

$$
\begin{aligned}
\text{let } x &= a\tan u \\
\int\frac{\mathrm{d}x}{x\sqrt{x^2+a^2}} &= \int\frac{\mathrm{d}(a\tan u)}{a\tan u\sqrt{a^2\tan^2 u+a^2}} \\
&= \frac{1}{a}\int\csc u\ \mathrm{d}u \\
\textbf{use}\ \int\csc x\ \mathrm{d}x &= \ln|\csc x-\cot x| + C \\
&= \frac{1}{a}\ln|\csc u-\cot u| + C \\
&= \frac{1}{a}\ln\left|\csc\arctan\frac{x}{a}-\cot\arctan\frac{x}{a}\right| + C \\
\textbf{use}\ \ \arctan\frac{x}{a} &= \operatorname{arccsc}\sqrt{\frac{x^2+a^2}{x^2}} \\
&= \frac{1}{a}\ln\left|\csc\operatorname{arccsc}\sqrt{\frac{x^2+a^2}{x^2}} - \frac{1}{\tan\arctan\frac{x}{a}}\right| + C \\
&= \frac{1}{a}\ln\left|\frac{\sqrt{x^2+a^2}}{x}-\frac{a}{x}\right| + C \\
&= \frac{1}{a}\ln\left|\frac{\sqrt{x^2+a^2}-a}{x}\right| + C \\
&= \frac{1}{a}\ln\frac{\sqrt{x^2+a^2}-a}{|x|} + C
\end{aligned}
$$

■

$$
\int\frac{\mathrm{d}x}{x^2\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{a^2x} + C \tag{2.38}
$$

Proof.

$$
\begin{aligned}
\text{let } x &= a\tan u \\
\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+a^2}} &= \int \frac{\mathrm{d}(a\tan u)}{a^2\tan^2 u\sqrt{a^2\tan^2 u+a^2}} \\
&= \frac{1}{a^2}\int \frac{\cos u}{\sin^2 u}\,\mathrm{d}u \\
&= \frac{1}{a^2}\int \frac{\mathrm{d}\sin u}{\sin^2 u} \\
&= -\frac{1}{a^2}\csc u + C \\
\textbf{use } \arctan\frac{x}{a} &= \operatorname{arccsc}\sqrt{\frac{x^2+a^2}{x^2}} \\
&= -\frac{1}{a^2}\csc \operatorname{arccsc}\sqrt{\frac{x^2+a^2}{x^2}} + C \\
&= -\frac{\sqrt{x^2+a^2}}{a^2x} + C
\end{aligned}
$$

■

$$
\int \sqrt{x^2+a^2}\,\mathrm{d}x = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right) + C \tag{2.39}
$$

Proof.

$$
\begin{aligned}
\text{let } x &= a\tan u \\
\int \sqrt{x^2+a^2}\,\mathrm{d}x &= \int \sqrt{a^2\tan^2 u+a^2}\,\mathrm{d}(a\tan u) \\
&= a^2\int \sec^3 u\,\mathrm{d}u \\
\textbf{use } \int \frac{\mathrm{d}x}{\cos^n x} &= \frac{1}{n-1}\frac{\sin x}{\cos^{n-1}x} + \frac{n-2}{n-1}\int \frac{\mathrm{d}x}{\cos^{n-2}x} \\
&= \frac{a^2}{2}\tan u\sec u + \frac{a^2}{2}\int \sec u\,\mathrm{d}u \\
\textbf{use } \int \sec x\,\mathrm{d}x &= \ln|\sec x+\tan x| + C \\
&= \frac{a^2}{2}\tan\arctan\frac{x}{a}\sec\arctan\frac{x}{a} + \frac{a^2}{2}\ln|\sec u+\tan u| + C \\
&= \frac{a^2}{2}\tan\arctan\frac{x}{a}\sec\arctan\frac{x}{a} \\
&+ \frac{a^2}{2}\ln\left|\sec\arctan\frac{x}{a}+\tan\arctan\frac{x}{a}\right| + C \\
\textbf{use } \arctan\frac{x}{a} &= \operatorname{arcsec}\sqrt{\frac{x^2}{a^2}+1} \\
&= \frac{a^2}{2}\frac{x}{a}\sqrt{\frac{x^2}{a^2}+1} + \frac{a^2}{2}\ln\left|\sqrt{\frac{x^2}{a^2}+1}+\frac{x}{a}\right| + C \\
&= \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right) + C
\end{aligned}
$$

■

$$\int \sqrt{\left(x^2+a^2\right)^3}\,\mathrm{d}x = \frac{x}{8}\left(2x^2+5a^2\right)\sqrt{x^2+a^2} + \frac{3}{8}a^4\ln\left(x+\sqrt{x^2+a^2}\right)+C \tag{2.40}$$

Proof.

$$\begin{aligned}
&\text{let } x = a\tan u\\
\int \sqrt{\left(x^2+a^2\right)^3}\,\mathrm{d}x &= \int \sqrt{\left(a^2\tan u+a^2\right)^3}\,\mathrm{d}(a\tan u)\\
&= a^4\int \sec^5 u\,\mathrm{d}u\\
\textbf{use } \int \frac{\mathrm{d}x}{\cos^n x} &= \frac{1}{n-1}\frac{\sin x}{\cos^{n-1}x}+\frac{n-2}{n-1}\int\frac{\mathrm{d}x}{\cos^{n-2}x}\\
&= \frac{a^4}{4}\tan u\sec^3 u + \frac{3a^4}{4}\int \sec^3 u\,\mathrm{d}u\\
&= \frac{a^4}{4}\tan u\sec^3 u + \frac{3a^4}{8}\tan u\sec u + \frac{3a^4}{8}\int \sec u\,\mathrm{d}u\\
\textbf{use } \int \sec x\,\mathrm{d}x &= \ln|\sec x+\tan x|+C\\
&= \frac{a^4}{4}\tan u\sec^3 u + \frac{3a^4}{8}\tan u\sec u + \frac{3a^4}{8}\ln|\sec u+\tan u|+C\\
&= \frac{a^4}{4}\tan\arctan\frac{x}{a}\sec^3\arctan\frac{x}{a} + \frac{3a^4}{8}\tan\arctan\frac{x}{a}\sec\arctan\frac{x}{a}\\
&+\frac{3a^4}{8}\ln|\sec\arctan\frac{x}{a}+\tan\arctan\frac{x}{a}|+C\\
\textbf{use } \arctan\frac{x}{a} &= \operatorname{arcsec}\sqrt{\frac{x^2}{a^2}+1}\\
&= \frac{x}{4}\sqrt{\left(x^2+a^2\right)^3} + \frac{3a^2x}{8}\sqrt{x^2+a^2} + \frac{3a^4}{8}\ln\left|\sqrt{\frac{x^2}{a^2}+1}+\frac{x}{a}\right|+C\\
&= \frac{x}{8}\left(2x^2+5a^2\right)\sqrt{x^2+a^2} + \frac{3}{8}a^4\ln\left(x+\sqrt{x^2+a^2}\right)+C
\end{aligned}$$

■

$$\int x\sqrt{x^2+a^2}\,\mathrm{d}x = \frac{1}{3}\sqrt{\left(x^2+a^2\right)^3}+C \tag{2.41}$$

Proof.

$$\begin{aligned}
\int x\sqrt{x^2+a^2}\,\mathrm{d}x &= \frac{1}{2}\int \sqrt{x^2+a^2}\,\mathrm{d}\left(x^2+a^2\right)\\
&= \frac{1}{3}\sqrt{\left(x^2+a^2\right)^3}+C
\end{aligned}$$

■

$$\int x^2\sqrt{x^2+a^2}\,\mathrm{d}x = \frac{x}{8}\left(2x^2+a^2\right)\sqrt{x^2+a^2} - \frac{a^4}{8}\ln\left(x+\sqrt{x^2+a^2}\right)+C \tag{2.42}$$

Proof.

$$
\begin{aligned}
&\text{let } x = a\tan u \\
\int x^2\sqrt{x^2+a^2}\,\mathrm{d}x &= \int a^2\tan^2 u\sqrt{a^2\tan^2 u+a^2}\,\mathrm{d}(a\tan u) \\
&= a^4\int \tan^2 u\sec^3 u\,\mathrm{d}u \\
&= a^4\int \left(\sec^2 u-1\right)\sec^3 u\,\mathrm{d}u \\
&= a^4\int \sec^5 u\,\mathrm{d}u - a^4\int \sec^3 u\,\mathrm{d}u \\
\textbf{use } \int \frac{\mathrm{d}x}{\cos^n x} &= \frac{1}{n-1}\frac{\sin x}{\cos^{n-1}x}+\frac{n-2}{n-1}\int\frac{\mathrm{d}x}{\cos^{n-2}x} \\
&= \frac{a^4}{4}\tan u\sec^3 u - \frac{a^4}{4}\int \sec^3 u\,\mathrm{d}u \\
&= \frac{a^4}{4}\tan u\sec^3 u - \frac{a^4}{8}\tan u\sec u - \frac{a^4}{8}\int \sec u\,\mathrm{d}u \\
\textbf{use } \int \sec x\,\mathrm{d}x &= \ln|\sec x+\tan x| + C \\
&= \frac{a^4}{4}\tan u\sec^3 u - \frac{a^4}{8}\tan u\sec u - \frac{a^4}{8}\ln|\sec u+\tan u| + C \\
&= \frac{a^4}{4}\tan\arctan\frac{x}{a}\sec^3\arctan\frac{x}{a} - \frac{a^4}{8}\tan\arctan\frac{x}{a}\sec\arctan\frac{x}{a} \\
&- \frac{a^4}{8}\ln\left|\sec\arctan\frac{x}{a}+\tan\arctan\frac{x}{a}\right| + C \\
\textbf{use } \arctan\frac{x}{a} &= \operatorname{arcsec}\sqrt{\frac{x^2}{a^2}+1} \\
&= \frac{a^4}{4}\frac{x}{a}\sqrt{\frac{x^2}{a^2}+1}^{\,3} - \frac{a^4}{8}\frac{x}{a}\sqrt{\frac{x^2}{a^2}+1} - \frac{a^4}{8}\ln\left|\sqrt{\frac{x^2}{a^2}+1}+\frac{x}{a}\right| + C \\
&= \frac{x}{8}\left(2x^2+a^2\right)\sqrt{x^2+a^2} - \frac{a^4}{8}\ln\left(x+\sqrt{x^2+a^2}\right) + C
\end{aligned}
$$

■

$$
\int \frac{\sqrt{x^2+a^2}}{x}\,\mathrm{d}x = \sqrt{x^2+a^2} + a\ln\frac{\sqrt{x^2+a^2}-a}{|x|} + C \tag{2.43}
$$

Proof.

$$
\begin{aligned}
&\text{let } x = a\tan u \\
\int \frac{\sqrt{x^2+a^2}}{x}\,\mathrm{d}x &= \int \frac{\sqrt{a^2\tan^2 u+a^2}}{a\tan u}\,\mathrm{d}(a\tan u) \\
&= a\int \csc u\sec^2 u\,\mathrm{d}u \\
&= a\int \csc u\left(\tan^2 u+1\right)\,\mathrm{d}u \\
&= a\int \tan u\sec u\,\mathrm{d}u + a\int \csc u\,\mathrm{d}u
\end{aligned}
$$

$$
\begin{aligned}
&\textbf{use} \int \sec x \tan x \ \mathrm{d}x = \sec x + C \\
&\quad\textbf{use} \int \csc x \ \mathrm{d}x = \ln|\csc x - \cot x| + C \\
&= a \sec u + a \ln|\csc x - \cot x| + C \\
&= a \sec \arctan \frac{x}{a} + a \ln\left|\csc \arctan \frac{x}{a} - \cot \arctan \frac{x}{a}\right| + C \\
&\textbf{use } \arctan \frac{x}{a} = \operatorname{arcsec} \sqrt{\frac{x^2}{a^2} + 1} \\
&\textbf{use } \arctan \frac{x}{a} = \operatorname{arccsc} \sqrt{\frac{x^2 + a^2}{x^2}} \\
&= a\sqrt{\frac{x^2}{a^2} + 1} + a \ln\left|\sqrt{\frac{x^2 + a^2}{x^2}} - \frac{a}{x}\right| + C \\
&= \sqrt{x^2 + a^2} + a \ln\left|\frac{\sqrt{x^2 + a^2} - a}{x}\right| + C \\
&= \sqrt{x^2 + a^2} + a \ln \frac{\sqrt{x^2 + a^2} - a}{|x|} + C
\end{aligned}
$$

■

$$
\int \frac{\sqrt{x^2 + a^2}}{x^2} \ \mathrm{d}x = -\frac{\sqrt{x^2 + a^2}}{x} + \ln\left(x + \sqrt{x^2 + a^2}\right) + C \tag{2.44}
$$

Proof.

$$
\begin{aligned}
&\text{let } x = a \tan u \\
\int \frac{\sqrt{x^2 + a^2}}{x^2} \ \mathrm{d}x &= \int \frac{\sqrt{a^2 \tan^2 u + a^2}}{a^2 \tan^2 u} \ \mathrm{d}(a \tan u) \\
&= \int \csc^2 u \sec u \ \mathrm{d}u \\
&= \int \left(\cot^2 u + 1\right) \sec u \ \mathrm{d}u \\
&= \int \csc u \cot u \ \mathrm{d}u + \int \sec u \ \mathrm{d}u \\
&\textbf{use} \int \csc x \cot x \ \mathrm{d}x = -\csc x + C \\
&\quad\textbf{use} \int \sec x \ \mathrm{d}x = \ln|\sec x + \tan x| + C \\
&= -\csc u + \ln|\sec u + \tan u| + C \\
&= -\csc \arctan \frac{x}{a} + \ln\left|\sec \arctan \frac{x}{a} + \tan \arctan \frac{x}{a}\right| + C \\
&\textbf{use } \arctan \frac{x}{a} = \operatorname{arcsec} \sqrt{\frac{x^2 + a^2}{a^2}} \\
&\textbf{use } \arctan \frac{x}{a} = \operatorname{arccsc} \sqrt{\frac{x^2 + a^2}{x^2}} \\
&= -\sqrt{\frac{x^2 + a^2}{x^2}} + \ln\left|\sqrt{\frac{x^2}{a^2} + 1} + \frac{x}{a}\right| + C
\end{aligned}
$$

$$= -\frac{\sqrt{x^2+a^2}}{x} + \ln\left(x+\sqrt{x^2+a^2}\right) + C$$

■

2.7 含有 $\sqrt{x^2-a^2}\,(a>0)$ 的积分

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = \frac{x}{|x|}\operatorname{arccosh}\frac{|x|}{a} + C_1 = \ln\left|x+\sqrt{x^2-a^2}\right| + C \tag{2.45}$$

Proof. Domain of the function:$(-\infty,-a)\cup(a,+\infty)$

❶ $x>a:$ let $x = a\sec u\left(0<u<\frac{\pi}{2}\right)$

$$\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} &= \int \frac{\mathrm{d}(a\sec u)}{a\sqrt{\sec^2 u-1}} \\
&= \int \sec u\ \mathrm{d}u
\end{aligned}$$

use $\displaystyle\int \sec x\ \mathrm{d}x = \ln|\sec x+\tan x| + C$

$$\begin{aligned}
&= \ln(\sec u+\tan u) + C \\
&= \ln\left(\sec\ \operatorname{arcsec}\frac{x}{a} + \tan\ \operatorname{arcsec}\frac{x}{a}\right) + C
\end{aligned}$$

use $\displaystyle\operatorname{arcsec}\frac{x}{a} = \arctan\sqrt{\frac{x^2-a^2}{a^2}}$

$$\begin{aligned}
&= \ln\left(\frac{x}{a} + \tan\arctan\sqrt{\frac{x^2-a^2}{a^2}}\right) + C \\
&= \ln\left(\frac{x}{a} + \sqrt{\frac{x^2-a^2}{a^2}}\right) + C \\
&= \ln\left(x+\sqrt{x^2-a^2}\right) + C
\end{aligned}$$

❷ $x<-a:$ let $x = -v(v>0)$

$$\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} &= -\int \frac{\mathrm{d}v}{\sqrt{v^2-a^2}} \\
&= -\ln\left(v+\sqrt{v^2-a^2}\right) + C \\
&= \ln\left(\frac{1}{-x+\sqrt{x^2-a^2}}\right) + C \\
&= \ln\left[\frac{-x-\sqrt{x^2-a^2}}{\left(-x+\sqrt{x^2-a^2}\right)\left(-x-\sqrt{x^2-a^2}\right)}\right] + C \\
&= \ln\left(-x-\sqrt{x^2-a^2}\right) + C
\end{aligned}$$

after summing up:

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + C$$

■

another way

Proof. Domain of the function:$(-\infty,-a)\cup(a,+\infty)$

$$
\begin{aligned}
&\text{❶ } x>a:\ \text{let } x=a\cosh u (u>0)\\
&\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}=\frac{1}{a}\int\frac{\mathrm{d}(a\cosh u)}{\sinh u}\\
&=\int\frac{\sinh u}{\sinh u}\,\mathrm{d}u\\
&=u+C=\operatorname{arccosh}\frac{x}{a}+C\\
&=\ln\left(x+\sqrt{x^2-a^2}\right)+C
\end{aligned}
$$

$$
\begin{aligned}
&\text{❷ } x<-a:\ \text{let } x=-v(v>0)\\
&\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}=-\int\frac{\mathrm{d}v}{\sqrt{v^2-a^2}}\\
&=-\ln\left(v+\sqrt{v^2-a^2}\right)+C\\
&=\ln\left(\frac{1}{-x+\sqrt{x^2-a^2}}\right)+C\\
&=\ln\left[\frac{-x-\sqrt{x^2-a^2}}{\left(-x+\sqrt{x^2-a^2}\right)\left(-x-\sqrt{x^2-a^2}\right)}\right]+C\\
&=\ln\left(-x-\sqrt{x^2-a^2}\right)+C
\end{aligned}
$$

after summing up:

$$\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}=\frac{x}{|x|}\operatorname{arccosh}\frac{|x|}{a}+C_1=\ln\left|x+\sqrt{x^2-a^2}\right|+C$$

■

$$\int\frac{\mathrm{d}x}{\sqrt{(x^2-a^2)^3}}=-\frac{x}{a^2\sqrt{x^2-a^2}}+C \tag{2.46}$$

Proof. Domain of the function:$(-\infty,-a)\cup(a,+\infty)$

$$
\begin{aligned}
&\text{❶ } x>a:\ \text{let } x=a\sec u\left(0<u<\frac{\pi}{2}\right)\\
&\int\frac{\mathrm{d}x}{\sqrt{(x^2-a^2)^3}}=\int\frac{\mathrm{d}(a\sec u)}{\sqrt{(a^2\sec^2 u-a^2)^3}}\\
&=\frac{1}{a^2}\int\frac{\cos u}{\sin^2 u}\,\mathrm{d}u\\
&=\frac{1}{a^2}\int\frac{\mathrm{d}\sin u}{\sin^2 u}\\
&=-\frac{1}{a^2}\csc u+C\\
&=-\frac{1}{a^2}\csc\,\operatorname{arcsec}\frac{x}{a}+C\\
&\textbf{use}\ \operatorname{arcsec}\frac{x}{a}=\operatorname{arccsc}\sqrt{\frac{x^2}{x^2-a^2}}\\
&=-\frac{1}{a^2}\sqrt{\frac{x^2}{x^2-a^2}}+C
\end{aligned}
$$

$$= -\frac{x}{a^2\sqrt{x^2-a^2}} + C$$

❷ $x < -a$: let $x = -v (v > 0)$

$$\begin{aligned}\int \frac{\mathrm{d}x}{\sqrt{(x^2-a^2)^3}} &= -\int \frac{\mathrm{d}v}{\sqrt{(v^2-a^2)^3}} \\ &= \frac{v}{a^2\sqrt{v^2-a^2}} + C \\ &= -\frac{x}{a^2\sqrt{x^2-a^2}} + C\end{aligned}$$

after summing up:

$$\int \frac{\mathrm{d}x}{\sqrt{(x^2-a^2)^3}} = -\frac{x}{a^2\sqrt{x^2-a^2}} + C$$

■

$$\int \frac{x}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \sqrt{x^2-a^2} + C \tag{2.47}$$

Proof.

$$\begin{aligned}\int \frac{x}{\sqrt{x^2-a^2}}\,\mathrm{d}x &= \frac{1}{2}\int \frac{\mathrm{d}\left(x^2-a^2\right)}{\sqrt{x^2-a^2}} \\ &= \sqrt{x^2-a^2} + C\end{aligned}$$

■

$$\int \frac{x}{\sqrt{(x^2-a^2)^3}}\,\mathrm{d}x = -\frac{1}{\sqrt{x^2-a^2}} + C \tag{2.48}$$

Proof.

$$\begin{aligned}\int \frac{x}{\sqrt{(x^2-a^2)^3}}\,\mathrm{d}x &= \frac{1}{2}\int \frac{\mathrm{d}\left(x^2-a^2\right)}{\sqrt{(x^2-a^2)^3}} \\ &= -\frac{1}{\sqrt{x^2-a^2}} + C\end{aligned}$$

■

$$\int \frac{x^2}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \frac{x}{2}\sqrt{x^2-a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C \tag{2.49}$$

Proof. Domain of the function:$(-\infty,-a)\cup(a,+\infty)$

❶ $x>a:$ let $x=a\sec u\left(0<u<\dfrac{\pi}{2}\right)$

$$\begin{aligned}
\int\frac{x^2}{\sqrt{x^2-a^2}}\,\mathrm{d}x&=\int\frac{a^2\sec^2 u}{\sqrt{a^2\sec^2 u-a^2}}\,\mathrm{d}(a\sec u)\\
&=a^2\int\sec^3 u\,\mathrm{d}u\\
\textbf{use }\int\frac{\mathrm{d}x}{\cos^n x}&=\frac{1}{n-1}\frac{\sin x}{\cos^{n-1}x}+\frac{n-2}{n-1}\int\frac{\mathrm{d}x}{\cos^{n-2}x}\\
&=\frac{a^2}{2}\tan u\sec u+\frac{a^2}{2}\int\sec u\,\mathrm{d}u\\
\textbf{use }\int\sec x\,\mathrm{d}x&=\ln|\sec x+\tan x|+C\\
&=\frac{a^2}{2}\tan u\sec u+\frac{a^2}{2}\ln(\sec x+\tan x)+C\\
&=\frac{a^2}{2}\tan\operatorname{arcsec}\frac{x}{a}\sec\operatorname{arcsec}\frac{x}{a}\\
&+\frac{a^2}{2}\ln\left(\sec\operatorname{arcsec}\frac{x}{a}+\tan\operatorname{arcsec}\frac{x}{a}\right)+C\\
\textbf{use }\operatorname{arcsec}\frac{x}{a}&=\arctan\sqrt{\frac{x^2-a^2}{a^2}}\\
&=\frac{a^2}{2}\sqrt{\frac{x^2-a^2}{a^2}}\frac{x}{a}+\frac{a^2}{2}\ln\left(\frac{x}{a}+\sqrt{\frac{x^2-a^2}{a^2}}\right)+C\\
&=\frac{x}{2}\sqrt{x^2-a^2}+\frac{a^2}{2}\ln\left(x+\sqrt{x^2-a^2}\right)+C
\end{aligned}$$

❷ $x<-a:$ let $x=-v(v>0)$

$$\begin{aligned}
\int\frac{x^2}{\sqrt{x^2-a^2}}\,\mathrm{d}x&=-\int\frac{v^2}{\sqrt{v^2-a^2}}\,\mathrm{d}v\\
&=-\frac{v}{2}\sqrt{v^2-a^2}-\frac{a^2}{2}\ln\left(v+\sqrt{v^2-a^2}\right)+C\\
&=\frac{x}{2}\sqrt{x^2-a^2}+\frac{a^2}{2}\ln\left(\frac{1}{-x+\sqrt{x^2-a^2}}\right)+C\\
&=\frac{x}{2}\sqrt{x^2-a^2}+\frac{a^2}{2}\ln\left[\frac{-x-\sqrt{x^2-a^2}}{\left(-x+\sqrt{x^2-a^2}\right)\left(-x-\sqrt{x^2-a^2}\right)}\right]+C\\
&=\frac{x}{2}\sqrt{x^2-a^2}+\frac{a^2}{2}\ln\left(-x-\sqrt{x^2-a^2}\right)+C
\end{aligned}$$

after summing up:

$$\int\frac{x^2}{\sqrt{x^2-a^2}}\,\mathrm{d}x=\frac{x}{2}\sqrt{x^2-a^2}+\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+C$$

■

$$\int\frac{x^2}{\sqrt{\left(x^2-a^2\right)^3}}\,\mathrm{d}x=-\frac{x}{\sqrt{x^2-a^2}}+\ln\left|x+\sqrt{x^2-a^2}\right|+C \tag{2.50}$$

Proof. Domain of the function:$(-\infty,-a)\cup(a,+\infty)$

❶ $x>a:$ let $x=a\sec u\left(0<u<\dfrac{\pi}{2}\right)$

$$\begin{aligned}
\int\frac{x^2}{\sqrt{(x^2-a^2)^3}}\,\mathrm{d}x &= \int\frac{a^2\sec^2 u}{\sqrt{(a^2\sec^2 u-a^2)^3}}\,\mathrm{d}(a\sec u)\\
&=\int\csc^2 u\sec u\,\mathrm{d}u\\
&=\int\left(\cot^2 u+1\right)\sec u\,\mathrm{d}u\\
&=\int\csc u\cot u\,\mathrm{d}u+\int\sec u\,\mathrm{d}u
\end{aligned}$$

use $\displaystyle\int\csc x\cot x\,\mathrm{d}x=-\csc x+C$

use $\displaystyle\int\sec x\,\mathrm{d}x=\ln|\sec x+\tan x|+C$

$$\begin{aligned}
&=\ln(\sec u+\tan u)-\csc u+C\\
&=\ln\left(\sec\operatorname{arcsec}\frac{x}{a}+\tan\operatorname{arcsec}\frac{x}{a}\right)-\csc\operatorname{arcsec}\frac{x}{a}+C
\end{aligned}$$

use $\operatorname{arcsec}\dfrac{x}{a}=\arctan\sqrt{\dfrac{x^2-a^2}{a^2}}$

use $\operatorname{arcsec}\dfrac{x}{a}=\operatorname{arccsc}\sqrt{\dfrac{x^2}{x^2-a^2}}$

$$\begin{aligned}
&=\ln\left(\frac{x}{a}+\sqrt{\frac{x^2-a^2}{a^2}}\right)-\sqrt{\frac{x^2}{x^2-a^2}}+C\\
&=-\frac{x}{\sqrt{x^2-a^2}}+\ln\left(x+\sqrt{x^2-a^2}\right)+C
\end{aligned}$$

❷ $x<-a:$ let $x=-v(v>0)$

$$\begin{aligned}
\int\frac{x^2}{\sqrt{(x^2-a^2)^3}}\,\mathrm{d}x &= -\int\frac{v^2}{\sqrt{(v^2-a^2)^3}}\,\mathrm{d}v\\
&=\frac{v}{\sqrt{v^2-a^2}}-\ln\left(v+\sqrt{v^2-a^2}\right)+C\\
&=-\frac{x}{\sqrt{x^2-a^2}}+\ln\left(\frac{1}{-x+\sqrt{x^2-a^2}}\right)+C\\
&=-\frac{x}{\sqrt{x^2-a^2}}+\ln\left[\frac{-x-\sqrt{x^2-a^2}}{\left(-x+\sqrt{x^2-a^2}\right)\left(-x-\sqrt{x^2-a^2}\right)}\right]+C\\
&=-\frac{x}{\sqrt{x^2-a^2}}+\ln\left(-x-\sqrt{x^2-a^2}\right)+C
\end{aligned}$$

after summing up:

$$\int\frac{x^2}{\sqrt{(x^2-a^2)^3}}\,\mathrm{d}x=-\frac{x}{\sqrt{x^2-a^2}}+\ln\left|x+\sqrt{x^2-a^2}\right|+C$$

■

$$\int\frac{\mathrm{d}x}{x\sqrt{x^2-a^2}}=\frac{1}{a}\arccos\frac{a}{|x|}+C \tag{2.51}$$

Proof. Domain of the function:$(-\infty,-a)\cup(a,+\infty)$

$$\text{❶ } x>a: \text{ let } x=a\sec u\left(0<u<\frac{\pi}{2}\right)$$

$$\begin{aligned}\int\frac{\mathrm{d}x}{x\sqrt{x^2-a^2}}&=\int\frac{\mathrm{d}(a\sec u)}{a\sec u\sqrt{a^2\sec^2 u+a^2}}\\&=\frac{1}{a}\int\mathrm{d}u\\&=\frac{u}{a}+C\\&=\frac{1}{a}\operatorname{arcsec}\frac{x}{a}+C\\&=\frac{1}{a}\arccos\frac{a}{x}+C\end{aligned}$$

$$\text{❷ } x<-a: \text{ let } x=-v(v>0)$$

$$\begin{aligned}\int\frac{\mathrm{d}x}{x\sqrt{x^2-a^2}}&=\int\frac{\mathrm{d}v}{v\sqrt{v^2-a^2}}\\&=\frac{1}{a}\arccos\frac{a}{v}+C\\&=\frac{1}{a}\arccos\frac{a}{-x}+C\end{aligned}$$

after summing up:

$$\int\frac{\mathrm{d}x}{x\sqrt{x^2-a^2}}=\frac{1}{a}\arccos\frac{a}{|x|}+C$$

■

$$\int\frac{\mathrm{d}x}{x^2\sqrt{x^2-a^2}}=\frac{\sqrt{x^2-a^2}}{a^2x}+C \tag{2.52}$$

Proof. Domain of the function:$(-\infty,-a)\cup(a,+\infty)$

$$\text{❶ } x>a: \text{ let } x=a\sec u\left(0<u<\frac{\pi}{2}\right)$$

$$\begin{aligned}\int\frac{\mathrm{d}x}{x^2\sqrt{x^2-a^2}}&=\int\frac{\mathrm{d}(a\sec u)}{a^2\sec^2 u\sqrt{a^2\sec^2 u-a^2}}\\&=\frac{1}{a^2}\int\cos u\,\mathrm{d}u\\&=\frac{1}{a^2}\sin u+C\\&=\frac{1}{a^2}\sin\operatorname{arcsec}\frac{x}{a}+C\\\textbf{use }\operatorname{arcsec}\frac{x}{a}&=\arcsin\sqrt{\frac{x^2-a^2}{x^2}}\\&=\frac{\sqrt{x^2-a^2}}{a^2x}+C\end{aligned}$$

$$\text{❷ } x<-a: \text{ let } x=-v(v>0)$$

$$\begin{aligned}\int\frac{\mathrm{d}x}{x^2\sqrt{x^2-a^2}}&=-\int\frac{\mathrm{d}v}{v^2\sqrt{x^2-a^2}}\\&=-\frac{\sqrt{v^2-a^2}}{a^2v}+C\end{aligned}$$

$$= \frac{\sqrt{x^2-a^2}}{a^2x} + C$$

after summing up:

$$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x} + C$$

■

$$\int \sqrt{x^2-a^2}\,\mathrm{d}x = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C \tag{2.53}$$

Proof. Domain of the function:$(-\infty,-a)\cup(a,+\infty)$

❶ $x > a:\ \text{let } x = a\sec u\left(0 < u < \frac{\pi}{2}\right)$

$$\begin{aligned}
\int \sqrt{x^2-a^2}\,\mathrm{d}x &= \int \sqrt{a^2\sec^2 u - a^2}\,\mathrm{d}(a\sec u)\\
&= a^2\int \tan^2 u\sec u\,\mathrm{d}u\\
&= a^2\int \left(\sec^2 u - 1\right)\sec u\,\mathrm{d}u\\
&= a^2\int \sec^3 u\,\mathrm{d}u - a^2\int \sec u\,\mathrm{d}u\\
\textbf{use}\ \int \frac{\mathrm{d}x}{\cos^n x} &= \frac{1}{n-1}\frac{\sin x}{\cos^{n-1}x} + \frac{n-2}{n-1}\int\frac{\mathrm{d}x}{\cos^{n-2}x}\\
&= \frac{a^2}{2}\tan u\sec u - \frac{a^2}{2}\int \sec u\,\mathrm{d}u\\
\textbf{use}\ \int \sec x\,\mathrm{d}x &= \ln|\sec x + \tan x| + C\\
&= \frac{a^2}{2}\tan u\sec u - \frac{a^2}{2}\ln(\sec u + \tan u) + C\\
&= \frac{a^2}{2}\tan\,\mathrm{arcsec}\,\frac{x}{a}\sec\,\mathrm{arcsec}\,\frac{x}{a}\\
&\quad - \frac{a^2}{2}\ln\left(\sec\,\mathrm{arcsec}\,\frac{x}{a} + \tan\,\mathrm{arcsec}\,\frac{x}{a}\right) + C\\
\textbf{use}\ \mathrm{arcsec}\,\frac{x}{a} &= \arctan\sqrt{\frac{x^2-a^2}{a^2}}\\
&= \frac{a^2}{2}\sqrt{\frac{x^2-a^2}{a^2}}\frac{x}{a} - \frac{a^2}{2}\ln\left(\frac{x}{a} + \sqrt{\frac{x^2-a^2}{a^2}}\right) + C\\
&= \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left(x+\sqrt{x^2-a^2}\right) + C
\end{aligned}$$

❷ $x < -a:\ \text{let } x = -v(v>0)$

$$\begin{aligned}
\int \sqrt{x^2-a^2}\,\mathrm{d}x &= -\int \sqrt{v^2-a^2}\,\mathrm{d}v\\
&= -\frac{v}{2}\sqrt{v^2-a^2} + \frac{a^2}{2}\ln\left(v+\sqrt{v^2-a^2}\right) + C\\
&= \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left(\frac{1}{-x+\sqrt{x^2-a^2}}\right) + C
\end{aligned}$$

$$= \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left[\frac{-x-\sqrt{x^2-a^2}}{\left(-x+\sqrt{x^2-a^2}\right)\left(-x-\sqrt{x^2-a^2}\right)}\right] + C$$

$$= \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left(-x-\sqrt{x^2-a^2}\right) + C$$

after summing up:

$$\int \sqrt{x^2-a^2}\,\mathrm{d}x = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C$$

■

$$\begin{aligned}\int \sqrt{\left(x^2-a^2\right)^3}\,\mathrm{d}x = &\frac{x}{8}\left(2x^2-5a^2\right)\sqrt{x^2-a^2}\\ &+\frac{3}{8}a^4\ln\left|x+\sqrt{x^2-a^2}\right| + C\end{aligned} \tag{2.54}$$

Proof. Domain of the function:$(-\infty,-a)\cup(a,+\infty)$

❶ $x>a:$ let $x = a\sec u\left(0<u<\frac{\pi}{2}\right)$

$$\begin{aligned}\int \sqrt{\left(x^2-a^2\right)^3}\,\mathrm{d}x &= \int \sqrt{\left(a^2\sec^2 u-a^2\right)^3}\,\mathrm{d}(a\sec u)\\ &= a^4\int \tan^4 u\sec u\,\mathrm{d}u\\ &= a^4\int\left(\sec^2 u-1\right)^2\sec u\,\mathrm{d}u\\ &= a^4\int\sec^5 u\,\mathrm{d}u - 2a^4\int\sec^3 u\,\mathrm{d}u + a^4\int\sec u\,\mathrm{d}u\end{aligned}$$

use $\int \frac{\mathrm{d}x}{\cos^n x} = \frac{1}{n-1}\frac{\sin x}{\cos^{n-1}x} + \frac{n-2}{n-1}\int\frac{\mathrm{d}x}{\cos^{n-2}x}$

$$\begin{aligned}&= \frac{a^4}{4}\tan u\sec^3 u - \frac{5a^4}{4}\int\sec^3 u\,\mathrm{d}u + a^4\int\sec u\,\mathrm{d}u\\ &= \frac{a^4}{4}\tan u\sec^3 u - \frac{5a^4}{8}\tan u\sec u + \frac{3a^4}{8}\int\sec u\,\mathrm{d}u\end{aligned}$$

use $\int \sec x\,\mathrm{d}x = \ln|\sec x+\tan x| + C$

$$\begin{aligned}&= \frac{a^4}{4}\tan u\sec^3 u - \frac{5a^4}{8}\tan u\sec u + \frac{3a^4}{8}\ln(\sec u+\tan u) + C\\ &= \frac{a^4}{4}\tan\,\mathrm{arcsec}\,\frac{x}{a}\sec^3\,\mathrm{arcsec}\,\frac{x}{a} - \frac{5a^4}{8}\tan\,\mathrm{arcsec}\,\frac{x}{a}\sec\,\mathrm{arcsec}\,\frac{x}{a}\\ &+\frac{3a^4}{8}\ln\left(\sec\,\mathrm{arcsec}\,\frac{x}{a}+\tan\,\mathrm{arcsec}\,\frac{x}{a}\right) + C\end{aligned}$$

use $\mathrm{arcsec}\,\frac{x}{a} = \arctan\sqrt{\frac{x^2-a^2}{a^2}}$

$$\begin{aligned}&= \frac{a^4}{4}\sqrt{\frac{x^2-a^2}{a^2}}\left(\frac{x}{a}\right)^3 - \frac{5a^4}{8}\sqrt{\frac{x^2-a^2}{a^2}}\frac{x}{a} + \frac{3a^4}{8}\ln\left(\frac{x}{a}+\sqrt{\frac{x^2-a^2}{a^2}}\right) + C\\ &= \frac{x}{8}\left(2x^2-5a^2\right)\sqrt{x^2-a^2} + \frac{3}{8}a^4\ln\left(x+\sqrt{x^2-a^2}\right) + C\end{aligned}$$

❷ $x<-a:$ let $x=-v(v>0)$

$$\int \sqrt{\left(x^2-a^2\right)^3}\,\mathrm{d}x = -\int\sqrt{\left(v^2-a^2\right)^3}\,\mathrm{d}v$$

$$
\begin{aligned}
&= -\frac{v}{8}\left(2v^2-5a^2\right)\sqrt{v^2-a^2}-\frac{3}{8}a^4\ln\left(v+\sqrt{v^2-a^2}\right)+C\\
&= \frac{x}{8}\left(2x^2-5a^2\right)\sqrt{x^2-a^2}+\frac{3}{8}a^4\ln\left(\frac{1}{-x+\sqrt{x^2-a^2}}\right)+C\\
&= \frac{x}{8}\left(2x^2-5a^2\right)\sqrt{x^2-a^2}\\
&+\frac{3}{8}a^4\ln\left[\frac{-x-\sqrt{x^2-a^2}}{\left(-x+\sqrt{x^2-a^2}\right)\left(-x-\sqrt{x^2-a^2}\right)}\right]+C\\
&= \frac{x}{8}\left(2x^2-5a^2\right)\sqrt{x^2-a^2}+\frac{3}{8}a^4\ln\left(-x-\sqrt{x^2-a^2}\right)+C
\end{aligned}
$$

after summing up:

$$
\int\sqrt{\left(x^2-a^2\right)^3}\,\mathrm{d}x=\frac{x}{8}\left(2x^2-5a^2\right)\sqrt{x^2-a^2}+\frac{3}{8}a^4\ln\left|x+\sqrt{x^2-a^2}\right|+C
$$

■

$$
\int x\sqrt{x^2-a^2}\,\mathrm{d}x=\frac{1}{3}\sqrt{\left(x^2-a^2\right)^3}+C \tag{2.55}
$$

Proof.

$$
\begin{aligned}
\int x\sqrt{x^2-a^2}\,\mathrm{d}x &= \frac{1}{2}\int\sqrt{x^2-a^2}\,\mathrm{d}\left(\sqrt{x^2-a^2}\right)\\
&= \frac{1}{3}\sqrt{\left(x^2-a^2\right)^3}+C
\end{aligned}
$$

■

$$
\begin{aligned}
\int x^2\sqrt{x^2-a^2}\,\mathrm{d}x &= \frac{x}{8}\left(2x^2-a^2\right)\sqrt{x^2-a^2}\\
&-\frac{a^4}{8}\ln\left|x+\sqrt{x^2-a^2}\right|+C
\end{aligned} \tag{2.56}
$$

Proof. Domain of the function:$(-\infty,-a)\cup(a,+\infty)$

❶ $x>a:$ let $x=a\sec u\left(0<u<\frac{\pi}{2}\right)$

$$
\begin{aligned}
\int x^2\sqrt{x^2-a^2}\,\mathrm{d}x &= \int a^2\sec^2 u\sqrt{a^2\sec^2 u-a^2}\,\mathrm{d}(a\sec u)\\
&= a^4\int\tan^2 u\sec^3 u\,\mathrm{d}u\\
&= a^4\int\left(\sec^2 u-1\right)\sec^3 u\,\mathrm{d}u\\
&= a^4\int\sec^5 u\,\mathrm{d}u-a^4\int\sec^3 u\,\mathrm{d}u
\end{aligned}
$$

use $\displaystyle\int\frac{\mathrm{d}x}{\cos^n x}=\frac{1}{n-1}\frac{\sin x}{\cos^{n-1}x}+\frac{n-2}{n-1}\int\frac{\mathrm{d}x}{\cos^{n-2}x}$

$$
\begin{aligned}
&= \frac{a^4}{4}\tan u\sec^3 u-\frac{a^4}{4}\int\sec^3 u\,\mathrm{d}u\\
&= \frac{a^4}{4}\tan u\sec^3 u-\frac{a^4}{8}\tan u\sec u-\frac{a^4}{8}\int\sec u\,\mathrm{d}u
\end{aligned}
$$

$$\textbf{use} \int \sec x\ \mathrm{d}x = \ln|\sec x + \tan x| + C$$

$$\begin{aligned}
&= \frac{a^4}{4}\tan u\sec^3 u - \frac{a^4}{8}\tan u\sec u - \frac{a^4}{8}\ln(\sec u + \tan u) + C \\
&= \frac{a^4}{4}\tan\ \mathrm{arcsec}\,\frac{x}{a}\sec^3\ \mathrm{arcsec}\,\frac{x}{a} - \frac{a^4}{8}\tan\ \mathrm{arcsec}\,\frac{x}{a}\sec\ \mathrm{arcsec}\,\frac{x}{a} \\
&- \frac{a^4}{8}\ln\left(\sec\ \mathrm{arcsec}\,\frac{x}{a} + \tan\ \mathrm{arcsec}\,\frac{x}{a}\right) + C
\end{aligned}$$

$$\textbf{use}\ \ \mathrm{arcsec}\,\frac{x}{a} = \arctan\sqrt{\frac{x^2-a^2}{a^2}}$$

$$\begin{aligned}
&= \frac{a^4}{4}\sqrt{\frac{x^2-a^2}{a^2}}\left(\frac{x}{a}\right)^3 - \frac{a^4}{8}\sqrt{\frac{x^2-a^2}{a^2}}\frac{x}{a} - \frac{a^4}{8}\ln\left(\frac{x}{a} + \sqrt{\frac{x^2-a^2}{a^2}}\right) + C \\
&= \frac{x}{8}\left(2x^2-a^2\right)\sqrt{x^2-a^2} - \frac{a^4}{8}\ln\left(x + \sqrt{x^2-a^2}\right) + C
\end{aligned}$$

❷ $x < -a:\ \text{let } x = -v(v > 0)$

$$\begin{aligned}
\int x^2\sqrt{x^2-a^2}\ \mathrm{d}x &= -\int v^2\sqrt{v^2-a^2}\ \mathrm{d}v \\
&= -\frac{v}{8}\left(2v^2-a^2\right)\sqrt{v^2-a^2} + \frac{a^4}{8}\ln\left(v + \sqrt{v^2-a^2}\right) + C \\
&= \frac{x}{8}\left(2x^2-a^2\right)\sqrt{x^2-a^2} - \frac{a^4}{8}\ln\left(\frac{1}{-x+\sqrt{x^2-a^2}}\right) + C \\
&= \frac{x}{8}\left(2x^2-a^2\right)\sqrt{x^2-a^2} \\
&- \frac{a^4}{8}\ln\left[\frac{-x-\sqrt{x^2-a^2}}{\left(-x+\sqrt{x^2-a^2}\right)\left(-x-\sqrt{x^2-a^2}\right)}\right] + C \\
&= \frac{x}{8}\left(2x^2-a^2\right)\sqrt{x^2-a^2} - \frac{a^4}{8}\ln\left(-x - \sqrt{x^2-a^2}\right) + C
\end{aligned}$$

after summing up:

$$\int x^2\sqrt{x^2-a^2}\ \mathrm{d}x = \frac{x}{8}\left(2x^2-a^2\right)\sqrt{x^2-a^2} - \frac{a^4}{8}\ln\left|x + \sqrt{x^2-a^2}\right| + C$$

■

$$\int \frac{\sqrt{x^2-a^2}}{x}\ \mathrm{d}x = \sqrt{x^2-a^2} - a\arccos\frac{a}{|x|} + C \tag{2.57}$$

Proof. Domain of the function:$(-\infty, -a) \cup (a, +\infty)$

❶ $x > a:\ \text{let } x = a\sec u\left(0 < u < \frac{\pi}{2}\right)$

$$\begin{aligned}
\int \frac{\sqrt{x^2-a^2}}{x}\ \mathrm{d}x &= \int \frac{\sqrt{a^2\sec^2 u - a^2}}{a\sec u}\ \mathrm{d}(a\sec u) \\
&= a\int \tan^2 u\ \mathrm{d}u \\
&= a\int \left(\sec^2 u - 1\right)\ \mathrm{d}u \\
&= a\int \sec^2 u\ \mathrm{d}u - au \\
&= a\tan u - au + C
\end{aligned}$$

$$= a\tan\,\mathrm{arcsec}\,\frac{x}{a} - a\,\mathrm{arcsec}\,\frac{x}{a} + C$$

$$\textbf{use}\ \ \mathrm{arcsec}\,\frac{x}{a} = \arctan\sqrt{\frac{x^2-a^2}{a^2}}$$

$$= a\sqrt{\frac{x^2-a^2}{a^2}} - a\arccos\frac{a}{x} + C$$

$$= \sqrt{x^2-a^2} - a\arccos\frac{a}{x} + C$$

❷ $x < -a$: let $x = -v(v > 0)$

$$\int\frac{\sqrt{x^2-a^2}}{x}\,\mathrm{d}x = \int\frac{\sqrt{v^2-a^2}}{v}\,\mathrm{d}v$$

$$= \sqrt{v^2-a^2} - a\arccos\frac{a}{v} + C$$

$$= \sqrt{x^2-a^2} - a\arccos\frac{a}{-x} + C$$

after summing up:

$$\int\frac{\sqrt{x^2-a^2}}{x}\,\mathrm{d}x = \sqrt{x^2-a^2} - a\arccos\frac{a}{|x|} + C$$

■

$$\int\frac{\sqrt{x^2-a^2}}{x^2}\,\mathrm{d}x = -\frac{\sqrt{x^2-a^2}}{x} + \ln\left|x+\sqrt{x^2-a^2}\right| + C \tag{2.58}$$

Proof. Domain of the function:$(-\infty,-a)\cup(a,+\infty)$

❶ $x > a$: let $x = a\sec u\left(0 < u < \frac{\pi}{2}\right)$

$$\int\frac{\sqrt{x^2-a^2}}{x^2}\,\mathrm{d}x = \int\frac{\sqrt{a^2\sec^2 u - a^2}}{a^2\sec^2 u}\,\mathrm{d}(a\sec u)$$

$$= \int\frac{\sin^2 u}{\cos u}\,\mathrm{d}u$$

$$= \int\frac{1-\cos^2 u}{\cos u}\,\mathrm{d}u$$

$$= \int\sec u\,\mathrm{d}u - \int\cos u\,\mathrm{d}u$$

$$\textbf{use}\ \ \int\sec x\,\mathrm{d}x = \ln|\sec x + \tan x| + C$$

$$= \ln(\sec u + \tan u) - \sin u + C$$

$$= \ln\left(\sec\,\mathrm{arcsec}\,\frac{x}{a} + \tan\,\mathrm{arcsec}\,\frac{x}{a}\right) - \sin\,\mathrm{arcsec}\,\frac{x}{a} + C$$

$$\textbf{use}\ \ \mathrm{arcsec}\,\frac{x}{a} = \arctan\sqrt{\frac{x^2-a^2}{a^2}}$$

$$\textbf{use}\ \ \mathrm{arcsec}\,\frac{x}{a} = \arcsin\sqrt{\frac{x^2-a^2}{x^2}}$$

$$= \ln\left(\frac{x}{a} + \sqrt{\frac{x^2-a^2}{a^2}}\right) - \sqrt{\frac{x^2-a^2}{x^2}} + C$$

$$= -\frac{\sqrt{x^2-a^2}}{x} + \ln\left(x+\sqrt{x^2-a^2}\right) + C$$

❷ $x < -a$: let $x = -v (v > 0)$

$$\begin{aligned}
\int \frac{\sqrt{x^2-a^2}}{x^2}\,\mathrm{d}x &= -\int \frac{\sqrt{v^2-a^2}}{v^2}\,\mathrm{d}v \\
&= \frac{\sqrt{v^2-a^2}}{v} - \ln\left(v+\sqrt{v^2-a^2}\right) + C \\
&= -\frac{\sqrt{x^2-a^2}}{x} + \ln\left(\frac{1}{-x-\sqrt{x^2-a^2}}\right) + C \\
&= -\frac{\sqrt{x^2-a^2}}{x} + \ln\left[\frac{-x-\sqrt{x^2-a^2}}{\left(-x+\sqrt{x^2-a^2}\right)\left(-x-\sqrt{x^2-a^2}\right)}\right] + C \\
&= -\frac{\sqrt{x^2-a^2}}{x} + \ln\left(-x-\sqrt{x^2-a^2}\right) + C
\end{aligned}$$

after summing up:

$$\int \frac{\sqrt{x^2-a^2}}{x^2}\,\mathrm{d}x = -\frac{\sqrt{x^2-a^2}}{x} + \ln\left|x+\sqrt{x^2-a^2}\right| + C$$

■

2.8 含有 $\sqrt{a^2-x^2}\,(a>0)$ 的积分

$$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C \tag{2.59}$$

Proof.

$$\begin{aligned}
&\text{let } x = a\sin u \\
\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} &= \int \frac{\mathrm{d}(a\sin u)}{\sqrt{a^2-a^2\sin^2 u}} \\
&= \int \mathrm{d}u \\
&= \arcsin\frac{x}{a} + C
\end{aligned}$$

■

$$\int \frac{\mathrm{d}x}{\sqrt{(a^2-x^2)^3}} = \frac{x}{a^2\sqrt{a^2-x^2}} + C \tag{2.60}$$

Proof.

$$
\begin{aligned}
&\text{let } x = a\sin u \\
\int \frac{\mathrm{d}x}{\sqrt{(a^2-x^2)^3}} &= \int \frac{\mathrm{d}(a\sin u)}{\sqrt{\left(a^2-a^2\sin^2 u\right)^3}} \\
&= \frac{1}{a^2}\int \sec^2 u\,\mathrm{d}u \\
&= \frac{1}{a^2}\tan u + C \\
&= \frac{1}{a^2}\tan\arcsin\frac{x}{a} + C \\
\textbf{use } \arcsin\frac{x}{a} &= \arctan\sqrt{\frac{x^2}{a^2-x^2}} \\
&= \frac{1}{a^2}\sqrt{\frac{x^2}{a^2-x^2}} + C \\
&= \frac{x}{a^2\sqrt{a^2-x^2}} + C
\end{aligned}
$$

■

$$
\int \frac{x}{\sqrt{a^2-x^2}}\,\mathrm{d}x = -\sqrt{a^2-x^2} + C \tag{2.61}
$$

Proof.

$$
\begin{aligned}
\int \frac{x}{\sqrt{a^2-x^2}}\,\mathrm{d}x &= -\frac{1}{2}\int \frac{\mathrm{d}\left(a^2-x^2\right)}{\sqrt{a^2-x^2}} \\
&= -\sqrt{a^2-x^2} + C
\end{aligned}
$$

■

$$
\int \frac{x}{\sqrt{(a^2-x^2)^3}}\,\mathrm{d}x = \frac{1}{\sqrt{a^2-x^2}} + C \tag{2.62}
$$

Proof.

$$
\begin{aligned}
\int \frac{x}{\sqrt{(a^2-x^2)^3}}\,\mathrm{d}x &= -\frac{1}{2}\int \frac{\mathrm{d}\left(a^2-x^2\right)}{\sqrt{(a^2-x^2)^3}} \\
&= \frac{1}{\sqrt{a^2-x^2}} + C
\end{aligned}
$$

■

$$
\int \frac{x^2}{\sqrt{a^2-x^2}}\,\mathrm{d}x = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C \tag{2.63}
$$

Proof.

$$
\begin{aligned}
&\text{let } x = a\sin u \\
\int \frac{x^2}{\sqrt{a^2-x^2}}\,\mathrm{d}x &= \int \frac{a^2\sin^2 u}{\sqrt{a^2-a^2\sin^2 u}}\,\mathrm{d}(a\sin u) \\
&= a^2\int \sin^2 u\,\mathrm{d}u \\
\textbf{use } \int \sin^2 x\,\mathrm{d}x &= \frac{x}{2}-\frac{1}{4}\sin 2x + C \\
&= \frac{a^2}{2}u - \frac{a^2}{4}\sin 2u + C \\
&= \frac{a^2}{2}\arcsin\frac{x}{a} - \frac{a^2}{2}\sin\arcsin\frac{x}{a}\cos\arcsin\frac{x}{a} + C \\
\textbf{use } \arcsin\frac{x}{a} &= \arccos\sqrt{\frac{a^2-x^2}{a^2}} \\
&= \frac{a^2}{2}\arcsin\frac{x}{a} - \frac{a^2}{2}\frac{x}{a}\sqrt{\frac{a^2-x^2}{a^2}} + C \\
&= -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C
\end{aligned}
$$

■

$$\int \frac{x^2}{\sqrt{(a^2-x^2)^3}}\,\mathrm{d}x = \frac{x}{\sqrt{a^2-x^2}} - \arcsin\frac{x}{a} + C \tag{2.64}$$

Proof.

$$
\begin{aligned}
&\text{let } x = a\sin u \\
\int \frac{x^2}{\sqrt{(a^2-x^2)^3}}\,\mathrm{d}x &= \int \frac{a^2\sin^2 u}{\sqrt{\left(a^2-a^2\sin^2 u\right)^3}}\,\mathrm{d}(a\sin u) \\
&= \int \tan^2 u\,\mathrm{d}u \\
&= \int \left[\sec^2 u - 1\right]\,\mathrm{d}u \\
&= \tan u - u + C \\
&= \tan\arcsin\frac{x}{a} - \arcsin\frac{x}{a} + C \\
\textbf{use } \arcsin\frac{x}{a} &= \arctan\sqrt{\frac{x^2}{a^2-x^2}} \\
&= \sqrt{\frac{x^2}{a^2-x^2}} - \arcsin\frac{x}{a} + C \\
&= \frac{x}{\sqrt{a^2-x^2}} - \arcsin\frac{x}{a} + C
\end{aligned}
$$

■

$$\int \frac{\mathrm{d}x}{x\sqrt{a^2-x^2}} = \frac{1}{a}\ln\frac{a-\sqrt{a^2-x^2}}{|x|} + C \tag{2.65}$$

Proof.

$$
\begin{aligned}
&\text{let } x = a\sin u \\
\int \frac{\mathrm{d}x}{x\sqrt{a^2-x^2}} &= \int \frac{\mathrm{d}(a\sin u)}{a\sin u\sqrt{a^2-a^2\sin^2 u}} \\
&= \frac{1}{a}\int \csc u \,\mathrm{d}u \\
\textbf{use } \int \csc x\,\mathrm{d}x &= \ln|\csc x - \cot x| + C \\
&= \frac{1}{a}\ln|\csc u - \cot u| + C \\
&= \frac{1}{a}\ln\left|\csc\arcsin\frac{x}{a} - \cot\arcsin\frac{x}{a}\right| + C \\
\textbf{use } \arcsin\frac{x}{a} &= \operatorname{arccot}\sqrt{\frac{a^2-x^2}{x^2}} \\
&= \frac{1}{a}\ln\left|\frac{a}{x} - \sqrt{\frac{a^2-x^2}{x^2}}\right| + C \\
&= \frac{1}{a}\ln\frac{a-\sqrt{a^2-x^2}}{|x|} + C
\end{aligned}
$$

■

$$
\int \frac{\mathrm{d}x}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2 x} + C \tag{2.66}
$$

Proof.

$$
\begin{aligned}
&\text{let } x = a\sin u \\
\int \frac{\mathrm{d}x}{x^2\sqrt{a^2-x^2}} &= \int \frac{\mathrm{d}(a\sin u)}{a^2\sin^2 u\sqrt{a^2-a^2\sin^2 u}} \\
&= \frac{1}{a^2}\int \csc^2 u \,\mathrm{d}u \\
&= -\frac{1}{a^2}\cot u + C \\
&= -\frac{1}{a^2}\cot\arcsin\frac{x}{a} + C \\
\textbf{use } \arcsin\frac{x}{a} &= \operatorname{arccot}\sqrt{\frac{a^2-x^2}{x^2}} \\
&= -\frac{1}{a^2}\sqrt{\frac{a^2-x^2}{x^2}} + C \\
&= -\frac{\sqrt{a^2-x^2}}{a^2 x} + C
\end{aligned}
$$

■

$$
\int \sqrt{a^2-x^2}\,\mathrm{d}x = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C \tag{2.67}
$$

Proof.

$$
\begin{aligned}
&\text{let } x = a\sin u \\
\int \sqrt{a^2-x^2}\,\mathrm{d}x &= \int \sqrt{a^2-a^2\sin^2 u}\,\mathrm{d}(a\sin u) \\
&= a^2 \int \cos^2 u\,\mathrm{d}u \\
\textbf{use } \int \cos^2 x\,\mathrm{d}x &= \frac{x}{2}+\frac{1}{4}\sin 2x + C \\
&= \frac{a^2}{4}\sin 2u + \frac{a^2}{2}u + C \\
&= \frac{a^2}{2}\sin\arcsin\frac{x}{a}\cos\arcsin\frac{x}{a} + \frac{a^2}{2}\arcsin\frac{x}{a} + C \\
\textbf{use } \arcsin\frac{x}{a} &= \arccos\sqrt{\frac{a^2-x^2}{a^2}} \\
&= \frac{a^2}{2}\frac{x}{a}\sqrt{\frac{a^2-x^2}{a^2}} + \frac{a^2}{2}\arcsin\frac{x}{a} + C \\
&= \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C
\end{aligned}
$$

■

$$
\int \sqrt{(a^2-x^2)^3}\,\mathrm{d}x = \frac{x}{8}\left(5a^2-2x^2\right)\sqrt{a^2-x^2} + \frac{3}{8}a^4\arcsin\frac{x}{a} + C \tag{2.68}
$$

Proof.

$$
\begin{aligned}
&\text{let } x = a\sin u \\
\int \sqrt{(a^2-x^2)^3}\,\mathrm{d}x &= \int \sqrt{\left(a^2-a^2\sin^2 u\right)^3}\,\mathrm{d}(a\sin u) \\
&= a^4 \int \cos^4 u\,\mathrm{d}u \\
\textbf{use } \int \cos^n x\,\mathrm{d}x &= \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,\mathrm{d}x \\
&= \frac{a^4}{4}\sin u\cos^3 u + \frac{3a^4}{4}\int \cos^2 u\,\mathrm{d}u \\
\textbf{use } \int \cos^2 x\,\mathrm{d}x &= \frac{x}{2}+\frac{1}{4}\sin 2x + C \\
&= \frac{a^4}{4}\sin u\cos^3 u + \frac{3a^4}{16}\sin 2u + \frac{3a^4}{8}u + C \\
&= \frac{a^4}{4}\sin\arcsin\frac{x}{a}\cos^3\arcsin\frac{x}{a} \\
&+ \frac{3a^4}{8}\sin\arcsin\frac{x}{a}\cos\arcsin\frac{x}{a} + \frac{3a^4}{8}\arcsin\frac{x}{a} + C \\
\textbf{use } \arcsin\frac{x}{a} &= \arccos\sqrt{\frac{a^2-x^2}{a^2}} \\
&= \frac{a^4}{4}\frac{x}{a}\sqrt{\frac{a^2-x^2}{a^2}}^3 + \frac{3a^4}{8}\frac{x}{a}\sqrt{\frac{a^2-x^2}{a^2}} + \frac{3a^4}{8}\arcsin\frac{x}{a} + C \\
&= \frac{x}{8}\left(5a^2-2x^2\right)\sqrt{a^2-x^2} + \frac{3}{8}a^4\arcsin\frac{x}{a} + C
\end{aligned}
$$

■

$$\int x\sqrt{a^2-x^2}\,\mathrm{d}x = -\frac{1}{3}\sqrt{\left(a^2-x^2\right)^3}+C \tag{2.69}$$

Proof.

$$\begin{aligned}\int x\sqrt{a^2-x^2}\,\mathrm{d}x &= -\frac{1}{2}\int\sqrt{a^2-x^2}\,\mathrm{d}\left(a^2-x^2\right)\\ &= -\frac{1}{3}\sqrt{\left(a^2-x^2\right)^3}+C\end{aligned}$$

■

$$\int x^2\sqrt{a^2-x^2}\,\mathrm{d}x = \frac{x}{8}\left(2x^2-a^2\right)\sqrt{a^2-x^2}+\frac{a^4}{8}\arcsin\frac{x}{a}+C \tag{2.70}$$

Proof.

$$\begin{aligned}
&\text{let } x = a\sin u\\
\int x^2\sqrt{a^2-x^2}\,\mathrm{d}x &= \int a^2\sin^2 u\sqrt{a^2-a^2\sin^2 u}\,\mathrm{d}(a\sin u)\\
&= a^4\int \sin^2 u\cos^2 u\,\mathrm{d}u\\
&= a^4\int \sin^2 u\left(1-\sin^2 u\right)\,\mathrm{d}u\\
&= a^4\int \sin^2 u\,\mathrm{d}u - a^4\int\sin^4 u\,\mathrm{d}u\\
\textbf{use}\ \int \sin^n x\,\mathrm{d}x &= -\frac{1}{n}\sin^{n-1}x\cos x+\frac{n-1}{n}\int\sin^{n-2}x\,\mathrm{d}x\\
&= \frac{a^4}{4}\sin^3 u\cos u+\frac{a^4}{4}\int\sin^2 u\,\mathrm{d}u\\
\textbf{use}\ \int \sin^2 x\,\mathrm{d}x &= \frac{x}{2}-\frac{1}{4}\sin 2x+C\\
&= \frac{a^4}{4}\sin^3 u\cos u+\frac{a^4}{8}u-\frac{a^4}{16}\sin 2u+C\\
&= \frac{a^4}{4}\sin^3\arcsin\frac{x}{a}\cos\arcsin\frac{x}{a}+\frac{a^4}{8}\arcsin\frac{x}{a}\\
&-\frac{a^4}{8}\sin\arcsin\frac{x}{a}\cos\arcsin\frac{x}{a}+C\\
\textbf{use}\ \arcsin\frac{x}{a} &= \arccos\sqrt{\frac{a^2-x^2}{a^2}}\\
&= \frac{a^4}{4}\left(\frac{x}{a}\right)^3\sqrt{\frac{a^2-x^2}{a^2}}+\frac{a^4}{8}\arcsin\frac{x}{a}-\frac{a^4}{8}\frac{x}{a}\sqrt{\frac{a^2-x^2}{a^2}}+C\\
&= \frac{x}{8}\left(2x^2-a^2\right)\sqrt{a^2-x^2}+\frac{a^4}{8}\arcsin\frac{x}{a}+C
\end{aligned}$$

■

$$\int\frac{\sqrt{a^2-x^2}}{x}\,\mathrm{d}x = \sqrt{a^2-x^2}+a\ln\frac{a-\sqrt{a^2-x^2}}{|x|}+C \tag{2.71}$$

Proof.

$$\begin{aligned}
&\text{let } x = a\sin u \\
\int \frac{\sqrt{a^2-x^2}}{x}\,\mathrm{d}x &= \int \frac{\sqrt{a^2-a^2\sin^2 u}}{a\sin u}\,\mathrm{d}(a\sin u) \\
&= a\int \frac{\cos^2 u}{\sin u}\,\mathrm{d}u \\
&= a\int \frac{1-\sin^2 u}{\sin u}\,\mathrm{d}u \\
&= a\int \csc u\,\mathrm{d}u - a\int \sin u\,\mathrm{d}u \\
\textbf{use } \int \csc x\,\mathrm{d}x &= \ln|\csc x-\cot x| + C \\
&= a\ln|\csc u-\cot u| + a\cos u \\
&= a\ln\left|\csc\arcsin\frac{x}{a} - \cot\arcsin\frac{x}{a}\right| + a\cos\arcsin\frac{x}{a} \\
\textbf{use } \arcsin\frac{x}{a} &= \arccos\sqrt{\frac{a^2-x^2}{a^2}} \\
\textbf{use } \arcsin\frac{x}{a} &= \operatorname{arccot}\sqrt{\frac{a^2-x^2}{x^2}} \\
&= a\ln\left|\frac{a}{x} - \sqrt{\frac{a^2-x^2}{x^2}}\right| + a\sqrt{\frac{a^2-x^2}{a^2}} \\
&= \sqrt{a^2-x^2} + a\ln\frac{a-\sqrt{a^2-x^2}}{|x|} + C
\end{aligned}$$

■

$$\int \frac{\sqrt{a^2-x^2}}{x^2}\,\mathrm{d}x = -\frac{\sqrt{a^2-x^2}}{x} - \arcsin\frac{x}{a} + C \tag{2.72}$$

Proof.

$$\begin{aligned}
&\text{let } x = a\sin u \\
\int \frac{\sqrt{a^2-x^2}}{x^2}\,\mathrm{d}x &= \int \frac{\sqrt{a^2-a^2\sin^2 u}}{a^2\sin^2 u}\,\mathrm{d}(a\sin u) \\
&= \int \cot^2 u\,\mathrm{d}u \\
&= \int \left(\csc^2 u - 1\right)\,\mathrm{d}u \\
&= -\cot u - u + C \\
&= -\cot\arcsin\frac{x}{a} - \arcsin\frac{x}{a} + C \\
\textbf{use } \arcsin\frac{x}{a} &= \operatorname{arccot}\sqrt{\frac{a^2-x^2}{x^2}} \\
&= -\sqrt{\frac{a^2-x^2}{x^2}} - \arcsin\frac{x}{a} + C \\
&= -\frac{\sqrt{a^2-x^2}}{x} - \arcsin\frac{x}{a} + C
\end{aligned}$$

■

2.9 含有 $\sqrt{\pm ax^2+bx+c}\,(a>0)$ 的积分

$$\int \frac{\mathrm{d}x}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C \tag{2.73}$$

Proof.

$$\text{as } f(x)=ax^2+bx+c>0(a>0), f(-\frac{b}{2a})\in\mathbb{R}, \Delta=b^2-4ac\in\mathbb{R}$$

$$\text{to fit the minimum demand we suppose } \Delta=b^2-4ac>0$$

$$\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{ax^2+bx+c}} &= \int \frac{\mathrm{d}x}{\sqrt{\left(\frac{b}{2\sqrt{a}}+\sqrt{a}x\right)^2-\frac{b^2-4ac}{4a}}} \\
&= 2\sqrt{a}\int \frac{\mathrm{d}x}{\sqrt{(2ax+b)^2-\sqrt{b^2-4ac}^2}} \\
&= \frac{1}{\sqrt{a}}\int \frac{\mathrm{d}(2ax+b)}{\sqrt{(2ax+b)^2-\sqrt{b^2-4ac}^2}} \\
\textbf{use} \int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} &= \ln\left|x+\sqrt{x^2-a^2}\right|+C \\
&= \frac{1}{\sqrt{a}}\ln\left|(2ax+b)+\sqrt{(2ax+b)^2-\sqrt{b^2-4ac}^2}\right|+C \\
&= \frac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C
\end{aligned}$$

■

$$\begin{aligned}
\int \sqrt{ax^2+bx+c}\,\mathrm{d}x = &\frac{2ax+b}{4a}\sqrt{ax^2+bx+c} \\
&+\frac{4ac-b^2}{8\sqrt{a^3}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C
\end{aligned} \tag{2.74}$$

Proof.

$$\text{as } f(x)=ax^2+bx+c>0(a>0), f(-\frac{b}{2a})\in\mathbb{R}, \Delta=b^2-4ac\in\mathbb{R}$$

$$\text{to fit the minimum demand we suppose } \Delta=b^2-4ac>0$$

$$\begin{aligned}
\int \sqrt{ax^2+bx+c}\,\mathrm{d}x &= \int \sqrt{\left(\frac{b}{2\sqrt{a}}+\sqrt{a}x\right)^2-\frac{b^2-4ac}{4a}}\,\mathrm{d}x \\
&= \frac{1}{2\sqrt{a}}\int \sqrt{(2ax+b)^2-\sqrt{b^2-4ac}^2}\,\mathrm{d}x \\
&= \frac{1}{4\sqrt{a^3}}\int \sqrt{(2ax+b)^2-\sqrt{b^2-4ac}^2}\,\mathrm{d}(2ax+b) \\
\textbf{use} \int \sqrt{x^2-a^2}\,\mathrm{d}x &= \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+C \\
&= \frac{(2ax+b)}{8\sqrt{a^3}}\sqrt{(2ax+b)^2-\sqrt{b^2-4ac}^2}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\sqrt{b^2-4ac}^2}{8\sqrt{a^3}}\ln\left|(2ax+b)+\sqrt{(2ax+b)^2-\sqrt{b^2-4ac}^2}\right|+C\\
&=\frac{2ax+b}{4a}\sqrt{ax^2+bx+c}\\
&+\frac{4ac-b^2}{8\sqrt{a^3}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C
\end{aligned}
$$

■

$$
\begin{aligned}
\int\frac{x}{\sqrt{ax^2+bx+c}}\,\mathrm{d}x=&\frac{1}{a}\sqrt{ax^2+bx+c}\\
&-\frac{b}{2\sqrt{a^3}}\ln\left(2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right)+C
\end{aligned}
\tag{2.75}
$$

Proof.

$$
\begin{aligned}
\int\frac{x}{\sqrt{ax^2+bx+c}}\,\mathrm{d}x&=\frac{1}{2a}\int\frac{2ax+b-b}{\sqrt{ax^2+bx+c}}\,\mathrm{d}x\\
&=\frac{1}{2a}\int\frac{\mathrm{d}\left(ax^2+bx+c\right)}{\sqrt{ax^2+bx+c}}-\frac{b}{2a}\int\frac{\mathrm{d}x}{\sqrt{ax^2+bx+c}}\\
\textbf{use}\quad\int\frac{\mathrm{d}x}{\sqrt{ax^2+bx+c}}&=\frac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C\\
&=\frac{1}{a}\sqrt{ax^2+bx+c}\\
&-\frac{b}{2\sqrt{a^3}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C
\end{aligned}
$$

■

$$
\int\frac{\mathrm{d}x}{\sqrt{c+bx-ax^2}}=\frac{1}{\sqrt{a}}\arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C
\tag{2.76}
$$

Proof.

$$
\begin{aligned}
&\text{as } f(x)=-ax^2+bx+c>0(a>0),f(\frac{b}{2a})>0,\Delta=b^2+4ac>0\\
\int\frac{\mathrm{d}x}{\sqrt{c+bx-ax^2}}&=\int\frac{\mathrm{d}x}{\sqrt{\frac{b^2+4ac}{4a}-\left(\sqrt{a}x-\frac{b}{2\sqrt{a}}\right)^2}}\\
&=2\sqrt{a}\int\frac{\mathrm{d}x}{\sqrt{\sqrt{b^2+4ac}^2-(2ax-b)^2}}\\
&=\frac{1}{\sqrt{a}}\int\frac{\mathrm{d}(2ax-b)}{\sqrt{\sqrt{b^2+4ac}^2-(2ax-b)^2}}\\
\textbf{use}\quad\int\frac{\mathrm{d}x}{\sqrt{a^2-x^2}}&=\arcsin\frac{x}{a}+C\\
&=\frac{1}{\sqrt{a}}\arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C
\end{aligned}
$$

■

$$\int \sqrt{c+bx-ax^2}\,\mathrm{d}x = \frac{2ax-b}{4a}\sqrt{c+bx-ax^2} + \frac{b^2+4ac}{8\sqrt{a^3}}\arcsin\frac{2ax-b}{\sqrt{b^2+4ac}} + C \tag{2.77}$$

Proof.

$$\text{as } f(x) = -ax^2+bx+c > 0 (a>0), f(\frac{b}{2a}) > 0, \Delta = b^2+4ac > 0$$

$$\begin{aligned}
\int \sqrt{c+bx-ax^2}\,\mathrm{d}x &= \frac{1}{2\sqrt{a}}\int\sqrt{\frac{b^2+4ac}{4a} - \left(\sqrt{a}x - \frac{b}{2\sqrt{a}}\right)^2}\,\mathrm{d}x \\
&= \frac{1}{2\sqrt{a}}\sqrt{\sqrt{b^2+4ac}^2 - (2ax-b)^2}\,\mathrm{d}x \\
&= \frac{1}{4\sqrt{a^3}}\sqrt{\sqrt{b^2+4ac}^2 - (2ax-b)^2}\,\mathrm{d}(2ax+b) \\
\int\sqrt{a^2-x^2}\,\mathrm{d}x &= \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C \\
&= \frac{(2ax-b)}{8\sqrt{a^3}}\sqrt{\sqrt{b^2+4ac}^2 - (2ax-b)^2} \\
&+ \frac{\sqrt{b^2+4ac}^2}{8\sqrt{a^3}}\arcsin\frac{(2ax-b)}{\sqrt{b^2+4ac}} + C \\
&= \frac{2ax-b}{4a}\sqrt{c+bx-ax^2} \\
&+ \frac{b^2+4ac}{8\sqrt{a^3}}\arcsin\frac{2ax-b}{\sqrt{b^2+4ac}} + C
\end{aligned}$$

■

$$\int\frac{x}{\sqrt{c+bx-ax^2}}\,\mathrm{d}x = -\frac{1}{a}\sqrt{c+bx-ax^2} + \frac{b}{2\sqrt{a^3}}\arcsin\frac{2ax-b}{\sqrt{b^2+4ac}} + C \tag{2.78}$$

Proof.

$$\begin{aligned}
\int\frac{x}{\sqrt{c+bx-ax^2}}\,\mathrm{d}x &= -\frac{1}{2a}\int\frac{b-2ax+b}{\sqrt{c+bx-ax^2}}\,\mathrm{d}x \\
&= -\frac{1}{2a}\int\frac{\mathrm{d}\left(c+bx-ax^2\right)}{\sqrt{c+bx-ax^2}} + \frac{b}{2a}\int\frac{\mathrm{d}x}{\sqrt{c+bx-ax^2}} \\
\textbf{use}\ \int\frac{\mathrm{d}x}{\sqrt{c+bx-ax^2}} &= \frac{1}{\sqrt{a}}\arcsin\frac{2ax-b}{\sqrt{b^2+4ac}} + C \\
&= -\frac{1}{a}\sqrt{c+bx-ax^2} + \frac{b}{2\sqrt{a^3}}\arcsin\frac{2ax-b}{\sqrt{b^2+4ac}} + C
\end{aligned}$$

■

2.10 含有 $\sqrt{\pm\frac{x-a}{x-b}}$ 或 $\sqrt{(x-a)(b-x)}$ 的积分

$$\int\sqrt{\frac{x-a}{x-b}}\,\mathrm{d}x=(x-b)\sqrt{\frac{x-a}{x-b}}+(b-a)\ln\left(\sqrt{|x-a|}+\sqrt{|x-b|}\right)+C \tag{2.79}$$

Proof.

$$\text{let } u=\sqrt{\frac{x-a}{x-b}}\text{ ,then } x=\frac{bu^2-a}{u^2-1}$$

$$\begin{aligned}
\int\sqrt{\frac{x-a}{x-b}}\,\mathrm{d}x &= \int u\,\mathrm{d}\left(\frac{bu^2-a}{u^2-1}\right)\\
&=(2a-2b)\int\frac{u^2}{\left(u^2-1\right)^2}\,\mathrm{d}u\\
&=(2a-2b)\int\frac{u^2-1+1}{\left(u^2-1\right)^2}\,\mathrm{d}u\\
&=(2a-2b)\int\frac{\mathrm{d}u}{u^2-1}+(2a-2b)\int\frac{\mathrm{d}u}{\left(u^2-1\right)^2}\\
\textbf{use }\int\frac{\mathrm{d}x}{\left(ax^2+b\right)^2}&=\frac{x}{2b\left(ax^2+b\right)}+\frac{1}{2b}\int\frac{\mathrm{d}x}{ax^2+b}\\
&=(a-b)\int\frac{\mathrm{d}u}{u^2-1}-\frac{(a-b)u}{u^2-1}\\
\textbf{use }\int\frac{\mathrm{d}x}{ax^2+b}&=\begin{cases}\dfrac{1}{\sqrt{ab}}\arctan\sqrt{\dfrac{a}{b}}x+C & \text{if } (b>0)\\ \dfrac{1}{2\sqrt{-ab}}\ln\left|\dfrac{\sqrt{a}x-\sqrt{-b}}{\sqrt{a}x+\sqrt{-b}}\right|+C & \text{if } (b<0)\end{cases}\\
&=\frac{a-b}{2}\ln\left|\frac{u-1}{u+1}\right|-\frac{(a-b)u}{u^2-1}+C\\
&=(a-b)\ln\left|\frac{\sqrt{u^2-1}}{u+1}\right|+(x-b)\sqrt{\frac{x-a}{x-b}}+C\\
&=(a-b)\ln\left|\frac{\sqrt{\frac{|b-a|}{|x-b|}}}{\frac{\sqrt{|x-a|}+\sqrt{|x-b|}}{\sqrt{|x-b|}}}\right|+(x-b)\sqrt{\frac{x-a}{x-b}}+C\\
&=(x-b)\sqrt{\frac{x-a}{x-b}}+(a-b)\ln\left|\frac{\sqrt{|b-a|}}{\sqrt{|x-a|}+\sqrt{|x-b|}}\right|+C\\
&=(x-b)\sqrt{\frac{x-a}{x-b}}+(a-b)\ln\sqrt{|b-a|}\\
&\quad+(b-a)\ln\left(\sqrt{|x-a|}+\sqrt{|x-b|}\right)+C\\
&=(x-b)\sqrt{\frac{x-a}{x-b}}+(b-a)\ln\left(\sqrt{|x-a|}+\sqrt{|x-b|}\right)+C
\end{aligned}$$

■

$$\int\sqrt{\frac{x-a}{b-x}}\,\mathrm{d}x=(x-b)\sqrt{\frac{x-a}{b-x}}+(b-a)\arcsin\sqrt{\frac{x-a}{b-a}}+C \tag{2.80}$$

Proof.

$$\begin{aligned}
&\text{let } u=\sqrt{\frac{x-a}{b-x}}\ \text{,then } x=\frac{bu^2+a}{u^2+1}\\
\int\sqrt{\frac{x-a}{b-x}}\,\mathrm{d}x &= \int u\,\mathrm{d}\left(\frac{bu^2+a}{u^2+1}\right)\\
&=(2b-2a)\int\frac{u^2}{\left(u^2+1\right)^2}\,\mathrm{d}u\\
&=(2b-2a)\int\frac{u^2+1-1}{\left(u^2+1\right)^2}\,\mathrm{d}u\\
&=(2b-2a)\int\frac{1}{u^2+1}\,\mathrm{d}u-(2b-2a)\int\frac{1}{\left(u^2+1\right)^2}\,\mathrm{d}u\\
\textbf{use }\int\frac{\mathrm{d}x}{\left(ax^2+b\right)^2} &= \frac{x}{2b\left(ax^2+b\right)}+\frac{1}{2b}\int\frac{\mathrm{d}x}{ax^2+b}\\
&=(b-a)\int\frac{1}{u^2+1}\,\mathrm{d}u+(a-b)\frac{u}{u^2+1}+C\\
\textbf{use }\int\frac{\mathrm{d}x}{ax^2+b} &= \begin{cases}\dfrac{1}{\sqrt{ab}}\arctan\sqrt{\dfrac{a}{b}}x+C & \text{if } (b>0)\\ \dfrac{1}{2\sqrt{-ab}}\ln\left|\dfrac{\sqrt{a}x-\sqrt{-b}}{\sqrt{a}x+\sqrt{-b}}\right|+C & \text{if } (b<0)\end{cases}\\
&=(x-b)\sqrt{\frac{x-a}{b-x}}+(b-a)\arctan u+C\\
&=(x-b)\sqrt{\frac{x-a}{b-x}}+(b-a)\arctan\sqrt{\frac{x-a}{b-x}}+C\\
\textbf{use }\arctan\frac{x}{a} &= \arcsin\sqrt{\frac{x^2}{x^2+a^2}}\\
&=(x-b)\sqrt{\frac{x-a}{b-x}}+(b-a)\arcsin\sqrt{\frac{x-a}{b-a}}+C
\end{aligned}$$

■

$$\int\frac{\mathrm{d}x}{\sqrt{(x-a)(b-x)}}=2\arcsin\sqrt{\frac{x-a}{b-a}}+C(a<b) \tag{2.81}$$

Proof.

$$\begin{aligned}
&\text{let } x-a=(b-a)\sin^2 u\ \text{,then } u=\arcsin\sqrt{\frac{x-a}{b-a}}\\
\int\frac{\mathrm{d}x}{\sqrt{(x-a)(b-x)}} &= \int\frac{\mathrm{d}\left[(b-a)\sin^2 u+a\right]}{\sqrt{(b-a)\sin^2 u\left[b-(b-a)\sin^2 u-a\right]}}\\
&=2\int\frac{(b-a)\sin u\cos u}{(b-a)\sqrt{\sin^2 u\cos^2 u}}\,\mathrm{d}u\\
&=2u+C\\
&=2\arcsin\sqrt{\frac{x-a}{b-a}}+C
\end{aligned}$$

■

$$\int \sqrt{(x-a)(b-x)}\,\mathrm{d}x = \frac{2x-a-b}{4}\sqrt{(x-a)(b-x)} + \frac{(b-a)^2}{4}\arcsin\sqrt{\frac{x-a}{b-a}} + C(a<b) \tag{2.82}$$

Proof.

$$\begin{aligned}
\int \sqrt{(x-a)(b-x)}\,\mathrm{d}x &= \int \sqrt{\left(\frac{a+b}{2}\right)^2 - ab - \left(x^2-(a+b)x+\left(\frac{a+b}{2}\right)^2\right)}\,\mathrm{d}x \\
&= \int \sqrt{\left(\frac{b-a}{2}\right)^2 - \left(x-\frac{a+b}{2}\right)^2}\,\mathrm{d}\left(x-\frac{a+b}{2}\right) \\
\textbf{use } \int \sqrt{a^2-x^2}\,\mathrm{d}x &= \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C \\
&= \frac{x-\frac{a+b}{2}}{2}\sqrt{\left(\frac{b-a}{2}\right)^2 - \left(x-\frac{a+b}{2}\right)^2} + \frac{\left(\frac{b-a}{2}\right)^2}{2}\arcsin\frac{x-\frac{a+b}{2}}{\frac{b-a}{2}} + C \\
&= \frac{2x-a-b}{4}\sqrt{(x-a)(b-x)} + \frac{(b-a)^2}{8}\arcsin\frac{2x-a-b}{b-a} + C
\end{aligned}$$

■

```
Manipulate[Plot[{ArcSin[(2 x - a - b)/(b - a)] + 1.5, 2 ArcSin[Sqrt[(x - a)/(b - a)]]}, {x, 0, 6}], {a, 0, 4}, {b, 0, 4}]
```

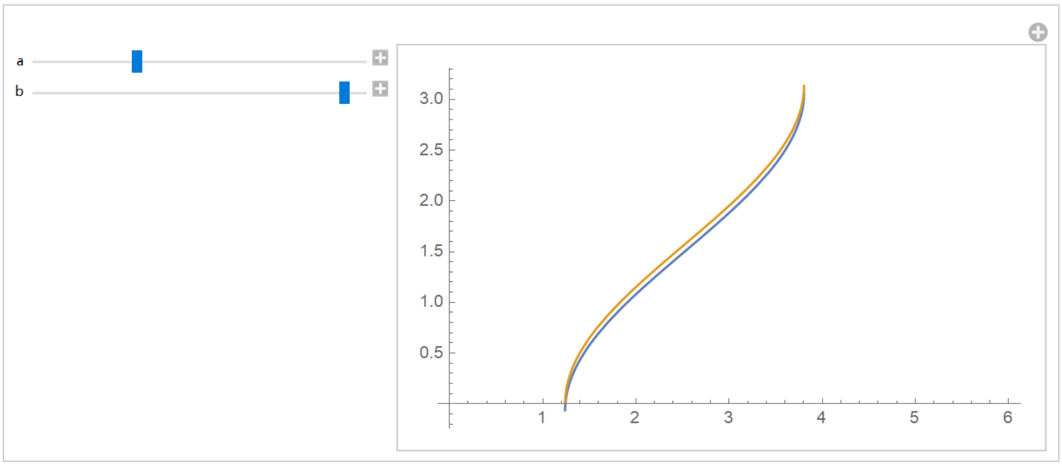

图 1: 利用 wolfram mathematica 进行比较

It's different from the answer.But they're really the same.Only a constant difference($\frac{\pi}{2}$).

another way:

Proof.

$$\text{let } x=a\cos^2 u+b\sin^2 u \text{ ,then } u=\arcsin\sqrt{\frac{x-a}{b-a}}, \sin u=\sqrt{\frac{x-a}{b-a}}, \cos u=\sqrt{\frac{b-x}{b-a}}$$

$$\begin{aligned}
\int\sqrt{(x-a)(b-x)}\,\mathrm{d}x &= \int\sqrt{\left(a\cos^2 u+b\sin^2 u-a\right)\left(b-a\cos^2 u-b\sin^2 u\right)}\,\mathrm{d}\left(a\cos^2 u+b\sin^2 u\right)\\
&= 2(b-a)^2\int\sin^2 u\cos^2 u\,\mathrm{d}u\\
&= 2(b-a)^2\int\sin^2 u\left(1-\sin^2 u\right)\,\mathrm{d}u\\
&= 2(b-a)^2\left[\int\sin^2 u\,\mathrm{d}u-\int\sin^4 u\,\mathrm{d}u\right]\\
\textbf{use }\int\sin^n x\,\mathrm{d}x &= -\frac{1}{n}\sin^{n-1}x\cos x+\frac{n-1}{n}\int\sin^{n-2}x\,\mathrm{d}x\\
&= \frac{(b-a)^2}{2}\left[\sin^3 u\cos u+\int\sin^2 u\,\mathrm{d}u\right]\\
\textbf{use }\int\sin^2 x\,\mathrm{d}x &= \frac{x}{2}-\frac{1}{4}\sin 2x+C\\
&= \frac{(b-a)^2}{2}\left[\sin^3 u\cos u+\frac{u}{2}-\frac{1}{4}\sin 2u\right]+C\\
&= \frac{(b-a)^2}{2}\left[\sqrt{\frac{x-a}{b-a}}^3\sqrt{\frac{b-x}{b-a}}+\frac{\arcsin\sqrt{\frac{x-a}{b-a}}}{2}-\frac{1}{2}\sqrt{\frac{x-a}{b-a}}\sqrt{\frac{b-x}{b-a}}\right]+C\\
&= \frac{(b-a)^2}{2}\left[\frac{x-a}{(b-a)^2}\sqrt{(x-a)(b-x)}-\frac{1}{2(b-a)}\sqrt{(x-a)(b-x)}+\frac{\arcsin\sqrt{\frac{x-a}{b-a}}}{2}\right]+C\\
&= \frac{2x-a-b}{4}\sqrt{(x-a)(b-x)}+\frac{(b-a)^2}{4}\arcsin\sqrt{\frac{x-a}{b-a}}+C
\end{aligned}$$

■

2.11 含有三角函数的积分

$$\int\sin x\,\mathrm{d}x=-\cos x+C \tag{2.83}$$

Proof.

$$\begin{aligned}
\int\sin x\,\mathrm{d}x &= \int\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots\right]\,\mathrm{d}x\\
&= \frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\cdots+(-1)^n\frac{x^{2n+2}}{(2n+2)!}+\cdots+C_1\\
&= -\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}+C\\
&= -\cos x+C
\end{aligned}$$

■

It's more easy if you know $(\cos x)' = -\sin x$

$$\int \cos x \, \mathrm{d}x = \sin x + C \tag{2.84}$$

Proof.

$$\begin{aligned}
\int \cos x \, \mathrm{d}x &= \int \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \right] \mathrm{d}x \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots + C \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} + C \\
&= \sin x + C
\end{aligned}$$

■

It's more easy if you know $(\sin x)' = \cos x$

$$\int \tan x \, \mathrm{d}x = -\ln|\cos x| + C \tag{2.85}$$

Proof.

$$\begin{aligned}
\int \tan x \, \mathrm{d}x &= \int \frac{\sin x}{\cos x} \, \mathrm{d}x \\
&= -\int \frac{\mathrm{d}(\cos x)}{\cos x} \\
&= -\ln|\cos x| + C
\end{aligned}$$

■

$$\int \cot x \, \mathrm{d}x = \ln|\sin x| + C \tag{2.86}$$

Proof.

$$\begin{aligned}
\int \cot x \, \mathrm{d}x &= \int \frac{\cos x}{\sin x} \, \mathrm{d}x \\
&= \int \frac{\mathrm{d}(\sin x)}{\sin x} \\
&= \ln|\sin x| + C
\end{aligned}$$

■

$$\int \sec x \, \mathrm{d}x = \ln\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C = \ln|\sec x + \tan x| + C \tag{2.87}$$

Proof.

$$\begin{aligned}\int \sec x \, \mathrm{d}x &= \int \frac{\cos x}{\cos^2 x} \, \mathrm{d}x = \int \frac{1}{1-\sin^2 x} \, \mathrm{d}(\sin x) \\ &= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C = \frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{1-\sin^2 x}\right| + C \\ &= \ln\left|\frac{1+\sin x}{\cos x}\right| + C = \ln|\sec x + \tan x| + C\end{aligned}$$

$$\begin{aligned}\ln\left|\frac{1+\sin x}{\cos x}\right| + C &= \ln\left|\frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{1-2\sin^2\frac{x}{2}}\right| + C = \ln\left|\frac{sec^2\frac{x}{2}+2\tan\frac{x}{2}}{sec^2\frac{x}{2}-2\tan^2\frac{x}{2}}\right| + C \\ &= \ln\left|\frac{\left(1+\tan\frac{x}{2}\right)^2}{1-\tan^2\frac{x}{2}}\right| + C = \ln\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| + C \\ &= \ln\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + C\end{aligned}$$

■

we have another way:

Proof.

$$\begin{aligned}\int \sec x \, \mathrm{d}x &= \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \, \mathrm{d}x \\ &= \int \frac{\mathrm{d}(\sec x + \tan x)}{\sec x + \tan x} \\ &= \ln|\sec x + \tan x| + C\end{aligned}$$

■

$$\int \csc x \, \mathrm{d}x = \ln\left|\tan\frac{x}{2}\right| + C = \ln|\csc x - \cot x| + C \tag{2.88}$$

Proof.

$$\begin{aligned}\int \csc x \, \mathrm{d}x &= \int \frac{\csc x(\csc x - \cot x)}{\csc x - \cot x} \, \mathrm{d}x \\ &= \int \frac{\mathrm{d}(\csc x - \cot x)}{\csc x - \cot x} \\ &= \ln|\csc x - \cot x| + C\end{aligned}$$

$$\begin{aligned}\ln|\csc x - \cot x| + C &= \ln\left|\frac{1-\cos x}{\sin x}\right| + C \\ &= \ln\left|\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right| + C \\ &= \ln\left|\tan\frac{x}{2}\right| + C\end{aligned}$$

■

$$\int \sec^2 x \, \mathrm{d}x = \tan x + C \tag{2.89}$$

Proof.

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x$$

∎

$$\int \csc^2 x \, \mathrm{d}x = -\cot x + C \tag{2.90}$$

Proof.

$$(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\csc^2 x$$

∎

$$\int \sec x \tan x \, \mathrm{d}x = \sec x + C \tag{2.91}$$

Proof.

$$(\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{\sin x}{\cos^2 x} = \sec x \tan x$$

∎

$$\int \csc x \cot x \, \mathrm{d}x = -\csc x + C \tag{2.92}$$

Proof.

$$(\csc x)' = \left(\frac{1}{\sin x}\right)' = -\frac{\cos x}{\sin^2 x} = -\csc x \cot x$$

∎

$$\int \sin^2 x \, \mathrm{d}x = \frac{x}{2} - \frac{1}{4}\sin 2x + C \tag{2.93}$$

Proof.

$$\begin{aligned}\int \sin^2 x \, \mathrm{d}x &= \int \left[\frac{1}{2} - \frac{1}{2}\cos 2x\right] \, \mathrm{d}x \\ &= \frac{x}{2} - \frac{1}{4}\sin 2x + C\end{aligned}$$

∎

$$\int \cos^2 x \, \mathrm{d}x = \frac{x}{2} + \frac{1}{4}\sin 2x + C \tag{2.94}$$

Proof.

$$\begin{aligned}\int \cos^2 x \, \mathrm{d}x &= \int \left[\frac{1}{2} + \frac{1}{2}\cos 2x\right] \, \mathrm{d}x \\ &= \frac{x}{2} + \frac{1}{4}\sin 2x + C\end{aligned}$$

■

$$\int \sin^n x \, \mathrm{d}x = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x \, \mathrm{d}x \tag{2.95}$$

Proof.

$$\begin{aligned}I_n &= \int \sin^n x \, \mathrm{d}x = \int \sin x \sin^{n-1} x \, \mathrm{d}x \\ &= -\cos x \sin^{n-1} x + (n-1)\int \cos^2 x \sin^{n-2} x \, \mathrm{d}x \\ &= -\cos x \sin^{n-1} x + (n-1)\int \left[\sin^{n-2} x - \sin^n x\right] \, \mathrm{d}x \\ &= -\cos x \sin^{n-1} x + (n-1)I_{n-2} - (n-1)I_n \\ I_n &= -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x \, \mathrm{d}x\end{aligned}$$

■

$$\int \cos^n x \, \mathrm{d}x = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x \, \mathrm{d}x \tag{2.96}$$

Proof.

$$\begin{aligned}I_n &= \int \cos^n x \, \mathrm{d}x = \int \cos x \cos^{n-1} x \, \mathrm{d}x \\ &= \sin x \cos^{n-1} x + (n-1)\int \sin^2 x \cos^{n-2} x \, \mathrm{d}x \\ &= \sin x \cos^{n-1} x + (n-1)\int \left[\cos^{n-2} x - \cos^n x\right] \, \mathrm{d}x \\ &= \sin x \cos^{n-1} x + (n-1)I_{n-2} - (n-1)I_n \\ I_n &= \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x \, \mathrm{d}x\end{aligned}$$

■

$$\int \frac{\mathrm{d}x}{\sin^n x} = -\frac{1}{n-1}\frac{\cos x}{\sin^{n-1} x} + \frac{n-2}{n-1}\int \frac{\mathrm{d}x}{\sin^{n-2} x} \tag{2.97}$$

***Proof*.**

$$
\begin{aligned}
I_n &= \int \frac{\mathrm{d}x}{\sin^n x} = \int \csc^n x \ \mathrm{d}x \\
&= \int \csc^2 x \csc^{n-2} x \ \mathrm{d}x \\
&= -\cot x \csc^{n-2} x - (n-2) \int \cot^2 x \csc^{n-2} x \ \mathrm{d}x \\
&= -\cot x \csc^{n-2} x - (n-2) \int \left[\csc^n x - \csc^{n-2} x\right] \ \mathrm{d}x \\
&= -\cot x \csc^{n-2} x - (n-2) I_n + (n-2) I_{n-2} \\
I_n &= -\frac{1}{n-1} \frac{\cos x}{\sin^{n-1} x} + \frac{n-2}{n-1} \int \frac{\mathrm{d}x}{\sin^{n-2} x}
\end{aligned}
$$

■

$$
\int \frac{\mathrm{d}x}{\cos^n x} = \frac{1}{n-1} \frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1} \int \frac{\mathrm{d}x}{\cos^{n-2} x} \tag{2.98}
$$

***Proof*.**

$$
\begin{aligned}
I_n &= \int \frac{\mathrm{d}x}{\cos^n x} = \int \sec^n x \ \mathrm{d}x \\
&= \int \sec^2 x \sec^{n-2} x \ \mathrm{d}x \\
&= \tan x \sec^{n-2} x - (n-2) \int \tan^2 x \sec^{n-2} x \ \mathrm{d}x \\
&= \tan x \sec^{n-2} x - (n-2) \int \left[\sec^n x - \sec^{n-2} x\right] \ \mathrm{d}x \\
&= \tan x \sec^{n-2} x - (n-2) I_n + (n-2) I_{n-2} \\
I_n &= \frac{1}{n-1} \frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1} \int \frac{\mathrm{d}x}{\cos^{n-2} x}
\end{aligned}
$$

■

$$
\begin{aligned}
\int \cos^m x \sin^n x \ \mathrm{d}x &= \frac{1}{m+n} \cos^{m-1} x \sin^{n+1} x + \frac{m-1}{m+n} \int \cos^{m-2} x \sin^n x \ \mathrm{d}x \\
&= -\frac{1}{m+n} \cos^{m+1} x \sin^{n-1} x + \frac{n-1}{m+n} \int \cos^m x \sin^{n-2} x \ \mathrm{d}x
\end{aligned} \tag{2.99}
$$

***Proof*.**

$$
\begin{aligned}
I_n^m &= \int \cos^m x \sin^n x \,\mathrm{d}x = \int \cos x \cos^{m-1} x \sin^n x \,\mathrm{d}x \\
&= \cos^{m-1} x \sin^{n+1} x - \int \sin x \left(\cos^{m-1} x \sin^n x\right)' \,\mathrm{d}x \\
&= \cos^{m-1} x \sin^{n+1} x + (m-1) \int \cos^{m-2} \sin^{n+2} \,\mathrm{d}x - n \int \cos^m x \sin^n x \,\mathrm{d}x \\
(n+1) I_n^m &= \cos^{m-1} x \sin^{n+1} x + (m-1) \int \sin^n x \left(1-\cos^2 x\right) \cos^{m-2} x \,\mathrm{d}x \\
&= \cos^{m-1} x \sin^{n+1} x + (m-1) \int \sin^n x \cos^{m-2} x \,\mathrm{d}x - (m-1) \int \cos^m x \sin^n x \,\mathrm{d}x \\
&= \cos^{m-1} x \sin^{n+1} x + (m-1) I_n^{m-2} - (m-1) I_n^m \\
I_n^m &= \frac{1}{m+n} \cos^{m-1} x \sin^{n+1} x + \frac{m-1}{m+n} \int \cos^{m-2} x \sin^n x \,\mathrm{d}x \\
&\text{or:} \\
I_n^m &= \int \cos^m x \sin^n x \,\mathrm{d}x = \int \sin x \cos^m x \sin^{n-1} x \,\mathrm{d}x \\
&= -\cos^{m+1} x \sin^{n-1} x + \int \cos x \left(\cos^m x \sin^{n-1}\right)' \,\mathrm{d}x \\
&= -\cos^{m+1} x \sin^{n-1} x + (n-1) \int \cos^{m+2} x \sin^{n-2} x \,\mathrm{d}x - m \int \cos^m x \sin^n x \,\mathrm{d}x \\
(m+1) I_n^m &= -\cos^{m+1} x \sin^{n-1} x + (n-1) \int \cos^m x \left(1-\sin^2 x\right) \sin^{n-2} x \,\mathrm{d}x \\
&= -\cos^{m+1} x \sin^{n-1} x + (n-1) \int \cos^m x \sin^{n-2} x \,\mathrm{d}x - (n-1) \int \cos^m x \sin^n x \,\mathrm{d}x \\
I_n^m &= -\frac{1}{m+n} \cos^{m+1} x \sin^{n-1} x + \frac{n-1}{m+n} \int \cos^m x \sin^{n-2} x \,\mathrm{d}x
\end{aligned}
$$

■

$$
\int \sin ax \cos bx \,\mathrm{d}x = -\frac{1}{2(a+b)} \cos{(a+b)x} - \frac{1}{2(a-b)} \cos{(a-b)x} + C \tag{2.100}
$$

***Proof*.**

$$
\begin{aligned}
\textbf{use } \sin\alpha \cdot \cos\beta &= \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)] \\
\int \sin ax \cos bx \,\mathrm{d}x &= \frac{1}{2} \int [\sin(ax+bx) + \sin(ax-bx)] \,\mathrm{d}x \\
&= -\frac{1}{2(a+b)} \cos{(a+b)x} - \frac{1}{2(a-b)} \cos{(a-b)x} + C
\end{aligned}
$$

■

$$
\int \sin ax \sin bx \,\mathrm{d}x = -\frac{1}{2(a+b)} \sin{(a+b)x} + \frac{1}{2(a-b)} \sin{(a-b)x} + C \tag{2.101}
$$

Proof.

$$
\begin{aligned}
\textbf{use}\ \ \sin\alpha\cdot\sin\beta &= -\frac{1}{2}[\cos(\alpha+\beta)-\cos(\alpha-\beta)]\\
\int \sin ax\sin bx\ \mathrm{d}x &= -\frac{1}{2}\int\ [\cos(ax+bx)-\cos(ax-bx)]\ \ \mathrm{d}x\\
&= -\frac{1}{2(a+b)}\sin{(a+b)x}+\frac{1}{2(a-b)}\sin{(a-b)x}+C
\end{aligned}
$$

■

$$
\int \cos ax\cos bx\ \mathrm{d}x = \frac{1}{2(a+b)}\sin{(a+b)x}+\frac{1}{2(a-b)}\sin{(a-b)x}+C \tag{2.102}
$$

Proof.

$$
\begin{aligned}
\textbf{use}\ \ \cos\alpha\cdot\cos\beta &= \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]\\
\int \cos ax\cos bx\ \mathrm{d}x &= \frac{1}{2}\int\ [\cos(ax+bx)+\cos(ax-bx)]\ \ \mathrm{d}x\\
&= \frac{1}{2(a+b)}\sin{(a+b)x}+\frac{1}{2(a-b)}\sin{(a-b)x}+C
\end{aligned}
$$

■

$$
\int \frac{\mathrm{d}x}{a+b\sin x} = \frac{2}{\sqrt{a^2-b^2}}\arctan\frac{a\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}}+C(a^2>b^2) \tag{2.103}
$$

Proof.

$$
\begin{aligned}
\text{let }u=\tan\frac{x}{2}\text{ ,then }\ \mathrm{d}x &= \frac{2}{1+u^2}\ \mathrm{d}u,\sin x=\frac{2u}{1+u^2}\\
\int\frac{\mathrm{d}x}{a+b\sin x} &= \int\frac{1}{a+b\frac{2u}{1+u^2}}\frac{2}{1+u^2}\ \mathrm{d}u\\
&= \int\frac{2}{au^2+2bu+a}\ \mathrm{d}u\\
&= \int\frac{2}{\left(\sqrt{a}u+\frac{b}{\sqrt{a}}\right)^2+\frac{a^2-b^2}{a}}\ \mathrm{d}u\\
\text{let }\sqrt{a}u+\frac{b}{\sqrt{a}}=\frac{\sqrt{a^2-b^2}}{\sqrt{a}}\tan v\text{ ,then }u &= \frac{\sqrt{a^2-b^2}\tan v-b}{a}, v=\arctan\frac{\sqrt{a}\left(\sqrt{a}u+\frac{b}{\sqrt{a}}\right)}{\sqrt{a^2-b^2}}\\
&= \frac{2a}{a^2-b^2}\int\frac{1}{\tan^2 v+1}\ \mathrm{d}\left(\frac{\sqrt{a^2-b^2}\tan v-b}{a}\right)\\
&= \frac{2}{\sqrt{a^2-b^2}}\int\ \mathrm{d}v\\
&= \frac{2}{\sqrt{a^2-b^2}}\arctan\frac{\sqrt{a}\left(\sqrt{a}u+\frac{b}{\sqrt{a}}\right)}{\sqrt{a^2-b^2}}+C\\
&= \frac{2}{\sqrt{a^2-b^2}}\arctan\frac{a\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}}+C
\end{aligned}
$$

■

$$\int \frac{\mathrm{d}x}{a+b\sin x} = \frac{1}{\sqrt{b^2-a^2}} \ln\left|\frac{a\tan\frac{x}{2}+b-\sqrt{b^2+a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2+a^2}}\right| + C(a^2<b^2) \tag{2.104}$$

Proof.

$$\begin{aligned}
\text{let } u = \tan\frac{x}{2} \text{ ,then } \mathrm{d}x &= \frac{2}{1+u^2}\,\mathrm{d}u, \sin x = \frac{2u}{1+u^2} \\
\int \frac{\mathrm{d}x}{a+b\sin x} &= \int \frac{1}{a+b\frac{2u}{1+u^2}} \frac{2}{1+u^2}\,\mathrm{d}u \\
&= \int \frac{2}{au^2+2bu+a}\,\mathrm{d}u \\
&= \int \frac{2}{a\left(u-\frac{-b-\sqrt{b^2-a^2}}{a}\right)\left(u-\frac{-b+\sqrt{b^2-a^2}}{a}\right)}\,\mathrm{d}u \\
&= \frac{1}{a}\int \left[\frac{A}{u+\frac{b+\sqrt{b^2-a^2}}{a}} + \frac{B}{u+\frac{b-\sqrt{b^2-a^2}}{a}}\right]\mathrm{d}u \\
&= \frac{1}{\sqrt{b^2-a^2}}\int \left[\frac{1}{u+\frac{b-\sqrt{b^2-a^2}}{a}} - \frac{1}{u+\frac{b+\sqrt{b^2-a^2}}{a}}\right]\mathrm{d}u \\
&= \frac{1}{\sqrt{b^2-a^2}}\left[\ln\left|au+b-\sqrt{b^2-a^2}\right| - \ln\left|au+b+\sqrt{b^2-a^2}\right|\right] \\
&= \frac{1}{\sqrt{b^2-a^2}} \ln\left|\frac{a\tan\frac{x}{2}+b-\sqrt{b^2+a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2+a^2}}\right| + C
\end{aligned}$$

■

$$\int \frac{\mathrm{d}x}{a+b\cos x} = \frac{2}{a+b}\sqrt{\frac{a+b}{a-b}} \arctan\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right) + C(a^2>b^2) \tag{2.105}$$

Proof.

$$\begin{aligned}
\text{let } u = \tan\frac{x}{2} \text{ ,then } \mathrm{d}x &= \frac{2}{1+u^2}\,\mathrm{d}u, \cos x = \frac{1-u^2}{1+u^2} \\
\int \frac{\mathrm{d}x}{a+b\cos x} &= \int \frac{1}{a+b\frac{1-u^2}{1+u^2}} \frac{2}{1+u^2}\,\mathrm{d}u \\
&= \int \frac{2}{(a-b)u^2+(a+b)}\,\mathrm{d}u \\
\text{let } \sqrt{\frac{a+b}{a-b}}\tan v = u \text{ ,then } v &= \arctan\left(\sqrt{\frac{a-b}{a+b}}u\right) \\
&= \frac{2}{a+b}\int \frac{1}{\tan^2 v+1}\,\mathrm{d}\left(\sqrt{\frac{a+b}{a-b}}\tan v\right) \\
&= \frac{2}{a+b}\sqrt{\frac{a+b}{a-b}}\int \mathrm{d}v \\
&= \frac{2}{a+b}\sqrt{\frac{a+b}{a-b}} \arctan\left(\sqrt{\frac{a-b}{a+b}}u\right) + C \\
&= \frac{2}{a+b}\sqrt{\frac{a+b}{a-b}} \arctan\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right) + C
\end{aligned}$$

■

$$\int \frac{\mathrm{d}x}{a+b\cos x} = \frac{1}{a+b}\sqrt{\frac{a+b}{b-a}} \ln \left| \frac{\tan\frac{x}{2} + \sqrt{\frac{a+b}{b-a}}}{\tan\frac{x}{2} - \sqrt{\frac{a+b}{b-a}}} \right| + C(a^2 < b^2) \tag{2.106}$$

Proof.

$$\begin{aligned}
\text{let } u = \tan\frac{x}{2} \text{ ,then } \ \mathrm{d}x &= \frac{2}{1+u^2}\,\mathrm{d}u, \cos x = \frac{1-u^2}{1+u^2} \\
\int \frac{\mathrm{d}x}{a+b\cos x} &= \int \frac{1}{a+b\frac{1-u^2}{1+u^2}} \frac{2}{1+u^2}\,\mathrm{d}u \\
&= \int \frac{2}{(a-b)u^2+(a+b)}\,\mathrm{d}u \\
&= \int \frac{2}{(a-b)\left(u^2 - \sqrt{\frac{a+b}{b-a}}^2\right)}\,\mathrm{d}u \\
&= \frac{1}{a-b}\int \left[\frac{A}{u-\sqrt{\frac{a+b}{b-a}}} + \frac{B}{u+\sqrt{\frac{a+b}{b-a}}}\right]\mathrm{d}u \\
&= \frac{1}{a-b}\int \left[\frac{\sqrt{\frac{b-a}{a+b}}}{u-\sqrt{\frac{a+b}{b-a}}} - \frac{\sqrt{\frac{b-a}{a+b}}}{u+\sqrt{\frac{a+b}{b-a}}}\right]\mathrm{d}u \\
&= \frac{1}{b-a}\sqrt{\frac{b-a}{a+b}}\int \left[\frac{1}{u+\sqrt{\frac{a+b}{b-a}}} - \frac{1}{u-\sqrt{\frac{a+b}{b-a}}}\right]\mathrm{d}u \\
&= \frac{1}{a+b}\sqrt{\frac{a+b}{b-a}}\left[\ln\left|u+\sqrt{\frac{a+b}{b-a}}\right| - \ln\left|u-\sqrt{\frac{a+b}{b-a}}\right|\right] + C \\
&= \frac{1}{a+b}\sqrt{\frac{a+b}{b-a}} \ln \left| \frac{\tan\frac{x}{2} + \sqrt{\frac{a+b}{b-a}}}{\tan\frac{x}{2} - \sqrt{\frac{a+b}{b-a}}} \right| + C
\end{aligned}$$

■

$$\int \frac{\mathrm{d}x}{a^2\cos^2 x + b^2\sin^2 x} = \frac{1}{ab}\arctan\left(\frac{b}{a}\tan x\right) + C \tag{2.107}$$

Proof.

$$\begin{aligned}
\int \frac{\mathrm{d}x}{a^2\cos^2 x + b^2\sin^2 x} &= \int \frac{\sin^2 x + \cos^2 x}{a^2\cos^2 x + b^2\sin^2 x}\,\mathrm{d}x \\
&= \int \frac{\tan^2 x + 1}{a^2 + b^2\tan^2 x}\,\mathrm{d}x \\
\text{let } \ \tan x = u \text{ ,then } x &= \arctan u \\
&= \int \frac{u^2+1}{a^2+b^2u^2}\,\mathrm{d}(\arctan u) \\
&= \int \frac{1}{a^2+b^2u^2}\,\mathrm{d}u
\end{aligned}$$

$$\begin{aligned}
\text{let } u &= \frac{a}{b}v \text{ ,then } v = \frac{b}{a}u \\
&= \frac{1}{ab}\int \frac{1}{1+v^2}\,\mathrm{d}v \\
&= \frac{1}{ab}\arctan v + C \\
&= \frac{1}{ab}\arctan\left(\frac{b}{a}\tan x\right) + C
\end{aligned}$$

■

$$\int \frac{\mathrm{d}x}{a^2\cos^2 x - b^2\sin^2 x} = \frac{1}{2ab}\ln\left|\frac{b\tan x + a}{b\tan x - a}\right| + C \tag{2.108}$$

Proof.

$$\begin{aligned}
\int \frac{\mathrm{d}x}{a^2\cos^2 x - b^2\sin^2 x} &= \int \frac{\sin^2 x + \cos^2 x}{a^2\cos^2 x - b^2\sin^2 x}\,\mathrm{d}x \\
&= \int \frac{\tan^2 x + 1}{a^2 - b^2\tan^2 x}\,\mathrm{d}x \\
\text{let } \tan x &= u \text{ ,then } x = \arctan u \\
&= \int \frac{u^2+1}{a^2 - b^2u^2}\,\mathrm{d}(\arctan u) \\
&= \int \frac{1}{a^2 - b^2u^2}\,\mathrm{d}u \\
\text{let } u &= \frac{a}{b}v \text{ ,then } v = \frac{b}{a}u \\
&= \frac{1}{ab}\int \frac{1}{1-v^2}\,\mathrm{d}v \\
&= \frac{1}{2ab}\left(\ln|v-1| - \ln|v+1|\right) + C \\
&= \frac{1}{2ab}\ln\left|\frac{b\tan x + a}{b\tan x - a}\right| + C
\end{aligned}$$

■

$$\int x\sin ax\,\mathrm{d}x = \frac{1}{a^2}\sin ax - \frac{1}{a}x\cos ax + C \tag{2.109}$$

Proof.

$$\begin{aligned}
\int x\sin ax\,\mathrm{d}x &= -\frac{1}{a}x\cos ax + \frac{1}{a}\int \cos ax\,\mathrm{d}x \\
&= \frac{1}{a^2}\sin ax - \frac{1}{a}x\cos ax + C
\end{aligned}$$

■

$$\int x^2\sin ax\,\mathrm{d}x = -\frac{1}{a}x^2\cos ax + \frac{2}{a^2}x\sin ax + \frac{2}{a^3}\cos ax + C \tag{2.110}$$

Proof.

$$
\begin{aligned}
\int x^2 \sin ax \,\mathrm{d}x &= -\frac{1}{a}x^2 \cos ax + \frac{2}{a}\int x \cos ax \,\mathrm{d}x \\
&= -\frac{1}{a}x^2 \cos ax + \frac{2}{a^2}x \sin ax - \frac{2}{a^2}\int \sin ax \,\mathrm{d}x \\
&= -\frac{1}{a}x^2 \cos ax + \frac{2}{a^2}x \sin ax + \frac{2}{a^3}\cos ax + C
\end{aligned}
$$

■

$$
\int x \cos ax \,\mathrm{d}x = \frac{1}{a^2}\cos ax + \frac{1}{a}x \sin ax + C \tag{2.111}
$$

Proof.

$$
\begin{aligned}
\int x \cos ax \,\mathrm{d}x &= \frac{1}{a}x \sin ax - \frac{1}{a}\int \sin ax \,\mathrm{d}x \\
&= \frac{1}{a^2}\cos ax + \frac{1}{a}x \sin ax + C
\end{aligned}
$$

■

$$
\int x^2 \cos ax \,\mathrm{d}x = \frac{1}{a}x^2 \sin ax + \frac{2}{a^2}x \cos ax - \frac{2}{a^3}\sin ax + C \tag{2.112}
$$

Proof.

$$
\begin{aligned}
\int x^2 \cos ax \,\mathrm{d}x &= \frac{1}{a}x^2 \sin ax - \frac{2}{a}\int x \sin ax \,\mathrm{d}x \\
&= \frac{1}{a}x^2 \sin ax + \frac{2}{a^2}x \cos ax - \frac{2}{a^2}\int \cos ax \,\mathrm{d}x \\
&= \frac{1}{a}x^2 \sin ax + \frac{2}{a^2}x \cos ax - \frac{2}{a^3}\sin ax + C
\end{aligned}
$$

■

2.12 含有反三角函数的积分 $(a > 0)$

$$
\int \arcsin\frac{x}{a}\,\mathrm{d}x = x \arcsin\frac{x}{a} + \sqrt{a^2 - x^2} + C \tag{2.113}
$$

Proof.

$$
\begin{aligned}
\int \arcsin\frac{x}{a}\,\mathrm{d}x &= \int 1 \cdot \arcsin\frac{x}{a}\,\mathrm{d}x \\
&= x \arcsin\frac{x}{a} - \int x\left(\arcsin\frac{x}{a}\right)' \,\mathrm{d}x \\
&= x \arcsin\frac{x}{a} + \frac{1}{2}\int \frac{\mathrm{d}\left(a^2 - x^2\right)}{\sqrt{a^2 - x^2}} \\
&= x \arcsin\frac{x}{a} + \sqrt{a^2 - x^2} + C
\end{aligned}
$$

■

$$\int x \arcsin\frac{x}{a}\,\mathrm{d}x = \left(\frac{x^2}{2}-\frac{a^2}{4}\right)\arcsin\frac{x}{a}+\frac{x}{4}\sqrt{a^2-x^2}+C \tag{2.114}$$

Proof.

$$\begin{aligned}
\int x \arcsin\frac{x}{a}\,\mathrm{d}x &= \frac{x^2}{2}\arcsin\frac{x}{a}-\frac{1}{2}\int x^2\left(\arcsin\frac{x}{a}\right)'\,\mathrm{d}x\\
&= \frac{x^2}{2}\arcsin\frac{x}{a}-\frac{1}{2}\int\frac{x^2}{\sqrt{a^2-x^2}}\,\mathrm{d}x\\
\textbf{use}\ \int\frac{x^2}{\sqrt{a^2-x^2}}\,\mathrm{d}x &= -\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}+C\\
&= \left(\frac{x^2}{2}-\frac{a^2}{4}\right)\arcsin\frac{x}{a}+\frac{x}{4}\sqrt{a^2-x^2}+C
\end{aligned}$$

■

$$\int x^2 \arcsin\frac{x}{a}\,\mathrm{d}x = \frac{x^3}{3}\arcsin\frac{x}{a}+\frac{1}{9}\left(x^2+2a^2\right)\sqrt{a^2-x^2}+C \tag{2.115}$$

Proof.

$$\begin{aligned}
\int x^2 \arcsin\frac{x}{a}\,\mathrm{d}x &= \frac{x^3}{3}\arcsin\frac{x}{a}-\frac{1}{3}\int x^3\left(\arcsin\frac{x}{a}\right)'\,\mathrm{d}x\\
&= \frac{x^3}{3}\arcsin\frac{x}{a}-\frac{1}{3}\int\frac{x^3}{\sqrt{a^2-x^2}}\,\mathrm{d}x\\
\text{let } u &= x^2\\
&= \frac{x^3}{3}\arcsin\frac{x}{a}-\frac{1}{6}\int\frac{u}{\sqrt{a^2-u}}\,\mathrm{d}u\\
\text{let } v &= \sqrt{a^2-u}\text{ ,then } u=a^2-v^2\\
&= \frac{x^3}{3}\arcsin\frac{x}{a}+\frac{1}{3}\int\left[a^2-v^2\right]\,\mathrm{d}v\\
&= \frac{x^3}{3}\arcsin\frac{x}{a}+\frac{a^2}{3}\sqrt{a^2-x^2}-\frac{a^2-x^2}{9}\sqrt{a^2-x^2}+C\\
&= \frac{x^3}{3}\arcsin\frac{x}{a}+\frac{1}{9}\left(x^2+2a^2\right)\sqrt{a^2-x^2}+C
\end{aligned}$$

■

$$\int \arccos\frac{x}{a}\,\mathrm{d}x = x\arccos\frac{x}{a}-\sqrt{a^2-x^2}+C \tag{2.116}$$

Proof.

$$\begin{aligned}
\int \arccos\frac{x}{a}\,\mathrm{d}x &= \int 1\cdot\arccos\frac{x}{a}\,\mathrm{d}x\\
&= x\arccos\frac{x}{a}-\int x\left(\arccos\frac{x}{a}\right)'\,\mathrm{d}x\\
&= x\arccos\frac{x}{a}-\frac{1}{2}\int\frac{\mathrm{d}\left(a^2-x^2\right)}{\sqrt{a^2-x^2}}\\
&= x\arccos\frac{x}{a}-\sqrt{a^2-x^2}+C
\end{aligned}$$

■

$$\int x \arccos \frac{x}{a} \, \mathrm{d}x = \left(\frac{x^2}{2} - \frac{a^2}{4} \right) \arccos \frac{x}{a} - \frac{x}{4} \sqrt{a^2 - x^2} + C \tag{2.117}$$

Proof.

$$\begin{aligned}
\int x \arccos \frac{x}{a} \, \mathrm{d}x &= \frac{x^2}{2} \arccos \frac{x}{a} - \frac{1}{2} \int x^2 \left(\arccos \frac{x}{a} \right)' \, \mathrm{d}x \\
&= \frac{x^2}{2} \arccos \frac{x}{a} + \frac{1}{2} \int \frac{x^2}{\sqrt{a^2 - x^2}} \, \mathrm{d}x \\
\textbf{use} \int \frac{x^2}{\sqrt{a^2 - x^2}} \, \mathrm{d}x &= -\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} + C \\
&= \left(\frac{x^2}{2} - \frac{a^2}{4} \right) \arccos \frac{x}{a} - \frac{x}{4} \sqrt{a^2 - x^2} + C
\end{aligned}$$

■

$$\int x^2 \arccos \frac{x}{a} \, \mathrm{d}x = \frac{x^3}{3} \arccos \frac{x}{a} - \frac{1}{9} \left(x^2 + 2a^2 \right) \sqrt{a^2 - x^2} + C \tag{2.118}$$

Proof.

$$\begin{aligned}
\int x^2 \arccos \frac{x}{a} \, \mathrm{d}x &= \frac{x^3}{3} \arccos \frac{x}{a} - \frac{1}{3} \int x^3 \left(\arccos \frac{x}{a} \right)' \, \mathrm{d}x \\
&= \frac{x^3}{3} \arccos \frac{x}{a} + \frac{1}{3} \int \frac{x^3}{\sqrt{a^2 - x^2}} \, \mathrm{d}x \\
\text{let } u = x^2 & \\
&= \frac{x^3}{3} \arccos \frac{x}{a} + \frac{1}{6} \int \frac{u}{\sqrt{a^2 - u}} \, \mathrm{d}u \\
\text{let } v = \sqrt{a^2 - u} \text{ ,then } u = a^2 - v^2 & \\
&= \frac{x^3}{3} \arccos \frac{x}{a} - \frac{1}{3} \int \left[a^2 - v^2 \right] \, \mathrm{d}v \\
&= \frac{x^3}{3} \arccos \frac{x}{a} - \frac{a^2}{3} \sqrt{a^2 - x^2} + \frac{a^2 - x^2}{9} \sqrt{a^2 - x^2} + C \\
&= \frac{x^3}{3} \arccos \frac{x}{a} - \frac{1}{9} \left(x^2 + 2a^2 \right) \sqrt{a^2 - x^2} + C
\end{aligned}$$

■

$$\int \arctan \frac{x}{a} \, \mathrm{d}x = x \arctan \frac{x}{a} - \frac{a}{2} \ln \left(a^2 + x^2 \right) + C \tag{2.119}$$

Proof.

$$\begin{aligned}
\int \arctan \frac{x}{a} \, \mathrm{d}x &= \int 1 \cdot \arctan \frac{x}{a} \, \mathrm{d}x \\
&= x \arctan \frac{x}{a} - \int x \left(\arctan \frac{x}{a} \right)' \, \mathrm{d}x \\
&= x \arctan \frac{x}{a} - \frac{a}{2} \int \frac{\mathrm{d} \left(a^2 + x^2 \right)}{a^2 + x^2} \\
&= x \arctan \frac{x}{a} - \frac{a}{2} \ln \left(a^2 + x^2 \right) + C
\end{aligned}$$

■

$$\int x \arctan\frac{x}{a}\,\mathrm{d}x = \frac{1}{2}\left(a^2+x^2\right)\arctan\frac{x}{a} - \frac{a}{2}x + C \tag{2.120}$$

Proof.

$$\begin{aligned}
\int x \arctan\frac{x}{a}\,\mathrm{d}x &= \frac{x^2}{2}\arctan\frac{x}{a} - \frac{1}{2}\int x^2\left(\arctan\frac{x}{a}\right)'\,\mathrm{d}x \\
&= \frac{x^2}{2}\arctan\frac{x}{a} - \frac{a}{2}\int \frac{x^2}{a^2+x^2}\,\mathrm{d}x \\
&= \frac{x^2}{2}\arctan\frac{x}{a} - \frac{a}{2}\int \left(1-\frac{a^2}{a^2+x^2}\right)\,\mathrm{d}x \\
&= \frac{x^2}{2}\arctan\frac{x}{a} - \frac{ax}{2} + \frac{a^3}{2}\int \frac{1}{a^2+x^2}\,\mathrm{d}x \\
\textbf{use } \int \frac{\mathrm{d}x}{x^2+a^2} &= \frac{1}{a}\arctan\frac{x}{a} + C \\
&= \frac{1}{2}\left(a^2+x^2\right)\arctan\frac{x}{a} - \frac{a}{2}x + C
\end{aligned}$$

■

$$\int x^2 \arctan\frac{x}{a}\,\mathrm{d}x = \frac{x^3}{3}\arctan\frac{x}{a} - \frac{a}{6}x^2 + \frac{a^3}{6}\ln\left(a^2+x^2\right) + C \tag{2.121}$$

Proof.

$$\begin{aligned}
\int x^2 \arctan\frac{x}{a}\,\mathrm{d}x &= \frac{x^3}{3}\arctan\frac{x}{a} - \frac{1}{3}\int x^3\left(\arctan\frac{x}{a}\right)'\,\mathrm{d}x \\
&= \frac{x^3}{3}\arctan\frac{x}{a} - \frac{a}{3}\int \frac{x^3}{a^2+x^2}\,\mathrm{d}x \\
\text{let } u &= x^2 \\
&= \frac{x^3}{3}\arctan\frac{x}{a} - \frac{a}{6}\int \frac{u}{a^2+u}\,\mathrm{d}u \\
&= \frac{x^3}{3}\arctan\frac{x}{a} - \frac{a}{6}\int \left(1-\frac{a^2}{a^2+u}\right)\,\mathrm{d}u \\
&= \frac{x^3}{3}\arctan\frac{x}{a} - \frac{a}{6}x^2 + \frac{a^3}{6}\int \frac{1}{a^2+u}\,\mathrm{d}u \\
&= \frac{x^3}{3}\arctan\frac{x}{a} - \frac{a}{6}x^2 + \frac{a^3}{6}\ln\left(a^2+u\right) + C \\
&= \frac{x^3}{3}\arctan\frac{x}{a} - \frac{a}{6}x^2 + \frac{a^3}{6}\ln\left(a^2+x^2\right) + C
\end{aligned}$$

■

2.13 含有指数函数的积分

$$\int a^x\,\mathrm{d}x = \frac{1}{\ln a}a^x + C \tag{2.122}$$

Proof.

$$\begin{aligned}\int a^x \,\mathrm{d}x &= \int \frac{1}{\ln a} \,\mathrm{d}\left(a^x\right) \\ &= \frac{1}{\ln a} a^x + C\end{aligned}$$

■

$$\int \mathrm{e}^{ax} \,\mathrm{d}x = \frac{1}{a} \mathrm{e}^{ax} + C \tag{2.123}$$

Proof.

$$\begin{aligned}\int \mathrm{e}^{ax} \,\mathrm{d}x &= \int \frac{1}{a} \,\mathrm{d}\left(\mathrm{e}^{ax}\right) \\ &= \frac{1}{a} \mathrm{e}^{ax} + C\end{aligned}$$

■

$$\int x \,\mathrm{e}^{ax} \,\mathrm{d}x = \frac{1}{a^2}(ax-1)\,\mathrm{e}^{ax} + C \tag{2.124}$$

Proof.

$$\begin{aligned}\int x \,\mathrm{e}^{ax} \,\mathrm{d}x &= \frac{1}{a} x \,\mathrm{e}^{ax} - \frac{1}{a} \int \mathrm{e}^{ax} \,\mathrm{d}x \\ &= \frac{1}{a} x \,\mathrm{e}^{ax} - \frac{1}{a^2} \mathrm{e}^{ax} + C \\ &= \frac{1}{a^2}(ax-1)\,\mathrm{e}^{ax} + C\end{aligned}$$

■

$$\int x^n \,\mathrm{e}^{ax} \,\mathrm{d}x = \frac{1}{a} x^n \,\mathrm{e}^{ax} - \frac{n}{a} \int x^{n-1} \,\mathrm{e}^{ax} \,\mathrm{d}x \tag{2.125}$$

Proof.

$$\begin{aligned}\int x^n \,\mathrm{e}^{ax} \,\mathrm{d}x &= \frac{1}{a} x^n \,\mathrm{e}^{ax} - \frac{1}{a} \int \left(x^n\right)' \,\mathrm{e}^{ax} \,\mathrm{d}x \\ &= \frac{1}{a} x^n \,\mathrm{e}^{ax} - \frac{n}{a} \int x^{n-1} \,\mathrm{e}^{ax} \,\mathrm{d}x\end{aligned}$$

■

$$\int x a^x \,\mathrm{d}x = \frac{x}{\ln a} a^x - \frac{1}{(\ln a)^2} a^x + C \tag{2.126}$$

Proof.

$$\begin{aligned}\int x a^x \,\mathrm{d}x &= \frac{1}{\ln a} x a^x - \frac{1}{\ln a} \int a^x \,\mathrm{d}x \\ &= \frac{x}{\ln a} a^x - \frac{1}{(\ln a)^2} a^x + C\end{aligned}$$

■

$$\int x^n a^x \,\mathrm{d}x = \frac{1}{\ln a} x^n a^x - \frac{n}{\ln a}\int x^{n-1} a^x \,\mathrm{d}x \tag{2.127}$$

Proof.

$$\begin{aligned}\int x^n a^x \,\mathrm{d}x &= \frac{1}{\ln a} x^n a^x - \frac{1}{\ln a}\int \left(x^n\right)' a^x \,\mathrm{d}x \\ &= \frac{1}{\ln a} x^n a^x - \frac{n}{\ln a}\int x^{n-1} a^x \,\mathrm{d}x\end{aligned}$$

■

$$\int \mathrm{e}^{ax} \sin bx \,\mathrm{d}x = \frac{1}{a^2+b^2}\,\mathrm{e}^{ax}\left(a\sin bx - b\cos bx\right) + C \tag{2.128}$$

Proof.

$$\begin{aligned}I = \int \mathrm{e}^{ax}\sin bx \,\mathrm{d}x &= \frac{1}{a}\,\mathrm{e}^{ax}\sin bx - \frac{b}{a}\int \mathrm{e}^{ax}\cos bx \,\mathrm{d}x \\ &= \frac{1}{a}\,\mathrm{e}^{ax}\sin bx - \frac{b}{a^2}\,\mathrm{e}^{ax}\cos bx - \frac{b^2}{a^2}\int \mathrm{e}^{ax}\sin bx \,\mathrm{d}x \\ &= \frac{1}{a}\,\mathrm{e}^{ax}\sin bx - \frac{b}{a^2}\,\mathrm{e}^{ax}\cos bx - \frac{b^2}{a^2} I \\ I &= \frac{1}{a^2+b^2}\,\mathrm{e}^{ax}\left(a\sin bx - b\cos bx\right) + C\end{aligned}$$

■

$$\int \mathrm{e}^{ax} \cos bx \,\mathrm{d}x = \frac{1}{a^2+b^2}\,\mathrm{e}^{ax}\left(b\sin bx + a\cos bx\right) + C \tag{2.129}$$

Proof.

$$\begin{aligned}I = \int \mathrm{e}^{ax}\cos bx \,\mathrm{d}x &= \frac{1}{a}\,\mathrm{e}^{ax}\cos bx + \frac{b}{a}\int \mathrm{e}^{ax}\sin bx \,\mathrm{d}x \\ &= \frac{1}{a}\,\mathrm{e}^{ax}\cos bx + \frac{b}{a^2}\,\mathrm{e}^{ax}\sin bx - \frac{b^2}{a^2}\int \mathrm{e}^{ax}\cos bx \,\mathrm{d}x \\ &= \frac{1}{a}\,\mathrm{e}^{ax}\cos bx + \frac{b}{a^2}\,\mathrm{e}^{ax}\sin bx - \frac{b^2}{a^2} I \\ I &= \frac{1}{a^2+b^2}\,\mathrm{e}^{ax}\left(b\sin bx + a\cos bx\right) + C\end{aligned}$$

■

$$\begin{aligned}\int \mathrm{e}^{ax}\sin^n bx \,\mathrm{d}x = \frac{1}{a^2+b^2n^2}\,\mathrm{e}^{ax}\sin^{n-1} bx\left(a\sin bx - nb\cos bx\right) \\ + \frac{n(n-1)b^2}{a^2+b^2n^2}\int \mathrm{e}^{ax}\sin^{n-2} bx \,\mathrm{d}x\end{aligned} \tag{2.130}$$

Proof.

$$
\begin{aligned}
I_n &= \int \mathrm{e}^{ax} \sin^n bx \, \mathrm{d}x = \frac{1}{a} \mathrm{e}^{ax} \sin^n bx - \frac{bn}{a} \int \mathrm{e}^{ax} \sin^{n-1} bx \cos bx \, \mathrm{d}x \\
&= \frac{1}{a} \mathrm{e}^{ax} \sin^n bx - \frac{bn}{a^2} \mathrm{e}^{ax} \sin^{n-1} bx \cos bx + \frac{bn}{a^2} \int \mathrm{e}^{ax} \left(\sin^{n-1} bx \cos bx\right)' \, \mathrm{d}x \\
&= \frac{1}{a} \mathrm{e}^{ax} \sin^n bx - \frac{bn}{a^2} \mathrm{e}^{ax} \sin^{n-1} bx \cos bx \\
&+ \frac{b^2 n}{a^2} \int \mathrm{e}^{ax} \left[(n-1) \sin^{n-2} bx \cos^2 bx - \sin^n bx\right] \, \mathrm{d}x \\
&= \frac{1}{a} \mathrm{e}^{ax} \sin^n bx - \frac{bn}{a^2} \mathrm{e}^{ax} \sin^{n-1} bx \cos bx \\
&+ \frac{b^2 n}{a^2} \int \mathrm{e}^{ax} \left[(n-1) \sin^{n-2} bx \left(1 - sin^2 bx\right) - \sin^n bx\right] \, \mathrm{d}x \\
&= \frac{1}{a} \mathrm{e}^{ax} \sin^n bx - \frac{bn}{a^2} \mathrm{e}^{ax} \sin^{n-1} bx \cos bx \\
&+ \frac{b^2 n(n-1)}{a^2} \int \mathrm{e}^{ax} \sin^{n-2} bx \, \mathrm{d}x - \frac{b^2 n^2}{a^2} \int \mathrm{e}^{ax} \sin^n bx \, \mathrm{d}x \\
I_n &= \frac{1}{a} \mathrm{e}^{ax} \sin^n bx - \frac{bn}{a^2} \mathrm{e}^{ax} \sin^{n-1} bx \cos bx + \frac{b^2 n(n-1)}{a^2} I_{n-2} - \frac{b^2 n^2}{a^2} I_n \\
&= \frac{1}{a^2 + b^2 n^2} \mathrm{e}^{ax} \sin^{n-1} bx \left(a \sin bx - nb \cos bx\right) \\
&+ \frac{n(n-1) b^2}{a^2 + b^2 n^2} \int \mathrm{e}^{ax} \sin^{n-2} bx \, \mathrm{d}x
\end{aligned}
$$

■

$$
\begin{aligned}
\int \mathrm{e}^{ax} \cos^n bx \, \mathrm{d}x &= \frac{1}{a^2 + b^2 n^2} \mathrm{e}^{ax} \cos^{n-1} bx \left(a \cos bx + nb \sin bx\right) \\
&+ \frac{n(n-1) b^2}{a^2 + b^2 n^2} \int \mathrm{e}^{ax} \cos^{n-2} bx \, \mathrm{d}x
\end{aligned}
\tag{2.131}
$$

Proof.

$$
\begin{aligned}
I_n &= \int \mathrm{e}^{ax} \cos^n bx \, \mathrm{d}x = \frac{1}{a} \mathrm{e}^{ax} \cos^n bx + \frac{bn}{a} \int \mathrm{e}^{ax} \cos^{n-1} bx \sin bx \, \mathrm{d}x \\
&= \frac{1}{a} \mathrm{e}^{ax} \cos^n bx + \frac{bn}{a^2} \mathrm{e}^{ax} \cos^{n-1} bx \sin bx - \frac{bn}{a^2} \int \mathrm{e}^{ax} \left(\cos^{n-1} bx \sin bx\right)' \, \mathrm{d}x \\
&= \frac{1}{a} \mathrm{e}^{ax} \sin^n bx + \frac{bn}{a^2} \mathrm{e}^{ax} \cos^{n-1} bx \sin bx \\
&- \frac{b^2 n}{a^2} \int \mathrm{e}^{ax} \left[-(n-1) \cos^{n-2} bx \sin^2 bx + \cos^n bx\right] \, \mathrm{d}x \\
&= \frac{1}{a} \mathrm{e}^{ax} \cos^n bx + \frac{bn}{a^2} \mathrm{e}^{ax} \cos^{n-1} bx \sin bx \\
&- \frac{b^2 n}{a^2} \int \mathrm{e}^{ax} \left[-(n-1) \cos^{n-2} bx \left(1 - cos^2 bx\right) + \cos^n bx\right] \, \mathrm{d}x \\
&= \frac{1}{a} \mathrm{e}^{ax} \cos^n bx + \frac{bn}{a^2} \mathrm{e}^{ax} \cos^{n-1} bx \sin bx \\
&+ \frac{b^2 n(n-1)}{a^2} \int \mathrm{e}^{ax} \cos^{n-2} bx \, \mathrm{d}x - \frac{b^2 n^2}{a^2} \int \mathrm{e}^{ax} \cos^n bx \, \mathrm{d}x \\
I_n &= \frac{1}{a} \mathrm{e}^{ax} \cos^n bx + \frac{bn}{a^2} \mathrm{e}^{ax} \cos^{n-1} bx \sin bx + \frac{b^2 n(n-1)}{a^2} I_{n-2} - \frac{b^2 n^2}{a^2} I_n
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{a^2+b^2n^2}\,\mathrm{e}^{ax}\cos^{n-1}bx\,(a\cos bx + nb\sin bx) \\
&+ \frac{n(n-1)b^2}{a^2+b^2n^2}\int \mathrm{e}^{ax}\cos^{n-2}bx\,\mathrm{d}x
\end{aligned}
$$

■

2.14 含有对数函数的积分

$$
\int \ln x\,\mathrm{d}x = x\ln x - x + C \tag{2.132}
$$

Proof.

$$
\begin{aligned}
\int \ln x\,\mathrm{d}x &= \int 1\cdot\ln x\,\mathrm{d}x \\
&= x\ln x - \int \mathrm{d}x \\
&= x\ln x - x + C
\end{aligned}
$$

■

$$
\int \frac{\mathrm{d}x}{x\ln x} = \ln|\ln x| + C \tag{2.133}
$$

Proof.

$$
\begin{aligned}
\int \frac{\mathrm{d}x}{x\ln x} &= \int \frac{\mathrm{d}(\ln x)}{\ln x} \\
&= \ln|\ln x| + C
\end{aligned}
$$

■

$$
\int x^n\ln x\,\mathrm{d}x = \frac{1}{n+1}x^{n+1}\left(\ln x - \frac{1}{n+1}\right) + C \tag{2.134}
$$

Proof.

$$
\begin{aligned}
\int x^n\ln x\,\mathrm{d}x &= \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{n+1}\int x^n\,\mathrm{d}x \\
&= \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{(n+1)^2}x^{n+1} + C \\
&= \frac{1}{n+1}x^{n+1}\left(\ln x - \frac{1}{n+1}\right) + C
\end{aligned}
$$

■

$$
\int (\ln x)^n\,\mathrm{d}x = x(\ln x)^n - n\int(\ln x)^{n-1}\,\mathrm{d}x \tag{2.135}
$$

Proof.

$$\begin{aligned}\int(\ln x)^n \,\mathrm{d}x &= \int 1\cdot(\ln x)^n \,\mathrm{d}x\\ &= x(\ln x)^n - \int x\left[(\ln x)^n\right]' \,\mathrm{d}x\\ &= x(\ln x)^n - n\int(\ln x)^{n-1} \,\mathrm{d}x\end{aligned}$$

■

$$\int x^m(\ln x)^n \,\mathrm{d}x = \frac{1}{m+1}x^{m+1}(\ln x)^n - \frac{n}{m+1}\int x^m(\ln x)^{n-1} \,\mathrm{d}x \tag{2.136}$$

Proof.

$$\begin{aligned}\int x^m(\ln x)^n \,\mathrm{d}x &= \frac{1}{m+1}x^{m+1}(\ln x)^n - \frac{1}{m+1}\int x^{m+1}\left[(\ln x)^n\right]' \,\mathrm{d}x\\ &= \frac{1}{m+1}x^{m+1}(\ln x)^n - \frac{n}{m+1}\int x^m(\ln x)^{n-1} \,\mathrm{d}x\end{aligned}$$

■

2.15 含有双曲函数的积分

$$\int \sinh x \,\mathrm{d}x = \cosh x + C \tag{2.137}$$

Proof.

$$\begin{aligned}\int \sinh x \,\mathrm{d}x &= \int \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} \,\mathrm{d}x\\ &= \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2} + C\\ &= \cosh x + C\end{aligned}$$

■

$$\int \cosh x \,\mathrm{d}x = \sinh x + C \tag{2.138}$$

Proof.

$$\begin{aligned}\int \cosh x \,\mathrm{d}x &= \int \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2} \,\mathrm{d}x\\ &= \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} + C\\ &= \sinh x + C\end{aligned}$$

■

$$\int \tanh x \,\mathrm{d}x = \ln\cosh x + C \tag{2.139}$$

Proof.

$$\begin{aligned}\int \tanh x \, \mathrm{d}x &= \int \frac{\sinh x}{\cosh x} \, \mathrm{d}x \\ &= \int \frac{\mathrm{d}(\cosh x)}{\cosh x} \\ &= \ln \cosh x + C\end{aligned}$$

■

$$\int \sinh^2 x \, \mathrm{d}x = -\frac{x}{2} + \frac{1}{4} \sinh 2x + C \tag{2.140}$$

Proof.

$$\begin{aligned}\int \sinh^2 x \, \mathrm{d}x &= \int \frac{(\mathrm{e}^x - \mathrm{e}^{-x})^2}{4} \, \mathrm{d}x \\ &= \int \frac{\mathrm{e}^{2x} + \mathrm{e}^{-2x} - 2}{4} \, \mathrm{d}x \\ &= -\frac{x}{2} + \frac{1}{4} \frac{\mathrm{e}^{2x} - \mathrm{e}^{-2x}}{2} + C \\ &= -\frac{x}{2} + \frac{1}{4} \sinh 2x + C\end{aligned}$$

■

$$\int \cosh^2 x \, \mathrm{d}x = \frac{x}{2} + \frac{1}{4} \sinh 2x + C \tag{2.141}$$

Proof.

$$\begin{aligned}\int \cosh^2 x \, \mathrm{d}x &= \int \frac{(\mathrm{e}^x + \mathrm{e}^{-x})^2}{4} \, \mathrm{d}x \\ &= \int \frac{\mathrm{e}^{2x} + \mathrm{e}^{-2x} + 2}{4} \, \mathrm{d}x \\ &= \frac{x}{2} + \frac{1}{4} \frac{\mathrm{e}^{2x} - \mathrm{e}^{-2x}}{2} + C \\ &= \frac{x}{2} + \frac{1}{4} \sinh 2x + C\end{aligned}$$

■

2.16 定积分 $(m, n \in \mathbb{Z})$

$$\int_{-\pi}^{\pi} \cos nx \, \mathrm{d}x = \int_{-\pi}^{\pi} \sin nx \, \mathrm{d}x = 0 \tag{2.142}$$

Proof.

$$\int_{-\pi}^{\pi} \cos nx \, \mathrm{d}x = \left[\frac{1}{n} \sin nx\right]_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi} \sin nx \, \mathrm{d}x = \left[-\frac{1}{n} \cos nx\right]_{-\pi}^{\pi} = 0$$

■

$$\int_{-\pi}^{\pi} \cos mx \sin nx \,\mathrm{d}x = 0 \tag{2.143}$$

Proof.

$$\begin{aligned}
f(x) &= \cos mx \sin nx \\
f(-x) &= \cos(-mx)\sin(-nx) = -f(x) \\
f(x) \text{ is an odd fanction } \int_{-a}^{a} f(x) \,\mathrm{d}x &= 0 \\
\int_{-\pi}^{\pi} \cos mx \sin nx \,\mathrm{d}x &= 0
\end{aligned}$$

■

$$\int_{-\pi}^{\pi} \cos mx \cos nx \,\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases} \tag{2.144}$$

Proof.

$$\begin{aligned}
&\text{if } m \neq n \\
\int_{-\pi}^{\pi} \cos mx \cos nx \,\mathrm{d}x &= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m-n)x + \cos(m+n)x] \,\mathrm{d}x \\
&= \frac{1}{2}\left[\frac{1}{m-n}\sin(m-n)x\right]_{-\pi}^{\pi} + \frac{1}{2}\left[\frac{1}{m+n}\sin(m+n)x\right]_{-\pi}^{\pi} \\
&= \frac{\sin\pi(m-n)}{m-n} + \frac{\sin\pi(m+n)}{m+n} = 0 \\
&\text{if } m = n \\
\int_{-\pi}^{\pi} \cos mx \cos nx \,\mathrm{d}x &= \int_{-\pi}^{\pi} \cos^2 mx \,\mathrm{d}x = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(2mx) + 1] \,\mathrm{d}x \\
&= \frac{1}{2}\left[\frac{1}{2m}\sin(2mx) + x\right]_{-\pi}^{\pi} = \pi
\end{aligned}$$

■

$$\int_{-\pi}^{\pi} \sin mx \sin nx \,\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases} \tag{2.145}$$

Proof.

$$
\begin{aligned}
&\text{if } m \neq n \\
\int_{-\pi}^{\pi} \sin mx \sin nx \, \mathrm{d}x &= \frac{1}{2}\int_{-\pi}^{\pi} [\cos{(m-n)x} - \cos{(m+n)x}] \, \mathrm{d}x \\
&= \frac{1}{2}\left[\frac{1}{m-n}\sin{(m-n)x}\right]_{-\pi}^{\pi} - \frac{1}{2}\left[\frac{1}{m+n}\sin{(m+n)x}\right]_{-\pi}^{\pi} \\
&= \frac{\sin \pi(m-n)}{m-n} - \frac{\sin \pi(m+n)}{m+n} = 0 \\
&\text{if } m = n \\
\int_{-\pi}^{\pi} \sin mx \sin nx \, \mathrm{d}x &= \int_{-\pi}^{\pi} \sin^2 mx \, \mathrm{d}x = \frac{1}{2}\int_{-\pi}^{\pi} [1 - \cos{(2mx)}] \, \mathrm{d}x \\
&= \frac{1}{2}\left[x - \frac{1}{2m}\sin(2mx)\right]_{-\pi}^{\pi} = \pi
\end{aligned}
$$

■

$$
\int_{0}^{\pi} \sin mx \sin nx \, \mathrm{d}x = \int_{0}^{\pi} \cos mx \cos nx \, \mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{\pi}{2} & \text{if } m = n \end{cases} \tag{2.146}
$$

Proof.

$$
\begin{aligned}
&\text{if } m \neq n \\
\int_{0}^{\pi} \sin mx \sin nx \, \mathrm{d}x &= \frac{1}{2}\int_{0}^{\pi} [\cos{(m-n)x} - \cos{(m+n)x}] \, \mathrm{d}x \\
&= \frac{1}{2}\left[\frac{1}{m-n}\sin{(m-n)x}\right]_{0}^{\pi} - \frac{1}{2}\left[\frac{1}{m+n}\sin{(m+n)x}\right]_{0}^{\pi} \\
&= \frac{1}{2}\frac{\sin \pi(m-n)}{m-n} - \frac{1}{2}\frac{\sin \pi(m+n)}{m+n} = 0 \\
\int_{0}^{\pi} \cos mx \cos nx \, \mathrm{d}x &= \frac{1}{2}\int_{0}^{\pi} [\cos{(m-n)x} + \cos{(m+n)x}] \, \mathrm{d}x \\
&= \frac{1}{2}\left[\frac{1}{m-n}\sin{(m-n)x}\right]_{0}^{\pi} + \frac{1}{2}\left[\frac{1}{m+n}\sin{(m+n)x}\right]_{0}^{\pi} \\
&= \frac{1}{2}\frac{\sin \pi(m-n)}{m-n} + \frac{1}{2}\frac{\sin \pi(m+n)}{m+n} = 0 \\
&\text{if } m = n \\
\int_{0}^{\pi} \sin mx \sin nx \, \mathrm{d}x &= \int_{0}^{\pi} \sin^2 mx \, \mathrm{d}x = \frac{1}{2}\int_{0}^{\pi} [1 - \cos{(2mx)}] \, \mathrm{d}x \\
&= \frac{1}{2}\left[x - \frac{1}{2m}\sin(2mx)\right]_{0}^{\pi} = \frac{\pi}{2} \\
\int_{0}^{\pi} \cos mx \cos nx \, \mathrm{d}x &= \int_{0}^{\pi} \cos^2 mx \, \mathrm{d}x = \frac{1}{2}\int_{0}^{\pi} [\cos{(2mx)} + 1] \, \mathrm{d}x \\
&= \frac{1}{2}\left[\frac{1}{2m}\sin(2mx) + x\right]_{0}^{\pi} = \frac{\pi}{2}
\end{aligned}
$$

■

$$
\begin{aligned}
I_n &= \int_0^{\frac{\pi}{2}} \sin^n x \,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^n x \,\mathrm{d}x \\
I_n &= \frac{n-1}{n} I_{n-2} = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \ldots \cdot \frac{4}{5} \cdot \frac{2}{3} & \text{if } \{x = 2n+1, n > 0\} I_1 = 1 \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \ldots \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} & \text{if } \{x = 2n, n > 0\} I_0 = \frac{\pi}{2} \end{cases}
\end{aligned} \tag{2.147}
$$

***Proof*.**

$$
\begin{aligned}
I_n &= \int_0^{\frac{\pi}{2}} \sin^n x \,\mathrm{d}x \\
&= \left[-\cos x \sin^{n-1} x\right]_0^{\frac{\pi}{2}} + (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \,\mathrm{d}x \\
&= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \left(1 - \sin^2 x\right) \,\mathrm{d}x \\
&= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \,\mathrm{d}x - (n-1) \int_0^{\frac{\pi}{2}} \sin^n x \,\mathrm{d}x \\
&= (n-1) I_{n-2} - (n-1) I_n \\
I_n &= \frac{n-1}{n} I_{n-2} \\
I_{2m} &= \frac{2m-1}{2m} \cdot \frac{2m-3}{2m-2} \cdot \ldots \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} I_0 \\
I_{2m+1} &= \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} \cdot \ldots \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} I_1 \\
I_0 &= \int_0^{\frac{\pi}{2}} \,\mathrm{d}x = \frac{\pi}{2}, I_1 = \int_0^{\frac{\pi}{2}} \sin x \,\mathrm{d}x = 1 \\
\text{let } x &= \frac{\pi}{2} - t \\
\int_0^{\frac{\pi}{2}} \sin^n x \,\mathrm{d}x &= -\int_{\frac{\pi}{2}}^0 \sin^n \left(\frac{\pi}{2} - t\right) \,\mathrm{d}t \\
&= \int_0^{\frac{\pi}{2}} \cos^n t \,\mathrm{d}t = \int_0^{\frac{\pi}{2}} \cos^n x \,\mathrm{d}x
\end{aligned}
$$

■

3 在证明中使用的引理及其证明

$$
\arctan \frac{x}{a} = \arcsin \sqrt{\frac{x^2}{x^2 + a^2}} \tag{3.1}
$$

Proof.

$$\begin{aligned} u &= \arctan\frac{x}{a} \\ \tan u &= \frac{x}{a} \\ \tan^2 u + 1 &= \frac{x^2}{a^2} + 1 \\ \sec^2 u &= \frac{x^2+a^2}{a^2} \\ 1 - \cos^2 u &= 1 - \frac{a^2}{x^2+a^2} \\ \sin u &= \sqrt{\frac{x^2}{x^2+a^2}} \\ \arctan\frac{x}{a} &= u = \arcsin\sqrt{\frac{x^2}{x^2+a^2}} \end{aligned}$$

■

$$\arctan\frac{x}{a} = \arccos\sqrt{\frac{a^2}{x^2+a^2}} \tag{3.2}$$

Proof.

$$\begin{aligned} u &= \arctan\frac{x}{a} \\ \tan u &= \frac{x}{a} \\ \tan^2 u + 1 &= \frac{x^2}{a^2} + 1 \\ \sec^2 u &= \frac{x^2+a^2}{a^2} \\ \cos u &= \sqrt{\frac{a^2}{x^2+a^2}} \\ \arctan\frac{x}{a} &= u = \arccos\sqrt{\frac{a^2}{x^2+a^2}} \end{aligned}$$

■

$$\arctan\frac{x}{a} = \operatorname{arcsec}\sqrt{\frac{x^2+a^2}{a^2}} \tag{3.3}$$

Proof.

$$
\begin{aligned}
u &= \arctan \frac{x}{a} \\
\tan u &= \frac{x}{a} \\
\tan^2 u + 1 &= \frac{x^2}{a^2} + 1 \\
\sec^2 u &= \frac{x^2 + a^2}{a^2} \\
\sec u &= \sqrt{\frac{x^2 + a^2}{a^2}} \\
\arctan \frac{x}{a} &= u = \operatorname{arcsec} \sqrt{\frac{x^2 + a^2}{a^2}}
\end{aligned}
$$

■

$$
\arctan \frac{x}{a} = \operatorname{arccsc} \sqrt{\frac{x^2 + a^2}{x^2}} \tag{3.4}
$$

Proof.

$$
\begin{aligned}
u &= \arctan \frac{x}{a} \\
\tan u &= \frac{x}{a} \\
\cot^2 u + 1 &= \frac{a^2}{x^2} + 1 \\
\csc u &= \sqrt{\frac{x^2 + a^2}{x^2}} \\
\arctan \frac{x}{a} &= u = \operatorname{arccsc} \sqrt{\frac{x^2 + a^2}{x^2}}
\end{aligned}
$$

■

$$
\operatorname{arcsec} \frac{x}{a} = \arcsin \sqrt{\frac{x^2 - a^2}{x^2}} \tag{3.5}
$$

Proof.

$$
\begin{aligned}
u &= \operatorname{arcsec} \frac{x}{a} \\
\sec u &= \frac{x}{a} \\
1 - \cos^2 u &= 1 - \frac{a^2}{x^2} \\
\sin u &= \sqrt{\frac{x^2 - a^2}{x^2}} \\
\operatorname{arcsec} \frac{x}{a} &= u = \arcsin \sqrt{\frac{x^2 - a^2}{x^2}}
\end{aligned}
$$

■

$$\operatorname{arcsec}\frac{x}{a} = \operatorname{arccsc}\sqrt{\frac{x^2}{x^2-a^2}} \tag{3.6}$$

Proof.

$$\begin{aligned}
u =&\ \operatorname{arcsec}\frac{x}{a}\\
\sec u =&\ \frac{x}{a}\\
1-\cos^2 u =&\ 1-\frac{a^2}{x^2}\\
\sin u =&\ \sqrt{\frac{x^2-a^2}{x^2}}\\
\csc u =&\ \sqrt{\frac{x^2}{x^2-a^2}}\\
\operatorname{arcsec}\frac{x}{a} =&\ u = \operatorname{arccsc}\sqrt{\frac{x^2}{x^2-a^2}}
\end{aligned}$$

■

$$\operatorname{arcsec}\frac{x}{a} = \arctan\sqrt{\frac{x^2-a^2}{a^2}} \tag{3.7}$$

Proof.

$$\begin{aligned}
u =&\ \operatorname{arcsec}\frac{x}{a}\\
\sec u =&\ \frac{x}{a}\\
\sec^2 u-1 =&\ \frac{x^2}{a^2}-1\\
\tan u =&\ \sqrt{\frac{x^2-a^2}{a^2}}\\
\operatorname{arcsec}\frac{x}{a} =&\ u = \arctan\sqrt{\frac{x^2-a^2}{a^2}}
\end{aligned}$$

■

$$\operatorname{arcsec}\frac{x}{a} = \operatorname{arccot}\sqrt{\frac{a^2}{x^2-a^2}} \tag{3.8}$$

Proof.

$$\begin{aligned} u &= \operatorname{arcsec}\frac{x}{a} \\ \sec u &= \frac{x}{a} \\ \sec^2 u - 1 &= \frac{x^2}{a^2} - 1 \\ \tan u &= \sqrt{\frac{x^2-a^2}{a^2}} \\ \cot u &= \sqrt{\frac{a^2}{x^2-a^2}} \\ \operatorname{arcsec}\frac{x}{a} &= u = \operatorname{arccot}\sqrt{\frac{a^2}{x^2-a^2}} \end{aligned}$$

∎

$$\arcsin\frac{x}{a} = \arccos\sqrt{\frac{a^2-x^2}{a^2}} \tag{3.9}$$

Proof.

$$\begin{aligned} u &= \arcsin\frac{x}{a} \\ \sin u &= \frac{x}{a} \\ 1 - \sin^2 u &= 1 - \frac{x^2}{a^2} \\ \cos u &= \sqrt{\frac{a^2-x^2}{a^2}} \\ \arcsin\frac{x}{a} &= u = \arccos\sqrt{\frac{a^2-x^2}{a^2}} \end{aligned}$$

∎

$$\arcsin\frac{x}{a} = \operatorname{arcsec}\sqrt{\frac{a^2}{a^2-x^2}} \tag{3.10}$$

Proof.

$$\begin{aligned} u &= \arcsin\frac{x}{a} \\ \sin u &= \frac{x}{a} \\ 1 - \sin^2 u &= 1 - \frac{x^2}{a^2} \\ \cos u &= \sqrt{\frac{a^2-x^2}{a^2}} \\ \sec u &= \sqrt{\frac{a^2}{a^2-x^2}} \end{aligned}$$

$$\arcsin\frac{x}{a} = u = \operatorname{arcsec}\sqrt{\frac{a^2}{a^2-x^2}}$$

■

$$\arcsin\frac{x}{a} = \arctan\sqrt{\frac{x^2}{a^2-x^2}} \tag{3.11}$$

Proof.

$$\begin{aligned}
u &= \arcsin\frac{x}{a} \\
\sin u &= \frac{x}{a} \\
\csc^2 u - 1 &= \frac{a^2}{x^2} - 1 \\
\cot u &= \sqrt{\frac{a^2-x^2}{x^2}} \\
\tan u &= \sqrt{\frac{x^2}{a^2-x^2}} \\
\arcsin\frac{x}{a} = u &= \arctan\sqrt{\frac{x^2}{a^2-x^2}}
\end{aligned}$$

■

$$\arcsin\frac{x}{a} = \operatorname{arccot}\sqrt{\frac{a^2-x^2}{x^2}} \tag{3.12}$$

Proof.

$$\begin{aligned}
u &= \arcsin\frac{x}{a} \\
\sin u &= \frac{x}{a} \\
\csc^2 u - 1 &= \frac{a^2}{x^2} - 1 \\
\cot u &= \sqrt{\frac{a^2-x^2}{x^2}} \\
\arcsin\frac{x}{a} = u &= \operatorname{arccot}\sqrt{\frac{a^2-x^2}{x^2}}
\end{aligned}$$

■

$$\sin\alpha\cdot\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)] \tag{3.13}$$

Proof.

$$\begin{aligned}
\sin\alpha\cdot\cos\beta &= \frac{1}{2}(2\sin\alpha\cos\beta) \\
&= \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]
\end{aligned}$$

■

$$\cos\alpha \cdot \cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)] \tag{3.14}$$

Proof.

$$\begin{aligned}
\cos\alpha \cdot \cos\beta &= \frac{1}{2}(2\cos\alpha\cos\beta) \\
&= \frac{1}{2}[\cos\alpha\cos\beta - \sin\alpha\sin\beta + (\cos\alpha\cos\beta + \sin\alpha\sin\beta)] \\
&= \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]
\end{aligned}$$

■

$$\sin\alpha \cdot \sin\beta = -\frac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)] \tag{3.15}$$

Proof.

$$\begin{aligned}
\sin\alpha \cdot \sin\beta &= -\frac{1}{2}(-2\sin\alpha\sin\beta) \\
&= -\frac{1}{2}[\cos\alpha\cos\beta - \sin\alpha\sin\beta - (\cos\alpha\cos\beta + \sin\alpha\sin\beta)] \\
&= -\frac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)]
\end{aligned}$$

■