3D Vision

Lab 3: Camera calibration theory

We start with a set of corresponding points between the planar pattern and an image. We say that the point (x, y) in the pattern (which may be measured in millimeters for example) corresponds to the pixel (u, v) in the image. From these correspondeces, we can estimate an homography that transforms the coordinates of the planar pattern onto the corresponding coordinates in the image,

$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \sim H \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}. \tag{1}$$

The way this is done has been covered in a previous session.

Now given the projection matrix of the camera, $P = K(R|t)$, we can derive an expression for this homography that relates it to the camera matrix. Assume the world reference is attached to the pattern so that the pattern plane is $z = 0$. The projection equation writes

$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \sim P \begin{pmatrix} x \\ y \\ 0 \\ 1 \end{pmatrix} \sim K(R|t) \begin{pmatrix} x \\ y \\ 0 \\ 1 \end{pmatrix} \sim \underbrace{K \begin{pmatrix} r_1 & r_2 & t \end{pmatrix}}_{H} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}, \tag{2}$$

where r_1 and r_2 are the first two columns of R. From this and the homography defitition above we deduce that $H \sim K \begin{pmatrix} r_1 & r_2 & t \end{pmatrix}$. It will be convinient to write the columns of H separately as

$$\begin{pmatrix} h_1 & h_2 & h_3 \end{pmatrix} \sim \begin{pmatrix} Kr_1 & Kr_2 & Kt \end{pmatrix} \tag{3}$$

and equivalently

$$\begin{pmatrix} K^{-1}h_1 & K^{-1}h_2 & K^{-1}h_3 \end{pmatrix} \sim \begin{pmatrix} r_1 & r_2 & t \end{pmatrix}. \tag{4}$$

We know H and our goal is to find K and eventually R and t. For this, we need to enforce some constraints on the unknowns. To start, we will use the fact that R is an orthonormal matrix. That is, its columns are unit vectors orthogonal to each other. In particular, $r_1 \cdot r_2 = 0$ and $r_1 \cdot r_1 = r_2 \cdot r_2 = 1$. From this we get the following equations on K,

$$h_1^T K^{-T} K^{-1} h_2 = 0 \tag{5}$$

$$h_1^T K^{-T} K^{-1} h_1 = h_2^T K^{-T} K^{-1} h_2 \tag{6}$$

Here, $K^{-T}K^{-1}$ turns out to be the fancy *Image of the Absolute Conic*, which is usually written as ω. For computational pruposes, it will be simpler to first find ω and then recover the camera matrix K from it. The above constraints are in fact linear on ω

$$h_1^T \omega h_2 = 0 \tag{7}$$

$$h_1^T \omega h_1 = h_2^T \omega h_2 \tag{8}$$

Lets write the terms in ω as

$$\omega = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{pmatrix} \tag{9}$$

and

$$\mathbf{x} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{pmatrix}^T. \tag{10}$$

Now we can rewrite the expressions in (7) and (8) as

$$h_i^T \omega h_j = \begin{pmatrix} h_{1i} & h_{2i} & h_{3i} \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{pmatrix} \begin{pmatrix} h_{1j} \\ h_{2j} \\ h_{3j} \end{pmatrix} = \mathbf{v}_{ij}^T \mathbf{x} \tag{11}$$

with

$$\mathbf{v}_{ij}^T = (h_{1i}h_{1j}, \quad h_{1i}h_{2j} + h_{2i}h_{1j}, \quad h_{1i}h_{3j} + h_{3i}h_{1j}, \quad h_{2i}h_{2j}, \quad h_{2i}h_{3j} + h_{3i}h_{2j}, \quad h_{3i}h_{3j}) \tag{12}$$

so that the equations write as

$$\begin{aligned} \mathbf{v}_{12}^T \mathbf{x} &= 0 \\ (\mathbf{v}_{11}^T - \mathbf{v}_{22}^T)\mathbf{x} &= 0. \end{aligned} \tag{13}$$

These are two linear equations on the coeficients of ω. By using more than one image, we get more equations. Since there are 6 unknows, we need at least 3 images of the planar pattern (there are actually only 5 unknowns since the scale of ω is irrellevant). It is also possible to add additional linear equations based on prior knowledge on K, such as 0 skew, aspect ration 1 or known principal point.

The result is an homogeneous linear system that we can solve using the SVD decomposition, and get ω. From that, we can recover de camera calibration matrix K by solving $\omega = K^{-T}K^{-1}$ using the Cholesky factorization.

Then, knowing K, the camera rotation and translation can be recovered from equation (4).