First Order Relations

Consider the recurrence relation

$$a_n = c\,a_{n-1} + f(n) \quad \text{for } n \geq 1$$

where c is a known constant and f(n) is a known function. Such a relation is called a linear recurrence relation of first order with constant coefficient.

If $f(n) = 0$ the relation is called homogeneous otherwise it is called non homogeneous recurrence relation

$$a_n = c\,a_{n-1} + f(n)$$

Change n to n+1

$$a_{n+1} = c\,a_n + f(n+1) \quad \text{for } n \geq 0$$

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$a_1 = c a_0 + f(1)$

$a_2 = c a_1 + f(2) = c(c a_0 + f(1)) + f(2)$

$a_2 = c^2 a_0 \; c f(1) + f(2)$

$a_3 = c^3 a_0 + c^2 f(1) + c f(2) + f(3)$

and so on

$a_n = c^n a_0 + c^{n-1} f(n-1) + \cdots f(n)$

$= c^n a_0 + \sum_{k=1}^{n} c^{n-k} f(k)$ for $n \geq 1$

This is the general solution of the recurrence relation

If $f(n) = 0$ then

$a_n = c^n a_0$ for $n \geq 1$

a_0 is called the ~~suff~~ initial condition

1. Solve the recurrence relation

$a_{n+1} = 4a_n$ for $n \geq 0$ given $a_0 = 3$

$a_n = 4a_{n-1}$

a_{n+1} $\cancel{a_{n+1}}$ $= 4(4a_{n-1})$

$a_{n+1} = 4^2 a_{n-1}$

$a_{n-1} = 4a_{n-2}$

$a_{n+1} = 4^3 a_{n-2}$

$\vdots$

$a_{n+1} = 4^{n+1} a_0$ $\quad a_0 = 3$

$a_{n+1} = 3 \times 4^{n+1}$

2. Solve the recurrence relation

$a_n = 7a_{n-1}$ where $n \geq 0$ given that $a_2 = 98$

$a_n = 7a_{n-1}$

$a_{n-1} = 7a_{n-2}$

$a_n = 7^2 a_{n-2}$

$a_n = 7^3 a_{n-3}$ $\quad n - k = 2$

$k = n - 2$

$a_n = 7^{n-2} a_2$ $\quad n \geq 2$

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$a_2 = 7^2 a_0$

$98 = 49 a_0$

$a_0 = 2$

$a_n = 2 \times 7^n$ for $n \geq 1$

This is the solution given relation under the condition $a_2 = 98$

3. Solve the recurrence relation $n = n a_{n-1}$ for $n \geq 1$ given that $a_0 = 1$

$a_n = n a_{n-1}$

$a_1 = 1 \times a_0$

$a_2 = 2 a_1 = (2 \times 1) a_0$

$a_3 = 3 a_2 = (3 \times 2 \times 1) a_0$

$\vdots$

$a_n = n! \, a_0$

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If a_n is a solution of the recurrence relation $a_{n+1} = K a_n$ for $n \geq 0$ and $a_3 = \frac{153}{49}$ and $a_5 = \frac{1377}{2401}$,

What is K.

$$a_{n+1} = K a_n$$

$$a_1 = K a_0$$

$$a_2 = K a_1$$

$$a_2 = K^2 a_0$$

$$\vdots$$

$$a_n = K^n a_0$$

$$a_3 = K^3 a_0$$

$$a_5 = K^5 a_0$$

$$\frac{a_5}{a_3} = K^2 \qquad K = \frac{1377}{2401} \times \frac{49}{153} = \frac{9}{49}$$

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Solve the recurrence relation $a_n - 3a_{n-1} = 5 \times 3^n$ for $n \geq 1$ given $a_0 = 2$

$$a_n = 3a_{n-1} + 5 \times 3^n$$

$f(n) = 5 \times 3^n$

$f(1) = 5 \times 3$

Let $f(n) = 5 \times 3^n$

$$a_n = 3^n a_0 + \sum_{k=1}^{n} 3^{n-k} f(k)$$

Given $a_0 = 2$

$$a_n = 2 \times 3^n + 3^{n-1} f(1) + 3^{n-2} f(2) + \cdots 3^0 f(n)$$

$$a_n = 2 \times 3^n + 5 \times [3^{n-1} \times 3 + 3^{n-2} \times 3^2 + \cdots 3^0 \times 3^n]$$

$$= 2 \times 3^n + 5n3^n = (2+5n)3^n$$

Second order linear homogeneous recurrence relations

Consider the recurrence relation

$$C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} = 0 \quad \text{for } n \geq 2 \rightarrow (1).$$

where C_n, C_{n-1} and C_{n-2} are real constants with $C_n \neq 0$.

A relation of this types is called a second order linear homogeneous recurrence relation with constant coefficients.

We seek a solution in the form $a_n = CK^n$ where $C \neq 0$ and $K \neq 0$

put $a_n = CK^n$

$$C_n CK^n + C_{n-1} CK^{n-1} + C_{n-2} C.K^{n-2} = 0$$

: Divide the above eqn CK^{n-2}

$$C_n K^2 + C_{n-1} K + C_{n-2} = 0. \rightarrow (2)$$

Thus $a_n = CK^n$ is a solution of (1) if K satisfies the quadratic equation in (2)

This quadratic equation is called the auxiliary equation or the characteristic equation.

Three cases arise:

Case 1: The two roots K_1 and K_2 of eqn (2) are real and distinct.

Then $a_n = AK_1^n + BK_2^n$

where A and B are arbitrary real constants as the general solution.

Case 2: The two roots K_1 and K_2 of equation (2) are real and equal. Then

$$a_n = (A+Bn)K^n$$

where A and B are arbitrary real constants as the general solution.

Case 3: The two roots K_1 and K_2 of equation (2) are complex. Then K_1 and K_2 are complex conjugates of each other

if $K_1 = p+iq$ then $K_2 = p-iq$

$$a_n = r^n(A\cos n\theta + B\sin n\theta)$$

where A and B are arbitrary complex constants

$r = |K_1| = |K_2| = \sqrt{p^2+q^2}$ $\theta = \tan^{-1}(q/p)$ as the general solution

1) Solve the recurrence relation

$a_n + a_{n-1} - 6a_{n-2} = 0$ for $n \geq 2$

given that $a_0 = -1$ and $a_1 = 8$.

The coefficients of a_n is $C_n = 1$ $C_{n-1} = 1$ $C_{n-2} = -6$

The characteristic equation is

$$k^2 + k - 6 = 0$$

$$(k+3)(k-2) = 0$$

$k_1 = -3$ and $k_2 = 2$ which are real and distinct.

Therefore the general solution of the given relation is

$$a_n = A(-3)^n + B(2^n)$$

$$a_0 = A + B$$

$$a_1 = -3A + 2B$$

$$-1 = A + B$$

$$8 = -3A + 2B$$

$$-3 = 3A + 3B$$

$$8 = -3A + 2B$$

$B = 1$

$A = -2$

1) Solve the recurrence relation

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$$a_0 = A + B$$

$$a_1 = -3A + 2B$$

$$-1 = A + B$$

$$8 = -3A + 2B$$

$$-3 = \cancel{3A} + 3B$$

$$8 = -\cancel{3A} + 2B$$

$$5 = 5B$$

$B = 1$

$A = -2$

$$\underline{a_n = -2(-3)^n + 2^n}$$

2. Solve the recurrence relation

$$a_n = 3a_{n-1} - 2a_{n-2} \quad \text{for } n \geq 2$$

given $a_1 = 5$ and $a_2 = 3$.

$$a_n - 3a_{n-1} + 2a_{n-2} = 0$$

$C_n = 1$, $C_{n-1} = -3$ $C_{n-2} = +2$

The characteristic equation is

$$K^2 - 3K + 2 = 0$$

$$(K-2)(K-1) = 0 \qquad K_1 = 2 \quad K_2 = 1$$

Therefore the general solution for a_n is

$$a_n = A2^n + B1^n$$

using the conditions $a_1 = 5$ and $a_2 = 3$

$$a_1 = 2A + B$$

$$5 = 2A + B$$

$$a_2 = 4A + B$$

$$3 = 4A + B$$

$$\underline{5 = 2A + B}$$

$$-2 = 2A \qquad \underline{A = -1}$$

$$5 = 2A + B$$

$$5 = -2 + B \qquad \underline{B = 7}$$

$$a_n = -2^n + 7$$

$a_n = 7 - 2^n$ is the Solution under the given conditions

3) Solve the recurrence relations

$$a_{n+2} = 4a_{n+1} - 4a_n \quad n \geq 0$$

$$a_0 = 1 \quad a_1 = 3$$

$$a_{n+2} - 4a_{n+1} + 4a_n = 0 \quad n \geq 0$$

$$C_{n+2} = 1 \quad C_{n+1} = -4 \quad C_n = 4$$

$$K^2 - 4K + 4 = 0.$$

$$K^2 - 2K - 2K + 4 = 0$$

$$K(K-2) - 2(K-2) = 0$$

$$(K-2)(K-2) = 0 \quad K = 2$$

The two roots K_1 and K_2 are real and equal

$$a_n = (A + Bn)K^n$$

$$= (A + Bn)2^n$$

$$a_n = A2^n + Bn2^n$$

$$a_0 = A + B(0)$$

$$a_0 = A \quad \underline{\underline{A = 1}}.$$

$$a_1 = 2A + 2B$$

$$3 = 2A + 2B$$

$$B = 1/2$$

$$a_n = A2^n + Bn2^n$$

$$= 2^n + n2^{n-1}$$

4) $a_n - 4a_{n-1} + 4a_{n-2} = 0 \quad n \geq 2 \quad a_0 = 5/2$

$a_1 = 8$

$C_n = 1 \quad C_{n-1} = -4 \quad C_{n-2} = 4$

$$K^2 - 4K + \cancel{4} = 0$$

$$K^2 - 2K - 2K + 4 = 0$$

$$K(K-2) - 2(K-2) = 0$$

$$K = 2$$

The two roots K_1 and K_2 are real and equal

$$a_n = (A + Bn)K^n$$

$$= (A + Bn)2^n$$

$$a_n = A2^n + Bn2^n$$

$a_0 = A \qquad A = 5/2$

$$a_1 = 2A + 2B$$

$$8 = \cancel{2} * \frac{5}{\cancel{2}} + 2B$$

$$B = \frac{3}{2}$$

$$a_n = \left(\frac{5}{2} + \frac{3}{2} * \cancel{n}\right) 2^n$$

$$= \frac{5}{2} * 2^n + \left(\frac{3}{2}\right) n 2^n$$

7) Solve the recurrence relation

$$a_n - 6a_{n-1} + 9a_{n-2} = 0 \quad \text{for } n \geq 2$$

given that $a_0 = 5$ $a_1 = 12$

The characteristic equation is

$$K^2 - 6K + 9 = 0 \qquad (K-3)^2 = 0$$

$$a_n = (A + Bn)3^n$$

$$a_0 = A \qquad A = 5$$

$$a_1 = 3A + 3B = 12$$

$$B = -1$$

$$a_n = (5-n)(3^n)$$

Solve the recurrence relation

$$a_n = 2(a_{n-1} - a_{n-2}) \quad \text{for } n \geq 2$$

$$a_0 = 1 \text{ and } a_1 = 2$$

The characteristic equation is $K^2 - 2K + 2 = 0$

$$K = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

Therefore the general solution for a_n is

$$a_n = r^n [A \cos n\theta + B \sin n\theta]$$

Where A and B are arbitrary constants

$r = |1 \pm i| = \sqrt{2}$ and $\tan\theta = 1/1 = 1$

$\theta = \pi/4$

$$a_n = (\sqrt{2})^n \left[A\cos\frac{n\pi}{4} + B\sin\frac{n\pi}{4}\right]$$

$a_0 = A \quad \therefore A = 1$

$$a_1 = 2 = (\sqrt{2})\left[A\cos\frac{\pi}{4} + B\sin\frac{\pi}{4}\right] = A + B$$

$B = 1$

$$a_n = (\sqrt{2})^n \left[\cos\frac{n\pi}{4} + \sin\frac{n\pi}{4}\right]$$

This is the solution of the given relation under the initial conditions.

If $a_0 = 0$ $a_1 = 1$, $a_2 = 4$ and $a_3 = 37$ satisfy the recurrence relation

$$a_{n+2} + b a_{n+1} + c a_n = 0 \quad \text{for } n \geq 0$$

determine the constants b and c and then solve the relation for a_n.

For $n = 0$

$$a_2 + b a_1 + c a_0 = 0$$

For $n = 1$

$$a_3 + b a_2 + c a_1 = 0$$

Substitute the given values of a_0, a_1, a_2 and a_3

$$4 + b + 0 = 0 \qquad 37 + 4b + c = 0$$

$$b = -4 \qquad c = -21.$$

With these values of b and c

$$a_{n+2} - 4 a_{n+1} - 21 a_n = 0 \quad \text{for } n \geq 0$$

$$a_n - 4 a_{n-1} - 21 a_{n-2} = 0 \quad \text{for } n \geq 2$$

The characteristic equation is

$$K^2 - 4K - 21 = 0$$

$$K^2 + 3K - 7K - 21 = 0$$

If $a_0 = 0$ $a_1 = 1$, $a_2 = 4$ and $a_3 = 37$ satisfy the recurrence relation

$$a_{n+2} + b a_{n+1} + C a_n = 0 \quad \text{for } n \geq 0$$

determine the constants b and c and then solve the relation for a_n.

For $n = 0$

$$a_2 + b a_1 + C a_0 = 0$$

For $n = 1$

$$a_3 + b a_2 + C a_1 = 0$$

Substitute the given values of a_0, a_1, a_2 and a_3

$$4 + b + 0 = 0 \qquad 37 + 4b + C = 0$$

$$b = -4 \qquad C = -21.$$

with these values of b and c

$$a_{n+2} - 4 a_{n+1} - 21 a_n = 0 \quad \text{for } n \geq 0$$

$$a_n - 4 a_{n-1} - 21 a_{n-2} = 0 \quad \text{for } n \geq 2$$

The characteristic equation is

$$K^2 - 4K - 21 = 0$$

$$K^2 \cancel{-4K} + 3K - 7K - 21 = 0$$

$$K(K+3) - 7(K+3) = 0$$

$$K_1 = 7 \qquad K_2 = -3$$

$a_n = A * 7^n + B(-3)^n$

where A and B are arbitrary constants

$a_0 = A + B = 0$

$a_1 = 7A - 3B = 1$

$A = -B = -\frac{1}{10}$

$a_n = \frac{1}{10}\left[7^n - (-3)^n\right]$

Third and higher order linear Homogeneous Recurrence Relations

consider the higher order recurrence relation

$$C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} + C_{n-3} a_{n-3} + \cdots C_{n-k} a_{n-k} = 0$$

$$n \geq k \geq 3$$

where $C_n, C_{n-1} \cdots C_{n-k}$ are real constants with $C_n \neq 0$. The method of solution is analogous to that of solving recurrence relations of the second order.

1. Solve the recurrence relation

$$2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n \quad \text{for } n \geq 0$$

with $a_0 = 0 \quad a_1 = 1$ and $a_2 = 2$.

The given relation is the same as

$$2a_n - a_{n-1} - 2a_{n-2} + a_{n-3} = 0 \qquad n \geq 3.$$

This is a third order relation with

$C_n = 2 \quad C_{n-1} = -1 \quad C_{n-2} = -2 \quad C_{n-3} = 1$

The characteristic equation is

$$2K^3 - K^2 - 2K + 1 = 0$$

$$2K^3 - 2K - K^2 + 1 = 0$$

$$2K(K-1) - 1(K^2-1) = 0$$

$$\cancel{K_1 = 1/2} \quad (2K-1)(K^2-1) = 0$$

$$K_1 = 1/2 \quad K_2 = 1 \quad K_3 = -1$$

All roots are real and distinct.

The solution is of the form

$a_n = A K_1^n + B K_2^n + C K_3^n$

$= A\left(\frac{1}{2}\right)^n + B(1)^n + C(-1)^n$

A, B, C are constants

$a_0 = A\left(\frac{1}{2}\right)^0 + B(1)^0 + C(-1)^0$

$0 = A + B + C$

$a_1 = A\left(\frac{1}{2}\right) + B(1) + C(-1)$

$1 = \frac{A}{2} + B - C$

$a_2 = A\left(\frac{1}{2}\right)^2 + B(1)^2 + C(-1)^2$

$2 = \frac{A}{4} + B + C$

$\therefore$

$A + B + C = 0 \rightarrow (1)$

$\frac{A}{2} + B - C = 1 \rightarrow (2)$

$\frac{A}{4} + B + C = 2 \rightarrow (3)$

$A + B + C = 0$

$\frac{A}{2} + B - C = 1$

$\frac{3A}{2} + 2B = 1$

$\frac{3A}{4} + 2B = 3$

$\frac{3A}{2} + 2B = 1$

$-\frac{3}{4}A = 2$

$A = -8/3$

$\frac{3}{4}\left(-\frac{8}{3}\right) + 2B = 3$

$-2 + 2B = 3$

$\frac{3}{4} - \frac{3}{2} = \frac{3-6}{4} = \frac{-3}{4}$

$2B = 5 \quad B = 5/2$

$A + B + C = 0$

$-\frac{8}{3} + \frac{5}{2} + C = 0$

$\frac{-16+15}{6} + C = 0$.

$-\frac{1}{6} + C = 0 \quad C = +1/6$.

$A = -\frac{8}{3}$

$B = 5/2$

$C = 1/6$.

$a_n = \left(-\frac{8}{3}\right)\left(\frac{1}{2}\right)^n + \frac{5}{2}(1)^n + \frac{1}{6}(-1)^n$

a_n

3. Solve the recurrence relation

$$a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$$

given $a_0 = 1$ $a_1 = 4,$ $a_2 = 28$

The characteristic equation for the given relation is

$$K^3 - 6K^2 + 12K - 8 = 0$$

$$(K-2)^3 = 0$$

whose roots are $K_1 = K_2 = K_3 = 2$

The general solution for an

$$a_n = (A + Bn + Cn^2)(2^n)$$

where A, B, C are constants.

$$a_0 = 1 = A$$

$$a_1 = 4 = 2(A+B+C)$$

$$a_2 = 28 = 4(A+2B+4C)$$

$A = 1 \quad B = -1 \quad C = 2$

$a_n = (1 - n + 2n^2) 2^n$.

Find the general solution of the recurrence relation.

$a_n + a_{n-3} = 0 \qquad n \geq 3$.

The characteristic equation for the recurrence relation is

$K^3 + 1 = 0 \qquad (K+1)(K^2 - K + 1) = 0$

$K_1 = -1 \qquad K_2 = \frac{1}{2}(1 + i\sqrt{3})$

$K_2 = \frac{1}{2}(1 - i\sqrt{3})$

$a_n = A(-1)^n + r^n [C_1 \cos n\theta + C_2 \sin n\theta]$

$r = |K_2| = |K_3| = \frac{1}{2} \sqrt{1^2 + (\sqrt{3})^2}$

$\tan\theta = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$

$\theta = \pi/3$

$a_n = A(-1)^n \left(C_1 \left\{ \cos \frac{n\pi}{3} + C_2 \sin \frac{n\pi}{3} \right)\right.$

Linear Non homogenous relations of Second and higher orders.

$$C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} + \cdots C_{n-k} a_{n-k} = f(n)$$

for $n \geq k \geq 2$ $\rightarrow$ ①.

where $C_n, C_{n-1}, \cdots C_{n-k}$ are real constants $C_n \neq 0$ $f(n)$ is a given real valued function of n.

A general solution of the recurrence relation is given by

$$a_n = a_n^{(h)} + a_n^{(P)}$$

where $a_n^{(h)}$ is the general solution of the homogenous part of the relation with $f(n) = 0$ and $a_n^{(P)}$ is any particular solution of the relation.

Determination of $a_n^{(P)}$ for arbitrary $f(n)$ is tedious. It is only in some special cases that we can find $a_n^{(P)}$ in a straight forward manner.

Following are some of these special cases

(1) Suppose $f(n)$ is a polynomial of degree q and 1 is not a root of the characteristic equation of the homogeneous part of the relation (1). In this case, $a_n^{(p)}$ is taken in the form

$$a_n^{(p)} = A_0 + A_1 n + A_2 n^2 + \cdots A_q n^q.$$

where $A_0, A_1, A_2, \cdots A_q$ are constants to be evaluated using $a_n = a_n^{(p)}$

(2) Suppose $f(n)$ is a polynomial of degree q. and 1 is a root of multiplicity m of characteristic equation of the homogenous part of the relation then

$$a_n^{(p)} = n^m \{A_0 + A_1 n + A_2 n^2 + \cdots A_q n^q\}$$

where $A_0, A_1, \cdots A_q$ are constants to be evaluated by using the fact that $a_n = a_n^{(p)}$

(3) Suppose $f(n) = \alpha b^n$ where α is a constant and b is not a root of the characteristic equation of the homogenous part then $a_n^{(p)}$ is

$$a_n^{p} = A_0 b^n$$

where A_0 is a constant to be evaluated by using the fact that $a_n = a_n^{(p)}$