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① Determine the number of positive integers n such that $1 \leq n \leq 100$ and n is not divisible by 2, 3 or 5.

$|S| = 100$

$|A_1|$ = No of elements in S that are divisible by 2

$|A_1| = \lfloor 100/2 \rfloor = 50$

$|A_2| = \lfloor 100/3 \rfloor = 33$

$|A_3| = \lfloor 100/5 \rfloor = 20$

$|A_1 \cap A_2| = \lfloor 100/6 \rfloor = 16$ LCM of 2 & 3 is 6

$|A_1 \cap A_3| = \lfloor 100/10 \rfloor = 10$

$|A_2 \cap A_3| = \lfloor 100/15 \rfloor = 6$

$|A_1 \cap A_2 \cap A_3| = \lfloor 100/30 \rfloor = 3$

$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

$= 50 + 33 + 20 - 16 - 10 - 6 + 3$

$= 106 - 32$

$= 74$

$|\bar{A_1} \cap \bar{A_2} \cap \bar{A_3}| = 100 - 74$

$= 26.$

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(2) How many integers between 1 and 300 are (i) divisible by atleast one of 5, 6, 8 (ii) divisible by none of 5, 6, 8

$|S| = 300$

$|A_1| = \lfloor 300/5 \rfloor = 60$

$|A_2| = \lfloor 300/6 \rfloor = 50$

$|A_3| = \lfloor 300/8 \rfloor = 37$

$|A_1 \cap A_2| = \lfloor 300/30 \rfloor = 10$

$|A_1 \cap A_3| = \lfloor 300/40 \rfloor = 7$

$|A_2 \cap A_3| = \lfloor 300/24 \rfloor = 12$

$|A_1 \cap A_2 \cap A_3| = \lfloor 300/120 \rfloor = 2$

$|A_1 \cup A_2 \cup A_3| = 60 + 50 + 37 - 10 - 7 - 12 + 2$

$= 145 - 29$

$= 122$

$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = |S| - |A_1 \cup A_2 \cup A_3|$

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(3) Determine the number of positive integers n such that $1 \le n \le 300$ and n is divisible by atleast one of 3, 5, 7.

$|S| = 300$

$|A_1| = \lfloor 300/3 \rfloor = 100$

$|A_2| = \lfloor 300/5 \rfloor = 60$

$|A_3| = \lfloor 300/7 \rfloor = 42$

$|A_1 \cap A_2| = \lfloor 300/15 \rfloor = 20$

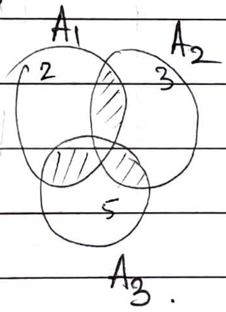
$|A_1 \cap A_3| = \lfloor 300/21 \rfloor = 14$

$|A_2 \cap A_3| = \lfloor 300/35 \rfloor = 8$

$|A_1 \cap A_2 \cap A_3| = \lfloor 300/105 \rfloor = 2$

$|A_1 \cup A_2 \cup A_3| = 100 + 60 + 42 - 20 - 14 - 8 + 2$
$= 162.$

(4) Determine the number of integers between 1 and 100 which are (i) divisible by exactly two of 2, 3, 5 and divisible by atleast two of 2, 3, 5



$|A_1| = \lfloor 100/2 \rfloor = 50.$

$|A_2| = \lfloor 100/3 \rfloor = 33$

$|A_3| = \lfloor 100/5 \rfloor = 20.$

$|A_1 \cap A_2| = \lfloor 100/6 \rfloor = 16$

$|A_1 \cap A_3| = \lfloor 100/10 \rfloor = 10$

$|A_2 \cap A_3| = \lfloor 100/15 \rfloor = 6$

$|A_1 \cap A_2 \cap A_3| = \lfloor 100/30 \rfloor = 3$

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$|A_1 \cap A_2| - |A_1 \cap A_2 \cap A_3|$
$= 16 - 3 = 13$

$|A_1 \cap A_3| - |A_1 \cap A_2 \cap A_3|$
$10 - 3 = 7$

$|A_2 \cap A_3| - |A_1 \cap A_2 \cap A_2|$
$6 - 3 = 3.$

Divisible by exactly two of 2, 3, 5

$= 13 + 7 + 3 = 23.$

Divisible by atleast two of 2, 3, 5

$= 23 + 3 = 26$

(5) Find the number of integers between 1 and 1000 that are divisible by none of 5, 6 and 8.

$|A_1| = \lfloor 1000/5 \rfloor = 200$

$|A_2| = \lfloor 1000/6 \rfloor = 166$

$|A_3| = \lfloor 1000/8 \rfloor = 125$

$$\begin{array}{r|l} 2 & 6, 8 \\ 2 & 3, 4 \\ & 3, 2 \end{array}$$

$|A_1 \cap A_2| = \lfloor 1000/30 \rfloor = 33$

$|A_1 \cap A_3| = \lfloor 1000/40 \rfloor = 25$

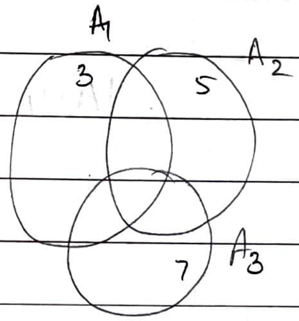
$|A_2 \cap A_3| = \lfloor 1000/24 \rfloor = 41$

$|A_1 \cap A_2 \cap A_3| = 1000/120 = 8$

$|A_1 \cup A_2 \cup A_3| = 200 + 166 + 125 - 33 - 25 - 41 + 8$
$= 400$

$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = 1000 - 400$
$= 600.$

(6) How many integers between 1 and 300 are
(a) divisible by atleast one of 3, 5, 7
(b) divisible by 3 and 5 but not 7
(c) divisible by 5 but neither 3 nor 7



$|A_1| = \lfloor 300/3 \rfloor = 100$

$|A_2| = \lfloor 300/5 \rfloor = 60$

$|A_3| = \lfloor 300/7 \rfloor = 42$

$|A_1 \cap A_2| = \lfloor 300/15 \rfloor = 20$

$|A_1 \cap A_3| = \lfloor 300/21 \rfloor = 14$

$|A_2 \cap A_3| = \lfloor 300/35 \rfloor = 8$

$|A_1 \cap A_2 \cap A_3| = \lfloor 300/105 \rfloor = 2$

(a) $|A_1 \cup A_2 \cup A_3| = 100 + 60 + 42 - 20 - 14 - 8 + 2$
$= 162$

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(b) $|A_1 \cap A_2| - |A_1 \cap A_2 \cap A_3|$

$= 20 - 2 = 18$

(c) $|A_2| - [|A_1 \cap A_2| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|]$

$60 - [20 + 8 - 2]$

$= 60 - 26 = 34$

How many integers between 1 and 500 are (i) divisible by 3 or 5
(ii) divisible by 3 but not by 5?

$|A \cup B| = |A| + |B| - |A \cap B|$

$= \lfloor 500/3 \rfloor + \lfloor 500/5 \rfloor - \lfloor 500/15 \rfloor$

$= 166 + 100 - 33$

$= 233$

(ii) 66

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(1) Find the number of nonnegative integer solutions of the equation.

$x_1 + x_2 + x_3 + x_4 = 18$

under the condition $x_i \leq 7$ for $i = 1, 2, 3, 4$.

Let S denote the set of all nonnegative integer solutions of the given equation.

The number of such solutions is

$C(4+18-1, 18) = C(21, 18)$

$|S| = C(21, 18)$

Let A_1 be the subset of S that contain the nonnegative integer solution of the given equation under the conditions $x_1 > 7$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$.

$A_1 = ((x_1, x_2, x_3, x_4) \in S \mid x_1 > 7)$

$A_2 = ((x_1, x_2, x_3, x_4) \in S \mid x_2 > 7)$

$A_3 = ((x_1, x_2, x_3, x_4) \in S \mid x_3 > 7)$

$A_4 = ((x_1, x_2, x_3, x_4) \in S \mid x_4 > 7)$

Then the required number of solutions would be $(\bar{A_1} \cap \bar{A_2} \cap \bar{A_3} \cap \bar{A_4})$

$x_1 + x_2 + x_3 + x_4 = 18$

$x_1' + x_2 + x_3 + x_4 = 18 - 8$ $\quad x_1' \geq 0$

$x_1' + x_2 + x_3 + x_4 = 10$

The number of nonnegative integer solutions of this equation is $C(4 + 10 - 1, 10) = C(13, 10)$

$|A_2| = C(13, 10)$

$|A_3| = C(13, 10)$

$|A_4| = C(13, 10)$

$x_1 + x_2 + x_3 + x_4 = 18$

$x_1' + x_2' + x_3 + x_4 = 18 - 8 - 8$ $\quad x_1' \geq 0 \; x_2' \geq 0$

$x_1' + x_2' + x_3 + x_4 = 2$

The number of nonnegative integer solutions of this equation is $C(4+2-1, 2)$

$|A_1 \cap A_2| = C(5, 2)$

$|A_1 \cap A_3| = C(5, 2)$ $\quad |A_2 \cap A_3| = C(5, 2)$

In the given equation, more than two x_i's cannot be greater than 7 simultaneously.

21 × 20 × 19

$nC_r = \cancel{n!}\,\cancel{r!(n-r)!}$

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nC_r

$|A_1 \cap A_2 \cap A_3| = 0 \quad |A_1 \cap A_3 \cap A_4| = 0$

$|A_2 \cap A_3 \cap A_4| = 0 \quad |A_1 \cap A_2 \cap A_3 \cap A_4| = 0$

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4| = |S| - (|A_1| + |A_2| + |A_3| + |A_4|)$$
$$+ |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|$$
$$+ |A_1 \cap A_4| + |A_2 \cap A_4| + |A_3 \cap A_4|$$
$$- [|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|]$$
$$+ |A_1 \cap A_2 \cap A_3 \cap A_4|$$

$$= C(21,18) - [C(13,10) + C(13,10) + C(13,10) + C(13,10)]$$
$$+ C(5,2) + C(5,2) + C(5,2) + C(5,2)$$
$$- 0 + 0$$

=

$$nC_r = \frac{n!}{r!\,(n-r)!} \qquad 21C_{18} = \frac{21!}{18!\,(21-18)!}$$

$$= \frac{21 \times 20 \times 19 \times \cancel{18!}}{\cancel{18!} \times 3!} = \frac{7980}{6} = 1330$$

$$13C_{10} = \frac{13!}{10!\,3!} = \frac{13 \times 12 \times 11}{6} = 286$$

$4C_2$

$$5C_2 = \frac{5!}{2!\,3!} = \frac{5 \times 4 \times 3}{6} = 10$$

$\frac{4!}{2!\,2!} \quad \frac{\overset{2}{\cancel{4}} \times 3}{\cancel{2}}$

$$= 1330 - 4 \times 286 + 6 \times 10$$
$$= 366$$

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Find the number of integer solutions of the equation

$x_1 + x_2 + x_3 = 20$

such that $2 \leq x_1 \leq 5$ $4 \leq x_2 \leq 7$

$-2 \leq x_3 \leq 9$

Let $y_1 = x_1 - 2$

$y_2 = x_2 - 4$

$y_3 = x_3 + 2$ we have $y_1 \geq 0$ $y_2 \geq 0$

$y_3 \geq 0$.

~~$y_1 + y_2 + y_3$~~

$y_1 + \cancel{2} + y_2 + 4 + y_3 - \cancel{2} = 20$

$y_1 + y_2 + y_3 = 16$

The number of non negative integer solutions of this equation is $C(3+16-1, 16)$

$= C(18, 16)$

If S is the set of solutions $|S| = C(18, 16)$

when $x_1 \leq 5$ we have $y_1 \leq 3$

$x_2 \leq 7$ we have $y_2 \leq 3$

$x_3 \leq 9$ we have $y_3 \leq 11$

$A_1 = \{(y_1, y_2, y_3) \in S \mid y_1 > 3\}$

$A_2 = \{(y_1, y_2, y_3) \in S \mid y_2 > 3\}$

$A_3 = \{(y_1, y_2, y_3) \in S \mid y_3 > 11\}$

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we have to find $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3|$

$$y_1 + y_2 + y_3 = 16$$

$$y_1' + y_2 + y_3 = 16 - 4 \qquad y_1' \geq 0$$

$$y_1' + y_2 + y_3 = 12$$

The number of non negative integer solutions of this equation is $C(3+12-1, 12)$

$$|A| = C(14, 12)$$

$$y_1 + y_2' + y_3 = 16 - 4 \qquad y_2' \geq 0$$

$$y_1 + y_2' + y_3 = 12$$

The number of non negative integer solutions of this equation is $C(3+12-1, 12)$

$$|A_2| = C(14, 12)$$

$$y_1 + y_2 + y_3' = 16 - 12$$

$$y_1 + y_2 + y_3' = 4$$

The number of non negative integer solutions of this equation is $C(3+4-1, 4)$

$$|A_3| = C(6, 4)$$

$$y_1' + y_2' + y_3 = 16 - 8$$

$$y_1' + y_2' + y_3 = 8$$

$$|A_1 \cap A_2| = C(3+8-1, 8)$$

$$= C(10, 8)$$

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$$y_1' + y_2 + y_3' = 16 - 4 - 12$$
$$y_1' + y_2 + y_3' = 0$$

$$|A_1 \cap A_3| = C(3+0-1, 0) = C(2,0)$$

$$|A_2 \cap A_3| = C(3+0-1, 0) = C(2,0)$$

$$y_1 + y_2' + y_3' = 16 - 4 - 12$$

$$|A_1 \cap A_2 \cap A_3| = 0.$$

$$|\bar{A_1} \cap \bar{A_2} \cap \bar{A_3}| = |S| - [|A_1| + |A_2| + |A_3|] + |A_1 \cap A_2| + |A_2 \cap A_3| + |A_1 \cap A_3| - |A_1 \cap A_2 \cap A_3|$$

$$= C(18,16) - (C(14,12) + C(14,12) + C(6,4)) + C(10,8) + C(2,0) + C(2,0) - 0$$

$$= 153 - (2 \times 91 + 15) + (45 + 2)$$
$$= 3$$

Thus under the given conditions the given equation exactly has three integer solutions.