AUTOMATA FORMAL LANGUAGES AND LOGIC

Lecture Notes
on
Functions & Relations

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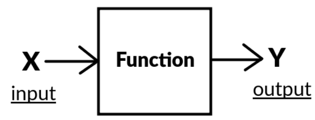
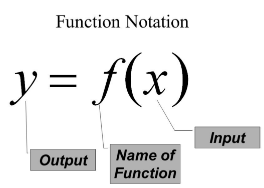
1.What is a Function?

Functions are central to mathematics. A function can be thought of as a rule that assigns a set of inputs to a set of outputs.



We use the notation f(x) where f is the function (a general notation) x is the input. For example, we could define f(x) = {x + 4 for 0 < x < 5}. So, if we give the input as 2 then , f(2) = 2 + 4 = 6. We denote the output as y, so we can say y is a function of x which is equal to {x + 4 for 0 < x < 5}.



A function also is called a mapping, and, if f(x) = y, we say that f maps x to y.

In a function, it is imperative that each input is assigned to only one output. The reason for this is because if an input were assigned to two different outputs, then given the input, we will not be able to determine the output. Given an input, we must be able to determine the output. Thus, each input of a function has only one output.

2.Domain & Range of a Function

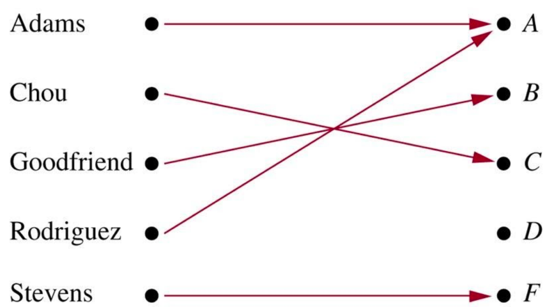
The set of inputs of a function is the domain of the function and the set of outputs of a function the range of a function.

For example, consider the function defined by the rule
f(x) = {x + 4 for 0 < x < 5}

Here domain = 1, 2, 3, 4
and range = 5, 6,7,8

The Notation :
$f:$ D -> R
A function maps a set of inputs (domain) to set of outputs(range).

A function may not necessarily use all elements in a range. Consider the following example:
$f:$ Student -> Grades



Here we are mapping Set of students to their corresponding grade. Each student will receive a grade for sure although few students might receive the same grade. However, there could be a possibility that none of the students got a particular grade, in this example, no student got grade D. So we have not used all the elements in a range.

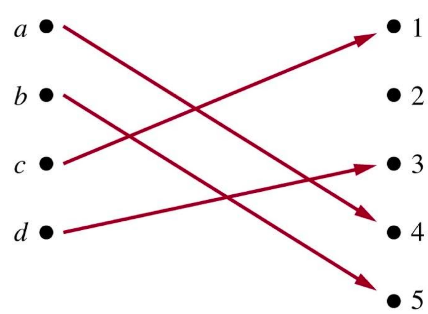
3.Different types of functions

We discuss the following types of functions:

1) One to One (injective)
2) Onto (Surjective)
3) Bijective

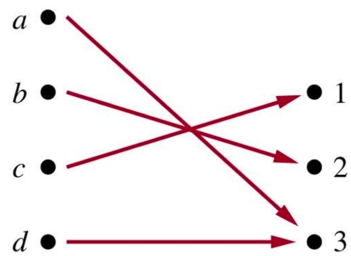
1)One to One Function :

A function f: A → B is One to One if for each element of A there is a distinct element of B. It is also known as Injective. or we can say every element in A has a unique image.
Here, Image means to which value does an element map to.

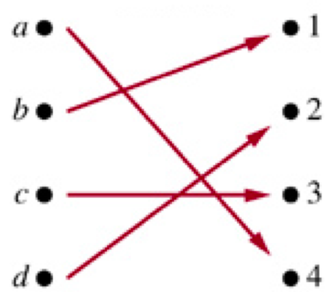


In the given example, Image(a) = 4 and we can in turn say pre-image(4) = a.

2)Onto Function :
A function where, for every element of set B there exists a pre-image in set A, it is called Onto Function. Onto is also referred to as Surjective Function.
For example:



3)Bijective Function :
A function, Bijective if the function is both One to One and Onto function. In other words, the function associates each element of A with a distinct element of B and every element of B has a pre-image in A.
For example :



4.What is a Relation?

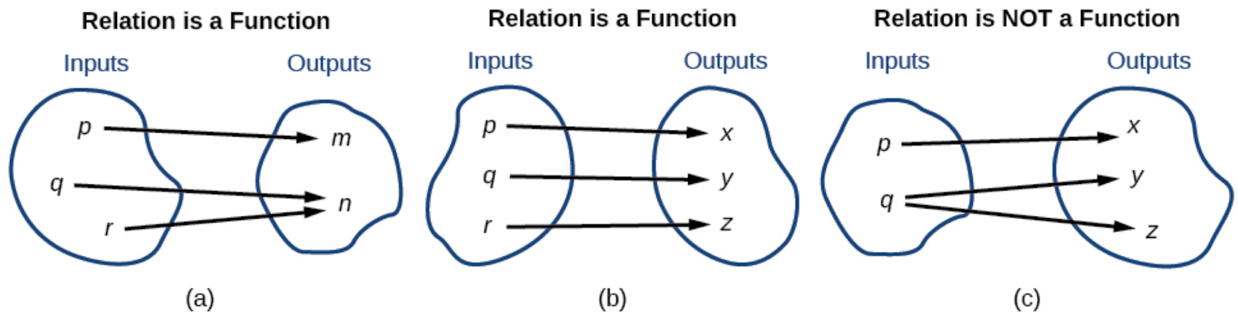
A relation between two sets is a collection of ordered pairs (x,y), containing one object from each set. Here x is from the first set and y is from the second set.

The set of the first components of each ordered pair is called the domain and the set of the second components of each ordered pair is called the range.
For example :
If we define relation R as,
R = {(1,2), (2,4), (3,6)}

then
Domain is {1,2,3}
Range is {2,4,6}

5.When is a Relation called a Function?

A function is a special relation in which each possible input maps to exactly one output value.
Given the following relations lets see which one is a function.



In example a, Each input maps to exactly one output hence this relation is a function.

In example b, again Each input element maps to exactly one output hence this relation is a function.

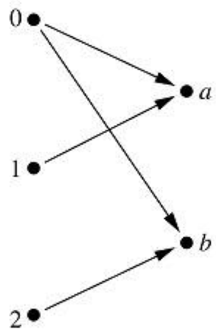
In example c, input q maps to two outputs y and z, which is not desirable from function point of view, where the output for each input is determined and is a single value with no confusion. Hence this one is not a function.

6.Binary Relation

A binary relation is a relationship between two sets lets say X and Y and formally is a subset of the Cartesian product of X and Y;

We denote an Element a is related to element b as aRb. which means the pair $(a, b) \in R$

Here if we consider Set A as containing {0, 1, 2} and Set B as {a, b} then the cartesian product of A and B will contain elements like
A x B = { (0, a), (1, a), (0, b), (1, b), (2, a), (2, b) }
We can define a relation as containing only few pairs as shown in the figure below and is a subset of the cartesian product of A and B



R	a	b
0	×	×
1	×	
2		×

7.Representing a Relation

We can represent a relation in two ways -

1) Matrix form
2) Directed Graph (Digraph)

Let's Consider a Set A = {a, b, c, d}

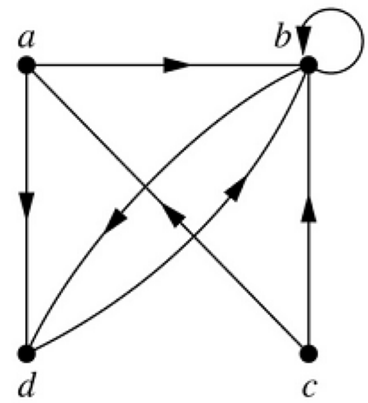
Relation R on Set A is a relation from A to A (⊆ A x A)

R = {(a, d),(a, b),(b, b),(b, d), (c, b), (c, a), (d, b)}

We can represent this relation as a matrix where the row and the column headings are the elements of the set. Value 1 for a pair indicates they are related. For example here (a,b) belongs to R hence is marked 1 whereas (a,a) and (a,c) does not and therefore these entries are marked 0.

	a	b	c	d
a	0	1	0	1
b	0	1	0	1
c	1	1	0	0
d	0	1	0	0

In the form of digraph, an edge is drawn from element x to y for each ordered pair (x,y) as shown in the figure below for R = {(a, d),(a, b),(b, b),(b, d), (c, b), (c, a), (d, b)}.



8.Properties of a Relation :

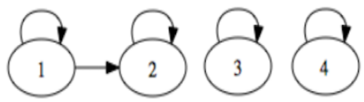
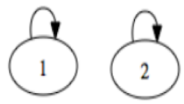
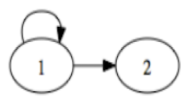
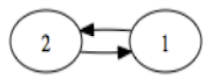
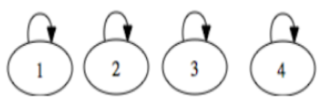
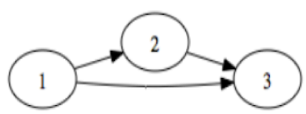
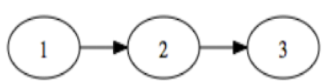
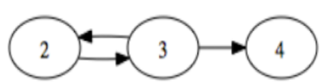
A relation R on a set A (A = {1, 2, 3, 4}) is

Reflexive : iff (a,a)∈R for every element a∈A.
R = {(1, 1), (2, 2), (3, 3), (4, 4), (1, 2)} is reflexive

Symmetric : iff (b,a)∈R whenever (a,b)∈R, for all a,b∈A.
R = {(1, 2), (2, 1), (1, 4), (4, 1), (3, 3)} symmetric

Transitive : iff (a,c)∈R whenever (a,b)∈R and (b,c)∈R, for all a,b,c∈R.
R = {(1,2), (2,1), (1,1)}

Provided are a few examples of relation with the properties identified:

Relation	Reflexive	Symmetric	Transitive
	Y	N	Y
	Y	Y	Y
	N	N	Y
	N	Y	N
	Y	Y	Y
	N	N	Y
	N	N	N
	N	N	N

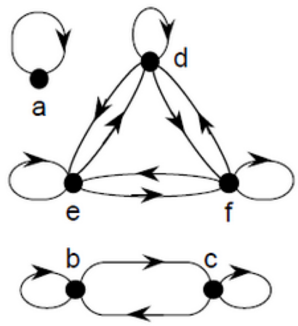
9.Equivalence Relation

Denoted by x ≡ y

A relation R on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

Example: Let S = {a,b,c,d,e,f}.

Relation R on set S be
{(a,a),(b,b), (b,c),(c,b),(c,c),(d,d),(d,e),(d,f),(e,d),(e,e),(e,f),(f,d),(f,e),(f,f)}



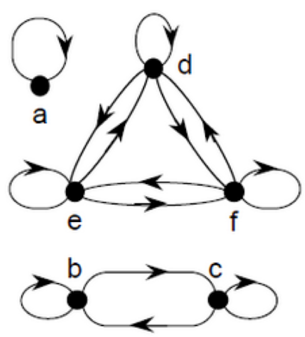
	a	b	c	d	e	f
a	1					
b		1	1			
c		1	1			
d				1	1	1
e				1	1	1
f				1	1	1

10.Equivalence Class

Let R be an equivalence relation on a set A.

The set of all the elements that are related to an element ‘a’ of A is called the equivalence class of ‘a’.

Denoted by $[a]_R$ or [a] when the relation is implicit.
Given the relation :



	a	b	c	d	e	f
a	1					
b		1	1			
c		1	1			
d				1	1	1
e				1	1	1
f				1	1	1

We can say,
[b] = {b, c}
[d] = {d, e, f}