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Generating Functions

Consider a sequence of real numbers $a_0, a_1, a_2, a_3 \ldots$ Let us denote this sequence by $\langle a_r \rangle$ $r = 0, 1, 2, \ldots$ or $\langle a_r \rangle$

Given this sequence suppose there exists a function $f(x)$ whose expansion in a series of powers of x is given by

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_{n-1} x^{n-1} + \cdots$$

$$\sum_{r=0}^{\infty} a_r x^r$$

Given a sequence $\langle a_r \rangle$ if there exists a function $f(x)$ such that a_r is the coefficient of x^r in the expansion of $f(x)$ in a series of powers of x, then $f(x)$ is called a generating function of $\langle a_r \rangle$

If $f(x)$ is a generating function of the sequence $\langle a_r \rangle$ we say that $f(x)$ generates the sequence $\langle a_r \rangle$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 \cdots \quad \sum_{r=0}^{\infty} x^r$$

$f(x) = (1-x)^{-1}$ is the generating function for the sequence $1, 1, 1, 1 \ldots$

Since

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots \sum_{r=0}^{\infty} (-1)^r x^r$$

$f(x) = (1+x)^{-1}$ is the generating function for the sequence $1, -1, 1, -1, 1 \ldots\ldots$

In general

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2} x^2 + \frac{n(n-1)(n-2)}{1.2.3} x^3 + \cdots$$

$$= \sum_{r=0}^{\infty} \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} x^r$$

for any real number n

we note that $f(x) = (1+x)^n$ is the generating function for the sequence

$$1, \frac{n}{1!}, \frac{n(n-1)}{2!}, \frac{n(n-1)(n-2)}{3!} \cdots\cdots$$

If n is a positive integer, the expansion terminates with the term containing x^n. In this case, $(1+x)^n$ generates the sequence

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots \binom{n}{n}, 0, 0, 0 \ldots$$

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Properties

The following are the properties of generating functions

1. If $f(x)$ is the generating function for a sequence $\langle a_r \rangle$ and $g(x)$ is the generating function for a sequence $\langle b_r \rangle$ then $p f(x) + q g(x)$ is the generating function for the sequence $\langle p a_r + q b_r \rangle$ where p and q are any two real numbers.

2. If $f(x)$ is the generating function for a sequence $\langle a_r \rangle$ then $x f'(x)$ (where $f'(x)$ is the derivative of $f(x)$) is the generating function for the sequence $\langle r a_r \rangle$

1. Find the sequences generated by the following functions

(i) $(3+x)^3$

$$(3+x)^3 = 27\left(1+\frac{x}{3}\right)^3$$

$$= 27 \times \left\{1 + \binom{3}{1}\left(\frac{x}{3}\right) + \binom{3}{2}\left(\frac{x}{3}\right)^2 + \binom{3}{3}\left(\frac{x}{3}\right)^3\right\}$$

$$= 27\left(1 + x + \frac{x^2}{3} + \frac{x^3}{27}\right)$$

$$= 27 + 27x + 9x^2 + x^3$$

This shows that the sequence generated by $(3+x)^3$ is

$$27, 27, 9, 1, 0, \ldots 0 \ldots$$

(ii) $2x^2(1-x)^{-1} = 2x^2\left(1+x+x^2+x^3+\ldots\right)$

$$= 0 + 0x + 2x^2 + 2x^3 + 2x^4 + \ldots$$

This shows that the sequence generated by $2x^2(1-x)^{-1}$ is

$$0, 0, 2, 2, 2 \ldots$$

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(iii) $\frac{1}{1-x} + 2x^3 = (1-x)^{-1} + 2x^3$

$= (1+x+x^2+x^3+\cdots) + 2x^3$

$= 1+x+x^2+3x^3+x^4+x^5+\cdots$

This shows that the sequence generated by $\frac{1}{1-x}+2x^3$ is

1, 1, 1, 3, 1, 1, ----.

2. Find the generating functions for the following sequences.

(i) 1, 2, 3, 4

$(1-x)^{-2} = 1+2x+3x^2+4x^3+\cdots$

$\frac{1}{(1-x)^2}$ It follows that $f(x) = (1-x)^{-2}$ is the generating function for the sequence 1, 2, 3, 4 ----.

(ii) 1, −2, 3, −4

$(1+x)^{-2} = 1-2x+3x^2-4x^3+\cdots$

$f(x) = (1+x)^{-2}$ is the generating function for the sequence 1, −2, 3, −4 ...

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(iii) 0, 1, 2, 3 - - - - - .

$0+x+2x^2+3x^3+ - - - .$

$= x(1+2x+3x^2+ - - -) = x(1-x)^{-2}$

$\therefore f(x) = x(1-x)^{-2}$ is the generating function for the sequence 0, 1, 2, 3 - - .

(iv) 0, 1, -2, 3, -4

$0+x-2x^2+3x^3-4x^4+ - - -$

$= x(1-2x+3x^2-4x^3+ - - -)$

$= x(1+x)^{-2}$

$\therefore f(x) = x(1+x)^{-2}$ is the generating function for the sequence

0, 1, -2, 3, -4, - - - - .

3. Find the generating functions for the following sequences

(i) $1^2, 2^2, 3^2, - - - .$

We have

$0+1x+2x^2+3x^3+ - - - .$

$= x(1-x)^{-2}$

Differentiate the above sequence

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$$= \frac{d}{dx}\left[\frac{x}{(1-x)^2}\right] = \frac{1+x}{(1-x)^3}$$

$$f(x) = \frac{1+x}{(1-x)^3}$$

This shows that $f(x) = \frac{1+x}{(1-x)^3}$ is the generating function for the sequence $1^2, 2^2, 3^2, 4^2, \ldots\ldots$

(ii) $0^2, 1^2, 2^2, 3^2, \ldots\ldots$

$$0^2 + 1^2 x + 2^2 x^2 + 3^2 x^3 + \cdots = x\left(1^2 + 2^2 x + 3^2 x^2 + \cdots\right) \rightarrow (2)$$

$$= \frac{x(1+x)}{(1-x)^3} = f(x)$$

This shows that $f(x) = \frac{x(1+x)}{(1-x)^3}$ is the generating function for the sequence $0^2, 1^2, 2^2, 3^2, \ldots$

(iii) $1^3, 2^3, 3^3, \ldots\ldots$

differentiate eqn (2)

$$1 + 2^3 x + 3^3 x^2 + \cdots = \frac{d}{dx}\left[\frac{x(1+x)}{(1-x)^3}\right]$$

$$f(x) = \frac{x^2 + 4x + 1}{(1-x)^4}$$

is the generating sequence for the sequence $1^3, 2^3, 3^3, \ldots$

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(iv) $0^3, 1^3, 2^3, 3^3, \ldots$

$0^3 + 1^3 x + 2^3 x^2 + 3^3 x^3 + \ldots$

$= x\left(1^3 + 2^3 x + 3^3 x^2 + \ldots\right)$

$= \frac{x\left(x^2 + 4x + 1\right)}{(1-x)^4}$

4. Find the generating function for each of the following sequences.

(i) $1, 1, 0, 1, 1, \ldots$

$a_0 = 1, \quad a_1 = 1, \quad a_2 = 0 \quad a_3 = 1, \quad a_4 = 1$

$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

$1 + x + 0 \cdot x^2 + x^3 + x^4 + \ldots$

$= (1-x)^{-1} - x^2$

(ii) $0, 2, 6, 12, 20, 30, 42, \ldots$

$a_0 = 0 = 0 + 0$
$a_1 = 2 = 1 + 1$
$a_2 = 6 = 2 + 2^2$
$a_3 = 12 = 3 + 3^2$
$a_4 = 20 = 4 + 4^2$

$a_r = r + r^2$ for $r = 0, 1, 2 \ldots$

$\langle r \rangle = 0, 1, 2, 3, \ldots$ is

$$f(x) = \frac{x}{(1-x)^2}$$

$\langle r^2 \rangle = 0^2, 1^2, 2^2, 3^2 \ldots$ is

$$g(x) = \frac{x(1+x)}{(1-x)^3}$$

The generating function for the given sequence is

$$f(x) + g(x) = \frac{x}{(1-x)^2} + \frac{x(1+x)}{(1-x)^3} = \frac{2x}{(1-x)^3}$$

(iii) $8, 26, 54, 92, \ldots$

$a_r = 3(r+1) + 5(r+1)^2$ for $r = 0, 1, 2, \ldots$

$3 \times 2 + 5 \times 4$

$6 + 20$

Generating function for the sequence

$\langle r+1 \rangle = 1, 2, 3, \ldots$ is

$$f(x) = \frac{1}{(1-x)^2}$$

$\langle (r+1)^2 \rangle = 1^2, 2^2, 3^2, \ldots$

$$g(x) = \frac{(1+x)}{(1-x)^3}$$

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Therefore the generating function for the given sequence

$$3f(x) + 5g(x) = \frac{3}{(1-x)^2} + \frac{5(1+x)}{(1-x)^3}$$

$$= \frac{8+2x}{(1-x)^3}$$

Find the generating function for a_r = the number of non negative integral solution of $e_1+e_2+e_3+e_4+e_5=$ where $0 \le e_1 \le 3$, $0 \le e_2 \le 3$, $2 \le e_3 \le 6$, $2 \le e_4 \le 6$, e_5 is odd and $1 \le e_5 \le 9$

$A_1(x) = 1+x+x^2+x^3$

$A_2(x) = 1+x+x^2+x^3$

$A_3(x) = x^2+x^3+x^4+x^5+x^6$

$A_4(x) = x^2+x^3+x^4+x^5+x^6$

$A_5(x) = x+x^3+x^5+x^7+x^9$

Thus the generating function we want is

$A_1(x)\, A_2(x)\, A_3(x)\, A_4(x)\, A_5(x)$

$= (1+x+x^2+x^3)^2 (x^2+x^3+x^4+x^5+x^6)^2$
$(x+x^3+x^5+x^7+x^9)$

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Generating function as a counting Technique

Suppose we wish to determine the number of integer solutions of the equation

$x_1 + x_2 + x_3 + \cdots x_n = r$ where $n \geq r \geq 0$

under the constraints that

x_1 can take the integer values $P_{11}, P_{12}, P_{13} \ldots$
x_2 can take the integer values $P_{21}, P_{22}, P_{23} \ldots$
$\vdots$
x_n can take the integer values $P_{n1}, P_{n2} \ldots$

we first define the functions $f_1(x), f_2(x) \ldots f_n(x)$ such that

$f_1(x) = x^{P_{11}} + x^{P_{12}} \ldots$

$f_2(x) = x^{P_{21}} + x^{P_{22}} + x^{P_{23}} + \ldots$

$\vdots$

$f_n(x) = x^{P_{n1}} + x^{P_{n2}} + \ldots$

$f(x)$ is defined by

$f(x) = f_1(x) \cdot f_2(x)\, f_3(x) \ldots f_n(x)$

and determine the Coefficient of x^r in this function. This coefficient happens to be equal to the number of solutions that we desired to find. The function $f(x)$ is called the generating

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function for the problem.

1. Find the generating function that determines the number of non negative integer solutions of the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

under the constraints $x_1 \leq 3$, $x_2 \leq 4$, $2 \leq x_3 \leq 6$ $2 \leq x_4 \leq 5$ x_5 is odd and $x_5 \leq 9$.

$$f_1(x) = x^0 + x^1 + x^2 + x^3$$

$$f_2(x) = x^0 + x^1 + x^2 + x^3 + x^4$$

$$f_3(x) = x^2 + x^3 + x^4 + x^5 + x^6$$

$$f_4(x) = x^2 + x^3 + x^4 + x^5$$

$$f_5(x) = x^1 + x^3 + x^5 + x^7 + x^9$$

$$f(x) = f_1(x)\, f_2(x)\, f_3(x)\, f_4(x)\, f_5(x)$$

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2. Using generating functions find the number of nonnegative and positive integer solutions of the equation

$x_1 + x_2 + x_3 + x_4 = 25$

$f_i(x) = x^0 + x^1 + x^2 + x^3 + \cdots$ for $i = 1, 2, 3, 4$

$f(x) = f_1(x)\, f_2(x)\, f_3(x)\, f_4(x)$

$= (x^0 + x^1 + x^2 + x^3 + \cdots)^4$

$= (1 + x + x^2 + \cdots)^4 = \left((1-x)^{-1}\right)^4$

$= (1-x)^{-4} = \sum_{r=0}^{\infty} \binom{3+r}{r} x^r$

The coefficient of x^{25} is

$\binom{3+25}{25} = \frac{28!}{25!\,3!} = 3276$

Thus the given equation has 3276 nonnegative integer solutions

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ii) In case of positive integer solutions x_i's can take values 1, 2, 3, 4

$$f_i(x) = x + x^2 + x^3 + \cdots \text{ for } i = 1, 2, 3, 4$$

$$f(x) = x^4 (1 + x + x^2 + x^3 + \cdots)^4$$

$$= x^4 \left((1-x)^{-1}\right)^4$$

$$= x^4 (1-x)^{-4} = x^4 \sum_{r=0}^{\infty} \binom{3+r}{r} x^r$$

The coefficient of x^{25} is

$$\binom{3+21}{21} = \frac{24!}{21!\,3!} = 2024$$

Thus the given equation has 2024 positive integer solutions

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3. Find the number of integer solutions of the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 30$$

$x_i \geq 0$ for $i = 1, 2, 3, 4, 5$ x_2 is even
x_3 is odd.

$$f_1(x) = x^0 + x^1 + x^2 + \cdots = (1-x)^{-1}$$

$$f_2(x) = x^0 + x^2 + x^4 \cdots = (1-x^2)^{-1}$$

$$f_3(x) = x + x^3 + x^5 + \cdots = x(1-x^2)^{-1}$$

$$f_4(x) = x^0 + x^1 + x^2 + \cdots = (1-x)^{-1}$$
$$f_5(x) = x^0 + x^1 + x^2 + \cdots = (1-x)^{-1}$$

$$f(x) = f_1(x)\, f_2(x)\, f_3(x)\, f_4(x)\, f_5(x)$$

$$= x(1-x^2)^{-2}(1-x)^{-3}$$

$$= x \sum_{r=0}^{\infty} \binom{2+r-1}{r} (x^2)^r \times \sum_{s=0}^{\infty} \binom{2+s}{s} x^s$$

$$= \sum_{r=0}^{\infty} \binom{1+r}{r} x^{2r+1} \times \sum_{s=0}^{\infty} \binom{2+s}{s} x^s$$

The coefficient of x^{30} is

$$\binom{1}{0}\binom{31}{29} + \binom{2}{1}\binom{29}{27} + \binom{3}{2}\binom{27}{25} + \cdots$$

$$\binom{14}{13}\binom{5}{3} + \binom{15}{14}\binom{3}{1}$$

$= 455 - 366 + \ldots$

A bag contains large number of red, green white and black marbles with atleast 24 of each color. In how many ways can one select 24 of these marbles, so that there are even number of white marbles and atleast six black marbles?

$x_1 + x_2 + x_3 + x_4 = 24$

$x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 = 0, 2, 4 \ldots \quad x_4 \geq 6$

$f_1(x) = x^0 + x^1 + x^2 + \ldots$

$f_2(x) = x^0 + x^1 + x^2 + \ldots$

$f_3(x) = x^0 + x^2 + x^4 + \ldots$

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$f_4(x) = x^6 + x^7 + x^8 + \dots$

$f(x) = (1+x+x^2+\dots)(1+x+x^2+\dots)(1+x^2+x^4+\dots)$
$(x^6+x^7+x^8+\dots)$

$= (1-x)^{-1}(1-x)^{-1}(1-x^2)^{-1} \times x^6(1-x)^{-1}$

$= x^6 (1-x)^{-3} (1-x^2)^{-1}$

$= x^6 \sum_{r=0}^{\infty} \binom{2+r}{r} x^r \sum_{s=0}^{\infty} (x^2)^s$

the coefficient of x^{24} is

$\binom{20}{18} + \binom{18}{16} + \binom{16}{14} + \binom{14}{12} + \binom{12}{10} + \binom{10}{8}$

$+ \binom{8}{6} + \binom{6}{4} + \binom{4}{2} + \binom{2}{0}$

$= 190 + 153 + 120 + 91 + 66 + 45 + 28 + 15$
$+ 6 + 1$
$= 715$

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Exponential Generating function :

Given a sequence $\langle a_r \rangle$ suppose there exists a function $E(x)$ such that the expansion of $E(x)$ is a series of powers of x given by

$$E(x) = a_0 + a_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + a_n \frac{x^n}{n!}$$

$$= \sum_{r=0}^{\infty} a_r \frac{x^r}{r!}$$

Then $E(x)$ is called the exponential generating function for the sequence $\langle a_r \rangle$

In other words, given a sequence $\langle a_r \rangle$ if there exists a function $E(x)$ such that a_r is the coefficient of $x^r/r!$ then $E(x)$ is called the generating function of the sequence $\langle a_r \rangle$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$E(x) = e^x$ is the exponential generating function for the sequence $1, 1, 1, \ldots$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$E(e^{-x})$ is the generating function for the sequence $1, -1, 1, -1, \ldots$

$$\frac{1}{2}(e^{x} + e^{-x}) = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)$$

$\therefore E(x) = \frac{1}{2}(e^{x} + e^{-x})$ is the exponential generating function for the sequence $1, 0, 1, 0, \ldots$

$\frac{1}{2}(e^{x} - e^{-x})$ is the exponential generating function for the sequence $0, 1, 0, 1, \ldots$

Properties

The following are some properties of exponential generating functions

1. If $E(x)$ is the exponential generating function for the sequence $\langle a_r \rangle$ then $xE'(x)$ is the exponential generating function for the sequence $\langle r a_r \rangle$

2. If $E_1(x)$ is the exponential generating function for a sequence $\langle a_r \rangle$ and $E_2(x)$ is the exponential generating function for the sequence $\langle b_r \rangle$ then $pE_1(x) + qE_2(x)$ is the exponential

generating function for the sequence $\langle pa_r + q b_r \rangle$ for real numbers p and q.

Determine the sequences of

(1) $6e^{5x} - 3e^{2x}$

$$= 6\sum_{r=0}^{\infty} \frac{(5x)^r}{r!} - 3\sum_{r=0}^{\infty} \frac{(2x)^r}{r!}$$

$$= \sum_{r=0}^{\infty} (6\times 5^r - 3\times 2^r) \frac{x^r}{r!}$$

Therefore the sequence generated by $6e^{5x} - 3e^{2x}$ is

$$\langle 6\times 5^r - 3\times 2^r \rangle$$

(2) $e^{2x} - 3x^3 + 5x^2 + 7x$

$$= \sum_{r=0}^{\infty} \frac{(2x)^r}{r!} - 3x^3 + 5x^2 + 7x$$

$$= 1 + (2+7)x + \left(\frac{2^2}{2!} + 5\right)x^2 + \left(\frac{2^3}{3!} - 3\right)$$

$$+ \sum_{r=4}^{\infty} \frac{2^r x^r}{r!}$$

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$a_0 = 1 \quad a_1 = 9 \quad a_2 = 14 \quad a_3 = -10$

$a_r = 2^r$ for $r \geq 4$

(3) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

$= \sum_{r=0}^{\infty} x^r = \sum_{r=0}^{\infty} r! \cdot \frac{x^r}{r!}$

Therefore the sequence $\langle a_r \rangle = \langle r! \rangle$

4) $\frac{3}{1-2x} + e^x = 3(1-2x)^{-1} + e^x$

$3 \times \sum_{r=0}^{\infty} (2x)^r + \sum_{r=0}^{\infty} \frac{x^r}{r!}$

$= \sum_{r=0}^{\infty} ((3 \times 2^r \times r!) + 1) \frac{x^r}{r!}$

$\langle a_r \rangle = \langle 1 + 3 \times 2^r \times r! \rangle$

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Find the exponential generating function for the following sequences

(1) $1, 2, 3, 0, 0, 0, \ldots$

$E(x) = 1 + 2x + 3 \cdot \frac{x^2}{2!} + 0 + 0 + 0 \ldots$

$= 1 + 2x + \frac{3}{2} x^2$

(2) $0, 0, 1, 1, 1, \ldots$

$E(x) = 0 + 0 + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

$= e^x - 1 - x$

(3) $1, a, a^2, a^3, a^4, \ldots$

$E(x) = 1 + ax + \frac{a^2 x^2}{2!} + \frac{a^3 x^3}{3!} + \ldots$

$E(x) = e^{ax}$

(4) $1, -a, a^2, -a^3 \ldots\ldots$

$1 - ax + a^2\dfrac{x^2}{2!} - a^3\dfrac{x^3}{3!} + \ldots$

$= e^{-ax}$

(5) $0, 1, 2a, 3a^2, 4a^3, \ldots$

$0 + x + 2a\dfrac{x^2}{2!} + 3a^2\dfrac{x^3}{3!} + \ldots$

$= \cancel{x\left(1 + \dfrac{2ax}{2!} + 3a^2x^2\right.}$

$= x\left(1 + ax + a^2\dfrac{x^2}{2!} + a^3\dfrac{x^3}{3!} + \ldots\right)$

$= x e^{ax}$

(6) $a, a^3, a^5, a^7, \ldots$

$= a + a^3x + a^5\dfrac{x^2}{2!} + a^7\dfrac{x^3}{3!} + \ldots$

$a\left(1 + a^2x + \dfrac{(a^2x)^2}{2!} + \dfrac{(a^2x)^3}{3!} + \ldots\right)$

$= a e^{a^2x}$