(4) Suppose $f(n) = \alpha b^n$ where $\alpha$ is a constant and b is a root of multiplicity m of the characteristic equation of the homogenous part of the relation

$$a_n^{(P)} = A_0 n^m b^n$$

where $A_0$ is a constant to be evaluated using the fact that $a_n = a_n^{(P)}$

Solve the recurrence relation

$$a_n + 4a_{n-1} + 4a_{n-2} = 8 \quad \text{for } n \geq 2$$

$a_0 = 1 \quad a_1 = 2$

The characteristic equation for the homogenous part is

$$K^2 + 4K + 4 = 0$$

$$(K+2)^2 = 0 \quad K = -2$$

$$a_n^{(h)} = (A + Bn)(-2)^n$$

A and B are arbitrary constants.

The RHS of the Recurrence relation is 8

$f(n) = 8$ is a polynomial of degree $q = 0$ and 1 is not a root of the characteristic equation. hence

$$a_n^{(p)} = A_0$$

put this $a_n^{(p)}$ in the given recurrence relation

$$A_0 + 4A_0 + 4A_0 = 8$$

$$A_0 = \frac{8}{9} = a_n^{(p)}$$

The general solution for $a_n$ is

$$a_n = a_n^{(h)} + a_n^{(p)}$$
$$= (A + Bn)(-2)^n + \frac{8}{9}$$

$a_0 = A + \frac{8}{9} = 1$ $\qquad 9A + 8 = 9$

$9A = 1 \qquad A = 1/9$

$B = -2/3$

$a_1 = -2A - 2B + \frac{8}{9} = 2$

hence $a_n = \left(\frac{1}{9} - \frac{2}{3}n\right)(-2)^n + \frac{8}{9}$

4 3 72

Solve the recurrence relation

$$a_{n+2}^2 - 5a_{n+1}^2 + 6a_n^2 = 7n \quad \text{for } n \geq 0$$

let $b_{n+2} - 5b_{n+1} + 6b_n = 7n$

The characteristic equation is

~~Let~~

$$K^2 - 5K + 6 = 0$$

$$K^2 - 2K - 3K + 6 = 0$$

$$(K-2)(K-3) = 0 \qquad K_1 = 2 \quad K_2 = 3$$

$$b_n^{(h)} = A \times 3^n + B \times 2^n$$

$$b_n^{(p)} = A_0 + A_1 n$$

$$[A_0 + A_1(n+2)] - 5[A_0 + A_1(n+1)] + 6(A_0 + A_1 n) = 7n$$

$$A_0 + A_1 n + 2A_1 - 5A_0 - 5A_1 n - 5A_1 + 6A_0 + 6A_1 n = 7n$$

$$A_0 + 2A_1 - 5A_0 - 5A_1 + 6A_0 = 0$$

$$2A_0 - 3A_1 = 0$$

$A_1 n \cancel{0} - 5 A_1 n + 6 A_1 n)$

$2 A_1 = 7 \qquad A_1 = 7/2$

$2 A_0 - 3 \times \frac{7}{2} = 0$

$2 A_0 = \frac{21}{2} \qquad A_0 = \frac{21}{4}$

$b_n^{(p)} = \frac{7}{2} n + \frac{21}{4}$

$b_n = b_n^{(h)} + b_n^{(p)}$

$= A(3)^n + B(2^n) + \frac{7}{2} n + \frac{21}{4}$

given $a_0 = 1 \qquad b_0 = a_0^2 = 1$

$a_1 = 1 \qquad b_1 = a_1^2 = 1$

$b_0 = A + B + \frac{21}{4} = 1$

$4A + 4B + 21 = 4 \qquad 4A + 4B = -17$

$b_1 = 3A + 2B + \frac{7}{2} + \frac{21}{4} = 1$

$= \frac{12A + 8B + 14 + 21}{4} = 1$

$=$

$$4 = 12A + 8B + 35$$

$$\begin{array}{rcr} 12A + \cancel{8B} & = & -31 \\ 8A + \cancel{8B} & = & -34 \\ - \quad\;\; + & & + \\ \hline 4A & = & 3 \end{array}$$

$A = 3/4 \quad B = -5$

$$b_n = \frac{3}{4}(3)^n - 5(2^n) + \frac{7}{2}n + \frac{21}{4}.$$

$$a_n = \pm\sqrt{b_n} = \pm\left[\frac{3}{4}3^n - 5(2)^n + \frac{7}{2}n + \frac{21}{4}\right]^{1/2}$$

# Solving Recurrence Relations by Generating Functions

Consider a recurrence relation of first order

$$a_n = c\,a_{n-1} + f(n) \quad \text{for } n \geq 1$$

$$a_{n+1} = c\,a_n + f(n+1) \quad \text{for } n \geq 0$$

$$a_{n+1} = c\,a_n + \phi(n) \quad \text{where } \phi(n) = f(n+1)$$

Multiply both sides by $x^{n+1}$

$$a_{n+1} x^{n+1} = c\,a_n x^{n+1} + \phi(n) x^{n+1} \quad n \geq 0.$$

Sum all relations got from this relation for $n = 0, 1, 2, 3, \ldots$

$$\sum_{n=0}^{\infty} a_{n+1} x^{n+1} = \sum_{n=0}^{\infty} c\,a_n x^{n+1} + \sum_{n=0}^{\infty} \phi(n) x^{n+1} \quad n \geq 0$$

$$= \sum_{n=1}^{\infty} a_n x^n - \sum_{n=0}^{\infty} c\,a_n x^{n+1} = x \sum_{n=0}^{\infty} \phi(n) x^n$$

$$= \sum_{n=1}^{\infty} a_n x^n - c x \sum_{n=0}^{\infty} a_n x^n = x \sum_{n=0}^{\infty} \phi(n) x^n$$

$$\therefore \left[\sum_{n=0}^{\infty} a_n x^n - a_0\right] - cx \sum_{n=0}^{\infty} a_n x^n = x \sum_{n=0}^{\infty} \phi(n) x^n$$

$$(1-cx) \sum_{n=0}^{\infty} a_n x^n = a_0 + x \sum_{n=0}^{\infty} \phi(n) x^n$$

If $f(x)$ is the generating function for $\sum_{n=0}^{\infty} a_n x^n$ and $g(x)$ is the generating function for $\sum_{n=0}^{\infty} \phi(n) x^n$

$$(1-cx) f(x) = a_0 + x\, g(x)$$

$$f(x) = \frac{a_0 + x\, g(x)}{1-cx}.$$

where $c$ is an arbitrary constant.

1) Find the generating function for the recurrence relation

$$a_{n+1} - a_n = 3^n \quad n \geq 0.$$

and $a_0 = 1$ Hence solve the relation.

$$a_{n+1} = a_n + 3^n \quad n \geq 0$$

$$c = 1 \quad \phi(n) = 3^n$$

$$f(x) = \frac{a_0 + x\,g(x)}{1 - cx} = \frac{1 + x\,g(x)}{1 - x}$$

$$g(x) = \sum_{n=0}^{\infty} \phi(n)\,x^n = \sum_{n=0}^{\infty} 3^n x^n$$

$$= \sum_{n=0}^{\infty} (3x)^n = \frac{1}{1-3x}$$

$$f(x) = \frac{1 + \frac{x}{1-3x}}{1-x} = \frac{1 - 3x + x}{(1-x)(1-3x)}$$

$$= \frac{1 - 2x}{(1-x)(1-3x)}$$

$$\frac{1-2x}{(1-3x)(1-x)} = \frac{A}{1-x} + \frac{B}{1-3x}$$

$$1-2x = A(1-3x) + B(1-x)$$

$$1-2x = A - 3Ax + B - Bx$$

$$1-2x = A + B - x(3A+B)$$

$$A + B = 1$$

$$-3A - B = -2$$

$$-2A = -1$$

$$A = 1/2 \qquad B = 1/2$$

$$f(x) = \frac{1}{2}\left[\frac{1}{1-x} + \frac{1}{1-3x}\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} (3x)^n\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} (1)^n x^n + \sum_{n=0}^{\infty} 3^n x^n\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} (1+3^n)\, x^n$$

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{2}\sum_{n=0}^{\infty} (1+3^n)\, x^n$$

$$a_n = \frac{1}{2}(1+3^n)$$

②. Using ~~the~~ generating f

the recurrence relation

$a_n - 3a_{n-1} = n \quad n \geq 1 \quad a_0 = 1$

$a_{n+1} = 3a_n + (n+1)$ for $n \geq 0$ $a_0 = 1$

$\phi(n) = n+1 \quad c = 3 \quad a_0 = 1$

$$f(x) = \frac{a_0 + x\, g(x)}{1 - cx} = \frac{1 + x\, g(x)}{1 - 3x}$$

$$g(x) = \sum_{n=0}^{\infty} (n+1)\, x^n$$

$g(x)$ is the generating function for the sequence $1, 2, 3, \ldots$

$g(x) = (1-x)^2$

$$f(x) = \cancel{\frac{1}{1-3x}} \cdot \frac{1 + \frac{x}{(1-x)^2}}{1-3x}$$

$$= \frac{(1-x)^2 + x}{(1-3x)(1-x)^2} = \frac{1 + x^2 - 2x + x}{(1-3x)(1-x)^2}$$

$$= \frac{1 + x^2 - x}{(1-3x)(1-x^2)}$$

$$\frac{x^2-x+1}{(1-3x)(1-x)^2} = \frac{A}{(1-x)} + \frac{B}{(1-x)^2} + \frac{C}{1-3x}$$

$$x^2-x+1 = A(1-x)(1-3x) + B(1-3x) + C(1-x)^2$$

$$x^2-x+1 = A(1-3x-x+3x^2) + B - B3x + C(1+x^2-2x)$$

$$= \cancel{A-3Ax} \quad A(1-4x+3x^2) + B - 3Bx + C(1+x^2-2x)$$

$$= A - 4Ax + 3Ax^2 + B - 3Bx + C + Cx^2 - 2Cx$$

$$x^2-x+1 = (A+B+C) + x(-4A-3B-2C) +$$

$$x^2(3A+C)$$

$A+B+C = 1 \longrightarrow (1).$

$-4A-3B-2C = -1$ a $4A+3B+2C=1 \longrightarrow (2).$

$3A+C = 1 \longrightarrow (3).$

$$\begin{aligned} A+B+C &= 1 \\ 3A+C &= 1 \\ - \quad - \quad &- \\ \hline -2A+B &= 0 \end{aligned}$$

$$\begin{aligned} 4A+2B+2C &= 1 \\ 6A \quad +2C &= 2 \\ - \quad - \quad &- \\ \hline -2A+2B &= -1 \end{aligned}$$

$-2A + B = 0$

$-2A + 2B = -1$

$-B = 1$

$B = -1$

$-2A + B = 0$

$-2A - 1 = 0$

$-2A = 1$

$A = -1/2$

~~$f(x) = -\frac{1}{2}(1-x)^{-1}$~~

$A + B + C = 1$

$-\frac{1}{2} - 1 + C = 1$

$C = 1 + 1 + \frac{1}{2}$

$= 5/2$

$$f(x) = -\frac{1}{2}(1-x)^{-1} - 1(1-x)^{-2} + \frac{5}{2}(1-3x)^{-1}$$

$$\sum_{n=0}^{\infty} a_n x^n = -\frac{1}{2}\sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty}(n+1)x^n + \frac{5}{2}\sum_{n=0}^{\infty}(3x)^n$$

$$a_n = -\frac{1}{2} - (n+1) + \frac{5}{2}3^n$$

$$= -\frac{1}{2} - n - 1 + \frac{5}{2}3^n$$

$$= -\frac{3}{2} - n + \frac{5}{2}3^n$$

$$a_n = \frac{5}{2}3^n - n - \frac{3}{2}$$

(3). $a_n - 4a_{n-1} = 0$ for $n \geq 1$ $a_0 = 1$

$a_{n+1} = 4a_n$ for $n \geq 0$

$c = 4$ $\phi(n) = 0$ $a_0 = 1$

$$f(x) = \frac{a_0 + x\,g(x)}{1 - cx} = \frac{1 + x(0)}{1 - 4x}$$

$$f(x) = \frac{1}{1-4x}$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (4x)^n$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 4^n x^n$$

$a_n = 4^n$.

(4). $a_n - a_{n-1} = 2$ $n \geq 1$ $a_0 = 6$

$a_{n+1} = a_n + 2$ $n \geq 0$ $a_0 = 6$

$c = 1$ $\phi(n) = 2$ $a_0 = 6$

$$f(x) = \frac{a_0 + x\,g(x)}{1 - cx} = \frac{6 + x(2)}{1 - x}$$

$$= \frac{2x+6}{1-x} = \frac{A}{1-x} + \frac{B}{1-x}$$

$\cancel{A} =$

$$\frac{2x}{1-x} + \frac{6}{1-x}.$$

$$= (2x+6)(1-x)^{-1}$$

$$2x \sum_{n=0}^{\infty} x^n + 6 \sum_{n=0}^{\infty} x^n$$

$$= (2x+6) \sum_{n=0}^{\infty} x^n$$

$$2 \sum_{n=0}^{\infty} \cancel{x} x^{n+1} + 6 \sum_{n=0}^{\infty} x^n$$

$$a_n = 2n+6.$$