Derangements

A permutation of n distinct objects in which none of the objects is in its natural place is called a derangement.

For eg. a permutation of integers $1, 2, 3, 4 \cdots n$ in which 1 is not in the first place, 2 is not in the second place, 3 is not in the third place and so on and n is not in the n^{th} place is a derangement.

The number of possible derangements of n distinct objects $1, 2, 3, \cdots n$ is denoted by d_n.

If there is only one object it continues to be in its original place in every derangement therefore $d_1 = 0$.

If there are two objects $d_2 = 1$

$d_3 = 2$

Following is the formula for d_n for $n \geq 1$

$$d_n = n! \left\{ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \frac{(-1)^n}{n!} \right\}$$

$$= n! \sum_{K=0}^{n} \frac{(-1)^K}{K!}$$

Proof: Let S be the set of all permutations of $1, 2, 3 --- n$. Then $|S| = n!$
Let A_1 be the set of all permutations of $1, 2, 3, --- n$ where 1 is in its natural place.
A_2 be the set of all permutations where 2 is in its natural place and so on.
Then the set of all derangements of $1, 2, 3 --- n$ is $\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cdots \bar{A}_n$

$$d_n = |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \cdots \bar{A}_n|$$

$$d_n = |S| - \Sigma|A_i| + \Sigma|A_i \cap A_j| - \Sigma|A_i \cap A_j \cap A_k| + \cdots (-1)^n |A_1 \cap A_2 \cdots A_n|$$

$$= (n!) - S_1 + S_2 - S_3 + \cdots (-1)^n S_n$$

permutations in A_1 are all of the form $1 b_2 b_3 \cdots b_n$ where $b_2 b_3 \cdots b_n$ is a permutation of $2, 3, --- n$. Thus A_1 consists of all permutations of $2, 3, --- n$ $\quad |A_1| = (n-1)!$

Similarly $|A_2| = |A_3| = \cdots |A_n| = (n-1)!$

$$S_1 = \Sigma|A_i| = n(n-1)! = c(n,1)(n-1)!$$

The permutations $A_1 \cap A_2 = 1 2 b_3 b_4 \cdots b_n$

$$|A_1 \cap A_2| = (n-2)!$$

$$S_2 = c(n,2)(n-2)!$$

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$S_3 = C(n,3)(n-3)!$

$S_n = C(n,n)(n-n)! = C(n,n)$

$d_n = (n!) - C(n,1)(n-1)! + C(n,2)(n-2)!$
$- C(n,3)(n-3)! + \cdots (-1)^n C(n,n)$

$= n! \cancel{1 - \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3}}$

$= n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots (-1)^n \frac{n!}{n!}$

$= n! \left\{ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \frac{(-1)^n}{n!} \right\}$

$= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$

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Find the number of derangements of 1, 2, 3, 4

$$d_4 = 4!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right]$$

$$= 24\left[1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right]$$

$$= 12 - 4 + 1 = 9.$$

Nine derangements are

2143	4312
3142	2413
4123	3421
2341	4321
3412	

Eleven books are arranged on a shelf in alphabetical order by author name. In how many can your little sister rearrange these books so that no book is in its original position

$$d_{11} = 11!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \frac{1}{9!} + \frac{1}{10!} - \frac{1}{11!}\right]$$

2. Twenty people check their hats at a theater. In how many ways can their hats to be returned so that exactly one person receives his or her own hat

$$^5C_4 = \frac{5!}{4!}$$

$^{20}C_1 \, d_{19} = 20 d_{19}$.

3. In how many ways can the integers 1 through 9 be permuted such that no odd integer will be in its natural position

If we select one of odd integers to be in its natural position, we can do so in $^5C_1 = 5$ ways and permute the remaining 8 numbers in $8!$ ways for a total of $5 \times 8!$ ways.

If we select 2 odd integers to be in their natural position, we can do so in $^5C_2 = 10$ ways and permute the remaining 7 numbers in $7!$ ways for a total of $10 \times 7!$ ways.

If we select 3 odd integers to be in their natural position we can do so in $^5C_3 = 10$ ways and permute the remaining 6 numbers in $6!$ ways for a total of $10 \times 6!$ ways

Similarly 4 odd integers $= 5 \times 5!$
5 odd integers $= 5 \times 4!$

$\therefore$ Total permutations $= 9! - [5 \times 8! - 10 \times 7! + 10 \times 6! - 5 \times 5! + 5 \times 4!]$

Rook polynomial

consider a board that resembles a full chess board or a part of a chess board. Let n be the number of squares present in the board. Pawns are placed in the squares of the board such that not more than one pawn occupies a square. Two pawns placed on a board having 2 or more squares are said to capture each other if they are in the same row or in the same column of the board.

For $2 \le k \le n$ let r_k denote the number of ways in which k pawns can be placed on a board such that no two pawns capture each other - that is no two pawns are in the same row or in the same column of the board. Then the polynomial

$$1 + nx + r_2 x^2 + \cdots r_n x^n \rightarrow (1)$$

is called the rook polynomial for the board considered.

In trivial case $n = 1$ $r_2, r_3 \cdots = 0$

$$r(C, x) = 1 + x \rightarrow (2)$$

If r_1 is the number of squares

$$r(C, x) = 1 + r_1 x + r_2 x^2 + \cdots r_n x^n$$

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1) Consider the board which contains 4 squares

1	2
3	4

$r_1 = 4$

The number of ways in which two rooks can be placed on this board such that no two of them capture each other is 2. $r_2 = 2$ $r_3 = 0$ $r_4 = 0$

$$r(C,x) = 1 + r_1 x + r_2 x^2$$

$$= 1 + 4x + 2x^2$$

2) consider the board containing 5 squares

1	2	3
4		5

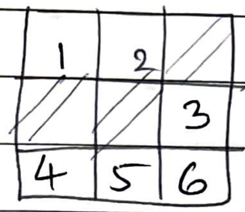
$r_1 = 5$

r_2 $(1,5)$ $(2,5)$ $(2,4)$ $(3,4)$ $= 4$

$r_3 = 0$, $r_4 = 0$ $r_5 = 0$

$$r(C,x) = 1 + 5x + 4x^2$$

3. Consider the board containg 6 squares



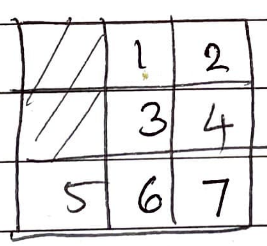
$r_1 = 6$ $\quad r_2$ $(1,3)\ (2,3)\ (1,5)\ (1,6)\ (2,4)\ (2,6)$
$(3,4)\ (3,5) = 8$

$r_3 :$ $\cancel{(1,3,4)}\ (1,3,5)\ (2,3,4)\ \cancel{(2,3,5)} = 2$

$r_4 = 0 \quad r_5 = 0 \quad r_6 = 0$

$r(C,x) = 1 + 6x + 8x^2 + 2x^3$

4. consider the board with 7 squares



$r_1 = 7$ $\quad r_2$ $(1,4)\ (1,5)\ (1,7)\ (2,3)\ (2,5)\ (2,6)$
$(3,5)\ (3,7)\ (4,5)\ (4,6) = 10$

r_3 $(1,4,5)\ (2,3,5) = 2$ $\quad r_4 = 0 \quad r_5 = 0 \quad r_6 = 0 \quad r_7 = 0$

$r(C,x) = 1 + 7x + 10x^2 + 2x^3$

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consider the board with 8 squares

1	2	3
4		5
6	7	8

$r_1 = 8$

r_2 (1,5) (1,7) (1,8) (2,4) (2,5) (2,6) (2,8)
(3,4) (3,6), (3,7) : ~~10~~ (4,7) (4,8) (5,6) (5,7)
= 14

r_3 (1,5,7) (2,4,8), (3,4,7) (2,5,6) = 4

$r_4 = 0 \quad r_5 = r_6 = r_7 = r_8 = 0$

$r(C,x) = 1 + 8x + 14x^2 + 4x^3$

1	2	3
4	5	6
7	8	9

$r_1 = 9 \quad r_2 =$ (1,5) (1,6) (1,8) (1,9)
(2,4) (2,6) (2,7) (2,9)
(3,4) (3,5) (3,7) (3,8)
(4,8) (4,9) (5,7) (5,9)
(6,7) (6,8)
= 18

$r_3 =$ (1,5,9) (3,5,7) (2,4,9) (2,6,7) (1,6,8)
(3,4,8)
$r_3 = 6$

$r_4 = 0$

$r(C,x) = 1 + 9x + 18x^2 + 6x^3$

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Rook polynomial for $n \times n$ board

consider a $n \times n$ board

$r_1 = n^2$

In this board, one can choose K ($2 \le K \le n$) rows out of n rows in $C(n, K)$ ways. After that K rooks can be placed in these K rows so that there is exactly 1 rook in each row and column in $P(n, K)$ ways.

Thus $r_K = C(n, K) \times P(n, K)$

$$= \frac{n!}{(n-K)!\,K!} \times \frac{n!}{(n-K)!}$$

$$= \frac{n!}{(n-K)!\,K!} \times \frac{n!}{(n-K)!} \cdot \frac{K!}{K!}$$

$$= K!\; C(n, K)^2$$

$$r(C_{n \times n}, x) = 1 + n^2 x + \sum_{K=2}^{n} K! \{C(n, K)\}^2 x^K$$

$$= 1 + \binom{n}{1}^2 x + 2! \times \binom{n}{2}^2 x^2 + 3! \binom{n}{3}^2 x^3$$

$$+ \cdots n! \binom{n}{n}^2 x^n$$

If $n = 2$

$$r(C_{2 \times 2}, x) = 1 + \binom{2}{1}^2 x + 2! \binom{2}{2}^2 x^2$$

$$= 1 + 4x + 2x^2$$

~~The rook polynom~~

In a given board C, suppose we choose a particular square and mark it as $\circledast$. Let D be the board obtained from C by deleting the row and the column containing $\circledast$. Let E be the board obtained from C by deleting only the square $\circledast$

$$r(C,x) = x\, r(D,x) + r(E,x)$$

This is known as the expansion formula for $r(C,x)$

1. Find the rook polynomial for the 2x2 board by using the expansion formula

*1	2
3	4

Let us mark the square numbered 1 as (*). Then the boards D and E appear as shown below

4

D

	2
3	4

E

$$r(D, x) = 1 + x$$

For E

$$r_1 = 3 \quad r_2 = 1 \quad r_3 = 0$$

$$r(E, x) = 1 + 3x + x^2$$

$$r(C_{2\times 2}, x) = x\, r(D, x) + r(E, x)$$

$$= x(1+x) + 1 + 3x + x^2$$

$$= x + x^2 + 1 + 3x + x^2$$

$$= 1 + 4x + 2x^2$$

By using the expansion formula obtain the

Let us mark the topmost square 1 in the given board as * Then the boards D and E appear as shown below

D

E

For D we have $r_1 = 5$ $r_2 = 4$ (2,4) (2,7) (4,8) (5,7)

$r_3 = r_4 = r_5 = 0$

$r(D, x) = 1 + 5x + 4x^2$

For the board E, we have $r_1 = 7$, $r_2 = 11$ (2,4) (2,6) (2,7) (3,4) (3,5) (3,7) (3,8) (4,8) (5,7) (6,7) (6,8) $r_3 = 3$ (2,6,7) (3,4,8) (3,5,7) $r_K = 0$ for $K = 4, 5, 6, 7$

$r(E, x) = 1 + 7x + 11x^2 + 3x^3$

$r(C, x) = x\, r(D, x) + r(E, x) = x(1 + 5x + 4x^2) + 1 + 7x + 11x^2 + 3x^3$

$= 1 + 8x + 16x^2 + 7x^3$

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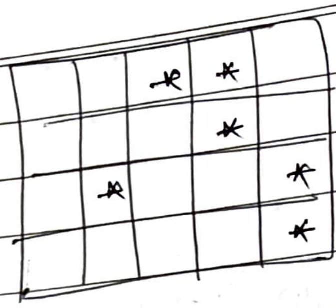
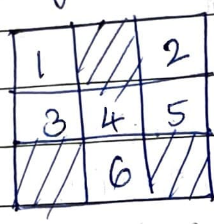
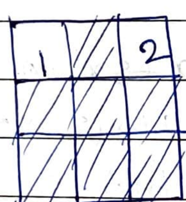


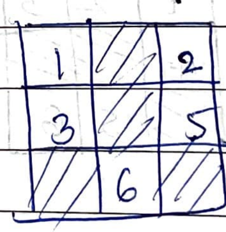
Using the expansion formula, find the rook polynomial for the board C



Square numbered * as * we get boards D and E



D



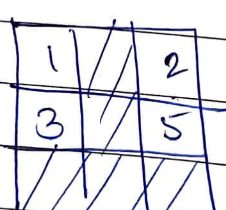
E

For Board D

$r_1 = 2 \quad r_2 = 0$

$r(D,x) = 1+2x$

In Board E let us mark square numbered 6 as *



D'

E'

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D' and E' are identical

$$r(D', x) = 1 + 4x + 2x^2$$
$$r(E', x) = 1 + 4x + 2x^2$$

$$r(E, x) = x\, r(D', x) + r(E', x)$$
$$= x(1 + 4x + 2x^2) + 1 + 4x + 2x^2$$
$$= x + 4x^2 + 2x^3 + 1 + 4x + 2x^2$$
$$= 1 + 5x + 6x^2 + 2x^3.$$

Now

$$r(C, x) = x\, r(D, x) + r(E, x)$$
$$= x(1 + 2x) + 1 + 5x + 6x^2 + 2x^3$$
$$= x + 2x^2 + 1 + 5x + 6x^2 + 2x^3$$
$$= 1 + 6x + 8x^2 + 2x^3$$

$$r(C, x) = 1 + 6x + 8x^2 + 2x^3$$

<u>Product Formula</u>

Suppose a board C is made up of two parts C_1 and C_2 where C_1 and C_2 have no squares in same row or column of C. Such parts of C are called disjoint sub boards of C.
Then the rook polynomial $r(C, x)$ for the board C has the following

property

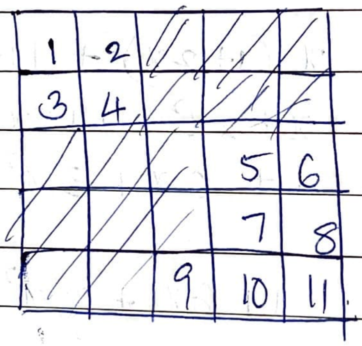
$$r(C,x) = r(C_1,x) \times r(C_2,x)$$

This is known as the product formula for $r(C,x)$.

If board C is made up of pairwise disjoint sub boards $C_1, C_2, C_3 \cdots C_n$ then

$$r(C,x) = r(C_1,x) \times r(C_2,x) \cdots r(C_n,x)$$

1. Find the rook polynomial for the board shown below.



We note that the board C is made up of two disjoint sub boards C_1 and C_2 where C_1 is 2×2 board and C_2 is a board numbered from 5 to 11

$$r(C_2,x) = 1 + 7x + 10x^2 + 2x^3$$

$$r(C_1,x) = 1 + 4x + 2x^3$$

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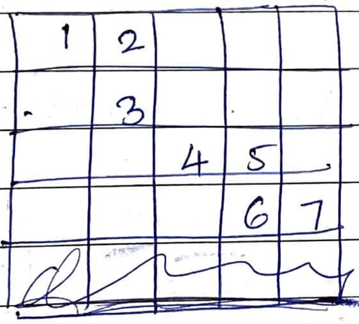
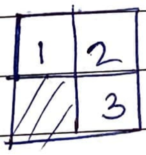
$r(C,x) = r(C_1,x)\ r(C_2,x)$

$= (1+4x+2x^2)(1+7x+10x^2+2x^3)$

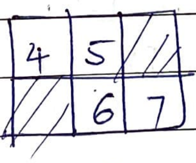
$= 1+11x+40x^2+56x^3+28x^4+4x^5$

2. A board consists of 7 squares as shown below. Find its rook polynomial





C_1



C_2

For C_1 $r_1 = 3$ $r_2 = 1$ $r_3 = 0$.

$r(C_1 x) = 1+3x+x^2$

For C_2 $r_1 = 4$ $r_2 = 3$ $r_3 = 0$ $r_4 = 0$

$r(C_2 x) = (1+4x+3x^2)$

$r(C,x) = (1+3x+x^2)(1+4x+3x^2)$

$= 1+7x+16x^2+13x^3+3x^4$

Arrangements with forbidden positions

Suppose m objects are to be arranged in n places where $n \geq m$. Suppose there are constraints under which some objects cannot occupy certain places. Such places are called the forbidden positions for the objects.

The number of ways of carrying out this task is given by

$$\bar{N} = S_0 - S_1 + S_2 - S_3 + \cdots (-1)^n S_n$$

where $S_0 = n!$

and $S_K = (n-K)!\, r_K \quad K = 1, 2, 3 \cdots n$

Here r_K is the coefficient of x^K in the rook polynomial of the board of m rows and n columns whose squares represent the forbidden places.

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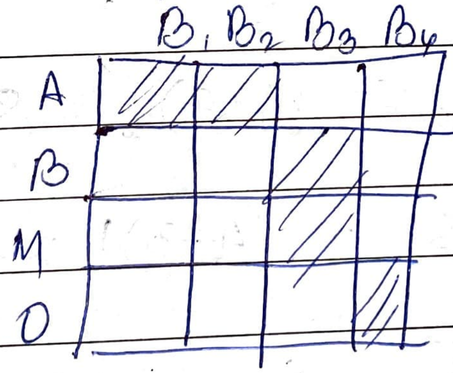
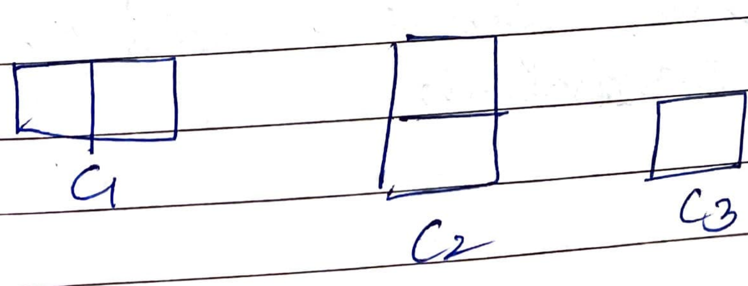
1. An apple, a banana, a mango and an orange are to be distributed to four boys B_1, B_2, B_3 and B_4. The B_1 and B_2 do not wish to have apple, B_3 does not want Banana or mango and B_4 refuses orange. In how many ways the distribution can be made so that no boy is displeased.

The situation can be described by the board shown above in which rows represent apple banana mango and orange, and the columns represent the boys B_1, B_2, B_3, B_4 respectively. Shaded squares represent forbidden places.



Let us consider the board C consists of shaded squares.
C is formed by the mutually disjoint boards C_1, C_2, C_3.



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$r(C, x) = r(C_1 x)\, r(C_2 x)\, r(C_3 x)$

$r(C_1 x) = 1 + 2x$

$r(C_2 x) = 1 + 2x$

$r(C_3, x) = (1 + x)$

$r(C, x) = (1+2x)^2(1+x) = 1 + 5x + 8x^2 + 4x^3$

for C

$r_1 = 5 \quad r_2 = 8 \quad r_3 = 4$

$S_0 = 4! = 24 \qquad S_k = (n-k)!\, r_k$

$S_1 = (4-1)! \times 5 = 30$

$S_2 = (4-2)! \times 8 = 16$

$S_3 = (4-3)! \times 4 = 4$

$\bar{N} = S_0 - S_1 + S_2 - S_3$

$= 24 - 30 + 16 - 4 = 6$

This is the number of ways of distributing the fruits under the given constraints