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Principle of Inclusion - Exclusion

If S is a finite set then the number of elements in S is called the cardinality of S and is denoted by $|S|$. If A and B are subsets of S, then the cardinality of $A \cup B$ is given by the formula

$$|A \cup B| = |A| + |B| - |A \cap B| \rightarrow (1).$$

Thus for determining the number of elements that are in $A \cup B$ we include all elements in A and B but exclude all elements common to A and B.

Note that $\bar{A} \cap \bar{B} = \overline{(A \cup B)}$ and

$$|\overline{(A \cup B)}| = |S| - |A \cup B|$$

$$|\bar{A} \cap \bar{B}| = |\overline{(A \cup B)}| = |S| - |A \cup B|$$
$$= |S| - |A| - |B| + |A \cap B| \rightarrow (2).$$

Equations (1) and (2) are referred to as The principle of inclusion - exclusion for two sets.

Principle of Inclusion - Exclusion for n sets

Let S be a finite set and $A_1, A_2, \dots A_n$ be subsets of S. Then the principle of inclusion - exclusion for $A_1, A_2 \dots A_n$ states that

$$|A_1 \cup A_2 \cup \dots A_n| = \Sigma |A_i| - \Sigma |A_i \cap A_j| + \Sigma |A_i \cap A_j \cap A_k| + \dots$$
$$(-1)^{n-1} |A_1 \cap A_2 \dots \cap A_n|$$

Proof: Take any $x \in A_1 \cup A_2 \cdots \cup A_n$.
Then x is in m of the sets $A_1, A_2, \cdots A_n$
where $1 \leq m \leq n$.
without loss of generality let us
assume that $x \in A_i$ for $1 \leq i \leq m$
and $x \notin A_i$ for $i > m$. Then x will
be counted once in each of the terms
$|A_i|$, $i = 1, 2, \cdots m$. Thus x will be counted
m times in $\Sigma |A_i|$

We note that there are $C(m,2)$ pairs
of sets A_i, A_j where x is in both
A_i and A_j. As such x will be
counted $C(m,2)$ times in $\Sigma |A_i \cap A_j|$
Similarly x will be counted $C(m,3)$
times in $\Sigma |A_i \cap A_j \cap A_k|$ and so on.

Continuing in this manner x is counted

$$m - C(m,2) + C(m,3) + \cdots (-1)^{m-1} C(m,m)$$

number of times.

$$m - C(m,2) + C(m,3) + \cdots (-1)^{m-1} C(m,m)$$

$$= 1 - \{1 - m + C(m,2) - C(m,3) + \cdots (-1)^{m} C(m,m)\}$$

$$= 1 - (1 + (-1))^{m}$$

By Binomial theorem

$$= 1$$

Thus x is counted exactly once.

By De Morgan Law

$$\overline{(A_1 \cup A_2 \cdots \cup A_n)} = \bar{A_1} \cap \bar{A_2} \cap \cdots \bar{A_n}$$

Since $\bar{A} = |S| - |A|$ for any subset A of S, this yields

$$|\bar{A_1} \cap \bar{A_2} \cap \bar{A_3} \cap \cdots \bar{A_n}| = |S| - |A_1 \cup A_2 \cdots A_n|$$

$$|\bar{A_1} \cap \bar{A_2} \cap \bar{A_3} \cdots \bar{A_n}| = |S| - \Sigma|A_i| + \Sigma|A_i \cap A_j| - \cdots$$
$$\Sigma|A_i \cap A_j \cap A_k| + \cdots$$
$$+ (-1)^n |A_1 \cap A_2 \cap \cdots A_n|$$

Problems

1. Among the students in a hostel, 12 students study Mathematics (A), 20 study Physics (B) 20 study chemistry (C) and 8 study Biology (D). There are 5 students for A and B 7 students for A and C, 4 students for A and D 16 students for B and C, 4 students for B and D and 3 students for C and D. There are 3 students for A, B and C 2 for A, B and D, 2 for B, C and D. 3 for A, C and D. Finally there are 2 who study all these subjects. Furthermore there are 71 students who do not study any of these subjects. Find the total number of students

Given

$|A| = 12$ $|B| = 20$ $|C| = 20$ $|D| = 8$

$|A \cap B| = 5$ $|A \cap C| = 7$ $|A \cap D| = 4$

$|B \cap C| = 16$ $|B \cap D| = 4$ $|C \cap D| = 3$

$|A \cap B \cap C| = 3$ $|A \cap B \cap D| = 2$ $|B \cap C \cap D| = 2$

$|A \cap C \cap D| = 3$ $|A \cap B \cap C \cap D| = 2$

$|\bar{A} \cap \bar{B} \cap \bar{C} \cap \bar{D}| = 71$

$S = ?$

$$71 = |S| - (12+20+20+8) + (5+7+4+16+4+3) - (3+2+2+3 \cancel{+2}) + 2$$

$$71 = |S| - 60 + 39 - 10 + 2$$

$\cancel{|S| = 71}$ $71 = |S| - 70 + 41$

$$|S| = 71 + 70 - 41$$

$$|S| = 100.$$

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2. Of 30 personal computers owned by faculty members in a certain university dept 20 run windows, eight have 21 inch monitors, 25 have CD-ROM drives, 20 have atleast two these features and six have all three

(a) How many PCs have atleast one of these features?

(b) How many have none of these features

(c) How many have exactly one feature

$|S| = 30$

$|A| = 20 \quad |B| = 8 \quad |C| = 25$

$\big| |A \cap B| \cup |A \cap C| \cup |B \cap C| \big| = 20$

$|A \cap B \cap C| = 6$

$\cancel{30 = |A| + |B| + |C| - \big| |A \cap B| \cup |A \cap C| \cup |B \cap C| \big| + |A \cap B \cap C|}$

$20 = |A \cap B| + |A \cap C| + |B \cap C|$
$- |A \cap B \cap C| - |A \cap B \cap C| - |A \cap B \cap C|$
$+ |A \cap B \cap C|$

$20 = |A \cap B| + |A \cap C| + |B \cap C|$
$- 2|A \cap B \cap C|$

$|A \cap B| + |A \cap C| + |B \cap C| = 32$

(3)

(a) The number of PCs with atleast one feature is

$|A \cup B \cup C| = |A| + |B| + |C| - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C|$

$= 20 + 8 + 25 - 32 + 6$

$= 27$

(b) It follows that $30 - 27 = 3$ PCs have none of the specified features.

(c) Since the number of computers with exactly one feature is number of PC with atleast one feature less the number with atleast two

$27 - 20 = 7$.

(d) How many have computers with exactly two of the three features described ?

The number of computers with exactly two of the features is the number with atleast two less the number with exactly three

ie, $20 - 6 = 14$.

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(3) At a university 60% of the teachers play tennis, 50% of them play Bridge, 70% jog, 20% play tennis and Bridge, 30% play tennis and jog and 40% play Bridge and jog. What is the percentage of teachers who jog, play tennis and play Bridge.

Let us assume that there are 100 teachers in the university. Let T, B and J be the set of teachers who play tennis, who play Bridge and who jog respectively.

$|T| = 60 \quad |B| = 50 \quad |J| = 70$

$|T \cap B| = 20 \quad |T \cap J| = 30 \quad |B \cap J| = 40$

$|T \cap B \cap J| = ?$

$|T \cup B \cup J| = |T| + |B| + |J| - |T \cap B| - |T \cap J|$
$- |B \cap J| + |T \cap B \cap J|$

$100 = 60 + 50 + 70 - (20 + 30 + 40)$
$+ |T \cap B \cap J|$

$= 180 - 90 + T \cap B \cap J$

$100 = 90 + T \cap B \cap J$

$T \cap B \cap J = 10.$

Therefore 10% of the teachers who jog play tennis and play Bridge

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4. A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian and 14 have taken courses in both French and Russian. If 2092 students have taken at least one of Spanish, French and Russian how many have taken a course in all three languages.

Let S be the set of students who have taken a course in Spanish, F the set of students who have taken a course in French, and R the set of students who have taken a course in Russian. Then

$|S| = 1232 \quad |F| = 879 \quad |R| = 114$

$|S \cap F| = 103 \quad |S \cap R| = 23 \quad |F \cap R| = 14$

$\cancel{|S| + |F| + |R|} \quad |S \cup F \cup R| = 2092$

$|S \cap F \cap R| = ?$

$|S \cup F \cup R| = |S| + |F| + |R| - (|S \cap F| + |S \cap R| + |F \cap R|) + |S \cap F \cap R|$

$2092 = 1232 + 879 + 114 - (103 + 23 + 14) + S \cap F \cap R$

$2092 = 2225 - 140 + |S \cap F \cap R|$

$2092 = 2085 + |S \cap F \cap R|$

$|S \cap F \cap R| = 7$

5. Suppose that there are 1807 freshman at your school. of these 453 are taking a course in computer Science 567 are taking a course in mathematics and 299 are taking courses in both computer science and Mathematics How many are not taking a course either in computer science or in Mathematics.

To find the number of freshman who are not taking a course in either mathematics or computer Science, subtract the number that are taking a course in either of these ~~sub~~ courses from total number of freshman

Let A be the set of all freshman who are Taking a Course in Computer Science Let B be the set of all freshmen taking a course in mathematics

$|A| = 453$ $|B| = 567$ $|A \cap B| = 299$

$|A \cup B| = |A| + |B| - |A \cap B|$

$= 453 + 567 - 299 = 721$

There are $1807 - 721 = 1086$ freshman who are not taking a course in computer Science or Mathematics.

6. A discrete mathematics class contains 25 students majoring in computer science 13 students majoring in Mathematics and eight join Mathematics and computer Science majors. How many students are in this class, if every student is majoring in Mathematics, computer Science or both Mathematics and Computer Science?

Let A be the set of students in the class majoring in Computer Science and B be the set of students in the class majoring in Mathematics

Then $A \cap B$ is the set of students in the class who are Majoring in Mathematics and computer Science.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= 25 + 13 - 8 = 30$$

Therefore there are 30 students in the class.

7. In a survey of students at Florida State university the following information was obtained; 260 were taking a statistics course, 208 were taking a mathematics course, 160 were taking a computer programming course, 76 were taking statistics and mathematics, 48 were taking statistics and computer programming, 62 were taking mathematics courses and computer programming, 30 were taking all 3 kinds of courses, and 150 were taking none of the 3 courses.

(a) How many students were surveyed?

(b) How many students were taking a statistics and mathematics but not computer programming course?

(c) How many were taking a statistics and computer course but not a mathematics course.

(d) How many were taking a computer programming and mathematics course but not statistics course

(e) How many were taking a statistics course but not taking a course in mathematics or in computer programming?

(f) How many were taking a ~~stats~~ mathematics course but not taking a statistics course or a computer programming course?

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How many were taking a computer programming course but not taking a course in mathematics or in statistics?

$|S| = 260 \quad |M| = 208 \quad |C| = 160$

$|S \cap M| = 76 \quad |S \cap C| = 48$

$|S \cap M \cap C| = 30$

$|M \cap C| = 62$

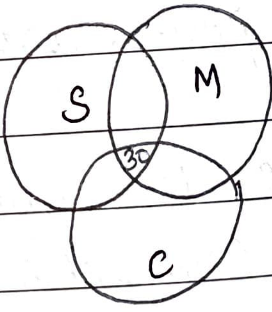
$|\overline{S \cup M \cup C}| = 150$

(a)

$|S \cup M \cup C| = |S| + |M| + |C| - |S \cap M| - |S \cap C| - |M \cap C| + |S \cap M \cap C|$

$= 260 + 208 + 160 - 76 - 48 - 62 + 30$

$= 622$



(b) $|S \cap M| - |S \cap M \cap C|$

$= 76 - 30 = 46$

(c) $|S \cap C| - |S \cap M \cap C|$

$= 48 - 30 = 18$

(d) $|C \cap M| - |S \cap M \cap C|$

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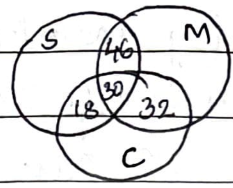
(e)



$|S \cap \bar{M} \cap \bar{C}| = 260 - 46 - 18 - 30 = 166$

(f) $|\bar{S} \cap \bar{C} \cap M| = 208 - 46 - 32 - 30$

$= 100$

208
108

(g) $|\bar{C} \cap \bar{M} \cap \bar{S}| = 160 - 32 - 18 - 30$

80

46
32
30
108

(8) If there are 200 faculty members that speak French, 50 that speak Russian, 100 that speak Spanish, 20 that speak French and Russian, 60 that speak French and Spanish, 35 that speak Russian and Spanish while only 10 speak French, Russian and Spanish, how many speak either French or Russian or Spanish?

32
18
30
80

$|F| = 200 \quad |R| = 50 \quad |S| = 100$

$|F \cap R| = 20 \quad |F \cap S| = 60 \quad |R \cap S| = 35$

$|F \cap R \cap S| = 10$

$|F \cup R \cup S| = |F| + |R| + |S| - |F \cap R| - |F \cap S|$

$- |R \cap S| + |F \cap R \cap S|$

$= 200 + 50 + 100 - 20 - 60 - 35 + 10$

$= 245.$