Stein-Weiss inequality revisit on Heisenberg group

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Abstract

We study a family of fractional integral operators defined as

$$\mathbf{I}_{\alpha\beta\vartheta} f(u,v,t) = \iiint_{\mathbb{R}^{2n+1}} f(\xi,\eta,\tau)\mathbf{V}^{\alpha\beta\vartheta}\Big[(u,v,t)\odot(\xi,\eta,\tau)^{-1}\Big]d\xi d\eta d\tau$$

where $\odot$ denotes the multiplication law of a Heisenberg group. $\mathbf{V}^{\alpha\beta\vartheta}$ is a distribution satisfying Zygmund dilation.

Let $\omega(u,v) = \sqrt{|u|^2+|v|^2}^{-\gamma}$, $\sigma(u,v) = \sqrt{|u|^2+|v|^2}^{\delta}$. A characterization is established between $\omega\mathbf{I}_{\alpha\beta\vartheta}\sigma^{-1}\colon \mathbf{L}^p(\mathbb{R}^{2n+1}) \longrightarrow \mathbf{L}^q(\mathbb{R}^{2n+1})$ and the necessary constraints consisting of $\alpha,\beta,\vartheta,\gamma,\delta\in\mathbb{R}$ for $1<p<q<\infty$.

1 Introduction

To begin, we recall two classical results of fractional integration on Euclidean space. Define

$$\mathbf{T}_{\mathbf{a}} f(x) = \int_{\mathbb{R}^{\mathbf{N}}} f(y)\left[\frac{1}{|x-y|}\right]^{\mathbf{N}-\mathbf{a}} dy, \qquad 0<\mathbf{a}<\mathbf{N}. \tag{1. 1}$$

In 1928, Hardy and Littlewood [1] first established an regularity theorem for $\mathbf{T}_{\mathbf{a}}$ for $\mathbf{N}=1$. Ten years later, Sobolev [2] made extensions on every higher dimensional space.

Hardy-Littlewood-Sobolev theorem *Let $\mathbf{T}_{\mathbf{a}}$ defined in (1. 1) for $0<\mathbf{a}<\mathbf{N}$. We have*

$$\begin{gathered}\left\|\mathbf{T}_{\mathbf{a}} f\right\|_{\mathbf{L}^q(\mathbb{R}^{\mathbf{N}})} \leq \mathfrak{B}_{p\,q}\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{\mathbf{N}})}, \qquad 1<p<q<\infty \\ \textit{if and only if} \qquad \frac{\mathbf{a}}{\mathbf{N}} = \frac{1}{p}-\frac{1}{q}.\end{gathered} \tag{1. 2}$$

In 1958, Stein and Weiss [3] obtained a weighted analogue of the above regularity theorem by considering the *weights* to be suitable powers.

Stein-Weiss theorem *Let $\mathbf{T}_{\mathbf{a}}$ defined in (1. 1) for $0<\mathbf{a}<\mathbf{N}$ and $\omega(x)=|x|^{-\gamma}, \sigma(x)=|x|^{\delta}$ for $\gamma,\delta\in\mathbb{R}$ whenever $x\neq 0$. We have*

$$\left\|\omega\mathbf{T}_{\mathbf{a}} f\right\|_{\mathbf{L}^q(\mathbb{R}^{\mathbf{N}})} \leq \mathfrak{B}_{p\,q\,\gamma\,\delta}\left\|f\sigma\right\|_{\mathbf{L}^p(\mathbb{R}^{\mathbf{N}})}, \qquad 1<p\leq q<\infty \tag{1. 3}$$

if and only if

$$\gamma<\frac{\mathbf{N}}{q}, \qquad \delta<\mathbf{N}\left(\frac{p-1}{p}\right), \qquad \gamma+\delta\geq 0, \qquad \frac{\mathbf{a}}{\mathbf{N}} = \frac{1}{p}-\frac{1}{q}+\frac{\gamma+\delta}{\mathbf{N}}. \tag{1. 4}$$

$\diamond$ Throughout, $\mathfrak{B}>0$ is a generic constant depending on its sub-indices.

Remark 1.1. *In the original paper of Stein and Weiss [3], (1. 4) is given as a sufficient condition. Conversely, it turns out to be necessary as well. See the appendix in [14].*

Hardy-Littlewood-Sobolev theorem was first re-investigated by Folland and Stein [4] on Heisenberg group. We work on its real variable representation with a multiplication law:

$$(u,v,t)\odot(\xi,\eta,\tau) \;=\; \Big[u+\xi, v+\eta, t+\tau+\mu(u\cdot\eta - v\cdot\xi)\Big], \qquad \mu\in\mathbb{R} \tag{1. 5}$$

for every $(u,v,t)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$ and $(\xi,\eta,\tau)^{-1}=(-\xi,-\eta,-\tau)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$.

Let $0<\rho<n+1$. Consider

$$\mathbf{S}_\rho f(u,v,t) \;=\; \iiint_{\mathbb{R}^{2n+1}} f(\xi,\eta,\tau)\Omega^\rho\Big[(u,v,t)\odot(\xi,\eta,\tau)^{-1}\Big]d\xi d\eta d\tau. \tag{1. 6}$$

Ω^ρ is a distribution in $\mathbb{R}^{2n+1}$ agree with

$$\Omega^\rho(u,v,t) \;=\; \left[\frac{1}{|u|^2+|v|^2+|t|}\right]^{n+1-\rho}, \qquad (u,v,t)\neq(0,0,0). \tag{1. 7}$$

Folland-Stein theorem *Let* $\mathbf{S}_\rho$ *defined in (1. 6)-(1. 7) for* $0<\rho<n+1$. *We have*

$$\begin{gathered}\Big\|\mathbf{S}_\rho f\Big\|_{\mathbf{L}^q(\mathbb{R}^{n+1})} \;\leq\; \mathfrak{B}_{p\,q}\,\Big\|f\Big\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})}, \qquad 1<p<q<\infty \\ \textit{if and only if} \qquad \frac{\rho}{n+1} \;=\; \frac{1}{p}-\frac{1}{q}.\end{gathered} \tag{1. 8}$$

The best constant for the $\mathbf{L}^p\longrightarrow\mathbf{L}^q$-norm inequality in (1. 8) is found by Frank and Lieb [11]. A discrete analogue of this result has been obtained by Pierce [12]. Recently, the regarding commutator estimates are established by Fanelli and Roncal [13].

Stein-Weiss theorem has been re-investigated on Heisenberg group by Han, Lu and Zhu [10].

Han-Lu-Zhu theorem *Let* $\mathbf{S}_\rho$ *defined in (1. 6)-(1. 7) for* $0<\rho<n+1$. *Suppose* $\gamma,\delta\in\mathbb{R}$ *and* $\omega(u,v)=\sqrt{|u|^2+|v|^2}^{\,-\gamma}$, $\sigma(u,v)=\sqrt{|u|^2+|v|^2}^{\,\delta}$ *for* $(u,v)\neq(0,0)$. *We have*

$$\Big\|\omega\mathbf{S}_\rho f\Big\|_{\mathbf{L}^q(\mathbb{R}^{n+1})} \;\leq\; \mathfrak{B}_{p\,q}\,\Big\|f\sigma\Big\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})}, \qquad 1<p\leq q<\infty \tag{1. 9}$$

if

$$\gamma<\frac{2n}{q}, \qquad \delta<2n\left(\frac{p-1}{p}\right), \qquad \gamma+\delta\geq 0, \qquad \frac{\rho}{n+1} \;=\; \frac{1}{p}-\frac{1}{q}+\frac{\gamma+\delta}{2n+2}. \tag{1. 10}$$

Remark 1.2. *Note that the two power weights* ω,σ *are defined in the subspace* $\mathbb{R}^{2n}$. *An analogue two-weight* $\mathbf{L}^p\longrightarrow\mathbf{L}^q$*-norm inequality with*

$$\omega(u,v,t) \;=\; \sqrt{|u|^2+|v|^2+|t|}^{\,-\gamma}, \qquad \sigma(u,v,t) \;=\; \sqrt{|u|^2+|v|^2+|t|}^{\,\delta}$$

can be found in the paper of Han, Lu and Zhu [10].

The proof of **Han-Lu-Zhu theorem** was accomplished by using the language of fractional integrals defined in homogeneous spaces. In this paper, we first show that the constraints inside (1. 10) are also necessary conditions for the $\mathbf{L}^p \longrightarrow \mathbf{L}^q$-norm inequality in (1. 9). Conversely, we give a new proof of (1. 10) implying (1. 9) for $1 < p < q < \infty$ with a more direct approach.

Theorem One *Let $\mathbf{S}_\rho$ defined in (1. 6)-(1. 7) for $0 < \rho < n+1$. Suppose $\gamma, \delta \in \mathbb{R}$ and $\omega(u,v) = \sqrt{|u|^2+|v|^2}^{-\gamma}, \sigma(u,v) = \sqrt{|u|^2+|v|^2}^{\delta}$ for $(u,v) \neq (0,0)$. We have*

$$\left\|\omega \mathbf{S}_\rho f\right\|_{\mathbf{L}^q(\mathbb{R}^{n+1})} \leq \mathfrak{B}_{p\,q} \left\|f\sigma\right\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})}, \qquad 1<p<q<\infty \tag{1. 11}$$

if and only if

$$\gamma < \frac{2n}{q}, \qquad \delta < 2n\left(\frac{p-1}{p}\right), \qquad \gamma+\delta \geq 0, \qquad \frac{\rho}{n+1} = \frac{1}{p}-\frac{1}{q}+\frac{\gamma+\delta}{2n+2}. \tag{1. 12}$$

Next, we extend **Theorem One** to a multi-parameter setting by replacing Ω^ρ with a larger kernel having singularity on every coordinate subspace.

Observe that

$$\Omega^\rho(u,v,t) \leq \left[\frac{1}{|u||v|+|t|}\right]^{n+1-\rho}, \qquad (u,t)\neq(0,0) \text{ or } (v,t)\neq(0,0). \tag{1. 13}$$

Furthermore, we find

$$\begin{aligned}\left[\frac{1}{|u||v|+|t|}\right]^{n+1-\rho} &\approx \left[\frac{1}{|u|^2|v|^2+t^2}\right]^{\frac{n+1}{2}-\frac{\rho}{2}} \\ &= |u|^{\frac{\rho}{2}-\frac{n+1}{2}}|v|^{\frac{\rho}{2}-\frac{n+1}{2}}|t|^{\frac{\rho}{2}-\frac{n+1}{2}}\left[\frac{|u||v||t|}{|u|^2|v|^2+t^2}\right]^{\frac{n+1}{2}-\frac{\rho}{2}} \\ &= |u|^{\left[\frac{\rho}{2}+\frac{n-1}{2}\right]-n}|v|^{\left[\frac{\rho}{2}+\frac{n-1}{2}\right]-n}|t|^{\left[\frac{\rho}{2}-\frac{n-1}{2}\right]-1}\left[\frac{|u||v|}{|t|}+\frac{|t|}{|u||v|}\right]^{-\left[\frac{n+1}{2}-\frac{\rho}{2}\right]}.\end{aligned} \tag{1. 14}$$

Above estimates lead us to the following assertion. Let $\alpha, \beta \in \mathbb{R}$ and $\vartheta \geq 0$. $\mathbf{V}^{\alpha\beta\vartheta}$ is a distribution in $\mathbb{R}^{2n+1}$ agree with

$$\mathbf{V}^{\alpha\beta\vartheta}(u,v,t) = |u|^{\alpha-n}|v|^{\alpha-n}|t|^{\beta-1}\left[\frac{|u||v|}{|t|}+\frac{|t|}{|u||v|}\right]^{-\vartheta}, \qquad u\neq 0, v\neq 0, t\neq 0. \tag{1. 15}$$

Define

$$\mathbf{I}_{\alpha\beta\vartheta} f(u,v,t) = \iiint_{\mathbb{R}^{2n+1}} f(\xi,\eta,\tau)\, \mathbf{V}^{\alpha\beta\vartheta}\Big[(u,v,t)\odot(\xi,\eta,\tau)^{-1}\Big] d\xi d\eta d\tau. \tag{1. 16}$$

This fractional integral operator is associated with Zygmund dilation, whereas

$$\mathbf{V}^{\alpha\beta\vartheta}\Big[(ru,sv,rst)\odot(r\xi,s\eta,rs\tau)^{-1}\Big] = r^{\alpha+\beta-n-1}s^{\alpha+\beta-n-1}\mathbf{V}^{\alpha\beta\vartheta}\Big[(u,v,t)\odot(\xi,\eta,\tau)^{-1}\Big], \qquad r,s>0.$$

Singular integral operators with kernels having certain multi-parameter structures defined on Heisenberg group have been systematically studied, for example by Phong and Stein [5], Ricci and Stein [6] and Müller, Ricci and Stein [7]. Much less is known in this direction for fractional integration.

Theorem Two *Let $\mathbf{I}_{\alpha\beta\vartheta}$ defined in (1. 15)-(1. 16) for $\alpha, \beta \in \mathbb{R}$ and $\vartheta \geq 0$. Suppose $\gamma, \delta \in \mathbb{R}$ and $\omega(u,v) = \sqrt{|u|^2+|v|^2}^{-\gamma}, \sigma(u,v) = \sqrt{|u|^2+|v|^2}^{\delta}$ for $(u,v) \neq (0,0)$. We have*

$$\left\|\omega \mathbf{I}_{\alpha\beta\vartheta} f\right\|_{\mathbf{L}^q(\mathbb{R}^{2n+1})} \leq \mathfrak{B}_{p\,q\,\gamma\,\delta} \left\|f\sigma\right\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})}, \qquad 1<p<q<\infty \tag{1. 17}$$

if and only if

$$\begin{gathered}
\gamma < \frac{2n}{q}, \qquad \delta < 2n\left(\frac{p-1}{p}\right), \qquad \gamma+\delta \geq 0, \qquad \frac{\alpha+\beta}{n+1} = \frac{1}{p}-\frac{1}{q}+\frac{\gamma+\delta}{2n+2} \\
\vartheta \geq \left|\frac{\alpha-n\beta}{n+1}-\frac{\gamma+\delta}{2n+2}\right|; \\
n\left[\frac{\alpha+\beta}{n+1}\right]+\frac{\gamma+\delta}{2n+2}-\frac{n}{p} < \delta \qquad for \qquad \gamma \geq 0,\ \delta \leq 0; \\
n\left[\frac{\alpha+\beta}{n+1}\right]+\frac{\gamma+\delta}{2n+2}-n\left(\frac{q-1}{q}\right) < \gamma \qquad for \qquad \gamma \leq 0,\ \delta \geq 0.
\end{gathered} \tag{1. 18}$$

Remark 1.3. *Recall (1. 13)-(1. 14). By taking into account $\alpha = \frac{\rho}{2}+\frac{n-1}{2}$, $\beta = \frac{\rho}{2}-\frac{n-1}{2}$ and $\vartheta^* = \left|\frac{\alpha-n\beta}{n+1}-\frac{\gamma+\delta}{2n+2}\right|$ for $\rho, \gamma, \delta, p, q$ satisfying (1. 12). We find*

$$\left[\frac{1}{|u||v|+|t|}\right]^{n+1-\rho} \lesssim \mathbf{V}^{\alpha\beta\vartheta^*}(u,v,t), \qquad u \neq 0, v \neq 0, t \neq 0.$$

This is equivalent to verify $\vartheta^* \leq \frac{n+1}{2}-\frac{\rho}{2}$. We omit the regarding computations.

The remaining paper is organized as follows. First, we prove **Theorem One** in section 2. Section 3-5 are devoted to the proof of **Theorem Two**. In section 3, we show (1. 17) implying (1. 18). In section 4, after a reformulation of $\mathbf{I}_{\alpha\beta\vartheta}$, we shall see that in the one-weight case, i.e : $\omega = \sigma$ occurred at $\gamma+\delta = 0$, the $\mathbf{L}^p \longrightarrow \mathbf{L}^q$-norm inequality in (1. 17) can be obtained by using an iteration argument. In contrast, this idea of iteration does not apply to $\omega \neq \sigma$ whenever $\gamma+\delta > 0$. In section 5, we develop a new framework to handle this two-weight case where the product space $\mathbb{R}^n \times \mathbb{R}^n$ is decomposed into an infinitely many dyadic cones. Each partial operator is defined on one of these dyadic cones. Essentially, it is a classical one-parameter fractional integral operator, satisfying the desired regularity. Moreover, its operator's norm decays as the eccentricity of the cone getting large.

2 Proof of Theorem One

Let $\omega(u,v) = \sqrt{|u|^2+|v|^2}^{-\gamma}$ and $\sigma(u,v) = \sqrt{|u|^2+|v|^2}^{\delta}$ for $\gamma, \delta \in \mathbb{R}$ and $(u,v) \neq (0,0)$.

Because $\mathbf{S}_\rho$ defined in (1. 6)-(1. 7) for $0<\rho<n+1$ is self-adjoint, it is essential to have $\omega^q, \sigma^{-\frac{p}{p-1}}$ locally integrable in $\mathbb{R}^{2n}$ for the $\mathbf{L}^p \longrightarrow \mathbf{L}^q$-norm inequality in (1. 9). Therefore,

$$\gamma<\frac{2n}{q}, \qquad \delta<2n\left(\frac{p-1}{p}\right) \tag{2. 1}$$

are necessities.

Denote $\mathbf{Q}\subset\mathbb{R}^{2n}$ to be a cube parallel to the coordinates and $I\subset\mathbb{R}$ to be an interval. Consider

$$f(u,v,t) \;=\; \sigma^{-\frac{p}{p-1}}(u,v)\chi_{\mathbf{Q}\times I}(u,v,t) \;=\; \sigma^{-\frac{p}{p-1}}(u,v)\chi_{\mathbf{Q}}(u,v)\chi_I(t), \qquad (u,v)\neq(0,0). \tag{2. 2}$$

Let $\mathbf{vol}\{\mathbf{Q}\}^{\frac{1}{n}}=\mathbf{vol}\{I\}$. By changing variable $\tau\longrightarrow\tau-\mu(u\cdot\eta-v\cdot\xi)$ in $\left\|\omega\mathbf{S}_\rho f\right\|_{\mathbf{L}^q(\mathbb{R}^{2n+1})}$, we have

$$\begin{aligned}
&\left\{\iiint_{\mathbb{R}^{2n+1}}\omega^q(u,v)\left\{\iiint_{\mathbb{R}^{2n+1}} f\left(\xi,\eta,\tau-\mu(u\cdot\eta-v\cdot\xi)\right)\Omega^\rho(u-\xi,v-\eta,t-\tau)d\xi d\eta d\tau\right\}^q dudvdt\right\}^{\frac{1}{q}}\\
&\geq\;\left\{\iiint_{\mathbf{Q}\times I}\omega^q(u,v)\left\{\iiint_{\mathbf{Q}\times\mathbb{R}}\sigma^{-\frac{p}{p-1}}(\xi,\eta)\chi_I(\tau-\mu(u\cdot\eta-v\cdot\xi))\right.\right.\\
&\qquad\qquad\left.\left.\left[\frac{1}{|u-\xi|^2+|v-\eta|^2+|t-\tau|}\right]^{n+1-\rho} d\xi d\eta d\tau\right\}^q dudvdt\right\}^{\frac{1}{q}}\\
&\geq\;\mathbf{vol}\{\mathbf{Q}\}^{\frac{\rho}{n}-\frac{n+1}{n}}\left\{\iiint_{\mathbf{Q}\times I}\omega^q(u,v)\left\{\iint_{\mathbf{Q}}\sigma^{-\frac{p}{p-1}}(\xi,\eta)\left\{\int_{I-\mu(u\cdot\eta-v\cdot\xi)}d\tau\right\}d\xi d\eta\right\}^q dudvdt\right\}^{\frac{1}{q}}\\
&=\;\mathbf{vol}\{\mathbf{Q}\}^{\frac{\rho}{n}-\frac{n+1}{n}+\left[1+\frac{1}{q}\right]\frac{1}{n}}\left\{\iint_{\mathbf{Q}}\omega^q(u,v)dudv\right\}^{\frac{1}{q}}\iint_{\mathbf{Q}}\sigma^{-\frac{p}{p-1}}(\xi,\eta)d\xi d\eta.
\end{aligned} \tag{2. 3}$$

The $\mathbf{L}^p\longrightarrow\mathbf{L}^q$-norm inequality in (1. 9) implies

$$\begin{aligned}
&\mathbf{vol}\{\mathbf{Q}\}^{\frac{\rho}{n}-\frac{n+1}{n}+\left[1+\frac{1}{q}\right]\frac{1}{n}}\left\{\iint_{\mathbf{Q}}\omega^q(u,v)dudv\right\}^{\frac{1}{q}}\iint_{\mathbf{Q}}\sigma^{-\frac{p}{p-1}}(u,v)dudv\\
&\leq\;\mathfrak{B}_{p\,q}\left\{\iiint_{\mathbf{Q}\times I}\sigma^{-\frac{p}{p-1}}(u,v)dudvdt\right\}^{\frac{1}{p}}\;=\;\mathfrak{B}_{p\,q}\,\mathbf{vol}\{\mathbf{Q}\}^{\frac{1}{n}\frac{1}{p}}\left\{\iint_{\mathbf{Q}}\sigma^{-\frac{p}{p-1}}(u,v)dudv\right\}^{\frac{1}{p}}.
\end{aligned} \tag{2. 4}$$

From (2. 3)-(2. 4), we find

$$\begin{aligned}
&\mathbf{vol}\{\mathbf{Q}\}^{\frac{\rho}{n}-\frac{n+1}{n}+\left[1+\frac{1}{q}-\frac{1}{p}\right]\frac{1}{n}}\left\{\iint_{\mathbf{Q}}\omega^q(u,v)dudv\right\}^{\frac{1}{q}}\left\{\iint_{\mathbf{Q}}\sigma^{-\frac{p}{p-1}}(u,v)dudv\right\}^{\frac{p-1}{p}}\;=\\
&\mathbf{vol}\{\mathbf{Q}\}^{\left[\frac{\rho}{n+1}-\frac{1}{p}+\frac{1}{q}\right]\frac{n+1}{n}}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}\}}\iint_{\mathbf{Q}}\omega^q(u,v)dudv\right\}^{\frac{1}{q}}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}\}}\iint_{\mathbf{Q}}\sigma^{-\frac{p}{p-1}}(u,v)dudv\right\}^{\frac{p-1}{p}}<\infty
\end{aligned} \tag{2. 5}$$

for every $\mathbf{Q}\subset\mathbb{R}^{2n}$.

A standard exercise of changing one-parameter dilation in (2. 5) shows that

$$\frac{\rho}{n+1} = \frac{1}{p}-\frac{1}{q}+\frac{\gamma+\delta}{2n+2} \tag{2. 6}$$

is an necessary homogeneity condition.

Let $\mathbf{Q}$ shrink to some $(u,v)\in\mathbf{Q}$ with $(u,v)\neq(0,0)$ inside (2. 5). We have

$$\lim_{\mathbf{vol}\{\mathbf{Q}\}\longrightarrow 0} \mathbf{vol}\{\mathbf{Q}\}^{\left[\frac{\rho}{n+1}-\frac{1}{p}+\frac{1}{q}\right]\frac{n+1}{n}}\omega(u,v)\sigma^{-1}(u,v) \tag{2. 7}$$

by applying Lebesgue differentiation theorem. In order to have this limit finite, we need

$$\frac{\rho}{n+1} \geq \frac{1}{p}-\frac{1}{q}. \tag{2. 8}$$

By putting together (2. 6) and (2. 8), we find

$$\gamma+\delta \geq 0. \tag{2. 9}$$

Recall $\Omega^\rho(u,v,t)$ defined in (1. 7). We have

$$\begin{aligned}
\Omega^\rho(u,v,t) &= \left[\frac{1}{|u|^2+|v|^2+|t|}\right]^{n+1-\rho} \\
&= \left[\frac{1}{|u|^2+|v|^2+|t|}\right]^{n-\left(\frac{n}{n+1}\right)\rho-\frac{\gamma+\delta}{2n+2}+1-\frac{\rho}{n+1}+\frac{\gamma+\delta}{2n+2}} \\
&\leq \left[\frac{1}{|u|^2+|v|^2}\right]^{n-n\left[\frac{\rho}{n+1}+\frac{1}{n}\frac{\gamma+\delta}{2n+2}\right]}|t|^{\left[\frac{\rho}{n+1}-\frac{\gamma+\delta}{2n+2}\right]-1}, \qquad (u,v)\neq(0,0),\ t\neq 0.
\end{aligned} \tag{2. 10}$$

Note that a direct computation shows

$$\begin{aligned}
\frac{\rho}{n+1}+\frac{1}{n}\frac{\gamma+\delta}{2n+2} &= \frac{1}{p}-\frac{1}{q}+\frac{\gamma+\delta}{2n} \qquad \left(\tfrac{\rho}{n+1}=\tfrac{1}{p}-\tfrac{1}{q}+\tfrac{\gamma+\delta}{2n+2}\right) \\
&< 1 \qquad \text{because } \gamma<\tfrac{2n}{q},\ \delta<2n\left(\tfrac{p-1}{p}\right).
\end{aligned} \tag{2. 11}$$

Let $\mathbf{S}_\rho$ defined in (1. 6)-(1. 7) for $0<\rho<n+1$. By changing variable $\tau\longrightarrow\tau-\mu(u\cdot\eta-v\cdot\xi)$, we find

$$\begin{aligned}
\mathbf{S}_\rho f(u,v,t) &= \iiint_{\mathbb{R}^{2n+1}} f\left(\xi,\eta,\tau-\mu(u\cdot\eta-v\cdot\xi)\right)\Omega^\rho(u-\xi,v-\eta,t-\tau)d\xi d\eta d\tau \\
&\leq \iiint_{\mathbb{R}^{2n+1}} f\left(\xi,\eta,\tau-\mu(u\cdot\eta-v\cdot\xi)\right) \\
&\qquad \left[\frac{1}{|u-\xi|^2+|v-\eta|^2}\right]^{n-n\left[\frac{\rho}{n+1}+\frac{1}{n}\frac{\gamma+\delta}{2n+2}\right]}|t-\tau|^{\left[\frac{\rho}{n+1}-\frac{\gamma+\delta}{2n+2}\right]-1}d\xi d\eta d\tau \qquad \text{by (2. 10)} \\
&\doteq \iint_{\mathbb{R}^{2n}}\left[\frac{1}{|u-\xi|^2+|v-\eta|^2}\right]^{n-n\left[\frac{\rho}{n+1}+\frac{1}{n}\frac{\gamma+\delta}{2n+2}\right]}\mathbf{F}_{\rho\gamma\delta}(\xi,\eta,u,v,t)d\xi d\eta
\end{aligned} \tag{2. 12}$$

where

$$\mathbf{F}_{\rho\gamma\delta}(\xi,\eta,u,v,t) \;=\; \int_{\mathbb{R}} f\left(\xi,\eta,\tau-\mu\left(u\cdot\eta-v\cdot\xi\right)\right)|t-\tau|^{\left[\frac{\rho}{n+1}-\frac{\gamma+\delta}{2n+2}\right]-1}d\tau. \tag{2. 13}$$

Because $\left[\frac{1}{|u|^2+|v|^2}\right]^{n-n\left[\frac{\rho}{n+1}+\frac{1}{n}\frac{\gamma+\delta}{2n+2}\right]}$, $|t|^{\left[\frac{\rho}{n+1}-\frac{\gamma+\delta}{2n+2}\right]-1}$ are positive definite, it is suffice to assert $f\geq 0$.

Recall **Hardy-Littlewood-Sobolev theorem** and **Stein-Weiss theorem** stated in the beginning of this paper. By applying (1. 2) with $\mathbf{a}=\frac{\rho}{n+1}-\frac{\gamma+\delta}{2n+2}=\frac{1}{p}-\frac{1}{q}$ and $\mathbf{N}=1$, we obtain

$$\begin{aligned}\left\{\int_{\mathbb{R}}\mathbf{F}^q_{\rho\gamma\delta}(\xi,\eta,u,v,t)dt\right\}^{\frac{1}{q}} \;&\leq\; \mathfrak{B}_{p\ q}\left\{\int_{\mathbb{R}}\Big[f\left(\xi,\eta,t-\mu\left(u\cdot\eta-v\cdot\xi\right)\right)\Big]^p dt\right\}^{\frac{1}{p}}\\ &=\; \mathfrak{B}_{p\ q}\ \left\|f(\xi,\eta,\cdot)\right\|_{\mathbf{L}^p(\mathbb{R})}\end{aligned} \tag{2. 14}$$

regardless of $(u,v)\in\mathbb{R}^n\times\mathbb{R}^n$.

From (2. 12)-(2. 13), we have

$$\begin{aligned}&\left\{\iiint_{\mathbb{R}^{2n+1}}\sqrt{|u|^2+|v|^2}^{-\gamma q}\Big(\mathbf{S}_\rho f\Big)^q(u,v,t)dudvdt\right\}^{\frac{1}{q}}\\
&\leq\;\Bigg\{\iiint_{\mathbb{R}^{2n+1}}\sqrt{|u|^2+|v|^2}^{-\gamma q}\\
&\qquad\left\{\iint_{\mathbb{R}^{2n}}\left[\frac{1}{|u-\xi|^2+|v-\eta|^2}\right]^{n-n\left[\frac{\rho}{n+1}+\frac{1}{n}\frac{\gamma+\delta}{2n+2}\right]}\mathbf{F}_{\rho\gamma\delta}(\xi,\eta,u,v,t)d\xi d\eta\right\}^q dudvdt\Bigg\}^{\frac{1}{q}}\\
&\leq\;\Bigg\{\iint_{\mathbb{R}^{2n}}\sqrt{|u|^2+|v|^2}^{-\gamma q}\\
&\qquad\left\{\iint_{\mathbb{R}^{2n}}\left[\frac{1}{|u-\xi|^2+|v-\eta|^2}\right]^{n-n\left[\frac{\rho}{n+1}+\frac{1}{n}\frac{\gamma+\delta}{2n+2}\right]}\left\{\int_{\mathbb{R}}\mathbf{F}^q_{\rho\gamma\delta}(\xi,\eta,u,v,t)dt\right\}^{\frac{1}{q}}d\xi d\eta\right\}^q dudv\Bigg\}^{\frac{1}{q}}\\
&\qquad\qquad\qquad\text{by Minkowski integral inequality}\\
&\leq\;\mathfrak{B}_{p\ q}\Bigg\{\iint_{\mathbb{R}^{2n}}\sqrt{|u|^2+|v|^2}^{-\gamma q}\\
&\qquad\left\{\iint_{\mathbb{R}^{2n}}\left[\frac{1}{|u-\xi|^2+|v-\eta|^2}\right]^{n-n\left[\frac{\rho}{n+1}+\frac{1}{n}\frac{\gamma+\delta}{2n+2}\right]}\left\|f(\xi,\eta,\cdot)\right\|_{\mathbf{L}^p(\mathbb{R})}d\xi d\eta\right\}^q dudv\Bigg\}^{\frac{1}{q}}\quad\text{by (2. 14)}\\
&\leq\;\mathfrak{B}_{p\ q\ \gamma\ \delta}\left\{\iint_{\mathbb{R}^{2n}}\left\|f(\xi,\eta,\cdot)\right\|^p_{\mathbf{L}^p(\mathbb{R})}\left[\sqrt{|\xi|^2+|\eta|^2}\right]^{p\delta}d\xi d\eta\right\}^{\frac{1}{p}}\\
&\qquad\text{by (2. 11) and applying (1. 3)-(1. 4) with }\mathbf{a}=n\Big[\tfrac{\rho}{n+1}+\tfrac{1}{n}\tfrac{\gamma+\delta}{2n+2}\Big]\text{ and }\mathbf{N}=n\\
&=\;\mathfrak{B}_{p\ q\ \gamma\ \delta}\left\{\iiint_{\mathbb{R}^{2n+1}}\Big[f(\xi,\eta,\tau)\Big]^p\left[\sqrt{|\xi|^2+|\eta|^2}\right]^{p\delta}d\xi d\eta d\tau\right\}^{\frac{1}{p}}.\end{aligned} \tag{2. 15}$$

3 Some necessary constraints

Recall $\mathbf{I}_{\alpha\beta\vartheta}$ defined in (1. 15)-(1. 16). By changing variable $\tau \longrightarrow \tau - \mu(u\cdot\eta - v\cdot\xi)$, we find

$$\begin{aligned}\mathbf{I}_{\alpha\beta\vartheta} f(u,v,t) \;=\;& \iiint_{\mathbb{R}^{2n+1}} f\left(\xi,\eta,\tau-\mu(u\cdot\eta-v\cdot\xi)\right)\\ & |u-\xi|^{\alpha-n}|v-\eta|^{\alpha-n}|t-\tau|^{\beta-1}\left[\frac{|u-\xi||v-\eta|}{|t-\tau|}+\frac{|t-\tau|}{|u-\xi||v-\eta|}\right]^{-\vartheta} d\xi d\eta d\tau.\end{aligned} \tag{3. 1}$$

Let $\omega(u,v) = \sqrt{|u|^2+|v|^2}^{-\gamma}$ and $\sigma(u,v) = \sqrt{|u|^2+|v|^2}^{\delta}$ for $\gamma,\delta\in\mathbb{R}$ and $(u,v)\neq(0,0)$.

By changing dilations $(u,v,t)\longrightarrow(ru,rv,r^2\lambda t)$ and $(\xi,\eta,\tau)\longrightarrow(r\xi,r\eta,r^2\lambda\tau)$ for $r>0$ and $0<\lambda<1$ or $\lambda>1$, we have

$$\begin{aligned}&\left\{\iiint_{\mathbb{R}^{2n+1}} \omega^q(u,v)\left\{\iiint_{\mathbb{R}^{2n+1}} f\left[r^{-1}\xi, r^{-1}\eta, r^{-2}\lambda^{-1}[\tau-\mu\lambda(u\cdot\eta-v\cdot\xi)]\right]\right.\right.\\ &\left.\left. |u-\xi|^{\alpha-n}|v-\eta|^{\alpha-n}|t-\tau|^{\beta-1}\left[\frac{|u-\xi||v-\eta|}{|t-\tau|}+\frac{|t-\tau|}{|u-\xi||v-\eta|}\right]^{-\vartheta} d\xi d\eta d\tau\right\}^q dudvdt\right\}^{\frac{1}{q}}\\ &= r^{2\alpha+2\beta} r^{-\gamma} r^{\frac{2n+2}{q}}\lambda^{\beta}\lambda^{\frac{1}{q}}\left\{\iiint_{\mathbb{R}^{2n+1}}\left[\sqrt{|u|^2+|v|^2}\right]^{-\gamma q}\left\{\iiint_{\mathbb{R}^{2n+1}} f\left(\xi,\eta,\tau-\mu(u\cdot\eta-v\cdot\xi)\right)\right.\right.\\ &\left.\left. |u-\xi|^{\alpha-n}|v-\eta|^{\alpha-n}|t-\tau|^{\beta-1}\left[\frac{|u-\xi||v-\eta|}{\lambda|t-\tau|}+\frac{\lambda|t-\tau|}{|u-\xi||v-\eta|}\right]^{-\vartheta} d\xi d\eta d\tau\right\}^q dudvdt\right\}^{\frac{1}{q}}\\ &\geq r^{2\alpha+2\beta} r^{-\gamma} r^{\frac{2n+2}{q}}\lambda^{\beta}\lambda^{\frac{1}{q}}\begin{cases}\lambda^{\vartheta}, & 0<\lambda<1,\\ \lambda^{-\vartheta}, & \lambda>1\end{cases}\\ &\left\{\iiint_{\mathbb{R}^{2n+1}}\left[\sqrt{|u|^2+|v|^2}\right]^{-\gamma q}\left\{\iiint_{\mathbb{R}^{2n+1}} f\left(\xi,\eta,\tau-\mu(u\cdot\eta-v\cdot\xi)\right)\right.\right.\\ &\left.\left. |u-\xi|^{\alpha-n}|v-\eta|^{\alpha-n}|t-\tau|^{\beta-1}\left[\frac{|u-\xi||v-\eta|}{|t-\tau|}+\frac{|t-\tau|}{|u-\xi||v-\eta|}\right]^{-\vartheta} d\xi d\eta d\tau\right\}^q dudvdt\right\}^{\frac{1}{q}}.\end{aligned} \tag{3. 2}$$

The $\mathbf{L}^p \longrightarrow \mathbf{L}^q$-norm inequality in (1. 11) implies that the last line of (3. 2) is bounded by

$$\begin{aligned}&\left\{\iiint_{\mathbb{R}^{2n+1}}\left|f\left(r^{-1}\xi, r^{-1}\eta, r^{-2}\lambda^{-1}\tau\right)\right|^p\left[\sqrt{|\xi|^2+|\eta|^2}\right]^{\delta p} d\xi d\eta d\tau\right\}^{\frac{1}{p}}\\ &= r^{\frac{2n+2}{p}} r^{\delta}\lambda^{\frac{1}{p}}\left\|f\sigma\right\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})}, \qquad (\xi,\eta,\tau)\longrightarrow(r\xi,r\eta,r^2\lambda\tau).\end{aligned} \tag{3. 3}$$

This must be true for every $r>0$ and $0<\lambda<1$ or $\lambda>1$. We necessarily have

$$\frac{\alpha+\beta}{n+1} = \frac{1}{p}-\frac{1}{q}+\frac{\gamma+\delta}{2n+2} \tag{3. 4}$$

and

$$\beta+\vartheta \geq \frac{1}{p}-\frac{1}{q} \qquad \text{or} \qquad \beta-\vartheta \leq \frac{1}{p}-\frac{1}{q}. \tag{3. 5}$$

By adding (3. 4) and (3. 5) together, we find

$$\vartheta \;\geq\; \frac{n\beta-\alpha}{n+1}+\frac{\gamma+\delta}{2n+2} \qquad \text{or} \qquad \vartheta \;\geq\; \frac{\alpha-n\beta}{n+1}-\frac{\gamma+\delta}{2n+2}.$$

This further implies

$$\vartheta \;\geq\; \left|\frac{\alpha-n\beta}{n+1}-\frac{\gamma+\delta}{2n+2}\right|. \tag{3. 6}$$

Because $\mathbf{I}_{\alpha\beta\vartheta}$ is self-adjoint, it is essential to have $\omega^q, \sigma^{-\frac{p}{p-1}}$ locally integrable. Therefore,

$$\gamma<\frac{2n}{q}, \qquad \delta<2n\left(\frac{p-1}{p}\right) \tag{3. 7}$$

are necessary.

Denote $\mathbf{R}=\mathbf{Q}_1\times\mathbf{Q}_2\times I\subset\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$ where $\mathbf{Q}_1,\mathbf{Q}_2$ are cubes in $\mathbb{R}^n$ parallel to the coordinates. Moreover, I is an interval. $\mathbf{R}'=\mathbf{Q}_1'\times\mathbf{Q}_2'\times I'$ is a translation of $\mathbf{R}$ defined as

$$\mathbf{R}' \;=\; \left\{(u,v,t)\colon \begin{array}{c} u_i=\xi_i+2\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}},\ v_i=\eta_i+2\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}},\ i=1,2,\ldots,n \\ t=\tau+2\mathbf{vol}\{I\} \end{array} \quad (\xi,\eta,\tau)\in\mathbf{R}\right\}. \tag{3. 8}$$

Consider

$$f(u,v,t) \;=\; \sigma^{-\frac{p}{p-1}}(u,v)\chi_{\mathbf{Q}_1\times\mathbf{Q}_2}(u,v)\chi_I(t), \qquad (u,v)\neq(0,0) \tag{3. 9}$$

where χ is an indicator function.

Let $\mathbf{vol}\{I\}=\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}}$. We have

$$\begin{aligned}
&\left\|\omega\mathbf{I}_{\alpha\beta\vartheta}f\right\|_{\mathbf{L}^q(\mathbb{R}^{2n+1})} \;\geq \\
&\Bigg\{\iiint_{\mathbf{R}'}\omega^q(u,v)\Bigg\{\iiint_{\mathbf{Q}_1\times\mathbf{Q}_2\times\mathbb{R}}\sigma^{-\frac{p}{p-1}}(\xi,\eta)\chi_I(\tau-\mu(u\cdot\eta-v\cdot\xi))\,|u-\xi|^{\alpha-n}\left|v-\eta\right|^{\alpha-n}|t-\tau|^{\beta-1} \\
&\qquad\left[\frac{|u-\xi|\left|v-\eta\right|}{|t-\tau|}+\frac{|t-\tau|}{|u-\xi|\left|v-\eta\right|}\right]^{-\vartheta}d\xi d\eta d\tau\Bigg\}^q dudvdt\Bigg\}^{\frac{1}{q}} \\
&\geq\; \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{I\}^{\beta-1} \\
&\quad\left\{\iiint_{\mathbf{Q}_1'\times\mathbf{Q}_2'\times I'}\omega^q(u,v)\left\{\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\sigma^{-\frac{p}{p-1}}(\xi,\eta)\left\{\int_{I-\mu(u\cdot\eta-v\cdot\xi)}d\tau\right\}d\xi d\eta\right\}^q dudvdt\right\}^{\frac{1}{q}} \\
&=\; \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{I\}^{\beta-1+\frac{1}{q}+1} \\
&\quad\left\{\iint_{\mathbf{Q}_1'\times\mathbf{Q}_2'}\omega^q(u,v)dudv\right\}^{\frac{1}{q}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\sigma^{-\frac{p}{p-1}}(\xi,\eta)d\xi d\eta \\
&=\; \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{I\}^{\beta+\frac{1}{q}}\left\{\iint_{\mathbf{Q}_1'\times\mathbf{Q}_2'}\omega^q(u,v)dudv\right\}^{\frac{1}{q}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\sigma^{-\frac{p}{p-1}}(u,v)dudv.
\end{aligned} \tag{3. 10}$$

The norm inequality in (1. 11) implies

$$\begin{aligned}&\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{I\}^{\beta+\frac{1}{q}}\left\{\iint_{\mathbf{Q}_1'\times\mathbf{Q}_2'}\omega^q(u,v)dudv\right\}^{\frac{1}{q}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\sigma^{-\frac{p}{p-1}}(u,v)dudv\\&\leq\ \mathfrak{B}_{\alpha\,\beta\,p\,q}\,\mathbf{vol}\{I\}^{\frac{1}{p}}\left\{\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\sigma^{-\frac{p}{p-1}}(u,v)dudv\right\}^{\frac{1}{p}}.\end{aligned}\tag{3. 11}$$

By taking into account $\mathbf{vol}\{I\} = \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}}$, we find

$$\begin{aligned}&\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{I\}^{\beta+\frac{1}{q}-\frac{1}{p}}\left\{\iint_{\mathbf{Q}_1'\times\mathbf{Q}_2'}\omega^q(u,v)dudv\right\}^{\frac{1}{q}}\left\{\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\sigma^{-\frac{p}{p-1}}(u,v)dudv\right\}^{\frac{p-1}{p}}\\&=\ \mathbf{vol}\{\mathbf{Q}_1\}^{\left[\frac{\alpha+\beta}{n+1}-\left(\frac{1}{p}-\frac{1}{q}\right)\right]\frac{n+1}{n}}\mathbf{vol}\{\mathbf{Q}_2\}^{\left[\frac{\alpha+\beta}{n+1}-\left(\frac{1}{p}-\frac{1}{q}\right)\right]\frac{n+1}{n}}\\&\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1'\}\mathbf{vol}\{\mathbf{Q}_2'\}}\iint_{\mathbf{Q}_1'\times\mathbf{Q}_2'}\omega^q(u,v)dudv\right\}^{\frac{1}{q}}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\sigma^{-\frac{p}{p-1}}(u,v)dudv\right\}^{\frac{p-1}{p}}\\&<\ \infty\end{aligned}\tag{3. 12}$$

for every $\mathbf{Q}_1\times\mathbf{Q}_2\subset\mathbb{R}^n\times\mathbb{R}^n$.

Note that (3. 12) holds for every $\mathbf{Q}_1\times\mathbf{Q}_2\subset\mathbb{R}^n\times\mathbb{R}^n$. Suppose $\mathbf{Q}_2$ centered on the origin and $\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}} = 1$. Let $\mathbf{Q}_1$ shrink to $u\in\mathbf{Q}_1$. Simultaneously, as defined in (3. 8), $\mathbf{Q}_1'$ shrinks to some $u'\in\mathbf{Q}_1'$ and $\mathbf{vol}\{\mathbf{Q}_2'\}^{\frac{1}{n}} = 1$. By applying Lebesgue differentiation theorem, we find

$$\lim_{\mathbf{vol}\{\mathbf{Q}_1\}\longrightarrow0}\mathbf{vol}\{\mathbf{Q}_1\}^{\left[\frac{\alpha+\beta}{n+1}-\left(\frac{1}{p}-\frac{1}{q}\right)\right]\frac{n+1}{n}}\left\{\int_{\mathbf{Q}_2'}\omega^q(u',v)dv\right\}^{\frac{1}{q}}\left\{\int_{\mathbf{Q}_2}\sigma^{-\frac{p}{p-1}}(u,v)dv\right\}^{\frac{p-1}{p}}\ <\ \infty.\tag{3. 13}$$

Clearly, the product of two integral terms in (3. 13) never vanishes. We must have $\frac{\alpha+\beta}{n+1}\geq\frac{1}{p}-\frac{1}{q}$ in order to bound the limit as $\mathbf{vol}\{\mathbf{Q}_1\}\longrightarrow0$. This together with the homogeneity condition in (3. 4) imply

$$\gamma+\delta\ \geq\ 0.\tag{3. 14}$$

For brevity of computation, denote

$$\zeta\ =\ n\left[\frac{\alpha+\beta}{n+1}\right]+\frac{\gamma+\delta}{2n+2}.\tag{3. 15}$$

We find

$$\begin{aligned}\zeta\ &=\ \frac{n}{p}-\frac{n}{q}+\frac{\gamma+\delta}{2}\qquad\left(\tfrac{\alpha+\beta}{n+1}=\tfrac{1}{p}-\tfrac{1}{q}+\tfrac{\gamma+\delta}{2n+2}\right);\\0\ <\ \zeta\ &=\ \frac{n}{p}-\frac{n}{q}+\frac{\gamma+\delta}{2}\qquad(\gamma+\delta\geq0,1<p<q<\infty)\\&<\ \frac{n}{p}-\frac{n}{q}+\frac{n}{q}+n\left(\frac{p-1}{p}\right)\ =\ n.\qquad\left(\gamma<\tfrac{2n}{q},\delta<2n\left(\tfrac{p-1}{p}\right)\right)\end{aligned}\tag{3. 16}$$

Moreover, a direct computation shows

$$\begin{aligned}\left[\frac{\alpha+\beta}{n+1}-\left(\frac{1}{p}-\frac{1}{q}\right)\right]\frac{n+1}{n} &= \frac{\alpha+\beta}{n+1}-\left(\frac{1}{p}-\frac{1}{q}\right)+\frac{1}{n}\frac{\gamma+\delta}{2n+2} \qquad \text{by (3. 4)} \\ &= \frac{\zeta}{n}-\left(\frac{1}{p}-\frac{1}{q}\right).\end{aligned} \tag{3. 17}$$

From (3. 12) and (3. 17), we obtain

$$\begin{aligned}\sup_{\mathbf{Q}_1\times\mathbf{Q}_2\subset\mathbb{R}^n\times\mathbb{R}^n} & \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\zeta}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\zeta}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)} \\ & \left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1'\}\mathbf{vol}\{\mathbf{Q}_2'\}}\iint_{\mathbf{Q}_1'\times\mathbf{Q}_2'}\left[\sqrt{|u|^2+|v|^2}\right]^{-\gamma q}dudv\right\}^{\frac{1}{q}} \\ & \left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\sqrt{|u|^2+|v|^2}\right]^{-\delta\frac{p}{p-1}}dudv\right\}^{\frac{p-1}{p}} < \infty.\end{aligned} \tag{3. 18}$$

3.1 Case One: $\gamma \geq 0$, $\delta \leq 0$

Suppose $\gamma+\delta=0$. Let ζ defined in (3. 15). From (3. 4) and (3. 17), we find

$$\frac{\zeta}{n} = \frac{1}{p}-\frac{1}{q}. \tag{3. 19}$$

Recall $\mathbf{R}'=\mathbf{Q}_1'\times\mathbf{Q}_2'\times I$ defined in (3. 8) which is a translation of $\mathbf{R}=\mathbf{Q}_1\times\mathbf{Q}_2\times I$. We consider $\mathbf{Q}_1'\times\mathbf{Q}_2'$ centered on the origin of $\mathbb{R}^n\times\mathbb{R}^n$. Let $\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}}=\mathbf{vol}\{\mathbf{Q}_2'\}^{\frac{1}{n}}=1$ and $\mathbf{Q}_1$ shrink to some $u\in\mathbf{Q}_1$ whereas $\mathbf{Q}_1'$ shrink to 0.

From (3. 18)-(3. 19), by applying Lebesgue differentiation theorem, we have

$$\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_2'\}}\int_{\mathbf{Q}_2'}|v|^{-\gamma q}dv\right\}^{\frac{1}{q}}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_2\}}\int_{\mathbf{Q}_2}\left[\sqrt{|u|^2+|v|^2}\right]^{-\delta\frac{p}{p-1}}dv\right\}^{\frac{p-1}{p}} < \infty \tag{3. 20}$$

for every $\mathbf{Q}_2\subset\mathbb{R}^n$. This suggests

$$\gamma < \frac{n}{q} \qquad \Longrightarrow \qquad \zeta-\frac{n}{p} = -\frac{n}{q}<-\gamma = \delta \tag{3. 21}$$

as an necessity.

Suppose $\gamma+\delta>0$. From (3. 4) and (3. 17), we find

$$\frac{\zeta}{n} > \frac{1}{p}-\frac{1}{q}. \tag{3. 22}$$

For every $\mathbf{Q}_1\times\mathbf{Q}_2\subset\mathbb{R}^n\times\mathbb{R}^n$, we define

$$\begin{aligned}\mathbf{A}_{p\,q}^{\zeta\,\gamma\,\delta}(\mathbf{Q}_1\times\mathbf{Q}_2) = {} & \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\zeta}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\zeta}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)} \\ & \left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1'\}\mathbf{vol}\{\mathbf{Q}_2'\}}\iint_{\mathbf{Q}_1'\times\mathbf{Q}_2'}\left[\sqrt{|u|^2+|v|^2}\right]^{-\gamma q}dudv\right\}^{\frac{1}{q}} \\ & \left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\sqrt{|u|^2+|v|^2}\right]^{-\delta\frac{p}{p-1}}dudv\right\}^{\frac{p-1}{p}}.\end{aligned} \tag{3. 23}$$

Moreover, denote

$$\mathbf{Q'}_1^k \;=\; \mathbf{Q}'_1 \cap \Big\{ 2^{-k-1} \le |u| < 2^{-k} \Big\}, \qquad \mathbf{Q'}_2^k \;=\; \mathbf{Q}'_2 \cap \Big\{ 2^{-k-1} \le |v| < 2^{-k} \Big\}, \qquad k \ge 0. \tag{3. 24}$$

Let $\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}} = \mathbf{vol}\{\mathbf{Q}'_2\}^{\frac{1}{n}} = 1$ and $\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}} = \mathbf{vol}\{\mathbf{Q}'_1\}^{\frac{1}{n}} = \lambda$ for $0 < \lambda < 1$. From (3. 23)-(3. 24), we have

$$\begin{aligned}
\Big[\mathbf{A}_{p\,q}^{\zeta\,\gamma\,\delta}(\mathbf{Q}_1\times\mathbf{Q}_2)\Big]^q \;&=\; \lambda^{q\left[\zeta-(\frac{n}{p}-\frac{n}{q})\right]} \left\{ \frac{1}{\lambda^n} \iint_{\mathbf{Q}'_1\times\mathbf{Q}'_2} \Big[\sqrt{|u|^2+|v|^2}\Big]^{-\gamma q} dudv \right\} \\
&\qquad \left\{ \frac{1}{\lambda^n} \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \Big[\sqrt{|u|^2+|v|^2}\Big]^{-\delta\frac{p}{p-1}} dudv \right\}^{\left[\frac{p-1}{p}\right]q} \\
&=\; \lambda^{q\left[\zeta-(\frac{n}{p}-\frac{n}{q})\right]} \sum_{k\ge0} \left\{ \frac{1}{\lambda^n} \iint_{\mathbf{Q}'_1\times\mathbf{Q'}_2^k} \Big[\sqrt{|u|^2+|v|^2}\Big]^{-\gamma q} dudv \right\} \\
&\qquad \left\{ \frac{1}{\lambda^n} \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \Big[\sqrt{|u|^2+|v|^2}\Big]^{-\delta\frac{p}{p-1}} dudv \right\}^{\left[\frac{p-1}{p}\right]q} \\
&\doteq\; \sum_{k\ge0} \mathbf{A}_k(\lambda).
\end{aligned} \tag{3. 25}$$

Lebesgue's Differentiation Theorem implies

$$\lim_{\lambda\longrightarrow0} \frac{1}{\lambda^n} \iint_{\mathbf{Q}'_1\times\mathbf{Q'}_2^k} \Big[\sqrt{|u|^2+|v|^2}\Big]^{-\gamma q} dudv \;=\; \int_{\mathbf{Q'}_2^k} |v|^{-\gamma q} dv. \tag{3. 26}$$

Because $\delta \le 0$ and $\zeta > \frac{n}{p} - \frac{n}{q}$, we find

$$\mathbf{A}_k(0) \;=\; 0, \qquad k \ge 0. \tag{3. 27}$$

Note that (3. 27) is true if $\zeta - (\frac{n}{p} - \frac{n}{q})$ in (3. 25) is replaced by any smaller positive number. Therefore, $\mathbf{A}_k(\lambda)$ is Hölder continuous *w.r.t* λ whose exponent remains strictly positive as $k \longrightarrow \infty$. Recall (3. 18). We have $\sum_{k\ge0}\mathbf{A}_k(\lambda) \le \mathfrak{B}_{\alpha\,\gamma\,\delta\,q}$ for every $\lambda > 0$. Consequently, $\sum_{k\ge0}\mathbf{A}_k(\lambda)$ is continuous at $\lambda = 0$ and

$$\lim_{\lambda\longrightarrow0} \sum_{k\ge0} \mathbf{A}_k(\lambda) \;=\; 0. \tag{3. 28}$$

A direct computation shows

$$\begin{aligned}
\Big[\mathbf{A}_{p\,q}^{\zeta\,\gamma\,\delta}(\mathbf{Q}_1\times\mathbf{Q}_2)\Big]^q \;&=\; \lambda^{q\left[\zeta-(\frac{n}{p}-\frac{n}{q})\right]} \left\{ \frac{1}{\lambda^n} \iint_{\mathbf{Q}'_1\times\mathbf{Q}'_2} \Big[\sqrt{|u|^2+|v|^2}\Big]^{-\gamma q} dudv \right\} \\
&\qquad \left\{ \frac{1}{\lambda^n} \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \Big[\sqrt{|u|^2+|v|^2}\Big]^{-\delta\frac{p}{p-1}} dudv \right\}^{\left[\frac{p-1}{p}\right]q} \\
&\ge\; \mathfrak{B}\lambda^{q\left[\zeta-(\frac{n}{p}-\frac{n}{q})\right]} \int_{\mathbf{Q}'_2} \Big[\sqrt{\lambda^2+|v|^2}\Big]^{-\gamma q} dv \qquad (\,\delta\le0,\ \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}} = \mathbf{vol}\{\mathbf{Q}'_1\}^{\frac{1}{n}} = \lambda\,) \\
&\ge\; \mathfrak{B}\lambda^{q\left[\zeta-(\frac{n}{p}-\frac{n}{q})\right]} \int_{0<|v|\le\lambda} \left(\frac{1}{\lambda}\right)^{\gamma q} dv \;=\; \mathfrak{B}_{\gamma\,q}\, \lambda^{n-\gamma q+q\left[\zeta-(\frac{n}{p}-\frac{n}{q})\right]}.
\end{aligned} \tag{3. 29}$$

From (3. 28)-(3. 29), by using $\zeta = \frac{n}{p} - \frac{n}{q} + \frac{\gamma+\delta}{2}$ as shown in (3. 16), we find

$$\begin{array}{lcl} \frac{n}{q} - \gamma + \zeta - \left(\frac{n}{p} - \frac{n}{q}\right) \; > \; 0 & & \Longrightarrow \\ \zeta \; < \; \frac{n}{q} - \gamma + 2\zeta - \left(\frac{n}{p} - \frac{n}{q}\right) & = & \frac{n}{q} - \gamma + \left(\frac{n}{p} - \frac{n}{q}\right) + \gamma + \delta \\ & = & \frac{n}{p} + \delta. \end{array} \tag{3. 30}$$

Recall $\zeta = n[\frac{\alpha+\beta}{n+1}] + \frac{\gamma+\delta}{2n+2}$. By putting together (3. 21) and (3. 30), we obtain

$$n\left[\frac{\alpha+\beta}{n+1}\right] + \frac{\gamma+\delta}{2n+2} - \frac{n}{p} \; < \; \delta \qquad \text{for} \qquad \gamma \geq 0, \; \delta \leq 0. \tag{3. 31}$$

3.2 Case Two: $\gamma \leq 0$, $\delta \geq 0$

Suppose $\gamma + \delta = 0$. From (3. 4) and (3. 17), we find $\frac{\zeta}{n} = \frac{1}{p} - \frac{1}{q} = \frac{q-1}{q} - \frac{p-1}{p}$ as shown in (3. 19). The estimate in (3. 20) suggests

$$\delta \; < \; n\left(\frac{p-1}{p}\right) \qquad \Longrightarrow \qquad \zeta - n\left(\frac{q-1}{q}\right) \; = \; -n\left(\frac{p-1}{p}\right) < -\delta \; = \; \gamma \tag{3. 32}$$

as an necessity.

Suppose $\gamma + \delta > 0$. From (3. 4) and (3. 17), we find $\frac{\zeta}{n} > \frac{1}{p} - \frac{1}{q}$ as (3. 22).

For every $\mathbf{Q}_1 \times \mathbf{Q}_2 \subset \mathbb{R}^n \times \mathbb{R}^n$, $\mathbf{A}_{p\,q}^{\zeta\,\gamma\,\delta}(\mathbf{Q}_1 \times \mathbf{Q}_2)$ is defined in (3. 23). Denote

$$\mathbf{Q}_1^k \; = \; \mathbf{Q}_1 \cap \left\{2^{-k-1} \leq |u| < 2^{-k}\right\}, \qquad \mathbf{Q}_2^k \; = \; \mathbf{Q}_2 \cap \left\{2^{-k-1} \leq |v| < 2^{-k}\right\}, \qquad k \geq 0. \tag{3. 33}$$

As before, suppose $\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}} = \mathbf{vol}\{\mathbf{Q}_2'\}^{\frac{1}{n}} = 1$ and $\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}} = \mathbf{vol}\{\mathbf{Q}_1'\}^{\frac{1}{n}} = \lambda$ for $0 < \lambda < 1$. From (3. 23) and (3. 33), we have

$$\begin{aligned} \left[\mathbf{A}_{p\,q}^{\zeta\,\gamma\,\delta}(\mathbf{Q}_1 \times \mathbf{Q}_2)\right]^{\frac{p}{p-1}} \; = \; & \lambda^{\frac{p}{p-1}[\zeta - (\frac{n}{p} - \frac{n}{q})]} \left\{\frac{1}{\lambda^n} \iint_{\mathbf{Q}_1' \times \mathbf{Q}_2'} \left[\sqrt{|u|^2 + |v|^2}\right]^{-\gamma q} dudv\right\}^{\frac{1}{q}\frac{p}{p-1}} \\ & \left\{\frac{1}{\lambda^n} \iint_{\mathbf{Q}_1 \times \mathbf{Q}_2} \left[\sqrt{|u|^2 + |v|^2}\right]^{-\delta\frac{p}{p-1}} dudv\right\} \\ = \; & \lambda^{\frac{p}{p-1}[\zeta - (\frac{n}{p} - \frac{n}{q})]} \left\{\frac{1}{\lambda^n} \iint_{\mathbf{Q}_1' \times \mathbf{Q}_2'} \left[\sqrt{|u|^2 + |v|^2}\right]^{-\gamma q} dudv\right\}^{\frac{1}{q}\frac{p}{p-1}} \\ & \sum_{k\geq 0} \left\{\frac{1}{\lambda^n} \iint_{\mathbf{Q}_1 \times \mathbf{Q}_2^k} \left[\sqrt{|u|^2 + |v|^2}\right]^{-\delta\frac{p}{p-1}} dudv\right\} \\ \doteq \; & \sum_{k\geq 0} \mathbf{B}_k(\lambda). \end{aligned} \tag{3. 34}$$

Lebesgue's Differentiation Theorem implies

$$\lim_{\lambda \longrightarrow 0} \frac{1}{\lambda^n} \iint_{\mathbf{Q}_1 \times \mathbf{Q}_2^k} \left[\sqrt{|u|^2 + |v|^2} \right]^{-\delta \frac{p}{p-1}} dudv \ = \ \int_{\mathbf{Q}_2^k} |v|^{-\delta \frac{p}{p-1}} dv. \tag{3. 35}$$

Because $\gamma \leq 0$ and $\zeta > \frac{n}{p} - \frac{n}{q}$, we find

$$\mathbf{B}_k(0) \ = \ 0, \qquad k \geq 0. \tag{3. 36}$$

As same as (3. 27), the estimate in (3. 36) is true if $\zeta - (\frac{n}{p} - \frac{n}{q})$ in (3. 34) is replaced by a smaller positive number. Therefore, $\mathbf{B}_k(\lambda)$ is Hölder continuous *w.r.t* λ whose exponent remains strictly positive as $k \longrightarrow \infty$.

Recall (3. 18). We have $\sum_{k \geq 0} \mathbf{B}_k(\lambda) \leq \mathfrak{B}_{\alpha\, \gamma\, \delta\, q}$ for every $\lambda > 0$. Consequently, $\sum_{k\geq 0} \mathbf{B}_k(\lambda)$ is continuous at $\lambda = 0$ and

$$\lim_{\lambda \longrightarrow 0} \sum_{k \geq 0} \mathbf{B}_k(\lambda) \ = \ 0. \tag{3. 37}$$

A direct computation shows

$$\begin{aligned}
\left[\mathbf{A}_{p\, q}^{\zeta\, \gamma\, \delta}(\mathbf{Q}_1 \times \mathbf{Q}_2) \right]^{\frac{p}{p-1}} \ &= \ \lambda^{\frac{p}{p-1}[\zeta - (\frac{n}{p} - \frac{n}{q})]} \left\{ \frac{1}{\lambda^n} \iint_{\mathbf{Q}_1' \times \mathbf{Q}_2'} \left[\sqrt{|u|^2 + |v|^2} \right]^{-\gamma q} dudv \right\}^{\frac{1}{q} \frac{p}{p-1}} \\
&\qquad \left\{ \frac{1}{\lambda^n} \iint_{\mathbf{Q}_1 \times \mathbf{Q}_2} \left[\sqrt{|u|^2 + |v|^2} \right]^{-\delta \frac{p}{p-1}} dudv \right\} \\
&\geq \ \mathfrak{B} \lambda^{\frac{p}{p-1}[\zeta - (\frac{n}{p} - \frac{n}{q})]} \int_{\mathbf{Q}_2} \left[\sqrt{\lambda^2 + |v|^2} \right]^{-\delta \frac{p}{p-1}} dv \qquad (\gamma \leq 0, \ \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}} = \mathbf{vol}\{\mathbf{Q}_1'\}^{\frac{1}{n}} = \lambda) \\
&\geq \ \mathfrak{B} \lambda^{\frac{p}{p-1}[\zeta - (\frac{n}{p} - \frac{n}{q})]} \int_{0 < |v| \leq \lambda} \left(\frac{1}{\lambda} \right)^{\delta \frac{p}{p-1}} dv \ = \ \mathfrak{B}_{\delta\, p}\, \lambda^{n - \delta(\frac{p}{p-1}) + \frac{p}{p-1}[\zeta - (\frac{n}{p} - \frac{n}{q})]}.
\end{aligned} \tag{3. 38}$$

From (3. 37)-(3. 38), by using $\zeta = \frac{n}{p} - \frac{n}{q} + \frac{\gamma + \delta}{2} = n[\frac{q-1}{q} - \frac{p-1}{p}] + \frac{\gamma+\delta}{2}$ in (3. 16), we find

$$\begin{aligned}
& n\left(\frac{p-1}{p} \right) - \delta + \zeta - \left(\frac{n}{p} - \frac{n}{q} \right) \ > \ 0 \qquad \Longrightarrow \\
& \zeta \ < \ n\left(\frac{p-1}{p} \right) - \delta + 2\zeta - n\left[\frac{q-1}{q} - \frac{p-1}{p} \right] \ = \ n\left(\frac{p-1}{p} \right) - \delta + n\left[\frac{q-1}{q} - \frac{p-1}{p} \right] + \gamma + \delta \\
& \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \ = \ n\left(\frac{q-1}{q} \right) + \gamma.
\end{aligned} \tag{3. 39}$$

Recall $\zeta = n[\frac{\alpha+\beta}{n+1}] + \frac{\gamma+\delta}{2n+2}$. By putting together (3. 32) and (3. 39), we obtain

$$n\left[\frac{\alpha + \beta}{n+1} \right] + \frac{\gamma + \delta}{2n+2} - n\left(\frac{q-1}{q} \right) \ < \ \gamma \qquad \text{for} \qquad \gamma \leq 0, \ \delta \geq 0. \tag{3. 40}$$

4 Reformulation of $\mathbf{I}_{\alpha\beta\vartheta}$

Recall $\mathbf{V}^{\alpha\beta\vartheta}(u,v,t)$ defined in (1. 15) for $u\neq 0, v\neq 0, t\neq 0$ and $\vartheta\geq\left|\frac{\alpha-n\beta}{n+1}-\frac{\gamma+\delta}{2n+2}\right|$. Suppose $2\alpha-2n\beta-\gamma-\delta\geq 0$. We have

$$\begin{aligned}
\mathbf{V}^{\alpha\beta\vartheta}(u,v,t) &\leq |u|^{\alpha-n}|v|^{\alpha-n}|t|^{\beta-1}\left[\frac{|u||v|}{|t|}+\frac{|t|}{|u||v|}\right]^{-\left[\frac{\alpha-n\beta}{n+1}-\frac{\gamma+\delta}{2n+2}\right]}\\
&\leq |u|^{\alpha-n}|v|^{\alpha-n}|t|^{\beta-1}\left[\frac{|u||v|}{|t|}\right]^{-\left[\frac{\alpha-n\beta}{n+1}-\frac{\gamma+\delta}{2n+2}\right]}\\
&= |u|^{n\left[\frac{\alpha+\beta}{n+1}\right]+\frac{\gamma+\delta}{2n+2}-n}|v|^{n\left[\frac{\alpha+\beta}{n+1}\right]+\frac{\gamma+\delta}{2n+2}-n}|t|^{\frac{\alpha+\beta}{n+1}-\frac{\gamma+\delta}{2n+2}-1}, \qquad u\neq 0, v\neq 0, t\neq 0.
\end{aligned} \tag{4. 1}$$

Suppose $2\alpha-2n\beta-\gamma-\delta\leq 0$. We find

$$\begin{aligned}
\mathbf{V}^{\alpha\beta\vartheta}(u,v,t) &\leq |u|^{\alpha-n}|v|^{\alpha-n}|t|^{\beta-1}\left[\frac{|u||v|}{|t|}+\frac{|t|}{|u||v|}\right]^{\frac{\alpha-n\beta}{n+1}-\frac{\gamma+\delta}{2n+2}}\\
&\leq |u|^{\alpha-n}|v|^{\alpha-n}|t|^{\beta-1}\left[\frac{|t|}{|u||v|}\right]^{\frac{\alpha-n\beta}{n+1}-\frac{\gamma+\delta}{2n+2}}\\
&= |u|^{n\left[\frac{\alpha+\beta}{n+1}\right]+\frac{\gamma+\delta}{2n+2}-n}|v|^{n\left[\frac{\alpha+\beta}{n+1}\right]+\frac{\gamma+\delta}{2n+2}-n}|t|^{\frac{\alpha+\beta}{n+1}-\frac{\gamma+\delta}{2n+2}-1}, \qquad u\neq 0, v\neq 0, t\neq 0.
\end{aligned} \tag{4. 2}$$

As (3. 15), we write $\zeta=n\left[\frac{\alpha+\beta}{n+1}\right]+\frac{\gamma+\delta}{2n+2}$ where $0<\zeta<n$. Let $\mathbf{I}_{\alpha\beta\vartheta}$ defined in (1. 15)-(1. 16). From now on, we assert $f\geq 0$. By changing variable $\tau\longrightarrow\tau-\mu(u\cdot\eta-v\cdot\xi)$, we have

$$\begin{aligned}
\mathbf{I}_{\alpha\beta\vartheta}f(u,v,t) &= \iiint_{\mathbb{R}^{2n+1}} f\left(\xi,\eta,\tau-\mu(u\cdot\eta-v\cdot\xi)\right)\mathbf{V}^{\alpha\beta\vartheta}(u-\xi,v-\eta,t-\tau)d\xi d\eta d\tau\\
&\leq \iiint_{\mathbb{R}^{2n+1}} f\left(\xi,\eta,\tau-\mu(u\cdot\eta-v\cdot\xi)\right)\\
&\qquad |u-\xi|^{\alpha-n}|v-\eta|^{\alpha-n}|t-\tau|^{\beta-1}\left[\frac{|u-\xi||v-\eta|}{|t-\tau|}+\frac{|t-\tau|}{|u-\xi||v-\eta|}\right]^{-\left|\frac{\alpha-n\beta}{n+1}-\frac{\gamma+\delta}{2n+2}\right|}d\xi d\eta d\tau\\
&\leq \iiint_{\mathbb{R}^{2n+1}} f\left(\xi,\eta,\tau-\mu(u\cdot\eta-v\cdot\xi)\right)\\
&\qquad |u-\xi|^{\zeta-n}|v-\eta|^{\zeta-n}|t-\tau|^{\frac{\alpha+\beta}{n+1}-\frac{\gamma+\delta}{2n+2}-1}d\xi d\eta d\tau \qquad \text{by (4. 1)-(4. 2)}\\
&\doteq \iint_{\mathbb{R}^{2n}} |u-\xi|^{\zeta-n}|v-\eta|^{\zeta-n}\mathbf{F}_{\alpha\beta\gamma\delta}(\xi,\eta,u,v,t)d\xi d\eta
\end{aligned} \tag{4. 3}$$

where

$$\mathbf{F}_{\alpha\beta\gamma\delta}(\xi,\eta,u,v,t) = \int_{\mathbb{R}} f\left(\xi,\eta,\tau-\mu\left(u\cdot\eta-v\cdot\xi\right)\right)|t-\tau|^{\left[\frac{\alpha+\beta}{n+1}-\frac{\gamma+\delta}{2n+2}\right]-1}d\tau. \tag{4. 4}$$

Recall the **Hardy-Littlewood-Sobolev theorem** stated in the beginning of this paper. By applying (1. 2) with $\mathbf{a} = \frac{\alpha+\beta}{n+1} - \frac{\gamma+\delta}{2n+2} = \frac{1}{p} - \frac{1}{q}$ and $\mathbf{N} = 1$, we find

$$\begin{aligned}\left\{\int_{\mathbb{R}} \mathbf{F}^q_{\alpha\beta\gamma\delta}(\xi,\eta,u,v,t)dt\right\}^{\frac{1}{q}} &\leq \mathfrak{B}_{p\,q}\left\{\int_{\mathbb{R}}\Big[f\left(\xi,\eta,t+\mu\left(u\cdot\eta-v\cdot\xi\right)\right)\Big]^p dt\right\}^{\frac{1}{p}} \\ &= \mathfrak{B}_{p\,q}\left\|f(\xi,\eta,\cdot)\right\|_{\mathbf{L}^p(\mathbb{R})}, \qquad (u,v)\in\mathbb{R}^n\times\mathbb{R}^n.\end{aligned} \tag{4. 5}$$

From (4. 3)-(4. 5), we find

$$\left\{\iiint_{\mathbb{R}^{2n+1}} \sqrt{|u|^2+|v|^2}^{-\gamma q}\Big(\mathbf{I}_{\alpha\beta\vartheta}f\Big)^q(u,v,t)dudvdt\right\}^{\frac{1}{q}}$$

$$\leq \left\{\iiint_{\mathbb{R}^{2n+1}} \sqrt{|u|^2+|v|^2}^{-\gamma q}\left\{\iint_{\mathbb{R}^{2n}} |u-\xi|^{\zeta-n}|v-\eta|^{\zeta-n}\mathbf{F}_{\alpha\beta\gamma\delta}(\xi,\eta,u,v,t)d\xi d\eta\right\}^q dudvdt\right\}^{\frac{1}{q}}$$

$$\leq \left\{\iint_{\mathbb{R}^{2n}} \sqrt{|u|^2+|v|^2}^{-\gamma q}\left\{\iint_{\mathbb{R}^{2n}} |u-\xi|^{\zeta-n}|v-\eta|^{\zeta-n}\left\{\int_{\mathbb{R}}\mathbf{F}^q_{\alpha\beta\gamma\delta}(\xi,\eta,u,v,t)dt\right\}^{\frac{1}{q}} d\xi d\eta\right\}^q dudv\right\}^{\frac{1}{q}}$$

by Minkowski integral inequality

$$\leq \mathfrak{B}_{p\,q}\left\{\iint_{\mathbb{R}^{2n}} \sqrt{|u|^2+|v|^2}^{-\gamma q}\left\{\iint_{\mathbb{R}^{2n}} |u-\xi|^{\zeta-n}|v-\eta|^{\zeta-n}\left\|f(\xi,\eta,\cdot)\right\|_{\mathbf{L}^p(\mathbb{R})}d\xi d\eta\right\}^q dudv\right\}^{\frac{1}{q}}. \tag{4. 6}$$

Define

$$\mathbf{II}_\zeta g(u,v) = \iint_{\mathbb{R}^{2n}} g(\xi,\eta)|u-\xi|^{\zeta-n}|v-\eta|^{\zeta-n}d\xi d\eta, \qquad 0<\zeta<n. \tag{4. 7}$$

Recall (1. 17)-(1. 18). As a consequence of (4. 6), we can finish the proof of **Theorem Two** by obtaining the next two results.

Proposition One *Let* $\mathbf{II}_\zeta$ *defined in (4. 7) for* $0<\zeta<n$. *Suppose* $\omega(u,v) = \sqrt{|u|^2+|v|^2}^{-\gamma}$, $\sigma(u,v) = \sqrt{|u|^2+|v|^2}^{\delta}$ *for* $(u,v)\neq(0,0)$ *and* $\gamma+\delta=0$. *We have*

$$\begin{aligned}\left\|\omega\mathbf{II}_\zeta g\right\|_{\mathbf{L}^q(\mathbb{R}^{2n})} &\leq \mathfrak{B}_{p\,q}\left\|g\omega\right\|_{\mathbf{L}^p(\mathbb{R}^{2n})}, \qquad 1<p<q<\infty \\ \textit{if}\qquad \frac{\zeta}{n} &= \frac{1}{p}-\frac{1}{q}, \qquad \gamma<\frac{n}{q}, \qquad \delta<n\left(\frac{p-1}{p}\right).\end{aligned} \tag{4. 8}$$

Proposition Two *Let* $\mathbf{II}_\zeta$ *defined in (4. 7) for* $0<\zeta<n$. *Suppose* $\omega(u,v) = \sqrt{|u|^2+|v|^2}^{-\gamma}$, $\sigma(u,v) = \sqrt{|u|^2+|v|^2}^{\delta}$ *for* $(u,v)\neq(0,0)$ *and* $\gamma+\delta>0$. *We have*

$$\left\|\omega\mathbf{II}_\zeta g\right\|_{\mathbf{L}^q(\mathbb{R}^{2n})} \leq \mathfrak{B}_{p\,q\,\gamma\,\delta}\left\|g\sigma\right\|_{\mathbf{L}^p(\mathbb{R}^{2n})}, \qquad 1<p\leq q<\infty$$

$$\textit{if}\qquad \gamma<\frac{2n}{q}, \qquad \delta<2n\left(\frac{p-1}{p}\right), \qquad \frac{\zeta}{n} = \frac{1}{p}-\frac{1}{q}+\frac{\gamma+\delta}{2n}; \tag{4. 9}$$

$$\zeta-\frac{n}{p}<\delta \;\; \textit{for} \;\; \gamma\geq 0,\ \delta\leq 0; \qquad \zeta-n\left(\frac{q-1}{q}\right)<\gamma \;\; \textit{for} \;\; \gamma\leq 0,\ \delta\geq 0.$$

4.1 Proof of Proposition One

Observe that when $\gamma + \delta = 0$, we have $\omega = \sigma$. Recall a classical one-weight theorem of fractional integrals due to Muckenhoupt and Wheeden [8].

Muckenhoupt-Wheeden theorem *Let* $\mathbf{T_a}$ *defined in (1. 1) for* $0 < \mathbf{a} < \mathbf{N}$. *Suppose* $\omega \geq 0$ *for a.e* $x \in \mathbb{R}^{\mathbf{N}}$. *Denote* $\mathbf{Q}$ *to be a cube in* $\mathbb{R}^{\mathbf{N}}$ *parallel to the coordinates. We have*

$$\left\|\omega \mathbf{T_a} f\right\|_{\mathbf{L}^q(\mathbb{R}^{\mathbf{N}})} \leq \mathfrak{B}_{p\,q} \left\|f\omega\right\|_{\mathbf{L}^p(\mathbb{R}^{\mathbf{N}})}, \qquad 1 < p < q < \infty \tag{4. 10}$$

if and only if

$$\frac{\mathbf{a}}{\mathbf{N}} = \frac{1}{p} - \frac{1}{q} \tag{4. 11}$$

and

$$\left\{\mathbf{vol}\{\mathbf{Q}\}^{-1}\int_{\mathbf{Q}} \omega^q(x)dx\right\}^{\frac{1}{q}} \left\{\mathbf{vol}\{\mathbf{Q}\}^{-1}\int_{\mathbf{Q}} \omega^{-\frac{p}{p-1}}(x)dx\right\}^{\frac{p-1}{p}} < \infty \tag{4. 12}$$

for every $\mathbf{Q} \subset \mathbb{R}^{\mathbf{N}}$.

Consider $\omega(u,v) = \sqrt{|u|^2+|v|^2}^{-\gamma}$ where $\gamma + \delta = 0$ for $\gamma < \frac{n}{q}$ and $\delta < n\left(\frac{p-1}{p}\right)$. Take into account for $\mathbf{a} = \zeta$ and $\mathbf{N} = n$. For every $\mathbf{Q} \subset \mathbb{R}^n$, we simultaneously find

$$\left\{\mathbf{vol}\{\mathbf{Q}\}^{-1}\int_{\mathbf{Q}} \left[\sqrt{|u|^2+|v|^2}\right]^{-\gamma q} du\right\}^{\frac{1}{q}} \left\{\mathbf{vol}\{\mathbf{Q}\}^{-1}\int_{\mathbf{Q}} \left[\sqrt{|u|^2+|v|^2}\right]^{-\delta\frac{p}{p-1}} du\right\}^{\frac{p-1}{p}} < \infty, \quad v \in \mathbb{R}^n; \tag{4. 13}$$

$$\left\{\mathbf{vol}\{\mathbf{Q}\}^{-1}\int_{\mathbf{Q}} \left[\sqrt{|u|^2+|v|^2}\right]^{-\gamma q} dv\right\}^{\frac{1}{q}} \left\{\mathbf{vol}\{\mathbf{Q}\}^{-1}\int_{\mathbf{Q}} \left[\sqrt{|u|^2+|v|^2}\right]^{-\delta\frac{p}{p-1}} dv\right\}^{\frac{p-1}{p}} < \infty, \quad u \in \mathbb{R}^n. \tag{4. 14}$$

Indeed, by using $\gamma + \delta = 0$, a standard one-parameter dilations in the left-hand -side of (4. 13) or (4. 14) shows that it is suffice to assume $\mathbf{vol}\{\mathbf{Q}\}^{\frac{1}{n}} = 1$. Moreover, $\omega^{q\gamma}(\cdot, v)$ and $\omega^{-\delta\frac{p}{p-1}}(\cdot, v)$ are locally integrable in $\mathbb{R}^n$ for every $v \in \mathbb{R}^n$ provided that $\gamma < \frac{n}{q}$ and $\delta < n\left(\frac{p-1}{p}\right)$. Vice versa for $\omega^{q\gamma}(u,\cdot)$ and $\omega^{-\delta\frac{p}{p-1}}(u,\cdot)$. Let $\mathbf{II}_\zeta$ defined in (4. 7) for $0 < \zeta < n$ and $g \geq 0$. By applying **Muckenhoupt-Wheeden theorem** two times, we have

$$\begin{aligned}
\left\|\omega \mathbf{II}_\zeta g\right\|_{\mathbf{L}^q(\mathbb{R}^{2n})} &= \left\{\iint_{\mathbb{R}^{2n}} \sqrt{|u|^2+|v|^2}^{-\gamma q} \left\{\iint_{\mathbb{R}^{2n}} |u-\xi|^{\zeta-n}|v-\eta|^{\zeta-n} g(\xi,\eta)d\xi d\eta\right\}^q dudv\right\}^{\frac{1}{q}} \\
&\leq \mathfrak{B}_{p\,q}\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n} |u-\xi|^{\zeta-n} g(\xi,v)d\xi\right\}^p \sqrt{|u|^2+|v|^2}^{\delta p} dv\right\}^{\frac{q}{p}} du\right\}^{\frac{1}{q}} \\
&\leq \mathfrak{B}_{p\,q}\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n} \sqrt{|u|^2+|v|^2}^{-\gamma q}\left\{\int_{\mathbb{R}^n} |u-\xi|^{\zeta-n} g(\xi,v)d\xi\right\}^q du\right\}^{\frac{p}{q}} dv\right\}^{\frac{1}{p}} \\
&\qquad \text{by Minkowski integral inequality and } \gamma = -\delta \\
&\leq \mathfrak{B}_{p\,q}\left\{\iint_{\mathbb{R}^{2n}} \left[g(u,v)\right]^p \sqrt{|u|^2+|v|^2}^{\delta p} dudv\right\}^{\frac{1}{p}} = \mathfrak{B}_{p\,q}\left\|g\omega\right\|_{\mathbf{L}^p(\mathbb{R}^{2n})}.
\end{aligned} \tag{4. 15}$$

5 Cone decomposition on $\mathbb{R}^n \times \mathbb{R}^n$

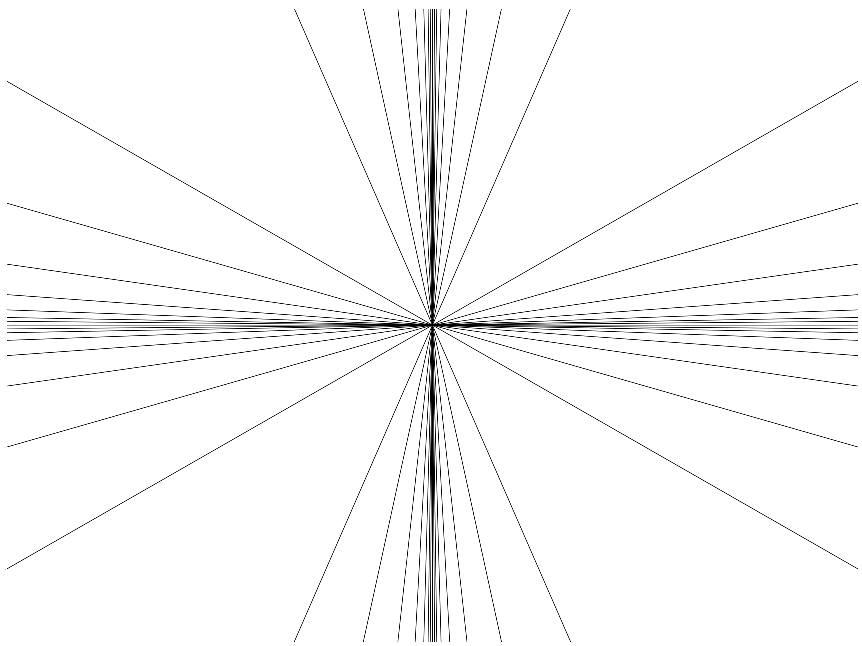
Let $\mathbf{II}_\zeta$ defined in (4. 7) for $0 < \zeta < n$. For every $j \in \mathbb{Z}$, we consider

$$\Delta_j \mathbf{II}_\zeta g(u,v) = \iint_{\Lambda_j(u,v)} g(\xi,\eta) \left(\frac{1}{|u-\xi|}\right)^{n-\zeta} \left(\frac{1}{|v-\eta|}\right)^{n-\zeta} d\xi d\eta \tag{5. 1}$$

where

$$\Lambda_j(u,v) = \left\{(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n : 2^{-j} \leq \frac{|u-\xi|}{|v-\eta|} < 2^{-j+1}\right\}. \tag{5. 2}$$

Observe that each $\Lambda_j(u,v)$ is a dyadic cone centered on $(u,v) \in \mathbb{R}^n \times \mathbb{R}^n$ with an eccentricity depending on $j \in \mathbb{Z}$.



Denote $\mathbf{Q}_i^j$ to be a dilated of $\mathbf{Q}_i \subset \mathbb{R}^n$ such that $\mathbf{vol}\{\mathbf{Q}_i^j\}^{\frac{1}{n}} = 2^{-j}\mathbf{vol}\{\mathbf{Q}_i\}^{\frac{1}{n}}$ for $i = 1,2$ and $j \in \mathbb{Z}$. Let $r > 1$. We have

$$\begin{aligned}
&\prod_{i=1}^2 \mathbf{vol}\{\mathbf{Q}_i\}^{\frac{\zeta}{n}-\frac{1}{p}+\frac{1}{q}} \left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}} \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \omega^{qr}\left(2^{-j}u,v\right) dudv\right\}^{\frac{1}{qr}} \\
&\qquad\qquad \left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}} \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}} \left(2^{-j}u,v\right) dudv\right\}^{\frac{p-1}{pr}} \\
&= 2^{j\left[\zeta-\frac{n}{p}+\frac{n}{q}\right]} \mathbf{vol}\{\mathbf{Q}_1^j\}^{\frac{\zeta}{n}-\frac{1}{p}+\frac{1}{q}} \mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\zeta}{n}-\frac{1}{p}+\frac{1}{q}} \\
&\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1^j\}\mathbf{vol}\{\mathbf{Q}_2\}} \iint_{\mathbf{Q}_1^j\times\mathbf{Q}_2} \omega^{qr}(u,v) dudv\right\}^{\frac{1}{qr}} \left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1^j\}\mathbf{vol}\{\mathbf{Q}_2\}} \iint_{\mathbf{Q}_1^j\times\mathbf{Q}_2} \left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}} (u,v) dudv\right\}^{\frac{p-1}{pr}}.
\end{aligned} \tag{5. 3}$$

Given $j \in \mathbb{Z}$, we define

$$\begin{aligned}\mathbf{A}^{\zeta}_{pqr}(j:\omega,\sigma) \;=\; & \sup_{\mathbf{Q}_1\times\mathbf{Q}_2\subset\mathbb{R}^n\times\mathbb{R}^n:\ \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}/\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}}=2^{-j}} \prod_{i=1}^{2} \mathbf{vol}\{\mathbf{Q}_i\}^{\frac{\zeta}{n}-\frac{1}{p}+\frac{1}{q}} \\ & \left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\omega^{qr}(u,v)\,dudv\right\}^{\frac{1}{qr}}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}}(u,v)\,dudv\right\}^{\frac{p-1}{pr}}.\end{aligned} \tag{5. 4}$$

Suppose $\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}} = \mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}}$. We find

$$\begin{aligned}& \prod_{i=1}^{2}\mathbf{vol}\{\mathbf{Q}_i\}^{\frac{\zeta}{n}-\frac{1}{p}+\frac{1}{q}}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\omega^{qr}\left(2^{-j}u,v\right)dudv\right\}^{\frac{1}{qr}} \\ & \qquad \left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}}\left(2^{-j}u,v\right)dudv\right\}^{\frac{p-1}{pr}} \\ & \le\; 2^{j\left[\zeta-\frac{n}{p}+\frac{n}{q}\right]}\mathbf{A}^{\zeta}_{pqr}(j:\omega,\sigma) \qquad \text{by (5. 3)-(5. 4).}\end{aligned} \tag{5. 5}$$

Next, recall a classical result due to Sawyer and Wheeden [9] for one-parameter fractional integral operators in weighted norms.

Suppose

$$\mathbf{A}^{\zeta}_{pqr}(0\colon\omega,\sigma) \;<\; \infty \qquad \text{for some } r>1. \tag{5. 6}$$

We have

$$\begin{aligned}& \left\{\iint_{\mathbb{R}^{2n}}\left\{\iint_{\mathbb{R}^{2n}} g(\xi,\eta)\left[\frac{1}{\sqrt{|u-\xi|^2+|v-\eta|^2}}\right]^{2n-2\zeta} d\xi d\eta\right\}^{q}\omega^{q}(u,v)dudv\right\}^{\frac{1}{q}} \\ & \le\; \mathfrak{B}_{p\;q\;r\;\zeta}\,\mathbf{A}^{\zeta}_{pqr}(0\colon\omega,\sigma)\left\{\iint_{\mathbb{R}^{2n}}\left(g\sigma\right)^{p}(u,v)dudv\right\}^{\frac{1}{p}}, \qquad 1<p\le q<\infty.\end{aligned} \tag{5. 7}$$

Remark 5.1. *The constant $\mathfrak{B}_{p\;q\;r\;\zeta}\,\mathbf{A}^{\zeta}_{pqr}(0\colon\omega,\sigma)$ in (5. 7) is not written explicitly in the original statement by Saywer and Wheeden [9] (Theorem 1). But, it can be computed directly by carrying out the proof given in section 2 of [9].*

By applying (5. 7) and using the estimate in (5. 5), we find

$$\begin{aligned}& \left\{\iint_{\mathbb{R}^{2n}}\left\{\iint_{\mathbb{R}^{2n}} g(2^{-j}\xi,\eta)\left[\frac{1}{\sqrt{|u-\xi|^2+|v-\eta|^2}}\right]^{2n-2\zeta} d\xi d\eta\right\}^{q}\omega^{q}(2^{-j}u,v)dudv\right\}^{\frac{1}{q}} \\ & \le\; \mathfrak{B}_{p\;q\;r\;\zeta}\,2^{j\left[\zeta-\frac{n}{p}+\frac{n}{q}\right]}\mathbf{A}^{\zeta}_{pqr}(j:\omega,\sigma)\left\{\iint_{\mathbb{R}^{2n}}\left(g\sigma\right)^{p}(2^{-j}u,v)dudv\right\}^{\frac{1}{p}}\end{aligned} \tag{5. 8}$$

for $1<p\le q<\infty$ and every $j \in \mathbb{Z}$.

Recall (5. 1)-(5. 2). By changing dilations $(u,v)\longrightarrow(2^{-j}u,v)$ and $(\xi,\eta)\longrightarrow(2^{-j}\xi,\eta)$, we have

$$
\begin{aligned}
&\left\{\iint_{\mathbb{R}^{2n}}\left(\Delta_j\mathbf{II}_\zeta g\right)^q(u,v)\omega^q(u,v)dudv\right\}^{\frac{1}{q}}\\
&=\ \left\{\iint_{\mathbb{R}^{2n}}\left\{\iint_{\Lambda_j(u,v)}g(\xi,\eta)\left(\frac{1}{|u-\xi|}\right)^{n-\zeta}\left(\frac{1}{|v-\eta|}\right)^{n-\zeta}d\xi d\eta\right\}^q\omega^q(u,v)dudv\right\}^{\frac{1}{q}}\\
&=\ \left\{\iint_{\mathbb{R}^{2n}}\left\{\iint_{\Lambda_0(u,v)}g(2^{-j}\xi,\eta)\left(\frac{1}{2^{-j}|u-\xi|}\right)^{n-\zeta}\left(\frac{1}{|v-\eta|}\right)^{n-\zeta}2^{-jn}d\xi d\eta\right\}^q\omega^q(2^{-j}u,v)2^{-jn}dudv\right\}^{\frac{1}{q}}\\
&\lesssim\ 2^{-j\left[\zeta+\frac{n}{q}\right]}\left\{\iint_{\mathbb{R}^{2n}}\left\{\iint_{\mathbb{R}^{2n}}g(2^{-j}\xi,\eta)\left[\frac{1}{\sqrt{|u-\xi|^2+|v-\eta|^2}}\right]^{2n-2\zeta}d\xi d\eta\right\}^q\omega^q(2^{-j}u,v)dudv\right\}^{\frac{1}{q}}\\
&\le\ \mathfrak{B}_{p\ q\ r\ \zeta}\,2^{-j\left[\zeta+\frac{n}{q}\right]}2^{j\left[\zeta-\frac{n}{p}+\frac{n}{q}\right]}\mathbf{A}^{\zeta}_{pqr}\,(j:\omega,\sigma)\left\{\iint_{\mathbb{R}^{2n}}\left(g\sigma\right)^p(2^{-j}u,v)dudv\right\}^{\frac{1}{p}}\qquad\text{by (5. 8)}\\
&=\ \mathfrak{B}_{p\ q\ r\ \zeta}\,\mathbf{A}^{\zeta}_{pqr}(j:\omega,\sigma)\,2^{-j\left[\zeta+\frac{n}{q}\right]}2^{j\left[\zeta-\frac{n}{p}+\frac{n}{q}\right]}\left\{\iint_{\mathbb{R}^{2n}}\left(g\sigma\right)^p(u,v)\,2^{jn}dudv\right\}^{\frac{1}{p}}\\
&=\ \mathfrak{B}_{p\ q\ r\ \zeta}\,\mathbf{A}^{\zeta}_{pqr}\,(j:\omega,\sigma)\left\{\iint_{\mathbb{R}^{2n}}\left(g\sigma\right)^p(u,v)dxudv\right\}^{\frac{1}{p}}.
\end{aligned}
\tag{5. 9}
$$

By using (5. 9) and Minkowski inequality, we obtain the $\mathbf{L}^p\longrightarrow\mathbf{L}^q$-norm inequality in (1. 17) provided that

$$\sum_{j\in\mathbb{Z}}\mathbf{A}^{\zeta}_{pqr}(j:\omega,\sigma)\ <\ \infty.$$

Principal Lemma *Suppose* $\omega(u,v)=\sqrt{|u|^2+|v|^2}^{-\gamma}$ *and* $\sigma(u,v)=\sqrt{|u|^2+|v|^2}^{\delta}$ *for* $(u,v)\neq(0,0)$. *Let* $\gamma+\delta>0$. *There exists* $\varepsilon=\varepsilon(p,q,\gamma,\delta)>0$ *such that*

$$\mathbf{A}^{\zeta}_{pqr}(j:\omega,\sigma)\ <\ \mathfrak{B}_{p\ q\ \gamma\ \delta}\,2^{-\varepsilon|j|}$$

$$
\begin{gathered}
\textit{if}\qquad\gamma<\frac{2n}{q},\qquad\delta<2n\left(\frac{p-1}{p}\right),\qquad\frac{\zeta}{n}\ =\ \frac{1}{p}-\frac{1}{q}+\frac{\gamma+\delta}{2n};\\
\zeta-\frac{n}{p}<\delta\qquad\textit{for}\qquad\gamma\ge0,\ \delta\le0;\\
\zeta-n\left(\frac{q-1}{q}\right)<\gamma\qquad\textit{for}\qquad\gamma\le0,\ \delta\ge0
\end{gathered}
\tag{5. 10}
$$

for some $r=r(p,q,\gamma,\delta)>1$ *and every* $j\in\mathbb{Z}$.

By symmetry, we consider $j > 0$ only. For every $\mathbf{Q}_1 \times \mathbf{Q}_2 \subset \mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}} / \mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}} \; = \; \lambda, \qquad 0 < \lambda \leq 1, \tag{5. 11}$$

we aim to show that the constraints of p, q, γ, δ inside (5. 10) imply

$$\begin{array}{r}\displaystyle\prod_{i=1}^{2} \mathbf{vol}\{\mathbf{Q}_i\}^{\frac{\zeta}{n}-\frac{1}{p}+\frac{1}{q}} \left\{ \frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}} \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \left(\frac{1}{|u|+|v|}\right)^{\gamma q r} dudv \right\}^{\frac{1}{qr}} \\ \displaystyle\left\{ \frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}} \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \left(\frac{1}{|u|+|v|}\right)^{\delta\frac{pr}{p-1}} dudv \right\}^{\frac{p-1}{pr}} \quad \leq \; \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\varepsilon}\end{array} \tag{5. 12}$$

where $\varepsilon > 0$ and $r > 1$ depend on p, q, γ, δ.

By using the homogeneity condition $\frac{\zeta}{n} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{2n}$, we find that the left-hand-side of (5. 12) is invariant by changing dilations in one-parameter. Therefore, it is suffice to assert $\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}} = 1$.

Remark 5.2. *Let $\mathbf{Q}_i^o$ and $\mathbf{Q}_i^* \subset \mathbb{R}^n$ be cubes centered on the origin of $\mathbb{R}^n$ and*

$$\mathbf{vol}\{\mathbf{Q}_i^o\}^{\frac{1}{n}} \; = \; \mathbf{vol}\{\mathbf{Q}_i\}^{\frac{1}{n}}, \qquad \mathbf{vol}\{\mathbf{Q}_i^*\}^{\frac{1}{n}} \; = \; 3\mathbf{vol}\{\mathbf{Q}_i\}^{\frac{1}{n}}, \qquad i \; = \; 1, 2. \tag{5. 13}$$

Suppose $\mathbf{Q}_i \cap \mathbf{Q}_i^o = \emptyset$. We must have $|x| \geq |x^o|/\sqrt{n}$ for every $x \in \mathbf{Q}_i$ and $x^o \in \mathbf{Q}_i^o$.

Otherwise, $\mathbf{Q}_i \subset \mathbf{Q}_i^*$ if $\mathbf{Q}_i$ intersects $\mathbf{Q}_i^o$.

Suppose $\mathbf{Q}_1 \times \mathbf{Q}_2$ centered on $(u_o, v_o) \in \mathbb{R}^n \times \mathbb{R}^n$ of which $\sqrt{|u_o|^2 + |v_o|^2} > 3$. Because $\mathbf{Q}_1 \times \mathbf{Q}_2$ has a diameter 1, we find

$$\frac{1}{2}\sqrt{|u_o|^2+|v_o|^2} \; \leq \; \sqrt{|u|^2+|v|^2} \; \leq \; 2\sqrt{|u_o|^2+|v_o|^2}, \qquad (u, v) \in \mathbf{Q}_1 \times \mathbf{Q}_2.$$

This further implies

$$\begin{array}{r}\displaystyle\prod_{i=1}^{2} \mathbf{vol}\{\mathbf{Q}_i\}^{\frac{\zeta}{n}-\frac{1}{p}+\frac{1}{q}} \left\{ \frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}} \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \left(\frac{1}{|u|+|v|}\right)^{\gamma q r} dudv \right\}^{\frac{1}{qr}} \\ \displaystyle\left\{ \frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}} \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \left(\frac{1}{|u|+|v|}\right)^{\delta\frac{pr}{p-1}} dudv \right\}^{\frac{p-1}{pr}}\end{array} \tag{5. 14}$$

$$\begin{aligned} &\leq \; \mathfrak{B}_{p\,q\,r\,\gamma\,\delta} \left[\sqrt{|u_o|^2+|v_o|^2} \right]^{-(\gamma+\delta)} \lambda^{\frac{\zeta}{n}-\frac{1}{p}+\frac{1}{q}} \\ &\leq \; \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\varepsilon}, \qquad \varepsilon = \tfrac{\zeta}{n} - \tfrac{1}{p} + \tfrac{1}{q} = \tfrac{\gamma+\delta}{2n} > 0. \end{aligned}$$

Remark 5.3. *From now on, we assume $\mathbf{Q}_1 \times \mathbf{Q}_2$ centered on some $(u_o, v_o) \in \mathbb{R}^n \times \mathbb{R}^n$ with $\sqrt{|u_o|^2 + |v_o|^2} \leq 3$.*

Let $\mathbf{Q}_1^*$ defined (5. 13). We have

$$\begin{aligned} &\int_{\mathbf{Q}_1} \left(\frac{1}{|u|}\right)^{\gamma q r} du \; \lesssim \; \int_{\mathbf{Q}_1^*} \left(\frac{1}{|u|}\right)^{\gamma q r} du, \qquad 0 < \gamma q r < n; \\ &\int_{\mathbf{Q}_1} \left(\frac{1}{|u|}\right)^{\gamma q r - n} du \; \lesssim \; \int_{\mathbf{Q}_1^*} \left(\frac{1}{|u|}\right)^{\gamma q r - n} du, \qquad n < \gamma q r < 2n \end{aligned} \tag{5. 15}$$

and

$$
\begin{aligned}
&\int_{\mathbf{Q}_1} \left(\frac{1}{|u|}\right)^{\delta \frac{pr}{p-1}} du \lesssim \int_{\mathbf{Q}_1^*} \left(\frac{1}{|u|}\right)^{\delta \frac{pr}{p-1}} du, \qquad 0 < \delta\left(\tfrac{p}{p-1}\right) r < n; \\
&\int_{\mathbf{Q}_1} \left(\frac{1}{|u|}\right)^{\delta \frac{pr}{p-1} - n} du \lesssim \int_{\mathbf{Q}_1^*} \left(\frac{1}{|u|}\right)^{\delta \frac{pr}{p-1} - n} du, \qquad n < \delta\left(\tfrac{p}{p-1}\right) r < 2n
\end{aligned} \tag{5. 16}
$$

The remaining proof is split into 3 cases, *w.r.t* $\gamma \geq 0, \delta \leq 0$; $\gamma \leq 0, \delta \geq 0$ and $\gamma > 0, \delta > 0$.

5.1 Case One: $\gamma \geq 0$, $\delta \leq 0$

By adjusting the value of $r > 1$, we find

$$
0 < \gamma q r < n \qquad \text{or} \qquad n < \gamma q r < 2n. \tag{5. 17}
$$

Suppose $0 < \gamma q r < n$. We have

$$
\begin{aligned}
&\prod_{i=1}^{2} \mathbf{vol}\{\mathbf{Q}_i\}^{\frac{\zeta}{n} - \frac{1}{p} + \frac{1}{q}} \left\{ \frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}} \iint_{\mathbf{Q}_1 \times \mathbf{Q}_2} \left(\frac{1}{|u| + |v|}\right)^{\gamma q r} du dv \right\}^{\frac{1}{qr}} \\
&\qquad \left\{ \frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}} \iint_{\mathbf{Q}_1 \times \mathbf{Q}_2} \left(\frac{1}{|u| + |v|}\right)^{\delta \frac{pr}{p-1}} du dv \right\}^{\frac{p-1}{pr}} \\
&\leq \mathfrak{B}_{p\,r\,\delta}\, \lambda^{\zeta - \frac{n}{p} + \frac{n}{q}} \left(\frac{1}{\lambda}\right)^{\frac{n}{qr}} \left\{ \int_{\mathbf{Q}_1} \left\{ \int_{\mathbf{Q}_2} \left(\frac{1}{|u| + |v|}\right)^{\gamma q r} dv \right\} du \right\}^{\frac{1}{qr}} \qquad \text{by } \textbf{Remark 5.3}\ (\delta \leq 0) \\
&\leq \mathfrak{B}_{p\,r\,\delta}\, \lambda^{\zeta - \frac{n}{p} + \frac{n}{q}} \left(\frac{1}{\lambda}\right)^{\frac{n}{qr}} \lambda^{\frac{n}{qr}} \left\{ \int_{\mathbf{Q}_2} \left(\frac{1}{|v|}\right)^{\gamma q r} dv \right\}^{\frac{1}{qr}} \\
&\leq \mathfrak{B}_{p\,r\,\delta}\, \lambda^{\zeta - \frac{n}{p} + \frac{n}{q}} \left(\frac{1}{\lambda}\right)^{\frac{n}{qr}} \lambda^{\frac{n}{qr}} \left\{ \int_{\mathbf{Q}_2^*} \left(\frac{1}{|v|}\right)^{\gamma q r} dv \right\}^{\frac{1}{qr}} \qquad \text{by (5. 15)} \\
&\leq \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\zeta - \frac{n}{p} + \frac{n}{q}} \left(\frac{1}{\lambda}\right)^{\frac{n}{qr}} \lambda^{\frac{n}{qr}} \\
&= \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\zeta - \frac{n}{p} + \frac{n}{q}} \\
&= \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\frac{\gamma+\delta}{2}} \qquad \left(\tfrac{\zeta}{n} = \tfrac{1}{p} - \tfrac{1}{q} + \tfrac{\gamma+\delta}{2n} \right) \\
&= \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\varepsilon}, \qquad \varepsilon = \tfrac{\gamma+\delta}{2} > 0.
\end{aligned} \tag{5. 18}
$$

Suppose $n < \gamma qr < 2n$. Recall $\zeta - \frac{n}{p} < \delta$ as an necessity. Together with the homogeneity condition $\frac{\zeta}{n} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{2n}$, we find

$$\begin{aligned} \zeta - \frac{n}{p} &= -\frac{n}{q} + \frac{\gamma+\delta}{2} < \delta \\ \Longrightarrow \qquad & \frac{n}{q} - \frac{\gamma}{2} + \frac{\delta}{2} > 0. \end{aligned} \tag{5. 19}$$

For r chosen sufficiently close to 1, we have

$$\begin{aligned}
&\prod_{i=1}^{2} \mathbf{vol}\{\mathbf{Q}_i\}^{\frac{\zeta}{n}-\frac{1}{p}+\frac{1}{q}} \left\{ \frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}} \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \left(\frac{1}{|u|+|v|}\right)^{\gamma qr} dudv \right\}^{\frac{1}{qr}} \\
&\qquad \left\{ \frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}} \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \left(\frac{1}{|u|+|v|}\right)^{\delta\frac{pr}{p-1}} dudv \right\}^{\frac{p-1}{pr}} \\
\le\ & \mathfrak{B}_{p\ r\ \delta}\ \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}} \left(\frac{1}{\lambda}\right)^{\frac{n}{qr}} \left\{ \int_{\mathbf{Q}_1} \left\{ \int_{\mathbf{Q}_2} \left(\frac{1}{|u|+|v|}\right)^{\gamma qr} dv \right\} du \right\}^{\frac{1}{qr}} \qquad \text{by } \textbf{Remark 5.3} \ (\delta \le 0) \\
\le\ & \mathfrak{B}_{p\ r\ \delta}\ \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}} \left(\frac{1}{\lambda}\right)^{\frac{n}{qr}} \left\{ \int_{\mathbf{Q}_1} \left\{ \int_{\mathbb{R}^n} \left(\frac{1}{|u|+|v|}\right)^{\gamma qr} dv \right\} du \right\}^{\frac{1}{qr}} \\
\approx\ & \mathfrak{B}_{p\ r\ \delta}\ \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}} \left(\frac{1}{\lambda}\right)^{\frac{n}{qr}} \\
&\left\{ \int_{\mathbf{Q}_1} \left\{ \int\cdots\int_{\mathbb{R}^n} \left(\frac{1}{|u|+|v_1|+\cdots+|v_n|}\right)^{\gamma qr} dv_1\cdots dv_n \right\} du \right\}^{\frac{1}{qr}} \qquad (5.\ 20) \\
\le\ & \mathfrak{B}_{p\ q\ r\ \gamma\ \delta}\ \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}} \left(\frac{1}{\lambda}\right)^{\frac{n}{qr}} \left\{ \int_{\mathbf{Q}_1} \left(\frac{1}{|u|}\right)^{\gamma qr-n} du \right\}^{\frac{1}{qr}} \\
\le\ & \mathfrak{B}_{p\ q\ r\ \gamma\ \delta}\ \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}} \left(\frac{1}{\lambda}\right)^{\frac{n}{qr}} \left\{ \int_{\mathbf{Q}_1^*} \left(\frac{1}{|u|}\right)^{\gamma qr-n} du \right\}^{\frac{1}{qr}} \qquad \text{by (5. 15)} \\
\le\ & \mathfrak{B}_{p\ q\ r\ \gamma\ \delta}\ \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}} \left(\frac{1}{\lambda}\right)^{\frac{n}{qr}} \lambda^{\frac{2n}{qr}-\gamma} \\
=\ & \mathfrak{B}_{p\ q\ r\ \gamma\ \delta}\ \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}} \lambda^{\frac{n}{qr}-\gamma} = \mathfrak{B}_{p\ q\ r\ \gamma\ \delta}\ \lambda^{\frac{\gamma+\delta}{2}} \lambda^{\frac{n}{qr}-\gamma} \qquad \left(\frac{\zeta}{n} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{2n} \right) \\
=\ & \mathfrak{B}_{p\ q\ r\ \gamma\ \delta}\ \lambda^{\frac{n}{qr}-\frac{\gamma}{2}+\frac{\delta}{2}} \\
=\ & \mathfrak{B}_{p\ q\ r\ \gamma\ \delta}\ \lambda^{\varepsilon}, \qquad \varepsilon = \frac{n}{qr} - \frac{\gamma}{2} + \frac{\delta}{2} > 0 \ \text{ by (5. 19).}
\end{aligned}$$

5.2 Case Two: $\gamma \le 0$, $\delta \ge 0$

By adjusting the value of $r > 1$, we find

$$0 < \delta\left(\tfrac{p}{p-1}\right) r < n \qquad \text{or} \qquad n < \delta\left(\tfrac{p}{p-1}\right) r < 2n. \tag{5. 21}$$

Suppose $0 < \delta\left(\frac{p}{p-1}\right) r < n$. We have

$$\begin{aligned}
&\prod_{i=1}^{2} \mathbf{vol}\{\mathbf{Q}_i\}^{\frac{\zeta}{n}-\frac{1}{p}+\frac{1}{q}} \left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}} \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \left(\frac{1}{|u|+|v|}\right)^{\gamma q r} dudv\right\}^{\frac{1}{qr}} \\
&\qquad\qquad \left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}} \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \left(\frac{1}{|u|+|v|}\right)^{\delta\frac{pr}{p-1}} dudv\right\}^{\frac{p-1}{pr}} \\
&\le\ \mathfrak{B}_{q\,r\,\gamma}\ \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}} \left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)} \left\{\int_{\mathbf{Q}_1} \left\{\int_{\mathbf{Q}_2} \left(\frac{1}{|u|+|v|}\right)^{\delta\frac{pr}{p-1}} dv\right\} du\right\}^{\frac{p-1}{pr}} \qquad \text{by } \textbf{Remark 5.3}\ (\gamma \le 0) \\
&\le\ \mathfrak{B}_{q\,r\,\gamma}\ \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}} \left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)} \lambda^{n\left(\frac{p-1}{pr}\right)} \left\{\int_{\mathbf{Q}_2} \left(\frac{1}{|v|}\right)^{\delta\frac{pr}{p-1}} dv\right\}^{\frac{p-1}{pr}} \\
&\le\ \mathfrak{B}_{q\,r\,\gamma}\ \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}} \left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)} \lambda^{n\left(\frac{p-1}{pr}\right)} \left\{\int_{\mathbf{Q}_2^*} \left(\frac{1}{|v|}\right)^{\delta\frac{pr}{p-1}} dv\right\}^{\frac{p-1}{pr}} \qquad \text{by (5. 16)} \\
&\le\ \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}} \left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)} \lambda^{n\left(\frac{p-1}{pr}\right)} \\
&=\ \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}} \\
&=\ \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\frac{\gamma+\delta}{2}} \qquad \left(\ \tfrac{\zeta}{n} = \tfrac{1}{p} - \tfrac{1}{q} + \tfrac{\gamma+\delta}{2n}\ \right) \\
&=\ \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\varepsilon}, \qquad \varepsilon = \tfrac{\gamma+\delta}{2} > 0.
\end{aligned} \tag{5. 22}$$

Suppose $n < \delta\left(\frac{p}{p-1}\right) r < 2n$. Recall $\zeta - n\left(\frac{q-1}{q}\right) < \gamma$ as an necessity. Together with the homogeneity condition $\frac{\zeta}{n} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{2n} = \frac{q-1}{q} - \frac{p-1}{p} + \frac{\gamma+\delta}{2n}$, we find

$$\begin{aligned}
\zeta - n\left(\frac{q-1}{q}\right) \ &= \ -n\left(\frac{p-1}{p}\right) + \frac{\gamma+\delta}{2}\ <\ \gamma \\
\Longrightarrow \qquad & n\left(\frac{p-1}{p}\right) + \frac{\gamma}{2} - \frac{\delta}{2}\ >\ 0.
\end{aligned} \tag{5. 23}$$

For r chosen sufficiently close to 1, we have

$$\begin{array}{l}
\displaystyle\prod_{i=1}^{2}\mathbf{vol}\{\mathbf{Q}_i\}^{\frac{\zeta}{n}-\frac{1}{p}+\frac{1}{q}}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left(\frac{1}{|u|+|v|}\right)^{\gamma qr}dudv\right\}^{\frac{1}{qr}}\\
\qquad\qquad\displaystyle\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left(\frac{1}{|u|+|v|}\right)^{\delta\frac{pr}{p-1}}dudv\right\}^{\frac{p-1}{pr}}\\
\displaystyle\le\ \mathfrak{B}_{q\ r\ \gamma}\ \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}}\left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)}\left\{\int_{\mathbf{Q}_1}\left\{\int_{\mathbf{Q}_2}\left(\frac{1}{|u|+|v|}\right)^{\delta\frac{pr}{p-1}}dv\right\}du\right\}^{\frac{p-1}{pr}}\qquad\text{by }\textbf{Remark 5.3}\ (\gamma\le 0)\\
\displaystyle\le\ \mathfrak{B}_{q\ r\ \gamma}\ \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}}\left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)}\left\{\int_{\mathbf{Q}_1}\left\{\int_{\mathbb{R}^n}\left(\frac{1}{|u|+|v|}\right)^{\delta\frac{pr}{p-1}}dv\right\}du\right\}^{\frac{p-1}{pr}}\\
\displaystyle\approx\ \mathfrak{B}_{q\ r\ \gamma}\ \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}}\left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)}\\
\qquad\displaystyle\left\{\int_{\mathbf{Q}_1}\left\{\int\cdots\int_{\mathbb{R}^n}\left(\frac{1}{|u|+|v_1|+\cdots+|v_n|}\right)^{\delta\frac{pr}{p-1}}dv_1\cdots dv_n\right\}du\right\}^{\frac{p-1}{pr}}\\
\displaystyle\le\ \mathfrak{B}_{p\ q\ r\ \gamma\ \delta}\ \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}}\left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)}\left\{\int_{\mathbf{Q}_1}\left(\frac{1}{|u|}\right)^{\delta\frac{pr}{p-1}-n}du\right\}^{\frac{p-1}{pr}}\\
\displaystyle\le\ \mathfrak{B}_{p\ q\ r\ \gamma\ \delta}\ \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}}\left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)}\left\{\int_{\mathbf{Q}_1^*}\left(\frac{1}{|u|}\right)^{\delta\frac{pr}{p-1}-n}du\right\}^{\frac{p-1}{pr}}\qquad\text{by (5. 16)}\\
\displaystyle\le\ \mathfrak{B}_{p\ q\ r\ \gamma\ \delta}\ \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}}\left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)}\lambda^{2n\left(\frac{p-1}{pr}\right)-\delta}\\
=\ \mathfrak{B}_{p\ q\ r\ \gamma\ \delta}\ \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}}\lambda^{n\left(\frac{p-1}{pr}\right)-\delta}\\
=\ \mathfrak{B}_{p\ q\ r\ \gamma\ \delta}\ \lambda^{\frac{\gamma+\delta}{2}}\lambda^{n\left(\frac{p-1}{pr}\right)-\delta}\qquad\left(\ \frac{\zeta}{n}=\frac{1}{p}-\frac{1}{q}+\frac{\gamma+\delta}{2n}\ \right)\\
=\ \mathfrak{B}_{p\ q\ r\ \gamma\ \delta}\ \lambda^{n\left(\frac{p-1}{pr}\right)+\frac{\gamma}{2}-\frac{\delta}{2}}\\
=\ \mathfrak{B}_{p\ q\ r\ \gamma\ \delta}\ \lambda^{\varepsilon},\qquad\varepsilon=n\left(\frac{p-1}{pr}\right)+\frac{\gamma}{2}-\frac{\delta}{2}>0\ \ \text{by (5. 23).}
\end{array}\tag{5. 24}$$

5.3 Case Three: $\gamma > 0$, $\delta > 0$

By adjusting the value of $r > 1$, we find

$$
\begin{gathered}
0 < \gamma q r < n, \qquad 0 < \delta\left(\frac{p}{p-1}\right) r < n;\\
n < \gamma q r < 2n, \qquad 0 < \delta\left(\frac{p}{p-1}\right) r < n \qquad \text{or} \qquad 0 < \gamma q r < n, \qquad n < \delta\left(\frac{p}{p-1}\right) r < 2n; \qquad (5.\ 25)\\
n < \gamma q r < 2n, \qquad n < \delta\left(\frac{p}{p-1}\right) r < 2n.
\end{gathered}
$$

Suppose $0 < \gamma q r < n$ and $0 < \delta\left(\frac{p}{p-1}\right) r < n$. We have

$$
\begin{aligned}
&\prod_{i=1}^{2} \mathbf{vol}\{\mathbf{Q}_i\}^{\frac{\zeta}{n}-\frac{1}{p}+\frac{1}{q}} \left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}} \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \left(\frac{1}{|u|+|v|}\right)^{\gamma q r} dudv\right\}^{\frac{1}{qr}}\\
&\qquad \left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}} \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \left(\frac{1}{|u|+|v|}\right)^{\delta\frac{pr}{p-1}} dudv\right\}^{\frac{p-1}{pr}}\\
&= \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}} \left(\frac{1}{\lambda}\right)^{\frac{n}{qr}} \left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)}\\
&\qquad \left\{\int_{\mathbf{Q}_1} \left\{\int_{\mathbf{Q}_2} \left(\frac{1}{|u|+|v|}\right)^{\gamma q r} dv\right\} du\right\}^{\frac{1}{qr}} \left\{\int_{\mathbf{Q}_1} \left\{\int_{\mathbf{Q}_2} \left(\frac{1}{|u|+|v|}\right)^{\delta\frac{pr}{p-1}} dv\right\} du\right\}^{\frac{p-1}{pr}}\\
&\le \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}} \left(\frac{1}{\lambda}\right)^{\frac{n}{qr}} \left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)} \lambda^{\frac{n}{qr}} \lambda^{n\left(\frac{p-1}{pr}\right)} \left\{\int_{\mathbf{Q}_2} \left(\frac{1}{|v|}\right)^{\gamma q r} dv\right\}^{\frac{1}{qr}} \left\{\int_{\mathbf{Q}_2} \left(\frac{1}{|v|}\right)^{\delta\frac{pr}{p-1}} dv\right\}^{\frac{p-1}{pr}} \qquad (5.\ 26)\\
&\le \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}} \left(\frac{1}{\lambda}\right)^{\frac{n}{qr}} \left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)} \lambda^{\frac{n}{qr}} \lambda^{n\left(\frac{p-1}{pr}\right)} \left\{\int_{\mathbf{Q}_2^*} \left(\frac{1}{|v|}\right)^{\gamma q r} dv\right\}^{\frac{1}{qr}} \left\{\int_{\mathbf{Q}_2^*} \left(\frac{1}{|v|}\right)^{\delta\frac{pr}{p-1}} dv\right\}^{\frac{p-1}{pr}}\\
&\qquad\qquad \text{by } (5.\ 15)\text{-}(5.\ 16)\\
&\le \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}} \left(\frac{1}{\lambda}\right)^{\frac{n}{qr}} \left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)} \lambda^{\frac{n}{qr}} \lambda^{n\left(\frac{p-1}{pr}\right)}\\
&= \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}}\\
&= \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\frac{\gamma+\delta}{2}} \qquad \left(\ \frac{\zeta}{n} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{2n}\ \right)\\
&= \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\varepsilon}, \qquad \varepsilon = \frac{\gamma+\delta}{2} > 0.
\end{aligned}
$$

Suppose $n < \gamma qr < 2n$ and $0 < \delta\left(\frac{p}{p-1}\right)r < n$. We have

$$\prod_{i=1}^{2}\mathbf{vol}\{\mathbf{Q}_i\}^{\frac{\zeta}{n}-\frac{1}{p}+\frac{1}{q}}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left(\frac{1}{|u|+|v|}\right)^{\gamma qr}dudv\right\}^{\frac{1}{qr}}$$
$$\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left(\frac{1}{|u|+|v|}\right)^{\delta\frac{pr}{p-1}}dudv\right\}^{\frac{p-1}{pr}}$$

$$= \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}}\left(\frac{1}{\lambda}\right)^{\frac{n}{qr}}\left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)}$$
$$\left\{\int_{\mathbf{Q}_1}\left\{\int_{\mathbf{Q}_2}\left(\frac{1}{|u|+|v|}\right)^{\gamma qr}dv\right\}du\right\}^{\frac{1}{qr}}\left\{\int_{\mathbf{Q}_1}\left\{\int_{\mathbf{Q}_2}\left(\frac{1}{|u|+|v|}\right)^{\delta\frac{pr}{p-1}}dv\right\}du\right\}^{\frac{p-1}{pr}}$$

$$\leq \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}}\left(\frac{1}{\lambda}\right)^{\frac{n}{qr}}\left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)}\left\{\int_{\mathbf{Q}_1}\left\{\int_{\mathbb{R}^n}\left(\frac{1}{|u|+|v|}\right)^{\gamma qr}dv\right\}du\right\}^{\frac{1}{qr}}\left\{\int_{\mathbf{Q}_1}\left\{\int_{\mathbf{Q}_2}\left(\frac{1}{|v|}\right)^{\delta\frac{pr}{p-1}}dv\right\}du\right\}^{\frac{p-1}{pr}}$$

$$\approx \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}}\left(\frac{1}{\lambda}\right)^{\frac{n}{qr}}\left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)}\lambda^{n\left(\frac{p-1}{pr}\right)}$$
$$\left\{\int_{\mathbf{Q}_1}\left\{\int\cdots\int_{\mathbb{R}^n}\left(\frac{1}{|u|+|v_1|+\cdots+|v_n|}\right)^{\gamma qr}dv_1\cdots dv_n\right\}du\right\}^{\frac{1}{qr}}\left\{\int_{\mathbf{Q}_2}\left(\frac{1}{|v|}\right)^{\delta\frac{pr}{p-1}}dv\right\}^{\frac{p-1}{pr}}$$

$$\leq \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\,\lambda^{\zeta-\frac{n}{p}+\frac{n}{q}}\left(\frac{1}{\lambda}\right)^{\frac{n}{qr}}\left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)}\lambda^{n\left(\frac{p-1}{pr}\right)}\left\{\int_{\mathbf{Q}_1}\left(\frac{1}{|u|}\right)^{\gamma qr-n}du\right\}^{\frac{1}{qr}}\left\{\int_{\mathbf{Q}_2}\left(\frac{1}{|v|}\right)^{\delta\frac{pr}{p-1}}dv\right\}^{\frac{p-1}{pr}}$$

$$\leq \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\,\lambda^{\zeta-\frac{n}{p}+\frac{n}{q}}\left(\frac{1}{\lambda}\right)^{\frac{n}{qr}}\left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)}\lambda^{n\left(\frac{p-1}{pr}\right)}\left\{\int_{\mathbf{Q}_1^*}\left(\frac{1}{|u|}\right)^{\gamma qr-n}du\right\}^{\frac{1}{qr}}\left\{\int_{\mathbf{Q}_2^*}\left(\frac{1}{|v|}\right)^{\delta\frac{pr}{p-1}}dv\right\}^{\frac{p-1}{pr}}$$
by (5. 15)-(5. 16)

$$\leq \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\,\lambda^{\zeta-\frac{n}{p}+\frac{n}{q}}\left(\frac{1}{\lambda}\right)^{\frac{n}{qr}}\left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)}\lambda^{n\left(\frac{p-1}{pr}\right)}\lambda^{\frac{2n}{qr}-\gamma}$$

$$= \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\,\lambda^{\zeta-\frac{n}{p}+\frac{n}{q}}\lambda^{\frac{n}{qr}-\gamma}$$

$$= \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\,\lambda^{\frac{\gamma+\delta}{2}}\lambda^{\frac{n}{qr}-\gamma}\qquad\left(\frac{\zeta}{n}=\frac{1}{p}-\frac{1}{q}+\frac{\gamma+\delta}{2n}\right)$$

$$= \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\,\lambda^{\frac{n}{qr}-\frac{\gamma}{2}}\lambda^{\frac{\delta}{2}}$$

$$= \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\,\lambda^{\varepsilon},\qquad \varepsilon=\frac{n}{qr}-\frac{\gamma}{2}+\frac{\delta}{2}>0. \tag{5. 27}$$

For $0 < \gamma qr < n$ and $n < \delta\left(\frac{p}{p-1}\right)r < 2n$, an analogue estimate to (5. 27) shows the same result with $\varepsilon = \frac{\gamma}{2} + n\left(\frac{p-1}{pr}\right) - \frac{\delta}{2} > 0$.

Suppose $n < \gamma qr < 2n$ and $n < \delta\left(\frac{p}{p-1}\right) r < 2n$. We have

$$
\begin{array}{lr}
\displaystyle\prod_{i=1}^{2} \mathbf{vol}\{\mathbf{Q}_i\}^{\frac{\zeta}{n}-\frac{1}{p}+\frac{1}{q}} \left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}} \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \left(\frac{1}{|u|+|v|}\right)^{\gamma qr} dudv\right\}^{\frac{1}{qr}} & \\
\qquad\qquad\displaystyle\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}} \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \left(\frac{1}{|u|+|v|}\right)^{\delta\frac{pr}{p-1}} dudv\right\}^{\frac{p-1}{pr}} & \\
\displaystyle = \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}} \left(\frac{1}{\lambda}\right)^{\frac{n}{qr}} \left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)} & \\
\displaystyle\left\{\int_{\mathbf{Q}_1}\left\{\int_{\mathbf{Q}_2}\left(\frac{1}{|u|+|v|}\right)^{\gamma qr} dv\right\}du\right\}^{\frac{1}{qr}} \left\{\int_{\mathbf{Q}_1}\left\{\int_{\mathbf{Q}_2}\left(\frac{1}{|u|+|v|}\right)^{\delta\frac{pr}{p-1}} dv\right\}du\right\}^{\frac{p-1}{pr}} & \\
\displaystyle \leq \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}} \left(\frac{1}{\lambda}\right)^{\frac{n}{qr}} \left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)} & \\
\displaystyle\left\{\int_{\mathbf{Q}_1}\left\{\int_{\mathbb{R}^n}\left(\frac{1}{|u|+|v|}\right)^{\gamma qr} dv\right\}du\right\}^{\frac{1}{qr}} \left\{\int_{\mathbf{Q}_1}\left\{\int_{\mathbb{R}^n}\left(\frac{1}{|u|+|v|}\right)^{\delta\frac{pr}{p-1}} dv\right\}du\right\}^{\frac{p-1}{pr}} & \\
\displaystyle \approx \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}} \left(\frac{1}{\lambda}\right)^{\frac{n}{qr}} \left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)} & \\
\displaystyle\quad\left\{\int_{\mathbf{Q}_1}\left\{\int\cdots\int_{\mathbb{R}^n}\left(\frac{1}{|u|+|v_1|+\cdots+|v_n|}\right)^{\gamma qr} dv_1\cdots dv_n\right\}du\right\}^{\frac{1}{qr}} & \\
\displaystyle\quad\left\{\int_{\mathbf{Q}_1}\left\{\int\cdots\int_{\mathbb{R}^n}\left(\frac{1}{|u|+|v_1|+\cdots+|v_n|}\right)^{\delta\frac{pr}{p-1}} dv_1\cdots dv_n\right\}du\right\}^{\frac{p-1}{pr}} & (5.\ 28) \\
\displaystyle \leq \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}} \left(\frac{1}{\lambda}\right)^{\frac{n}{qr}} \left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)} \left\{\int_{\mathbf{Q}_1}\left(\frac{1}{|u|}\right)^{\gamma qr-n} du\right\}^{\frac{1}{qr}} \left\{\int_{\mathbf{Q}_1}\left(\frac{1}{|u|}\right)^{\delta\frac{pr}{p-1}-n} du\right\}^{\frac{p-1}{pr}} & \\
\displaystyle \leq \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}} \left(\frac{1}{\lambda}\right)^{\frac{n}{qr}} \left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)} \left\{\int_{\mathbf{Q}_1^*}\left(\frac{1}{|u|}\right)^{\gamma qr-n} du\right\}^{\frac{1}{qr}} \left\{\int_{\mathbf{Q}_1^*}\left(\frac{1}{|u|}\right)^{\delta\frac{pr}{p-1}-n} du\right\}^{\frac{p-1}{pr}} & \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{by (5. 15)-(5. 16)} & \\
\displaystyle \leq \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}} \left(\frac{1}{\lambda}\right)^{\frac{n}{qr}} \left(\frac{1}{\lambda}\right)^{n\left(\frac{p-1}{pr}\right)} \lambda^{\frac{2n}{qr}-\gamma} \lambda^{2n\left(\frac{p-1}{pr}\right)-\delta} & \\
= \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\zeta-\frac{n}{p}+\frac{n}{q}} \lambda^{\frac{n}{qr}-\gamma} \lambda^{n\left(\frac{p-1}{pr}\right)-\delta} & \\
= \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\frac{\gamma+\delta}{2}} \lambda^{\frac{n}{qr}-\gamma} \lambda^{n\left(\frac{p-1}{pr}\right)-\delta} \qquad \left(\ \frac{\zeta}{n} = \frac{1}{p}-\frac{1}{q}+\frac{\gamma+\delta}{2n}\ \right) & \\
= \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\frac{n}{qr}-\frac{\gamma}{2}} \lambda^{n\left(\frac{p-1}{pr}\right)-\frac{\delta}{2}} & \\
= \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\varepsilon}, \qquad \varepsilon = \frac{n}{qr}-\frac{\gamma}{2}+n\left(\frac{p-1}{pr}\right)-\frac{\delta}{2} > 0. &
\end{array}
$$

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