Triebel-Lizorkin spaces in Dunkl setting

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Abstract

We establish Triebel-Lizorkin spaces in the Dunkl setting which are associated with finite reflection groups on the Euclidean space. The group structures induce two nonequivalent metrics: the Euclidean metric and the Dunkl metric. In this paper, the $\mathbf{L}^2$ space and the Dunkl-Calderón-Zygmund singular integral operator in the Dunkl setting play a fundamental role. The main tools used in this paper are as follows: (i) the Dunkl-Calderón-Zygmund singular integral operator and a new Calderón reproducing formula in $\mathbf{L}^2$ with the Triebel-Lizorkin space norms; (ii) new test functions in terms of the $\mathbf{L}^2$ functions and distributions; (iii) the Triebel-Lizorkin spaces in the Dunkl setting which are defined by the wavelet-type decomposition and the analogous atomic decomposition of the Hardy spaces.

1 Introduction

It is well-known that the Triebel-Lizorkin spaces play an important role in modern harmonic analysis. In the 1970's, the Triebel-Lizorkin space $\dot{F}_p^{\alpha,q}$ on $\mathbb{R}^n$ was introduced by several mathematicians. Lizorkin [14]-[15] and Triebel [21] independently investigated the Triebel-Lizorkin space $\dot{F}_p^{\alpha,q}(\mathbb{R}^n)$, $\alpha \in \mathbb{R}, 1 < p < \infty, 1 < q < \infty$. Peetre [16]-[18] extended the range of the admissible parameters p and q to values less than one. In [8], applying the basic representation formula of the form $f = \sum_Q \left\langle f, \varphi_Q \right\rangle \psi_Q$ for a distribution f on $\mathbb{R}^n$, the Triebel-Lizorkin space was defined through the Littlewood-Paley theory. More precisely, let φ, ψ be functions on $\mathbb{R}^n$ satisfying:

$$\varphi, \psi \in \mathscr{S}(\mathbb{R}^n), \tag{1. 1}$$

$$\textbf{supp } \hat{\varphi}, \textbf{supp } \hat{\psi} \subseteq \left\{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\right\}, \tag{1. 2}$$

$$|\hat{\varphi}(\xi)|, |\hat{\psi}(\xi)| \geq c > 0 \quad \text{if} \quad \frac{3}{5} \leq |\xi| \leq \frac{5}{3}, \tag{1. 3}$$

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and

$$\sum_{k\in\mathbb{Z}} \overline{\hat{\varphi}(2^k\xi)}\hat{\psi}(2^k\xi) = 1 \quad \text{if} \quad \xi \neq 0. \tag{1. 4}$$

Set $\varphi_k(x) = 2^{kn}\varphi(2^k x)$ and $\psi_k(x) = 2^{kn}\psi(2^k x)$, $k \in \mathbb{Z}$. The Triebel-Lizorkin space $\dot{F}_p^{\alpha,q}(\mathbb{R}^n)$, $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, was defined by the collection of all $f \in \mathscr{S}'/\mathscr{P}(\mathbb{R}^n)$ (tempered distributions modulo polynomials) such that

$$\|f\|_{\dot{F}_p^{\alpha,q}} = \left\| \left(\sum_{k\in\mathbb{Z}} \left(2^{k\alpha} \left|\varphi_k * f\right|\right)^q \right)^{\frac{1}{q}} \right\|_{\mathbf{L}^p(\mathbb{R}^n)} < \infty, \tag{1. 5}$$

and when $q = \infty$, $\|f\|_{\dot{F}_p^{\alpha,\infty}} = \left\| \sup_{k\in\mathbb{Z}} 2^{k\alpha} \left|\varphi_k * f(x)\right| \right\|_{\mathbf{L}^p(\mathbb{R}^n)}$.

As a generalization of $\mathbb{R}^n$, the space of homogeneous type was introduced by Coifman and Weiss in [4], which provides a natural setting for studying function spaces. The homogeneous Triebel-Lizorkin spaces on spaces of homogeneous type were studied in [10]-[13]. More precisely, the Triebel-Lizorkin spaces on homogeneous type were introduced by using the family of operators $\{\mathbf{D}_k\}_{k\in\mathbb{Z}}$ where $\mathbf{D}_k = \mathbf{S}_k - \mathbf{S}_{k-1}$ and $\{\mathbf{S}_k\}_{k\in\mathbb{Z}}$ is an approximation to the identity. For $|\alpha| < \varepsilon$, $1 < p, q < \infty$, the Triebel-Lizorkin space $\dot{F}_p^{\alpha,q}$ is the collection of all $f \in \left(\mathcal{M}^{(\beta,\gamma)}\right)'$ with $0 < \beta, \gamma < \varepsilon$ such that

$$\|f\|_{\dot{F}_p^{\alpha,q}} = \left\| \left\{ \sum_{k\in\mathbb{Z}} \left(2^{k\alpha} \left|\mathbf{D}_k(f)\right|\right)^q \right\}^{\frac{1}{q}} \right\|_{\mathbf{L}^p(\mathbb{R}^n)} < \infty \tag{1. 6}$$

where $\mathcal{M}^{(\beta,\gamma)}$ is the space of test functions. See [10]-[13] for more details.

In recent years, the Dunkl setting is more and more important. In [11], the authors introduced the Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}$ in the Dunkl setting for $1 < p, q < \infty$. In this paper, we extend the range of p, q to all $0 < p, q < \infty$. The key tool is the discrete Calderón reproducing formula derived from the Dunkl-Poisson kernel (see [19]). We mention that the Dunkl-Poisson kernel involves two nonequivalent metrics: the Euclidean metric and the Dunkl metric.

Now we recall the framework of the Dunkl setting, see [6], [7], and [20]. In $\mathbb{R}^n$, the reflection σ_α with respect to the hyperplane $\alpha_\perp$ orthogonal to a nonzero vector α is given by

$$\sigma_\alpha(x) = x - 2\frac{\langle x, \alpha\rangle}{\|\alpha\|^2}\alpha.$$

A finite set $R \subset \mathbb{R}^n \setminus \{0\}$ is called a root system if $\sigma_\alpha(R) = R$ for every $\alpha \in R$. Let R be a root system in $\mathbb{R}^n$ normalized so that $\langle\alpha, \alpha\rangle = 2$ for $\alpha \in R$ and G the finite reflection group generated by the reflections $\sigma_\alpha(\alpha \in R)$, where $\sigma_\alpha(x) = x - \langle\alpha, x\rangle\,\alpha$ for $x \in \mathbb{R}^n$. Corresponding to this reflection group, we denote by $\mathcal{O}(x)$ the G-orbit of a point $x \in \mathbb{R}^n$. There is a natural metric between two G-orbits $\mathcal{O}(x)$ and $\mathcal{O}(y)$, given by

$$d(x, y) := \min_{\sigma\in G} \|x - \sigma(y)\|. \tag{1. 7}$$

Obviously, $d(x, y) \leq \|x - y\|$, $d(x, y) = d(y, x)$ and $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in \mathbb{R}^n$.

A multiplicity function κ defined on R (invariant under G) is fixed throughout this paper. Let

$$d\omega(x) = \prod_{\alpha \in R} |\langle \alpha, x\rangle|^{\kappa(\alpha)}\, dx \tag{1. 8}$$

be the associated measure in $\mathbb{R}^n$, see [1], where, here and subsequently, dx stands for the Lebesgue measure in $\mathbb{R}^n$. We denote by $N = n + \sum_{\alpha \in R} \kappa(\alpha)$ the homogeneous dimension of the system, here N is also called the upper dimension. Observe that for $x \in \mathbb{R}^n$ and $r > 0$,

$$\omega(B(x,r)) \sim r^n \prod_{\alpha \in R} (|\langle \alpha, x\rangle| + r)^{\kappa(\alpha)}, \tag{1. 9}$$

hence $\inf_{x \in \mathbb{R}^n} \omega(B(x,1)) \geq C > 0$, and $\omega(B(x,r)) \geq Cr^N$. According to (1. 7), we also have $\omega(B(x,r)) \sim \omega(B(y,r))$ when $d(x,y) \sim r$, and $\omega(B(x,r)) \leq \omega(B_d(x,r)) \leq |G|\omega(B(x,r))$, where $B_d(x,r) := \{y \in \mathbb{R}^n : d(x,y) < r\}$.

Moreover, the measure $d\omega(x)$ satisfies

$$C^{-1}\left(\frac{r_2}{r_1}\right)^n \leq \frac{\omega(B(x,r_2))}{\omega(B(x,r_1))} \leq C\left(\frac{r_2}{r_1}\right)^N \qquad \text{for } 0 < r_1 < r_2. \tag{1. 10}$$

This implies that $d\omega(x)$ satisfies the doubling and reverse doubling properties, that is, there exists a constant $C > 0$ such that for all $x \in \mathbb{R}^n$, $r > 0$ and $\lambda \geq 1$,

$$C^{-1}\lambda^n \omega(B(x,r)) \leq \omega(B(x,\lambda r)) \leq C\lambda^N \omega(B(x,r)). \tag{1. 11}$$

Next we consider the Dunkl (differential) operators $\mathbf{T}_j$ defined by

$$\mathbf{T}_j f(x) = \partial_j f(x) + \sum_{\alpha \in R} \frac{\kappa(\alpha)}{2} \left\langle \alpha, e_j \right\rangle \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle}, \tag{1. 12}$$

where e_1, e_2, …, e_n are standard unit vectors of $\mathbb{R}^n$. The Dunkl Laplacian is then defined as $\Delta_D = \sum_{j=1}^n \mathbf{T}_j^2$, which is equivalent to

$$\Delta_D f(x) = \Delta_{\mathbb{R}^n} f(x) + \sum_{\alpha \in R} \kappa(\alpha) \left(\frac{\partial_\alpha f(x)}{\langle \alpha, x\rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle^2} \right). \tag{1. 13}$$

Here $\Delta_{\mathbb{R}^n}$ is the standard Euclidean Laplacian.

The operator Δ_D is self-adjoint on $\mathbf{L}^2(\mathbb{R}^n, \omega)$, see [1], and generates the heat semigroup

$$\mathbf{H}_t f(x) = e^{t\Delta_D} f(x) = \int_{\mathbb{R}^n} H_t(x,y) f(y) d\omega(y), \tag{1. 14}$$

where the heat kernel $H_t(x,y)$ is a C^∞ function for all $t > 0$, $x, y \in \mathbb{R}^n$ and satisfies $H_t(x,y) = H_t(y,x) > 0$ and $\int_{\mathbb{R}^n} H_t(x,y) d\omega(y) = 1$.

The Poisson semigroup $\mathbf{P}_t = e^{-t\sqrt{-\Delta_D}}$ is subordinated to the heat semigroup $\mathbf{H}_t = e^{t\Delta_D}$ by

$$\mathbf{P}_t f(x) = \pi^{-1/2} \int_0^\infty e^{-u} e^{\frac{t^2}{4u}\Delta_D} f(x) \frac{du}{\sqrt{u}} \tag{1. 15}$$

and correspondingly for their integral kernels

$$P_t(x,y) = \pi^{-1/2}\int_0^{2\pi} e^{-u} H_{\frac{t^2}{4u}}(x,y)\frac{du}{u}. \tag{1. 16}$$

Moreover, $u(x,t) = \mathbf{P}_t f(x)$, so-called the Dunkl Poisson integral, solves the boundary value problem

$$\begin{cases} (\partial_t^2 + \Delta_D)u(x,t) = 0 \\ u(x,0) = f(x) \end{cases} \tag{1. 17}$$

in the half-space $\mathbb{R}_+^{n+1}$, see [1].

Observe that $\{\mathbf{P}_{2^{-k}}\}_{k\in\mathbb{Z}}$ is an approximation to the identity on $\mathbf{L}^2(\mathbb{R}^n,\omega)$, that is, for $f\in\mathbf{L}^2(\mathbb{R}^n,\omega)$

$$\lim_{k\to+\infty}\mathbf{P}_{2^{-k}}(f)(x) = f(x), \quad \lim_{k\to-\infty}\mathbf{P}_{2^{-k}}(f)(x) = 0. \tag{1. 18}$$

$\diamond$ Throughout, we denote the operator $\mathbf{T}$ and its kernel $T(x,y)$ by the same letter with bold type text and plain text respectively, such that

$$\mathbf{T}(f)(x) = \int_{\mathbb{R}^n} T(x,y)f(y)d\omega(y).$$

Now we consider the Triebel-Lizorkin spaces in the Dunkl setting. In [2], by using the Dunkl Laplacian Δ_D, the author proved that the Triebel-Lizorkin spaces associated with the Dunkl Laplacian are identical to the Triebel-Lizorkin spaces defined in the space of homogeneous type $(\mathbb{R}^n, \|\cdot\|, \omega)$.

In this paper, we establish the Triebel-Lizorkin spaces in the Dunkl setting in a different way. The critical difference is we use the Calderón reproducing formula derived from two different kernels. Our the new method is using a new Calderón reproducing formula in $\mathbf{L}^2$ (see [19]) with the Triebel-Lizorkin space norms derived from the Dunkl-Poisson kernel. Set

$$\mathbf{D}_k = \mathbf{P}_{2^{-k}} - \mathbf{P}_{2^{-k-1}}$$

with the kernel $D_k(x,y) = P_{2^{-k}}(x,y) - P_{2^{-k-1}}(x,y)$. Then applying the Coifman's decomposition (see [5]) of the identity on $\mathbf{L}^2(\mathbb{R}^n,\omega)$, we have

$$\mathbf{I} = \left(\sum_{\ell\in\mathbb{Z}}\mathbf{D}_\ell\right)\left(\sum_{k\in\mathbb{Z}}\mathbf{D}_k\right) = \sum_{|k-\ell|\le M}\mathbf{D}_\ell\mathbf{D}_k + \sum_{|k-\ell|>M}\mathbf{D}_\ell\mathbf{D}_k = \sum_{k\in\mathbb{Z}}\mathbf{D}_k^M\mathbf{D}_k + \mathbf{R}_1 \tag{1. 19}$$

where M is a fixed constant, and

$$\mathbf{D}_k^M = \sum_{\ell:|k-\ell|\le M}\mathbf{D}_\ell, \qquad \mathbf{R}_1 = \sum_{|k-\ell|>M}\mathbf{D}_\ell\mathbf{D}_k. \tag{1. 20}$$

Therefore, for $f\in\mathbf{L}^2(\mathbb{R}^n,\omega)$ we have

$$f(x) = \sum_{k\in\mathbb{Z}}\mathbf{D}_k^M\mathbf{D}_k(f)(x) + \mathbf{R}_1(f)(x) = \mathbf{T}_M(f)(x) + \mathbf{R}_1(f)(x) + \mathbf{R}_2(f)(x), \tag{1. 21}$$

where

$$\mathbf{T}_M(f)(x) = \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \omega(Q) D_k^M(x, x_Q)\mathbf{D}_k(f)(x_Q), \tag{1. 22}$$

$$\mathbf{R}_2(f)(x) = \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \left(\int_Q D_k^M(x,y)\mathbf{D}_k(f)(y)d\omega(y) - \omega(Q) D_k^M(x, x_Q)\mathbf{D}_k(f)(x_Q)\right), \tag{1. 23}$$

where Q_d^k is the collection of all dyadic cubes Q with the side length 2^{-M-k}, M is some fixed large integer, and x_Q is any fixed point in the cube Q.

By showing the operator $\mathbf{T}_M$ is invertible, the authors in [19] proved the Calderón reproducing formula on $\mathbf{L}^2$ as following

Theorem 1.1. [19] *If $f \in \mathbf{L}^2(\mathbb{R}^n, \omega)$, then there exists a function $h \in \mathbf{L}^2(\mathbb{R}^n, \omega)$, such that $\|f\|_2 \sim \|h\|_2$ and*

$$f(x) = \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \omega(Q) D_k^M(x, x_Q)\boldsymbol{D}_k(h)(x_Q) \tag{1. 24}$$

where Q are dyadic cubes in $\mathbb{R}^n$, Q_d^k is the collection of Q with the side length $\ell(Q) = 2^{-k-M}$, and x_Q are any fixed point in Q.

Based on the above theorem, we introduce the following

Definition 1.1. *Suppose that $f \in \mathbf{L}^2(\mathbb{R}^n, \omega)$, $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q < \infty$, the Littlewood-Paley q-function $\boldsymbol{S}_q^\alpha(f)$ is defined by*

$$\boldsymbol{S}_q^\alpha(f)(x) := \left\{ \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \left(2^{k\alpha} \left|\boldsymbol{D}_k(f)(x_Q)\right|\right)^q \chi_Q(x) \right\}^{\frac{1}{q}}, \tag{1. 25}$$

where $\chi_Q(x)$ is the characteristic function of the cube Q.

To establish the Triebel-Lizorkin space in the Dunkl setting, we utilize the new Calderón reproducing formula in $\mathbf{L}^2$ with the Triebel-Lizorkin space norms derived from the Dunkl-Poisson kernel (see [1]). Next, we introduce the new test functions in terms of the $\mathbf{L}^2$ functions and distributions by the duality estimates, which is crucial for developing the Dunkl-Triebel-Lizorkin spaces. Then we consider the Dunkl-Triebel-Lizorkin spaces as the collection of some distributions on the test function, and define the spaces by the wavelet-type decomposition and the analogous atomic decomposition of the Hardy spaces. Finally, we prove the Dunkl-Triebel-Lizorkin spaces defined this way are complete.

2 Formulation on the main results

The discrete Littlewood-Paley q-function in Definition 1.1 leads to introduce the Dunkl-Triebel-Lizorkin space norm for $f \in \mathbf{L}^2(\mathbb{R}^n, \omega)$ as follows:

Definition 2.1. *For $f \in \mathbf{L}^2(\mathbb{R}^n, \omega)$, the Dunkl-Triebel-Lizorkin space norm of f is defined as*

$$\|f\|_{\dot{F}_p^{\alpha,q}} := \|\boldsymbol{S}_q^\alpha(f)\|_p \tag{2. 1}$$

for $\alpha \in \mathbb{R}$, $0 < p < \infty$ and $0 < q < \infty$.

Here and subsequently, for $0 < p < \infty$, we denote the norm of $f \in \mathbf{L}^p(\mathbb{R}^n, \omega)$ by

$$\|f\|_p := \|f\|_{\mathbf{L}^p(\mathbb{R}^n,\omega)} = \left\{ \int_{\mathbb{R}^n} |f(x)|^p d\omega(x) \right\}^{\frac{1}{p}} < \infty.$$

Our first main result, the discrete Calderón reproducing formula for $f \in \mathbf{L}^2(\mathbb{R}^n, \omega)$ with respect to the Dunkl-Triebel-Lizorkin space norm, is given by the following

Theorem 2.1. *If $f \in \mathbf{L}^2(R^n, \omega)$ with $\|f\|_{\dot{F}_p^{\alpha,q}} < \infty$, for $|\alpha| < 1$, $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < p < \infty$, $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < q < \infty$, where N is the upper dimension, then there exists a function $h \in \mathbf{L}^2(\mathbb{R}^n, \omega)$, such that $\|f\|_2 \sim \|h\|_2$, $\|f\|_{\dot{F}_p^{\alpha,q}} \sim \|h\|_{\dot{F}_p^{\alpha,q}}$ and*

$$f(x) = \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \omega(Q) D_k^M(x, x_Q) \boldsymbol{D}_k(h)(x_Q) \tag{2. 2}$$

where Q_d^k is the collection of all dyadic cubes Q with the side length 2^{-M-k}, M is some fixed large integer, and x_Q is any fixed point in the cube Q. The series converges in $\mathbf{L}^2(\mathbb{R}^n, \omega)$ norm and the Dunkl-Triebel-Lizorkin space norm.

Applying the above theorem, we provide the duality estimates which will be a key idea for developing the Dunkl-Triebel-Lizorkin space theory. Before we state the estimates, we first define some necessary space norms.

Definition 2.2. *Suppose that $|\alpha| < 1$, $0 < p \le 1$, $1 < q < \infty$ and $f \in \mathbf{L}^2(\mathbb{R}^n, \omega)$. The norm of f in the Dunkl-Carleson measure space $CMO_p^{\alpha,q}$ is defined by*

$$\|f\|_{CMO_p^{\alpha,q}} = \sup_P \left(\frac{1}{\omega(P)^{\frac{q}{p}-\frac{q}{q'}}} \sum_{Q\in P} \omega(Q) \left| 2^{k\alpha} \boldsymbol{D}_k^M(f)(x_Q) \right|^q \right)^{\frac{1}{q}} \tag{2. 3}$$

where P runs over all dyadic cubes.

Definition 2.3. *Suppose that $|\alpha| < 1$, $1 \le p < \infty$ and $f \in \mathbf{L}^2(\mathbb{R}^n, \omega)$. The norm of $f \in \dot{F}_p^{\alpha,\infty}(\mathbb{R}^n, \omega)$ is defined by*

$$\|f\|_{\dot{F}_p^{\alpha,\infty}} = \left\| \sup_{k\in\mathbb{Z}, Q\in Q_d^k} 2^{k\alpha} \left| \boldsymbol{D}_k^M(f)(x_Q) \right| \chi_Q(x) \right\|_p . \tag{2. 4}$$

Definition 2.4. *Suppose that $|\alpha| < 1$ and $f \in \mathbf{L}^2(\mathbb{R}^n, \omega)$. The norm of $f \in \dot{F}_\infty^{\alpha,\infty}(\mathbb{R}^n, \omega)$ is defined by*

$$\|f\|_{\dot{F}_\infty^{\alpha,\infty}} = \sup_{k\in\mathbb{Z}, Q\in Q_d^k} 2^{k\alpha} \left| \boldsymbol{D}_k^M(f)(x_Q) \right| . \tag{2. 5}$$

Now we return to state the duality estimates

Theorem 2.2. *Suppose that $f, g \in \mathbf{L}^2(\mathbb{R}^n, \omega)$, p', q' are the conjugates of $1 < p, q < \infty$, respectively.*

(A) $1 < p < \infty$, $1 < q < \infty$, *there exists a constant C such that*

$$\left| \langle f, g \rangle \right| \le C \|f\|_{\dot{F}_p^{\alpha,q}} \|g\|_{\dot{F}_{p'}^{-\alpha,q'}} \tag{2. 6}$$

(B) $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < p \leq 1$, $1 < q < \infty$, *there exists a constant C such that*

$$\left|\langle f, g\rangle\right| \leq C\|f\|_{\dot{F}_p^{\alpha,q}} \|g\|_{CMO_p^{-\alpha,q'}} \tag{2. 7}$$

where $\|g\|_{CMO_p^{-\alpha,q'}}$ *is defined as in Definition 2.2.*

(C) $1 < p < \infty$, $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < q \leq 1$, *there exists a constant C such that*

$$\left|\langle f, g\rangle\right| \leq C\|f\|_{\dot{F}_p^{\alpha,q}} \|g\|_{\dot{F}_{p'}^{-\alpha,\infty}} \tag{2. 8}$$

where $\|g\|_{\dot{F}_{p'}^{-\alpha,\infty}}$ *is defined as in Definition 2.3.*

(D) $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < p \leq 1$, $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < q \leq 1$, *there exists a constant C such that*

$$\left|\langle f, g\rangle\right| \leq C\|f\|_{\dot{F}_p^{\alpha,q}} \|g\|_{\dot{F}_\infty^{-\alpha+N(\frac{1}{p}-1),\infty}} \tag{2. 9}$$

where $\|g\|_{\dot{F}_\infty^{-\alpha+N(\frac{1}{p}-1),\infty}}$ *is defined as in Definition 2.4.*

The above Theorem 2.2 means that for $1 < p < \infty$, $1 < q < \infty$, each function $f \in \mathbf{L}^2(\mathbb{R}^n, \omega)$ with $\|f\|_{\dot{F}_p^{\alpha,q}} < \infty$ can be considered as a continuous linear functional on $\mathbf{L}^2(\mathbb{R}^n, \omega) \cap \dot{F}_{p'}^{-\alpha,q'}(\mathbb{R}^n, \omega)$, the subspace of $g \in \mathbf{L}^2(\mathbb{R}^n, \omega)$ with the norm $\|g\|_{\dot{F}_{p'}^{-\alpha,q'}} < \infty$.

Therefore, one can consider $\mathbf{L}^2(\mathbb{R}^n, \omega) \cap \dot{F}_{p'}^{-\alpha,q'}(\mathbb{R}^n, \omega)$ as a new test function space and write $f \in \left(\mathbf{L}^2(\mathbb{R}^n, \omega) \cap \dot{F}_{p'}^{\alpha,q'}(\mathbb{R}^n, \omega)\right)'$, where $\left(\mathbf{L}^2(\mathbb{R}^n, \omega) \cap \dot{F}_{p'}^{\alpha,q'}(\mathbb{R}^n, \omega)\right)'$ is the distribution space. Other ranges of p, q stated above have the same results. The following result describes an important property for such a distribution f. More precisely, we establish the following discrete Calderón reproducing formula in the distribution sense:

Theorem 2.3. *For* $|\alpha| < 1$, $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < p < \infty$, $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < q < \infty$, *suppose that* $\{f_n\}_{n\in\mathbb{Z}}$ *is a Cauchy sequence in* $\mathbf{L}^2(\mathbb{R}^n, \omega)$ *with* $\|\boldsymbol{S}_q^\alpha(f_n - f_m)\|_p \to 0$ *as* $m, n \to \infty$*. Then there exists a distribution* f *satisfies*

(A) *For* $1 < p < \infty$, $1 < q < \infty$, f *is a distribution in* $\left(\mathbf{L}^2(\mathbb{R}^n, \omega) \cap \dot{F}_{p'}^{-\alpha,q'}(\mathbb{R}^n, \omega)\right)'$ *such that*

$$\lim_{n\to\infty} \|f_n - f\|_{\dot{F}_p^{\alpha,q}} = 0,$$

and for each $g \in \mathbf{L}^2(\mathbb{R}^n, \omega) \cap \dot{F}_{p'}^{-\alpha,q'}(\mathbb{R}^n, \omega)$,

$$\langle f, g\rangle = \left\langle \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \omega(Q) D_k^M(\cdot, x_Q) \boldsymbol{D}_k(h)(x_Q), g(\cdot)\right\rangle$$

where $h \in \left(\mathbf{L}^2(\mathbb{R}^n, \omega) \cap \dot{F}_{p'}^{-\alpha,q'}(\mathbb{R}^n, \omega)\right)'$ *with* $\|f\|_{\dot{F}_p^{\alpha,q}} \sim \|h\|_{\dot{F}_p^{\alpha,q}}$;

(B) *For* $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < p \le 1, 1 < q < \infty$, *$f$ is a distribution in* $\left(\mathbf{L}^2(\mathbb{R}^n,\omega) \cap CMO_p^{-\alpha,q'}(\mathbb{R}^n,\omega)\right)'$, *such that*

$$\lim_{n\to\infty} \|f_n - f\|_{\dot{F}_p^{\alpha,q}} = 0,$$

and for each $g \in \mathbf{L}^2(\mathbb{R}^n,\omega) \cap CMO_p^{-\alpha,q'}(\mathbb{R}^n,\omega)$,

$$\langle f, g\rangle = \left\langle \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \omega(Q) D_k^M(\cdot, x_Q) \boldsymbol{D}_k(h)(x_Q), g(\cdot)\right\rangle$$

where $h \in \left(\mathbf{L}^2(\mathbb{R}^n,\omega) \cap CMO_p^{-\alpha,q'}(\mathbb{R}^n,\omega)\right)'$ *with* $\|f\|_{\dot{F}_p^{\alpha,q}} \sim \|h\|_{\dot{F}_p^{\alpha,q}}$;

(C) *For* $1 < p < \infty$, $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < q \le 1$, *$f$ is a distribution in* $\left(\mathbf{L}^2(\mathbb{R}^n,\omega) \cap \dot{F}_{p'}^{-\alpha,\infty}(\mathbb{R}^n,\omega)\right)'$ *such that*

$$\lim_{n\to\infty} \|f_n - f\|_{\dot{F}_p^{\alpha,q}} = 0,$$

and for each $g \in \mathbf{L}^2(\mathbb{R}^n,\omega) \cap \dot{F}_{p'}^{-\alpha,\infty}(\mathbb{R}^n,\omega)$,

$$\langle f, g\rangle = \left\langle \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \omega(Q) D_k^M(\cdot, x_Q) \boldsymbol{D}_k(h)(x_Q), g(\cdot)\right\rangle$$

where $h \in \left(\mathbf{L}^2(\mathbb{R}^n,\omega) \cap \dot{F}_{p'}^{-\alpha,\infty}(\mathbb{R}^n,\omega)\right)'$ *with* $\|f\|_{\dot{F}_p^{\alpha,q}} \sim \|h\|_{\dot{F}_p^{\alpha,q}}$;

(D) *For* $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < p \le 1$, $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < q \le 1$, *$f$ is a distribution in* $\left(\mathbf{L}^2(\mathbb{R}^n,\omega) \cap \dot{F}_{\infty}^{-\alpha+N(\frac{1}{p}-1),\infty}(\mathbb{R}^n,\omega)\right)'$ *such that*

$$\lim_{n\to\infty} \|f_n - f\|_{\dot{F}_p^{\alpha,q}} = 0$$

and for each $g \in \mathbf{L}^2(\mathbb{R}^n,\omega) \cap \dot{F}_{\infty}^{-\alpha+N(\frac{1}{p}-1),\infty}(\mathbb{R}^n,\omega)$,

$$\langle f, g\rangle = \left\langle \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \omega(Q) D_k^M(\cdot, x_Q) \boldsymbol{D}_k(h)(x_Q), g(\cdot)\right\rangle$$

where $h \in \left(\mathbf{L}^2(\mathbb{R}^n,\omega) \cap \dot{F}_{\infty}^{-\alpha+N(\frac{1}{p}-1),\infty}(\mathbb{R}^n,\omega)\right)'$ *with* $\|f\|_{\dot{F}_p^{\alpha,q}} \sim \|h\|_{\dot{F}_p^{\alpha,q}}$.

Then we define the Dunkl-Triebel-Lizorkin spaces by the following

Definition 2.5. *The Dunkl-Triebel-Linzorkin space* $\dot{F}_p^{\alpha,q}(\mathbb{R}^n,\omega)$, $|\alpha| < 1$, $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < p < \infty$, $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < q < \infty$ *is defined as follows:*

$$\dot{F}_p^{\alpha,q} = \left\{ f : f(x) = \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \omega(Q)\lambda_Q D_k^M(x, x_Q),\ \text{with}\ \left\| \left\{ \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \left(2^{k\alpha} \left|\lambda_Q\right|\right)^q \chi_Q(x) \right\}^{\frac{1}{q}} \right\|_p < \infty \right\} \tag{2. 10}$$

where the series converges in the following distribution sense:

(A) for $1 < p < \infty$, $1 < q < \infty$, *f converges in* $\left(\mathbf{L}^2(\mathbb{R}^n,\omega) \cap \dot{F}_{p'}^{-\alpha,q'}(\mathbb{R}^n,\omega)\right)'$;

(B) for $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < p \leq 1$, $1 < q < \infty$, *f converges in* $\left(\mathbf{L}^2(\mathbb{R}^n,\omega) \cap CMO_p^{-\alpha,q'}(\mathbb{R}^n,\omega)\right)'$;

(C) for $1 < p < \infty$, $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < q \leq 1$, *f converges in* $\left(\mathbf{L}^2(\mathbb{R}^n,\omega) \cap \dot{F}_{p'}^{-\alpha,\infty}(\mathbb{R}^n,\omega)\right)'$;

(D) for $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < p \leq 1$, $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < q \leq 1$, *f converges in* $\left(\mathbf{L}^2(\mathbb{R}^n,\omega) \cap \dot{F}_{\infty}^{-\alpha+N(\frac{1}{p}-1),\infty}(\mathbb{R}^n,\omega)\right)'$.

If $f \in \dot{F}_p^{\alpha,q}(\mathbb{R}^n,\omega)$, *the norm of f is defined by*

$$\|f\|_{\dot{F}_p^{\alpha,q}} := \inf\left\{ \left\| \left\{ \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \left(2^{k\alpha}\left|\lambda_Q\right|\right)^q \chi_Q(x) \right\}^{\frac{1}{q}} \right\|_p \right\} \tag{2. 11}$$

where the infimum is taken over all $f(x) = \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \omega(Q)\lambda_Q D_k^M(x,x_Q)$.

To clarify that the Dunkl-Triebel-Lizorkin space in Definition 2.5 is complete, we show the following

Theorem 2.4. *For* $|\alpha| < 1$, $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < p < \infty$, $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < q < \infty$,

$$\dot{F}_p^{\alpha,q}(\mathbb{R}^n,\omega) = \overline{\mathbf{L}^2(\mathbb{R}^n,\omega) \cap \dot{F}_p^{\alpha,q}(\mathbb{R}^n,\omega)}. \tag{2. 12}$$

More precisely,

(A) for $1 < p < \infty$, $1 < q < \infty$, $\overline{\mathbf{L}^2(\mathbb{R}^n,\omega) \cap \dot{F}_p^{\alpha,q}(\mathbb{R}^n,\omega)}$ *is the collection of all distributions* $f \in \left(\mathbf{L}^2(\mathbb{R}^n,\omega) \cap \dot{F}_{p'}^{-\alpha,q'}(\mathbb{R}^n,\omega)\right)'$ *such that there exists a sequence* $\{f_n\}_{n=1}^{\infty}$ *in* $\mathbf{L}^2(\mathbb{R}^n,\omega)$ *with* $\|f_n - f_m\|_{\dot{F}_p^{\alpha,q}} \to 0$ *as* $n,m \to \infty$. *Moreover,* f_n *converges to f in* $\left(\mathbf{L}^2(\mathbb{R}^n,\omega) \cap \dot{F}_{p'}^{-\alpha,q'}(\mathbb{R}^n,\omega)\right)'$;

(B) for $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < p \leq 1$, $1 < q < \infty$, $\overline{\mathbf{L}^2(\mathbb{R}^n,\omega) \cap \dot{F}_p^{\alpha,q}(\mathbb{R}^n,\omega)}$ *is the collection of all distributions* $f \in \left(\mathbf{L}^2(\mathbb{R}^n,\omega) \cap CMO_p^{-\alpha,q'}(\mathbb{R}^n,\omega)\right)'$ *such that there exists a sequence* $\{f_n\}_{n=1}^{\infty}$ *in* $\mathbf{L}^2(\mathbb{R}^n,\omega)$ *with* $\|f_n - f_m\|_{\dot{F}_p^{\alpha,q}} \to 0$ *as* $n,m \to \infty$. *Moreover,* f_n *converges to f in* $\left(\mathbf{L}^2(\mathbb{R}^n,\omega) \cap CMO_p^{-\alpha,q'}(\mathbb{R}^n,\omega)\right)'$;

(C) for $1 < p < \infty$, $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < q \leq 1$, $\overline{\mathbf{L}^2(\mathbb{R}^n,\omega) \cap \dot{F}_p^{\alpha,q}(\mathbb{R}^n,\omega)}$ *is the collection of all distributions* $f \in \left(\mathbf{L}^2(\mathbb{R}^n,\omega) \cap \dot{F}_{p'}^{-\alpha,\infty}(\mathbb{R}^n,\omega)\right)'$ *such that there exists a sequence* $\{f_n\}_{n=1}^{\infty}$ *in* $\mathbf{L}^2(\mathbb{R}^n,\omega)$ *with* $\|f_n - f_m\|_{\dot{F}_p^{\alpha,q}} \to 0$ *as* $n,m \to \infty$. *Moreover,* f_n *converges to f in* $\left(\mathbf{L}^2(\mathbb{R}^n,\omega) \cap \dot{F}_{p'}^{-\alpha,\infty}(\mathbb{R}^n,\omega)\right)'$;

(D) for $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < p \leq 1$, $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < q \leq 1$, $\overline{\mathbf{L}^2(\mathbb{R}^n,\omega) \cap \dot{F}_p^{\alpha,q}(\mathbb{R}^n,\omega)}$ *is the collection of all distributions* $f \in \left(\mathbf{L}^2(\mathbb{R}^n,\omega) \cap \dot{F}_{\infty}^{-\alpha+N(\frac{1}{p}-1),\infty}(\mathbb{R}^n,\omega)\right)'$ *such that there exists a*

sequence $\{f_n\}_{n=1}^{\infty}$ *in* $\mathbf{L}^2(\mathbb{R}^n,\omega)$ *with* $\|f_n - f_m\|_{\dot{F}_p^{\alpha,q}} \to 0$ *as* $n, m \to \infty$. *Moreover,* f_n *converges to* f *in* $\left(\mathbf{L}^2(\mathbb{R}^n,\omega) \cap \dot{F}_{\infty}^{-\alpha+N(\frac{1}{p}-1),\infty}(\mathbb{R}^n,\omega)\right)'$.

The paper is organized as follows. In the next section, we prove Theorem 2.1, which is the Calderón reproducing formula in $\mathbf{L}^2 \cap \dot{F}_p^{\alpha,q}$. The main tools are orthogonal estimates in the Dunkl setting. In section 4, we demonstrate the Theorem 2.2, the duality estimates which lead a way for developing the Dunkl-Triebel-Lizorkin space theory. In the last section, we define the Triebel-Lizorkin space in the Dunkl setting in Definition 2.5 and show the spaces is complete by Theorem 2.4.

3 Calderón reproducing formula in $\mathbf{L}^2(\mathbb{R}^n,\omega) \cap \dot{F}_p^{\alpha,q}(\mathbb{R}^n,\omega)$

As mentioned before, the Dunkl-Calderón-Zygmund operator theory plays a crucial role. To prove Theorem 2.1, we recall the Dunkl-Calderón-Zygmund singular integral operator and almost orthogonality estimates in the Dunkl setting. See [19] for more details.

Let $C_0^{\eta}(\mathbb{R}^n)$, $\eta > 0$, denote the space of continuous functions f with compact support and

$$\|f\|_{\eta} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x-y\|^{\eta}} < \infty. \tag{3. 1}$$

Definition 3.1. *An operator* $\boldsymbol{T} : C_0^{\eta}(\mathbb{R}^n) \to (C_0^{\eta}(\mathbb{R}^n))'$ *with* $\eta > 0$, *is said to be a Dunkl-Calderón-Zygmund singular integral operator if* $K(x,y)$, *the kernel of* $\boldsymbol{T}$, *satisfies the following estimates: for some* $0 < \varepsilon \leq 1$,

$$\left|K(x,y)\right| \leq \frac{C}{\omega(B(x,d(x,y)))}\left(\frac{d(x,y)}{\|x-y\|}\right)^{\varepsilon} \tag{3. 2}$$

for all $x \neq y$;

$$\left|K(x,y) - K(x',y)\right| \leq \left(\frac{\|x-x'\|}{\|x-y\|}\right)^{\varepsilon} \frac{C}{\omega(B(x,d(x,y)))} \tag{3. 3}$$

for $\|x - x'\| \leq d(x,y)/2$;

$$\left|K(x,y) - K(x,y')\right| \leq \left(\frac{\|y-y'\|}{\|x-y\|}\right)^{\varepsilon} \frac{C}{\omega(B(x,d(x,y)))} \tag{3. 4}$$

for $\|y - y'\| \leq d(x,y)/2$.

A Dunkl-Calderón-Zygmund singular integral operator is said to be the Dunkl-Calderón-Zygmund operator if it extends a bounded operator on $\mathbf{L}^2(\mathbb{R}^n)$. Suppose that $\mathbf{T}$ is the Dunkl-Calderón-Zygmund operator. We denote

$$\|\mathbf{T}\|_{dcz} = \|\mathbf{T}\|_{2,2} + \|K\|_{dcz} \tag{3. 5}$$

as the Dunkl-Calderón-Zygmund operator norm, where $\|K\|_{dcz}$ is the minimum of the constants in (3. 2)-(3. 4).

The following almost orthogonality estimates are important tools in the proof. See [19] for more details. Let $\{\mathbf{S}_k\}_{k\in\mathbb{Z}}$ be approximations to the identity and set $\mathbf{D}_k := \mathbf{S}_k - \mathbf{S}_{k-1}$, then

Lemma 3.1. [19] *For $k, j \in \mathbb{Z}$, $\varepsilon > 0$, γ, $\varepsilon' \in (0, \varepsilon)$,*

$$\left|\left(D_k D_j\right)(x,y)\right| \le C\, 2^{-|k-j|\varepsilon'} \frac{1}{V(x,y,2^{-k\vee -j}+d(x,y))}\left(\frac{2^{-k\vee -j}}{2^{-k\vee -j}+d(x,y)}\right)^{\gamma}, \tag{3. 6}$$

where $a \wedge b = \min\{a,b\}$, $a \vee b = \max\{a,b\}$, $V(x,y,r) := \max\{\omega(B(x,r)), \omega(B(y,r))\}$.

Lemma 3.2. [19] *Let $\boldsymbol{T}$ be a Dunkl-Calderón-Zygmund singular integral operator satisfying $\boldsymbol{T}(1) = \boldsymbol{T}^*(1) = 0$ and $\boldsymbol{T}$ is bounded in $\mathbf{L}^2(\mathbb{R}^n, \omega)$. Then*

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} D_k(x,u)K(u,v)D_j(v,y)d\omega(u)d\omega(v)\right|\\ &\le C\, 2^{-|k-j|\varepsilon'}\|\boldsymbol{T}\|_{dcz}\frac{1}{V(x,y,2^{-k\vee -j}+d(x,y))}\left(\frac{2^{-k\vee -j}}{2^{-k\vee -j}+d(x,y)}\right)^{\gamma},\end{aligned} \tag{3. 7}$$

where γ, $\varepsilon' \in (0,\varepsilon)$, ε is the regularity exponent of the kernel of $\boldsymbol{T}$ given in (3. 3) and (3. 4).

Proof of Theorem 2.1: First, as mentioned before, we decompose the identity operator on $\mathbf{L}^2(\mathbb{R}^n,\omega)$ by $\mathbf{I} = \mathbf{T}_M + \mathbf{R}_1 + \mathbf{R}_2$. To prove that $\mathbf{T}_M$ is invertible and $(\mathbf{T}_M)^{-1}$, the inverse of $\mathbf{T}_M$, is bounded on $\mathbf{L}^2 \cap \dot{F}^{\alpha,q}_p$, we need to estimate $\mathbf{R}_1$ and $\mathbf{R}_2$ on $\mathbf{L}^2 \cap \dot{F}^{\alpha,q}_p$ and show that the norm of $\mathbf{R}_1$ and $\mathbf{R}_2$ on $\mathbf{L}^2 \cap \dot{F}^{\alpha,q}_p$ are less than 1. To this end, we consider the Dunkl setting, $(\mathbb{R}^n, \|\cdot\|, \omega)$, as a space of homogeneous type in the sense of Coifman and Wiess. The discrete Calderón reproducing formula in the space of homogeneous type is given by the following (see [12])

Theorem 3.1. [12] *Let $\{\boldsymbol{S}_k\}_{k\in\mathbb{Z}}$ be a Coifman's approximations to the identity and set $\boldsymbol{E}_k := \boldsymbol{S}_k - \boldsymbol{S}_{k-1}$. Then there exists a family of operators $\{\widetilde{\boldsymbol{E}}_k\}_{k\in\mathbb{Z}}$ such that for any fixed $x_Q \in Q$ with $k \in \mathbb{Z}$ and $Q \in Q^k_{cw}$ are dyadic cubes with the side length 2^{-k-M_0},*

$$f(x) = \sum_{k\in\mathbb{Z}}\sum_{Q\in Q^k_{cw}} \omega(Q)\widetilde{E}_k(x,x_Q)\boldsymbol{E}_k(f)(x_Q), \tag{3. 8}$$

where the series converge in $\mathbf{L}^p(\mathbb{R}^n,\omega)$, $\mathcal{M}(\beta,\gamma,r,x_0)$ and $(\mathcal{M}(\beta,\gamma,r,x_0))'$, the dual space of $\mathcal{M}(\beta,\gamma,r,x_0)$.

Recall the Littlewood-Paley theory and the Triebel-Lizorkin spaces on space of homogeneous type $(\mathbb{R}^n, \|\cdot\|, \omega)$ in the sense of Coifman and Weiss. The discrete Calderón reproducing formula in Theorem 3.1 leads the following discrete q-function on space of homogeneous type $(\mathbb{R}^n, \|\cdot\|, \omega)$:

Definition 3.2. *Suppose that $f \in (\mathcal{M}(\beta,\gamma,r,x_0))'$, $\alpha \in \mathbb{R}$, define the Littlewood-Paley q-function $\boldsymbol{S}^{\alpha}_{q,cw}(f)$ for the space of homogeneous type $(\mathbb{R}^n, \|\cdot\|, \omega)$ as*

$$\boldsymbol{S}^{\alpha}_{q,cw}(f)(x) = \left\{\sum_{k\in\mathbb{Z}}\sum_{Q\in Q^k_{cw}}\left(2^{k\alpha}\left|\boldsymbol{E}_k(f)(x_Q)\right|\right)^q\chi_Q(x)\right\}^{\frac{1}{q}} \quad \text{for } 0 < q < \infty, \tag{3. 9}$$

where $\boldsymbol{E}_k$, Q^k_{cw} are given in Theorem 3.1.

Remark 3.1. *It is well known that $\mathbb{R}^n$ together with the Euclidean metric and the Dunkl measure is space of homogeneous type in the sense of Coifman and Weiss. See [12] for more details.*

Now we return to the estimates of $\mathbf{R}_1$ and $\mathbf{R}_2$. Note that $\mathbf{R}_1$ and $\mathbf{R}_2$ are Dunkl-Calderón-Zygmund operators with $\|\mathbf{R}_1\|_{dcz} + \|\mathbf{R}_2\|_{dcz} \le C\, 2^{-M\delta}$, $\delta > 0$. The boundedness of $\mathbf{R}_1$ and $\mathbf{R}_2$ on $\mathbf{L}^2$ follows from the Cotlar-Stein Lemma. Moreover, $\|\mathbf{R}_1 + \mathbf{R}_2\|_{2,2} < 1$. See [19] for more details.

To estimate $\mathbf{R}_1$ and $\mathbf{R}_2$ on $\dot{F}_p^{\alpha,q}(\mathbb{R}^n,\omega)$, we establish the following estimates

$$\|\mathbf{S}_q^\alpha(\mathbf{R}_1(f))\|_p \le C\,\|\mathbf{S}_{q,cw}^\alpha(\mathbf{R}_1(f))\|_p \le C\,\|\mathbf{R}_1\|_{dcz}\|\mathbf{S}_{q,cw}^\alpha(f)\|_p \le C\,\|\mathbf{R}_1\|_{dcz}\|\mathbf{S}_q^\alpha(f)\|_p \tag{3. 10}$$

and the similar estimates also hold for $\mathbf{R}_2$. Now we show the above estimates for $\mathbf{R}_1$ by the following steps:

Step 1: $\|\mathbf{S}_q^\alpha(\mathbf{R}_1(f))\|_p \le C\,\|\mathbf{S}_{q,cw}^\alpha(\mathbf{R}_1(f))\|_p$.

Indeed we only need to show that for each $f \in \mathbf{L}^2(\mathbb{R}^n,\omega)$,

$$\|\mathbf{S}_q^\alpha(f)\|_p \le C\,\|\mathbf{S}_{q,cw}^\alpha(f)\|_p \tag{3. 11}$$

since $\mathbf{R}_1$ is bounded on $\mathbf{L}^2$. By the discrete Calderón reproducing formula of $f \in \mathbf{L}^2$ given in Theorem 3.1 we have

$$\begin{aligned}\mathbf{S}_q^\alpha(f)(x) &= \left\{\sum_{k'\in\mathbb{Z}}\sum_{Q'\in Q_d^{k'}}\left(2^{k'\alpha}\left|\mathbf{D}_{k'}\left(\sum_{k\in\mathbb{Z}}\sum_{Q\in Q_{cw}^k}\omega(Q)\widetilde{E}_k(\cdot,x_Q)\mathbf{E}_k(f)(x_Q)\right)(x_{Q'})\right|\right)^q\chi_{Q'}(x)\right\}^{\frac{1}{q}}\\ &= \left\{\sum_{k'\in\mathbb{Z}}\sum_{Q'\in Q_d^{k'}}\left(\left|\sum_{k\in\mathbb{Z}}\sum_{Q\in Q_{cw}^k}2^{k'\alpha}\omega(Q)(D_{k'}\widetilde{E}_k)(x_{Q'},x_Q)\mathbf{E}_k(f)(x_Q)\right|\right)^q\chi_{Q'}(x)\right\}^{\frac{1}{q}}.\end{aligned} \tag{3. 12}$$

By the almost orthogonal estimate given in the Lemma 3.1, we choose $|\alpha| < \varepsilon < 1$ such that

$$\begin{aligned}&\left|(D_{k'}\widetilde{E}_k)(x_{Q'},x_Q)\right|\chi_{Q'}(x)\\ &\le C2^{-|k'-k|\varepsilon}\frac{1}{V(x_{Q'},x_Q,2^{-k'\vee -k}+d(x_{Q'},x_Q))}\left(\frac{2^{-k'\vee -k}}{2^{-k'\vee -k}+d(x_{Q'},x_Q)}\right)^\varepsilon\chi_{Q'}(x)\\ &\le C2^{-|k'-k|\varepsilon}\frac{1}{\omega(B(x_Q,2^{-k'\vee -k}+d(x,x_Q)))}\left(\frac{2^{-k'\vee -k}}{2^{-k'\vee -k}+d(x,x_Q)}\right)^\varepsilon\chi_{Q'}(x).\end{aligned} \tag{3. 13}$$

Since $d(x,y) = \min_{\sigma\in G}\|\sigma(x)-y\|$, then

$$\begin{aligned}&\left|(D_{k'}\widetilde{E}_k)(x_{Q'},x_Q)\right|\chi_{Q'}(x)\\ &\le C\sum_{\sigma\in G}2^{-|k'-k|\varepsilon}\frac{1}{\omega(B(x_Q,2^{-k'\vee -k}+\|\sigma(x)-x_Q\|))}\left(\frac{2^{-k'\vee -k}}{2^{-k'\vee -k}+\|\sigma(x)-x_Q\|}\right)^\varepsilon\chi_{Q'}(x)\\ &\le C\sum_{\sigma\in G}2^{-|k'-k|\varepsilon}\frac{1}{\omega(B(\sigma(x),2^{-k'\vee -k}+\|\sigma(x)-x_Q\|))}\left(\frac{2^{-k'\vee -k}}{2^{-k'\vee -k}+\|\sigma(x)-x_Q\|}\right)^\varepsilon\chi_{Q'}(x).\end{aligned} \tag{3. 14}$$

Let θ satisfies that $\max\left\{\frac{N}{N+\varepsilon}, \frac{N}{N+\alpha+\varepsilon}\right\} < \theta < \min\{p, q, 1\}$, then

$$
\begin{aligned}
&\left|\sum_{Q\in Q_{cw}^k} 2^{k'\alpha}\omega(Q)(D_{k'}\widetilde{E}_k)(x_{Q'}, x_Q)\mathbf{E}_k(f)(x_Q)\right|\chi_{Q'}(x)\\
&\le C\sum_{Q\in Q_{cw}^k}\sum_{\sigma\in G} 2^{k'\alpha}2^{-|k'-k|\varepsilon}\omega(Q)\frac{1}{\omega(B(\sigma(x), 2^{-k'\vee -k}+\|\sigma(x)-x_Q\|))}\left(\frac{2^{-k'\vee -k}}{2^{-k'\vee -k}+\|\sigma(x)-x_Q\|}\right)^{\varepsilon}\left|\mathbf{E}_k f(x_Q)\right|\chi_{Q'}(x)\\
&\le C\sum_{\sigma\in G}\left\{\sum_{Q\in Q_{cw}^k} 2^{-|k'-k|\varepsilon\theta}2^{(k'-k)\alpha\theta}\omega(Q)^{\theta}\frac{1}{\omega(B(\sigma(x), 2^{-k'\vee -k}+\|\sigma(x)-x_Q\|))^{\theta}}\right.\\
&\quad\left.\times\left(\frac{2^{-k'\vee -k}}{2^{-k'\vee -k}+\|\sigma(x)-x_Q\|}\right)^{\theta\varepsilon}\left|2^{k\alpha}\mathbf{E}_k f(x_Q)\right|^{\theta}\right\}^{\frac{1}{\theta}}\chi_{Q'}(x).
\end{aligned}
\tag{3. 15}
$$

Denote c_Q as the center of Q. Set

$$A_0 = \left\{Q\in Q_{cw}^k : \|c_Q - \sigma(x)\| \le 2^{-k\vee -k'}\right\}$$

and

$$A_\ell = \left\{Q\in Q_{cw}^k : 2^{\ell-1+(-k\vee -k')} < \|c_Q-\sigma(x)\| \le 2^{\ell+(-k\vee -k')}\right\}$$

for $\ell\in\mathbb{N}$.

For $Q\in Q_{cw}^k$, we have

$$\omega(Q)\chi_Q(z)\sim\omega(B(z, 2^{-k-M_0}))\chi_Q(z)\sim\omega(B(\sigma(z), 2^{-k-M_0}))\chi_Q(z),\ \text{for } \sigma\in G, \tag{3. 16}$$

and

$$\omega(B(\sigma(z), 2^{-k\vee -k'}))\le C\,2^{((-k\vee -k')-(-k-M_0))N}\omega(B(\sigma(z), 2^{-k-M_0})). \tag{3. 17}$$

Hence

$$\sum_{Q\in Q^k_{cw}} 2^{-|k'-k|\varepsilon\theta}2^{(k'-k)\alpha\theta}\omega(Q)^\theta \frac{1}{\omega(B(\sigma(x),2^{-k'\vee -k}+\|\sigma(x)-x_Q\|))^\theta}\left(\frac{2^{-k'\vee -k}}{2^{-k'\vee -k}+\|\sigma(x)-x_Q\|}\right)^{\theta\varepsilon}\left|2^{k\alpha}\mathbf{E}_k f(x_Q)\right|^\theta$$

$$= \sum_{\ell=0}^{\infty}\sum_{Q\in A_\ell} 2^{-|k'-k|\varepsilon\theta}2^{(k'-k)\alpha\theta}\frac{\omega(Q)^{\theta-1}}{\omega(B(\sigma(x),2^{-k'\vee -k}+\|\sigma(x)-x_Q\|))^{\theta-1}}\left(\frac{2^{-k'\vee -k}}{2^{-k'\vee -k}+\|\sigma(x)-x_Q\|}\right)^{\theta\varepsilon}$$

$$\times\ \frac{1}{\omega(B(\sigma(x),2^{-k'\vee -k}+\|\sigma(x)-x_Q\|))}\omega(Q)\left|2^{k\alpha}\mathbf{E}_k f(x_Q)\right|^\theta$$

$$\leq C\, 2^{-|k'-k|\varepsilon\theta}2^{(k'-k)\alpha\theta}2^{(-k-M_0-(-k'\vee -k))N(\theta-1)}\sum_{\ell=0}^{\infty}\frac{\omega(B(\sigma(x),2^{-k'\vee -k}))^{\theta-1}}{\omega(B(\sigma(x),2^{\ell-1+(-k'\vee -k)}))^{\theta-1}}\left(\frac{1}{2^{\ell-1}}\right)^{\theta\varepsilon}$$

$$\times\ \frac{1}{\omega(B(\sigma(x),2^{\ell-1+(-k'\vee -k)}))}\int_{\|\sigma(x)-z\|\leq 2\times 2^{\ell+(-k\vee -k')}}\sum_{Q\in A_\ell}\left|2^{k\alpha}\mathbf{E}_k f(x_Q)\right|^\theta\chi_Q(z)d\omega(z)$$

$$\leq C\, 2^{-M_0N(\theta-1)}2^{-|k'-k|\varepsilon\theta}2^{(-k-(-k'\vee -k))N(\theta-1)}2^{(k'-k)\alpha\theta}$$

$$\times\ \sum_{\ell=0}^{\infty}\frac{1}{2^{(\ell-1)(N(\theta-1)+\theta\varepsilon)}}\mathbf{M}\left(\sum_{Q\in Q^k_{cw}}\left|2^{k\alpha}\mathbf{E}_k f(x_Q)\right|^\theta\chi_Q\right)(\sigma(\cdot))$$

$$\leq C\, 2^{-M_0N(\theta-1)}2^{-|k'-k|\varepsilon\theta}2^{(-k-(-k'\vee -k))N(\theta-1)+(k'-k)\alpha\theta}\mathbf{M}\left(\sum_{Q\in Q^k_{cw}}\left|2^{k\alpha}\mathbf{E}_k f(x_Q)\right|^\theta\chi_Q\right)(\sigma(\cdot))\tag{3. 18}$$

where $\mathbf{M}$ denote the Hardy-Littlewood maximal operator on $(\mathbb{R}^n,\|\cdot\|,\omega)$. Therefore,

$$\left|\sum_{Q\in Q^k_{cw}} 2^{k'\alpha}\omega(Q)(D_{k'}\widetilde{E}_k)(x_{Q'},x_Q)\mathbf{E}_k(f)(x_Q)\right|\chi_{Q'}(x)$$

$$\leq C\sum_{\sigma\in G}2^{-M_0N(1-\frac{1}{\theta})}2^{-|k'-k|\varepsilon}2^{(-k-(-k'\vee -k))N(1-\frac{1}{\theta})+(k'-k)\alpha}\left\{\mathbf{M}\left(\sum_{Q\in Q^k_{cw}}\left|2^{k\alpha}\mathbf{E}_k f(x_Q)\right|^\theta\chi_Q\right)(\sigma(\cdot))\right\}^{\frac{1}{\theta}}\chi_{Q'}(x).\tag{3. 19}$$

For $|\alpha|<\varepsilon$, $\max\left\{\frac{N}{N+\varepsilon},\frac{N}{N+\alpha+\varepsilon}\right\}<\theta<\min\{p,q,1\}$, it is obvious that

$$\sup_{k'}\sum_{k\in\mathbb{Z}}2^{-|k-k'|\varepsilon}2^{(-k-(-k'\vee -k))N(1-\frac{1}{\theta})+(k'-k)\alpha}<\infty.\tag{3. 20}$$

For $1 < q < \infty$, by Hölder's inequality, we have

$$\begin{aligned}
&\left|\sum_{k\in\mathbb{Z}}\sum_{Q\in Q_{cw}^k} 2^{k'\alpha}\omega(Q)(D_{k'}\widetilde{E}_k)(x_{Q'},x_Q)\mathbf{E}_k(f)(x_Q)\right|^q \chi_{Q'}(x)\\
&\le C\,\left\{\sum_{\sigma\in G}\sum_{k\in\mathbb{Z}} 2^{-M_0N(1-\frac{1}{\theta})}2^{-|k-k'|\varepsilon}2^{(-k-(-k'\vee -k))N(1-\frac{1}{\theta})+(k'-k)\alpha}\right.\\
&\qquad\times\left.\left\{\mathbf{M}\left(\sum_{Q\in Q_{cw}^k}\left|2^{k\alpha}\mathbf{E}_k f(x_Q)\right|^\theta\chi_Q\right)(\sigma(\cdot))\right\}^{\frac{1}{\theta}}\right\}^q \chi_{Q'}(x)\\
&\le C\,2^{-M_0N(1-\frac{1}{\theta})q}\left\{\sum_{\sigma\in G}\sum_{k\in\mathbb{Z}}2^{-|k-k'|\varepsilon}2^{(-k-(-k'\vee -k))N(1-\frac{1}{\theta})+(k'-k)\alpha}\right\}^{\frac{q}{q'}}\\
&\qquad\times\left\{\sum_{\sigma\in G}\sum_{k\in\mathbb{Z}}2^{-|k-k'|\varepsilon}2^{(-k-(-k'\vee -k))N(1-\frac{1}{\theta})+(k'-k)\alpha}\right.\\
&\qquad\times\left.\left\{\mathbf{M}\left(\sum_{Q\in Q_{cw}^k}\left|2^{k\alpha}\mathbf{E}_k f(x_Q)\right|^\theta\chi_Q\right)(\sigma(\cdot))\right\}^{\frac{q}{\theta}}\right\}\chi_{Q'}(x)\\
&\le C\,2^{-M_0N(1-\frac{1}{\theta})q}\sum_{\sigma\in G}\sum_{k\in\mathbb{Z}}2^{-|k-k'|\varepsilon}2^{(-k-(-k'\vee -k))N(1-\frac{1}{\theta})+(k'-k)\alpha}\\
&\qquad\times\left\{\mathbf{M}\left(\sum_{Q\in Q_{cw}^k}\left|2^{k\alpha}\mathbf{E}_k f(x_Q)\right|^\theta\chi_Q\right)(\sigma(\cdot))\right\}^{\frac{q}{\theta}}\chi_{Q'}(x).
\end{aligned}\tag{3. 21}$$

For $\max\left\{\frac{N}{N+\varepsilon},\frac{N}{N+\alpha+\varepsilon}\right\} < q \le 1$, by q-inequality we have

$$\begin{aligned}
&\left|\sum_{k\in\mathbb{Z}}\sum_{Q\in Q_{cw}^k} 2^{k'\alpha}(D_{k'}\widetilde{E}_k)(x_{Q'},x_Q)\mathbf{E}_k(f)(x_Q)\right|^q \chi_{Q'}(x)\\
&\le C\,2^{-M_0N(1-\frac{1}{\theta})q}\sum_{\sigma\in G}\sum_{k\in\mathbb{Z}}2^{-|k-k'|\varepsilon q}2^{(-k-(-k'\vee -k))N(1-\frac{1}{\theta})q+(k'-k)\alpha q}\\
&\qquad\times\left\{\mathbf{M}\left(\sum_{Q\in Q_{cw}^k}\left|2^{k\alpha}\mathbf{E}_k f(x_Q)\right|^\theta\chi_Q\right)(\sigma(\cdot))\right\}^{\frac{q}{\theta}}\chi_{Q'}(x).
\end{aligned}\tag{3. 22}$$

For $|\alpha| < \varepsilon$, $\max\left\{\frac{N}{N+\varepsilon}, \frac{N}{N+\alpha+\varepsilon}\right\} < \theta < \min\{p, q, 1\}$, it is obvious that

$$\sup_{k} \sum_{k' \in \mathbb{Z}} 2^{-|k-k'|\varepsilon} 2^{(-k-(-k' \vee -k))N(1-\frac{1}{\theta})+(k'-k)\alpha} < \infty, \tag{3. 23}$$

then

$$\begin{aligned}
&\left\{\sum_{k' \in \mathbb{Z}} \sum_{Q' \in Q_d^{k'}} \left|\sum_{k \in \mathbb{Z}} \sum_{Q \in Q_{cw}^k} 2^{k'\alpha} \omega(Q)(D_{k'} \widetilde{E}_k)(x_{Q'}, x_Q) \mathbf{E}_k(f)(x_Q)\right|^q \chi_{Q'}(x)\right\}^{\frac{1}{q}} \\
&\leq C\, 2^{-M_0 N(1-\frac{1}{\theta})} \left\{\sum_{\sigma \in G} \sum_{k \in \mathbb{Z}} \left\{\mathbf{M}\left(\sum_{Q \in Q_{cw}^k} \left|2^{k\alpha} \mathbf{E}_k f(x_Q)\right|^\theta \chi_Q\right)(\sigma(\cdot))\right\}^{\frac{q}{\theta}}\right\}^{\frac{1}{q}}.
\end{aligned} \tag{3. 24}$$

By using Fefferman-Stein vectored maximal function inequality (see [9]) with $\theta < \min\{p, q, 1\}$, we have

$$\begin{aligned}
&\left\|\left\{\sum_{k' \in \mathbb{Z}} \sum_{Q' \in Q_d^{k'}} \left|\sum_{k \in \mathbb{Z}} \sum_{Q \in Q_{cw}^k} 2^{k'\alpha} \omega(Q)(D_{k'} \widetilde{E}_k)(x_{Q'}, x_Q) \mathbf{E}_k(f)(x_Q)\right|^q \chi_{Q'}(\cdot)\right\}^{\frac{1}{q}}\right\|_p \\
&\leq C\, 2^{-M_0 N(1-\frac{1}{\theta})} \sum_{\sigma \in G} \left\|\left\{\sum_{k \in \mathbb{Z}} \sum_{Q \in Q_{cw}^k} \left(2^{k\alpha} \left|\mathbf{E}_k(f)(x_Q)\right|\right)^q \chi_Q(\sigma(\cdot))\right\}^{\frac{1}{q}}\right\|_p \\
&\leq C\, 2^{-M_0 N(1-\frac{1}{\theta})} \left\|\left\{\sum_{k \in \mathbb{Z}} \sum_{Q \in Q_{cw}^k} \left(2^{k\alpha} \left|\mathbf{E}_k(f)(x_Q)\right|\right)^q \chi_Q(\cdot)\right\}^{\frac{1}{q}}\right\|_p
\end{aligned} \tag{3. 25}$$

where the last inequality follows from the fact that G is a finite group and

$$\int_{\mathbb{R}^n} f(\sigma(x)) d\omega(x) = \int_{\mathbb{R}^n} f(x) d\omega(x). \tag{3. 26}$$

Thus,

$$\|\mathbf{S}_q^\alpha(f)\|_p \leq C\, 2^{-M_0 N(1-\frac{1}{\theta})} \|\mathbf{S}_{q,cw}^\alpha(f)\|_p. \tag{3. 27}$$

Step 2: $\|\mathbf{S}_{q,cw}^\alpha(\mathbf{R}_1(f))\|_p \leq C \|\mathbf{R}_1\|_{dcz} \|\mathbf{S}_{q,cw}^\alpha(\mathbf{R}_1(f))\|_p$.

The $\mathbf{L}^2$ boundedness of $\mathbf{R}_1$ together with the discrete Calderón reproducing formula of $f \in \mathbf{L}^2$

on space of homogeneous type given in (3. 8) yields

$$
\begin{aligned}
\|\mathbf{S}^{\alpha}_{q,cw}(\mathbf{R}_1(f))\|_p &= \left\| \left\{ \sum_{k'\in\mathbb{Z}} \sum_{Q'\in Q^{k'}_{cw}} \left(2^{k'\alpha} \left| \mathbf{E}_{k'}(\mathbf{R}_1(f))(x_{Q'}) \right| \right)^q \chi_{Q'}(\cdot) \right\}^{\frac{1}{q}} \right\|_p \\
&= \left\| \left\{ \sum_{k'\in\mathbb{Z}} \sum_{Q'\in Q^{k'}_{cw}} \left(2^{k'\alpha} \left| \sum_{k\in\mathbb{Z}} \sum_{Q\in Q^{k}_{cw}} \omega(Q)(E_{k'}R_1\widetilde{E}_k)(x_{Q'},x_Q)\mathbf{E}_k(f)(x_Q) \right| \right)^q \chi_{Q'}(\cdot) \right\}^{\frac{1}{q}} \right\|_p .
\end{aligned}
\tag{3. 28}
$$

Applying the almost orthogonal estimate given in Lemma 3.2 to $(E_{k'}R_1\widetilde{E}_k)(x_{Q'},x_Q)$, we obtain that for $|\alpha| < \varepsilon < 1$,

$$
\begin{aligned}
&\left|(E_{k'}R_1\widetilde{E}_k)(x_{Q'},x_Q)\right| \chi_{Q'}(x) \\
&\le C\,\|\mathbf{R}_1\|_{dcz} 2^{-|k'-k|\varepsilon} \frac{1}{V(x_{Q'},x_Q,2^{-k'\vee -k}+d(x_{Q'},x_Q))} \left(\frac{2^{-k'\vee -k}}{2^{-k'\vee -k}+d(x_{Q'},x_Q)} \right)^{\varepsilon} \chi_{Q'}(x) \\
&\le C\,\|\mathbf{R}_1\|_{dcz} \sum_{\sigma\in G} 2^{-|k'-k|\varepsilon} \frac{1}{V(x_{Q'},x_Q,2^{-k'\vee -k}+\|\sigma(x_{Q'})-x_Q\|)} \left(\frac{2^{-k'\vee -k}}{2^{-k'\vee -k}+\|\sigma(x_{Q'})-x_Q\|} \right)^{\varepsilon} \chi_{Q'}(x) \\
&\le C\,\|\mathbf{R}_1\|_{dcz} \sum_{\sigma\in G} 2^{-|k'-k|\varepsilon} \frac{1}{\omega(B(\sigma(x),2^{-k'\vee -k}+\|\sigma(x)-x_Q\|))} \left(\frac{2^{-k'\vee -k}}{2^{-k'\vee -k}+\|\sigma(x)-x_Q\|} \right)^{\varepsilon} \chi_{Q'}(x)
\end{aligned}
\tag{3. 29}
$$

where we use the fact that if $x \in Q'$, then $2^{-k\vee -k'} + \|\sigma(x_{Q'}) - x_Q\| \sim 2^{-k\vee -k'} + \|\sigma(x) - x_Q\|$ and $\omega(B(x_{Q'}, 2^{-k\vee -k'} + \|\sigma(x) - x_Q\|))\chi_{Q'}(x) \sim \omega(B(\sigma(x), 2^{-k\vee -k'} + \|\sigma(x) - x_Q\|))\chi_{Q'}(x)$.

Similar to the estimate in **Step 1**, for $\max\left\{\frac{N}{N+\varepsilon}, \frac{N}{N+\varepsilon+\alpha}\right\} < \theta < \min\{p,q,1\}$ we have

$$
\begin{aligned}
&\left| \sum_{Q\in Q^k_{cw}} 2^{k'\alpha}\omega(Q)(E_{k'}R_1\widetilde{E}_k)(x_{Q'},x_Q)\mathbf{E}_k(f)(x_Q) \right| \chi_{Q'}(x) \\
&\le C\,\|\mathbf{R}_1\|_{dcz} \sum_{\sigma\in G} 2^{-M_0N(1-\frac{1}{\theta})} 2^{-|k'-k|\varepsilon} 2^{(-k-(-k\vee -k'))N(1-\frac{1}{\theta})+(k'-k)\alpha} \\
&\quad \times \left\{ \mathbf{M}\left(\sum_{Q\in Q^k_{cw}} \left| 2^{k\alpha}\mathbf{E}_{k'}(f)(x_Q) \right|^{\theta} \chi_Q \right)(\sigma(\cdot)) \right\}^{\frac{1}{\theta}} \chi_{Q'}(x).
\end{aligned}
\tag{3. 30}
$$

For $1 < q < \infty$, by Hölder's inequality, we have

$$\begin{aligned}
&\left|\sum_{k\in\mathbb{Z}}\sum_{Q\in Q^k_{cw}} 2^{k'\alpha}\omega(Q)(E_{k'}R_1\widetilde{E}_k)(x_{Q'},x_Q)\mathbf{E}_k(f)(x_Q)\right|^q \chi_{Q'}(x)\\
&\leq C\,2^{-M_0N(1-\frac{1}{\theta})q}\|\mathbf{R}_1\|^q_{dcz}\sum_{\sigma\in G}\sum_{k\in\mathbb{Z}} 2^{-|k-k'|\varepsilon}2^{(-k-(-k\vee -k'))N(1-\frac{1}{\theta})+(k'-k)\alpha} \\
&\times\left\{\mathbf{M}\left(\sum_{Q\in Q^k_{cw}}\left|2^{k\alpha}\mathbf{E}_k(f)(x_Q)\right|^\theta\chi_Q\right)(\sigma(\cdot))\right\}^{\frac{q}{\theta}}\chi_{Q'}(x).
\end{aligned} \tag{3. 31}$$

For $\max\left\{\frac{N}{N+\varepsilon},\frac{N}{N+\alpha+\varepsilon}\right\} < q \leq 1$, by q-inequality we have

$$\begin{aligned}
&\left|\sum_{k\in\mathbb{Z}}\sum_{Q\in Q^k_{cw}} 2^{k'\alpha}\omega(Q)(E_{k'}R_1\widetilde{E}_k)(x_{Q'},x_Q)\mathbf{E}_k(f)(x_Q)\right|^q \chi_{Q'}(x)\\
&\leq C\,2^{-M_0N(1-\frac{1}{\theta})q}\|\mathbf{R}_1\|^q_{dcz}\sum_{\sigma\in G}\sum_{k\in\mathbb{Z}} 2^{-|k-k'|\varepsilon q}2^{(-k-(-k\vee -k'))N(1-\frac{1}{\theta})q+(k'-k)\alpha q} \\
&\times\left\{\mathbf{M}\left(\sum_{Q\in Q^k_{cw}}\left|2^{k\alpha}\mathbf{E}_k(f)(x_Q)\right|^\theta\chi_Q\right)(\sigma(\cdot))\right\}^{\frac{q}{\theta}}\chi_{Q'}(x).
\end{aligned} \tag{3. 32}$$

Then (3. 31)-(3. 32) implies that

$$\begin{aligned}
&\left\{\sum_{k'\in\mathbb{Z}}\sum_{Q'\in Q^{k'}_{cw}}\left|\sum_{k\in\mathbb{Z}}\sum_{Q\in Q^k_{cw}} 2^{k'\alpha}\omega(Q)(E_{k'}R_1\widetilde{E}_k)(x_{Q'},x_Q)\mathbf{E}_k(f)(x_Q)\right|^q \chi_{Q'}(x)\right\}^{\frac{1}{q}}\\
&\leq C\,2^{-M_0N(1-\frac{1}{\theta})}\|\mathbf{R}_1\|_{dcz}\left\{\sum_{\sigma\in G}\sum_{k\in\mathbb{Z}}\left\{\mathbf{M}\left(\sum_{Q\in Q^k_{cw}}\left|2^{k\alpha}\mathbf{E}_k(f)(x_Q)\right|^\theta\chi_Q\right)(\sigma(\cdot))\right\}^{\frac{q}{\theta}}\right\}^{\frac{1}{q}}.
\end{aligned} \tag{3. 33}$$

The Fefferman-Stein vector valued maximal function inequality with $\theta < \min\{p,q,1\}$ yields

$$\begin{aligned}
\|\mathbf{S}^\alpha_{q,cw}(\mathbf{R}_1(f))\|_p &\leq C\,2^{-M_0N(1-\frac{1}{\theta})}\|\mathbf{R}_1\|_{dcz}\left\|\left\{\sum_{k\in\mathbb{Z}}\sum_{Q\in Q^k_{cw}}\left|2^{k\alpha}\mathbf{E}_k(f)(x_Q)\right|^q\chi_Q(\sigma(\cdot))\right\}^{\frac{1}{q}}\right\|_p \\
&\leq C\,2^{-M_0N(1-\frac{1}{\theta})}\|\mathbf{R}_1\|_{dcz}\|\mathbf{S}^\alpha_{q,cw}(f)\|_p.
\end{aligned} \tag{3. 34}$$

Applying the similar proof, we also have

$$\|\mathbf{S}^\alpha_{q,cw}(\mathbf{R}_2(f))\|_p \leq C\,2^{-M_0N(1-\frac{1}{\theta})}\|\mathbf{R}_2\|_{dcz}\|\mathbf{S}^\alpha_{q,cw}(f)\|_p. \tag{3. 35}$$

Step 3: $\|\mathbf{S}^{\alpha}_{q,cw}(f)\|_p \le C\|\mathbf{S}^{\alpha}_{q}(f)\|_p$.

To show this estimate, the key point is to write

$$f(x) = \mathbf{T}_M(f)(x) + \mathbf{R}_1(f)(x) + \mathbf{R}_2(f)(x). \tag{3. 36}$$

Recall the estimates in **Step 2** for $\max\left\{\frac{N}{N+\varepsilon}, \frac{N}{N+\alpha+\varepsilon}\right\} \le p < \infty$,

$$\|\mathbf{S}^{\alpha}_{q,cw}(\mathbf{R}_1(f))\|_p \le C\, 2^{-M_0 N(1-\frac{1}{\theta})}\|\mathbf{R}_1\|_{dcz}\|\mathbf{S}^{\alpha}_{q,cw}(f)\|_p, \tag{3. 37}$$

and

$$\|\mathbf{S}^{\alpha}_{q,cw}(\mathbf{R}_2(f))\|_p \le C\, 2^{-M_0 N(1-\frac{1}{\theta})}\|\mathbf{R}_2\|_{dcz}\|\mathbf{S}^{\alpha}_{q,cw}(f)\|_p. \tag{3. 38}$$

Since $\|\mathbf{R}_1\|_{dcz} + \|\mathbf{R}_2\|_{dcz} \le C\, 2^{-M\delta}$, $\delta > 0$, and we choose M is sufficiently larger than M_0, then we can set $C\, 2^{-M_0 N(1-\frac{1}{\theta})}(\|\mathbf{R}_1\|_{dcz} + \|\mathbf{R}_2\|_{dcz}) \le \min\left\{\left(\frac{1}{2}\right)^{\frac{1}{p}}, \frac{1}{2}\right\}$, then for $\max\left\{\frac{N}{N+\varepsilon}, \frac{N}{N+\alpha+\varepsilon}\right\} < p \le 1$, by using p-inequality, we have

$$\|\mathbf{S}^{\alpha}_{q,cw}(f)\|_p^p \le \|\mathbf{S}^{\alpha}_{q,cw}(\mathbf{T}_M(f) + \mathbf{R}_1(f) + \mathbf{R}_2(f))\|_p^p \le \|\mathbf{S}^{\alpha}_{q,cw}(\mathbf{T}_M(f))\|_p^p + \frac{1}{2}\|\mathbf{S}^{\alpha}_{q,cw}(f)\|_p^p, \tag{3. 39}$$

and for $1 < p < \infty$, by using Minkowski's inequality we have

$$\begin{aligned} \|\mathbf{S}^{\alpha}_{q,cw}(f)\|_p &\le \|\mathbf{S}^{\alpha}_{q,cw}(\mathbf{T}_M(f))\|_p + \|\mathbf{S}^{\alpha}_{q,cw}(\mathbf{R}_1(f))\|_p + \|\mathbf{S}^{\alpha}_{q,cw}(\mathbf{R}_2(f))\|_p \\ &\le C\, \|\mathbf{S}^{\alpha}_{q,cw}(\mathbf{T}_M(f))\|_p + \frac{1}{2}\|\mathbf{S}^{\alpha}_{q,cw}(f)\|_p. \end{aligned} \tag{3. 40}$$

Hence,

$$\|\mathbf{S}^{\alpha}_{q,cw}(f)\|_p \le C_p\|\mathbf{S}^{\alpha}_{q,cw}(\mathbf{T}_M(f))\|_p. \tag{3. 41}$$

Claim: $\|\mathbf{S}^{\alpha}_{q,cw}(\mathbf{T}_M(f))\|_p \le C_p\|\mathbf{S}^{\alpha}_{q}(f)\|_p$.

Indeed, observing that

$$\mathbf{T}_M(f)(x) = \sum_{k\in\mathbb{Z}}\sum_{Q\in Q_d^k} \omega(Q) D_k^M(x, x_Q)\mathbf{D}_k(f)(x_Q), \tag{3. 42}$$

and

$$\left|\mathbf{E}_{k'}\mathbf{T}_M(f)(x)\right| \le \sum_{k\in\mathbb{Z}}\sum_{Q\in Q_d^k} \omega(Q)\left|(E_{k'}D_k^M)(x, x_Q)\right|\, \left|\mathbf{D}_k(f)(x_Q)\right|. \tag{3. 43}$$

Following the same proof as in **Step 1**, there exists a constant $C > 0$, such that

$$\begin{aligned} &\left|(E_{k'}D_k^M)(x, x_Q)\right| \chi_{Q'}(x) \\ &\le C\sum_{\sigma\in G} 2^{-|k-k'|\varepsilon}\frac{1}{\omega(B(\sigma(x), 2^{-k\vee -k'} + \|\sigma(x) - x_Q\|))}\left(\frac{2^{-k\vee -k'}}{2^{-k\vee -k'} + \|\sigma(x) - x_Q\|}\right)^{\varepsilon} \chi_{Q'}(x). \end{aligned} \tag{3. 44}$$

Therefore for $\max\left\{\frac{N}{N+\varepsilon}, \frac{N}{N+\alpha+\varepsilon}\right\} < \theta < \min\{p, q, 1\}$, we have

$$
\begin{aligned}
&\left|\sum_{Q\in Q_d^k} 2^{k'\alpha}\omega(Q)(E_{k'}D_k^M)(x, x_Q)\mathbf{D}_k(f)(x_Q)\right|\chi_{Q'}(x)\\
&\le C\sum_{\sigma\in G} 2^{-MN(1-\frac{1}{\theta})}2^{-|k-k'|\varepsilon}2^{(-k-(-k'\vee -k))N(1-\frac{1}{\theta})+(k'-k)\alpha}\left\{\mathbf{M}\left(\sum_{Q\in Q_d^k}\left|2^{k\alpha}\mathbf{D}_k(f)(x_Q)\right|^\theta\right)(\sigma(\cdot))\right\}^{\frac{1}{\theta}}\chi_{Q'}(x).
\end{aligned}
\tag{3. 45}
$$

For $1 < q < \infty, \max\{\frac{N}{N+\varepsilon}, \frac{N}{N+\varepsilon+\alpha}\} < \theta < \min\{p, q, 1\}$, by Hölder's inequality, we obtain

$$
\begin{aligned}
&\left|\sum_{k\in\mathbb{Z}}\sum_{Q\in Q_d^k} 2^{k'\alpha}\omega(Q)(E_{k'}D_k^M)(x, x_Q)\mathbf{D}_k(f)(x_Q)\right|^q\chi_{Q'}(x)\\
&\le C\,2^{-MN(1-\frac{1}{\theta})q}\sum_{\sigma\in G}\sum_{k\in\mathbb{Z}} 2^{-|k-k'|\varepsilon}2^{(-k-(-k'\vee -k))N(1-\frac{1}{\theta})+(k'-k)\alpha}\\
&\quad\times\left\{\mathbf{M}\left(\sum_{Q\in Q_d^k}\left|2^{k\alpha}\mathbf{D}_k(f)(x_Q)\right|^\theta\right)(\sigma(\cdot))\right\}^{\frac{q}{\theta}}\chi_{Q'}(x).
\end{aligned}
\tag{3. 46}
$$

For $\max\left\{\frac{N}{N+\varepsilon}, \frac{N}{N+\alpha+\varepsilon}\right\} < q \le 1$, by q-inequaliaty, we have

$$
\begin{aligned}
&\left|\sum_{k\in\mathbb{Z}}\sum_{Q\in Q_d^k} 2^{k'\alpha}\omega(Q)(E_{k'}D_k^M)(x, x_Q)\mathbf{D}_k(f)(x_Q)\right|^q\chi_{Q'}(x)\\
&\le C\,2^{-MN(1-\frac{1}{\theta})q}\sum_{\sigma\in G}\sum_{k\in\mathbb{Z}} 2^{-|k-k'|\varepsilon q}2^{(-k-(-k'\vee -k))N(1-\frac{1}{\theta})q+(k'-k)\alpha q}\\
&\quad\times\left\{\mathbf{M}\left(\sum_{Q\in Q_d^k}\left|2^{k\alpha}\mathbf{D}_k(f)(x_Q)\right|^\theta\right)(\sigma(\cdot))\right\}^{\frac{q}{\theta}}\chi_{Q'}(x).
\end{aligned}
\tag{3. 47}
$$

Thus (3. 46)-(3. 47) implies that

$$
\begin{aligned}
&\left\{\sum_{k'\in\mathbb{Z}}\sum_{Q'\in Q_{cw}^{k'}}\left|\sum_{k\in\mathbb{Z}}\sum_{Q\in Q_d^k} 2^{k'\alpha}\omega(Q)(E_{k'}D_k^M)(x, x_Q)\mathbf{D}_k(f)(x_Q)\right|^q\chi_{Q'}(x)\right\}^{\frac{1}{q}}\\
&\le C\,2^{-MN(1-\frac{1}{\theta})}\left\{\sum_{\sigma\in G}\sum_{k\in\mathbb{Z}}\left\{\mathbf{M}\left(\sum_{Q\in Q_d^k}\left|2^{k\alpha}\mathbf{D}_k(f)(x_Q)\right|^\theta\right)(\sigma(\cdot))\right\}^{\frac{q}{\theta}}\right\}^{\frac{1}{q}}.
\end{aligned}
\tag{3. 48}
$$

The Fefferman-Stein vector valued maximal function inequality with $\theta < \min\{p, q, 1\}$ yields

$$\|\mathbf{S}^{\alpha}_{q,cw}(\mathbf{T}_M(f))\|_p \leq C\, 2^{-MN(1-\frac{1}{\theta})} \left\| \left\{ \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \left(2^{k\alpha} \left|\mathbf{D}_k(f)(x_Q)\right|\right)^q \chi_Q(\cdot) \right\}^{\frac{1}{q}} \right\|_p \leq C\, \|\mathbf{S}^{\alpha}_q(f)\|_p, \quad (3.\ 49)$$

where M is a fixed constant.

The proof of **Step 3** is complete.

Observing that $f(x) = \mathbf{T}_M(f)(x) + \mathbf{R}_1(f)(x) + \mathbf{R}_2(f)(x)$ and applying the above estimates, we have $\|\mathbf{S}^{\alpha}_q(\mathbf{R}_1 + \mathbf{R}_2)(f)\|_p \leq C\,(\|\mathbf{R}_1\|_{dcz} + \|\mathbf{R}_2\|_{dcz})\|\mathbf{S}^{\alpha}_q(f)\|_p$, so

$$\|\mathbf{S}^{\alpha}_q(\mathbf{I} - \mathbf{T}_M)(f)\|_p = C\,\|\mathbf{S}^{\alpha}_q(\mathbf{R}_1 + \mathbf{R}_2)(f)\|_p \leq \frac{1}{2}\,\|\mathbf{S}^{\alpha}_q(f)\|_p. \quad (3.\ 50)$$

Similarly, for $(\mathbf{T}_M)^{-1}(f)$, we also have $\|\mathbf{S}^{\alpha}_q(\mathbf{T}_M(f))^{-1}\|_p \leq C\|\mathbf{S}^{\alpha}_q(f)\|_p$. If there exist a function $h \in \mathbf{L}^2$ and set $h = (\mathbf{T}_M)^{-1} f$, we obtain

$$f(x) = \mathbf{T}_M(h)(x) = \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \omega(Q) D_k^M(x, x_Q)\mathbf{D}_k(h)(x_Q), \quad (3.\ 51)$$

where $\|f\|_2 \sim \|h\|_2$, and $\|f\|_{\dot{F}^{\alpha,q}_p} \sim \|h\|_{\dot{F}^{\alpha,q}_p}$, $|\alpha| < 1$, $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < p < \infty$, $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < q < \infty$.

It remains to show that the series (3. 51) converges in $\mathbf{L}^2 \cap \dot{F}^{\alpha,q}_p$. To this end, we only need to prove

$$\left\| \mathbf{S}^{\alpha}_q \left(\sum_{|k|>m} \sum_{Q\in Q_d^k} \omega(Q) D_k^M(x, x_Q)\mathbf{D}_k(h)(x_Q) \right) \right\|_p \to 0, \quad \text{as} \quad m \to \infty. \quad (3.\ 52)$$

Repeating the same proof in **Step 1**,

$$\left\| \mathbf{S}^{\alpha}_q \left(\sum_{|k|>m} \sum_{Q\in Q_d^k} \omega(Q) D_k^M(x, x_Q)\mathbf{D}_k(h)(x_Q) \right) \right\| \leq C \left\| \left\{ \sum_{|k|>m} \sum_{Q\in Q_d^k} \left(2^{k\alpha} \left|\mathbf{D}_k(h)(x_Q)\right|\right)^q \chi_Q(x) \right\}^{\frac{1}{q}} \right\| \quad (3.\ 53)$$

where by the fact $\|h\|_{\dot{F}^{\alpha,q}_p} \sim \|f\|_{\dot{F}^{\alpha,q}_p}$, the last term tends to 0 as $m \to \infty$.

The proof of Theorem 2.1 is complete.

4 Duality estimates

The duality estimates will be a key idea in developing the Dunkl-Triebel-Lizorkin space theory. Now we show Theorem 2.2.

Proof of Theorem 2.2 (A):

Applying the weak-type discrete Calderón-type reproducing formula given in Theorem 2.1 for $f \in \mathbf{L}^2 \cap \dot{F}_p^{\alpha,q}$, we write

$$f(x) = \sum_{k \in \mathbb{Z}} \sum_{Q \in Q_d^k} \omega(Q) D_k^M(x, x_Q) \mathbf{D}_k(h)(x_Q), \tag{4. 1}$$

where $\|h\|_2 \sim \|f\|_2$, and $\|\mathbf{S}_q^\alpha(h)\|_p \sim \|\mathbf{S}_q^\alpha(f)\|_p$.

For $1 < q < \infty$, by Hölder's inequality

$$\begin{aligned}
\left|\langle f, g\rangle\right| &= \left|\left\langle \sum_{k \in \mathbb{Z}} \sum_{Q \in Q_d^k} \omega(Q) D_k^M(\cdot, x_Q) \mathbf{D}_k(h)(x_Q), g(\cdot)\right\rangle\right| \\
&\leq \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} \sum_{Q \in Q_d^k} 2^{k\alpha} \left|\mathbf{D}_k(h)(x_Q)\right| 2^{-k\alpha} \left|\mathbf{D}_k^M(g)(x_Q)\right| \chi_Q(x) d\omega(x) \\
&\leq C \int_{\mathbb{R}^n} \left\{ \sum_{k \in \mathbb{Z}} \sum_{Q \in Q_d^k} \left(2^{k\alpha} \left|\mathbf{D}_k(h)(x_Q)\right|\right)^q \chi_Q(x) \right\}^{\frac{1}{q}} \\
&\quad \times \left\{ \sum_{k \in \mathbb{Z}} \sum_{Q \in Q_d^k} \left(2^{-k\alpha} \left|\mathbf{D}_k^M(g)(x_Q)\right|\right)^{q'} \chi_Q(x) \right\}^{\frac{1}{q'}} d\omega(x) \\
&\leq C \left\{ \int_{\mathbb{R}^n} \left|\mathbf{S}_q^\alpha(h)(x)\right|^p d\omega(x) \right\}^{\frac{1}{p}} \left\{ \int_{\mathbb{R}^n} \left|\mathbf{S}_{q'}^{-\alpha}(g)(x)\right|^{p'} d\omega(x) \right\}^{\frac{1}{p'}} \\
&\leq C \|f\|_{\dot{F}_p^{\alpha,q}} \|g\|_{\dot{F}_{p'}^{-\alpha,q'}}.
\end{aligned} \tag{4. 2}$$

Proof of Theorem 2.2 (B):

Set $\Omega_\ell = \left\{x \in \mathbb{R}^n : \mathbf{S}_q^\alpha(h)(x) > 2^\ell\right\}$, and $B_\ell = \left\{Q : \omega(Q \cap \Omega_\ell) > \frac{1}{2}\omega(Q) \text{ and } \omega(Q \cap \Omega_{\ell+1}) \leq \frac{1}{2}\omega(Q)\right\}$ where Q are dyadic cubes. Denote $B_\ell^* := \left\{Q_\ell^*\right\}$ as the maximal dyadic cubes in B_ℓ for $\ell \in \mathbb{Z}$.

Claim: the Calderón reproducing formula can be rewrite as:

$$f(x) = \sum_{\ell \in \mathbb{Z}} \sum_{Q_\ell^* \in B_\ell^*} \sum_{Q \subset Q_\ell^*} \omega(Q) D_k^M(x, x_Q) \mathbf{D}_k(h)(x_Q). \tag{4. 3}$$

In order to prove the above claim, we only need to show that if the dyadic cube $Q \notin B_\ell$ for all $\ell \in \mathbb{Z}$, then

$$\omega(Q) D_k^M(x, x_Q) \mathbf{D}_k(h)(x_Q) = 0. \tag{4. 4}$$

Observe that by the stopping time argument, each dyadic cube Q can be in one and only one B_ℓ, that is, if Q belongs to both B_ℓ and $B_{\ell'}$, then $\ell = \ell'$. We now assume that $\omega(Q) \neq 0$. Otherwise, the equality (4. 4) holds obviously. Note that $\omega(\Omega_\ell) < 2^{-2\ell}\|\mathbf{S}_q^\alpha(h)\|_2^2 \to 0$ as

$\ell \to +\infty$. As a consequence, if $Q \notin B_\ell$ for all $\ell \in \mathbb{Z}$, then $\omega(Q \cap \Omega_\ell) \le \frac{1}{2}\omega(Q)$ for all $\ell \in \mathbb{Z}$ since, otherwise, there exists an $\ell_0 \in \mathbb{Z}$, such that $\omega(Q \cap \Omega_{\ell_0}) > \frac{1}{2}\omega(Q)$. However, $\omega(Q \cap \Omega_\ell) \to 0$ as $\ell \to +\infty$ and $\{\omega(Q \cap \Omega_\ell)\}_\ell$ is a decreasing sequence. So there must be a critical index ℓ_1 such that $\omega(Q \cap \Omega_{\ell_1}) > \frac{1}{2}\omega(Q)$ and $\omega(Q \cap \Omega_{\ell_1+1}) \le \frac{1}{2}\omega(Q)$, that is $Q \in B_{\ell_1}$. This is contradict to the fact that Q is not in B_ℓ for all $\ell \in \mathbb{Z}$.

Since $\omega(Q \cap \Omega_\ell) \le \frac{1}{2}\omega(Q)$ for all $\ell \in \mathbb{Z}$, then $\omega(Q \cap \Omega_\ell^c) \ge \frac{1}{2}\omega(Q)$ for all $\ell \in \mathbb{Z}$. Set $K = \left\{x \in \mathbb{R}^n, \mathbf{S}_q^\alpha(h)(x) = 0\right\}$. Note that $\cap_{\ell\in\mathbb{Z}}\Omega_\ell^c = \cap_{\ell\in\mathbb{Z}}\left\{x \in \mathbb{R}^n : \mathbf{S}_q^\alpha(h)(x) \le 2^\ell\right\} = K$. Thus

$$\omega(Q \cap K) = \lim_{\ell\to-\infty} \omega(Q \cap \Omega_\ell^c) \ge \frac{1}{2}\omega(Q) > 0 \tag{4. 5}$$

for all $x \in K$, $0 = \mathbf{S}_q^\alpha(h)(x) = \left\{\sum_{k\in\mathbb{Z}}\sum_{Q\in Q_d^k}\left(2^{k\alpha}\left|\mathbf{D}_k(h)(x_Q)\right|\right)^q \chi_Q(x)\right\}^{\frac{1}{q}}$, then $\left|\mathbf{D}_k(h)(x_Q)\right| = 0$. Hence the claim is proved.

Then by Hölder's inequality and p-inequality with $p \le 1$, we have

$$\begin{aligned}
\left|\langle f, g\rangle\right| &= \left|\sum_{\ell\in\mathbb{Z}}\sum_{Q_\ell^*\in B_\ell^*}\sum_{Q\subset Q_\ell^*} \omega(Q)^{\frac{1}{q}}2^{k\alpha}\mathbf{D}_k(h)(x_Q)\omega(Q)^{\frac{1}{q'}}2^{-k\alpha}\mathbf{D}_k^M(g)(x_Q)\right| \\
&\le \sum_{\ell\in\mathbb{Z}}\sum_{Q_\ell^*\in B_\ell^*}\left(\sum_{Q\subset Q_\ell^*}\omega(Q)\left|2^{k\alpha}\mathbf{D}_k(h)(x_Q)\right|^q\right)^{\frac{1}{q}}\left(\sum_{Q\subset Q_\ell^*}\omega(Q)\left|2^{-k\alpha}\mathbf{D}_k^M(g)(x_Q)\right|^{q'}\right)^{\frac{1}{q'}} \\
&\le \left\{\sum_{\ell\in\mathbb{Z}}\sum_{Q_\ell^*\in B_\ell^*}\left(\sum_{Q\subset Q_\ell^*}\omega(Q)\left|2^{k\alpha}\mathbf{D}_k(h)(x_Q)\right|^q\right)^{\frac{p}{q}}\left(\sum_{Q\subset Q_\ell^*}\omega(Q)\left|2^{-k\alpha}\mathbf{D}_k^M(g)(x_Q)\right|^{q'}\right)^{\frac{p}{q'}}\right\}^{\frac{1}{p}} \\
&= \left\{\sum_{\ell\in\mathbb{Z}}\sum_{Q_\ell^*\in B_\ell^*}\left(\sum_{Q\subset Q_\ell^*}\omega(Q)\left|2^{k\alpha}\mathbf{D}_k(h)(x_Q)\right|^q\right)^{\frac{p}{q}}\right. \qquad (4.\ 6) \\
&\qquad \left.\times\,\omega(Q_\ell^*)^{1-\frac{p}{q}}\left(\frac{1}{\omega(Q_\ell^*)^{\frac{q'}{p}-\frac{q'}{q}}}\sum_{Q\subset Q_\ell^*}\omega(Q)\left|2^{-k\alpha}\mathbf{D}_k^M(g)(x_Q)\right|^{q'}\right)^{\frac{p}{q'}}\right\}^{\frac{1}{p}} \\
&\le \left\{\sum_{\ell\in\mathbb{Z}}\sum_{Q_\ell^*\in B_\ell^*}\left(\sum_{Q\subset Q_\ell^*}\omega(Q)\left|2^{k\alpha}\mathbf{D}_k(h)(x_Q)\right|^q\right)^{\frac{p}{q}}\omega(Q_\ell^*)^{1-\frac{p}{q}}\right\}^{\frac{1}{p}}\|g\|_{CMO_p^{-\alpha,q'}} \\
&\le \left\{\sum_{\ell\in\mathbb{Z}}\left\{\sum_{Q_\ell^*\in B_\ell^*}\sum_{Q\subset Q_\ell^*}\omega(Q)\left|2^{k\alpha}\mathbf{D}_k(h)(x_Q)\right|^q\right\}^{\frac{p}{q}}\left\{\sum_{Q_\ell^*\in B_\ell^*}\omega(Q_\ell^*)\right\}^{1-\frac{p}{q}}\right\}^{\frac{1}{p}}\|g\|_{CMO_p^{-\alpha,q'}}.
\end{aligned}$$

To estimate the term $\left\{\sum_{Q_\ell^*\in B_\ell^*}\omega(Q_\ell^*)\right\}^{1-\frac{p}{q}}$. Set $\widetilde{\Omega}_\ell = \left\{x\in\mathbb{R}^n : \mathbf{M}(\chi_{\Omega_\ell})(x) > \frac{1}{2}\right\}$, where $\mathbf{M}$ is the Hardy-Littlewood maximal function on $\mathbb{R}^n$ with the measure $d\omega$ and $\chi_{\Omega_\ell}(x)$ is the indicate function of Ω_ℓ. It is easily to see that if $Q\in B_\ell$, then $Q\subset\widetilde{\Omega}_\ell$. Since all Q_ℓ^* are disjoint, thus

$$\left\{\sum_{Q_\ell^*\in B_\ell^*}\omega(Q_\ell^*)\right\}^{1-\frac{p}{q}} \leq C\,\omega(\widetilde{\Omega}_\ell)^{1-\frac{p}{q}} \leq C\,\omega(\Omega_\ell)^{1-\frac{p}{q}}, \tag{4. 7}$$

where the first inequality follows from the fact that $\cup_{Q_\ell^*\in B_\ell^*}Q_\ell^*\subset\widetilde{\Omega}_\ell$ and $\sum_{Q_\ell^*\in B_\ell^*}\omega(Q_\ell^*)\leq\omega(\widetilde{\Omega}_\ell)$, and by the $\mathbf{L}^2$-boundedness of the Hardy-Littlewood maximal function, the last inequality follows from the estimate $\omega(\widetilde{\Omega}_\ell)\leq C\omega(\Omega_\ell)$.

We claim that

$$\sum_{Q_\ell^*\in B_\ell^*}\sum_{Q\subset Q_\ell^*}\omega(Q)\left|2^{k\alpha}\mathbf{D}_k(h)(x_Q)\right|^q \leq C\,2^{q\ell}\omega(\Omega_\ell). \tag{4. 8}$$

Under this claim (4. 8), we get

$$\begin{aligned}
\left|\langle f,g\rangle\right| &\leq C\left(\sum_{\ell\in\mathbb{Z}}\left(2^{q\ell}\omega(\Omega_\ell)\right)^{\frac{p}{q}}\omega(\Omega_\ell)^{1-\frac{p}{q}}\right)^{\frac{1}{p}}\|g\|_{CMO_p^{-\alpha,q'}}\\
&\leq C\left(\sum_{\ell\in\mathbb{Z}}2^{p\ell}\omega(\Omega_\ell)\right)^{\frac{1}{p}}\|g\|_{CMO_p^{-\alpha,q'}} \qquad (4.\ 9)\\
&\leq C\|\mathbf{S}_q^\alpha(h)\|_p\|g\|_{CMO_p^{-\alpha,q'}}\\
&\leq C\|f\|_{\dot{F}_p^{\alpha,q}}\|g\|_{CMO_p^{-\alpha,q'}}.
\end{aligned}$$

It remains to show the claim (4. 8). In order to do that, we begin with the following estimate

$$\int_{\widetilde{\Omega}_\ell\setminus\Omega_{\ell+1}}\mathbf{S}_q^\alpha(h)(x)^q d\omega(x) \leq C\,2^{q\ell}\omega(\widetilde{\Omega}_\ell) \leq C\,2^{q\ell}\omega(\Omega_\ell). \tag{4. 10}$$

Note that

$$\int_{\widetilde{\Omega}_\ell\setminus\Omega_{\ell+1}}\left|\mathbf{S}_q^\alpha(h)(x)\right|^q d\omega(x) \geq \sum_{Q\in B_\ell}\left(2^{k\alpha}\left|\mathbf{D}_k(h)(x_Q)\right|\right)^q\omega\left((\widetilde{\Omega}_\ell\setminus\Omega_{\ell+1})\cap Q\right). \tag{4. 11}$$

Since for each $Q\in B_\ell$ implies $Q\subseteq\widetilde{\Omega}_\ell$ and $\Omega_{\ell+1}\subset\Omega_\ell$. Thus

$$\omega\left((\widetilde{\Omega}_\ell\setminus\Omega_{\ell+1})\cap Q\right) = \omega(Q)-\omega(\Omega_{\ell+1}\cap Q) \geq \frac{1}{2}\omega(Q). \tag{4. 12}$$

Therefore,

$$\int_{\widetilde{\Omega}_\ell\setminus\Omega_{\ell+1}}\left|\mathbf{S}_q^\alpha(h)(x)\right|^q d\omega(x) \geq C\sum_{Q\in B_\ell}\omega(Q)\left(2^{k\alpha}\left|\mathbf{D}_k(h)(x_Q)\right|\right)^q. \tag{4. 13}$$

This implies the claim (4. 8). The prove of Theorem 2.2 (B) is complete.

Proof of Theorem 2.2 (C):

Recall Definition 2.3, for $g \in \mathbf{L}^2(\mathbb{R}^n, \omega)$, the norm of $g \in \dot{F}_p^{\alpha,\infty}(\mathbb{R}^n, \omega)$ is defined by

$$\|f\|_{\dot{F}_p^{\alpha,\infty}} = \left\| \sup_{k\in\mathbb{Z}, Q\in Q_d^k} 2^{k\alpha} \left| \mathbf{D}_k^M(f)(x_Q) \right| \chi_Q(x) \right\|_{\mathbf{L}^p(\mathbb{R}^n,\omega)}. \tag{4. 14}$$

By Hölder's inequality, we have

$$\begin{aligned}\left|\langle f, g\rangle\right| &= \left| \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \omega(Q) \mathbf{D}_k^M(g)(x_Q) \mathbf{D}_k(h)(x_Q) \right| \\ &\leq C \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} 2^{k\alpha} \left| \mathbf{D}_k(h)(x_Q) \right| \chi_Q(x) \mathbf{S}_\infty^{-\alpha}(g)(x) d\omega(x) \\ &\leq C \, \|\mathbf{S}_1^\alpha(h)\|_p \, \|\mathbf{S}_\infty^{-\alpha}(g)\|_{p'} \\ &\leq C \, \|f\|_{\dot{F}_p^{\alpha,1}} \, \|g\|_{\dot{F}_{p'}^{-\alpha,\infty}}.\end{aligned} \tag{4. 15}$$

To get Theorem 2.2 (C), it suffices to show $\|f\|_{\dot{F}_p^{\alpha,1}} \leq C \, \|f\|_{\dot{F}_p^{\alpha,q}}$.

According to the q-inequality with $q \leq 1$, we find

$$\begin{aligned}\|f\|_{\dot{F}_p^{\alpha,1}} &= C \left\{ \int_{\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} 2^{k\alpha} \left| \mathbf{D}_k(h)(x_Q) \right| \chi_Q(x) \right)^p d\omega(x) \right\}^{\frac{1}{p}} \\ &\leq C \left\{ \int_{\mathbb{R}^n} \left| \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \left(2^{k\alpha} \left| \mathbf{D}_k(h)(x_Q) \right| \right)^q \chi_Q(x) \right|^{\frac{p}{q}} d\omega(x) \right\}^{\frac{1}{p}} \\ &\leq C \, \|\mathbf{S}_q^\alpha(h)\|_p \leq C \, \|f\|_{\dot{F}_p^{\alpha,q}}.\end{aligned} \tag{4. 16}$$

As a result,

$$\left|\langle f, g\rangle\right| \leq C\|f\|_{\dot{F}_p^{\alpha,q}} \, \|g\|_{\dot{F}_{p'}^{-\alpha,\infty}}. \tag{4. 17}$$

Proof of Theorem 2.2 (D):

Recall Definition 2.4, for $g \in \mathbf{L}^2(\mathbb{R}^n, \omega)$, the norm of $g \in \dot{F}_\infty^{\alpha,\infty}(\mathbb{R}^n, \omega)$ is defined by

$$\|g\|_{\dot{F}_\infty^{\alpha,\infty}} = \sup_{k\in\mathbb{Z}, Q\in Q_d^k} 2^{k\alpha} \left| \mathbf{D}_k^M(g)(x_Q) \right|. \tag{4. 18}$$

Denote $\beta = -\alpha + N(\frac{1}{p} - 1)$, then we have

$$\begin{aligned}
\left|\langle f, g\rangle\right| &= \left|\sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \omega(Q)\mathbf{D}_k^M(g)(x_Q)\mathbf{D}_k(h)(x_Q)\right| \\
&\le C \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} 2^{-k\beta} \left|\mathbf{D}_k(h)(x_Q)\right| \chi_Q(x) d\omega(x) \sup_{k\in\mathbb{Z}, Q\in Q_d^k} 2^{k\beta} \left|\mathbf{D}_k^M(g)(x_Q)\right| \qquad (4.\ 19) \\
&= C \, \|S_1^{-\beta}(h)\|_1 \, \|g\|_{\dot{F}_\infty^{\beta,\infty}} \\
&\le C \, \|f\|_{\dot{F}_1^{\alpha-N(\frac{1}{p}-1),1}} \, \|g\|_{\dot{F}_\infty^{-\alpha+N(\frac{1}{p}-1),\infty}}.
\end{aligned}$$

Recall that $\omega(B(x,r)) \ge C\, r^N$ deduced by (1. 9). Since $p \le 1$, and $\ell(Q) = 2^{-k-M}$, where M is a fixed constant, then $2^{-kN(\frac{1}{p}-1)} \le C\omega(B(x, 2^{-k}))^{\frac{1}{p}-1} \sim C\, \omega(Q)^{\frac{1}{p}-1}$. Similar to the proof of Theorem 2.2 (C),it suffices to show $\|f\|_{\dot{F}_1^{\alpha-N(\frac{1}{p}-1),1}} \le C \, \|f\|_{\dot{F}_p^{\alpha,q}}$. By Minkowski's inequality, and p-inequality with $p \le 1$, we find

$$\begin{aligned}
\|f\|_{\dot{F}_1^{\alpha-N(\frac{1}{p}-1),1}} &= \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} 2^{k(\alpha-N(\frac{1}{p}-1))} \left|\mathbf{D}_k(f)(x_Q)\right| \chi_Q(x) d\omega(x) \\
&\le C \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} 2^{k\alpha} \omega(Q)^{\frac{1}{p}-1} \left|\mathbf{D}_k(f)(x_Q)\right| \omega(Q) \qquad (4.\ 20) \\
&= C \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} 2^{k\alpha} \left|\mathbf{D}_k(f)(x_Q)\right| \omega(Q)^{\frac{1}{p}}
\end{aligned}$$

If $\frac{p}{q} \ge 1$, applying Minkowski's inequality and p-inequality again, the right-hand side of (4.20) is further controlled by

$$\begin{aligned}
&\left(\sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} 2^{k\alpha p} \left|\mathbf{D}_k(f)(x_Q)\right|^p \omega(Q)\right)^{\frac{1}{p}} \\
&\le \left\{\int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \left(2^{k\alpha q} \left|\mathbf{D}_k(f)(x_Q)\right|^q \chi_Q(x)\right)^{\frac{p}{q}} d\omega(x)\right\}^{\frac{1}{p}} \qquad (4.\ 21) \\
&\le \left\{\int_{\mathbb{R}^n} \left\{\sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} 2^{k\alpha q} \left|\mathbf{D}_k(f)(x_Q)\right|^q \chi_Q(x)\right\}^{\frac{p}{q}} d\omega(x)\right\}^{\frac{1}{p}} \\
&= C \, \|\mathbf{S}_q^\alpha(f)\|_p = C \, \|f\|_{\dot{F}_p^{\alpha,q}}.
\end{aligned}$$

If $\frac{p}{q} < 1$, applying q-inequality and the Minkowski's inequality again, the estimates of right-hand side of (4.20) are following

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}\sum_{Q\in Q_d^k} 2^{k\alpha}\left|\mathbf{D}_k(f)(x_Q)\right|\omega(Q)^{\frac{1}{p}}\\
&=\sum_{k\in\mathbb{Z}}\sum_{Q\in Q_d^k}\left(2^{k\alpha p}\left|\mathbf{D}_k(f)(x_Q)\right|^p\omega(Q)\right)^{\frac{1}{p}}\\
&\le\left\{\sum_{k\in\mathbb{Z}}\sum_{Q\in Q_d^k}\left(2^{k\alpha p}\left|\mathbf{D}_k(f)(x_Q)\right|^p\omega(Q)\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}}\\
&=\left\{\left\{\sum_{k\in\mathbb{Z}}\sum_{Q\in Q_d^k}\left(\int_{\mathbb{R}^n}2^{k\alpha p}\left|\mathbf{D}_k(f)(x_Q)\right|^p\chi_Q(x)d\omega(x)\right)^{\frac{q}{p}}\right\}^{\frac{p}{q}}\right\}^{\frac{1}{p}} \qquad (4.\ 22)\\
&\le\left\{\int_{\mathbb{R}^n}\left\{\sum_{k\in\mathbb{Z}}\sum_{Q\in Q_d^k}2^{k\alpha q}\left|\mathbf{D}_k(f)(x_Q)\right|^q\chi_Q(x)\right\}^{\frac{p}{q}}d\omega(x)\right\}^{\frac{1}{p}}\\
&=\|\mathbf{S}_q^\alpha(f)\|_p=\|f\|_{\dot{F}_p^{\alpha,q}}.
\end{aligned}$$

The proof of Theorem 2.2 (D) is complete.

The Theorem 2.2 (A) indicates that if $\{f_n\}_{n=1}^{\infty}$ is a sequence in $\mathbf{L}^2$ with $\|\mathbf{S}_q^\alpha(f_n - f_m)\|_p \to 0$ as $n, m \to \infty$. Then for each $g \in \mathbf{L}^2$ with $\|g\|_{\dot{F}_{p'}^{-\alpha,q'}} < \infty$, $1 < p < \infty$, $1 < q < \infty$, we have $\lim\limits_{n,m\to\infty}\langle f_n - f_m, g\rangle = 0$. Therefore, there exists f, as a distribution on $\mathbf{L}^2 \cap \dot{F}_{p'}^{-\alpha,q'}$, such that for each $g \in \mathbf{L}^2$ with $\|g\|_{\dot{F}_{p'}^{-\alpha,q'}} < \infty$,

$$\langle f, g\rangle = \lim_{n\to\infty}\langle f_n, g\rangle. \qquad (4.\ 23)$$

Other ranges of p, q stated above have the same results.

5 Dunkl-Triebel-Lizorkin space

In this section, we define the Dunkl-Triebel-Lizorkin spaces, and show the spaces are complete. Before introducing the Dunkl-Triebel-Lizorkin space, we need the following

Lemma 5.1.

(A) For $1 < p < \infty$, $1 < q < \infty$. Then $D_k(\cdot, y)$ is in $\mathbf{L}^2 \cap \dot{F}_{p'}^{-\alpha,q'}$ for any fixed k and $y \in \mathbb{R}^n$. Moreover

$$\|S_{q'}^{-\alpha}\left(D_k(\cdot, y)\right)\|_{p'} \le C, \qquad (5.\ 1)$$

where the constant C only depends on k.

(B) *For* $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < p \le 1$, $1 < q < \infty$. *Then* $D_k(\cdot, y)$ *is in* $\mathbf{L}^2 \cap CMO_p^{-\alpha,q'}$ *for any fixed k and* $y \in \mathbb{R}^n$. *Moreover*

$$\sup_P \left(\frac{1}{\omega(P)^{\frac{q'}{p}-\frac{q'}{q}}} \sum_{Q\subset P} \omega(Q) \left(2^{-j\alpha} \left|\boldsymbol{D}_j(D_k(\cdot, y))(x_Q)\right|\right)^{q'} \right)^{\frac{1}{q'}} \le C \tag{5. 2}$$

where both P and Q are dyadic cubes on $\mathbb{R}^n$ *and the constant C which depends on k but is independent of y.*

(C) *For* $1 < p < \infty$, $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < q \le 1$. *Then* $D_k(\cdot, y)$ *is in* $\mathbf{L}^2 \cap \dot{F}_{p'}^{-\alpha,\infty}$ *for any fixed k and* $y \in \mathbb{R}^n$. *Moreover*

$$\|\boldsymbol{S}_\infty^{-\alpha}(D_k(\cdot, y))\|_{p'} = \left\| \sup_{j\in\mathbb{Z}, Q\in Q_d^j} 2^{-j\alpha} \left|\boldsymbol{D}_j(D_k(\cdot, y))(x_Q)\right| \chi_Q(x) \right\|_{p'} < C. \tag{5. 3}$$

where the constant C only depends on k.

(D) *For* $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < p \le 1$, $\max\left\{\frac{N}{N+1}, \frac{N}{N+\alpha+1}\right\} < q \le 1$. *Then* $D_k(\cdot, y)$ *is in* $\mathbf{L}^2 \cap \dot{F}_\infty^{-\alpha+N(\frac{1}{p}-1),\infty}$ *for any fixed k and* $y \in \mathbb{R}^n$. *Moreover*

$$\sup_{j\in\mathbb{Z}, Q\in Q_d^j} 2^{(-\alpha+N(\frac{1}{p}-1))j} \left|\boldsymbol{D}_j(D_k(\cdot, y))(x_Q)\right| \le C. \tag{5. 4}$$

Proof of Lemma 5.1 (A):

According to Lemma 3.1, we choose $|\alpha| < \varepsilon < 1$ such that

$$\left|\mathbf{D}_j(D_k(\cdot, y))(x_Q)\right| \le C\, 2^{-|j-k|\varepsilon} \frac{1}{V(x_Q, y, 2^{-j\vee -k} + d(x_Q, y))} \left(\frac{2^{-j\vee -k}}{2^{-j\vee -k} + \|x_Q - y\|}\right)^{\varepsilon}. \tag{5. 5}$$

Then,

$$\|\mathbf{S}_{q'}^{-\alpha}\,(D_k(\cdot, y))\,\|_{p'}$$

$$= \left\{ \int_{R^n} \left\{ \sum_{j\in\mathbb{Z}} \sum_{Q\in Q_d^j} \left(2^{-j\alpha} \left|\mathbf{D}_j\,(D_k(\cdot, y))\,(x_Q)\right| \chi_Q(x)\right)^{q'} \right\}^{\frac{p'}{q'}} d\omega(x) \right\}^{\frac{1}{p'}}$$

$$\le C \left\{ \int_{R^n} \left\{ \sum_{j\in\mathbb{Z}} \left(\sum_{Q\in Q_d^j} 2^{-j\alpha} 2^{-|j-k|\varepsilon} \frac{1}{\omega(B(x, 2^{-j\vee -k} + d(x, y)))} \left(\frac{2^{-j\vee -k}}{2^{-j\vee -k} + \|x - y\|}\right)^{\varepsilon} \chi_Q(x) \right)^{q'} \right\}^{\frac{p'}{q'}} d\omega(x) \right\}^{\frac{1}{p'}} \tag{5. 6}$$

where the inequality also follows from the fact that $q' > 1$ and $x \in Q$.

If $\frac{p'}{q'} \geq 1$, by Minkowski's inequality, we have

$$\begin{aligned}
&\|\mathbf{S}_{q'}^{-\alpha}(D_k(\cdot, y))\|_{p'} \\
&\leq C \left\{ \sum_{j\in\mathbb{Z}} \left\{ \int_{\mathbb{R}^n} \left(2^{-j\alpha} 2^{-|j-k|\varepsilon} \frac{1}{\omega(B(x, 2^{-j\vee -k} + d(x,y)))} \left(\frac{2^{-j\vee -k}}{2^{-j\vee -k} + \|x-y\|} \right)^{\varepsilon} \right)^{p'} d\omega(x) \right\}^{\frac{q'}{p'}} \right\}^{\frac{1}{q'}} \qquad (5.\ 7) \\
&\leq C \sum_{j\in\mathbb{Z}} 2^{-j\alpha} 2^{-|j-k|\varepsilon} \left\{ \int_{\mathbb{R}^n} \left(\frac{1}{\omega\left(x, 2^{-j\vee -k} + d(x,y)\right)} \left(\frac{2^{-j\vee -k}}{2^{-j\vee -k} + \|x-y\|} \right)^{\varepsilon} \right)^{p'} d\omega(x) \right\}^{\frac{1}{p'}} .
\end{aligned}$$

If $\frac{p'}{q'} < 1$, applying Minkowski's inequality again, and $\frac{1}{p'}$-inequality with $p' > 1$, we have

$$\begin{aligned}
&\|\mathbf{S}_{q'}^{-\alpha}(D_k(\cdot, y))\|_{p'} \\
&\leq C \left\{ \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} \left(2^{-j\alpha} 2^{-|j-k|\varepsilon} \frac{1}{\omega(B(x, 2^{-j\vee -k} + d(x,y)))} \left(\frac{2^{-j\vee -k}}{2^{-j\vee -k} + \|x-y\|} \right)^{\varepsilon} \right)^{p'} d\omega(x) \right\}^{\frac{1}{p'}} \\
&\leq C \left\{ \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} \left(2^{-j\alpha} 2^{-|j-k|\varepsilon} \frac{1}{\omega(B(x, 2^{-j\vee -k} + d(x,y)))} \left(\frac{2^{-j\vee -k}}{2^{-j\vee -k} + \|x-y\|} \right)^{\varepsilon} \right)^{p'} d\omega(x) \right\}^{\frac{1}{p'}} \qquad (5.\ 8) \\
&\leq C \sum_{j\in\mathbb{Z}} 2^{-j\alpha} 2^{-|j-k|\varepsilon} \left\{ \int_{\mathbb{R}^n} \left(\frac{1}{\omega(B(x, 2^{-j\vee -k} + d(x,y)))} \left(\frac{2^{-j\vee -k}}{2^{-j\vee -k} + \|x-y\|} \right)^{\varepsilon} \right)^{p'} d\omega(x) \right\}^{\frac{1}{p'}} .
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|\mathbf{S}_{q'}^{-\alpha}\left(D_k(\cdot, y)\right)\|_{p'} \\
&\leq C \sum_{j\in\mathbb{Z}} 2^{-j\alpha} 2^{-|j-k|\varepsilon} \left\{ \int_{\mathbb{R}^n} \left(\frac{1}{\omega(B(x, 2^{-j\vee -k} + d(x,y)))} \left(\frac{2^{-j\vee -k}}{2^{-j\vee -k} + \|x-y\|} \right)^{\varepsilon} \right)^{p'} d\omega(x) \right\}^{\frac{1}{p'}} . \qquad (5.\ 9)
\end{aligned}$$

For $j \geq k$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(\frac{1}{\omega(B(x, 2^{-j\vee -k} + d(x,y)))} \left(\frac{2^{-j\vee -k}}{2^{-j\vee -k} + \|x-y\|} \right)^{\varepsilon} \right)^{p'} d\omega(x) \\
&\leq C \int_{d(x,y)<2^{-k}} \left(\frac{1}{\omega(B(x, 2^{-j\vee -k} + d(x,y)))} \left(\frac{2^{-j\vee -k}}{2^{-j\vee -k} + \|x-y\|} \right)^{\varepsilon} \right)^{p'} d\omega(x) \qquad (5.\ 10) \\
&\quad + \sum_{m=1}^{\infty} \int_{2^{-k+(m-1)} \leq d(x,y) < 2^{-k+m}} \left(\frac{1}{\omega(B(x, 2^{-j\vee -k} + d(x,y)))} \left(\frac{2^{-j\vee -k}}{2^{-j\vee -k} + \|x-y\|} \right)^{\varepsilon} \right)^{p'} d\omega(x).
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_{d(x,y)<2^{-k}} \left(\frac{1}{\omega(B(x, 2^{-j\vee -k} + d(x,y)))} \left(\frac{2^{-j\vee -k}}{2^{-j\vee -k} + \|x-y\|} \right)^{\varepsilon} \right)^{p'} d\omega(x) \\
&\leq C \sum_{\sigma\in G} \int_{\|\sigma(y)-x\|<2^{-k}} \frac{1}{\omega(B(x,2^{-k}))^{p'}} d\omega(x) \\
&\leq C \sum_{\sigma\in G} \int_{\|\sigma(y)-x\|<2^{-k}} \frac{1}{\omega(B(\sigma(y),2^{-k}))^{p'}} d\omega(x) \\
&\leq C\, \omega(B(\sigma(y),2^{-k}))^{1-p'} \\
&\leq C\, 2^{-kN(1-p')}.
\end{aligned} \tag{5. 11}$$

And for $m \geq 1$,

$$\begin{aligned}
&\int_{2^{-k+(m-1)}\leq d(x,y)<2^{-k+m}} \left(\frac{1}{\omega(B(x, 2^{-j\vee -k} + d(x,y)))} \left(\frac{2^{-j\vee -k}}{2^{-j\vee -k} + \|x-y\|} \right)^{\varepsilon} \right)^{p'} d\omega(x) \\
&\leq C\, 2^{-m\varepsilon} \sum_{\sigma\in G} \int_{\|\sigma(y)-x\|<2^{-k+m}} \frac{1}{\omega(B(\sigma(y),2^{-k+m}))^{p'}} d\omega(x) \\
&\leq C\, 2^{-m\left(\varepsilon+(p'-1)N\right)} 2^{-kN(1-p')}.
\end{aligned} \tag{5. 12}$$

Thus, for $j \geq k$,

$$\int_{\mathbb{R}^n} \left(\frac{1}{\omega(B(x, 2^{-j\vee -k} + d(x,y)))} \left(\frac{2^{-j\vee -k}}{2^{-j\vee -k} + \|x-y\|} \right)^{\varepsilon} \right)^{p'} d\omega(x) \leq C\, 2^{-kN(1-p')} \leq C_k, \tag{5. 13}$$

where C_k is a constant depending on k.

Similarly, for $j < k$,

$$\int_{\mathbb{R}^n} \left(\frac{1}{\omega(B(x, 2^{-j\vee -k} + d(x,y)))} \left(\frac{2^{-j\vee -k}}{2^{-j\vee -k} + \|x-y\|} \right)^{\varepsilon} \right)^{p'} d\omega(x) \leq C\, 2^{-jN(1-p')}. \tag{5. 14}$$

Together with (5. 9) and $|\alpha| < \varepsilon$, we have

$$\|\mathbf{S}_{q'}^{-\alpha}\left(D_k(\cdot, y)\right)\|_{p'} \leq C_k \sum_{j:j\geq k} 2^{-j\alpha} 2^{-(j-k)\varepsilon} + \sum_{j:j<k} 2^{-j\alpha} 2^{(j-k)\varepsilon} 2^{-jN(\frac{1}{p'}-1)} \leq C_k. \tag{5. 15}$$

Proof of Lemma 5.1 (B):

Fix a dyadic cube P with the side length 2^{-M-j_0} and the center x_P. For $Q \in \mathcal{Q}_{d'}^{j}$, applying

Lemma 3.1, we have

$$\begin{aligned}
\left|\mathbf{D}_j\left(D_k(\cdot,y)\right)(x_Q)\right| &\le C\,2^{-|j-k|\varepsilon}\frac{1}{V(x_Q,y,2^{-j\vee -k}+d(x_Q,y))}\left(\frac{2^{-j\vee -k}}{2^{-j\vee -k}+\|x_Q-y\|}\right)^{\varepsilon}\\
&\le C\,2^{-|j-k|\varepsilon}\frac{1}{\omega(B(x_Q,2^{-j\vee -k}))} \qquad (5.\ 16)\\
&\le C\,2^{-|j-k|\varepsilon}\frac{1}{\omega(B(x_Q,2^{-k}))}.
\end{aligned}$$

We have the estimate

$$\begin{aligned}
&\frac{1}{\omega(P)^{\frac{q'}{p}-\frac{q'}{q}}}\sum_{Q\subset P}\omega(Q)\left(2^{-j\alpha}\left|\mathbf{D}_j(D_k(\cdot,y))(x_Q)\right|\right)^{q'}\\
&=\frac{1}{\omega(P)^{\frac{q'}{p}-\frac{q'}{q}}}\sum_{j=j_0}^{\infty}\sum_{\{Q\in Q_d^j:Q\subset P\}}\omega(Q)\left(2^{-j\alpha}\left|\mathbf{D}_j(D_k(\cdot,y))(x_Q)\right|\right)^{q'}\\
&\le C\frac{1}{\omega(P)^{\frac{q'}{p}-\frac{q'}{q}}}\sum_{j=j_0}^{\infty}\sum_{\{Q\in Q_d^j:Q\subset P\}}\omega(Q)2^{-j\alpha q'}2^{-|j-k|\varepsilon q'}\frac{1}{\omega(B(x_Q,2^{-k}))^{q'}}\\
&\le C\sup_{x\in P}\frac{1}{\omega(B(x,2^{-k}))^{q'}}\frac{1}{\omega(P)^{\frac{q'}{p}-\frac{q'}{q}}}\sum_{j=j_0}^{\infty}\sum_{\{Q\in Q_d^j:Q\subset P\}}\omega(Q)2^{-(j\alpha+|j-k|\varepsilon)q'} \qquad (5.\ 17)\\
&\le C\sup_{x\in P}\frac{1}{\omega(B(x,2^{-k}))^{q'}}\frac{1}{\omega(P)^{\frac{q'}{p}-\frac{q'}{q}}}\sum_{j=j_0}^{\infty}\omega(P)2^{-(j\alpha+|j-k|\varepsilon)q'}\\
&\le C\sup_{x\in P}\frac{1}{\omega(B(x,2^{-k}))^{q'}}\frac{1}{\omega(P)^{\frac{q'}{p}-\frac{q'}{q}-1}}\sum_{j=j_0}^{\infty}2^{-(j\alpha+|j-k|\varepsilon)q'}.
\end{aligned}$$

(i) For $j_0\ge k$, the doubling property of the measure ω implies

$$\omega(B(x_P,2^{-k}))\le C\,2^{(-k+j_0)N}\omega(B(x_P,2^{-j_0}))\sim C\,2^{(-k+j_0)N}\omega(P). \qquad (5.\ 18)$$

Thus, for $|\alpha|<\varepsilon$,

$$\begin{aligned}
&\frac{1}{\omega(P)^{\frac{q'}{p}-\frac{q'}{q}}}\sum_{Q\subset P}\omega(Q)\left(2^{-j\alpha}\left|\mathbf{D}_j(D_k(\cdot,y))(x_Q)\right|\right)^{q'}\\
&\le C\sup_{x\in P}\frac{1}{\omega(B(x,2^{-k}))^{q'}}\frac{1}{\omega(B(x_P,2^{-k}))^{\frac{q'}{p}-\frac{q'}{q}-1}}2^{(-k+j_0)N(\frac{q'}{p}-\frac{q'}{q}-1)}\sum_{j=j_0}^{\infty}2^{-((j-k)\varepsilon+j\alpha)q'} \qquad (5.\ 19)\\
&\le C_k\sup_{x\in P}\frac{1}{\omega(B(x,2^{-k}))^{\frac{q'}{p}}}.
\end{aligned}$$

(ii) For $j_0 < k$, since

$$\omega(B(x_Q, 2^{-k})) \le \omega(B(x_Q, 2^{-j_0})) \sim \omega(P), \tag{5. 20}$$

we have

$$\begin{aligned}
&\frac{1}{\omega(P)^{\frac{q'}{p}-\frac{q'}{q}}} \sum_{Q\subset P} \omega(Q) \left(2^{-j\alpha} \left|\mathbf{D}_j(D_k(\cdot, y))(x_Q)\right|\right)^{q'} \\
&\le C \sup_{x\in P} \frac{1}{\omega(B(x, 2^{-k}))^{q'}} \frac{1}{\omega(P)^{\frac{q'}{p}-\frac{q'}{q}-1}} \sum_{j=j_0}^{\infty} 2^{-(j\alpha+|j-k|\varepsilon)q'} \qquad (5.\ 21) \\
&\le C_k \sup_{x\in P} \frac{1}{\omega(B(x, 2^{-k}))^{\frac{q'}{p}}}.
\end{aligned}$$

Taking the supremum over all dyadic cubes P and using $\inf_{x\in\mathbb{R}^n} \omega(B(x,1)) \ge C$, we obtain

$$\sup_P \left(\frac{1}{\omega(P)^{\frac{q'}{p}-\frac{q'}{q}}} \sum_{Q\subset P} \omega(Q) \left(2^{-j\alpha} \left|\mathbf{D}_j(D_k(\cdot, x_Q))(y)\right|\right)^{q'} \right)^{\frac{1}{q'}} \le C_k \tag{5. 22}$$

and the proof of Lemma 5.1 (B) is complete.

Proof of Lemma 5.1 (C):

$$\begin{aligned}
&\|\mathbf{S}_\infty^{-\alpha}(D_k(\cdot, y))\|_{p'} \\
&= \left\{ \int_{\mathbb{R}^n} \left| \sup_{j\in\mathbb{Z}, Q\in Q_d^j} 2^{-j\alpha} \left|\mathbf{D}_j\left(D_k(\cdot, y)\right)(x_Q)\right| \chi_Q(x) \right|^{p'} d\omega(x) \right\}^{\frac{1}{p'}} \\
&\le C \left\{ \int_{\mathbb{R}^n} \left| \sup_{j\in\mathbb{Z}, Q\in Q_d^j} 2^{-j\alpha} 2^{-|j-k|\varepsilon} \frac{1}{V(x_Q, y, 2^{-j\vee -k} + d(x_Q, y))} \left(\frac{2^{-j\vee -k}}{2^{-j\vee -k} + \|x_Q - y\|} \right)^{\varepsilon} \chi_Q(x) \right|^{p'} d\omega(x) \right\}^{\frac{1}{p'}} \\
&\le C \left\{ \sup_{j\in\mathbb{Z}, Q\in Q_d^j} 2^{-j\alpha p'} 2^{-|j-k|\varepsilon p'} \int_Q \left(\frac{1}{\omega(B(x, 2^{-j\vee -k} + d(x, y)))} \left(\frac{2^{-j\vee -k}}{2^{-j\vee -k} + \|x - y\|} \right)^{\varepsilon} \right)^{p'} d\omega(x) \right\}^{\frac{1}{p'}} \\
&\le C \sup_{j\in\mathbb{Z}} 2^{-j\alpha} 2^{-|j-k|\varepsilon} \left\{ \int_{\mathbb{R}^n} \left(\frac{1}{\omega(B(x, 2^{-j\vee -k} + d(x, y)))} \left(\frac{2^{-j\vee -k}}{2^{-j\vee -k} + \|x - y\|} \right)^{\varepsilon} \right)^{p'} d\omega(x) \right\}^{\frac{1}{p'}}
\end{aligned} \tag{5. 23}$$

Similar to the proof of Lemma 5.1 (A), we have

$$\|\mathbf{S}_\infty^{-\alpha}(D_k(\cdot, y))\|_{p'} \le C_k \sup_{j: j\ge k} 2^{-j\alpha} 2^{-(j-k)\varepsilon} + \sup_{j: j<k} 2^{-j\alpha} 2^{(j-k)\varepsilon} 2^{-jN(\frac{1}{p'}-1)} \le C_k. \tag{5. 24}$$

Proof of Lemma 5.1 (D):

Recall (5. 16), we have

$$\left|\mathbf{D}_j(D_k(\cdot,y))(x_Q)\right| \le C\, 2^{-|j-k|\varepsilon}\frac{1}{\omega(B(x_Q,2^{-k}))}. \tag{5. 25}$$

(i) For $j \ge k$, since $p > \frac{N}{N+\alpha+\varepsilon}$, then

$$\sup_{j:j\ge k;\ Q\in Q_d^j} 2^{(-\alpha+N(\frac{1}{p}-1))j}\left|\mathbf{D}_j(D_k(\cdot,y))(x_Q)\right| \le C \sup_{j:j\ge k;\ Q\in Q_d^j} 2^{(-\alpha+N(\frac{1}{p}-1))j}2^{-(j-k)\varepsilon}\frac{1}{\omega(B(x_Q,2^{-k}))} \le C_k, \tag{5. 26}$$

(ii) For $j < k$, since $p \le 1$ and $|\alpha| < \varepsilon$, then

$$\sup_{j:j<k;\ Q\in Q_d^j} 2^{(-\alpha+N(\frac{1}{p}-1))j}\left|\mathbf{D}_j(D_k(\cdot,y))(x_Q)\right| \le C \sup_{j:j<k;\ Q\in Q_d^j} 2^{(-\alpha+N(\frac{1}{p}-1))j}2^{(j-k)\varepsilon}\frac{1}{\omega(B(x_Q,2^{-k}))} \le C_k. \tag{5. 27}$$

The proof of Lemma 5.1 is complete.

For $1 < p < \infty$, $1 < q < \infty$, we denote $\mathbf{L}^2 \cap \dot{F}_{p'}^{-\alpha,q'}$, as the subspace of $f \in \mathbf{L}^2$, with the norm $\|f\|_{\dot{F}_{p'}^{-\alpha,q'}} < \infty$. Based on the above Lemma 5.1, if $f \in \left(\mathbf{L}^2 \cap \dot{F}_{p'}^{-\alpha,q'}\right)'$, then $\mathbf{D}_k(f)(x)$ is well defined since for each fixed x, $D_k(x,y) \in \mathbf{L}^2 \cap \dot{F}_{p'}^{-\alpha,q'}$. Other ranges of p, q have similar results.

Theorem 2.3 describes an important property for each distribution f. More precisely, it establishes the weak-type discrete Calderón reproducing formula in the distribution sense. Now we prove Theorem 2.3.

Proof Theorem 2.3 (A)

By the Theorem 2.2, there exists $f \in \left(\mathbf{L}^2 \cap \dot{F}_{p'}^{-\alpha,q'}\right)'$ such that for each $g \in \mathbf{L}^2 \cap \dot{F}_{p'}^{-\alpha,q'}$,

$$\langle f, g\rangle = \lim_{n\to\infty} \langle f_n, g\rangle. \tag{5. 28}$$

Observing that

$$\|\mathbf{S}_q^\alpha(f - f_n)\|_p = \|\mathbf{S}_q^\alpha(\lim_{m\to\infty}(f_m - f_n))\|_p \le \liminf_{m\to\infty}\|\mathbf{S}_q^\alpha(f_m - f_n)\|_p, \tag{5. 29}$$

hence $\|\mathbf{S}_q^\alpha(f - f_n)\|_p \to 0$ as $n \to \infty$. This implies that

$$\|f\|_{\dot{F}_p^{\alpha,q}} = \|\mathbf{S}_q^\alpha(f)\|_p = \lim_{n\to\infty}\|\mathbf{S}_q^\alpha(f_n)\|_p = \lim_{n\to\infty}\|f_n\|_{\dot{F}_p^{\alpha,q}} < \infty. \tag{5. 30}$$

Applying Theorem 2.1, for each f_n there exists an h_n such that $\|f_n\|_2 \sim \|h_n\|_2$ and $\|f_n\|_{\dot{F}_p^{\alpha,q}} \sim \|h_n\|_{\dot{F}_p^{\alpha,q}}$. Thus by Theorem 2.2, there exists $h \in \left(\mathbf{L}^2 \cap \dot{F}_{p'}^{-\alpha,q'}\right)'$ such that for each $g \in \mathbf{L}^2 \cap \dot{F}_{p'}^{-\alpha,q'}$, we have

$$\langle h, g\rangle = \lim_{n\to\infty} \langle h_n, g\rangle. \tag{5. 31}$$

Therefore,

$$\|\mathbf{S}_q^\alpha(h_n - h_m)\|_p \to 0, \tag{5. 32}$$

and

$$\|h\|_{\dot{F}_p^{\alpha,q}} = \|\mathbf{S}_q^\alpha(h)\|_p = \lim_{n\to\infty} \|\mathbf{S}_q^\alpha(h_n)\|_p \sim \lim_{n\to\infty} \|\mathbf{S}_q^\alpha(f_n)\|_p = \|\mathbf{S}_q^\alpha(f)\|_p = \|f\|_{\dot{F}_p^{\alpha,q}}. \tag{5. 33}$$

For each $g \in \mathbf{L}^2 \cap \dot{F}_{p'}^{-\alpha,q'}$, we know that

$$\left|\sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \omega(Q)\mathbf{D}_k^M(g)(x_Q)\mathbf{D}_k(h)(x_Q)\right| \le C\,\|f\|_{\dot{F}_p^{\alpha,q}}\|g\|_{\dot{F}_{p'}^{-\alpha,q'}}, \tag{5. 34}$$

which implies that the series $\sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \omega(Q)D_k^M(x,x_Q)\mathbf{D}_k(h)(x_Q)$ is a distribution in $\left(\mathbf{L}^2 \cap \dot{F}_{p'}^{-\alpha,q'}\right)'$. Moreover, by the reproducing formula of f_n in Theorem 2.1, for each $g \in \mathbf{L}^2 \cap \dot{F}_{p'}^{-\alpha,q'}$,

$$\langle f, g\rangle = \lim_{n\to\infty} \langle f_n, g\rangle = \lim_{n\to\infty} \left\langle \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \omega(Q)D_k^M(\cdot,x_Q)\mathbf{D}_k(h_n)(x_Q), g(\cdot)\right\rangle, \tag{5. 35}$$

where $\|f_n\|_2 \sim \|h_n\|_2$ and $\|f_n\|_{\dot{F}_p^{\alpha,q}} \sim \|h_n\|_{\dot{F}_p^{\alpha,q}}$.

By the same proof of Theorem 2.2, we have

$$\left|\left\langle \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \omega(Q)D_k^M(\cdot,x_Q)\mathbf{D}_k(h-h_n)(x_Q), g(\cdot)\right\rangle\right| \le C\,\|h_n - h\|_{\dot{F}_p^{\alpha,q}}\,\|g\|_{\dot{F}_{p'}^{-\alpha,q'}}. \tag{5. 36}$$

Since $\|h_n - h\|_{\dot{F}_p^{\alpha,q}} \to 0$, as $n \to \infty$, we have

$$\langle f, g\rangle = \lim_{n\to\infty} \langle f_n, g\rangle = \left\langle \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \omega(Q)D_k^M(\cdot,x_Q)\mathbf{D}_k(h)(x_Q), g(\cdot)\right\rangle. \tag{5. 37}$$

The proof of Theorem 2.3 (B), (C), (D) is similar, so we omit the details.

The Theorem 2.3 (A) indicates that one can consider $\mathbf{L}^2 \cap \dot{F}_{p'}^{-\alpha,q'}$, $|\alpha| < 1$, $1 < p < \infty$, $1 < q < \infty$, the subspace of $f \in \mathbf{L}^2$ with the norm $\|f\|_{\dot{F}_{p'}^{-\alpha,q'}} < \infty$, as the test function space and $\left(\mathbf{L}^2 \cap \dot{F}_{p'}^{-\alpha,q'}\right)'$ as the distribution space. The Dunkl-Triebel-Lizorkin space is defined by Definition 2.5. We remark that in the Definition 2.5, the series $\sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \omega(Q)\lambda_Q D_k^M(x,x_Q)$ with $\left\|\left\{\sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \left(2^{k\alpha}\left|\lambda_Q\right|\right)^q \chi_Q\right\}^{\frac{1}{q}}\right\|_p < \infty$ defines a distribution in $\left(\mathbf{L}^2 \cap \dot{F}_{p'}^{-\alpha,q'}\right)'$. Indeed, applying the proof of Theorem 2.2 for each $g \in \mathbf{L}^2 \cap \dot{F}_{p'}^{-\alpha,q'}$,

$$\left|\sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \omega(Q)\lambda_Q\mathbf{D}_k^M(g)(x_Q)\right| \le C\left\|\left\{\sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \left(2^{k\alpha}\left|\lambda_Q\right|\right)^q \chi_Q\right\}^{\frac{1}{q}}\right\|_p \|g\|_{\dot{F}_{p'}^{-\alpha,q'}}. \tag{5. 38}$$

Other ranges of p, q has the same results.

Proof of Theorem 2.4 (A)

Suppose $f \in \dot{F}_p^{\alpha,q}(\mathbb{R}^n,\omega)$. Then $f \in \left(\mathbf{L}^2 \cap \dot{F}_{p'}^{-\alpha,q'}\right)'$ and f has a wavelet-type decomposition $f(x) = \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \omega(Q)\lambda_Q D_k^M(x,x_Q)$ in $\left(\mathbf{L}^2 \cap \dot{F}_{p'}^{-\alpha,q'}\right)'$ with its norm $\left\|\left\{\sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \left(2^{k\alpha}\left|\lambda_Q\right|\right)^q \chi_Q(x)\right\}^{\frac{1}{q}}\right\|_p < \infty$. Set

$$f_n(x) = \sum_{|k|\le n} \sum_{\substack{Q\in Q_d^k \\ Q\subseteq B(0,n)}} \omega(Q)\lambda_Q D_k^M(x,x_Q). \tag{5. 39}$$

Then $f_n \in \mathbf{L}^2 \cap \dot{F}_p^{\alpha,q}$ and f_n converges to f in $\left(\mathbf{L}^2 \cap \dot{F}_{p'}^{-\alpha,q'}\right)'$ as $n \to \infty$.

To see that $f \in \overline{\mathbf{L}^2 \cap \dot{F}_p^{\alpha,q}}$, by Theorem 2.3, it suffices to show that $\|f_n - f_m\|_{\dot{F}_p^{\alpha,q}} \to 0$ as $n, m \to \infty$. Indeed, if let $E_n = \left\{(k,Q) : |k| \le n, Q \in Q_d^k \subseteq B(0,n)\right\}$ and $E_{n,m}^c = E_n \setminus E_m$ with $n \ge m$,

$$\begin{aligned}
\|f_n - f_m\|_{\dot{F}_p^{\alpha,q}} &= \left\| \left(\sum_{k'\in\mathbb{Z}} \sum_{Q'\in Q_d^k} \left(2^{k'\alpha}\left|\mathbf{D}_{k'}^M(f_n - f_m)(x_{Q'})\right|\right)^q \chi_{Q'}(x)\right)^{\frac{1}{q}} \right\|_p \\
&\le \left\| \left(\sum_{k'\in\mathbb{Z}} \sum_{Q'\in Q_d^k} \left(2^{k'\alpha}\left|\mathbf{D}_{k'}^M\left(\sum_{E_{n,m}^c} \omega(Q)\lambda_Q D_k^M(\cdot,x_Q)\right)(x_{Q'})\right|\right)^q \chi_{Q'}(x)\right)^{\frac{1}{q}} \right\|_p \qquad (5.\ 40) \\
&\le C \left\| \left\{\sum_{E_{n,m}^c} \left(2^{k\alpha}\left|\lambda_Q\right|\right)^q \chi_Q(x)\right\}^{\frac{1}{q}} \right\|_p \to 0,
\end{aligned}$$

as $n, m \to \infty$, where the last inequality follows from the same proof of **Step 1** in the Theorem 2.1 and hence, $f \in \overline{\mathbf{L}^2 \cap \dot{F}_p^{\alpha,q}}$.

Conversely, if $f \in \overline{\mathbf{L}^2 \cap \dot{F}_p^{\alpha,q}}$ by Theorem 2.2, then there exists $h \in \left(\mathbf{L}^2 \cap \dot{F}_{p'}^{-\alpha,q'}\right)'$ with $\|\mathbf{S}_q^\alpha(h)\|_p \sim \|\mathbf{S}_q^\alpha(f)\|_p$ such that for each $g \in \mathbf{L}^2 \cap \dot{F}_{p'}^{-\alpha,q'}$,

$$\langle f, g\rangle = \left\langle \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \omega(Q) D_k^M(\cdot,x_Q)\mathbf{D}_k(h)(x_Q), g(\cdot)\right\rangle. \tag{5. 41}$$

Set $\lambda_Q = \mathbf{D}_k(h)(x_Q)$ with $Q \in Q_d^k$. We obtain a wavelet-type decomposition of f in $\left(\mathbf{L}^2 \cap \dot{F}_{p'}^{-\alpha,q'}\right)'$ in the distribution sense:

$$f(x) = \sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \omega(Q)\lambda_Q D_k^M(x,x_Q). \tag{5. 42}$$

Hence, $f \in \dot{F}_p^{\alpha,q}(\mathbb{R}^n,\omega)$. Moreover

$$\|f\|_{\dot{F}_p^{\alpha,q}} = \inf\left\{ \left\| \left\{\sum_{k\in\mathbb{Z}} \sum_{Q\in Q_d^k} \left(2^{k\alpha}\left|\lambda_Q\right|\right)^q \chi_Q(x)\right\}^{\frac{1}{q}} \right\|_p \right\} \le C\,\|\mathbf{S}_q^\alpha(h)\|_p \le C\,\|\mathbf{S}_q^\alpha(f)\|_p. \tag{5. 43}$$

The proof of Theorem 2.4 (B), (C), (D) are similar, so we omit the details.

6 Declaration

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