Fractional integration associated with multi-parameter dilation

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Preface

This thesis was accomplished under the supervision of Professor Zipeng Wang. The work focuses on the fractional integral operator associated with different dilations and contains three parts.

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text.

It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification as the Westlake University or any other University or similar institution except as declared in the Preface and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at Westlake University or any other University of similar institution except as declared in the Preface and specified in the text.

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Acknowledgements

I would like to express my deepest appreciation to my advisor, Professor Zipeng Wang.

The work in the thesis have been finished under his guidance, patience and encouragement. His insightful advice has shaped the way I think about mathematics and research.

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Abstract

This thesis contains three main objectives.

First, we improve the classical Stein-Weiss inequality to include the end-point $p = 1$.

Second, we give an extension of Stein-Weiss inequality to the multi-parameter setting by proving a typical bi-parameter case.

Third, we prove a number of results for fractional integration on Heisenberg groups. This includes a $\mathbf{L}^p \longrightarrow \mathbf{L}^q$-regularity estimate for the strong fractional maximal operator $\mathbf{M}_\gamma, 0 \leq \gamma < 1$ defined on a Heisenberg group; a bi-parameter extension of Folland-Stein theorem and a bi-parameter extension of Stein-Weiss inequality on Heisenberg group.

Chapter 1: Introduction

1.1 A brief history of fractional integrals

Let $0 < \alpha < n$. A fractional integral operator $\mathbf{I}_\alpha$ is initially defined as

$$\mathbf{I}_\alpha f(x) \;=\; \int_{\mathbb{R}^n} f(u) \left(\frac{1}{|x-u|} \right)^{n-\alpha} du. \tag{1. 1}$$

In 1928, Hardy and Littlewood [14] established a regularity theorem for $\mathbf{I}_\alpha$ when $n = 1$. Ten years later, Sobolev [28] extended this result to every higher dimensional space. Today, it is known as Hardy-Littlewood-Sobolev theorem.

$\diamond$ Throughout, $\mathfrak{C} > 0$ is regarded as a generic constant depending on its sub-indices.

Hardy-Littlewood-Sobolev theorem, 1938 *Let $\mathbf{I}_\alpha$ defined in (1. 1) for $0 < \alpha < n$. We have*

$$\left\|\mathbf{I}_\alpha f\right\|_{\mathbf{L}^q(\mathbb{R}^n)} \;\leq\; \mathfrak{C}_{p\,q} \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad 1 < p < q < \infty \tag{1. 2}$$

if and only if

$$\frac{\alpha}{n} \;=\; \frac{1}{p} - \frac{1}{q}. \tag{1. 3}$$

In 1958, Stein and Weiss [30] obtained a weighted analogue of the above $\mathbf{L}^p \longrightarrow \mathbf{L}^q$-norm inequality by considering the *weights* to be suitable powers.

Stein-Weiss theorem, 1958 *Let $\mathbf{I}_\alpha$ defined in (1. 1) for $0 < \alpha < n$ and $\omega(x) = |x|^{-\gamma}, \sigma(x) = |x|^{\delta}$ for $\gamma, \delta \in \mathbb{R}$ whenever $x \neq 0$. We have*

$$\left\|\omega \mathbf{I}_\alpha f\right\|_{\mathbf{L}^q(\mathbb{R}^n)} \;\leq\; \mathfrak{C}_{p\,q\,\alpha\,\gamma\,\delta} \left\|f\sigma\right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad 1 < p \leq q < \infty \tag{1. 4}$$

if and only if

$$\gamma < \frac{n}{q}, \qquad \delta < n\left(\frac{p-1}{p}\right), \qquad \gamma + \delta \geq 0, \qquad \frac{\alpha}{n} \;=\; \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{n}. \tag{1. 5}$$

Remark 1.1.1. *In the original paper of Stein and Weiss [30], (1. 5) is given as a sufficient condition. These constrains of α, γ, δ, p, q in (1. 5) are in fact necessary.*

The theory of fractional integration in weighted norms has been substantially developed during the second half of 20th century. See Coifman and Fefferman [4], Fefferman and Muckenhoupt [6], Muckenhoupt and Wheeden [17], Pérez [21] and Sawyer and Wheeden [27].

Hardy-Littlewood-Sobolev theorem was first re-investigated by Folland and Stein [11] on

Heisenberg group. Consider its real variable representation with a multiplication law:

$$(x, y, t) \odot (u, v, s) \;=\; \Big[x + u, y + v, t + s + \mu\big(x \cdot v - y \cdot u\big)\Big], \qquad \mu \in \mathbb{R} \tag{1. 6}$$

for every $(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and $(u, v, s)^{-1} = (-u, -v, -s) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$.

Let $0 < \rho < n + 1$. Define

$$\mathbf{S}_\rho f(x, y, t) \;=\; \iiint_{\mathbb{R}^{2n+1}} f(u, v, s)\Omega^\rho\Big[(x, y, t) \odot (u, v, s)^{-1}\Big]dudvds. \tag{1. 7}$$

Ω^ρ is a distribution in $\mathbb{R}^{2n+1}$ agree with

$$\Omega^\rho(x, y, t) \;=\; \left[\frac{1}{|x|^2 + |y|^2 + |t|}\right]^{n+1-\rho}, \qquad (x, y, t) \neq (0, 0, 0). \tag{1. 8}$$

Folland-Stein theorem, 1974 *Let* $\mathbf{S}_\rho$ *defined in (1. 7)-(1. 8) for* $0 < \rho < n + 1$. *We have*

$$\Big\|\mathbf{S}_\rho f\Big\|_{\mathbf{L}^q(\mathbb{R}^{2n+1})} \;\leq\; \mathfrak{C}_{p\ q}\ \Big\|f\Big\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})}, \qquad 1 < p < q < \infty \tag{1. 9}$$

if and only if

$$\frac{\rho}{n + 1} \;=\; \frac{1}{p} - \frac{1}{q}. \tag{1. 10}$$

The best constant for the $\mathbf{L}^p \longrightarrow \mathbf{L}^q$-norm inequality in (1. 9) is found by Frank and Lieb [12]. A discrete analogue of this result has been obtained by Pierce [23]. Recently, the regarding commutator estimates are established by Fanelli and Roncal [16].

Next, **Stein-Weiss theorem** has been refined on Heisenberg group by Han, Lu and Zhu [13].

Han-Lu-Zhu theorem, 2012 *Let* $\mathbf{S}_\rho$ *defined in (1. 7)-(1. 8) for* $0 < \rho < n + 1$. *Suppose* $\gamma, \delta \in \mathbb{R}$ *and* $\omega(x, y) = \left(\sqrt{|x|^2 + |y|^2}\right)^{-\gamma}$, $\sigma(x, y) = \left(\sqrt{|x|^2 + |y|^2}\right)^{\delta}$ *for* $(x, y) \neq (0, 0)$. *We have*

$$\Big\|\omega\mathbf{S}_\rho f\Big\|_{\mathbf{L}^q(\mathbb{R}^{2n+1})} \;\leq\; \mathfrak{C}_{p\ q\ \rho\ \gamma\ \delta}\ \Big\|f\sigma\Big\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})}, \qquad 1 < p \leq q < \infty \tag{1. 11}$$

if

$$\gamma < \frac{2n}{q}, \qquad \delta < 2n\left(\frac{p-1}{p}\right), \qquad \gamma + \delta \geq 0, \qquad \frac{\rho}{n+1} \;=\; \frac{1}{p} - \frac{1}{q} + \frac{\gamma + \delta}{2n + 2}. \tag{1. 12}$$

Remark 1.1.2. *Note that the two power weights* ω, σ *are defined in the subspace* $\mathbb{R}^{2n}$. *An analogue two-weight* $\mathbf{L}^p \longrightarrow \mathbf{L}^q$*-norm inequality with*

$$\omega(x, y, t) \;=\; \left(\sqrt{|x|^2 + |y|^2 + |t|}\right)^{-\gamma}, \qquad \sigma(x, y, t) \;=\; \left(\sqrt{|x|^2 + |y|^2 + |t|}\right)^{\delta}$$

can be found in the paper of Han, Lu and Zhu [13].

1.2 Formulation on the main results

1.2.1 On the end-point of Stein-Weiss inequality

Our first main result is to show that the classical **Stein-Weiss theorem** is true for $p = 1$.

Theorem One *Let* $\mathbf{I}_\alpha$ *defined in (1. 1) for* $0 < \alpha < n$ *and* $\omega(x) = |x|^{-\gamma}, \sigma(x) = |x|^{\delta}$ *for* $\gamma, \delta \in \mathbb{R}$ *whenever* $x \neq 0$*. We have*

$$\left\|\omega \mathbf{I}_\alpha f\right\|_{\mathbf{L}^q(\mathbb{R}^n)} \leq \mathfrak{C}_{p\,q\,\alpha\,\gamma\,\delta} \left\|f\sigma\right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad 1 \leq p \leq q < \infty \tag{1. 13}$$

if and only if

$$\gamma < \frac{n}{q}, \qquad \delta < n\left(\frac{p-1}{p}\right), \qquad \gamma + \delta \geq 0, \qquad \frac{\alpha}{n} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{n}. \tag{1. 14}$$

Remark 1.2.1. *For* $1 = p \leq q < \infty$*, the necessary constraints in (2. 3) become*

$$\gamma < \frac{n}{q}, \qquad \delta < 0, \qquad \gamma + \delta \geq 0, \qquad \frac{\alpha}{n} = 1 - \frac{1}{q} + \frac{\gamma+\delta}{n}.$$

1.2.2 Bi-parameter Stein-Weiss inequality

Our second main result gives an extension of **Theorem One** to the bi-parameter setting.

Let $0 < \alpha < n$, $0 < \beta < m$. We define

$$\mathbf{I}_{\alpha\beta} f(x,y) = \iint_{\mathbb{R}^n \times \mathbb{R}^m} f(u,v) \left(\frac{1}{|x-u|}\right)^{n-\alpha} \left(\frac{1}{|y-v|}\right)^{m-\beta} dudv. \tag{1. 15}$$

Observe that the kernel of $\mathbf{I}_{\alpha\beta}$ has singularity on the coordinate subspace $\mathbb{R}^n$ and $\mathbb{R}^m$.

The study of certain operators which commute with a multi-parameter family of dilations dates back to the time of Jessen, Marcinkiewicz and Zygmund. Over the several past decades, a number of remarkable results have been accomplished, for example, by Fefferman [7], Cordoba and Fefferman [5], Fefferman and Stein [9], Müller, Ricci and Stein [18], Journé [15] and Pipher [24].

Theorem Two *Let* $\mathbf{I}_{\alpha\beta}$ *defined in (1. 15). Suppose* $\omega(x,y) = \sqrt{|x|^2+|y|^2}^{\,-\gamma}$, $\sigma(x,y) = \sqrt{|x|^2+|y|^2}^{\,\delta}$ *for* $\gamma, \delta \in \mathbb{R}$ *and* $(x,y) \neq (0,0)$*. The following two conditions are equivalent.*

1.

$$\left\|\omega \mathbf{I}_{\alpha\beta} f\right\|_{\mathbf{L}^q(\mathbb{R}^n \times \mathbb{R}^m)} \leq \mathfrak{C}_{p\,q\,\alpha\,\beta\,\gamma\,\delta} \left\|f\sigma\right\|_{\mathbf{L}^p(\mathbb{R}^n \times \mathbb{R}^m)}, \qquad 1 \leq p \leq q < \infty. \tag{1. 16}$$

2.

$$\gamma < \frac{n+m}{q}, \qquad \delta < (n+m)\left(\frac{p-1}{p}\right), \qquad \gamma + \delta \geq 0 \tag{1. 17}$$

and

$$\frac{\alpha+\beta}{n+m} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{n+m}. \tag{1. 18}$$

For $\gamma \geq 0, \delta \leq 0,$
$$\alpha - \frac{n}{p} < \delta, \qquad \beta - \frac{m}{p} < \delta. \tag{1. 19}$$
For $\gamma \leq 0, \delta \geq 0,$
$$\alpha - n\left(\frac{q-1}{q}\right) < \gamma, \qquad \beta - m\left(\frac{q-1}{q}\right) < \gamma. \tag{1. 20}$$
For $\gamma > 0, \delta > 0,$
$$\begin{gathered}
\alpha - \frac{n}{p} < \delta \qquad \text{if} \qquad \alpha - \frac{n}{p} \geq 0, \quad \beta - \frac{m}{p} < 0; \\
\beta - \frac{m}{p} < \delta \qquad \text{if} \qquad \alpha - \frac{n}{p} < 0, \quad \beta - \frac{m}{p} \geq 0; \\
\alpha - \frac{n}{p} + \beta - \frac{m}{p} < \delta \qquad \text{if} \qquad \alpha - \frac{n}{p} \geq 0, \quad \beta - \frac{m}{p} \geq 0. \\
\alpha - n\left(\frac{q-1}{q}\right) < \gamma \qquad \text{if} \qquad \alpha - n\left(\frac{q-1}{q}\right) \geq 0, \quad \beta - m\left(\frac{q-1}{q}\right) < 0; \\
\beta - m\left(\frac{q-1}{q}\right) < \gamma \qquad \text{if} \quad \alpha - n\left(\frac{q-1}{q}\right) < 0, \quad \beta - m\left(\frac{q-1}{q}\right) \geq 0; \\
\alpha - n\left(\frac{q-1}{q}\right) + \beta - m\left(\frac{q-1}{q}\right) < \gamma \qquad \text{if} \qquad \alpha - n\left(\frac{q-1}{q}\right) \geq 0, \quad \beta - m\left(\frac{q-1}{q}\right) \geq 0.
\end{gathered} \tag{1. 21}$$

Remark 1.2.2. *For* $1 < p \leq q < \infty$, **Theorem Two** *is proved by Wang [32]. The characterization between the weighted norm inequality in (1. 16) and necessary constraints in (1. 17)-(1. 21) is now extended to include* $p = 1$.

Remark 1.2.3. *Let* $\alpha = \beta = \zeta$ *and* $n = m$ *in* **Theorem Two**. *The necessary constraints in (1. 21) can be implied by (1. 17) and (1. 18).*

Consider (1. 17)-(1. 18) with $\alpha = \beta = \zeta$ and $n = m$. We have
$$\gamma < \frac{2n}{q}, \qquad \delta < 2n\left(\frac{p-1}{p}\right), \qquad \gamma + \delta \geq 0 \tag{1. 22}$$
and
$$\frac{\zeta}{n} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma + \delta}{2n}. \tag{1. 23}$$
For $\gamma > 0, \delta > 0$, we find
$$\begin{gathered}
2\zeta - \frac{2n}{p} \;=\; -\frac{2n}{q} + (\gamma + \delta) \;<\; -\gamma + (\gamma + \delta) \;<\; \delta, \\
2\zeta - 2n\left(\frac{q-1}{q}\right) \;=\; -2n\left(\frac{p-1}{p}\right) + (\gamma + \delta) \;<\; -\delta + (\gamma + \delta) \;<\; \gamma.
\end{gathered} \tag{1. 24}$$

1.2.3 Fractional integration on Heisenberg group

In this subsection, we introduce a number of results associated to fractional integration on Heisenberg group. Recall the multiplication law $\odot$ defined in (1. 6).

Let $0 \leq \gamma < 1$. A strong fractional maximal operator $\mathbf{M}_\gamma$ is defined on Heisenberg group as

$$\mathbf{M}_\gamma f(x,y,t) = \sup_{\mathbf{R}\subset\mathbb{R}^{2n+1}} \mathbf{vol}\{\mathbf{R}\}^{\gamma-1} \iiint_{\mathbf{R}} \left| f[(x,y,t)\odot(u,v,s)^{-1}] \right| dudvds \tag{1. 25}$$

where $\mathbf{R}$ denotes a rectangle centered on the origin with sides parallel to the coordinates.

Theorem Three *Let* $\mathbf{M}_\gamma$ *defined in (1. 25) for* $0 \leq \gamma < 1$. *We have*

$$\left\|\mathbf{M}_\gamma f\right\|_{\mathbf{L}^q(\mathbb{R}^{2n+1})} \leq \mathfrak{C}_{p\,q} \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})}, \qquad 1 < p \leq q < \infty \tag{1. 26}$$

if and only if

$$\gamma = \frac{1}{p} - \frac{1}{q}. \tag{1. 27}$$

As a special case, consider $\mathbf{R} = \mathbf{Q}_1\times\mathbf{Q}_2\times\mathbf{Q}_3 \subset \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$: $\mathbf{Q}_1, \mathbf{Q}_2$ and $\mathbf{Q}_3$ are cubes centered on the origin of regarding subspaces. For $\alpha, \beta \in \mathbb{R}$, we define

$$\begin{aligned}\mathbf{M}_{\alpha\beta} f(x,y,t) = \sup_{\mathbf{R}:\, \mathbf{vol}\{\mathbf{Q}_3\}=\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}}} & \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{\mathbf{Q}_3\}^{\beta-1} \\ & \iiint_{\mathbf{R}} \left| f[(x,y,t)\odot(u,v,s)^{-1}] \right| dudvds.\end{aligned} \tag{1. 28}$$

This is known as the fractional maximal function associated with Zygmund dilation defined on Heisenberg group. Later, we shall find

$$\mathbf{M}_{\alpha\beta} f(x,y,t) \leq \mathbf{M}_\gamma f(x,y,t), \qquad \gamma = \tfrac{\alpha+\beta}{n+1}. \tag{1. 29}$$

Corollary 1.2.1. *Let* $\mathbf{M}_{\alpha\beta}$ *defined in (1. 28) for* $\alpha, \beta \in \mathbb{R}$. *We have*

$$\left\|\mathbf{M}_{\alpha\beta} f\right\|_{\mathbf{L}^q(\mathbb{R}^{2n+1})} \leq \mathfrak{C}_{p\,q} \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})}, \qquad 1 < p \leq q < \infty \tag{1. 30}$$

if and only if

$$\frac{\alpha+\beta}{n+1} = \frac{1}{p} - \frac{1}{q}. \tag{1. 31}$$

Recall $\Omega^\rho, 0 < \rho < n+1$ given in (1. 8). We extend **Folland-Stein theorem** by replacing Ω^ρ with a larger kernel having singularity on every coordinate subspace. First, it is clear

$$\Omega^\rho(x,y,t) \leq \left[\frac{1}{|x||y|+|t|}\right]^{n+1-\rho}, \qquad (x,t)\neq(0,0) \quad \text{or} \quad (y,t)\neq(0,0). \tag{1. 32}$$

A direct computation shows

$$\left[\frac{1}{|x||y|+|t|}\right]^{n+1-\rho} \approx \left[\frac{1}{|x|^2|y|^2+|t|^2}\right]^{\frac{n+1}{2}-\frac{\rho}{2}}$$

$$= |x|^{\frac{\rho}{2}-\frac{n+1}{2}}|y|^{\frac{\rho}{2}-\frac{n+1}{2}}|t|^{\frac{\rho}{2}-\frac{n+1}{2}}\left[\frac{|x||y||t|}{|x|^2|y|^2+|t|^2}\right]^{\frac{n+1}{2}-\frac{\rho}{2}} \tag{1. 33}$$

$$= |x|^{\left[\frac{\rho}{2}+\frac{n-1}{2}\right]-n}|y|^{\left[\frac{\rho}{2}+\frac{n-1}{2}\right]-n}|t|^{\left[\frac{\rho}{2}-\frac{n-1}{2}\right]-1}\left[\frac{|x||y|}{|t|}+\frac{|t|}{|x||y|}\right]^{-\left[\frac{n+1}{2}-\frac{\rho}{2}\right]}, \qquad x\neq 0, y\neq 0, t\neq 0.$$

Remark 1.2.4. *By taking into account $\alpha = \frac{\rho}{2}+\frac{n-1}{2}$ and $\beta = \frac{\rho}{2}-\frac{n-1}{2}$ for $0<\rho<n+1$, we find*

$$\alpha > n\beta, \qquad \frac{\alpha-n\beta}{n+1} < \frac{n+1}{2}-\frac{\rho}{2}.$$

Above estimates lead us to the following assertion. Let $\alpha,\beta\in\mathbb{R}$ and $\vartheta\geq 0$. $\mathbf{V}^{\alpha\beta\vartheta}$ is a distribution in $\mathbb{R}^{2n+1}$ agree with

$$\mathbf{V}^{\alpha\beta\vartheta}(x,y,t) = |x|^{\alpha-n}|y|^{\alpha-n}|t|^{\beta-1}\left[\frac{|x||y|}{|t|}+\frac{|t|}{|x||y|}\right]^{-\vartheta}, \qquad x\neq 0, y\neq 0, t\neq 0. \tag{1. 34}$$

Define

$$\mathbf{I}_{\alpha\beta\vartheta}f(x,y,t) = \iiint_{\mathbb{R}^{2n+1}} f(u,v,s)\,\mathbf{V}^{\alpha\beta\vartheta}\Big[(x,y,t)\odot(u,v,s)^{-1}\Big]dudvds. \tag{1. 35}$$

This fractional integral operator is associated with Zygmund dilation whereas

$$\mathbf{V}^{\alpha\beta\vartheta}\Big[(\lambda_1 x,\lambda_2 y,\lambda_1\lambda_2 t)\odot(\lambda_1 u,\lambda_2 v,\lambda_1\lambda_2 s)^{-1}\Big]$$

$$= \lambda_1^{\alpha+\beta-n-1}\lambda_2^{\alpha+\beta-n-1}\mathbf{V}^{\alpha\beta\vartheta}\Big[(x,y,t)\odot(u,v,s)^{-1}\Big], \qquad \lambda_1,\ \lambda_2>0.$$

Singular integral operators with kernels carrying certain multi-parameter structures defined on Heisenberg group have been systematically studied, for instance by Phong and Stein [22], Ricci and Stein [25] and Müller, Ricci and Stein [19]. Much less is known in this direction for fractional integration.

Theorem Four *Let $\mathbf{I}_{\alpha\beta\vartheta}$ defined in (1. 34)-(1. 35) for $\alpha,\beta\in\mathbb{R}$ and $\vartheta\geq 0$. We have*

$$\left\|\mathbf{I}_{\alpha\beta\vartheta}f\right\|_{\mathbf{L}^q(\mathbb{R}^{2n+1})} \leq \mathfrak{C}_{p\,q\,\alpha\,\beta}\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})}, \qquad 1<p<q<\infty \tag{1. 36}$$

if and only if

$$\vartheta \geq \frac{|\alpha-n\beta|}{n+1}, \qquad \frac{\alpha+\beta}{n+1} = \frac{1}{p}-\frac{1}{q}. \tag{1. 37}$$

Remark 1.2.5. $\vartheta = \frac{|\alpha-n\beta|}{n+1}$ *is the smallest (best) exponent for which (1. 36)-(1. 37) holds.*

Lastly, we give an extension of **Han-Lu-Zhu theorem** to a bi-parameter setting.

Theorem Five *Let* $\mathbf{I}_{\alpha\beta\vartheta}$ *defined in (1. 34)-(1. 35) for* $\alpha, \beta \in \mathbb{R}$ *and* $\vartheta \geq 0$. *Suppose* $\gamma, \delta \in \mathbb{R}$ *and* $\omega(x,y) = \left(\sqrt{|x|^2+|y|^2}\right)^{-\gamma}$, $\sigma(x,y) = \left(\sqrt{|x|^2+|y|^2}\right)^{\delta}$ *for* $(x,y) \neq (0,0)$. *The following two conditions are equivalent.*

1.

$$\left\|\omega \mathbf{I}_{\alpha\beta\vartheta} f\right\|_{\mathbf{L}^q(\mathbb{R}^{2n+1})} \leq \mathfrak{C}_{p\,q\,\gamma\,\delta} \left\|f\sigma\right\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})}, \qquad 1<p<q<\infty. \tag{1. 38}$$

2.

$$\gamma < \frac{2n}{q}, \qquad \delta < 2n\left(\frac{p-1}{p}\right), \qquad \gamma+\delta \geq 0, \qquad \frac{\alpha+\beta}{n+1} = \frac{1}{p}-\frac{1}{q}+\frac{\gamma+\delta}{2n+2} \tag{1. 39}$$

and

$$\vartheta \geq \left|\frac{\alpha-n\beta}{n+1}-\frac{\gamma+\delta}{2n+2}\right|. \tag{1. 40}$$

For $\gamma \geq 0$, $\delta \leq 0$, *we have*

$$n\left[\frac{\alpha+\beta}{n+1}\right]+\frac{\gamma+\delta}{2n+2}-\frac{n}{p} < \delta. \tag{1. 41}$$

For $\gamma \leq 0$, $\delta \geq 0$, *we have*

$$n\left[\frac{\alpha+\beta}{n+1}\right]+\frac{\gamma+\delta}{2n+2}-n\left(\frac{q-1}{q}\right) < \gamma. \tag{1. 42}$$

Note that **Theorem Four** is a special case of **Theorem Five** at $\gamma = \delta = 0$.

1.3 Chapter summary

Chapter 2: We prove **Theorem One** in the same spirt of Stein and Weiss [30] by splitting the kernel $\left(\frac{1}{|x-u|}\right)^{n-\alpha}$ into three cases *w.r.t* $0 \leq \frac{|u|}{|x|} \leq \frac{1}{2}$, $\frac{|u|}{|x|} > 2$ and $\frac{1}{2} < \frac{|u|}{|x|} < 2$. The crucial estimate occurs at $\frac{1}{2} < \frac{|u|}{|x|} < 2$ when $1 = p < q < \infty$. In this situation, we need to go through an interpolation argument of changing measures.

Chapter 3: We prove **Theorem Two**. The proof contains two major parts for $1 = p \leq q < \infty$ and $1 < p \leq q < \infty$. In order to obtain the regarding two-weight $\mathbf{L}^p \longrightarrow \mathbf{L}^q$-norm inequality in (1. 16), we develop a new framework where the product space $\mathbb{R}^n \times \mathbb{R}^m$ is decomposed into an infinitely many dyadic cones.

Namely, we define

$$\Delta_\ell \mathbf{I}_{\alpha\beta} f(x,y) = \iint_{\Gamma_\ell(x,y)} f(u,v)\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta} dudv, \qquad \ell \in \mathbb{Z}$$

where

$$\Gamma_\ell(x,y) = \left\{(u,v) \in \mathbb{R}^n \times \mathbb{R}^m : 2^{\ell-1} \leq \frac{|y-v|}{|x-u|} < 2^\ell\right\}, \quad \ell \in \mathbb{Z}.$$

Observe that $\Gamma_\ell(x,y)$ is a dyadic cone vertex on (x,y) whose eccentricity depends on $\ell \in \mathbb{Z}$.

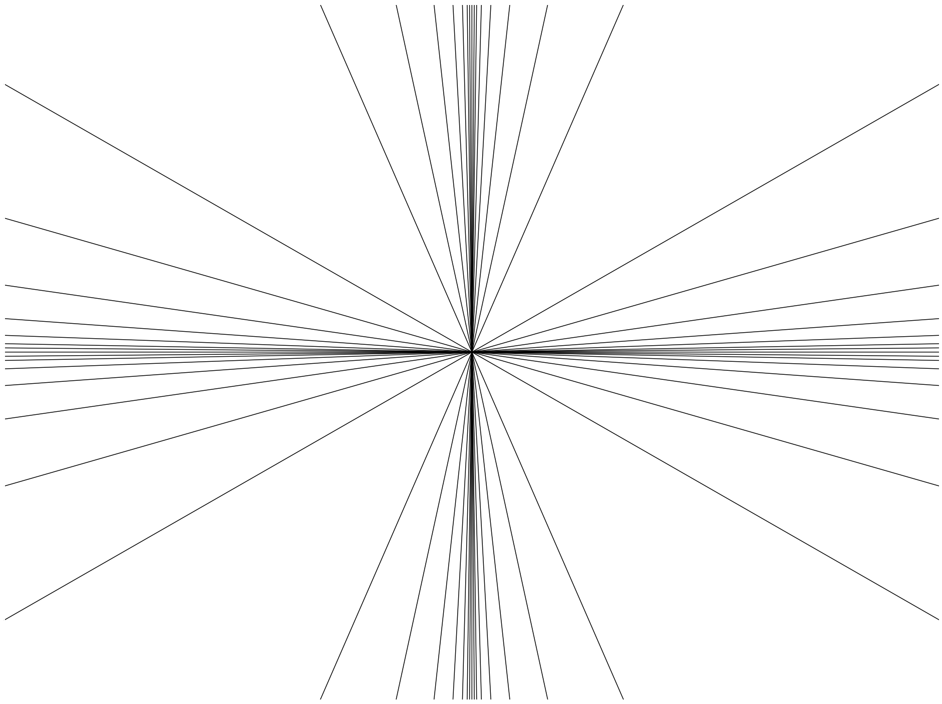

Figure 1.1: Figure 1

Each $\Delta_\ell \mathbf{I}_{\alpha\beta}$ is essentially an one-parameter fractional integral operator, satisfying the desired two-weight $\mathbf{L}^p \longrightarrow \mathbf{L}^q$-regularity estimate. Furthermore, under certain circumstances, its operator's norm decays exponentially as the eccentricity of the dyadic cone getting large.

Chapter 4: We prove **Theorem Three**, **Theorem Four** and **Theorem Five** in the direction of fractional integration on Heisenberg groups.

Let $\mathbf{M}_\gamma, 0 \leq \gamma < 1$ defined in (1. 25). For $\gamma = 0$, $\mathbf{M}_0 \doteq \mathbf{M}$ is the strong maximal operator defined on Heisenberg group. The $\mathbf{L}^p$-boundedness of $\mathbf{M}$ extensively defined on general Nilpotent Lie groups has been proved by Christ [1]. Thereby, the elegant work is done by using a number of " ingredients" developed previously by Ricci and Stein [26] and Christ [2]-[3]. We prove **Theorem Three** with a more direct approach by applying a multi-parameter covering lemma of Córdoba and Fefferman [5].

Let $\mathbf{I}_{\alpha\beta\vartheta}$ defined in (1. 34)-(1. 35). After a reformulation on the kernel $\mathbf{V}^{\alpha\beta\vartheta}$, we prove **Theorem Four** within an iteration argument.

Finally, we prove **Theorem Five** by reducing the problem into a two-weight $\mathbf{L}^p \longrightarrow \mathbf{L}^q$-regularity estimate of a bi-parameter fractional integral operator defined in $\mathbb{R}^n \times \mathbb{R}^n$. The proof is then completed as an application of **Theorem Two**.

Remark 1.3.1. *Every operator introduced above is positive definite. From now on, we assume $f \geq 0$.*

Chapter 2: Fractional integration associated with one-parameter dilation

In this chapter, we prove **Theorem One** which improves the classical **Stein-Weiss theorem** to include the case $p = 1$.

For $0 < \alpha < n$ and $\gamma, \delta < n$. We define

$$\mathbf{I}_{\alpha\gamma\delta} f(x) \;=\; \int_{\mathbb{R}^n} f(u) \left(\frac{1}{|x|}\right)^{\gamma} \left(\frac{1}{|x-u|}\right)^{n-\alpha} \left(\frac{1}{|u|}\right)^{\delta} du, \qquad x \neq 0. \tag{2. 1}$$

Theorem One can be equivalently stated as follows:

Theorem One* *Let* $\mathbf{I}_{\alpha\gamma\delta}$ *be defined in (2. 1) for* $0 < \alpha < n$ *and* $\gamma, \delta < n$. *We have*

$$\left\|\mathbf{I}_{\alpha\gamma\delta} f\right\|_{\mathbf{L}^q(\mathbb{R}^n)} \;\leq\; \mathfrak{C}_{\alpha\;\gamma\;\delta}\; \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad 1 \leq p \leq q < \infty \tag{2. 2}$$

if and only if

$$\gamma \;<\; \frac{n}{q}, \qquad \delta \;<\; n\left(\frac{p-1}{p}\right), \qquad \gamma + \delta \;\geq\; 0, \qquad \frac{\alpha}{n} \;=\; \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{n}. \tag{2. 3}$$

The following two fundamental lemmas were initially given by Stein and Weiss [30] for $p > 1$. These results are mollified now to become applicable to $p \geq 1$.

2.1 Two fundamental lemmas

Lemma One *Let* $\Omega(u, v) \geq 0$ *be defined in the quadrant* $\{(u, v) : u \geq 0,\; v \geq 0\}$ *which is homogeneous of degree* $-n$ *and for* $p \geq 1$,

$$\mathbf{A} \;\doteq\; \int_0^{\infty} \Omega(1, t) t^{n\left(\frac{p-1}{p}\right)-1} dt \;<\; \infty. \tag{2. 4}$$

Consider

$$\mathbf{U} f(x) \;=\; \int_{\mathbb{R}^n} \Omega\left(|x|, |u|\right) f(u) du. \tag{2. 5}$$

We have

$$\left\|\mathbf{U} f(x)\right\|_{\mathbf{L}^p(\mathbb{R}^n)} \;\leq\; \mathfrak{C}_{\mathbf{A}}\; \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)}. \tag{2. 6}$$

Proof Let $R = |x|$ and $r = |u|$. For $n \geq 2$, write $x = R\xi$ and $u = r\eta$ of which ξ, η are unit vectors. We have

$$\mathbf{U} f(x) \;=\; \int_{\mathbb{S}^{n-1}} \int_0^{\infty} \Omega(R, r) f(r\eta) r^{n-1} dr d\sigma(\eta) \tag{2. 7}$$

where σ denotes the surface measure on $\mathbb{S}^{n-1}$.

Consider

$$\left(\int_0^\infty \left|\int_0^\infty \Omega(R,r)f(r\eta)r^{n-1}dr\right|^p R^{n-1}dR\right)^{\frac{1}{p}}$$

$$= \left(\int_0^\infty \left|\int_0^\infty \Omega(1,t)f(tR\eta)t^{n-1}dt\right|^p R^{n-1}dR\right)^{\frac{1}{p}} \qquad r = tR \text{ and } \Omega \text{ is homogeneous of degree } -n$$

$$\leq \int_0^\infty \Omega(1,t)t^{n-1}\left(\int_0^\infty \left|f(tR\eta)\right|^p R^{n-1}dR\right)^{\frac{1}{p}} dt \qquad \text{by Minkowski integral inequality}$$

$$= \int_0^\infty \Omega(1,t)t^{n\left(\frac{p-1}{p}\right)-1}\left(\int_0^\infty \left|f(r\eta)\right|^p r^{n-1}dr\right)^{\frac{1}{p}} dt$$

$$= \mathbf{A}\left(\int_0^\infty \left|f(r\eta)\right|^p r^{n-1}dr\right)^{\frac{1}{p}}. \tag{2. 8}$$

We find

$$\begin{aligned}
\left\|\mathbf{U}f\right\|_{\mathbf{L}^p(\mathbb{R}^n)} &= \left(\int_{\mathbb{S}^{n-1}}\int_0^\infty \left|\int_{\mathbb{S}^{n-1}}\int_0^\infty \Omega(R,r)f(r\eta)r^{n-1}drd\sigma(\eta)\right|^p R^{n-1}dRd\sigma(\xi)\right)^{\frac{1}{p}} \\
&= \omega_{n-1}^{\frac{1}{p}}\left(\int_0^\infty \left|\int_{\mathbb{S}^{n-1}}\int_0^\infty \Omega(R,r)f(r\eta)r^{n-1}drd\sigma(\eta)\right|^p R^{n-1}dR\right)^{\frac{1}{p}} \\
&\leq \omega_{n-1}^{\frac{1}{p}}\int_{\mathbb{S}^{n-1}}\left(\int_0^\infty \left|\int_0^\infty \Omega(R,r)f(r\eta)r^{n-1}dr\right|^p R^{n-1}dR\right)^{\frac{1}{p}} d\sigma(\eta) \\
&\qquad\qquad \text{by Minkowski integral inequality} \\
&\leq \omega_{n-1}^{\frac{1}{p}}\mathbf{A}\int_{\mathbb{S}^{n-1}}\left(\int_0^\infty \left|f(r\eta)\right|^p r^{n-1}dr\right)^{\frac{1}{p}} d\sigma(\eta) \qquad \text{by (2. 8)} \\
&\leq \alpha\omega_{n-1}^{\frac{1}{p}}\left(\int_{\mathbb{S}^{n-1}}\int_0^\infty \left|f(r\eta)\right|^p drd\sigma(\eta)\right)^{\frac{1}{p}}\left(\int_{\mathbb{S}^{n-1}} d\sigma(\eta)\right)^{\frac{p-1}{p}} \\
&\qquad\qquad \text{by Hölder inequality} \\
&= \mathbf{A}\,\omega_{n-1}\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)}
\end{aligned} \tag{2. 9}$$

where $\omega_{n-1} = 2\pi^{\frac{n}{2}}\Gamma^{-1}\left(\frac{n}{2}\right)$ is the area of $\mathbb{S}^{n-1}$. □

When $n = 1$, simply take $d\sigma$ to be the point measure on 1 and -1. The same estimates hold in (2. 8)-(2. 9)

Lemma Two *Let $n \geq 2$. Define $\Delta(t,\xi,\eta) = \left|1 - 2t\xi\cdot\eta + t^2\right|^{\frac{1}{2}}$ for $t > 0$ and $\xi,\eta \in \mathbb{S}^{n-1}$. We have*

$$\int_{\mathbb{S}^{n-1}} \frac{1}{\Delta^{n-\alpha}(t,\xi,\eta)}d\sigma(\xi) = \int_{\mathbb{S}^{n-1}} \frac{1}{\Delta^{n-\alpha}(t,\xi,\eta)}d\sigma(\eta) \leq \mathfrak{C}\ |1-t|^{-\frac{n-\alpha}{n}}, \quad t \neq 1, \quad \xi,\eta \in \mathbb{S}^{n-1}. \tag{2. 10}$$

Proof Observe that $\Delta(t,\xi,\eta)$ is symmetric *w.r.t* ξ and η. For $0<t<1$, we have

$$\mathbf{P}(\xi,t\eta) \;=\; \frac{1-|t\eta|^2}{|\xi-t\eta|^n} \;=\; \frac{1-t^2}{\Delta^n(t,\xi,\eta)}, \qquad \xi,\eta\in\mathbb{S}^{n-1} \tag{2. 11}$$

which is the Poisson kernel on the unit sphere $\mathbb{S}^{n-1}$. A direct computation shows

$$\Delta_\eta \mathbf{P}(\xi,t\eta) \;=\; 0, \qquad \xi\in\mathbb{S}^{n-1} \tag{2. 12}$$

where Δ_η is the Laplacian operator *w.r.t* η.
By using the mean value property of harmonic functions, we find

$$1 \;=\; \mathbf{P}(\xi,0) \;=\; \frac{1}{\omega_{n-1}}\int_{\mathbb{S}^{n-1}}\mathbf{P}(\xi,t\eta)d\sigma(\eta), \qquad 0<t<1. \tag{2. 13}$$

This further implies

$$\frac{1}{\omega_{n-1}}\int_{\mathbb{S}^{n-1}}\frac{1-t^2}{\Delta^n(t,\xi,\eta)}d\sigma(\eta) \;=\; 1, \qquad 0<t<1. \tag{2. 14}$$

On the other hand, write $0<s=t^{-1}<1$ for $t>1$. From (2. 11), we have

$$\begin{aligned}\frac{1-t^2}{\Delta^n(t,\xi,\eta)} \;&=\; \frac{1-t^2}{\left|1-2t\xi\cdot\eta+t^2\right|^{\frac{n}{2}}} \;=\; \frac{t^2(s^2-1)}{t^n\left|s^2-2s\xi\cdot\eta+1\right|^{\frac{n}{2}}}\\ &=\; -t^{2-n}\frac{1-s^2}{\Delta^n(s,\xi,\eta)} \;=\; -t^{2-n}\mathbf{P}(\xi,s\eta).\end{aligned} \tag{2. 15}$$

By using (2. 13) and (2. 15), we find

$$\frac{1}{\omega_{n-1}}\int_{\mathbb{S}^{n-1}}\frac{1-t^2}{\Delta^n(t,\xi,\eta)}d\sigma(\eta) \;=\; -t^{2-n}\,\frac{1}{\omega_{n-1}}\int_{\mathbb{S}^{n-1}}\mathbf{P}(\xi,s\eta)d\sigma(\eta) \;=\; -t^{2-n}, \quad t>1. \tag{2. 16}$$

By putting together (2. 14) and (2. 16), we obtain

$$\frac{1}{\omega_{n-1}}\int_{\mathbb{S}^{n-1}}\frac{1}{\Delta^n(t,\xi,\eta)}d\sigma(\eta) \;\le\; \frac{1}{|1-t^2|} \;<\; \frac{1}{|1-t|}. \tag{2. 17}$$

By applying Hölder inequality, we have

$$\int_{\mathbb{S}^{n-1}}\frac{1}{\Delta^{n-\alpha}(t,\xi,\eta)}d\sigma(\eta) \;\le\; \left\{\int_{\mathbb{S}^{n-1}}\left[\frac{1}{\Delta^{n-\alpha}(t,\xi,\eta)}\right]^{\frac{n}{n-\alpha}}d\sigma(\eta)\right\}^{\frac{n-\alpha}{n}}\left\{\int_{\mathbb{S}^{n-1}}d\sigma(\eta)\right\}^{\frac{\alpha}{n}} \tag{2. 18}$$

Then, from (2. 17) and (2. 18), we have

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} \frac{1}{\Delta^{n-\alpha}(t,\xi,\eta)} d\sigma(\eta) &\leq \left\{\int_{\mathbb{S}^{n-1}} \frac{1}{\Delta^{n}(t,\xi,\eta)} d\sigma(\eta)\right\}^{\frac{n-\alpha}{n}} (\omega_{n-1})^{\frac{\alpha}{n}} \\ &\leq \left(\frac{1}{|1-t|}\right)^{\frac{n-\alpha}{n}} (\omega_{n-1})^{\frac{n-\alpha}{n}} (\omega_{n-1})^{\frac{\alpha}{n}} \qquad \text{by (2. 17)} \\ &= \omega_{n-1}\, |1-t|^{-\frac{n-\alpha}{n}}.\end{aligned} \tag{2. 19}$$

2.2 Proof of Theorem One: necessary conditions

We show the weighted norm inequality in (2. 2) implying the constraints in (2. 3).

Case 1 Let $p = 1$. Denote $\mathbf{Q}$ as any cube in $\mathbb{R}^n$. Choose $f = \chi_{\mathbf{Q}}$ which is an indicator function supported in $\mathbf{Q}$. The norm inequality in (2. 2) implies

$$\sup_{\mathbf{Q}\subset\mathbb{R}^n} \mathbf{vol}\{\mathbf{Q}\}^{\frac{\alpha}{n}-1+\frac{1}{q}} \left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}\}} \int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\gamma q} dx\right\}^{\frac{1}{q}} \left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}\}} \int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\delta} dx\right\} < \infty. \tag{2. 20}$$

A standard exercise of changing dilations inside (2. 20) shows that $\frac{\alpha}{n} = 1 - \frac{1}{q} + \frac{\gamma+\delta}{n}$ is an necessary condition. Moreover, it is essential to have $\gamma < \frac{n}{q}$ for the local integrability of $|x|^{-\gamma q}$. We claim that $\frac{\alpha}{n} - 1 + \frac{1}{q} \geq 0$. Together with $\frac{\alpha}{n} = 1 - \frac{1}{q} + \frac{\gamma+\delta}{n}$, we must have $\gamma + \delta \geq 0$. Suppose $\frac{\alpha}{n} - 1 + \frac{1}{q} < 0$. Let $\mathbf{Q}$ be centered on some $x_o \neq 0$. By shrinking $\mathbf{Q}$ to x_o and applying Lebesgue Differentiation Theorem, we find

$$\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}\}} \int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\gamma q} dx\right\}^{\frac{1}{q}} \left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}\}} \int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\delta} dx\right\} = |x_o|^{-(\gamma+\delta)} > 0. \tag{2. 21}$$

On the other hand, $\mathbf{vol}\{\mathbf{Q}\}^{\frac{\alpha}{n}-1+\frac{1}{q}} \longrightarrow \infty$. This contradicts to (2. 20).

Let $I_{\alpha\gamma\delta} f$ be defined in (2. 1). Assert $f \geq 0$, $f \in \mathbf{L}^1(\mathbb{R}^n)$ supported in the unit ball, denoted by $\mathbf{B}$. We have

$$\begin{aligned} I_{\alpha\gamma\delta} f(x) &\geq \chi_{(|x|>10)} \int_{\mathbf{B}} f(u) \left(\frac{1}{|x|}\right)^{\gamma} \left(\frac{1}{|x-u|}\right)^{n-\alpha} \left(\frac{1}{|u|}\right)^{\delta} du \\ &> 2^{\alpha-n} \left(\frac{1}{|x|}\right)^{n-\alpha+\gamma} \chi_{(|x|>10)} \int_{\mathbf{B}} f(u) \left(\frac{1}{|u|}\right)^{\delta} du. \end{aligned} \tag{2. 22}$$

Observe that if $\delta \geq 0$, then $\frac{\alpha}{n} = 1 - \frac{1}{q} + \frac{\gamma+\delta}{n}$ implies $n - \alpha + \gamma \leq \frac{n}{q}$. Consequently, $\left(I_{\alpha\gamma\delta} f\right)^q$ cannot be integrable in $\mathbb{R}^n$. Therefore, we also need $\delta < 0$.

Case 2 Let $p > 1$. Consider $f(x) = \chi_{\mathbf{Q}}(x)|x|^{-\delta\left(\frac{1}{p-1}\right)}$. The norm inequality in (2. 2) implies

$$\sup_{\mathbf{Q}\subset\mathbb{R}^n} \mathbf{vol}\{\mathbf{Q}\}^{\frac{\alpha}{n}-\frac{1}{p}+\frac{1}{q}} \left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}\}} \int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\gamma q} dx\right\}^{\frac{1}{q}} \left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}\}} \int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\delta\left(\frac{p}{p-1}\right)} dx\right\}^{\frac{p-1}{p}} < \infty. \tag{2. 23}$$

We essentially need $\gamma < n/q$ and $\delta < n\left(\frac{p-1}{p}\right)$ for which $|x|^{-\gamma q}$ and $|x|^{-\delta\left(\frac{p}{p-1}\right)}$ are locally integrable.

By changing dilations inside (2. 23), we find $\frac{\alpha}{n}=\frac{1}{p}-\frac{1}{q}+\frac{\gamma+\delta}{n}$ is an necessity. Moreover, we claim $\frac{\alpha}{n}\geq\frac{1}{p}-\frac{1}{q}$. Together with $\frac{\alpha}{n}=\frac{1}{p}-\frac{1}{q}+\frac{\gamma+\delta}{n}$, we must have $\gamma+\delta\geq 0$.
Suppose $\frac{\alpha}{n}<\frac{1}{p}-\frac{1}{q}$. Let $\mathbf{Q}$ be centered on some $x_o\neq 0$. By shrinking $\mathbf{Q}$ to x_o and applying Lebesgue Differentiation Theorem, we find

$$\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}\}}\int_{\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\gamma q}dx\right\}^{\frac{1}{q}}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}\}}\int_{\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\delta\left(\frac{p}{p-1}\right)}\right\}^{\frac{p-1}{p}}=|x_o|^{-(\gamma+\delta)}>0. \tag{2. 24}$$

On the other hand, $\mathbf{vol}\{\mathbf{Q}\}^{\frac{\alpha}{n}-\frac{1}{p}+\frac{1}{q}}\longrightarrow\infty$. We reach a contradiction to (2. 23).

2.3 Proof of Theorem One: sufficient conditions

Consider

$$\mathbf{I}_{\alpha\gamma\delta}f(x)=\mathbf{U}_1f(x)+\mathbf{U}_2f(x)+\mathbf{U}_3f(x),\quad f\geq 0 \tag{2. 25}$$

where

$$\begin{aligned}\mathbf{U}_1f(x)&=\int_{\mathbb{R}^n}f(y)\Omega_1(x,u)du,\\ \Omega_1(x,u)&=\begin{cases}\left(\frac{1}{|x|}\right)^{\gamma}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|u|}\right)^{\delta}, & 0\leq\frac{|u|}{|x|}\leq\frac{1}{2},\\ 0, & \frac{|u|}{|x|}>\frac{1}{2};\end{cases}\end{aligned} \tag{2. 26}$$

$$\begin{aligned}\mathbf{U}_2f(x)&=\int_{\mathbb{R}^n}f(u)\Omega_2(x,u)du,\\ \Omega_2(x,u)&=\begin{cases}\left(\frac{1}{|x|}\right)^{\gamma}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|u|}\right)^{\delta}, & \frac{|u|}{|x|}\geq 2,\\ 0, & 0\leq\frac{|u|}{|x|}<2;\end{cases}\end{aligned} \tag{2. 27}$$

$$\begin{aligned}\mathbf{U}_3f(x)&=\int_{\mathbb{R}^n}f(u)\Omega_3(x,u)du,\\ \Omega_3(x,u)&=\begin{cases}\left(\frac{1}{|x|}\right)^{\gamma}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|u|}\right)^{\delta}, & \frac{1}{2}<\frac{|u|}{|x|}<2,\\ 0, & 0\leq\frac{|u|}{|x|}\leq\frac{1}{2}\text{ or }\frac{|u|}{|x|}\geq 2.\end{cases}\end{aligned} \tag{2. 28}$$

2.3.1 For $1\leq p=q<\infty$

Let $1\leq p=q<\infty$. The homogeneity condition $\frac{\alpha}{n}=\frac{1}{p}-\frac{1}{q}+\frac{\gamma+\delta}{n}$ implies $\alpha=\gamma+\delta$.
Note that $|x-u|\geq\frac{1}{2}|x|$ if $\frac{|u|}{|x|}\leq\frac{1}{2}$. From (2. 26), we find

$$\Omega_1(x,u)\leq 2^{n-\alpha}\left(\frac{1}{|x|}\right)^{n-\alpha+\gamma}\left(\frac{1}{|u|}\right)^{\delta},\qquad 0\leq\frac{|u|}{|x|}\leq\frac{1}{2}; \tag{2. 29}$$

and $\Omega_1(x,u)=0$ for $\frac{|u|}{|x|}>\frac{1}{2}$.

Because $\delta < n\left(\frac{p-1}{p}\right)$, we have

$$\mathbf{A}_1 \doteq \int_0^\infty \Omega_1(1,t)t^{n\left(\frac{p-1}{p}\right)-1}dt \leq 2^{n-\alpha}\int_0^{\frac{1}{2}} t^{n\left(\frac{p-1}{p}\right)-\delta-1}dt < \infty. \tag{2. 30}$$

By applying **Lemma One**, we obtain

$$\left\|\mathbf{U}_1 f\right\|_{\mathbf{L}^p(\mathbb{R}^n)} \leq \mathfrak{C}_{\alpha\,\delta\,p}\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad 1 \leq p < \infty. \tag{2. 31}$$

On the other hand, $|x-u| \geq \frac{1}{2}|u|$ if $\frac{|u|}{|x|} \geq 2$. From (2. 27), we find

$$\Omega_2(x,u) \leq 2^{n-\alpha}\left(\frac{1}{|x|}\right)^\gamma \left(\frac{1}{|u|}\right)^{n-\alpha+\delta}, \qquad \frac{|u|}{|x|} \geq 2; \tag{2. 32}$$

and $\Omega_2(x,u) = 0$ for $0 \leq \frac{|u|}{|x|} < 2$.
Because $\gamma < n/q = n/p$ and $\alpha = \gamma + \delta$, we have

$$\begin{aligned}\mathbf{A}_2 &\doteq \int_0^\infty \Omega_2(1,t)t^{n\left(\frac{p-1}{p}\right)-1}dt \\ &\leq 2^{n-\alpha}\int_2^\infty t^{n\left(\frac{p-1}{p}\right)-n+\alpha-\delta-1}dt = \int_2^\infty t^{-\frac{n}{p}+\gamma-1}dt < \infty.\end{aligned} \tag{2. 33}$$

By applying **Lemma One**, we obtain

$$\left\|\mathbf{U}_2 f\right\|_{\mathbf{L}^p(\mathbb{R}^n)} \leq \mathfrak{C}_{\alpha\,\gamma\,p}\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad 1 \leq p < \infty. \tag{2. 34}$$

For $n \geq 2$. Write $x = R\xi$ and $u = r\eta$ for $\xi, \eta \in \mathbb{S}^{n-1}$. Recall Ω_3 defined in (2. 28). We have

$$\begin{aligned}\mathbf{U}_3 f(x) &= \int_{\mathbb{R}^n}\Omega_3(x,u)f(u)du \\ &= \int_{\mathbb{S}^{n-1}}\int_0^\infty \Omega_3(R\xi, r\eta)f(r\eta)r^{n-1}drd\sigma(\eta) \\ &= \int_{\mathbb{S}^{n-1}}\int_0^\infty \Omega_3(\xi, t\eta)f(tR\eta)t^{n-1}dtd\sigma(\eta) \\ &\qquad\qquad r = tR,\ \Omega_3 \text{ is homogeneous of degree } -n \\ &= \int_{\mathbb{S}^{n-1}}\int_{\frac{1}{2}}^2 \frac{1}{|\xi - t\eta|^{n-\alpha}}f(tR\eta)t^{n-1-\delta}dtd\sigma(\eta) \\ &= \int_{\mathbb{S}^{n-1}}\int_{\frac{1}{2}}^2 \frac{1}{\left|(\xi-t\eta)\cdot(\xi-t\eta)\right|^{\frac{n-\alpha}{2}}}t^{n-1-\delta}dtd\sigma\eta \\ &= \int_{\mathbb{S}^{n-1}}\int_{\frac{1}{2}}^2 \frac{1}{|1-2t\xi\cdot\eta+t^2|^{\frac{n-\alpha}{2}}}f(tR\eta)t^{n-1-\delta}dtd\sigma(\eta) \\ &\leq \mathfrak{C}\int_{\mathbb{S}^{n-1}}\int_{\frac{1}{2}}^2 \frac{1}{\Delta^{n-\alpha}(t,\xi,\eta)}f(tR\eta)dtd\sigma(\eta).\end{aligned} \tag{2. 35}$$

For $n=1$, take $d\sigma$ to be the point measure on 1 and -1 inside (2. 35). We find

$$\mathbf{U}_3 f(x) \leq \mathfrak{C} \int_{\frac{1}{2}}^{2} |1-t|^{\alpha-1} \left[f(tR)+f(-tR)\right] dt. \tag{2. 36}$$

From (2. 35), we have

$$\begin{aligned}
&\left\{\int_{\mathbb{R}^n} \left|\mathbf{U}_3 f(x)\right|^p dx\right\}^{\frac{1}{p}} \\
\leq\ & \mathfrak{C}\left\{\int_{\mathbb{S}^{n-1}}\int_0^\infty \left\{\int_{\mathbb{S}^{n-1}}\int_{\frac{1}{2}}^2 \frac{1}{\Delta^{n-\alpha}(t,\xi,\eta)} f(tR\eta)dtd\sigma(\eta)\right\}^p R^{n-1}dRd\sigma(\xi)\right\}^{\frac{1}{p}} \\
\leq\ & \int_{\frac{1}{2}}^2 \left\{\int_0^\infty \int_{\mathbb{S}^{n-1}} \left\{\int_{\mathbb{S}^{n-1}} \frac{1}{\Delta^{n-\alpha}(t,\xi,\eta)} f(tR\eta)d\sigma(\eta)\right\}^p d\sigma(\xi)R^{n-1}dR\right\}^{\frac{1}{p}} dt \\
& \qquad\qquad\qquad \text{by Minkowski integral inequality} \\
\leq\ & \mathfrak{C}\int_{\frac{1}{2}}^2 \left\{\int_0^\infty \int_{\mathbb{S}^{n-1}} \left\{\int_{\mathbb{S}^{n-1}} \frac{\left|f(tR\eta)\right|^p}{\Delta^{n-\alpha}(t,\xi,\eta)} d\sigma(\eta)\right\}\left\{\int_{\mathbb{S}^{n-1}} \frac{1}{\Delta^{n-\alpha}(t,\xi,\eta)} d\sigma(\eta)\right\}^{p-1} d\sigma(\xi)R^{n-1}dR\right\}^{\frac{1}{p}} dt \\
& \qquad\qquad\qquad \text{by Hölder inequality} \\
\leq\ & \mathfrak{C}\int_{\frac{1}{2}}^2 \left\{\int_0^\infty \int_{\mathbb{S}^{n-1}} \left\{\int_{\mathbb{S}^{n-1}} \frac{\left|f(tR\eta)\right|^p}{\Delta^{n-\alpha}(t,\xi,\eta)} d\sigma(\eta)\right\} |1-t|^{\left[-\frac{n-\alpha}{n}\right](p-1)} d\sigma(\xi)R^{n-1}dR\right\}^{\frac{1}{p}} dt \\
& \qquad\qquad\qquad \text{by } \textbf{Lemma Two} \\
=\ & \mathfrak{C}\int_{\frac{1}{2}}^2 \left\{\int_0^\infty \int_{\mathbb{S}^{n-1}} \left\{|1-t|^{\left[-\frac{n-\alpha}{n}\right](p-1)} \int_{\mathbb{S}^{n-1}} \frac{1}{\Delta^{n-\alpha}(t,\xi,\eta)} d\sigma(\xi)\right\} \left[f(tR\eta)\right]^p R^{n-1}dRd\sigma(\eta)\right\}^{\frac{1}{p}} dt \\
\leq\ & \mathfrak{C}\int_{\frac{1}{2}}^2 \left\{|1-t|^{\left[-\frac{n-\alpha}{n}\right]p} \int_0^\infty \int_{\mathbb{S}^{n-1}} \left[f(tR\eta)\right]^p R^{n-1}dRd\sigma(\eta)\right\}^{\frac{1}{p}} dt \\
& \qquad\qquad\qquad \text{by } \textbf{Lemma Two} \\
=\ & \mathfrak{C}\int_{\frac{1}{2}}^2 \left\{\int_0^\infty \int_{\mathbb{S}^{n-1}} \left[f(tR\eta\right]^p d\sigma(\eta)R^{n-1}dR\right\}^{\frac{1}{p}} |1-t|^{\frac{\alpha-n}{n}} dt \\
=\ & \mathfrak{C}\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)} \int_{\frac{1}{2}}^2 |1-t|^{\frac{\alpha-n}{n}} dt \\
\leq\ & \mathfrak{C}_\alpha \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad 1\leq p<\infty.
\end{aligned} \tag{2. 37}$$

Moreover, by using (2. 36), we find

$$
\begin{aligned}
\left\|\mathbf{U}_3 f\right\|_{\mathbf{L}^p(\mathbb{R}^n)} &\leq \mathfrak{C}\left\{\int_0^\infty \left\{\int_{\frac{1}{2}}^2 |1-t|^{\alpha-1}\left[f(tR)+f(-tR)\right]dt\right\}^p dR\right\}^{\frac{1}{p}} \\
&\leq \mathfrak{C}\int_{\frac{1}{2}}^2 \left\{\int_0^\infty \left[f(tR)+f(-tR)\right]^p dR\right\}^{\frac{1}{p}} |1-t|^{\alpha-1}dt \\
&\qquad \text{by Minkowski integral inequality} \qquad (2.\ 38) \\
&= \mathfrak{C}\left\|f\right\|_{\mathbf{L}^p(\mathbb{R})}\int_{\frac{1}{2}}^2 |1-t|^{\alpha-1}dt \\
&\leq \mathfrak{C}_\alpha \left\|f\right\|_{\mathbf{L}^p(\mathbb{R})}, \qquad 1\leq p<\infty.
\end{aligned}
$$

2.3.2 For $1 \leq p < q < \infty$

Consider $1\leq p<q<\infty$. Assert

$$
\mathbf{V}_\delta f(x) = |x|^{-n+\delta}\int_{|u|<|x|} |u|^{-\delta} f(u)du, \qquad \delta < n\left(\frac{p-1}{p}\right). \tag{2. 39}
$$

We claim

$$
\left\|\mathbf{V}_\delta f\right\|_{\mathbf{L}^p(\mathbb{R}^n)} \leq \mathfrak{C}_{\delta\, p}\, ||f||_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad 1\leq p<\infty. \tag{2. 40}
$$

Write

$$
\mathbf{V}_\delta f(x) = \int_{\mathbb{R}^n} \Omega(|x|,|u|)f(u)du, \qquad \Omega(R,r) = \begin{cases} R^{-n+\delta}r^{-\delta} & \text{if } r<R \\ 0 & \text{otherwise.}\end{cases} \tag{2. 41}
$$

Observe that Ω in (2. 41) is homogeneous of degree $-n$. Moreover,

$$
\int_0^\infty \Omega(1,t)t^{n\left(\frac{p-1}{p}\right)-1}dt = \int_0^1 t^{n\left(\frac{p-1}{p}\right)-\delta-1}dt < \infty \tag{2. 42}
$$

provided by $\delta < n\left(\frac{p-1}{p}\right)$. **Lemma One** implies (2. 40).
On the other hand, $\mathbf{V}_\delta f$ defined in (2. 39) satisfies

$$
\begin{aligned}
\mathbf{V}_\delta f(x) &\leq |x|^{-n+\delta}\left\{\int_{|u|<|x|} |u|^{-\delta\left(\frac{p}{p-1}\right)}du\right\}^{\frac{p-1}{p}} \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)} \qquad \text{by Hölder inequality} \\
&\leq \mathfrak{C}_{\delta\, p}\, |x|^{-\frac{n}{p}} \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad 1\leq p<\infty.
\end{aligned} \tag{2. 43}
$$

Recall $\mathbf{U}_1 f$ defined in (2. 26). Note that $0\leq \frac{|u|}{|x|}\leq \frac{1}{2}$ implies $\frac{1}{2}|x|\leq |x|-|u|\leq |x-u|$. Let $f,g\geq 0$

and $f\in\mathbf{L}^p(\mathbb{R}^n)$, $g\in\mathbf{L}^{\frac{q}{q-1}}(\mathbb{R}^n)$. We have

$$\begin{array}{lr}
\displaystyle\int_{\mathbb{R}^n}\mathbf{U}_1 f(x)g(x)dx ~=~ \int_{\mathbb{R}^n}\left\{\int_{|u|\leq\frac{1}{2}|x|}\frac{f(u)g(x)}{|x|^{\gamma}\,|x-u|^{n-\alpha}\,|u|^{\delta}}du\right\}dx \\
\displaystyle\leq~\mathfrak{C}\int_{\mathbb{R}^n}\left\{\int_{|u|<|x|}\frac{f(u)g(x)}{|x|^{\gamma+n-\alpha}\,|u|^{\delta}}du\right\}dx \\
\displaystyle=~\mathfrak{C}\int_{\mathbb{R}^n}|x|^{\alpha-(\gamma+\delta)}g(x)\left\{|x|^{-n+\delta}\int_{|u|<|x|}f(u)|u|^{-\delta}du\right\}dx \\
\displaystyle=~\mathfrak{C}\int_{\mathbb{R}^n}|x|^{\alpha-(\gamma+\delta)}g(x)\mathbf{V}_{\delta}f(x)dx \\
\displaystyle\leq~\mathfrak{C}\left\{\int_{\mathbb{R}^n}|x|^{\left[\alpha-(\gamma+\delta)\right]q}\Big(\mathbf{V}_{\delta}f\Big)^{q}(x)dx\right\}^{\frac{1}{q}}\left\|g\right\|_{\mathbf{L}^{\frac{q}{q-1}}(\mathbb{R}^n)}\quad\text{by Hölder inequality.} \\
& (2.\ 44)
\end{array}$$

Let $\frac{\alpha}{n}=\frac{1}{p}-\frac{1}{q}+\frac{\gamma+\delta}{n}$, $1\leq p<q<\infty$. We find

$$\begin{array}{lr}
\displaystyle\left\{\int_{\mathbb{R}^n}|x|^{\left[\alpha-(\gamma+\delta)\right]q}\Big(\mathbf{V}_{\delta}f\Big)^{q}(x)dx\right\}^{\frac{1}{q}} \\
\displaystyle\leq~\left\{\int_{\mathbb{R}^n}|x|^{\left[\alpha-(\gamma+\delta)\right]q}|x|^{-n(\frac{q}{p}-1)}\left\|f\right\|^{q-p}_{\mathbf{L}^p(\mathbb{R}^n)}\left(\mathbf{V}_{\delta}f\right)^{p}(x)dx\right\}^{\frac{1}{q}}\qquad\text{by (2. 43)} \\
& (2.\ 45) \\
\displaystyle=~\left\|f\right\|^{1-\frac{p}{q}}_{\mathbf{L}^p(\mathbb{R}^n)}\left\{\int_{\mathbb{R}^n}\left(\mathbf{V}_{\delta}f\right)^{p}(x)dx\right\}^{\frac{1}{q}} \\
\displaystyle\leq~\mathfrak{C}_{\delta\ p}\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)}\qquad\text{by (2. 40).}
\end{array}$$

From (2. 44)-(2. 45), we conclude

$$\left\|\mathbf{U}_1 f\right\|_{\mathbf{L}^q(\mathbb{R}^n)}~\leq~\mathfrak{C}\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)},\qquad 1\leq p<q<\infty.\tag{2. 46}$$

Consider

$$\mathbf{V}_{\gamma}g(x)~=~|x|^{-n+\gamma}\int_{|u|<|x|}|u|^{-\gamma}g(u)du,\qquad \gamma<\frac{n}{q}.\tag{2. 47}$$

We claim

$$\left\|\mathbf{V}_{\gamma}g\right\|_{\mathbf{L}^{\frac{q}{q-1}}(\mathbb{R}^n)}~\leq~\mathfrak{C}_{\gamma\ q}\left\|g\right\|_{\mathbf{L}^{\frac{q}{q-1}}(\mathbb{R}^n)},\qquad 1<q<\infty.\tag{2. 48}$$

Write

$$\mathbf{V}_{\gamma}g(x)~=~\int_{\mathbb{R}^n}\Omega(|x|,|u|)g(u)du,\qquad \Omega(R,r)~=~\begin{cases}R^{-n+\gamma}r^{-\gamma} & \text{if } r<R\\ 0 & \text{otherwise.}\end{cases}\tag{2. 49}$$

Observe that Ω in (2. 49) is homogeneous of degree $-n$. Moreover,

$$\int_0^\infty \Omega(1,t) t^{\frac{n}{q}-1} dt \;=\; \int_0^1 t^{\frac{n}{q}-\gamma-1} dt \;<\; \infty \tag{2. 50}$$

provided by $\gamma < \frac{n}{q}$. **Lemma One** implies (2. 48).
On the other hand, $\mathbf{V}_\gamma g$ defined in (2. 47) satisfies

$$\begin{aligned} \mathbf{V}_\gamma g(u) \;&=\; |u|^{-n+\gamma} \int_{|x|<|u|} |x|^{-\gamma}\, g(x) dx \\ &\leq\; |u|^{-n+\gamma} \left\{ \int_{|x|<|u|} |x|^{-\gamma q} dx \right\}^{\frac{1}{q}} \left\| g \right\|_{\mathbf{L}^{\frac{q}{q-1}}(\mathbb{R}^n)} \qquad \text{by Hölder inequality} \\ &\leq\; \mathfrak{C}_{\gamma\, q} |u|^{-n\left(\frac{q-1}{q}\right)} \left\| g \right\|_{\mathbf{L}^{\frac{q}{q-1}}(\mathbb{R}^n)}. \end{aligned} \tag{2. 51}$$

Recall $\mathbf{U}_2 f$ defined in (2. 27). Note that $\frac{|u|}{|x|} \geq 2$ implies $\frac{1}{2}|u| \leq |u| - |x| \leq |x-u|$. We have

$$\begin{aligned} \int_{\mathbb{R}^n} \mathbf{U}_2 f(x) g(x) dx \;&=\; \int_{\mathbb{R}^n} \left\{ \int_{|u|\geq 2|x|} \frac{f(u)}{|x|^\gamma\, |x-u|^{n-\alpha} |u|^\delta} du \right\} g(x) dx \\ &\leq\; \mathfrak{C} \int_{\mathbb{R}^n} \left\{ \int_{|u|>|x|} \frac{f(u) g(x)}{|x|^\gamma\, |u|^{n-\alpha+\delta}} du \right\} dx. \end{aligned} \tag{2. 52}$$

By using (2. 52) and Tonelli's theorem, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \mathbf{U}_2 f(x) g(x) dx \;&\leq\; \mathfrak{C} \int_{\mathbb{R}^n} |u|^{\alpha-(\gamma+\delta)} f(u) \left\{ |u|^{-n+\gamma} \int_{|x|<|u|} g(x) |x|^{-\gamma} dx \right\} du \\ &=\; \mathfrak{C} \int_{\mathbb{R}^n} |u|^{\alpha-(\gamma+\delta)} f(u) \mathbf{V}_\gamma g(u) du \qquad \text{by (2. 47)} \\ &\leq\; \mathfrak{C} \left\| f \right\|_{\mathbf{L}^p(\mathbb{R}^n)} \left\{ \int_{\mathbb{R}^n} |u|^{\left[\alpha-(\gamma+\delta)\right]\frac{p}{p-1}} \left(\mathbf{V}_\gamma g \right)^{\frac{p}{p-1}} (u) du \right\}^{\frac{p-1}{p}} \qquad \text{by Hölder inequality.} \end{aligned} \tag{2. 53}$$

Let $\frac{\alpha}{n} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{n}$, $1 \leq p < q < \infty$. For $p = 1$, we find

$$\begin{aligned} \left\| |u|^{\alpha-(\gamma+\delta)}\, \mathbf{V}_\gamma g(u) \right\|_{\mathbf{L}^\infty(\mathbb{R}^n)} &\leq \mathfrak{C}_{\gamma\, q}\, |u|^{\alpha-(\gamma+\delta)}\, |u|^{-n\left(\frac{q-1}{q}\right)} \left\| g \right\|_{\mathbf{L}^{\frac{q}{q-1}}(\mathbb{R}^n)} \qquad \text{by (2. 51)} \\ &=\; \mathfrak{C}_{\gamma\, q} \left\| g \right\|_{\mathbf{L}^{\frac{q}{q-1}}(\mathbb{R}^n)}. \end{aligned} \tag{2. 54}$$

For $p > 1$, write

$$\left\{ \int_{\mathbb{R}^n} |u|^{\left[\alpha-(\gamma+\delta)\right]} \left(\mathbf{V}_\gamma g \right)^{\frac{p}{p-1}} (u) du \right\}^{\frac{p-1}{p}} \;=\; \left\{ \int_{\mathbb{R}^n} |u|^{\left[\alpha-(\gamma+\delta)\right]\frac{p}{p-1}} \left(\mathbf{V}_\gamma g \right)^{\left[\frac{p}{p-1}-\frac{q}{q-1}\right]} (u) \left(\mathbf{V}_\gamma g \right)^{\frac{q}{q-1}} (u) du \right\}^{\frac{p-1}{p}} \tag{2. 55}$$

By using (2. 51) again, we have

$$
\begin{aligned}
|u|^{\left[\alpha-(\gamma+\delta)\right]\frac{p}{p-1}}\left(\mathbf{V}_\gamma g\right)^{\left[\frac{p}{p-1}-\frac{q}{q-1}\right]}(u) &\leq \mathfrak{C}_{\gamma\, q}\, |u|^{\left[\alpha-(\gamma+\delta)\right]\frac{p}{p-1}}\, |u|^{-n\left(\frac{q-1}{q}\right)\left[\frac{p}{p-1}-\frac{q}{q-1}\right]}\left\|g\right\|_{\mathbf{L}^{\frac{q}{q-1}}(\mathbb{R}^n)}^{\frac{p}{p-1}-\frac{q}{q-1}} \\
&= \mathfrak{C}_{\gamma\, q}\, |u|^{n\left[\frac{1}{p}-\frac{1}{q}\right]\frac{p}{p-1}-n\left(\frac{q-1}{q}\right)\left[\frac{p}{p-1}-\frac{q}{q-1}\right]}\left\|g\right\|_{\mathbf{L}^{\frac{q}{q-1}}(\mathbb{R}^n)}^{\frac{p}{p-1}-\frac{q}{q-1}} \qquad (2.\ 56) \\
&= \mathfrak{C}_{\gamma\, q}\left\|g\right\|_{\mathbf{L}^{\frac{q}{q-1}}(\mathbb{R}^n)}^{\frac{p}{p-1}-\frac{q}{q-1}}.
\end{aligned}
$$

From (2. 53)-(2. 56), together with the $\mathbf{L}^{\frac{q}{q-1}}$-estimate in (2. 48), we conclude

$$
\left\|\mathbf{U}_2 f\right\|_{\mathbf{L}^q(\mathbb{R}^n)} \leq \mathfrak{C}\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad 1 \leq p < q < \infty. \qquad (2.\ 57)
$$

Recall $\mathbf{U}_3 f$ defined in (2. 28). We have

$$
\begin{aligned}
\mathbf{U}_3 f(x) &= \int_{\frac{1}{2}|x|<|u|<2|x|} f(u)\left(\frac{1}{|x|}\right)^{\gamma}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|u|}\right)^{\delta} du \\
&\leq \mathfrak{C}\int_{\mathbb{R}^n} f(u)\left(\frac{1}{|x-u|}\right)^{n-(\alpha-\gamma-\delta)} du.
\end{aligned}
\qquad (2.\ 58)
$$

Note that $\frac{\alpha-\gamma-\delta}{n} = \frac{1}{p}-\frac{1}{q}$, $1 \leq p < q < \infty$. Define

$$
\mathbf{I}_{\alpha-\gamma-\delta} f(x) = \int_{\mathbb{R}^n} f(u)\left(\frac{1}{|x-u|}\right)^{n-(\alpha-\gamma-\delta)} du. \qquad (2.\ 59)
$$

For $p > 1$, **Hardy-Littlewood-Sobolev theorem** implies

$$
\left\|\mathbf{I}_{\alpha-\gamma-\delta} f\right\|_{\mathbf{L}^q(\mathbb{R}^n)} \leq \mathfrak{C}_{p\, q}\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)}. \qquad (2.\ 60)
$$

Remark 2.3.1. *For $p = 1$, we have $\mathbf{I}_{\alpha-\gamma-\delta}\colon \mathbf{L}^1(\mathbb{R}^n) \longrightarrow \mathbf{L}^{q,\infty}(\mathbb{R}^n)$. See chapter V of Stein [29].*

Given $E \subset \mathbb{R}^n$, denote $\mathbf{vol}\{E\} = \int_E dx$. From (2. 58)-(2. 59) and **Remark 2.3.1**, we have

$$
\begin{aligned}
\lambda^q \mathbf{vol}\left\{x \in \mathbb{R}^n \colon \mathbf{U}_3 f(x) > \lambda\right\} &\leq \lambda^q \mathbf{vol}\left\{x \in \mathbb{R}^n \colon \mathbf{I}_{\alpha-\gamma-\delta} f(x) > \lambda\right\} \\
&\leq \mathfrak{C}\left\|f\right\|_{\mathbf{L}^1(\mathbb{R}^n)}^q, \qquad \lambda > 0.
\end{aligned}
\qquad (2.\ 61)
$$

By replacing $f(x)$ with $f(x)|x|^{\delta}$ inside (2. 61), we obtain

$$
\lambda\, \mathbf{vol}\left\{x \in \mathbb{R}^n \colon \int_{\frac{1}{2}|x|<|u|<2|x|} f(y)\left(\frac{1}{|x|}\right)^{\gamma}\left(\frac{1}{|x-u|}\right)^{n-\alpha} du > \lambda\right\}^{\frac{1}{q}} \leq \mathfrak{C}\int_{\mathbb{R}^n} f(x)|x|^{\delta} dx, \qquad (2.\ 62)
$$

for every $\lambda > 0$.

Recall $\delta < 0$. Let $\delta_1 < \delta < \delta_2 < 0$ of which $\delta_i, i = 1,2$ are close to δ. We find

$$\frac{\alpha}{n} = 1 - \frac{1}{q_i} + \frac{\gamma + \delta_i}{n}, \qquad i = 1,2 \tag{2. 63}$$

for some $q_1 > q > q_2 > 1$.

By carrying out the same argument as (2. 58)-(2. 62), we simultaneously have

$$\lambda \, \mathbf{vol} \left\{ x \in \mathbb{R}^n : \int_{\frac{1}{2}|x|<|u|<2|x|} f(u) \left(\frac{1}{|x|}\right)^{\gamma} \left(\frac{1}{|x-u|}\right)^{n-\alpha} du > \lambda \right\}^{\frac{1}{q_i}} \leq \mathfrak{C} \int_{\mathbb{R}^n} f(x)|x|^{\delta_i} dx, \qquad i = 1,2 \tag{2. 64}$$

for every $\lambda > 0$.

Next, we need to apply a Marcinkiewicz interpolation theorem of changing measures, due to Stein and Weiss [31].

Let $\mu_i, i = 1,2$ be two absolutely continuous measures satisfying

$$\mu_i(E) = \int_E |x|^{\delta_i} dx, \qquad i = 1,2. \tag{2. 65}$$

Define

$$\mu_t(E) = \int_E |x|^{\delta_1(1-t)}|x|^{\delta_2 t} dx, \qquad \frac{1}{q_t} = \frac{1-t}{q_t} + \frac{t}{q_2}, \qquad 0 \leq t \leq 1. \tag{2. 66}$$

Stein-Weiss interpolation theorem of changing measures, 1958

Let $\mathbf{T}$ *be a sub-linear operator, having the following properties:*

(1) The domain of $\mathbf{T}$ *includes* $\mathbf{L}^1(\mathbb{R}^n, d\mu_1) \cap \mathbf{L}^1(\mathbb{R}^n, d\mu_2)$.

(2) If $f \in \mathbf{L}^1(\mathbb{R}^n, d\mu_i)$, $i = 1,2$, *we have*

$$\lambda \, \mathbf{vol} \left\{ x \in \mathbb{R}^n : \left|\mathbf{T}f(x)\right| > \lambda \right\}^{\frac{1}{q_i}} \leq \mathfrak{C} \int_{\mathbb{R}^n} |f(x)| d\mu_i(x), \qquad i = 1,2. \tag{2. 67}$$

Then,

$$\left\|\mathbf{T}f\right\|_{\mathbf{L}^{q_t}(\mathbb{R}^n)} \leq \mathfrak{C} \int_{\mathbb{R}^n} |f(x)| d\mu_t(x), \qquad 0 < t < 1. \tag{2. 68}$$

Recall $\frac{\alpha}{n} = 1 - \frac{1}{q} + \frac{\gamma+\delta}{n}$ and (2. 63). There is a $0 < t < 1$ such that

$$\delta = (1-t)\delta_1 + t\delta_2, \qquad \frac{1}{q} = \frac{1-t}{q_1} + \frac{t}{q_2}. \tag{2. 69}$$

By using (2. 64) and applying **Stein-Weiss interpolation theorem of changing measures**, we obtain

$$\left\|\mathbf{U}_3 f\right\|_{\mathbf{L}^q(\mathbb{R}^n)} \leq \left\|f\right\|_{\mathbf{L}^1(\mathbb{R}^n)}. \tag{2. 70}$$

Chapter 3: Fractional integration associated with bi-parameter dilation

In this chapter, we prove **Theorem Two** which extends **Theorem One** to the bi-parameter setting. Our proof is mainly split into two parts with respect to $p = 1$ and $p > 1$.

3.1 Proof of Theorem Two at $p = 1$

Let $0 < \alpha < n$, $0 < \beta < m$ and $\gamma, \delta < n + m$. We define

$$\mathbf{I}_{\alpha\beta\gamma\delta} f(x,y) \ = $$

$$\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta}\left[\frac{1}{\sqrt{|u|^2+|v|^2}}\right]^{\delta} dudv, \qquad (x,y)\neq(0,0). \tag{3. 1}$$

For $p = 1$, **Theorem Two** is equivalent to the following result:

Theorem Two* ($p = 1$) *Let $\mathbf{I}_{\alpha\beta\gamma\delta}$ be defined in (3. 1) for $0 < \alpha < n$, $0 < \beta < m$ and $\gamma, \delta < n + m$. We have*

$$\left\|\mathbf{I}_{\alpha\beta\gamma\delta} f\right\|_{\mathbf{L}^q(\mathbb{R}^n\times\mathbb{R}^m)} \ \leq\ \mathfrak{C}_{\alpha\,\gamma\,\delta}\ \left\|f\right\|_{\mathbf{L}^1(\mathbb{R}^n\times\mathbb{R}^m)}, \qquad 1 \leq q < \infty \tag{3. 2}$$

if and only if

$$\gamma \ <\ \frac{n+m}{q}, \qquad \delta \ <\ 0, \qquad \gamma+\delta \ \geq\ 0, \qquad \frac{\alpha+\beta}{n+m} \ =\ 1-\frac{1}{q}+\frac{\gamma+\delta}{n+m} \tag{3. 3}$$

and

$$\alpha - n < \delta, \qquad \beta - m < \delta. \tag{3. 4}$$

3.1.1 The $\mathbf{L}^1 \longrightarrow \mathbf{L}^q$-norm inequality in (3. 2) implies (3. 3)-(3. 4)

Observe that

$$\left[\frac{1}{\sqrt{|x-u|^2+|y-v|^2}}\right]^{n+m-\alpha-\beta} \ \leq\ \left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta} \tag{3. 5}$$

for $0 < \alpha < n$, $0 < \beta < m$.

Because of (3. 5), we find $\mathbf{I}_{\alpha+\beta} f \leq \mathbf{I}_{\alpha\beta} f$ of which $\mathbf{I}_{\alpha+\beta}$, $0 < \alpha + \beta < n + m$ is defined in (1. 1). **Theorem One** for $p = 1$ suggests that

$$\gamma < \frac{n+m}{q}, \qquad \delta < 0, \qquad \gamma+\delta \geq 0, \qquad \frac{\alpha+\beta}{n+m} \ =\ 1-\frac{1}{q}+\frac{\gamma+\delta}{n+m}$$

are necessary conditions. Next, we show $\alpha - n < \delta$ and $\beta - m < \delta$ as two extra ones.

Assert $\mathbf{Q}_1 \times \mathbf{Q}_2 \subset \mathbb{R}^n \times \mathbb{R}^m$ for which $\mathbf{Q}_1, \mathbf{Q}_2$ are cubes in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively parallel to the coordinates. Let f be an indicator function supported on $\mathbf{Q}$. The $\mathbf{L}^1 \longrightarrow \mathbf{L}^q$-norm inequality in (3. 2) implies

$$\begin{array}{lr}\mathbf{A}_q^{\alpha\,\beta\,\gamma\,\delta}\Big[\mathbf{Q}_1\times\mathbf{Q}_2\Big] ~=~ \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-1+\frac{1}{q}}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\beta}{m}-1+\frac{1}{q}} & \\ \left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q}dxdy\right\}^{\frac{1}{q}} & \\ \left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\delta}dxdy\right\} & (3.\ 6) \\ \doteq~ \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-1+\frac{1}{q}}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\beta}{m}-1+\frac{1}{q}}~\mathbf{B}_q^{\alpha\,\beta\,\gamma\,\delta}\Big[\mathbf{Q}_1\times\mathbf{Q}_2\Big] ~\leq~ \mathfrak{C}_{\alpha\,\gamma\,\delta\,q}. & \end{array}$$

We claim

$$\frac{\alpha}{n}-1+\frac{1}{q} ~\geq~ 0,\qquad \frac{\beta}{m}-1+\frac{1}{q}\geq 0. \tag{3. 7}$$

Suppose $\frac{\beta}{m}-1+\frac{1}{q}<0$. Consider $\mathbf{Q}_1\times\mathbf{Q}_2$ centered on the origin of $\mathbb{R}^n\times\mathbb{R}^m$. Let $0<\lambda<1$ and $\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}=1$, $\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{m}}=\lambda$. By shrinking $\mathbf{Q}_2$ to the origin of $\mathbb{R}^m$ and then applying Lebesgue differentiation theorem, we find

$$\lim_{\lambda\longrightarrow 0}\mathbf{B}_q^{\alpha\,\beta\,\gamma\,\delta}\Big[\mathbf{Q}_1\times\mathbf{Q}_2\Big] ~=~ \left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}}\int_{\mathbf{Q}_1}\left(\frac{1}{|y|}\right)^{\gamma q}dy\right\}^{\frac{1}{q}}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}}\int_{\mathbf{Q}_1}\left(\frac{1}{|y|}\right)^{\delta}dy\right\} ~>~ 0. \tag{3. 8}$$

Consequently, $\mathbf{A}_q^{\alpha\,\beta\,\gamma\,\delta}\Big[\mathbf{Q}_1\times\mathbf{Q}_2\Big]$ in (3. 6) diverges to infinity as $\lambda\longrightarrow 0$.

Denote $\mathbf{Q}_1^k=\mathbf{Q}_1\cap\Big\{2^{-k-1}\leq|x|<2^{-k}\Big\}$ for $k\geq 0$. We assert $\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}=1$ and $\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{m}}=\lambda$. Write

$$\begin{array}{lr}\mathbf{vol}\{\mathbf{Q}_1\}^{q\left[\frac{\alpha}{n}-1+\frac{1}{q}\right]}\mathbf{vol}\{\mathbf{Q}_2\}^{q\left[\frac{\beta}{m}-1+\frac{1}{q}\right]}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q}dxdy\right\} & \\ \left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\delta}dxdy\right\}^{q} & \\ =~\sum_{k\geq 0}\mathbf{vol}\{\mathbf{Q}_2\}^{q\left[\frac{\beta}{m}-1+\frac{1}{q}\right]}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1^k\times\mathbf{Q}_2}\sqrt{|x|^2+|y|^2}^{\,-\gamma q}dxdy\right\} & \\ \left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\delta}dxdy\right\}^{q} & \\ \doteq~\sum_{k\geq 0}\mathbf{A}_k(\lambda). & \\ & (3.\ 9)\end{array}$$

By applying Lebesgue differentiation theorem, we have

$$\lim_{\lambda\longrightarrow 0}\frac{1}{\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1^k\times\mathbf{Q}_2}\left(\sqrt{|x|^2+|y|^2}\right)^{-\gamma q}dxdy ~=~ \int_{\mathbf{Q}_1^k}\left(\frac{1}{|x|}\right)^{\gamma q}dx.$$

Suppose $\frac{\beta}{m}-1+\frac{1}{q}>0$. We find $\mathbf{A}_k(0)=0$ for every $k\geq 0$. Moreover, this is true if $\frac{\beta}{m}-1+\frac{1}{q}$ is replaced by any smaller positive number. Therefore, each $\mathbf{A}_k(\lambda)$ is Hölder continuous for $\lambda\geq 0$ whose exponent is strict positive depending on $\frac{\beta}{m}-1+\frac{1}{q}$. For every $\lambda>0$, (3. 6) shows $\sum_{k\geq 0}\mathbf{A}_k(\lambda)\leq \mathfrak{C}_{\alpha\,\gamma\,\delta\,q}$. Consequently, $\sum_{k\geq 0}\mathbf{A}_k(\lambda)$ is continuous at $\lambda=0$. We have

$$\lim_{\lambda\longrightarrow 0}\sum_{k\geq 0}\mathbf{A}_k(\lambda)~=~0. \tag{3. 10}$$

A direct computation shows

$$\begin{aligned}&\mathbf{vol}\{\mathbf{Q}_1\}^{q\left[\frac{\alpha}{n}-1+\frac{1}{q}\right]}\mathbf{vol}\{\mathbf{Q}_2\}^{q\left[\frac{\beta}{m}-1+\frac{1}{q}\right]}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left(\frac{1}{\sqrt{|x|^2+|y|^2}}\right)^{\gamma q}dxdy\right\}\\&\qquad\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left(\frac{1}{\sqrt{|x|^2+|y|^2}}\right)^{\delta}dxdy\right\}^q\\&\geq~\mathfrak{C}_{\delta\,q}\lambda^{q\left[\beta-m+\frac{m}{q}\right]}\int_{\mathbf{Q}_1}\left(\sqrt{|x|^2+\lambda^2}\right)^{-\gamma q}dx\\&\geq~\mathfrak{C}_{\delta\,q}\lambda^{q\left[\beta-m+\frac{m}{q}\right]}\int_{0<|x|\leq\lambda}\left(\frac{1}{\lambda}\right)^{\gamma q}dx~=~\mathfrak{C}_{\gamma\,\delta\,q}\,\lambda^{q\left[\frac{n}{q}-\gamma+\left(\beta-m+\frac{m}{q}\right)\right]}.\end{aligned}\tag{3. 11}$$

From (3. 10)-(3. 11), by using $\frac{\alpha+\beta}{n+m}=1-\frac{1}{q}+\frac{\gamma+\delta}{n+m}$, we find

$$\frac{n}{q}-\gamma+\beta-m+\frac{m}{q}~>~0\qquad\Longrightarrow\qquad\alpha~<~n+\delta. \tag{3. 12}$$

On the other hand, suppose $\frac{\beta}{m}-1+\frac{1}{q}=0$. Similar to (3. 11), we have

$$\begin{aligned}&\mathbf{vol}\{\mathbf{Q}_1\}^{q\left[\frac{\alpha}{n}-1+\frac{1}{q}\right]}\mathbf{vol}\{\mathbf{Q}_2\}^{q\left[\frac{\beta}{m}-1+\frac{1}{q}\right]}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q}dxdy\right\}\\&\qquad\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\delta}dxdy\right\}^q\\&\geq~\mathfrak{C}_{\delta\,q}\lambda^{q\left[\beta-m+\frac{m}{q}\right]}\int_{\mathbf{Q}_1}\left(\sqrt{|x_i|^2+\lambda^2}\right)^{-\gamma q}dx~\geq~\mathfrak{C}_{\delta\,q}\int_{\lambda<|x_i|\leq 1}\left(\frac{1}{|x|}\right)^{\gamma q}dx.\end{aligned}\tag{3. 13}$$

The last integral in (3. 13) converges as $\lambda\longrightarrow 0$. We must have $\gamma q<n$. Together with $\frac{\alpha+\beta}{n+m}=1-\frac{1}{q}+\frac{\gamma+\delta}{n+m}$ and take into account $\frac{\beta}{m}-1+\frac{1}{q}=0$, we find

$$\alpha~=~n-\frac{n}{q}+\gamma+\delta\qquad\Longrightarrow\qquad\alpha~<~n+\delta. \tag{3. 14}$$

A repeat estimate of (3. 7)-(3. 14) by switching the roles of $\mathbf{Q}_1$ and $\mathbf{Q}_2$ shows $\beta<m+\delta$.

3.1.2 Constraints in (3. 3)-(3. 4) imply (3. 2)

Let $\ell \in \mathbb{Z}$. We define the partial operator

$$\Delta_\ell \mathbf{I}_{\alpha\beta\gamma\delta} f(x,y) = \iint_{\Gamma_\ell(x,y)} f(u,v) \left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma} \left(\frac{1}{|x-u|}\right)^{n-\alpha} \left(\frac{1}{|y-v|}\right)^{m-\beta} \left[\frac{1}{\sqrt{|u|^2+|v|^2}}\right]^{\delta} dudv \tag{3. 15}$$

for $(x,y) \neq (0,0)$ where

$$\Gamma_\ell(x,y) = \left\{(u,v) \in \mathbb{R}^n \times \mathbb{R}^m : 2^{\ell-1} < \frac{|y-v|}{|x-u|} \leq 2^\ell\right\}. \tag{3. 16}$$

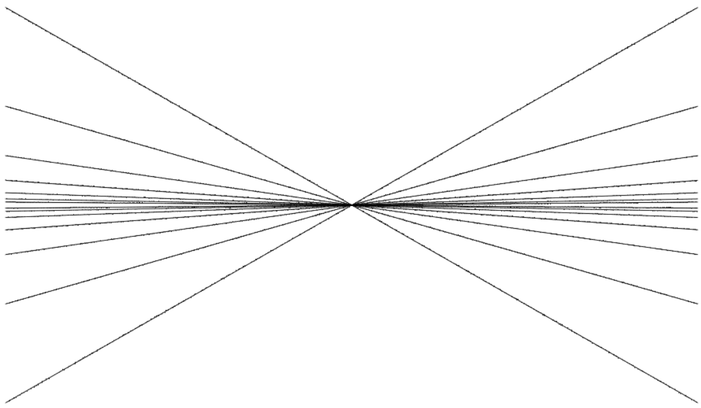

Figure 3.1: Figure 2

Consider $\ell \leq 0$. By changing variables $x \longrightarrow 2^{-\ell}x$ and $u \longrightarrow 2^{-\ell}u$, we have

$$\begin{aligned}
&\Delta_\ell \mathbf{I}_{\alpha\beta\gamma\delta} f(x,y) \\
&= \iint_{\Gamma_\ell(x,y)} f(u,v) \left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma} \left(\frac{1}{|x-u|}\right)^{n-\alpha} \left(\frac{1}{|y-v|}\right)^{m-\beta} \left[\frac{1}{\sqrt{|u|^2+|v|^2}}\right]^{\delta} dudv \\
&= \iint_{\Gamma_o(x,y)} f(2^{-\ell}u,v) \left[\frac{1}{\sqrt{|2^{-\ell}x|^2+|y|^2}}\right]^{\gamma} \left(\frac{1}{|2^{-\ell}x-2^{-\ell}u|}\right)^{n-\alpha} \left(\frac{1}{|y-v|}\right)^{m-\beta} \\
&\qquad\qquad \left[\frac{1}{\sqrt{|2^{-\ell}u|^2+|v|^2}}\right]^{\delta} 2^{-n\ell} dudv \\
&= 2^{-\alpha\ell} \iint_{\Gamma_o(x,y)} f(2^{-\ell}u,v) \left[\frac{1}{\sqrt{|2^{-\ell}x|^2+|y|^2}}\right]^{\gamma} \left(\frac{1}{|x-u|}\right)^{n-\alpha} \left(\frac{1}{|y-v|}\right)^{m-\beta} \left[\frac{1}{\sqrt{|2^{-\ell}u|^2+|v|^2}}\right]^{\delta} dudv.
\end{aligned} \tag{3. 17}$$

Recall $\gamma+\delta\geq 0$ and $\delta<0$. Because $\ell\leq 0$, we find

$$\left(\sqrt{|2^{-\ell}x|^2+|y|^2}\right)^{\gamma}\ \geq\ \left(\sqrt{|x|^2+|y|^2}\right)^{\gamma},\qquad \left(\sqrt{|2^{-\ell}u|^2+|v|^2}\right)^{\delta}\ \geq\ 2^{-\ell\delta}\left(\sqrt{|u|^2+|v|^2}\right)^{\delta}. \tag{3. 18}$$

From (3. 17) to (3. 18), we further have

$$\begin{aligned}
&\Delta_\ell \mathbf{I}_{\alpha\beta\gamma\delta}f(x,y)\\
&\leq\ 2^{-(\alpha-\delta)\ell}\iint_{\Gamma_o(x,y)} f(2^{-\ell}u,v)\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta}\left[\frac{1}{\sqrt{|u|^2+|v|^2}}\right]^{\delta}dudv\\
&\leq\ 2^{-(\alpha-\delta)\ell}\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(2^{-\ell}u,v)\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma}\left[\frac{1}{\sqrt{|x-u|^2+|y-v|^2}}\right]^{n+m-\alpha-\beta}\left[\frac{1}{\sqrt{|u|^2+|v|^2}}\right]^{\delta}dudv.
\end{aligned}\tag{3. 19}$$

By using (3. 19) and applying **Theorem One*** for $p=1$, we obtain

$$\begin{aligned}
\left\|\Delta_\ell \mathbf{I}_{\alpha\beta\gamma\delta}f\right\|_{\mathbf{L}^q(\mathbb{R}^{n+m})}\ &\leq\ \mathfrak{C}\,2^{-(\alpha-\delta)\ell}\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(2^{-\ell}u,v)dudv\\
&=\ \mathfrak{C}_{\alpha\,\gamma\,\delta}\,2^{-(\alpha-n-\delta)\ell}\left\|f\right\|_{\mathbf{L}^1(\mathbb{R}^{n+m})}\qquad \ell\leq 0.
\end{aligned}\tag{3. 20}$$

By carrying out a similar estimate to (3. 17)-(3. 19), we get

$$\left\|\Delta_\ell \mathbf{I}_{\alpha\beta\gamma\delta}f\right\|_{\mathbf{L}^q(\mathbb{R}^{n+m})}\ \leq\ \mathfrak{C}_{\alpha\,\gamma\,\delta}\,2^{(\beta-m-\delta)\ell}\left\|f\right\|_{\mathbf{L}^1(\mathbb{R}^{n+m})}\qquad \ell>0. \tag{3. 21}$$

Recall $\alpha-n<\delta$ and $\beta-n<\delta$. Let $\varepsilon=\min\{n-\alpha+\delta, m-\beta+\delta\}>0$. We obtain

$$\left\|\Delta_\ell \mathbf{I}_{\alpha\beta\gamma\delta}f\right\|_{\mathbf{L}^q(\mathbb{R}^{n+m})}\ \leq\ \mathfrak{C}\,2^{-\varepsilon|\ell|}\ \left\|f\right\|_{\mathbf{L}^1(\mathbb{R}^{n+m})}. \tag{3. 22}$$

By using (3. 22) and applying Minkowski inequality, we finish the proof of **Theorem Two*** **(**$p=1$**)**.

3.2 Proof of Theorem Two for $p>1$: necessary condition

Let $\omega(x,y)=\left(\sqrt{|x|^2+|y|^2}\right)^{-\gamma}$, $\sigma(x,y)=\left(\sqrt{|x|^2+|y|^2}\right)^{\delta}$ for $(x,y)\neq(0,0)$. Choose $f=\sigma^{-\frac{p}{p-1}}\chi_{\mathbf{Q}_1\times\mathbf{Q}_2}$ where χ is an indicator function. The two-weight $\mathbf{L}^p\longrightarrow\mathbf{L}^q$-norm inequality in (1. 16) implies

$$\begin{aligned}
\mathbf{A}^{\alpha\beta}_{pq}(\omega,\sigma)\ \doteq\ &\sup_{\mathbf{Q}\subset\mathbb{R}^{n+m}}\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\beta}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)}\\
&\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q}dxdy\right\}^{\frac{1}{q}}\\
&\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\frac{\delta p}{p-1}}dxdy\right\}^{\frac{p-1}{p}}\ <\ \infty.
\end{aligned}\tag{3. 23}$$

Let $\lambda > 0$ and $\mathbf{Q}^\lambda$ be a dilated of $\mathbf{Q}$ such that $\mathbf{vol}\{\mathbf{Q}_1^\lambda\}^{\frac{1}{n}} = \lambda \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}$, $\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{m}} = \lambda \mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{m}}$. We have

$$\begin{array}{l}
\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\beta}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q}dxdy\right\}^{\frac{1}{q}} \\
\qquad\qquad\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\frac{\delta p}{p-1}}dxdy\right\}^{\frac{p-1}{p}} \\
= \lambda^{\gamma+\delta-(\alpha+\beta)+(n+m)\left(\frac{1}{p}-\frac{1}{q}\right)}\mathbf{vol}\{\mathbf{Q}_1^\lambda\}^{\frac{\alpha}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)}\mathbf{vol}\{\mathbf{Q}_2^\lambda\}^{\frac{\beta}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)} \\
\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1^\lambda\}\mathbf{vol}\{\mathbf{Q}_2^\lambda\}}\iint_{\mathbf{Q}_1^\lambda\times\mathbf{Q}_2^\lambda}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q}dxdy\right\}^{\frac{1}{q}} \\
\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1^\lambda\}\mathbf{vol}\{\mathbf{Q}_2^\lambda\}}\iint_{\mathbf{Q}_1^\lambda\times\mathbf{Q}_2^\lambda}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\frac{\delta p}{p-1}}dxdy\right\}^{\frac{p-1}{p}} \\
\leq \lambda^{\gamma+\delta-(\alpha+\beta)+(n+m)\left(\frac{1}{p}-\frac{1}{q}\right)}\mathbf{A}^{\alpha\beta}_{pq}(\omega,\sigma).
\end{array} \tag{3. 24}$$

Let $\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}} = \mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{m}} = 1$. Note that the first line of (3. 24) is bounded from below. Suppose $\gamma+\delta-(\alpha+\beta)+(n+m)\left(\frac{1}{p}-\frac{1}{q}\right) \neq 0$. By either taking $\lambda \longrightarrow 0$ or $\lambda \longrightarrow \infty$, the last line of (3. 24) is vanished. Hence, we must have $\gamma+\delta-(\alpha+\beta)+(n+m)\left(\frac{1}{p}-\frac{1}{q}\right) = 0$ which is (1. 18).

Shrink $\mathbf{Q}_1$ to some $x_o \in \mathbf{Q}_1$ and $\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{m}} = 1$ in (3. 23). Suppose $x_o \neq 0$ in $\mathbb{R}^n$. By applying the Lebesgue differentiation theorem, we find

$$\begin{array}{l}
\lim_{\mathbf{vol}\{\mathbf{Q}_1\}\longrightarrow 0}\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)} \\
\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_2\}}\int_{\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x_o|^2+|y|^2}}\right]^{\gamma q}dy\right\}^{\frac{1}{q}}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_2\}}\int_{\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x_o|^2+|y|^2}}\right]^{\frac{\delta p}{p-1}}dy\right\}^{\frac{p}{p-1}} \\
\leq \mathbf{A}^{\alpha\beta}_{pq}(\omega,\sigma) < \infty.
\end{array} \tag{3. 25}$$

This requires

$$\frac{\alpha}{n} \geq \frac{1}{p}-\frac{1}{q}. \tag{3. 26}$$

A vice versa estimate of above with $\mathbf{Q}_1$, $\mathbf{Q}_2$ switched in roles shows

$$\frac{\beta}{m} \geq \frac{1}{p}-\frac{1}{q}. \tag{3. 27}$$

By putting (3. 26)-(3. 27) and (1. 18), we find $\gamma+\delta \geq 0$. On the other hand, it is essential to require $\gamma q < n+m$ and $\delta\left(\frac{p}{p-1}\right) < n+m$ for the local integrability of $\left(\sqrt{|x|^2+|y|^2}\right)^{-\gamma q}$ and $\left(\sqrt{|x|^2+|y|^2}\right)^{-\delta\left(\frac{p}{p-1}\right)}$. These are the constraints in (1. 17).

Let $\mathbf{Q}_1\times\mathbf{Q}_2$ centered on the origin of $\mathbb{R}^n\times\mathbb{R}^m$. Denote

$$\mathbf{Q}_1^k ~=~ \mathbf{Q}_1\cap\Big\{2^{-k-1}\leq|x|<2^{-k}\Big\},\qquad \mathbf{Q}_2^k ~=~ \mathbf{Q}_2\cap\Big\{2^{-k-1}\leq|x|<2^{-k}\Big\},\qquad k\geq 0.$$

Consider $\frac{\beta}{m}>\frac{1}{p}-\frac{1}{q}$. Let $\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}=1$ and $\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{m}}=\lambda$ for $0<\lambda<1$. We have

$$\begin{array}{lr}
& \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-\frac{1}{p}+\frac{1}{q}}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\beta}{m}-\frac{1}{p}+\frac{1}{q}}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q}dxdy\right\}^{\frac{1}{q}} \\
& \left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\delta\frac{p}{p-1}}dxdy\right\}^{\frac{p-1}{p}} \\
= & \left\{\mathbf{vol}\{\mathbf{Q}_2\}^{q\left[\frac{\beta}{m}-\frac{1}{p}+\frac{1}{q}\right]\frac{1}{2}}\sum_{k\geq 0}\frac{1}{\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1^k\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q}dxdy\right\}^{\frac{1}{q}} \\
& \left\{\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{p}{p-1}\left[\frac{\beta}{m}-\frac{1}{p}+\frac{1}{q}\right]\frac{1}{2}}\sum_{k\geq 0}\frac{1}{\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1^k\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\delta\frac{p}{p-1}}dxdy\right\}^{\frac{p-1}{p}} \\
= & \left\{\sum_{k\geq 0}\lambda^{q\left[\beta-\frac{m}{p}+\frac{m}{q}\right]\frac{1}{2}}\frac{1}{\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1^k\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q}dxdy\right\}^{\frac{1}{q}} \\
& \left\{\sum_{k\geq 0}\lambda^{\frac{p}{p-1}\left[\beta-\frac{m}{p}+\frac{m}{q}\right]\frac{1}{2}}\frac{1}{\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1^k\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\delta\frac{p}{p-1}}dxdy\right\}^{\frac{p-1}{p}} \\
\doteq & \left\{\sum_{k\geq 0}\mathbf{A}_k(\lambda)\right\}^{\frac{1}{q}}\left\{\sum_{k\geq 0}\mathbf{B}_k(\lambda)\right\}^{\frac{p-1}{p}}.
\end{array}\tag{3. 28}$$

Lebesgue differentiation theorem implies

$$\begin{array}{lr}
\lim_{\lambda\longrightarrow 0}\frac{1}{\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1^k\times\mathbf{Q}_2}\left[\sqrt{|x|^2+|y|^2}\right]^{-\gamma q}dxdy ~=~ \int_{\mathbf{Q}_1^k}\left(\frac{1}{|x|}\right)^{\gamma q}dx, \\
\lim_{\lambda\longrightarrow 0}\frac{1}{\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1^k\times\mathbf{Q}_2}\left[\sqrt{|x|^2+|y|^2}\right]^{-\delta\frac{p}{p-1}}dxdy ~=~ \int_{\mathbf{Q}_1^k}\left(\frac{1}{|x|}\right)^{\delta\frac{p}{p-1}}dx.
\end{array}\tag{3. 29}$$

Because $\frac{\beta}{m}-\frac{1}{p}+\frac{1}{q}>0$, we find $\mathbf{A}_k(0)=0$ and $\mathbf{B}_k(0)=0$ for every $k\geq 0$. Moreover, $\mathbf{A}_k(0)=0=\mathbf{B}_k(0)$ remains to be true if $\frac{\beta}{m}-\frac{1}{p}+\frac{1}{q}$ is replaced by any smaller positive number. Therefore, each $\mathbf{A}_k(\lambda)$ and $\mathbf{B}_k(\lambda)$ is Hölder continuous for $\lambda\geq 0$ whose exponent is strict positive depending on $\frac{\beta}{m}-\frac{1}{p}+\frac{1}{q}$. Furthermore, for every $\lambda>0$, $\sum_{k\geq 0}\mathbf{A}_k(\lambda)\leq\mathfrak{C}_{\beta\,\gamma\,q}$ and $\sum_{k\geq 0}\mathbf{B}_k(\lambda)\leq\mathfrak{C}_{\beta\,\delta\,p}$. Consequently, both $\sum_{k\geq 0}\mathbf{A}_k(\lambda)$ and $\sum_{k\geq 0}\mathbf{B}_k(\lambda)$ are continuous at $\lambda=0$. We have

$$\lim_{\lambda\longrightarrow 0}\sum_{k\geq 0}\mathbf{A}_k(\lambda) ~=~ 0,\qquad \lim_{\lambda\longrightarrow 0}\sum_{k\geq 0}\mathbf{B}_k(\lambda) ~=~ 0.\tag{3. 30}$$

Consider $\frac{\alpha}{n} > \frac{1}{p} - \frac{1}{q}$. Let $\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}} = 1$ and $\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{m}} = \lambda$ for $0 < \lambda < 1$. A repeat estimate of (3. 28)-(3. 30) gives us

$$\begin{aligned}&\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-\frac{1}{p}+\frac{1}{q}}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\beta}{m}-\frac{1}{p}+\frac{1}{q}}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q}dxdy\right\}^{\frac{1}{q}}\\ &\qquad\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\delta\frac{p}{p-1}}dxdy\right\}^{\frac{p-1}{p}} \qquad (3.\ 31)\\ &\longrightarrow 0 \qquad \text{as } \lambda\longrightarrow 0.\end{aligned}$$

Consider $\frac{\beta}{m} = \frac{1}{p} - \frac{1}{q}$. By shrinking $\mathbf{Q}_2$ to the origin of $\mathbb{R}^m$ in (3. 23) and then applying Lebesgue differrentiation theorem, we have

$$\begin{aligned}&\lim_{\mathbf{vol}\{\mathbf{Q}_2\}\longrightarrow 0}\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\beta}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q}dxdy\right\}^{\frac{1}{q}}\\ &\qquad\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\frac{\delta p}{p-1}}dxdy\right\}^{\frac{p-1}{p}}\\ &= \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}}\int_{\mathbf{Q}_1}\left(\frac{1}{|x|}\right)^{\gamma q}dx\right\}^{\frac{1}{q}}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}}\int_{\mathbf{Q}_1}\left(\frac{1}{|x|}\right)^{\frac{\delta p}{p-1}}dx\right\}^{\frac{p-1}{p}} < \infty \qquad (3.\ 32)\end{aligned}$$

for every $\mathbf{Q}_1 \subset \mathbb{R}^n$. This implies

$$\gamma < \frac{n}{q}, \qquad \delta < n\left(\frac{p-1}{p}\right), \qquad \frac{\alpha}{n} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{n}. \qquad (3.\ 33)$$

Similarly, assert $\frac{\alpha}{n} = \frac{1}{p} - \frac{1}{q}$. By shrinking $\mathbf{Q}_1$ to the origin of $\mathbb{R}^n$ in (3. 23) and applying Lebesgue differrentiation theorem, we also have

$$\begin{aligned}&\lim_{\mathbf{vol}\{\mathbf{Q}_1\}\longrightarrow 0}\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\beta}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q}dxdy\right\}^{\frac{1}{q}}\\ &\qquad\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\frac{\delta p}{p-1}}dxdy\right\}^{\frac{p-1}{p}}\\ &= \mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\beta}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_2\}}\int_{\mathbf{Q}_2}\left(\frac{1}{|y|}\right)^{\gamma q}dy\right\}^{\frac{1}{q}}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_2\}}\int_{\mathbf{Q}_2}\left(\frac{1}{|y|}\right)^{\frac{\delta p}{p-1}}dy\right\}^{\frac{p-1}{p}} < \infty \qquad (3.\ 34)\end{aligned}$$

for every $\mathbf{Q}_2 \subset \mathbb{R}^m$. This implies

$$\gamma < \frac{m}{q}, \qquad \delta < m\left(\frac{p-1}{p}\right), \qquad \frac{\alpha}{m} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{m}. \qquad (3.\ 35)$$

In order to prove (1. 19)-(1. 21), we develop a 3-fold estimate with respect to $\gamma \geq 0, \delta \leq 0$; $\gamma \leq 0, \delta \geq 0$ and $\gamma > 0, \delta > 0$.

Case 1. Consider $\gamma \geq 0, \delta \leq 0$. Suppose $\frac{\beta}{m} = \frac{1}{p} - \frac{1}{q}$. Let $\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}} = 1$ and $\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{m}} = \lambda$. We have

$$\begin{aligned}
&\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-\frac{1}{p}+\frac{1}{q}}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\beta}{m}-\frac{1}{p}+\frac{1}{q}}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q}dxdy\right\}^{\frac{1}{q}}\\
&\qquad\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\frac{\delta p}{p-1}}dxdy\right\}^{\frac{p-1}{p}}
\end{aligned}\tag{3. 36}$$

$$\geq \mathfrak{C}_{q\,\gamma}\left\{\int_{\mathbf{Q}_1}\left[\frac{1}{\sqrt{|x|^2+\lambda^2}}\right]^{\gamma q}dx\right\}^{\frac{1}{q}}\left\{\int_{\mathbf{Q}_1}\left(\frac{1}{|x|}\right)^{\frac{\delta p}{p-1}}dx\right\}^{\frac{p-1}{p}} \qquad (\delta \leq 0)$$

$$\geq \mathfrak{C}_{p\,q\,\gamma\,\delta}\left\{\int_{\lambda<|x|\leq 1}\left(\frac{1}{|x|+\lambda}\right)^{\gamma q}dx\right\}^{\frac{1}{q}}$$

where

$$\int_{\lambda<|x|\leq 1}\left(\frac{1}{|x|+\lambda}\right)^{\gamma q}dx \gtrsim \int_{\lambda<|x|\leq 1}\left(\frac{1}{|x|}\right)^{\gamma q}dx = \mathfrak{C}\begin{cases}2^{-\gamma q}\ln\left(\frac{1}{\lambda}\right) & \text{if } \gamma = \frac{n}{q},\\ \frac{2^{-\gamma q}}{\gamma q - n}\left[\left(\frac{1}{\lambda}\right)^{\gamma q - n} - 1\right] & \text{if } \gamma > \frac{n}{q}.\end{cases}\tag{3. 37}$$

Because of (3. 23), as $\lambda \longrightarrow 0$ in (3. 36)-(3. 37), we need

$$\gamma < \frac{n}{q} \implies \alpha - \frac{n}{p} < \delta \tag{3. 38}$$

by using the homogeneity condition in (1. 18): $\frac{\alpha+\beta}{n+m} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{n+m}$.

Suppose $\frac{\beta}{m} > \frac{1}{p} - \frac{1}{q}$. Let $\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}} = 1$ and $\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{m}} = \lambda$. We have

$$\begin{aligned}
&\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-\frac{1}{p}+\frac{1}{q}}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\beta}{m}-\frac{1}{p}+\frac{1}{q}}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q}dxdy\right\}^{\frac{1}{q}}\\
&\qquad\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\frac{\delta p}{p-1}}dxdy\right\}^{\frac{p-1}{p}}
\end{aligned}\tag{3. 39}$$

$$\geq \mathfrak{C}_{q\,\gamma}\lambda^{\beta-m\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\int_{\mathbf{Q}_1}\left[\frac{1}{\sqrt{|x|^2+\lambda^2}}\right]^{\gamma q}dx\right\}^{\frac{1}{q}}\left\{\int_{\mathbf{Q}_1}\left(\frac{1}{|x|}\right)^{\frac{\delta p}{p-1}}dx\right\}^{\frac{p-1}{p}} \qquad (\delta \leq 0)$$

$$\geq \mathfrak{C}_{p\,q\,\gamma\,\delta}\,\lambda^{\beta-m\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\int_{0<|x|\leq\lambda}\left(\frac{1}{\lambda}\right)^{\gamma q}dx\right\}^{\frac{1}{q}} = \mathfrak{C}_{p\,q\,\gamma\,\delta}\,\lambda^{\frac{n}{q}-\gamma+\beta-m\left(\frac{1}{p}-\frac{1}{q}\right)}.$$

From (3. 28)-(3. 30), we know that (3. 39) converges to zero as $\lambda \longrightarrow 0$. This requires $\frac{n}{q} - \gamma + \beta - m\left(\frac{1}{p} - \frac{1}{q}\right) > 0$. Together with the homogeneity condition in (1. 18), we find

$$\gamma < \frac{n}{q} + \beta - m\left(\frac{1}{p} - \frac{1}{q}\right) \implies \alpha - \frac{n}{p} < \delta. \tag{3. 40}$$

On the other hand, consider $\frac{\alpha}{n} \geq \frac{1}{p} - \frac{1}{q}$. By carrying out a repeat estimate of (3. 36)-(3. 39) with $\mathbf{Q}_1, \mathbf{Q}_2$ switched in roles, we obtain

$$\beta - \frac{m}{p} < \delta. \tag{3. 41}$$

Case 2. Consider $\gamma \leq 0$, $\delta \geq 0$. Suppose $\frac{\beta}{m} = \frac{1}{p} - \frac{1}{q}$.

Let $\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}} = 1$ and $\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{m}} = \lambda$. We have

$$\begin{aligned}
&\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-\frac{1}{p}+\frac{1}{q}}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\beta}{m}-\frac{1}{p}+\frac{1}{q}}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q}dxdy\right\}^{\frac{1}{q}}\\
&\qquad\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\frac{\delta p}{p-1}}dxdy\right\}^{\frac{p-1}{p}}\\
&\geq \mathfrak{C}_{q\,\gamma}\left\{\int_{\mathbf{Q}_1}\left(\frac{1}{|x|}\right)^{\gamma q}dx\right\}^{\frac{1}{q}}\left\{\int_{\mathbf{Q}_1}\left[\frac{1}{\sqrt{|x|^2+\lambda^2}}\right]^{\frac{\delta p}{p-1}}dx\right\}^{\frac{p-1}{p}} \qquad (\gamma \leq 0)\\
&\geq \mathfrak{C}_{p\,q\,\gamma\,\delta}\left\{\int_{\lambda<|x|\leq 1}\left(\frac{1}{|x|+\lambda}\right)^{\frac{\delta p}{p-1}}dx\right\}^{\frac{p-1}{p}}
\end{aligned} \tag{3. 42}$$

where

$$\begin{aligned}
&\int_{\lambda<|x|\leq 1}\left(\frac{1}{|x|+\lambda}\right)^{\delta\left(\frac{p}{p-1}\right)}dx \gtrsim\\
&\int_{\lambda<|x|\leq 1}\left(\frac{1}{|x|}\right)^{\frac{\delta p}{p-1}}dx = \mathfrak{C}\begin{cases} 2^{-\frac{\delta p}{p-1}}\ln\left(\frac{1}{\lambda}\right) & \text{if } \delta = n\left(\frac{p}{p-1}\right)\\ \frac{2^{-\frac{\delta p}{p-1}}}{\delta(\frac{p}{p-1})-n}\left[\left(\frac{1}{\lambda}\right)^{\delta\left(\frac{p}{p-1}-n\right)}-1\right] & \text{if } \delta > n\left(\frac{p-1}{p}\right).\end{cases}
\end{aligned} \tag{3. 43}$$

Because of (3. 23), as $\lambda \longrightarrow 0$ in (3. 42)-(3. 43), we need

$$\delta < n\left(\frac{p-1}{p}\right) \implies \alpha - n\left(\frac{q-1}{q}\right) < \gamma \tag{3. 44}$$

by using the homogeneity condition in (1. 18): $\frac{\alpha+\beta}{n+m} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{n+m}$.

Suppose $\frac{\beta}{m} > \frac{1}{p} - \frac{1}{q}$. We have

$$\begin{array}{l}
\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-\frac{1}{p}+\frac{1}{q}}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\beta}{m}-\frac{1}{p}+\frac{1}{q}}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q}dxdy\right\}^{\frac{1}{q}}\\
\qquad\qquad\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\frac{\delta p}{p-1}}dxdy\right\}^{\frac{p-1}{p}}\\[2ex]
\geq\ \mathfrak{C}_{q\,\delta}\lambda^{\beta-m\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\int_{\mathbf{Q}_1}\left(\frac{1}{|x|}\right)^{\gamma q}dx\right\}^{\frac{1}{q}}\left\{\int_{\mathbf{Q}_1}\left[\frac{1}{\sqrt{|x|^2+\lambda^2}}\right]^{\frac{\delta p}{p-1}}dx\right\}^{\frac{p-1}{p}}\qquad(\gamma\leq 0)\\[2ex]
\geq\ \mathfrak{C}_{p\,q\,\gamma\,\delta}\ \lambda^{\beta-m\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\int_{0<|x|\leq\lambda}\left(\frac{1}{\lambda}\right)^{\frac{\delta p}{p-1}}dx\right\}^{\frac{p-1}{p}}\ =\ \mathfrak{C}_{p\,q\,\gamma\,\delta}\ \lambda^{n\left(\frac{p-1}{p}\right)-\delta+\beta-m\left(\frac{1}{p}-\frac{1}{q}\right)}.
\end{array}\tag{3. 45}$$

From (3. 28)-(3. 30), we know that (3. 45) converges to zero as $\lambda \longrightarrow 0$. This requires $n\left(\frac{p-1}{p}\right) - \delta + \beta - m\left(\frac{1}{p} - \frac{1}{q}\right) > 0$. Together with the homogeneity condition in (1. 18), we find

$$\delta < n\left(\frac{p-1}{p}\right) + \beta - m\left(\frac{1}{p} - \frac{1}{q}\right) \implies \alpha - n\left(\frac{q-1}{q}\right) < \gamma. \tag{3. 46}$$

On the other hand, consider $\frac{\alpha}{n} \geq \frac{1}{p} - \frac{1}{q}$. By carrying out a repeat estimate of (3. 42)-(3. 45) with $\mathbf{Q}_1, \mathbf{Q}_2$ switched in roles, we obtain

$$\beta - m\left(\frac{q-1}{q}\right) < \gamma. \tag{3. 47}$$

Case 3. Consider $\gamma > 0$, $\delta > 0$. Note that (3. 23) is invariant by changing one-parameter dilation as shown in (3. 24). Suppose $\alpha - \frac{n}{p} \geq 0$, $\beta - \frac{m}{p} \geq 0$. Let $\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}} = \mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{m}} = \lambda^{-1}$. We have

$$\begin{array}{l}
\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\beta}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q}dxdy\right\}^{\frac{1}{q}}\\
\qquad\qquad\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\frac{\delta p}{p-1}}dxdy\right\}^{\frac{p-1}{p}}\\[2ex]
\geq \mathfrak{C}_{p\,q\,\gamma\,\delta}\left(\frac{1}{\lambda}\right)^{\alpha-\frac{n}{p}+\beta-\frac{m}{p}}\left\{\iint_{\{(x,y):0<|x|<1;0<|y|<1\}}dxdy\right\}^{\frac{1}{q}}\left\{\lambda^{n+m}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\lambda^{\frac{\delta p}{p-1}}dxdy\right\}^{\frac{p-1}{p}}\\[2ex]
=\ \mathfrak{C}_{p\,q\,\gamma\,\delta}\left(\frac{1}{\lambda}\right)^{\alpha-\frac{n}{p}+\beta-\frac{m}{p}-\delta}.
\end{array}\tag{3. 48}$$

Because $\gamma < \frac{n+m}{q}$ and $\frac{\alpha+\beta}{n+m} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{n+m}$, we have

$$\begin{aligned}\delta \ = \ \frac{n+m}{q} - \gamma + \alpha + \beta - \frac{n+m}{p} \ > \ \alpha + \beta - \frac{n+m}{p}\\ = \ \alpha - \frac{n}{p} + \beta - \frac{m}{p}.\end{aligned} \tag{3. 49}$$

Suppose $\alpha - \frac{n}{p} \geq 0$, $\beta - \frac{m}{p} < 0$. Let $\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}} = \lambda^{-1}$ and $\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{m}} = 1$. If $\frac{\beta}{m} > \frac{1}{p} - \frac{1}{q}$, we have

$$\begin{aligned}&\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-\frac{1}{p}+\frac{1}{q}}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\beta}{m}-\frac{1}{p}+\frac{1}{q}}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q}dxdy\right\}^{\frac{1}{q}}\\ &\qquad\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\frac{\delta p}{p-1}}dxdy\right\}^{\frac{p-1}{p}}\\ &\geq \ \mathfrak{C}_{p\,q\,\gamma\,\delta}\left(\frac{1}{\lambda}\right)^{\alpha-\frac{n}{p}}\left\{\int_{\mathbf{Q}_1}\left(\frac{1}{1+|x|}\right)^{\gamma q}dx\right\}^{\frac{1}{q}}\left\{\lambda^n\int_{\mathbf{Q}_1}\lambda^{\frac{\delta p}{p-1}}dx\right\}^{\frac{p-1}{p}}\\ &\geq \ \mathfrak{C}_{p\,q\,\gamma\,\delta}\left(\frac{1}{\lambda}\right)^{\alpha-\frac{n}{p}}\left\{\int_{0<|x|\leq 1}dx\right\}^{\frac{1}{q}}\left\{\lambda^n\int_{\mathbf{Q}_1}\lambda^{\frac{\delta p}{p-1}}dx\right\}^{\frac{p-1}{p}}\\ &= \ \mathfrak{C}_{p\,q\,\gamma\,\delta}\left(\frac{1}{\lambda}\right)^{\alpha-\frac{n}{p}-\delta}.\end{aligned} \tag{3. 50}$$

Recall (3. 28)-(3. 30). Note that (3. 50) converges to zero as $\lambda \longrightarrow 0$. We must have

$$\alpha - \frac{n}{p} < \delta. \tag{3. 51}$$

Suppose $\frac{\beta}{m} = \frac{1}{p} - \frac{1}{q}$. Recall (3. 32)-(3. 33). We find

$$\gamma < \frac{n}{q}, \qquad \delta < n\left(\frac{p-1}{p}\right), \qquad \frac{\alpha}{n} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{n}.$$

This further implies

$$\delta \ = \ \frac{n}{q} - \gamma + \alpha - \frac{n}{p} \ > \ \alpha - \frac{n}{p}. \tag{3. 52}$$

Suppose $\alpha - \frac{n}{p} < 0$ and $\beta = \frac{m}{p} \geq 0$. A repeat estimate of (3. 50)-(3. 52) with $\mathbf{Q}_1, \mathbf{Q}_2$ switched in roles and using (3. 35) instead if (3. 33) gives us

$$\beta - \frac{m}{p} \ < \ \delta. \tag{3. 53}$$

Suppose $\alpha - n\left(\frac{q-1}{q}\right) \geq 0$, $\beta - m\left(\frac{q-1}{q}\right) \geq 0$. Let $\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}} = \mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{m}} = \lambda^{-1}$. We have

$$
\begin{array}{lr}
\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\beta}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q}dxdy\right\}^{\frac{1}{q}} \\
\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\frac{\delta p}{p-1}}dxdy\right\}^{\frac{p-1}{p}} \\
\geq \mathfrak{C}_{p\,q\,\gamma\,\delta\,n\,m}\left(\frac{1}{\lambda}\right)^{\alpha-n\left(\frac{q-1}{q}\right)+\beta-m\left(\frac{q-1}{q}\right)}\left\{\lambda^{n+m}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\lambda^{\gamma q}dxdy\right\}^{\frac{1}{q}}\left\{\iint_{\{(x,y):0<|x|<1;0<|y|<1\}}dxdy\right\}^{\frac{p-1}{p}} \\
= \mathfrak{C}_{p\,q\,\gamma\,\delta\,n\,m}\left(\frac{1}{\lambda}\right)^{\alpha-n\left(\frac{q-1}{q}\right)+\beta-m\left(\frac{q-1}{q}\right)-\gamma}.
\end{array}
\tag{3. 54}
$$

Because $\delta < (n+m)\frac{p-1}{p}$ and $\frac{\alpha+\beta}{n+m} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{n+m}$, we have

$$
\begin{aligned}
\gamma &= (n+m)\frac{p-1}{p} - \delta + \alpha + \beta - (n+m)\frac{q-1}{q} \\
&> \alpha + \beta - (n+m)\frac{q-1}{q} = \alpha - n\left(\frac{q-1}{q}\right) + \beta - m\left(\frac{q-1}{q}\right).
\end{aligned}
\tag{3. 55}
$$

Suppose $\alpha - n\left(\frac{q-1}{q}\right) \geq 0$, $\beta - m\left(\frac{q-1}{q}\right) < 0$. Let $\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}} = \lambda^{-1}$ and $\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{m}} = 1$. If $\frac{\beta}{m} > \frac{1}{p} - \frac{1}{q}$, we have

$$
\begin{array}{lr}
\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-\frac{1}{p}+\frac{1}{q}}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\beta}{m}-\frac{1}{p}+\frac{1}{q}}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q}dxdy\right\}^{\frac{1}{q}} \\
\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\frac{\delta p}{p-1}}dxdy\right\}^{\frac{p-1}{p}} \\
\geq \mathfrak{C}_{p\,q\,\gamma\,\delta}\left(\frac{1}{\lambda}\right)^{\alpha-n\left(\frac{q-1}{q}\right)}\left\{\lambda^{n}\int_{\mathbf{Q}_1}\lambda^{\gamma q}dx\right\}^{\frac{1}{q}}\left\{\int_{\mathbf{Q}_1}\left(\frac{1}{1+|x|}\right)^{\frac{\delta p}{p-1}}dx\right\}^{\frac{p-1}{p}} \\
\geq \mathfrak{C}_{p\,q\,\gamma\,\delta}\left(\frac{1}{\lambda}\right)^{\alpha-n\left(\frac{q-1}{q}\right)}\left\{\lambda^{n}\int_{\mathbf{Q}_1}\lambda^{\gamma q}dx\right\}^{\frac{1}{q}}\left\{\int_{0<|x|\leq 1}dx\right\}^{\frac{p-1}{p}} \\
= \mathfrak{C}_{p\,q\,\gamma\,\delta}\left(\frac{1}{\lambda}\right)^{\alpha-n\left(\frac{q-1}{q}\right)-\gamma}.
\end{array}
\tag{3. 56}
$$

Recall (3. 28)-(3. 30). Note that (3. 56) converges to zero as $\lambda \longrightarrow 0$. We must have

$$
\alpha - n\left(\frac{q-1}{q}\right) < \gamma. \tag{3. 57}
$$

Suppose $\frac{\beta}{m} = \frac{1}{p} - \frac{1}{q}$. Recall (3. 32)-(3. 33). We find

$$\gamma < \frac{n}{q}, \quad \delta < n\left(\frac{p-1}{p}\right), \qquad \frac{\alpha}{n} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{n}.$$

This further implies

$$\gamma \ = \ n\left(\frac{p-1}{p}\right) - \delta + \alpha - n\left(\frac{q-1}{q}\right) \ > \ \alpha - n\left(\frac{q-1}{q}\right). \tag{3. 58}$$

Suppose $\alpha - n\left(\frac{q-1}{q}\right) < 0$, $\beta = m\left(\frac{q-1}{q}\right) \geq 0$. A repeat estimate of (3. 56)-(3. 58) with $\mathbf{Q}_1, \mathbf{Q}_2$ switched in roles and using (3. 35) instead if (3. 33) gives us

$$\beta - m\left(\frac{q-1}{q}\right) \ < \ \delta. \tag{3. 59}$$

3.3 Proof of Theorem Two for $p > 1$: sufficient condition in certain cases

We show (1. 17)-(1. 20) implying (1. 16) for **Case 1**: $\gamma \geq 0, \delta \leq 0$, **Case 2**: $\gamma \leq 0, \delta \geq 0$ and **Case 3**: $\gamma > 0, \delta > 0$ whenever $\frac{\alpha}{n} = \frac{\beta}{m}$.
Let $\rho = \gamma + \delta \geq 0$. From (1. 17)-(1. 18), we have

$$\rho \ = \ \alpha - n\left(\frac{1}{p} - \frac{1}{q}\right) + \beta - m\left(\frac{1}{p} - \frac{1}{q}\right). \tag{3. 60}$$

Write $\rho = \rho_1 + \rho_2$ for which

$$\rho_1 \ = \ \alpha - n\left(\frac{1}{p} - \frac{1}{q}\right) \ \geq \ 0, \qquad \rho_2 \ = \ \beta - m\left(\frac{1}{p} - \frac{1}{q}\right) \ \geq \ 0 \tag{3. 61}$$

as shown in (3. 26)-(3. 27). By applying Young's inequality, we find

$$\left[\frac{1}{\sqrt{|x|^2 + |y|^2}}\right]^{\rho} \ = \ \left[\frac{1}{|x|^2 + |y|^2}\right]^{\frac{\rho_1+\rho_2}{2}} \ \lesssim \ \left(\frac{1}{|x|}\right)^{\rho_1}\left(\frac{1}{|y|}\right)^{\rho_2}.$$

3.3.1 Case 1: $\gamma \geq 0, \delta \leq 0$

Let $\eta = -\delta \geq 0$. Recall that δ satisfies the strict inequality in (1. 19). By using (3. 61), we have

$$\rho_1 + \eta \ = \ \alpha - n\left(\frac{1}{p} - \frac{1}{q}\right) - \delta \ < \ \frac{n}{q}, \qquad \rho_2 + \eta \ = \ \beta - m\left(\frac{1}{p} - \frac{1}{q}\right) - \delta \ < \ \frac{m}{q}. \tag{3. 62}$$

From (3. 61) and (3. 62), the two pairs of weights

$$\left(\frac{1}{|x|}\right)^{\rho_i+\eta}, \quad \left(\frac{1}{|x|}\right)^{\eta} \qquad \text{and} \qquad \left(\frac{1}{|x|}\right)^{\rho_i}, \quad 1, \qquad i = 1, 2$$

both satisfy (2. 3) in **Theorem One**.

Let $\omega(x,y)=\left[\sqrt{|x|^2+|y|^2}\right]^{-\gamma}$ and $\sigma(x,y)=\left[\sqrt{|x|^2+|y|^2}\right]^{\delta}$. The two-weight $\mathbf{L}^p \longrightarrow \mathbf{L}^q$-norm inequality in (1. 16) is equivalent to $\left\|\omega\mathbf{I}_{\alpha\beta}\sigma^{-1}\right\|_{\mathbf{L}^q(\mathbb{R}^{n+m})} \leq \mathfrak{C}_{p\,q\,\alpha\,\beta\,\gamma\,\delta}\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{n+m})}, 1<p\leq q<\infty$. Consider

$$\begin{array}{l}\omega\mathbf{I}_{\alpha\beta}\sigma^{-1}(x,y)\ =\\ \displaystyle\iint_{|u|\leq|v|} f(u,v)\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta}\left[\frac{1}{\sqrt{|u|^2+|v|^2}}\right]^{\delta} dudv\\ \displaystyle+\iint_{|u|>|v|} f(u,v)\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta}\left[\frac{1}{\sqrt{|u|^2+|v|^2}}\right]^{\delta} dudv.\end{array}$$

Recall $\rho=\gamma+\delta$ and $\eta=-\delta$. We have

$$\begin{array}{l}\displaystyle\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left\{\iint_{|u|\leq|v|} f(u,v)\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\rho+\eta}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta}\left[\sqrt{|u|^2+|v|^2}\right]^{\eta} dudv\right\}^q dxdy\right\}^{\frac{1}{q}}\\[2ex]
\displaystyle\lesssim\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left\{\iint_{|u|\leq|v|} f(u,v)\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\rho+\eta}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta}|v|^{\eta} dudv\right\}^q dxdy\right\}^{\frac{1}{q}}\\[2ex]
\displaystyle\leq\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\rho}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta}\left(\frac{|v|}{|y|}\right)^{\eta} dudv\right\}^q dxdy\right\}^{\frac{1}{q}}\\[2ex]
\displaystyle\leq\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)\left(\frac{1}{|x|}\right)^{\rho_1}\left(\frac{1}{|y|}\right)^{\rho_2}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta}\left(\frac{|v|}{|y|}\right)^{\eta} dudv\right\}^q dxdy\right\}^{\frac{1}{q}}\\[2ex]
\displaystyle=\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^m}\left\{\int_{\mathbb{R}^m}\left\{\int_{\mathbb{R}^n} f(u,v)\left(\frac{1}{|x|}\right)^{\rho_1}\left(\frac{1}{|x-u|}\right)^{n-\alpha} du\right\}\left(\frac{1}{|y|}\right)^{\rho_2+\eta}\left(\frac{1}{|y-v|}\right)^{m-\beta}|v|^{\eta}dv\right\}^q dydx\right\}^{\frac{1}{q}}\\[2ex]
\displaystyle\leq\mathfrak{C}_{p\,q\,\beta\,\gamma\,\delta}\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^m}\left\{\int_{\mathbb{R}^n} f(u,y)\left(\frac{1}{|x|}\right)^{\rho_1}\left(\frac{1}{|x-u|}\right)^{n-\alpha} du\right\}^p dy\right\}^{\frac{q}{p}} dx\right\}^{\frac{1}{q}}\\
\qquad\qquad\qquad\qquad\text{by } \textbf{Stein-Weiss theorem} \text{ on } \mathbb{R}^m\\[2ex]
\displaystyle\leq\mathfrak{C}_{p\,q\,\beta\,\gamma\,\delta}\left\{\int_{\mathbb{R}^m}\left\{\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} f(u,y)\left(\frac{1}{|x|}\right)^{\rho_1}\left(\frac{1}{|x-u|}\right)^{n-\alpha} du\right]^q dx\right\}^{\frac{p}{q}} dy\right\}^{\frac{1}{p}}\\
\qquad\qquad\qquad\qquad\text{by Minkowski integral inequality}\\[2ex]
\displaystyle\leq\mathfrak{C}_{p\,q\,\alpha\,\beta\,\gamma\,\delta}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\big(f(x,y)\big)^p dxdy\right\}^{\frac{1}{p}}\qquad \text{by } \textbf{Stein-Weiss theorem} \text{ on } \mathbb{R}^n.\end{array}\tag{3. 63}$$

On the other hand, we have

$$\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left\{\iint_{|u|\geq|v|} f(u,v)\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\rho+\eta}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta}\left[\sqrt{|u|^2+|v|^2}\right]^{\eta} dudv\right\}^q dxdy\right\}^{\frac{1}{q}}$$

$$\lesssim \left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left\{\iint_{|u|\geq|v|} f(u,v)\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\rho+\eta}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta}|u|^{\eta} dudv\right\}^q dxdy\right\}^{\frac{1}{q}}$$

$$\leq \left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\rho}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta}\left(\frac{|u|}{|x|}\right)^{\eta} dudv\right\}^q dxdy\right\}^{\frac{1}{q}}$$

$$\leq \left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)\left(\frac{1}{|x|}\right)^{\rho_1}\left(\frac{1}{|y|}\right)^{\rho_2}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta}\left(\frac{|u|}{|x|}\right)^{\eta} dudv\right\}^q dxdy\right\}^{\frac{1}{q}}$$

$$= \left\{\int_{\mathbb{R}^m}\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^m} f(u,v)\left(\frac{1}{|y|}\right)^{\rho_2}\left(\frac{1}{|y-v|}\right)^{m-\beta} dv\right\}\left(\frac{1}{|x|}\right)^{\rho_1+\eta}\left(\frac{1}{|x-u|}\right)^{n-\alpha}|u|^{\eta} du\right\}^q dxdy\right\}^{\frac{1}{q}}$$

$$\leq \mathfrak{C}_{p\ q\ \alpha\ \gamma\ \delta}\left\{\int_{\mathbb{R}^m}\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^m} f(x,v)\left(\frac{1}{|y|}\right)^{\rho_2}\left(\frac{1}{|y-v|}\right)^{m-\beta} dv\right\}^p dx\right\}^{\frac{q}{p}} dy\right\}^{\frac{1}{q}}$$

by **Stein-Weiss theorem** on $\mathbb{R}^n$

$$\leq \mathfrak{C}_{p\ q\ \alpha\ \gamma\ \delta}\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^m}\left\{\int_{\mathbb{R}^m} f(x,v)\left(\frac{1}{|y|}\right)^{\rho_2}\left(\frac{1}{|y-v|}\right)^{m-\beta} dv\right\}^q dy\right\}^{\frac{p}{q}} dx\right\}^{\frac{1}{p}}$$

by Minkowski integral inequality

$$\leq \mathfrak{C}_{p\ q\ \alpha\ \beta\ \gamma\ \delta}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\Big(f(x,y)\Big)^p dxdy\right\}^{\frac{1}{p}} \qquad \text{by } \textbf{Stein-Weiss theorem} \text{ on } \mathbb{R}^m. \tag{3. 64}$$

3.3.2 Case 2: $\gamma \leq 0$, $\delta \geq 0$

Let $\eta = -\gamma \geq 0$. Recall that γ satisfies the strict inequality in (1. 20). By using (3. 61), we have

$$\begin{aligned}
\rho_1+\eta &= \alpha - n\left(\frac{1}{p}-\frac{1}{q}\right)-\gamma\\
&= \left[\alpha - n\left(\frac{q-1}{q}\right)-\gamma\right]+n\left(\frac{p-1}{p}\right) \ < \ n\left(\frac{p-1}{p}\right),\\
\rho_2+\eta &= \beta - m\left(\frac{1}{p}-\frac{1}{q}\right)-\gamma\\
&= \left[\beta - m\left(\frac{q-1}{q}\right)-\gamma\right]+m\left(\frac{p-1}{p}\right) \ < \ m\left(\frac{p-1}{p}\right).
\end{aligned} \tag{3. 65}$$

Observe that by (3. 61) and (3. 65), the two pairs of weights

$$|x|^{\eta}, \quad |x|^{\rho_i+\eta} \qquad \text{and} \qquad 1, \quad |x|^{\rho_i}, \qquad i=1,2$$

both satisfy (2. 3) in **Theorem One**. Let χ be an indicator function. Consider

$$\begin{aligned}
&\omega \mathbf{I}_{\alpha\beta}\sigma^{-1}(x,y) \;=\\
&\chi_{|x|\le|y|}\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta}\left[\frac{1}{\sqrt{|u|^2+|v|^2}}\right]^{\delta} dudv\\
&+\chi_{|x|>|y|}\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta}\left[\frac{1}{\sqrt{|u|^2+|v|^2}}\right]^{\delta} dudv.
\end{aligned}$$

Recall $\rho=\gamma+\delta$ and $\eta=-\gamma$. We have

$$\left\{\iint_{|x|\le|y|}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)\left[\sqrt{|x|^2+|y|^2}\right]^{\eta}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta}\left[\frac{1}{\sqrt{|u|^2+|v|^2}}\right]^{\rho+\eta} dudv\right\}^q dxdy\right\}^{\frac{1}{q}}$$

$$\lesssim \left\{\iint_{|x|\le|y|}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)|y|^{\eta}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta}\left[\frac{1}{\sqrt{|u|^2+|v|^2}}\right]^{\rho+\eta} dudv\right\}^q dxdy\right\}^{\frac{1}{q}}$$

$$\le \left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)\left(\frac{|y|}{|v|}\right)^{\eta}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta}\left[\frac{1}{\sqrt{|u|^2+|v|^2}}\right]^{\rho} dudv\right\}^q dxdy\right\}^{\frac{1}{q}}$$

$$\le \left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)\left(\frac{|y|}{|v|}\right)^{\eta}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta}\left(\frac{1}{|u|}\right)^{\rho_1}\left(\frac{1}{|v|}\right)^{\rho_2} dudv\right\}^q dxdy\right\}^{\frac{1}{q}}$$

$$= \left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^m}\left\{\int_{\mathbb{R}^m}\left\{\int_{\mathbb{R}^n} f(u,v)\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|u|}\right)^{\rho_1} du\right\}|y|^{\eta}\left(\frac{1}{|y-v|}\right)^{m-\beta}\left(\frac{1}{|v|}\right)^{\rho_2+\eta} dv\right\}^q dydx\right\}^{\frac{1}{q}}$$

$$\le \mathfrak{C}_{p\,q\,\beta\,\gamma\,\delta}\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^m}\left\{\int_{\mathbb{R}^n} f(u,y)\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|u|}\right)^{\rho_1} du\right\}^p dy\right\}^{\frac{q}{p}} dx\right\}^{\frac{1}{q}}$$

by **Stein-Weiss theorem** on $\mathbb{R}^m$

$$\le \mathfrak{C}_{p\,q\,\beta\,\gamma\,\delta}\left\{\int_{\mathbb{R}^m}\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n} f(u,y)\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|u|}\right)^{\rho_1} du\right\}^q dx\right\}^{\frac{p}{q}} dy\right\}^{\frac{1}{p}}$$

by Minkowski integral inequality

$$\le \mathfrak{C}_{p\,q\,\alpha\,\beta\,\gamma\,\delta}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\Big(f(x,y)\Big)^p dxdy\right\}^{\frac{1}{p}} \qquad \text{by } \textbf{Stein-Weiss theorem} \text{ on } \mathbb{R}^n. \tag{3. 66}$$

On the other hand, we have

$$
\left\{\iint_{|x|>|y|}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)\left[\sqrt{|x|^2+|y|^2}\right]^{\eta}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta}\left[\frac{1}{\sqrt{|u|^2+|v|^2}}\right]^{\rho+\eta} dudv\right\}^q dxdy\right\}^{\frac{1}{q}}
$$

$$
\lesssim\left\{\iint_{|x|>|y|}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)|x|^{\eta}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta}\left[\frac{1}{\sqrt{|u|^2+|v|^2}}\right]^{\rho+\eta} dudv\right\}^q dxdy\right\}^{\frac{1}{q}}
$$

$$
\leq\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)\left(\frac{|x|}{|u|}\right)^{\eta}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta}\left[\frac{1}{\sqrt{|u|^2+|v|^2}}\right]^{\rho} dudv\right\}^q dxdy\right\}^{\frac{1}{q}}
$$

$$
\leq\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)\left(\frac{|x|}{|u|}\right)^{\eta}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta}\left(\frac{1}{|u|}\right)^{\rho_1}\left(\frac{1}{|v|}\right)^{\rho_2} dudv\right\}^q dxdy\right\}^{\frac{1}{q}}
$$

$$
=\left\{\int_{\mathbb{R}^m}\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^m} f(u,v)\left(\frac{1}{|y-v|}\right)^{m-\beta}\left(\frac{1}{|v|}\right)^{\rho_2} dv\right\}|x|^{\eta}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|u|}\right)^{\rho_1+\eta} du\right\}^q dxdy\right\}^{\frac{1}{q}}
$$

$$
\leq \mathfrak{C}_{p\ q\ \alpha\ \gamma\ \delta}\left\{\int_{\mathbb{R}^m}\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^m} f(x,v)\left(\frac{1}{|y-v|}\right)^{m-\beta}\left(\frac{1}{|v|}\right)^{\rho_2} du\right\}^p dx\right\}^{\frac{q}{p}} dy\right\}^{\frac{1}{q}}
$$

by **Stein-Weiss theorem** on $\mathbb{R}^n$

$$
\leq \mathfrak{C}_{p\ q\ \alpha\ \gamma\ \delta}\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^m}\left\{\int_{\mathbb{R}^m} f(x,v)\left(\frac{1}{|y-v|}\right)^{m-\beta}\left(\frac{1}{|v|}\right)^{\rho_2} du\right\}^q dy\right\}^{\frac{p}{q}} dx\right\}^{\frac{1}{p}}
$$

by Minkowski integral inequality

$$
\leq \mathfrak{C}_{p\ q\ \alpha\ \beta\ \gamma\ \delta}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\Big(f(x,y)\Big)^p dxdy\right\}^{\frac{1}{p}} \qquad \text{by } \textbf{Stein-Weiss theorem} \text{ on } \mathbb{R}^m. \tag{3. 67}
$$

3.3.3 Case 3: $\gamma > 0$, $\delta > 0$ and $\frac{\alpha}{n} = \frac{\beta}{m}$

Recall (1. 18). We have

$$
\frac{\alpha}{n} = \frac{\beta}{m} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{n+m}. \tag{3. 68}
$$

Young's inequality implies

$$
\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma} \lesssim \left(\frac{1}{|x|}\right)^{\frac{n}{n+m}\gamma}\left(\frac{1}{|y|}\right)^{\frac{m}{n+m}\gamma}, \qquad \left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\delta} \lesssim \left(\frac{1}{|x|}\right)^{\frac{n}{n+m}\delta}\left(\frac{1}{|y|}\right)^{\frac{m}{n+m}\delta}.
$$

We have

$$\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma}\left[\frac{1}{\sqrt{|u|^2+|v|^2}}\right]^{\delta} dudv\right\}^q dxdy\right\}^{\frac{1}{q}}$$

$$\leq \left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)\right.\right.$$
$$\left.\left.\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta}\left(\frac{1}{|x|}\right)^{\frac{n}{n+m}\gamma}\left(\frac{1}{|y|}\right)^{\frac{m}{n+m}\gamma}\left(\frac{1}{|u|}\right)^{\frac{n}{n+m}\delta}\left(\frac{1}{|v|}\right)^{\frac{m}{n+m}\delta} dudv\right\}^q dxdy\right\}^{\frac{1}{q}}$$

$$= \left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^m}\left\{\int_{\mathbb{R}^m}\left\{\int_{\mathbb{R}^n} f(u,v)\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|x|}\right)^{\frac{n}{n+m}\gamma}\left(\frac{1}{|u|}\right)^{\frac{n}{n+m}\delta} du\right\}\right.\right.$$
$$\left.\left.\left(\frac{1}{|y-v|}\right)^{m-\beta}\left(\frac{1}{|y|}\right)^{\frac{m}{n+m}\gamma}\left(\frac{1}{|v|}\right)^{\frac{m}{n+m}\delta} dv\right\}^q dydx\right\}^{\frac{1}{q}}$$

$$\leq \mathfrak{C}_{p\,q\,\beta\,\gamma\,\delta}\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^n} f(u,y)\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|x|}\right)^{\frac{n}{n+m}\gamma}\left(\frac{1}{|u|}\right)^{\frac{n}{n+m}\delta} du\right)^p dy\right\}^{\frac{q}{p}} dx\right\}^{\frac{1}{q}}$$

by **Stein-Weiss theorem** on $\mathbb{R}^m$

$$\leq \mathfrak{C}_{p\,q\,\beta\,\gamma\,\delta}\left\{\int_{\mathbb{R}^m}\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n} f(u,y)\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|x|}\right)^{\frac{n}{n+m}\gamma}\left(\frac{1}{|u|}\right)^{\frac{n}{n+m}\delta} du\right\}^q dx\right\}^{\frac{p}{q}} dy\right\}^{\frac{1}{p}}$$

by Minkowski integral inequality

$$\leq \mathfrak{C}_{p\,q\,\alpha\,\beta\,\gamma\,\delta}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\big(f(x,y)\big)^p dxdy\right\}^{\frac{1}{p}} \qquad \text{by } \textbf{Stein-Weiss theorem} \text{ on } \mathbb{R}^n. \tag{3. 69}$$

3.4 Proof of Theorem Two for $p > 1$: sufficient condition for $\frac{\alpha}{n} > \frac{1}{p} - \frac{1}{q}$, $\frac{\beta}{m} > \frac{1}{p} - \frac{1}{q}$

Let $\ell \in \mathbb{Z}$. We define

$$\Delta_\ell \mathbf{I}_{\alpha\beta} f(x,y) = \iint_{\Gamma_\ell(x,y)} f(u,v)\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta} dudv \tag{3. 70}$$

where

$$\Gamma_\ell(x,y) = \left\{(u,v)\in\mathbb{R}^n\times\mathbb{R}^m :\ 2^{\ell-1} \leq \frac{|y-v|}{|x-u|} < 2^\ell\right\}. \tag{3. 71}$$

Observe that $\Gamma_\ell(x,y)$ is a dyadic cone vertex on (x,y) whose eccentricity depends on $\ell \in \mathbb{Z}$. In particular, we write $\Gamma_o(x,y) = \Gamma_\ell(x,y)$ for $\ell = 0$.

Recall $\omega(x,y)=\left[\sqrt{|x|^2+|y|^2}\right]^{-\gamma}$ and $\sigma(x,y)=\left[\sqrt{|x|^2+|y|^2}\right]^{\delta}$ for $(x,y)\neq(0,0)$. Define

$$
\begin{array}{lr}
{\bf A}^{\alpha\beta}_{pqr}(\ell:\omega,\sigma) = \sup_{{\bf Q}_1\times{\bf Q}_2\subset\mathbb{R}^n\times\mathbb{R}^m:{\bf vol}\{{\bf Q}_2\}^{\frac{1}{m}}/{\bf vol}\{{\bf Q}_1\}^{\frac{1}{n}}=2^\ell} {\bf vol}\{{\bf Q}_1\}^{\frac{\alpha}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)}{\bf vol}\{{\bf Q}_2\}^{\frac{\beta}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)} & \\
\qquad\left\{\frac{1}{{\bf vol}\{{\bf Q}_1\}{\bf vol}\{{\bf Q}_2\}}\iint_{{\bf Q}_1\times{\bf Q}_2}\omega^{qr}(x,y)dxdy\right\}^{\frac{1}{qr}} & (3.\ 72)\\
\qquad\left\{\frac{1}{{\bf vol}\{{\bf Q}_1\}{\bf vol}\{{\bf Q}_2\}}\iint_{{\bf Q}_1\times{\bf Q}_2}\left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}}(x,y)dxdy\right\}^{\frac{p-1}{p}} &
\end{array}
$$

for $r>1$ and $\ell\in\mathbb{Z}$.

Given $\ell\leq 0$. Let ${\bf Q}_i^\ell$ be a dilated of ${\bf Q}_i$ for $i=1,2$ such that ${\bf vol}\{{\bf Q}_1^\ell\}^{\frac{1}{n}}=2^\ell{\bf vol}\{{\bf Q}_1\}^{\frac{1}{n}}$ and ${\bf vol}\{{\bf Q}_2^\ell\}^{\frac{1}{m}}=2^\ell{\bf vol}\{{\bf Q}_2\}^{\frac{1}{m}}$. For every ${\bf Q}_1\times{\bf Q}_2\subset\mathbb{R}^n\times\mathbb{R}^m$, we have

$$
\begin{array}{lr}
{\bf vol}\{{\bf Q}_1\}^{\frac{\alpha}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)}{\bf vol}\{{\bf Q}_2\}^{\frac{\beta}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\frac{1}{{\bf vol}\{{\bf Q}_1\}{\bf vol}\{{\bf Q}_2\}}\iint_{{\bf Q}_1\times{\bf Q}_2}\omega^{qr}(x,2^\ell y)dxdy\right\}^{\frac{1}{qr}} & \\
\qquad\left\{\frac{1}{{\bf vol}\{{\bf Q}_1\}{\bf vol}\{{\bf Q}_2\}}\iint_{{\bf Q}_1\times{\bf Q}_2}\left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}}(x,2^\ell y)dxdy\right\}^{\frac{p-1}{p}} & \\
= 2^{-\ell\left[\beta-\left(\frac{m}{p}-\frac{m}{q}\right)\right]}{\bf vol}\{{\bf Q}_1\}^{\frac{\alpha}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)}{\bf vol}\{{\bf Q}_2^\ell\}^{\frac{\beta}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)} & (3.\ 73)\\
\quad\left\{\frac{1}{{\bf vol}\{{\bf Q}_1\}{\bf vol}\{{\bf Q}_2^\ell\}}\iint_{{\bf Q}_1\times{\bf Q}_2^\ell}\omega^{qr}(x,y)dxdy\right\}^{\frac{1}{qr}} & \\
\quad\left\{\frac{1}{{\bf vol}\{{\bf Q}_1\}{\bf vol}\{{\bf Q}_2^\ell\}}\iint_{{\bf Q}_1\times{\bf Q}_2^\ell}\left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}}(x,y)dxdy\right\}^{\frac{p-1}{p}}. &
\end{array}
$$

Suppose ${\bf vol}\{{\bf Q}_1\}^{\frac{1}{n}}={\bf vol}\{{\bf Q}_2\}^{\frac{1}{m}}$. By using (3. 72)-(3. 73), we find

$$
\begin{array}{lr}
{\bf vol}\{{\bf Q}_1\}^{\frac{\alpha}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)}{\bf vol}\{{\bf Q}_2\}^{\frac{\beta}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\frac{1}{{\bf vol}\{{\bf Q}_1\}{\bf vol}\{{\bf Q}_2\}}\iint_{{\bf Q}_1\times{\bf Q}_2}\omega^{qr}(x,2^\ell y)dxdy\right\}^{\frac{1}{qr}} & \\
\qquad\left\{\frac{1}{{\bf vol}\{{\bf Q}_1\}{\bf vol}\{{\bf Q}_2\}}\iint_{{\bf Q}_1\times{\bf Q}_2}\left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}}(x,2^\ell y)dxdy\right\}^{\frac{p-1}{p}} & \\
= 2^{-\ell\left[\beta-\left(\frac{m}{p}-\frac{m}{q}\right)\right]}{\bf vol}\{{\bf Q}_1\}^{\frac{\alpha}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)}{\bf vol}\{{\bf Q}_2^\ell\}^{\frac{\beta}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)} & \\
\quad\left\{\frac{1}{{\bf vol}\{{\bf Q}_1\}{\bf vol}\{{\bf Q}_2^\ell\}}\iint_{{\bf Q}_1\times{\bf Q}_2^\ell}\omega^{qr}(x,y)dxdy\right\}^{\frac{1}{qr}} & (3.\ 74)\\
\quad\left\{\frac{1}{{\bf vol}\{{\bf Q}_1\}{\bf vol}\{{\bf Q}_2^\ell\}}\iint_{{\bf Q}_1\times{\bf Q}_2^\ell}\left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}}(x,y)dxdy\right\}^{\frac{p-1}{p}} & \\
\leq 2^{-\ell\left[\beta-\left(\frac{m}{p}-\frac{m}{q}\right)\right]}\,{\bf A}^{\alpha\beta}_{pqr}(\ell,\omega,\sigma),\qquad \ell\leq 0. &
\end{array}
$$

A similar estimate shows that

$$\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\beta}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\omega^{qr}(2^{-\ell}x,y)dxdy\right\}^{\frac{1}{qr}}$$
$$\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}}(2^{-\ell}x,y)dxdy\right\}^{\frac{p-1}{p}} \qquad (3.\ 75)$$

$$\leq 2^{\ell\left[\alpha-\left(\frac{n}{p}-\frac{n}{q}\right)\right]}\mathbf{A}^{\alpha\beta}_{pqr}(\ell,\omega,\sigma),\qquad \ell\geq 0.$$

Now, we recall a classical result of Sawyer and Wheeden [27] for one-parameter fractional integrals, stated as follows.

Sawyer-Wheeden theorem, 1992 *Let $\mathbf{I}_{\alpha+\beta}$ defined in (1. 1) for $0<\alpha+\beta<n+m$. Suppose that $\omega^q(x,y)$, $\sigma^{-\frac{p}{p-1}}(x,y)$ are non-negative measurable functions on $\mathbb{R}^{n+m}$. We have*

$$\left\|\omega\mathbf{I}_{\alpha+\beta}f\right\|_{\mathbf{L}^q(\mathbb{R}^{n+m})}\leq \mathfrak{C}_{p\ q\ r\ \alpha\ \beta}\mathbf{A}^{\alpha+\beta}_{pqr}(\omega,\sigma)\left\|f\sigma\right\|_{\mathbf{L}^p(\mathbb{R}^{n+m})},\qquad 1<p\leq q<\infty \qquad (3.\ 76)$$

if

$$\mathbf{A}^{\alpha+\beta}_{pqr}(\omega,\sigma) = \sup_{\mathbf{Q}_1\times\mathbf{Q}_2\subset\mathbb{R}^n\times\mathbb{R}^m:\ \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}=\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{m}}}\left[\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}\right]^{\frac{\alpha+\beta}{n+m}-\left(\frac{1}{p}-\frac{1}{q}\right)}$$
$$\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\omega^{qr}(x,y)dxdy\right\}^{\frac{1}{qr}}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\sigma^{-\frac{pr}{p-1}}(x,y)dxdy\right\}^{\frac{p-1}{pr}} < \infty \qquad (3.\ 77)$$

for some $r>1$.

Remark 3.4.1. $\mathbf{Q}_1\times\mathbf{Q}_2$ *in (3. 77) is a cube, i.e:* $\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}=\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{m}}$. *The constant $\mathfrak{C}_{p\ q\ r\ \alpha\ \beta}\mathbf{A}^{\alpha}_{pqr}(\omega,\sigma)$ in (3. 76) is not written explicitly in the statement by Sawyer and Wheeden [27]. But it can be computed directly by carrying out its proof.*

By applying **Sawyer-Wheeden theorem** and using (3. 74)-(3. 75), we have

$$\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}f(u,2^\ell v)\left[\frac{1}{\sqrt{|x-u|^2+|y-v|^2}}\right]^{n+m-\alpha-\beta}\right\}^q\omega(x,2^\ell y)dxdy\right\}^{\frac{1}{q}}$$
$$\leq\ \mathfrak{C}_{p\ q\ r\ \alpha\ \beta}\,2^{-\ell\left[\beta-\left(\frac{m}{p}-\frac{m}{q}\right)\right]}\mathbf{A}^{\alpha\beta}_{pqr}(\ell,\omega,\sigma)\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}(f\sigma)^p\,(x,2^\ell y)dxdy\right\}^{\frac{1}{p}}\qquad \ell\leq 0; \qquad (3.\ 78)$$

$$\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}f(2^{-\ell}u,v)\left[\frac{1}{\sqrt{|x-u|^2+|y-v|^2}}\right]^{n+m-\alpha-\beta}\right\}^q\omega(2^{-\ell}x,y)dxdy\right\}^{\frac{1}{q}}$$
$$\leq\ \mathfrak{C}_{p\ q\ r\ \alpha\ \beta}\,2^{\ell\left[\alpha-\left(\frac{n}{p}-\frac{n}{q}\right)\right]}\mathbf{A}^{\alpha\beta}_{pqr}(\ell,\omega,\sigma)\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}(f\sigma)^p\,(2^{-\ell}x,y)dxdy\right\}^{\frac{1}{p}}\qquad \ell\geq 0. \qquad (3.\ 79)$$

From (3. 70)-(3. 71), by changing dilations $y \longrightarrow 2^\ell y$, $v \longrightarrow 2^\ell v$, $\ell \le 0$, we have

$$
\begin{aligned}
&\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} \left(\Delta_\ell \mathbf{I}_{\alpha\beta} f\right)^q (x,y)\omega^q(x,y)dxdy\right\}^{\frac{1}{q}} \\
&= \left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} \left\{\iint_{\Gamma_\ell(x,y)} f(u,v)\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta} dudv\right\}^q \omega^q(x,y)dxdy\right\}^{\frac{1}{q}} \\
&= \left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} \left\{\iint_{\Gamma_o(x,y)} f(u,2^\ell v)\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|2^\ell y-2^\ell v|}\right)^{m-\beta} 2^{\ell m}dudv\right\}^q \omega^q(x,2^\ell y)2^{\ell m}dxdy\right\}^{\frac{1}{q}} \\
&\le \mathfrak{C}_{\alpha\ \beta} 2^{\left(\beta+\frac{m}{q}\right)\ell}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} \left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,2^\ell v)\left[\frac{1}{\sqrt{|x-u|^2+|y-v|^2}}\right]^{n+m-\alpha-\beta} dudv\right\}^q \omega^q(x,2^\ell y)dxdy\right\}^{\frac{1}{q}} \\
&\le \mathfrak{C}_{p\ q\ r\ \alpha\ \beta}\ 2^{\left(\beta+\frac{m}{q}\right)\ell} 2^{-\left[\beta-\left(\frac{m}{p}-\frac{m}{q}\right)\right]\ell}\mathbf{A}^{\alpha\beta}_{pqr}(\ell:\omega,\sigma)\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} (f\sigma)^p (x,2^\ell y)dxdy\right\}^{\frac{1}{p}} \\
&= \mathfrak{C}_{p\ q\ r\ \alpha\ \beta}\ 2^{\frac{m}{p}\ell}\mathbf{A}^{\alpha\beta}_{pqr}(\ell:\omega,\sigma)\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} (f\sigma)^p (x,y)2^{-m\ell}dxdy\right\}^{\frac{1}{p}} \\
&= \mathfrak{C}_{p\ q\ r\ \alpha\ \beta}\mathbf{A}^{\alpha\beta}_{pqr}(\ell:\omega,\sigma)\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} (f\sigma)^p (x,y)dxdy\right\}^{\frac{1}{p}}, \qquad \ell\le 0.
\end{aligned}
\tag{3. 80}
$$

A similar estimate shows

$$
\begin{aligned}
&\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} \left(\Delta_\ell \mathbf{I}_{\alpha\beta} f\right)^q (x,y)\omega^q(x,y)dxdy\right\}^{\frac{1}{q}} \\
&\le\ \mathfrak{C}_{p\ q\ r\ \alpha\ \beta}\ \mathbf{A}^{\alpha\beta}_{pqr}(\ell:\omega,\sigma)\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} (f\sigma)^p (x,y)dxdy\right\}^{\frac{1}{p}}, \qquad \ell\ge 0.
\end{aligned}
\tag{3. 81}
$$

Observe that $\Delta_\ell \mathbf{I}_{\alpha\beta}$ is essentially an one-parameter fractional integral operator satisfying the desired regularity.
We claim that there is an $\varepsilon=\varepsilon(p,q,\alpha,\beta,\gamma,\delta)>0$ such that

$$
\mathbf{A}^{\alpha}_{pqr}(\ell:\omega,\sigma)\ <\ \mathfrak{C}_{p\ q\ \alpha\ \beta\ \gamma\ \delta}\, 2^{-\varepsilon|\ell|}, \qquad \ell\in\mathbb{Z}. \tag{3. 82}
$$

From (3. 80)-(3. 82), we obtain (1. 16) by applying Minkowski inequality.

Due to symmetry reason, we prove (3. 82) for $\ell\le 0$ only. This is formulated into the next result.

Principal Lemma *Let γ, δ satisfy (1. 18)-(1. 21). Suppose $\frac{\alpha}{n} > \frac{1}{p} - \frac{1}{q}$ and $\frac{\beta}{m} > \frac{1}{p} - \frac{1}{q}$. Consider*

$$\lambda \; = \; \mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{m}} / \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}, \qquad 0 < \lambda \leq 1. \tag{3. 83}$$

There exists an $\varepsilon = \varepsilon(p, q, \alpha, \beta, \gamma, \delta) > 0$ such that

$$\begin{array}{l} \sup_{\mathbf{Q}_1 \times \mathbf{Q}_2 \subset \mathbb{R}^n \times \mathbb{R}^m : \mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{m}} / \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}} = \lambda} \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n} - \left(\frac{1}{p} - \frac{1}{q}\right)} \mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\beta}{m} - \left(\frac{1}{p} - \frac{1}{q}\right)} \\ \left\{ \frac{1}{\mathbf{vol}\{\mathbf{Q}_1\} \mathbf{vol}\{\mathbf{Q}_2\}} \iint_{\mathbf{Q}_1 \times \mathbf{Q}_2} \left[\frac{1}{\sqrt{|x|^2 + |y|^2}} \right]^{\gamma q r} dx dy \right\}^{\frac{1}{qr}} \\ \left\{ \frac{1}{\mathbf{vol}\{\mathbf{Q}_1\} \mathbf{vol}\{\mathbf{Q}_2\}} \iint_{\mathbf{Q}_1 \times \mathbf{Q}_2} \left[\frac{1}{\sqrt{|x|^2 + |y|^2}} \right]^{\frac{\delta p r}{p-1}} dx dy \right\}^{\frac{p-1}{pr}} \leq \mathfrak{C}_{p\,q\,\alpha\,\beta\,\gamma\,\delta}\, \lambda^{\varepsilon} \end{array} \tag{3. 84}$$

for some $r = r(p, q, \alpha, \beta, \gamma, \delta) > 1$.

Proof Let $\omega(x, y) = \left[\sqrt{|x|^2 + |y|^2} \right]^{-\gamma}$ and $\sigma(x, y) = \left[\sqrt{|x|^2 + |y|^2} \right]^{\delta}$ for $(x, y) \neq (0, 0)$. Given $\mathbf{Q}_1 \times \mathbf{Q}_2 \subset \mathbb{R}^n \times \mathbb{R}^m$, we write

$$\begin{array}{l} \mathbf{A}^{\alpha\beta}_{pqr}(\omega, \sigma)\Big[\mathbf{Q}_1 \times \mathbf{Q}_2\Big] \; = \; \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n} - \left(\frac{1}{p} - \frac{1}{q}\right)} \mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\beta}{m} - \left(\frac{1}{p} - \frac{1}{q}\right)} \\ \qquad \left\{ \frac{1}{\mathbf{vol}\{\mathbf{Q}_1\} \mathbf{vol}\{\mathbf{Q}_2\}} \iint_{\mathbf{Q}_1 \times \mathbf{Q}_2} \left[\frac{1}{\sqrt{|x|^2 + |y|^2}} \right]^{\gamma q r} dx dy \right\}^{\frac{1}{qr}} \\ \qquad \left\{ \frac{1}{\mathbf{vol}\{\mathbf{Q}_1\} \mathbf{vol}\{\mathbf{Q}_2\}} \iint_{\mathbf{Q}_1 \times \mathbf{Q}_2} \left[\frac{1}{\sqrt{|x|^2 + |y|^2}} \right]^{\frac{\delta p r}{p-1}} dx dy \right\}^{\frac{p-1}{pr}}. \end{array} \tag{3. 85}$$

Recall (1. 17)-(1. 18). We have

$$\gamma \; < \; \frac{n+m}{q}, \qquad \delta \; < \; (n+m)\frac{p-1}{p}, \qquad \gamma + \delta \; \geq \; 0, \qquad \frac{\alpha + \beta}{n+m} \; = \; \frac{1}{p} - \frac{1}{q} + \frac{\gamma + \delta}{n+m}.$$

Observe that the r-bump characteristic in (3. 85) is invariant by changing one-parameter dilation. Therefore, it is suffice to assert $\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}} = 1$.

Let $\mathbf{Q}^o_1$, $\mathbf{Q}^*_1 \subset \mathbb{R}^n$ centered on the origin of $\mathbb{R}^n$ and $\mathbf{Q}^o_2$, $\mathbf{Q}^*_2 \subset \mathbb{R}^m$ centered on the origin of $\mathbb{R}^m$, such that $\mathbf{vol}\{\mathbf{Q}^o_1\}^{\frac{1}{n}} = \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}$, $\mathbf{vol}\{\mathbf{Q}^*_1\}^{\frac{1}{n}} = 3\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}$ and $\mathbf{vol}\{\mathbf{Q}^o_2\}^{\frac{1}{m}} = \mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{m}}$, $\mathbf{vol}\{\mathbf{Q}^*_2\}^{\frac{1}{m}} = 3\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{m}} = 3\lambda$.

Remark 3.4.2. *Suppose $\mathbf{Q}_1 \cap \mathbf{Q}^o_1 = \emptyset$. We must have $|x| \geq |x^o|/\sqrt{n}$ for every $x \in \mathbf{Q}_1$ and every $x^o \in \mathbf{Q}^o_1$. Otherwise, if $\mathbf{Q}_1$ intersects $\mathbf{Q}^o_1$, then $\mathbf{Q}_1 \subset \mathbf{Q}^*_1$.*

Suppose $\mathbf{Q}_1 \times \mathbf{Q}_2$ centered on some $(x_o, y_o) \in \mathbb{R}^n \times \mathbb{R}^m$ with $\sqrt{|x_o|^2 + |y_o|^2} \geq 3$. By using $\gamma + \delta \geq 0$, we can easily show

$$\mathbf{A}^{\alpha\beta}_{pqr}(\omega, \sigma)\Big[\mathbf{Q}_1 \times \mathbf{Q}_2\Big] \; \approx \; \lambda^{\beta - \frac{m}{p} + \frac{m}{q}}. \qquad (\beta - \tfrac{m}{p} + \tfrac{m}{q}) \tag{3. 86}$$

From now on, we assume $\mathbf{Q}_1 \times \mathbf{Q}_2$ centered on $(x_o, y_o) \in \mathbb{R}^n \times \mathbb{R}^m$ where $\sqrt{|x_o|^2 + |y_o|^2} < 3$.

3.4.1 Case 1: $\gamma \geq 0$, $\delta \leq 0$.

Recall (1. 19) of which $\gamma \geq 0, \delta \leq 0$ satisfy $\alpha - \frac{n}{p} < \delta$, $\beta - \frac{m}{p} < \delta$. By adjusting the value of r, we assume $0 < \gamma qr < n$ or $n < \gamma qr < n + m$.

Let $\mathbf{A}^{\alpha\beta}_{pqr}(\omega, \sigma)\Big[\mathbf{Q}_1 \times \mathbf{Q}_2\Big]$ defined in (3. 85). For $0 < \gamma qr < n$, we have

$$
\begin{aligned}
\mathbf{A}^{\alpha\beta}_{pqr}(\omega,\sigma)\Big[\mathbf{Q}_1 \times \mathbf{Q}_2\Big] \;\leq\; & \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left\{ \left(\frac{1}{\lambda}\right)^{m} \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma qr} dxdy \right\}^{\frac{1}{qr}} \qquad (\delta \leq 0) \\
\leq\; & \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{1}{\lambda}\right)^{\frac{m}{qr}} \left\{ \int_{\mathbf{Q}_2} \left\{ \int_{\mathbf{Q}_1} \left(\frac{1}{|x|}\right)^{\gamma qr} dx \right\} dy \right\}^{\frac{1}{qr}} \\
\leq\; & \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{1}{\lambda}\right)^{\frac{m}{qr}} \lambda^{\frac{m}{qr}} \left\{ \int_{\mathbf{Q}_1} \left(\frac{1}{|x|}\right)^{\gamma qr} dx \right\}^{\frac{1}{qr}} \\
\leq\; & \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{1}{\lambda}\right)^{\frac{m}{qr}} \lambda^{\frac{m}{qr}} \left\{ \int_{\mathbf{Q}^*_1} \left(\frac{1}{|x|}\right)^{\gamma qr} dx \right\}^{\frac{1}{qr}} \qquad \text{by } \textbf{Remark 3.4.2} \\
\leq\; & \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{1}{\lambda}\right)^{\frac{m}{qr}} \lambda^{\frac{m}{qr}} \;=\; \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\beta - \frac{m}{p} + \frac{m}{q}}. \qquad \left(\beta - \tfrac{m}{p} + \tfrac{m}{q} > 0\right)
\end{aligned}
\tag{3. 87}
$$

For $n < \gamma qr < n + m$, we have

$$
\begin{aligned}
\mathbf{A}^{\alpha\beta}_{pqr}(\omega,\sigma)\Big[\mathbf{Q}_1 \times \mathbf{Q}_2\Big] \;\leq\; & \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left\{ \left(\frac{1}{\lambda}\right)^{m} \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma qr} dxdy \right\}^{\frac{1}{qr}} \qquad (\delta \leq 0) \\
\leq\; & \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{1}{\lambda}\right)^{\frac{m}{qr}} \left\{ \int_{\mathbf{Q}_2} \left\{ \int_{\mathbb{R}^n} \left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma qr} dx \right\} dy \right\}^{\frac{1}{qr}} \\
\leq\; & \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{1}{\lambda}\right)^{\frac{m}{qr}} \left\{ \int_{\mathbf{Q}_2} \left(\frac{1}{|y|}\right)^{\gamma qr - n} dy \right\}^{\frac{1}{qr}} \\
\leq\; & \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{1}{\lambda}\right)^{\frac{m}{qr}} \left\{ \int_{\mathbf{Q}^*_2} \left(\frac{1}{|y|}\right)^{\gamma qr - n} dy \right\}^{\frac{1}{qr}} \qquad \text{See } \textbf{Remark 3.4.2} \\
\leq\; & \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{1}{\lambda_2}\right)^{\frac{m}{qr}} \lambda_2^{\frac{n+m}{qr} - \gamma} \;=\; \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\beta - \frac{m}{p} + \frac{m}{q} + \frac{n}{qr} - \gamma}.
\end{aligned}
\tag{3. 88}
$$

Note that $\alpha - \frac{n}{p} < \delta$, $\beta - \frac{m}{p} < \delta$ and $\frac{\alpha+\beta}{n+m} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{n+m}$. For r sufficiently close to 1, we find

$$
\beta - \frac{m}{p} + \frac{m}{q} + \frac{n}{qr} - \gamma \;=\; \delta - \alpha + \frac{n}{p} - \left(\frac{n}{q} - \frac{n}{qr}\right) \;>\; 0. \tag{3. 89}
$$

3.4.2 Case 2: $\gamma \leq 0, \delta \geq 0$

Recall (1. 20) of which $\gamma \leq 0$, $\delta \geq 0$ satisfy $\alpha - n\left(\frac{q-1}{q}\right) < \gamma$ and $\beta - m\left(\frac{q-1}{q}\right)$. By adjusting the value of r, we assume $0 < \delta\left(\frac{pr}{p-1}\right) < n$ or $n < \delta\left(\frac{pr}{p-1}\right) < n+m$.

Let $\mathbf{A}^{\alpha\beta}_{pqr}(\omega,\sigma)\Big[\mathbf{Q}_1 \times \mathbf{Q}_2\Big]$ defined in (3. 85). For $0 < \delta\left(\frac{pr}{p-1}\right) < n$, we have

$$\begin{aligned}
\mathbf{A}^{\alpha\beta}_{pqr}(\omega,\sigma)\Big[\mathbf{Q}_1 \times \mathbf{Q}_2\Big] &\leq \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left\{ \left(\frac{1}{\lambda}\right)^m \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\delta\frac{pr}{p-1}} dxdy \right\}^{\frac{p-1}{pr}} \qquad (\gamma \leq 0) \\
&\leq \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{1}{\lambda}\right)^{m\frac{p-1}{pr}} \left\{ \int_{\mathbf{Q}_2} \left\{ \int_{\mathbf{Q}_1} \left(\frac{1}{|x|}\right)^{\delta\frac{pr}{p-1}} dx \right\} dy \right\}^{\frac{p-1}{pr}} \\
&\leq \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{1}{\lambda}\right)^{m\frac{p-1}{pr}} \lambda^{m\frac{p-1}{pr}} \left\{ \int_{\mathbf{Q}_1} \left(\frac{1}{|x|}\right)^{\delta\frac{pr}{p-1}} dx \right\}^{\frac{p-1}{pr}} \\
&\leq \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{1}{\lambda}\right)^{m\frac{p-1}{pr}} \lambda^{m\frac{p-1}{pr}} \left\{ \int_{\mathbf{Q}_1^*} \left(\frac{1}{|x|}\right)^{\delta\frac{pr}{p-1}} dx \right\}^{\frac{p-1}{pr}} \qquad \text{by } \textbf{Remark 3.4.2} \\
&= \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\beta - \frac{m}{p} + \frac{m}{q}}. \qquad \left(\beta - \tfrac{m}{p} + \tfrac{m}{q} > 0\right)
\end{aligned} \tag{3. 90}$$

For $n < \delta\left(\frac{pr}{p-1}\right) < n+m$, we have

$$\begin{aligned}
\mathbf{A}^{\alpha\beta}_{pqr}(\omega,\sigma)\Big[\mathbf{Q}_1 \times \mathbf{Q}_2\Big] &\leq \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left\{ \left(\frac{1}{\lambda}\right)^m \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\delta\frac{pr}{p-1}} dxdy \right\}^{\frac{p-1}{pr}} \qquad (\gamma \leq 0) \\
&\leq \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{1}{\lambda}\right)^{m\frac{p-1}{pr}} \left\{ \int_{\mathbf{Q}_2} \left\{ \int_{\mathbb{R}^n} \left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\delta\frac{pr}{p-1}} dx \right\} dy \right\}^{\frac{p-1}{pr}} \\
&\leq \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{1}{\lambda}\right)^{m\frac{p-1}{pr}} \left\{ \int_{\mathbf{Q}_2} \left(\frac{1}{|y|}\right)^{\delta\frac{pr}{p-1}-n} dy \right\}^{\frac{p-1}{pr}} \\
&\leq \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{1}{\lambda}\right)^{m\frac{p-1}{pr}} \left\{ \int_{\mathbf{Q}_2^*} \left(\frac{1}{|y|}\right)^{\delta\frac{pr}{p-1}-n} dy \right\}^{\frac{p-1}{pr}} \qquad \text{See } \textbf{Remark 3.4.2} \\
&\leq \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{1}{\lambda}\right)^{m\frac{p-1}{pr}} \lambda^{(n+m)\frac{p-1}{pr}-\delta} = \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\, \lambda^{\beta - \frac{m}{p} + \frac{m}{q} + n\left(\frac{p-1}{pr}\right) - \delta}.
\end{aligned} \tag{3. 91}$$

Because $\alpha - n\left(\frac{q-1}{q}\right) < \gamma$, $\beta - m\left(\frac{q-1}{q}\right)$ and $\frac{\alpha+\beta}{n+m} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{n+m}$, we find

$$\beta - \frac{m}{p} + \frac{m}{q} + n\left(\frac{p-1}{pr}\right) - \delta \;=\; \gamma - \alpha + n\left(\frac{q-1}{q}\right) - n\left(\frac{p-1}{p} - \frac{p-1}{pr}\right) \;>\; 0 \qquad (3.\ 92)$$

for r chosen sufficiently close to 1.

3.4.3 Case 3: $\gamma > 0, \delta > 0$

Recall (1. 21) of which $\gamma > 0, \delta > 0$ satisfy

$$\alpha - \frac{n}{p} < \delta \quad \text{if} \quad \alpha - \frac{n}{p} \geq 0,\ \ \beta - \frac{m}{p} < 0; \qquad \beta - \frac{m}{p} < \delta \quad \text{if} \quad \alpha - \frac{n}{p} < 0,\ \ \beta - \frac{m}{p} \geq 0;$$

$$\alpha - \frac{n}{p} + \beta - \frac{m}{p} < \delta \quad \text{if} \quad \alpha - \frac{n}{p} \geq 0,\ \ \beta - \frac{m}{p} \geq 0.$$

$$\alpha - n\left(\frac{q-1}{q}\right) < \gamma \quad \text{if} \quad \alpha - n\left(\frac{q-1}{q}\right) \geq 0,\ \ \beta - m\left(\frac{q-1}{q}\right) < 0;$$

$$\beta - m\left(\frac{q-1}{q}\right) < \gamma \quad \text{if} \quad \alpha - n\left(\frac{q-1}{q}\right) < 0,\ \ \beta - m\left(\frac{q-1}{q}\right) \geq 0;$$

$$\alpha - n\left(\frac{q-1}{q}\right) + \beta - m\left(\frac{q-1}{q}\right) < \gamma \quad \text{if} \quad \alpha - n\left(\frac{q-1}{q}\right) \geq 0,\ \ \beta - m\left(\frac{q-1}{q}\right) \geq 0.$$

By adjusting the value of r, we assume $0 < \gamma qr < n$ or $n < \gamma qr < n+m$ and $0 < \delta\left(\frac{pr}{p-1}\right) < n$ or $n < \delta\left(\frac{pr}{p-1}\right) < n+m$.

Let $\mathbf{A}^{\alpha\beta}_{pqr}(\omega,\sigma)\big[\mathbf{Q}_1 \times \mathbf{Q}_2\big]$ defined in (3. 85).For $0 < \gamma qr < n$, $0 < \delta\left(\frac{pr}{p-1}\right) < n$, we have

$$\begin{aligned}
\mathbf{A}^{\alpha\beta}_{pqr}(\omega,\sigma)\big[\mathbf{Q}_1 \times \mathbf{Q}_2\big] \;\leq\; & \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{1}{\lambda}\right)^{\frac{m}{qr}} \left(\frac{1}{\lambda}\right)^{m\frac{p-1}{pr}} \\
& \left\{\int_{\mathbf{Q}_2}\left\{\int_{\mathbf{Q}_1}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma qr} dx\right\} dy\right\}^{\frac{1}{qr}} \left\{\int_{\mathbf{Q}_2}\left\{\int_{\mathbf{Q}_1}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\delta\frac{pr}{p-1}} dx\right\} dy\right\}^{\frac{p-1}{pr}} \\
\leq\; & \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{1}{\lambda}\right)^{\frac{m}{qr}} \left(\frac{1}{\lambda}\right)^{m\frac{p-1}{pr}} \lambda^{\frac{m}{qr}} \lambda^{m\frac{p-1}{pr}} \left\{\int_{\mathbf{Q}_1}\left(\frac{1}{|x|}\right)^{\gamma qr} dy\right\}^{\frac{1}{qr}} \left\{\int_{\mathbf{Q}_1}\left(\frac{1}{|x|}\right)^{\delta\frac{pr}{p-1}} dy\right\}^{\frac{p-1}{pr}} \\
\leq\; & \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{1}{\lambda}\right)^{\frac{m}{qr}} \left(\frac{1}{\lambda}\right)^{m\frac{p-1}{pr}} \lambda^{\frac{m}{qr}} \lambda^{m\frac{p-1}{pr}} \left\{\int_{\mathbf{Q}_1^*}\left(\frac{1}{|x|}\right)^{\gamma qr} dy\right\}^{\frac{1}{qr}} \left\{\int_{\mathbf{Q}_1^*}\left(\frac{1}{|x|}\right)^{\delta\frac{pr}{p-1}} dy\right\}^{\frac{p-1}{pr}} \\
& \qquad \text{by } \textbf{Remark 3.4.2} \\
\leq\; & \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{1}{\lambda}\right)^{\frac{m}{qr}} \left(\frac{1}{\lambda}\right)^{m\frac{p-1}{pr}} \lambda^{\frac{m}{qr}} \lambda^{m\frac{p-1}{pr}} \;=\; \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\beta - \frac{m}{p} + \frac{m}{q}}. \qquad \left(\beta - \tfrac{m}{p} + \tfrac{m}{q} > 0\right)
\end{aligned} \qquad (3.\ 93)$$

For $n < \gamma qr < n+m$, $0 < \delta\left(\frac{pr}{p-1}\right) < n$, we have

$$
\begin{aligned}
\mathbf{A}^{\alpha\beta}_{pqr}(\omega,\sigma)\Big[\mathbf{Q}_1\times\mathbf{Q}_2\Big] \ \le\ & \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\beta-m\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\frac{1}{\lambda}\right)^{\frac{m}{qr}}\left(\frac{1}{\lambda}\right)^{m\left(\frac{p-1}{pr}\right)}\\
& \left\{\int_{\mathbf{Q}_2}\left\{\int_{\mathbf{Q}_1}\left(\frac{1}{\sqrt{|x|^2+|y|^2}}\right)^{\gamma qr}dx\right\}dy\right\}^{\frac{1}{qr}}\left\{\int_{\mathbf{Q}_2}\left\{\int_{\mathbf{Q}_1}\left(\frac{1}{\sqrt{|x|^2+|y|^2}}\right)^{\delta\left(\frac{pr}{p-1}\right)}dx\right\}dy\right\}^{\frac{p-1}{pr}}\\
\le\ & \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\beta-m\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\frac{1}{\lambda}\right)^{\frac{m}{qr}}\left(\frac{1}{\lambda}\right)^{m\frac{p-1}{pr}}\ \lambda^{m\frac{p-1}{pr}}\\
& \left\{\int_{\mathbf{Q}_2}\left\{\int_{\mathbb{R}^n}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma qr}dx\right\}dy\right\}^{\frac{1}{qr}}\left\{\int_{\mathbf{Q}_1}\left(\frac{1}{|x|}\right)^{\delta\frac{pr}{p-1}}dy\right\}^{\frac{p-1}{pr}}\\
\le\ & \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\beta-m\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\frac{1}{\lambda}\right)^{\frac{m}{qr}}\left(\frac{1}{\lambda}\right)^{m\frac{p-1}{pr}}\ \lambda^{m\frac{p-1}{pr}}\\
& \left\{\int_{\mathbf{Q}_2^*}\left(\frac{1}{|y|}\right)^{\gamma qr-n}dy\right\}^{\frac{1}{qr}}\left\{\int_{\mathbf{Q}_1^*}\left(\frac{1}{|x|}\right)^{\delta\frac{pr}{p-1}}dy\right\}^{\frac{p-1}{pr}} \qquad \text{by } \textbf{Remark 3.4.2}\\
\le\ & \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\beta-m\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\frac{1}{\lambda}\right)^{\frac{m}{qr}}\left(\frac{1}{\lambda}\right)^{m\frac{p-1}{pr}}\ \lambda^{m\frac{p-1}{pr}}\,\lambda^{\frac{n+m}{qr}-\gamma}\\
=\ & \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\beta-\frac{m}{p}+\frac{m}{q}+\frac{n}{qr}-\gamma}.
\end{aligned}
\tag{3. 94}
$$

By using the homogeneity condition: $\frac{\alpha+\beta}{n+m} = \frac{1}{p}-\frac{1}{q}+\frac{\gamma+\delta}{n+m}$, we find

$$\beta-\frac{m}{p}+\frac{m}{q}+\frac{n}{qr}-\gamma \ =\ \delta-\left(\alpha-\frac{n}{p}\right)-\left(\frac{n}{q}-\frac{n}{qr}\right) \tag{3. 95}$$

Note that $\alpha-\frac{n}{p}<\delta$ if $\alpha-\frac{n}{p}\ge 0$. For r sufficiently close to 1, we have

$$\delta-\left(\alpha-\frac{n}{p}\right)-\left(\frac{n}{q}-\frac{n}{qr}\right)\ >\ 0. \tag{3. 96}$$

For $0<\gamma qr<n$, $n<\delta\left(\frac{pr}{p-1}\right)<n+m$, an analogue estimate of (3. 94) shows

$$\mathbf{A}^{\alpha\beta}_{pqr}(\omega,\sigma)\Big[\mathbf{Q}_1\times\mathbf{Q}_2\Big]\ \le\ \mathfrak{C}_{p\,q\,r\,\gamma\,\delta}\ \lambda^{\beta-\frac{m}{p}+\frac{m}{q}+n\left(\frac{p-1}{pr}\right)-\delta}. \tag{3. 97}$$

By using $\frac{\alpha+\beta}{n+m}=\frac{1}{p}-\frac{1}{q}+\frac{\gamma+\delta}{n+m}$ again, we find

$$\beta-\frac{m}{p}+\frac{m}{q}+n\left(\frac{p-1}{pr}\right)-\delta=\gamma-\left[\alpha-n\left(\frac{q-1}{q}\right)\right]-n\left(\frac{p-1}{p}-\frac{p-1}{pr}\right). \tag{3. 98}$$

Note that $\alpha - n\left(\frac{q-1}{q}\right) < \gamma$ if $\alpha - n\left(\frac{q-1}{q}\right) \geq 0$. For r sufficiently close to 1, we have

$$\gamma - \left[\alpha - n\left(\frac{q-1}{q}\right)\right] - n\left(\frac{p-1}{p} - \frac{p-1}{pr}\right) \; > \; 0. \tag{3. 99}$$

For $n < \gamma qr < n+m$, $n < \delta\left(\frac{pr}{p-1}\right) < n+m$, we have

$$\begin{aligned}
\mathbf{A}^{\alpha\beta}_{pqr}(\omega,\sigma)\Big[\mathbf{Q}_1 \times \mathbf{Q}_2\Big] \; &\leq \; \mathfrak{C}_{p\;q\;r\;\gamma\;\delta}\; \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{1}{\lambda}\right)^{\frac{m}{qr}} \left(\frac{1}{\lambda}\right)^{m\frac{p-1}{pr}} \\
&\left\{\int_{\mathbf{Q}_2}\left\{\int_{\mathbf{Q}_1}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma qr} dx\right\} dy\right\}^{\frac{1}{qr}} \left\{\int_{\mathbf{Q}_2}\left\{\int_{\mathbf{Q}_1}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\delta\frac{pr}{p-1}} dx\right\} dy\right\}^{\frac{p-1}{pr}} \\
&\leq \; \mathfrak{C}_{p\;q\;r\;\gamma\;\delta}\; \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{1}{\lambda}\right)^{\frac{m}{qr}} \left(\frac{1}{\lambda}\right)^{m\frac{p-1}{pr}} \\
&\left\{\int_{\mathbf{Q}_2}\left\{\int_{\mathbb{R}^n}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma qr} dx\right\} dy\right\}^{\frac{1}{qr}} \left\{\int_{\mathbf{Q}_2}\left\{\int_{\mathbb{R}^n}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\delta\frac{pr}{p-1}} dx\right\} dy\right\}^{\frac{p-1}{pr}} \\
&\leq \; \mathfrak{C}_{p\;q\;r\;\gamma\;\delta}\; \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{1}{\lambda}\right)^{\frac{m}{qr}} \left(\frac{1}{\lambda}\right)^{m\frac{p-1}{pr}} \left\{\int_{\mathbf{Q}_2}\left(\frac{1}{|y|}\right)^{\gamma qr - n} dy\right\}^{\frac{1}{qr}} \left\{\int_{\mathbf{Q}_2}\left(\frac{1}{|y|}\right)^{\delta\frac{pr}{p-1}-n} dy\right\}^{\frac{p-1}{pr}} \\
&\leq \; \mathfrak{C}_{p\;q\;r\;\gamma\;\delta}\; \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{1}{\lambda}\right)^{\frac{m}{qr}} \left(\frac{1}{\lambda}\right)^{m\frac{p-1}{pr}} \\
&\left\{\int_{\mathbf{Q}_2^*}\left(\frac{1}{|y|}\right)^{\gamma qr - n} dy\right\}^{\frac{1}{qr}} \left\{\int_{\mathbf{Q}_2^*}\left(\frac{1}{|y|}\right)^{\delta\frac{pr}{p-1}-n} dy\right\}^{\frac{p-1}{pr}} \qquad \text{by } \textbf{Remark 3.4.2} \\
&\leq \; \mathfrak{C}_{p\;q\;r\;\gamma\;\delta}\; \lambda^{\beta - m\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\frac{1}{\lambda}\right)^{\frac{m}{qr}} \left(\frac{1}{\lambda}\right)^{m\frac{p-1}{pr}} \lambda^{\frac{n+m}{qr}-\gamma} \lambda^{(n+m)\frac{p-1}{pr}-\delta} \\
&= \; \mathfrak{C}_{p\;q\;r\;\gamma\;\delta}\; \lambda^{\beta - \frac{m}{p} + \frac{m}{q} + \frac{n}{qr} + n\left(\frac{p-1}{pr}\right) - \gamma - \delta}.
\end{aligned} \tag{3. 100}$$

By using the homogeneity condition: $\frac{\alpha+\beta}{n+m} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{n+m}$, we find

$$\beta - \frac{m}{p} + \frac{m}{q} + \frac{n}{qr} + n\left(\frac{p-1}{pr}\right) - \gamma - \delta \; = \; \frac{n}{r} - \alpha + n\left(1 - \frac{1}{r}\right)\left(\frac{1}{p} - \frac{1}{q}\right). \tag{3. 101}$$

For r sufficiently close to 1, we have

$$\frac{n}{r} - \alpha + n\left(1 - \frac{1}{r}\right)\left(\frac{1}{p} - \frac{1}{q}\right) > 0. \tag{3. 102}$$

as $0 < \alpha < n$. □

3.5 Conclusion on the proof of sufficient conditions

In summary of the previous two sections, we prove the sufficient conditions for $\gamma \geq 0, \delta \leq 0$ and $\gamma \leq 0, \delta \geq 0$ in **Section 3.3**. Moreover, we prove the general case: $\gamma + \delta \geq 0$ whenever $\frac{\alpha}{n} > \frac{1}{p} - \frac{1}{q}, \frac{\beta}{m} > \frac{1}{p} - \frac{1}{q}$ in **Section 3.4**.

Recall (3. 26)-(3. 27) from **Section 3.2**. We have $\frac{\alpha}{n} \geq \frac{1}{p} - \frac{1}{q}$ and $\frac{\beta}{m} \geq \frac{1}{p} - \frac{1}{q}$ as necessities. Together with the homogeneity condition: $\frac{\alpha+\beta}{n+m} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{n+m}$, we find that $\frac{\alpha}{n} = \frac{1}{p} - \frac{1}{q} = \frac{\beta}{m}$ occurs only if $\gamma + \delta = 0$.

All together, what remains to be done is the case $\gamma > 0, \delta > 0$ when $\frac{\alpha}{n} = \frac{1}{p} - \frac{1}{q}, \frac{\beta}{m} > \frac{1}{p} - \frac{1}{q}$ or $\frac{\alpha}{n} > \frac{1}{p} - \frac{1}{q}, \frac{\beta}{m} = \frac{1}{p} - \frac{1}{q}$. Because $0 < \alpha < n, 0 < \beta < m$, we necessarily assert $p < q$.

For symmetry reason, consider only

$$\frac{\alpha}{n} = \frac{1}{p} - \frac{1}{q}, \qquad \frac{\beta}{m} > \frac{1}{p} - \frac{1}{q}.$$

Let $\omega(x,y) = \left(\sqrt{|x|^2+|y|^2}\right)^{-\gamma}$, $\sigma(x,y) = \left(\sqrt{|x|^2+|y|^2}\right)^{\delta}$ for $(x,y) \neq (0,0)$ which satisfy the Muckenhoupt characteristic in (3. 23):

$$\begin{aligned}\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\beta}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)}&\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q} dxdy\right\}^{\frac{1}{q}}\\ &\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\frac{\delta p}{p-1}} dxdy\right\}^{\frac{p-1}{p}} < \infty.\end{aligned}$$

for every $\mathbf{Q}_1 \times \mathbf{Q}_2 \subset \mathbb{R}^n \times \mathbb{R}^m$.

Suppose $\mathbf{Q}_1$ centered on the origin of $\mathbb{R}^n$. By taking $\mathbf{Q}_1$ shrink to the origin and applying Lebesgue differentiation theorem, we have

$$\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\beta}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_2\}}\int_{\mathbf{Q}_2}\left(\frac{1}{|y|}\right)^{\gamma q} dy\right\}^{\frac{1}{q}}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_2\}}\int_{\mathbf{Q}_2}\left(\frac{1}{|y|}\right)^{\frac{\delta p}{p-1}} dy\right\}^{\frac{p-1}{p}} < \infty. \qquad (3.\ 103)$$

This inequality holds for every $\mathbf{Q}_2 \subset \mathbb{R}^m$. By taking the supremum over $\mathbf{Q}_2 \subset \mathbb{R}^m$ in (3. 103), we find $\gamma q < m$, $\delta\left(\frac{p}{p-1}\right) < m$ and $\frac{\beta}{m} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{m}$ as necessary conditions.

By applying **Stein-Weiss theorem** on $\mathbb{R}^m$, we have

$$\left\{\int_{\mathbb{R}^m}\left\{\int_{\mathbb{R}^m} f(x,v)\left(\frac{1}{|y-v|}\right)^{m-\beta} dv\right\}^q \left(\frac{1}{|y|}\right)^{\gamma q} dy\right\}^{\frac{1}{q}} \leq \mathfrak{C}_{p\,q\,\beta\,\gamma\,\delta}\left\{\int_{\mathbb{R}^2} (f(x,y))^p\, |y|^{\delta p}\, dy\right\}^{\frac{1}{p}} \qquad (3.\ 104)$$

for $1 < p < q < \infty$.

Let $\gamma > 0$, $\delta > 0$ satisfying (1. 17)-(1. 18) and (1. 21). In particular, we find

$$\omega(x,y) = \left(\sqrt{|x|^2+|y|^2}\right)^{-\gamma} \leq |y|^{-\gamma}, \qquad \sigma(x,y) = \left(\sqrt{|x|^2+|y|^2}\right)^{\delta} \geq |y|^{\delta}.$$

Lastly, we have

$$\begin{aligned}
&\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} \left(\omega \mathbf{I}_{\alpha\beta} f\right)^q (x,y)dxdy\right\}^{\frac{1}{q}} \\
&= \left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} \left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left(\frac{1}{|y-v|}\right)^{m-\beta} dudv\right\}^q \omega^q(x,y)dxdy\right\}^{\frac{1}{q}} \\
&\leq \left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} \left\{\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^n} f(u,v)\left(\frac{1}{|x-u|}\right)^{n-\alpha} du\right)\left(\frac{1}{|y-v|}\right)^{m-\beta} dv\right\}^q \left(\frac{1}{|y|}\right)^{\gamma q} dxdy\right\}^{\frac{1}{q}} \\
&\leq \mathfrak{C}_{p\,q\,\beta\,\gamma\,\delta} \left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^m}\left\{\int_{\mathbb{R}^n} f(u,y)\left(\frac{1}{|x-u|}\right)^{n-\alpha} du\right\}^p |y|^{\delta p}dy\right\}^{\frac{q}{p}} dx\right\}^{\frac{1}{q}} \qquad \text{by (3. 104)} \\
&\leq \mathfrak{C}_{p\,q\,\beta\,\gamma\,\delta} \left\{\int_{\mathbb{R}^m}\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n} f(u,y)\left(\frac{1}{|x-u|}\right)^{n-\alpha} du\right\}^q dx\right\}^{\frac{p}{q}} |y|^{\delta p}dy\right\}^{\frac{1}{p}} \\
&\qquad\qquad \text{by Minkowski integral inequality} \\
&\leq \mathfrak{C}_{p\,q\,\alpha\,\gamma\,\delta} \left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} (f(x,y))^p |y|^{\delta p} dxdy\right\}^{\frac{1}{p}} \quad \text{by } \textbf{Hardy-Littlewood-Sobolev theorem } \left(\tfrac{\alpha}{n} = \tfrac{1}{p} - \tfrac{1}{q}\right) \\
&\leq \mathfrak{C}_{p\,q\,\alpha\,\gamma\,\delta} \left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} (f\sigma)^p (x,y)dxdy\right\}^{\frac{1}{p}}, \qquad 1 < p < q < \infty.
\end{aligned} \tag{3. 105}$$

Chapter 4: Fractional integration associated with Zygmund dilation

In this chapter, we prove **Theorem Three**, **Theorem Four** and **Theorem Five** regarding fractional integration on Heisenberg groups. We shall be working on the real variable representation with a multiplication law:

$$(x,y,t)\odot(u,v,s) \;=\; \Big[x+u, y+v, t+s+\mu\big(x\cdot v-y\cdot u\big)\Big], \qquad \mu\in\mathbb{R} \tag{4. 1}$$

for every $(x,y,t)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$ and $(u,v,s)^{-1}=(-u,-v,-s)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$.

4.1 Proof of Theorem Three

Recall $\mathbf{M}_\gamma$ defined in (1. 25) for $0\le\gamma<1$. Let $u\longrightarrow x-u$, $v\longrightarrow y-v$ and $s\longrightarrow t-s$, $\mathbf{M}_\gamma$ can be equivalently defined as

$$\mathbf{M}_\gamma f(x,y,t) \;=\; \sup_{\mathbf{R}\ni(x,y,t)} \mathbf{vol}\{\mathbf{R}\}^{\gamma-1}\iiint_{\mathbf{R}}\Big|f(u,v,s+\mu(x\cdot v-y\cdot u))\Big|\,dudvds \tag{4. 2}$$

where $\mathbf{R}\subset\mathbb{R}^{2n+1}$ is a rectangle centered with sides parallel to the coordinates.

As a special case, consider $\mathbf{R}=\mathbf{Q}_1\times\mathbf{Q}_2\times\mathbf{Q}_3\subset\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$: $\mathbf{Q}_1,\mathbf{Q}_2$ and $\mathbf{Q}_3$ are cubes in the regarding subspaces. $\mathbf{M}_{\alpha\beta}$ defined in (1. 28) for $0<\alpha<n$, $0<\beta<1$ is equivalent to

$$\begin{aligned}\mathbf{M}_{\alpha\beta}f(x,y,t) \;=&\\ \sup_{\substack{\mathbf{R}\ni(x,y,t)\\ \mathbf{vol}\{\mathbf{Q}_3\}=\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}}}} &\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{\mathbf{Q}_3\}^{\beta-1}\iiint_{\mathbf{R}}\Big|f(u,v,s+\mu(x\cdot v-y\cdot u))\Big|\,dudvds.\end{aligned} \tag{4. 3}$$

Note that $\mathbf{vol}\{\mathbf{Q}_3\}=\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}}$ implies $\mathbf{vol}\{\mathbf{R}\}=\mathbf{vol}\{\mathbf{Q}_1\}^{1+\frac{1}{n}}\mathbf{vol}\{\mathbf{Q}_2\}^{1+\frac{1}{n}}$. From (4. 3), we find

$$\begin{aligned}&\sup_{\substack{\mathbf{R}\ni(x,y,t)\\ \mathbf{vol}\{\mathbf{Q}_3\}=\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}}}} \mathbf{vol}\{\mathbf{Q}_1\}^{\left[\frac{\alpha+\beta}{n+1}-1\right](1+\frac{1}{n})}\mathbf{vol}\{\mathbf{Q}_2\}^{\left[\frac{\alpha+\beta}{n+1}-1\right](1+\frac{1}{n})}\iiint_{\mathbf{R}}\Big|f(u,v,s+\mu(x\cdot v-y\cdot u))\Big|\,dudvds\\
=&\sup_{\substack{\mathbf{R}\ni(x,y,t)\\ \mathbf{vol}\{\mathbf{Q}_3\}=\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}}}} \mathbf{vol}\{\mathbf{R}\}^{\frac{\alpha+\beta}{n+1}-1}\iiint_{\mathbf{R}}\Big|f(u,v,s+\mu(x\cdot v-y\cdot u))\Big|\,dudvds\\
\le&\sup_{\mathbf{R}\ni(x,y,t)}\mathbf{vol}\{\mathbf{R}\}^{\gamma-1}\iiint_{\mathbf{R}}\Big|f(u,v,s+\mu(x\cdot v-y\cdot u))\Big|\,dudvds \qquad \gamma=\tfrac{\alpha+\beta}{n+1}.\end{aligned} \tag{4. 4}$$

Hence, $\mathbf{M}_{\alpha\beta}$ is controlled by the strong fractional maximal operator $\mathbf{M}_\gamma$ whenever $\gamma=\frac{\alpha+\beta}{n+1}$.

Let
$$\gamma = \frac{1}{p} - \frac{1}{q} \qquad 1 < p \le q < \infty.$$
This required homogeneity condition can be found by changing one-parameter dilations inside (1. 26). In order to prove the converse, we need the following multi-parameter covering lemma due to Córdoba and Fefferman [5].

Córdoba-Fefferman covering lemma *Let $\left\{\mathbf{R}_j\right\}_{j=1}^{\infty}$ be a collection of rectangles in $\mathbb{R}^{2n+1}$ parallel to the coordinates. There is a subsequence $\left\{\widehat{\mathbf{R}}_k\right\}_{k=1}^{\infty}$ such that*
$$\mathbf{vol}\left\{\bigcup_j \mathbf{R}_j\right\} \lesssim \mathbf{vol}\left\{\bigcup_k \widehat{\mathbf{R}}_k\right\} \tag{4. 5}$$
and
$$\left\|\sum_k \chi_{\widehat{\mathbf{R}}_k}\right\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})}^p \lesssim \mathbf{vol}\left\{\bigcup_k \widehat{\mathbf{R}}_k\right\}, \qquad 1 < p < \infty \tag{4. 6}$$
where χ is an indicator function.

Remark 4.1.1. *This covering lemma is established within a much more general setting. Namely, the Lebesgue measure can be replaced by an absolutely continuous measure whose Nikodym derivative satisfies the rectangle A_∞-property. See the paper by Córdoba and Fefferman [5].*

Define
$$\mathbf{U}_\lambda = \left\{(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \colon \mathbf{M}_\gamma f(x,y,t) > \lambda\right\}. \tag{4. 7}$$
Given any $(x,y,t) \in \mathbf{U}_\lambda$, there is a rectangle $\mathbf{R}_j \subset \mathbb{R}^{2n+1}$ containing (x,y,t) such that
$$\mathbf{vol}\{\mathbf{R}_j\}^{\gamma-1} \iiint_{\mathbf{R}_j} \Big|f(u,v,s+\mu(x\cdot v - y\cdot u))\Big| dudvds > \frac{1}{2}\lambda. \tag{4. 8}$$
Let (x,y,t) run through the set $\mathbf{U}_\lambda$. We have
$$\mathbf{U}_\lambda \subset \bigcup_j \mathbf{R}_j.$$
By applying the covering lemma, we select a subsequence $\{\widehat{\mathbf{R}}_k\}_{k=1}^{\infty}$ from the union above and
$$\begin{aligned}
\mathbf{vol}\left\{\mathbf{U}_\lambda\right\} &\lesssim \mathbf{vol}\left\{\bigcup_j \mathbf{R}_j\right\} \lesssim \mathbf{vol}\left\{\bigcup_k \widehat{\mathbf{R}}_k\right\} \qquad \text{by (4. 5)}\\
&\le \sum_k \mathbf{vol}\left\{\widehat{\mathbf{R}}_k\right\} \qquad\qquad (4.\ 9)\\
&\le \sum_k \left\{\frac{2}{\lambda} \iiint_{\widehat{\mathbf{R}}_k} \Big|f(u,v,s+\mu(x\cdot v - y\cdot u))\Big| dudvds\right\}^{\frac{1}{1-\gamma}} \qquad \text{by (4. 8).}
\end{aligned}$$

Because $0 \le \gamma < 1$, we further have

$$
\begin{aligned}
\mathbf{vol}\left\{\bigcup_k \widehat{\mathbf{R}}_k\right\} &\lesssim \lambda^{-\frac{1}{1-\gamma}} \left\{\sum_k \iiint_{\widehat{\mathbf{R}}_k} \left|f(u,v,s+\mu(x\cdot v - y\cdot u))\right| dudvds\right\}^{\frac{1}{1-\gamma}} \\
&= \lambda^{-\frac{1}{1-\gamma}} \left\{\iiint_{\mathbb{R}^{2n+1}} \left|f(u,v,s+\mu(x\cdot v - y\cdot u)) \sum_k \chi_{\widehat{\mathbf{R}}_k}(u,v,s)\right| dudvds\right\}^{\frac{1}{1-\gamma}} \\
&\le \lambda^{-\frac{1}{1-\gamma}} \left\{\iiint_{\mathbb{R}^{2n+1}} \left|f(u,v,s+\mu(x\cdot v - y\cdot u))\right|^p dudvds\right\}^{\frac{1}{p}\frac{1}{1-\gamma}} \left\|\sum_k \chi_{\widehat{\mathbf{R}}_k}\right\|_{\mathbf{L}^{\frac{p}{p-1}}(\mathbb{R}^{2n+1})}^{\frac{1}{1-\gamma}} \\
&\qquad \text{by Hölder inequality} \\
&= \lambda^{-\frac{1}{1-\gamma}} \left\{\iint_{\mathbb{R}^{2n}} \left\|f(u,v,\cdot)\right\|_{\mathbf{L}^p(\mathbb{R})}^p dudv\right\}^{\frac{1}{p}\frac{1}{1-\gamma}} \left\|\sum_k \chi_{\widehat{\mathbf{R}}_k}\right\|_{\mathbf{L}^{\frac{p}{p-1}}(\mathbb{R}^{2n+1})}^{\frac{1}{1-\gamma}} \\
&\le \lambda^{-\frac{1}{1-\gamma}} \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})}^{\frac{1}{1-\gamma}} \mathbf{vol}\left\{\bigcup_k \widehat{\mathbf{R}}_k\right\}^{\frac{p-1}{p}\frac{1}{1-\gamma}} \qquad \text{by (4. 6).}
\end{aligned}
\tag{4. 10}
$$

By raising both sides of (4. 10) to the $(1-\gamma)$-th power and then taking into account for $1-\gamma-\frac{p-1}{p} = \frac{1}{p} - \left[\frac{1}{p} - \frac{1}{q}\right] = \frac{1}{q}$, we find

$$
\mathbf{vol}\left\{\bigcup_k \widehat{\mathbf{R}}_k\right\}^{\frac{1}{q}} \lesssim \frac{1}{\lambda} \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})}. \tag{4. 11}
$$

Let $\mathbf{U}_\lambda$ defined in (4. 7). From (4. 9), we obtain

$$
\begin{aligned}
\mathbf{vol}\left\{(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \colon \mathbf{M}_\gamma f(x,y,t) > \lambda\right\}^{\frac{1}{q}} &\lesssim \mathbf{vol}\left\{\bigcup_k \widehat{\mathbf{R}}_k\right\}^{\frac{1}{q}} \\
&\lesssim \frac{1}{\lambda} \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})} \qquad \text{by (4. 11)}.
\end{aligned}
\tag{4. 12}
$$

By using this weak type (p,q)-estimate and applying Marcinkiewicz interpolation theorem, we conclude that $\mathbf{M}_\gamma$ is bounded from $\mathbf{L}^p(\mathbb{R}^{2n+1})$ to $\mathbf{L}^q(\mathbb{R}^{2n+1})$ if $\gamma = \frac{1}{p} - \frac{1}{q}$, $1 < p \le q < \infty$.

4.1.1 Proof of Córdoba-Fefferman covering lemma

We re-arrange the order of $\{\mathbf{R}_j\}_{j=1}^\infty$ if necessary so that the side length of $\mathbf{R}_j$ parallel to the t-coordinate is decreasing as $j \longrightarrow \infty$. For brevity, we call it t-side length.

Denote $\mathbf{R}_j^*$ to be the rectangle co-centered with $\mathbf{R}_j$ having its t-side length tripled and keeping the others same. We select $\widehat{\mathbf{R}}_k$ from $\left\{\mathbf{R}_j\right\}_{j=1}^\infty$ as follows.

Let $\widehat{\mathbf{R}}_1 = \mathbf{R}_1$. Having chosen $\widehat{\mathbf{R}}_1, \widehat{\mathbf{R}}_2, \ldots, \widehat{\mathbf{R}}_{N-1}$, we pick $\widehat{\mathbf{R}}_N$ as the first rectangle $\mathbf{R}$ on the list of $\mathbf{R}_j$'s after $\widehat{\mathbf{R}}_{N-1}$ so that

$$\mathbf{vol}\left\{ \mathbf{R} \cap \left[\bigcup_{\substack{k=1 \\ \widehat{\mathbf{R}}_k^* \cap \mathbf{R} \neq \emptyset}}^{N-1} \widehat{\mathbf{R}}_k^* \right] \right\} < \frac{1}{2} \mathbf{vol}\left\{ \mathbf{R} \right\}. \tag{4. 13}$$

Suppose $\mathbf{R}$ is an unselected rectangle. There is a positive number M such that $\mathbf{R}$ is on the list of $\mathbf{R}_j$'s after $\widehat{\mathbf{R}}_M$ and

$$\mathbf{vol}\left\{ \mathbf{R} \cap \left[\bigcup_{\substack{k=1 \\ \widehat{\mathbf{R}}_k^* \cap \mathbf{R} \neq \emptyset}}^{M} \widehat{\mathbf{R}}_k^* \right] \right\} \geq \frac{1}{2} \mathbf{vol}\left\{ \mathbf{R} \right\}. \tag{4. 14}$$

Recall $\widehat{\mathbf{R}}_k^*$ whose t-side length is tripled. Moreover, the t-side length of $\{\mathbf{R}_j\}_{j=1}^{\infty}$ is decreasing as $j \longrightarrow \infty$. On the t-coordinate, the projection of $\mathbf{R}$ is covered by the projection of the union inside (4. 14).

Slice all rectangles with a plane perpendicular to the t-axis. Denote $\mathbf{S}$, $\widehat{\mathbf{S}}_k$ and $\widehat{\mathbf{S}}_k^*$ to be the slices regarding to $\mathbf{R}$, $\widehat{\mathbf{R}}_k$ and $\widehat{\mathbf{R}}_k^*$. Consequently, (4. 14) implies

$$\mathbf{vol}\left\{ \mathbf{S} \cap \left[\bigcup_{\substack{k \\ \widehat{\mathbf{S}}_k^* \cap \mathbf{S} \neq \emptyset}}^{M} \widehat{\mathbf{S}}_k^* \right] \right\} \geq \frac{1}{2} \mathbf{vol}\left\{ \mathbf{S} \right\}. \tag{4. 15}$$

Let $\mathbf{M}$ be the strong maximal operator defined in $\mathbb{R}^{2n}$. Observe that (4. 15) further implies

$$\mathbf{M}\chi_{\bigcup_k \widehat{\mathbf{S}}_k^*}(x,y) > \frac{1}{2}, \qquad (x,y) \in \bigcup_j \mathbf{S}_j. \tag{4. 16}$$

From (4. 15)-(4. 16), by applying the $\mathbf{L}^p$-boundedness of $\mathbf{M}$, we find

$$\mathbf{vol}\left\{ \bigcup_j \mathbf{S}_j \right\} \lesssim \mathbf{vol}\left\{ \bigcup_k \widehat{\mathbf{S}}_k^* \right\}. \tag{4. 17}$$

By using (4. 17) and integrating in the t-coordinate, we have

$$\mathbf{vol}\left\{ \bigcup_j \mathbf{R}_j \right\} \lesssim \mathbf{vol}\left\{ \bigcup_k \widehat{\mathbf{R}}_k^* \right\} \lesssim \mathbf{vol}\left\{ \bigcup_k \widehat{\mathbf{R}}_k \right\} \tag{4. 18}$$

which is (4. 5).

On the other hand, (4. 13) implies

$$\mathbf{vol}\left\{ \widehat{\mathbf{S}}_N \cap \left[\bigcup_{\substack{k=1 \\ \widehat{\mathbf{S}}_k^* \cap \mathbf{S} \neq \emptyset}}^{N-1} \widehat{\mathbf{S}}_k^* \right] \right\} < \frac{1}{2} \mathbf{vol}\left\{ \widehat{\mathbf{S}}_N \right\}. \tag{4. 19}$$

Denote $\widehat{\mathbf{E}}_N = \widehat{\mathbf{S}}_N \setminus \bigcup_{k<N}\widehat{\mathbf{S}}_k$. From (4. 19), we find

$$\mathbf{vol}\left\{\widehat{\mathbf{E}}_N\right\} > \frac{1}{2}\mathbf{vol}\left\{\widehat{\mathbf{S}}_N\right\}. \tag{4. 20}$$

Let $\phi \in \mathbf{L}^{\frac{p}{p-1}}(\mathbb{R}^{2n})$ and $\left\|\phi\right\|_{\mathbf{L}^{\frac{p}{p-1}}(\mathbb{R}^{2n})} = 1$. We have

$$\begin{aligned}
\iint_{\mathbb{R}^{2n}} \phi(x,y)\sum_k \chi_{\widehat{\mathbf{S}}_k}(x,y)dxdy &= \sum_k \iint_{\widehat{\mathbf{S}}_k} \phi(x,y)dxdy \\
&= \sum_k \left\{\frac{1}{\mathbf{vol}\{\widehat{\mathbf{S}}_k\}} \iint_{\widehat{\mathbf{S}}_k} \phi(x,y)dxdy\right\}\mathbf{vol}\left\{\widehat{\mathbf{S}}_k\right\} \\
&< 2\sum_k \left\{\frac{1}{\mathbf{vol}\{\widehat{\mathbf{S}}_k\}} \iint_{\widehat{\mathbf{S}}_k} \phi(x,y)dxdy\right\}\mathbf{vol}\left\{\widehat{\mathbf{E}}_k\right\} \qquad \text{by (4. 20)} \\
&\lesssim \sum_k \iint_{\widehat{\mathbf{E}}_k} \mathbf{M}\phi(x,y)dxdy \\
&= \iint_{\bigcup_k \widehat{\mathbf{S}}_k} \mathbf{M}\phi(x,y)dxdy.
\end{aligned} \tag{4. 21}$$

By applying Hölder inequality and the $\mathbf{L}^p$-boundedness of $\mathbf{M}$, we find

$$\iint_{\bigcup_k \widehat{\mathbf{S}}_k} \mathbf{M}\phi(x,y)dxdy \leq \left\|\mathbf{M}\phi\right\|_{\mathbf{L}^{\frac{p}{p-1}}(\mathbb{R}^{2n})} \mathbf{vol}\left\{\bigcup_k \widehat{\mathbf{S}}_k\right\}^{\frac{1}{p}} \leq \mathfrak{C}_p\, \mathbf{vol}\left\{\bigcup_k \widehat{\mathbf{S}}_k\right\}^{\frac{1}{p}}. \tag{4. 22}$$

By substituting (4. 22) to (4. 21) and taking the supremum of ϕ, we arrive at

$$\left\|\sum_k \chi_{\widehat{\mathbf{S}}_k}\right\|_{\mathbf{L}^p(\mathbb{R}^{2n})} \leq \mathfrak{C}_p\, \mathbf{vol}\left\{\bigcup_k \widehat{\mathbf{S}}_k\right\}^{\frac{1}{p}}. \tag{4. 23}$$

Raising both sides of (4. 23) to the p^{th} power and integrating over t give us (4. 6).

4.2 Proof of Theorem Four

Recall $\mathbf{I}_{\alpha\beta\vartheta}$ defined in (1. 34)-(1. 35) for $\alpha, \beta \in \mathbb{R}$ and $\vartheta \geq 0$. By changing variables $s \longrightarrow s - \mu(x \cdot v - y \cdot v)$, we find

$$
\begin{aligned}
\mathbf{I}_{\alpha\beta\vartheta} f(x, y, t) &= \iiint_{\mathbb{R}^{2n+1}} f(u, v, s - \mu(x \cdot v - y \cdot u)) \mathbf{V}^{\alpha\beta\vartheta}(x - u, y - v, t - s) dudvds \\
&= \iiint_{\mathbb{R}^{2n+1}} f\Big(u, v, s - \mu\big(x \cdot v - y \cdot u\big)\Big) \\
&\qquad |x - u|^{\alpha - n} |y - v|^{\alpha - n} |t - s|^{\beta - 1} \left[\frac{|x - u||y - v|}{|t - s|} + \frac{|t - s|}{|x - u||y - v|}\right]^{-\vartheta} dudvds.
\end{aligned}
\tag{4. 24}
$$

4.2.1 Some necessary constraints

Consider a more general situation by replacing $\mathbf{V}^{\alpha\beta\vartheta}(x, y, t)$ with

$$
|x|^{\alpha_1 - n} |y|^{\alpha_2 - n} |t|^{\beta - 1} \left[\frac{|x||y|}{|t|} + \frac{|t|}{|x||y|}\right]^{-\vartheta}, \quad \alpha_1, \alpha_2, \beta \in \mathbb{R}, \quad \vartheta \geq 0. \tag{4. 25}
$$

By changing dialtions $(x, y, t) \longrightarrow (\rho_1 x, \rho_2 y, \rho_1\rho_2 \lambda t)$ and $(u, v, s) \longrightarrow (\rho_1 u, \rho_2 v, \rho_1 \rho_2 \lambda s)$ for $\rho_1, \rho_2 > 0$ and $0 < \lambda < 1$ or $\lambda > 1$, we have

$$
\begin{aligned}
&\left\{\iiint_{\mathbb{R}^{2n+1}} \left\{\iiint_{\mathbb{R}^{2n+1}} f\left[\rho_1^{-1} u, \rho_2^{-1} v, \rho_1^{-1} \rho_2^{-1} \lambda^{-1}\Big(s - \mu\lambda(x \cdot v - y \cdot u)\Big)\right]\right.\right. \\
&\left.\left. |x - u|^{\alpha_1 - n} |y - v|^{\alpha_2 - n} |t - s|^{\beta - 1} \left[\frac{|x - u||y - v|}{|t - s|} + \frac{|t - s|}{|x - u||y - v|}\right]^{-\vartheta} dudvds\right\}^q dxdydt\right\}^{\frac{1}{q}} \\
&= \rho_1^{\alpha_1 + \beta} \rho_2^{\alpha_2 + \beta} \rho_1^{\frac{n+1}{q}} \rho_2^{\frac{n+1}{q}} \lambda^{\beta} \lambda^{\frac{1}{q}} \left\{\iiint_{\mathbb{R}^{2n+1}} \left\{\iiint_{\mathbb{R}^{2n+1}} f(u, v, s - \mu(x \cdot v - y \cdot u))\right.\right. \\
&\quad \left.\left. |x - u|^{\alpha_1 - n} |y - v|^{\alpha_2 - n} |t - s|^{\beta - 1} \left[\frac{|x - u||y - v|}{\lambda|t - s|} + \frac{\lambda|t - s|}{|x - u||y - v|}\right]^{-\vartheta} dudvds\right\}^q dxdydt\right\}^{\frac{1}{q}} \qquad (4.\ 26) \\
&\geq \rho_1^{\alpha_1 + \beta} \rho_2^{\alpha_2 + \beta} \rho_1^{\frac{n+1}{q}} \rho_2^{\frac{n+1}{q}} \lambda^{\beta} \lambda^{\frac{1}{q}} \begin{cases} \lambda^{\vartheta}, & 0 < \lambda < 1, \\ \lambda^{-\vartheta}, & \lambda > 1 \end{cases} \\
&\left\{\iiint_{\mathbb{R}^{2n+1}} \left\{\iiint_{\mathbb{R}^{2n+1}} f(u, v, s - \mu(x \cdot v - y \cdot u))\right.\right. \\
&\quad \left.\left. |x - u|^{\alpha_1 - n} |y - v|^{\alpha_1 - n} |t - s|^{\beta - 1} \left[\frac{|x - u||y - v|}{|t - s|} + \frac{|t - s|}{|x - u||y - v|}\right]^{-\vartheta} dudvds\right\}^q dxdydt\right\}^{\frac{1}{q}}.
\end{aligned}
$$

The $\mathbf{L}^p \longrightarrow \mathbf{L}^q$-norm inequality in (1. 36) implies that the last line of (4. 26) is bounded by

$$
\left\{\iiint_{\mathbb{R}^{2n+1}} \left[f\left(\rho_1^{-1} u, \rho_2^{-1} v, \rho^{-1} \rho_2^{-1} \lambda^{-1} s\right)\right]^p dudvds\right\}^{\frac{1}{p}} = \rho_1^{\frac{n+1}{p}} \rho_2^{\frac{n+1}{p}} \lambda^{\frac{1}{p}} \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})}. \tag{4. 27}
$$

This must be true for every $\rho_1, \rho_2 > 0$ and $0 < \lambda < 1$ or $\lambda > 1$. We necessarily have

$$\frac{\alpha_1 + \beta}{n+1} = \frac{1}{p} - \frac{1}{q} = \frac{\alpha_2 + \beta}{n+1}, \quad \beta + \vartheta \geq \frac{1}{p} - \frac{1}{q}, \quad \text{or} \quad \beta - \vartheta \leq \frac{1}{p} - \frac{1}{q}. \tag{4. 28}$$

The first constraint in (4. 28) forces to have $\alpha_1 = \alpha_2$. Therefore, write

$$\frac{\alpha + \beta}{n+1} = \frac{1}{p} - \frac{1}{q}, \quad \alpha = \alpha_1 = \alpha_2. \tag{4. 29}$$

By bringing (4. 29) to the two inequality in (4. 28), we find

$$\vartheta \geq \beta - \frac{\alpha + \beta}{n+1} = \frac{n\beta - \alpha}{n+1} \quad \text{or} \quad \vartheta \geq \frac{\alpha + \beta}{n+1} - \beta = \frac{\alpha - n\beta}{n+1}. \tag{4. 30}$$

Together, we conclude $\vartheta \geq \frac{|\alpha - n\beta|}{n+1}$.

4.2.2 Proof of sufficient conditions

Given $\alpha, \beta \in \mathbb{R}$, $\mathbf{V}^{\alpha\beta\vartheta}$ is a distribution in $\mathbb{R}^{2n+1}$ agree with $\mathbf{V}^{\alpha\beta\vartheta}(x, y, t)$ in (1. 34) whenever $x \neq 0$, $y \neq 0$, $t \neq 0$ and $\vartheta \geq \left|\frac{\alpha - n\beta}{n+1}\right|$.
Suppose $\alpha \geq n\beta$. We have $\vartheta \geq \left|\frac{\alpha - n\beta}{n+1}\right| = \frac{\alpha - n\beta}{n+1}$ and

$$\begin{aligned}
\mathbf{V}^{\alpha\beta\vartheta}(x, y, t) \ &\leq \ |x|^{\alpha - n}|y|^{\alpha - n}|t|^{\beta - 1}\left[\frac{|x||y|}{|t|} + \frac{|t|}{|x||y|}\right]^{-\frac{\alpha - n\beta}{n+1}} \\
&\leq |x|^{\alpha - n}|y|^{\alpha - n}|t|^{\beta - 1}\left[\frac{|x||y|}{|t|}\right]^{-\frac{\alpha - n\beta}{n+1}} \\
&= |x|^{n\left[\frac{\alpha+\beta}{n+1}\right] - n}|y|^{n\left[\frac{\alpha+\beta}{n+1}\right] - n}|t|^{\left[\frac{\alpha+\beta}{n+1}\right] - 1}, \qquad x \neq 0, y \neq 0, t \neq 0.
\end{aligned} \tag{4. 31}$$

Suppose $\alpha \leq n\beta$. We have $\vartheta \geq \left|\frac{\alpha - n\beta}{n+1}\right| = \frac{n\beta - \alpha}{n+1}$ and

$$\begin{aligned}
\mathbf{V}^{\alpha\beta\vartheta}(x, y, t) \ &\leq \ |x|^{\alpha - n}|y|^{\alpha - n}|t|^{\beta - 1}\left[\frac{|x||y|}{|t|} + \frac{|t|}{|x||y|}\right]^{-\frac{n\beta - \alpha}{n+1}} \\
&\leq |x|^{\alpha - n}|y|^{\alpha - n}|t|^{\beta - 1}\left[\frac{|t|}{|x||y|}\right]^{-\frac{n\beta - \alpha}{n+1}} \\
&= |x|^{n\left[\frac{\alpha+\beta}{n+1}\right] - n}|y|^{n\left[\frac{\alpha+\beta}{n+1}\right] - n}|t|^{\left[\frac{\alpha+\beta}{n+1}\right] - 1}, \qquad x \neq 0, y \neq 0, t \neq 0.
\end{aligned} \tag{4. 32}$$

Let $\mathbf{I}_{\alpha\beta\vartheta}$ defined in (4. 24) and

$$\frac{\alpha + \beta}{n+1} = \frac{1}{p} - \frac{1}{q}, \quad 1 < p < q < \infty. \tag{4. 33}$$

We have

$$\begin{aligned}
\mathbf{I}_{\alpha\beta\vartheta} f(x,y,t) &= \iiint_{\mathbb{R}^{2n+1}} f(u,v,s-\mu\,(x\cdot v-y\cdot u))\mathbf{V}^{\alpha\beta\vartheta}(x-u,y-v,t-s)dudvds \\
&\leq \iiint_{\mathbb{R}^{2n+1}} f\Big(u,v,s-\mu\Big(x\cdot v-y\cdot u\Big)\Big) \\
&\quad |x-u|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}|y-v|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}|t-s|^{\frac{\alpha+\beta}{n+1}-1}dudvds, \qquad \text{by (4. 31)-(4. 32)}
\end{aligned} \tag{4. 34}$$

Define

$$\mathbf{F}_{\alpha\beta}(u,v,x,y,t) = \int_{\mathbb{R}} f(u,v,s-\mu(x\cdot v-y\cdot u))|t-s|^{\frac{\alpha+\beta}{n+1}-1}ds. \tag{4. 35}$$

From (4. 34)-(4. 35), we find

$$\mathbf{I}_{\alpha\beta\vartheta} f(x,y,t) \leq \iint_{\mathbb{R}^{2n}} |x-u|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}|y-v|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}\mathbf{F}_{\alpha\beta}(u,v,x,y,t)dudv. \tag{4. 36}$$

Recall **Hardy-Littlewood-Sobolev theorem** stated in chapter 1. By applying (1. 2) with $\frac{\alpha+\beta}{n+1}=\frac{1}{p}-\frac{1}{q}$, we have

$$\begin{aligned}
\left\{\int_{\mathbb{R}} \mathbf{F}^q_{\alpha\beta}(u,v,x,y,t)dt\right\}^{\frac{1}{q}} &\leq \mathfrak{C}_{p\ q\ \alpha\ \beta}\left\{\int_{\mathbb{R}} \left[f(u,v,t-\mu(x\cdot v-y\cdot u))\right]^p dt\right\}^{\frac{1}{p}} \\
&= \mathfrak{C}_{p\ q\ \alpha\ \beta}\left\|f(u,v,\cdot)\right\|_{\mathbf{L}^p(\mathbb{R})}
\end{aligned} \tag{4. 37}$$

regardless of $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$.

On the other hand, by applying (1. 2) with $n\left[\frac{\alpha+\beta}{n+1}\right]/n=\frac{\alpha+\beta}{n+1}=\frac{1}{p}-\frac{1}{q}$, we find

$$\begin{aligned}
&\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n} |x-u|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}\left\|f(u,v,\cdot)\right\|_{\mathbf{L}^p(\mathbb{R})}du\right\}^q dx\right\}^{\frac{1}{q}} \\
&\leq \mathfrak{C}_{p\ q\ \alpha\ \beta}\left\{\int_{\mathbb{R}^n}\left\|f(x,v,\cdot)\right\|^p_{\mathbf{L}^p(\mathbb{R})}dx\right\}^{\frac{1}{p}}
\end{aligned} \tag{4. 38}$$

and

$$\begin{aligned}
&\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n} |y-v|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}\left\|f(u,v,\cdot)\right\|_{\mathbf{L}^p(\mathbb{R})}dv\right\}^q dy\right\}^{\frac{1}{q}} \\
&\leq \mathfrak{C}_{p\ q\ \alpha\ \beta}\left\{\int_{\mathbb{R}^n}\left\|f(u,y,\cdot)\right\|^p_{\mathbf{L}^p(\mathbb{R})}dy\right\}^{\frac{1}{p}}.
\end{aligned} \tag{4. 39}$$

From (4. 36), we have

$$
\begin{aligned}
&\left\|\mathbf{I}_{\alpha\beta\vartheta} f\right\|_{\mathbf{L}^q(\mathbb{R}^{2n+1})}\\
\leq\ &\left\{\iiint_{\mathbb{R}^{2n+1}}\left\{\iint_{\mathbb{R}^{2n}}|x-u|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}|y-v|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}\mathbf{F}_{\alpha\beta}(u,v,x,y,t)dudv\right\}^q dxdydt\right\}^{\frac{1}{q}}\\
\leq\ &\left\{\iint_{\mathbb{R}^{2n}}\left\{\iint_{\mathbb{R}^{2n}}|x-u|^{n[\alpha+\beta n+1]-n}|y-v|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}\left\{\int_{\mathbb{R}}\mathbf{F}^q_{\alpha\beta}(u,v,x,y,t)dt\right\}^{\frac{1}{q}} dudv\right\}^q dxdy\right\}^{\frac{1}{q}}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{by Minkowski integral inequality}\\
\leq\ &\mathfrak{C}_{p\ q\ \alpha\ \beta}\left\{\iint_{\mathbb{R}^{2n}}\left\{\iint_{\mathbb{R}^{2n}}|x-u|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}|y-v|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}\left\|f(u,v,\cdot)\right\|_{\mathbf{L}^p(\mathbb{R})}dudv\right\}^q dxdy\right\}^{\frac{1}{q}}\quad\text{by (4. 37)}\\
\leq\ &\mathfrak{C}_{p\ q\ \alpha\ \beta}\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}|y-v|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}\left\|f(x,v,\cdot)\right\|_{\mathbf{L}^p(\mathbb{R})}dv\right\}^p dx\right\}^{\frac{q}{p}} dy\right\}^{\frac{1}{q}}\quad\text{by (4. 38)}\\
\leq\ &\mathfrak{C}_{p\ q\ \alpha\ \beta}\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}|y-v|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}\left\|f(x,v,\cdot)\right\|_{\mathbf{L}^p(\mathbb{R})}dv\right\}^q dy\right\}^{\frac{p}{q}} dx\right\}^{\frac{1}{p}}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{by Minkowski integral inequality}\\
\leq\ &\mathfrak{C}_{p\ q\ \alpha\ \beta}\left\{\iint_{\mathbb{R}^{2n}}\left\|f(x,y,\cdot)\right\|^p_{\mathbf{L}^p(\mathbb{R})}dxdy\right\}^{\frac{1}{p}}\quad\text{by (4. 39)}\\
=\ &\mathfrak{C}_{p\ q\ \alpha\ \beta}\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})}.
\end{aligned}
\tag{4. 40}
$$

4.3 Proof of Theorem Five

Recall $\mathbf{I}_{\alpha\beta\vartheta}$ defined in (1. 34)-(1. 35) for $\alpha,\beta\in\mathbb{R}$ and $\vartheta\geq 0$. As before, by changing variables $s\longrightarrow s-\mu\,(x\cdot v-y\cdot v)$, we write

$$
\begin{aligned}
\mathbf{I}_{\alpha\beta\vartheta} f(x,y,t)\ &=\ \iiint_{\mathbb{R}^{2n+1}} f(u,v,s-\mu\,(x\cdot v-y\cdot u))\mathbf{V}^{\alpha\beta\vartheta}(x-u,y-v,t-s)dudvds\\
&=\ \iiint_{\mathbb{R}^{2n+1}} f\Big(u,v,s-\mu\Big(x\cdot v-y\cdot u\Big)\Big)\\
&\qquad |x-u|^{\alpha-n}|y-v|^{\alpha-n}|t-s|^{\beta-1}\left[\frac{|x-u||y-v|}{|t-s|}+\frac{|t-s|}{|x-u||y-v|}\right]^{-\vartheta}dudvds.
\end{aligned}
$$

4.3.1 Proof of necessary condition

Let $\omega(x,y)=\left(\sqrt{|x|^2+|y|^2}\right)^{-\gamma}$ and $\sigma(x,y)=\left(\sqrt{|x|^2+|y|^2}\right)^{\delta}$ for $\gamma,\delta\in\mathbb{R}$ and $(x,y)\neq(0,0)$.

By changing dilations $(x,y,t) \longrightarrow (\rho x, \rho y, \rho^2\lambda t)$ and $(u,v,s) \longrightarrow (\rho u, \rho v, \rho^2 \lambda s)$ for $\rho > 0$ and $0 < \lambda < 1$ or $\lambda > 1$, we have

$$\begin{aligned}
&\Bigg\{\iiint_{\mathbb{R}^{2n+1}} \omega^q(x,y)\Bigg\{\iiint_{\mathbb{R}^{2n+1}} f\left[\rho^{-1}u, \rho^{-1}v, \rho^{-2}\lambda^{-1}\left[s-\mu\lambda\big(x\cdot v - y\cdot u\big)\right]\right] \\
&\qquad |x-u|^{\alpha-n}|y-v|^{\alpha-n}|t-s|^{\beta-1}\left[\frac{|x-u||y-v|}{|t-s|}+\frac{|t-s|}{|x-u||y-v|}\right]^{-\vartheta} dudvds\Bigg\}^q dxdydt\Bigg\}^{\frac{1}{q}} \\
&= \rho^{2\alpha+2\beta}\rho^{-\gamma}\rho^{\frac{2n+2}{q}}\lambda^{\beta}\lambda^{\frac{1}{q}}\Bigg\{\iiint_{\mathbb{R}^{2n+1}}\left[\sqrt{|x|^2+|y|^2}\right]^{-\gamma q}\Bigg\{\iiint_{\mathbb{R}^{2n+1}} f\left(u,v,s-\mu\big(x\cdot v-y\cdot u\big)\right) \\
&\qquad |x-u|^{\alpha-n}|y-v|^{\alpha-n}|t-s|^{\beta-1}\left[\frac{|x-u||y-v|}{\lambda|t-s|}+\frac{\lambda|t-s|}{|x-u||y-v|}\right]^{-\vartheta} dudvds\Bigg\}^q dxdydt\Bigg\}^{\frac{1}{q}} \\
&\geq \rho^{2\alpha+2\beta}\rho^{-\gamma}\rho^{\frac{2n+2}{q}}\lambda^{\beta}\lambda^{\frac{1}{q}}\begin{cases}\lambda^{\vartheta}, & 0<\lambda<1, \\ \lambda^{-\vartheta}, & \lambda>1\end{cases} \\
&\quad \Bigg\{\iiint_{\mathbb{R}^{2n+1}}\left[\sqrt{|x|^2+|y|^2}\right]^{-\gamma q}\Bigg\{\iiint_{\mathbb{R}^{2n+1}} f\left(u,v,s-\mu\big(x\cdot v-y\cdot u\big)\right) \\
&\qquad |x-u|^{\alpha-n}|y-v|^{\alpha-n}|t-s|^{\beta-1}\left[\frac{|x-u||y-v|}{|t-s|}+\frac{|t-s|}{|x-u||y-v|}\right]^{-\vartheta} dudvds\Bigg\}^q dxdydt\Bigg\}^{\frac{1}{q}}.
\end{aligned} \tag{4. 41}$$

The $\mathbf{L}^p \longrightarrow \mathbf{L}^q$-norm inequality in (1. 38) implies that the last line of (4. 41) is bounded by

$$\begin{aligned}
&\left\{\iiint_{\mathbb{R}^{2n+1}}\left|f\left(\rho^{-1}x,\rho^{-1}y,\rho^{-2}\lambda^{-1}t\right)\right|^p\left[\sqrt{|x|^2+|y|^2}\right]^{\delta p} dxdydt\right\}^{\frac{1}{p}} = \rho^{\frac{2n+2}{p}}\rho^{\delta}\lambda^{\frac{1}{p}}\left\|f\sigma\right\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})}, \\
&\qquad (x,y,t)\longrightarrow(\rho x,\rho y,\rho^2\lambda t).
\end{aligned} \tag{4. 42}$$

This must be true for every $r>0$ and $0<\lambda<1$ or $\lambda>1$. We necessarily have

$$\frac{\alpha+\beta}{n+1} = \frac{1}{p}-\frac{1}{q}+\frac{\gamma+\delta}{2n+2} \tag{4. 43}$$

and

$$\beta+\vartheta \geq \frac{1}{p}-\frac{1}{q} \qquad \text{or} \qquad \beta-\vartheta \leq \frac{1}{p}-\frac{1}{q}. \tag{4. 44}$$

By adding (4. 43) and (4. 44) together, we find

$$\vartheta \geq \frac{n\beta-\alpha}{n+1}+\frac{\gamma+\delta}{2n+2} \qquad \text{or} \qquad \vartheta \geq \frac{\alpha-n\beta}{n+1}-\frac{\gamma+\delta}{2n+2}.$$

This further implies

$$\vartheta \geq \left|\frac{\alpha-n\beta}{n+1}-\frac{\gamma+\delta}{2n+2}\right|. \tag{4. 45}$$

Because $\mathbf{I}_{\alpha\beta\vartheta}$ is self-adjoint, it is essential to have $\omega^q, \sigma^{-\frac{p}{p-1}}$ locally integrable. Therefore, $\gamma < \frac{2n}{q}$, $\delta < 2n\left(\frac{p-1}{p}\right)$ are necessary.

Denote $\mathbf{R} = \mathbf{Q}_1 \times \mathbf{Q}_2 \times I \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ where $\mathbf{Q}_1, \mathbf{Q}_2$ are cubes parallel to the coordinates. Moreover, I is an interval. $\mathbf{R}' = \mathbf{Q}_1' \times \mathbf{Q}_2' \times I'$ is a translation of $\mathbf{R}$ defined as

$$\mathbf{R}' = \left\{ (x,y,t) \colon \begin{array}{c} x_i = u_i + 2\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}},\ y_i = v_i + 2\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}},\ i = 1,2,\ldots,n \\ t = s + 2\mathbf{vol}\{I\} \end{array} \quad (u,v,s) \in \mathbf{R} \right\}. \tag{4. 46}$$

Consider

$$f(x,y,t) = \sigma^{-\frac{p}{p-1}}(x,y)\chi_{\mathbf{Q}_1\times\mathbf{Q}_2}(x,y)\chi_I(t), \qquad (x,y) \neq (0,0) \tag{4. 47}$$

where χ is an indicator function.

Let $\mathbf{vol}\{I\} = \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}}$. We have

$$\begin{array}{l}
\left\| \omega \mathbf{I}_{\alpha\beta\vartheta} f \right\|_{\mathbf{L}^q(\mathbb{R}^{2n+1})} \geq \\[2ex]
\left\{ \iiint_{\mathbf{R}'} \omega^q(x,y) \left\{ \iiint_{\mathbf{Q}_1\times\mathbf{Q}_2\times\mathbb{R}} \sigma^{-\frac{p}{p-1}}(u,v)\chi_I(s - \mu(x\cdot v - y\cdot u)) \right.\right. \\
\left.\left. |x-u|^{\alpha-n} \left|y-v\right|^{\alpha-n} |t-s|^{\beta-1} \left[\frac{|x-u|\left|y-v\right|}{|t-s|} + \frac{|t-s|}{|x-u|\left|y-v\right|} \right]^{-\vartheta} dudvds \right\}^q dxdydt \right\}^{\frac{1}{q}} \\[3ex]
\geq \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{I\}^{\beta-1} \\
\quad \left\{ \iiint_{\mathbf{Q}_1'\times\mathbf{Q}_2'\times I'} \omega^q(x,y) \left\{ \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \sigma^{-\frac{p}{p-1}}(u,v) \left\{ \int_{I-\mu(x\cdot v - y\cdot u)} ds \right\} dudv \right\}^q dxdydt \right\}^{\frac{1}{q}} \\[3ex]
= \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{I\}^{\beta-1+\frac{1}{q}+1} \\
\quad \left\{ \iint_{\mathbf{Q}_1'\times\mathbf{Q}_2'} \omega^q(x,y)dxdy \right\}^{\frac{1}{q}} \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \sigma^{-\frac{p}{p-1}}(u,v)dudv \\[3ex]
= \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{I\}^{\beta+\frac{1}{q}} \left\{ \iint_{\mathbf{Q}_1'\times\mathbf{Q}_2'} \omega^q(x,y)dxdy \right\}^{\frac{1}{q}} \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \sigma^{-\frac{p}{p-1}}(x,y)dxdy.
\end{array} \tag{4. 48}$$

The norm inequality in (1. 38) implies

$$\begin{array}{l}
\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{I\}^{\beta+\frac{1}{q}} \left\{ \iint_{\mathbf{Q}_1'\times\mathbf{Q}_2'} \omega^q(x,y)dxdy \right\}^{\frac{1}{q}} \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \sigma^{-\frac{p}{p-1}}(x,y)dxdy \\[3ex]
\leq \mathfrak{C}_{\alpha\,\beta\,p\,q}\, \mathbf{vol}\{I\}^{\frac{1}{p}} \left\{ \iint_{\mathbf{Q}_1\times\mathbf{Q}_2} \sigma^{-\frac{p}{p-1}}(x,y)dxdy \right\}^{\frac{1}{p}}.
\end{array} \tag{4. 49}$$

Take into account $\mathbf{vol}\{I\} = \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}}$.

We find

$$\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{I\}^{\beta+\frac{1}{q}-\frac{1}{p}}\left\{\iint_{\mathbf{Q}_1'\times\mathbf{Q}_2'}\omega^q(x,y)dxdy\right\}^{\frac{1}{q}}\left\{\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\sigma^{-\frac{p}{p-1}}(x,y)dxdy\right\}^{\frac{p-1}{p}}$$

$$\begin{aligned}= \ &\mathbf{vol}\{\mathbf{Q}_1\}^{\left[\frac{\alpha+\beta}{n+1}-\left(\frac{1}{p}-\frac{1}{q}\right)\right]\frac{n+1}{n}}\mathbf{vol}\{\mathbf{Q}_2\}^{\left[\frac{\alpha+\beta}{n+1}-\left(\frac{1}{p}-\frac{1}{q}\right)\right]\frac{n+1}{n}}\\ &\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1'\}\mathbf{vol}\{\mathbf{Q}_2'\}}\iint_{\mathbf{Q}_1'\times\mathbf{Q}_2'}\omega^q(x,y)dxdy\right\}^{\frac{1}{q}}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\sigma^{-\frac{p}{p-1}}(x,y)dxdy\right\}^{\frac{p-1}{p}}\\ &< \ \infty\end{aligned} \tag{4. 50}$$

for every $\mathbf{Q}_1\times\mathbf{Q}_2\subset\mathbb{R}^n\times\mathbb{R}^n$.

Note that (4. 50) holds for every $\mathbf{Q}_1\times\mathbf{Q}_2\subset\mathbb{R}^n\times\mathbb{R}^n$. Suppose $\mathbf{Q}_2$ centered on the origin and $\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}}=1$. Let $\mathbf{Q}_1$ shrink to $x\in\mathbf{Q}_1$. Simultaneously, as defined in (4. 46), $\mathbf{Q}_1'$ shrinks to some $x'\in\mathbf{Q}_1'$ and $\mathbf{vol}\{\mathbf{Q}_2'\}^{\frac{1}{n}}=1$. By applying Lebesgue differentiation theorem, we find

$$\lim_{\mathbf{vol}\{\mathbf{Q}_1\}\longrightarrow 0}\mathbf{vol}\{\mathbf{Q}_1\}^{\left[\frac{\alpha+\beta}{n+1}-\left(\frac{1}{p}-\frac{1}{q}\right)\right]\frac{n+1}{n}}\left\{\int_{\mathbf{Q}_2'}\omega^q(x',y)dy\right\}^{\frac{1}{q}}\left\{\int_{\mathbf{Q}_2}\sigma^{-\frac{p}{p-1}}(x,y)dy\right\}^{\frac{p-1}{p}} \ < \ \infty. \tag{4. 51}$$

Clearly, the product of two integral terms in (4. 51) never vanishes. We must have $\frac{\alpha+\beta}{n+1}\geq\frac{1}{p}-\frac{1}{q}$ in order to bound the limit as $\mathbf{vol}\{\mathbf{Q}_1\}\longrightarrow 0$. This together with the homogeneity condition in (4. 43) imply

$$\gamma+\delta \ \geq \ 0. \tag{4. 52}$$

For brevity of computation, denote

$$\zeta \ = \ n\left[\frac{\alpha+\beta}{n+1}\right]+\frac{\gamma+\delta}{2n+2}. \tag{4. 53}$$

We find

$$\begin{aligned}\zeta \ &= \ \frac{n}{p}-\frac{n}{q}+\frac{\gamma+\delta}{2} \qquad \left(\tfrac{\alpha+\beta}{n+1}=\tfrac{1}{p}-\tfrac{1}{q}+\tfrac{\gamma+\delta}{2n+2}\right);\\ 0 \ < \ \zeta \ &= \ \frac{n}{p}-\frac{n}{q}+\frac{\gamma+\delta}{2} \qquad (\gamma+\delta\geq 0, 1<p<q<\infty)\\ &< \ \frac{n}{p}-\frac{n}{q}+\frac{n}{q}+n\left(\frac{p-1}{p}\right) \ = \ n. \qquad \left(\gamma<\tfrac{2n}{q}, \delta<2n\left(\tfrac{p-1}{p}\right)\right)\end{aligned} \tag{4. 54}$$

Moreover, a direct computation shows

$$\begin{aligned}\left[\frac{\alpha+\beta}{n+1}-\left(\frac{1}{p}-\frac{1}{q}\right)\right]\frac{n+1}{n} \ &= \ \frac{\alpha+\beta}{n+1}-\left(\frac{1}{p}-\frac{1}{q}\right)+\frac{1}{n}\frac{\gamma+\delta}{2n+2} \qquad \text{by (4. 43)}\\ &= \ \frac{\zeta}{n}-\left(\frac{1}{p}-\frac{1}{q}\right).\end{aligned} \tag{4. 55}$$

From (4. 50) and (4. 55), we obtain

$$\sup_{\mathbf{Q}_1\times\mathbf{Q}_2\subset\mathbb{R}^n\times\mathbb{R}^n} \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\zeta}{n}-\frac{1}{p}+\frac{1}{q}}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\zeta}{n}-\frac{1}{p}+\frac{1}{q}}$$
$$\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1'\}\mathbf{vol}\{\mathbf{Q}_2'\}}\iint_{\mathbf{Q}_1'\times\mathbf{Q}_2'}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q}dxdy\right\}^{\frac{1}{q}} \tag{4. 56}$$
$$\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\delta\frac{p}{p-1}}dxdy\right\}^{\frac{p-1}{p}} < \infty.$$

Observe that (4. 56) is an analogue of bi-parameter Muckenhoupt characteristic discussed earlier in **Section 3.2** whereas $\mathbf{Q}_1\times\mathbf{Q}_2$ inside the first bracket is replaced by its translation $\mathbf{Q}_1'\times\mathbf{Q}_2'$ defined in (4. 46). By carrying out the regarding estimates for **Case 1**: $\gamma\geq 0$, $\delta\leq 0$ and **Case 2**: $\gamma\leq 0$, $\delta\geq 0$ with $\alpha=\beta=\zeta$ and $m=n$, we obtain

$$\zeta-\frac{n}{p}<\delta, \quad \text{for} \quad \gamma\geq 0,\ \ \delta\leq 0 \qquad \text{and} \qquad \zeta-n\left(\frac{q-1}{q}\right)<\gamma \quad \text{for} \quad \gamma\leq 0,\ \ \delta\geq 0. \tag{4. 57}$$

In fact, certain estimates can be simplified as $\alpha=\beta=\zeta$ and $m=n$. For the sake of self-containedness, we prove (4. 57) below.

Case 1. Consider $\gamma\geq 0$, $\delta\leq 0$. Suppose $\gamma+\delta=0$.

Let ζ defined in (4. 53). From (4. 43) and (4. 55), we find

$$\frac{\zeta}{n} = \frac{1}{p}-\frac{1}{q}. \tag{4. 58}$$

Recall $\mathbf{R}'=\mathbf{Q}_1'\times\mathbf{Q}_2'\times I$ defined in (4. 46). We assert $\mathbf{Q}_1'\times\mathbf{Q}_2'$ centered on the origin of $\mathbb{R}^n\times\mathbb{R}^n$. Let $\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}}=\mathbf{vol}\{\mathbf{Q}_2'\}^{\frac{1}{n}}=1$ and $\mathbf{Q}_1$ shrink to some $u\in\mathbf{Q}_1$ whereas $\mathbf{Q}_1'$ shrink to 0.

From (4. 56)-(4. 58), by applying Lebesgue differentiation theorem, we have

$$\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_2'\}}\int_{\mathbf{Q}_2'}|y|^{-\gamma q}dy\right\}^{\frac{1}{q}}\left\{\frac{1}{\mathbf{vol}\{\mathbf{Q}_2\}}\int_{\mathbf{Q}_2}\left[\frac{1}{\sqrt{|u|^2+|y|^2}}\right]^{-\delta\frac{p}{p-1}}dy\right\}^{\frac{p-1}{p}} < \infty \tag{4. 59}$$

for every $\mathbf{Q}_2\subset\mathbb{R}^n$. This suggests

$$\gamma < \frac{n}{q} \qquad\Longrightarrow\qquad \zeta-\frac{n}{p} = -\frac{n}{q}<-\gamma = \delta \tag{4. 60}$$

as an necessity.

Suppose $\gamma+\delta>0$. From (4. 43) and (4. 55), we find

$$\frac{\zeta}{n} > \frac{1}{p}-\frac{1}{q}. \tag{4. 61}$$

For every $\mathbf{Q}_1 \times \mathbf{Q}_2 \subset \mathbb{R}^n \times \mathbb{R}^n$, we define

$$
\begin{aligned}
\mathbf{A}_{p\,q}^{\zeta\,\gamma\,\delta}(\mathbf{Q}_1 \times \mathbf{Q}_2) \; = \; & \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\zeta}{n}-\frac{1}{p}+\frac{1}{q}} \mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\zeta}{n}-\frac{1}{p}+\frac{1}{q}} \\
& \left\{ \frac{1}{\mathbf{vol}\{\mathbf{Q}_1'\}\mathbf{vol}\{\mathbf{Q}_2'\}} \iint_{\mathbf{Q}_1' \times \mathbf{Q}_2'} \left[\frac{1}{\sqrt{|u|^2+|v|^2}} \right]^{\gamma q} dudv \right\}^{\frac{1}{q}} \\
& \left\{ \frac{1}{\mathbf{vol}\{\mathbf{Q}_1\}\mathbf{vol}\{\mathbf{Q}_2\}} \iint_{\mathbf{Q}_1 \times \mathbf{Q}_2} \left[\frac{1}{\sqrt{|u|^2+|v|^2}} \right]^{\delta \frac{p}{p-1}} dudv \right\}^{\frac{p-1}{p}} .
\end{aligned}
\tag{4. 62}
$$

Denote

$$
\mathbf{Q'}_1^k \; = \; \mathbf{Q}_1' \cap \left\{ 2^{-k-1} \le |u| < 2^{-k} \right\}, \qquad \mathbf{Q'}_2^k \; = \; \mathbf{Q}_2' \cap \left\{ 2^{-k-1} \le |v| < 2^{-k} \right\}, \qquad k \ge 0.
$$

Let $\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}} = \mathbf{vol}\{\mathbf{Q}_2'\}^{\frac{1}{n}} = 1$ and $\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}} = \mathbf{vol}\{\mathbf{Q}_1'\}^{\frac{1}{n}} = \lambda$, $0 < \lambda < 1$. Then we have

$$
\begin{aligned}
\left[\mathbf{A}_{p\,q}^{\zeta\,\gamma\,\delta}(\mathbf{Q}_1 \times \mathbf{Q}_2) \right]^q \; = \; & \lambda^{q\left[\zeta - \left(\frac{n}{p}-\frac{n}{q}\right)\right]} \left\{ \frac{1}{\lambda^n} \iint_{\mathbf{Q}_1' \times \mathbf{Q}_2'} \left[\frac{1}{\sqrt{|x|^2+|y|^2}} \right]^{\gamma q} dxdy \right\} \\
& \left\{ \frac{1}{\lambda^n} \iint_{\mathbf{Q}_1 \times \mathbf{Q}_2} \left[\frac{1}{\sqrt{|x|^2+|y|^2}} \right]^{\delta \frac{p}{p-1}} dxdy \right\}^{\left[\frac{p-1}{p}\right] q} \\
= \; & \lambda^{q\left[\zeta - \left(\frac{n}{p}-\frac{n}{q}\right)\right]} \sum_{k \ge 0} \left\{ \frac{1}{\lambda^n} \iint_{\mathbf{Q}_1' \times \mathbf{Q'}_2^k} \left[\frac{1}{\sqrt{|x|^2+|y|^2}} \right]^{\gamma q} dudv \right\} \\
& \left\{ \frac{1}{\lambda^n} \iint_{\mathbf{Q}_1 \times \mathbf{Q}_2} \left[\frac{1}{\sqrt{|x|^2+|y|^2}} \right]^{\delta \frac{p}{p-1}} dudv \right\}^{\left[\frac{p-1}{p}\right] q} \\
\doteq \; & \sum_{k \ge 0} \mathbf{A}_k(\lambda).
\end{aligned}
\tag{4. 63}
$$

Lebesgue's differentiation theorem implies

$$
\lim_{\lambda \longrightarrow 0} \frac{1}{\lambda^n} \iint_{\mathbf{Q}_1' \times \mathbf{Q'}_2^k} \left[\frac{1}{\sqrt{|x|^2+|y|^2}} \right]^{\gamma q} dxdy \; = \; \int_{\mathbf{Q'}_2^k} |y|^{-\gamma q} dy. \tag{4. 64}
$$

Because $\delta \le 0$ and $\zeta > \frac{n}{p} - \frac{n}{q}$, we find $\mathbf{A}_k(0) = 0$, $k \ge 0$. This remains to be true if $\zeta - \frac{n}{p} + \frac{n}{q}$ in (4. 63) is replaced by any smaller positive number. Therefore, $\mathbf{A}_k(\lambda)$ is Hölder continuous *w.r.t* $\lambda \ge 0$ whose exponent is strictly positive for every $k \ge 0$. Recall (4. 56). We have $\sum_{k \ge 0} \mathbf{A}_k(\lambda) \le \mathfrak{C}_{\alpha\,\gamma\,\delta\,q}$ for $\lambda > 0$. Consequently, $\sum_{k \ge 0} \mathbf{A}_k(\lambda)$ is continuous at $\lambda = 0$ and

$$
\lim_{\lambda \longrightarrow 0} \sum_{k \ge 0} \mathbf{A}_k(\lambda) \; = \; 0. \tag{4. 65}
$$

A direct computation shows

$$\begin{aligned}\left[\mathbf{A}_{p\,q}^{\zeta\,\gamma\,\delta}(\mathbf{Q}_1\times\mathbf{Q}_2)\right]^q ~=~& \lambda^{q\left[\zeta-\frac{n}{p}+\frac{n}{q}\right]}\left\{\frac{1}{\lambda^n}\iint_{\mathbf{Q}_1'\times\mathbf{Q}_2'}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q}dxdy\right\}\\ &\left\{\frac{1}{\lambda^n}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\delta\frac{p}{p-1}}dxdy\right\}^{\left[\frac{p-1}{p}\right]q}\end{aligned} \tag{4. 66}$$

$$\geq~\mathfrak{C}\lambda^{q\left[\zeta-\frac{n}{p}+\frac{n}{q}\right]}\int_{\mathbf{Q}_2'}\left[\frac{1}{\sqrt{\lambda^2+|y|^2}}\right]^{\gamma q}dy \qquad (\,\delta\leq 0,~\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}=\mathbf{vol}\{\mathbf{Q}_1'\}^{\frac{1}{n}}=\lambda\,)$$

$$\geq~\mathfrak{C}\lambda^{q\left[\zeta-\frac{n}{p}+\frac{n}{q}\right]}\int_{0<|v|\leq\lambda}\left(\frac{1}{\lambda}\right)^{\gamma q}dv ~=~ \mathfrak{C}_{\gamma\,q}\,\lambda^{n-\gamma q+q\left[\zeta-\frac{n}{p}+\frac{n}{q}\right]}.$$

From (4. 65)-(4. 66), by using $\zeta=\frac{n}{p}-\frac{n}{q}+\frac{\gamma+\delta}{2}$ as shown in (4. 54), we find

$$\frac{n}{q}-\gamma+\zeta-\left(\frac{n}{p}-\frac{n}{q}\right)~>~0 \qquad \Longrightarrow$$

$$\begin{aligned}\zeta ~<~ \frac{n}{q}-\gamma+2\zeta-\left(\frac{n}{p}-\frac{n}{q}\right) ~&=~ \frac{n}{q}-\gamma+\left(\frac{n}{p}-\frac{n}{q}\right)+\gamma+\delta\\ &=~\frac{n}{p}+\delta.\end{aligned} \tag{4. 67}$$

Recall $\zeta=n\left[\frac{\alpha+\beta}{n+1}\right]+\frac{\gamma+\delta}{2n+2}$. By putting together (4. 60) and (4. 67), we obtain

$$n\left[\frac{\alpha+\beta}{n+1}\right]+\frac{\gamma+\delta}{2n+2}-\frac{n}{p}~<~\delta \qquad \text{for} \qquad \gamma\geq 0,~\delta\leq 0. \tag{4. 68}$$

Case 2. Consider $\gamma\leq 0$, $\delta\geq 0$ Suppose $\gamma+\delta=0$. From (4. 43) and (4. 55), we find $\frac{\zeta}{n}=\frac{1}{p}-\frac{1}{q}=\frac{q-1}{q}-\frac{p-1}{p}$ as shown in (4. 58). The estimate in (4. 59) suggests

$$\delta~<~n\left(\frac{p-1}{p}\right) \qquad \Longrightarrow \qquad \zeta-n\left(\frac{q-1}{q}\right)~=~-n\left(\frac{p-1}{p}\right)<-\delta~=~\gamma \tag{4. 69}$$

as an necessity.

Suppose $\gamma+\delta>0$. From (4. 43) and (4. 55), we find $\frac{\zeta}{n}>\frac{1}{p}-\frac{1}{q}$ as (4. 61).
For every $\mathbf{Q}_1\times\mathbf{Q}_2\subset\mathbb{R}^n\times\mathbb{R}^n$, $\mathbf{A}_{p\,q}^{\zeta\,\gamma\,\delta}(\mathbf{Q}_1\times\mathbf{Q}_2)$ is defined in (4. 62). Denote

$$\mathbf{Q}_1^k~=~\mathbf{Q}_1\cap\left\{2^{-k-1}\leq|u|<2^{-k}\right\}, \qquad \mathbf{Q}_2^k~=~\mathbf{Q}_2\cap\left\{2^{-k-1}\leq|v|<2^{-k}\right\}, \qquad k\geq 0.$$

Let $\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}}=\mathbf{vol}\{\mathbf{Q}_2'\}^{\frac{1}{n}}=1$ and $\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}=\mathbf{vol}\{\mathbf{Q}_1'\}^{\frac{1}{n}}=\lambda$ for $0<\lambda<1$.

From (4. 62), we have

$$\Big[\mathbf{A}_{p\,q}^{\zeta\,\gamma\,\delta}(\mathbf{Q}_1\times\mathbf{Q}_2)\Big]^{\frac{p}{p-1}} = \lambda^{\frac{p}{p-1}\left[\zeta-\frac{n}{p}+\frac{n}{q}\right]}\left\{\frac{1}{\lambda^n}\iint_{\mathbf{Q}_1'\times\mathbf{Q}_2'}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q}dxdy\right\}^{\frac{1}{q}\frac{p}{p-1}} \left\{\frac{1}{\lambda^n}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\delta\frac{p}{p-1}}dxdy\right\}$$

$$= \lambda^{\frac{p}{p-1}\left[\zeta-\frac{n}{p}+\frac{n}{q}\right]}\left\{\frac{1}{\lambda^n}\iint_{\mathbf{Q}_1'\times\mathbf{Q}_2'}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q}dxdy\right\}^{\frac{1}{q}\frac{p}{p-1}} \sum_{k\geq0}\left\{\frac{1}{\lambda^n}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2^k}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\delta\frac{p}{p-1}}dxdy\right\} \tag{4. 70}$$

$$\doteq \sum_{k\geq0}\mathbf{B}_k(\lambda).$$

Lebesgue's differentiation theorem implies

$$\lim_{\lambda\longrightarrow0}\frac{1}{\lambda^n}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2^k}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\delta\frac{p}{p-1}}dxdy = \int_{\mathbf{Q}_2^k}|y|^{-\delta\frac{p}{p-1}}dy. \tag{4. 71}$$

Because $\gamma\leq0$ and $\zeta>\frac{n}{p}-\frac{n}{q}$, we find $\mathbf{B}_k(0)=0, k\geq0$. This remains to be true if $\zeta-\frac{n}{p}+\frac{n}{q}$ in (4. 70) is replaced by a smaller positive number. Therefore, $\mathbf{B}_k(\lambda)$ is Hölder continuous *w.r.t* λ whose exponent remains strictly positive for every $k\geq0$. Recall (4. 56). We have $\sum_{k\geq0}\mathbf{B}_k(\lambda)\leq\mathfrak{C}_{\alpha\,\gamma\,\delta\,q}$ for every $\lambda>0$. Consequently, $\sum_{k\geq0}\mathbf{B}_k(\lambda)$ is continuous at $\lambda=0$ and

$$\lim_{\lambda\longrightarrow0}\sum_{k\geq0}\mathbf{B}_k(\lambda) = 0. \tag{4. 72}$$

A direct computation shows

$$\Big[\mathbf{A}_{p\,q}^{\zeta\,\gamma\,\delta}(\mathbf{Q}_1\times\mathbf{Q}_2)\Big]^{\frac{p}{p-1}} = \lambda^{\frac{p}{p-1}\left[\zeta-\frac{n}{p}+\frac{n}{q}\right]}\left\{\frac{1}{\lambda^n}\iint_{\mathbf{Q}_1'\times\mathbf{Q}_2'}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\gamma q}dxdy\right\}^{\frac{1}{q}\frac{p}{p-1}} \left\{\frac{1}{\lambda^n}\iint_{\mathbf{Q}_1\times\mathbf{Q}_2}\left[\frac{1}{\sqrt{|x|^2+|y|^2}}\right]^{\delta\frac{p}{p-1}}dxdy\right\}$$

$$\geq \mathfrak{C}\lambda^{\frac{p}{p-1}\left[\zeta-\frac{n}{p}+\frac{n}{q}\right]}\int_{\mathbf{Q}_2}\left[\frac{1}{\sqrt{\lambda^2+|y|^2}}\right]^{\delta\frac{p}{p-1}}dy \qquad (\,\gamma\leq0,\ \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}=\mathbf{vol}\{\mathbf{Q}_1'\}^{\frac{1}{n}}=\lambda\,) \tag{4. 73}$$

$$\geq \mathfrak{C}\lambda^{\frac{p}{p-1}\left[\zeta-\frac{n}{p}+\frac{n}{q}\right]}\int_{0<|v|\leq\lambda}\left(\frac{1}{\lambda}\right)^{\delta\frac{p}{p-1}}dv = \mathfrak{C}_{\delta\,p}\,\lambda^{n-\delta\left(\frac{p}{p-1}\right)+\frac{p}{p-1}\left[\zeta-\frac{n}{p}+\frac{n}{q}\right]}.$$

From (4. 72)-(4. 73), by using $\zeta = \frac{n}{p} - \frac{n}{q} + \frac{\gamma+\delta}{2} = n\Big[\frac{q-1}{q} - \frac{p-1}{p}\Big] + \frac{\gamma+\delta}{2}$ in (4. 54), we find

$$n\left(\frac{p-1}{p}\right) - \delta + \zeta - \left(\frac{n}{p} - \frac{n}{q}\right) > 0 \qquad \Longrightarrow$$

$$\begin{aligned} \zeta &< n\left(\frac{p-1}{p}\right) - \delta + 2\zeta - n\left[\frac{q-1}{q} - \frac{p-1}{p}\right] = n\left(\frac{p-1}{p}\right) - \delta + n\left[\frac{q-1}{q} - \frac{p-1}{p}\right] + \gamma + \delta \\ &= n\left(\frac{q-1}{q}\right) + \gamma. \end{aligned} \tag{4. 74}$$

Recall $\zeta = n\Big[\frac{\alpha+\beta}{n+1}\Big] + \frac{\gamma+\delta}{2n+2}$. By putting together (4. 69) and (3. 14), we obtain

$$n\left[\frac{\alpha+\beta}{n+1}\right] + \frac{\gamma+\delta}{2n+2} - n\left(\frac{q-1}{q}\right) < \gamma \qquad \text{for} \qquad \gamma \leq 0,\ \delta \geq 0. \tag{4. 75}$$

4.3.2 Proof of sufficient condition

Recall $\mathbf{V}^{\alpha\beta\vartheta}(u,v,t)$ defined in (1. 34) for $u \neq 0, v \neq 0, t \neq 0$ and $\vartheta \geq \left|\frac{\alpha-n\beta}{n+1} - \frac{\gamma+\delta}{2n+2}\right|$. Suppose $2\alpha - 2n\beta - \gamma - \delta \geq 0$. We have

$$\begin{aligned} \mathbf{V}^{\alpha\beta\vartheta}(x,y,t) &= |x|^{\alpha-n}|y|^{\alpha-n}|t|^{\beta-1}\left[\frac{|x||y|}{|t|} + \frac{|t|}{|x||y|}\right]^{-\vartheta} \\ &\leq |x|^{\alpha-n}|y|^{\alpha-n}|t|^{\beta-1}\left[\frac{|x||y|}{|t|} + \frac{|t|}{|x||y|}\right]^{-\left[\frac{\alpha-n\beta}{n+1} - \frac{\gamma+\delta}{2n+2}\right]} \\ &\leq |x|^{\alpha-n}|y|^{\alpha-n}|t|^{\beta-1}\left[\frac{|x||y|}{|t|}\right]^{-\left[\frac{\alpha-n\beta}{n+1} - \frac{\gamma+\delta}{2n+2}\right]} \\ &= |x|^{n\left[\frac{\alpha+\beta}{n+1}\right] + \frac{\gamma+\delta}{2n+2} - n}|y|^{n\left[\frac{\alpha+\beta}{n+1}\right] + \frac{\gamma+\delta}{2n+2} - n}|t|^{\frac{\alpha+\beta}{n+1} - \frac{\gamma+\delta}{2n+2} - 1}, \qquad x \neq 0, y \neq 0, t \neq 0. \end{aligned} \tag{4. 76}$$

Suppose $2\alpha - 2n\beta - \gamma - \delta \leq 0$. We find

$$\begin{aligned} \mathbf{V}^{\alpha\beta\vartheta}(x,y,t) &\leq |x|^{\alpha-n}|y|^{\alpha-n}|t|^{\beta-1}\left[\frac{|x||y|}{|t|} + \frac{|t|}{|x||y|}\right]^{\frac{\alpha-n\beta}{n+1} - \frac{\gamma+\delta}{2n+2}} \\ &\leq |x|^{\alpha-n}|y|^{\alpha-n}|t|^{\beta-1}\left[\frac{|t|}{|x||y|}\right]^{\frac{\alpha-n\beta}{n+1} - \frac{\gamma+\delta}{2n+2}} \\ &= |x|^{n\left[\frac{\alpha+\beta}{n+1}\right] + \frac{\gamma+\delta}{2n+2} - n}|y|^{n\left[\frac{\alpha+\beta}{n+1}\right] + \frac{\gamma+\delta}{2n+2} - n}|t|^{\frac{\alpha+\beta}{n+1} - \frac{\gamma+\delta}{2n+2} - 1}, \qquad x \neq 0, y \neq 0, t \neq 0. \end{aligned} \tag{4. 77}$$

Let $\zeta = n\left[\frac{\alpha+\beta}{n+1}\right] + \frac{\gamma+\delta}{2n+2}$ where $0 < \zeta < n$ as (4. 53). From (4. 24), we have

$$\begin{aligned}
\mathbf{I}_{\alpha\beta\vartheta} f(x,y,t) &= \iiint_{\mathbb{R}^{2n+1}} f\Big(u,v,s-\mu\big(x\cdot v - y\cdot u\big)\Big)\mathbf{V}^{\alpha\beta\vartheta}(x-u,y-v,t-s)dudvds \\
&\leq \iiint_{\mathbb{R}^{2n+1}} f\Big(u,v,s-\mu\big(x\cdot v - y\cdot u\big)\Big) \\
&\qquad |x-u|^{\alpha-n}|y-v|^{\alpha-n}|t-s|^{\beta-1}\left[\frac{|x-u||y-v|}{|t-s|} + \frac{|t-s|}{|x-u||y-v|}\right]^{-\left|\frac{\alpha-n\beta}{n+1}-\frac{\gamma+\delta}{2n+2}\right|} dudvds \\
&\leq \iiint_{\mathbb{R}^{2n+1}} f\Big(u,v,s-\mu\big(x\cdot v - y\cdot u\big)\Big) \\
&\qquad |x-u|^{\zeta-n}|y-v|^{\zeta-n}|t-s|^{\frac{\alpha+\beta}{n+1}-\frac{\gamma+\delta}{2n+2}-1}dudvds \qquad \text{by (4. 76)-(4. 77)} \\
&\doteq \iint_{\mathbb{R}^{2n}} |x-u|^{\zeta-n}|y-v|^{\zeta-n}\mathbf{F}_{\alpha\beta\gamma\delta}(u,v,x,y,t)dudv
\end{aligned} \tag{4. 78}$$

where

$$\mathbf{F}_{\alpha\beta\gamma\delta}(u,v,x,y,t) = \int_{\mathbb{R}} f\left(u,v,s-\mu\left(x\cdot v-y\cdot u\right)\right)|t-s|^{\left[\frac{\alpha+\beta}{n+1}-\frac{\gamma+\delta}{2n+2}\right]-1}ds. \tag{4. 79}$$

Recall **Hardy-Littlewood-Sobolev theorem** stated in **Chapter 1**. By applying (1. 2) with $\frac{\alpha+\beta}{n+1} - \frac{\gamma+\delta}{2n+2} = \frac{1}{p} - \frac{1}{q}$, we find

$$\begin{aligned}
\left\{\int_{\mathbb{R}} \mathbf{F}^q_{\alpha\beta\gamma\delta}(u,v,x,y,t)dt\right\}^{\frac{1}{q}} &\leq \mathfrak{C}_{p\,q}\left\{\int_{\mathbb{R}}\Big[f\left(u,v,t+\mu\left(x\cdot v-y\cdot u\right)\right)\Big]^p dt\right\}^{\frac{1}{p}} \\
&= \mathfrak{C}_{p\,q}\left\|f(u,v,\cdot)\right\|_{\mathbf{L}^p(\mathbb{R})}, \qquad (u,v)\in\mathbb{R}^n\times\mathbb{R}^n.
\end{aligned} \tag{4. 80}$$

From (4. 78)-(4. 80), we find

$$\begin{aligned}
&\left\{\iiint_{\mathbb{R}^{2n+1}}\left(\sqrt{|x|^2+|y|^2}\right)^{-\gamma q}\Big(\mathbf{I}_{\alpha\beta\vartheta}f\Big)^q(x,y,t)dxdydt\right\}^{\frac{1}{q}} \\
&\leq \left\{\iiint_{\mathbb{R}^{2n+1}}\left(\sqrt{|x|^2+|y|^2}\right)^{-\gamma q}\left\{\iint_{\mathbb{R}^{2n}}|x-u|^{\zeta-n}|y-v|^{\zeta-n}\mathbf{F}_{\alpha\beta\gamma\delta}(u,v,x,y,t)dudv\right\}^q dxdydt\right\}^{\frac{1}{q}} \\
&\leq \left\{\iint_{\mathbb{R}^{2n}}\left(\sqrt{|x|^2+|y|^2}\right)^{-\gamma q}\left\{\iint_{\mathbb{R}^{2n}}|x-u|^{\zeta-n}|y-v|^{\zeta-n}\left\{\int_{\mathbb{R}}\mathbf{F}^q_{\alpha\beta\gamma\delta}(u,v,x,y,t)dt\right\}^{\frac{1}{q}}dudv\right\}^q dxdy\right\}^{\frac{1}{q}} \\
&\qquad\qquad \text{by Minkowski integral inequality} \\
&\leq \mathfrak{C}_{p\,q}\left\{\iint_{\mathbb{R}^{2n}}\left(\sqrt{|x|^2+|y|^2}\right)^{-\gamma q}\left\{\iint_{\mathbb{R}^{2n}}|x-u|^{\zeta-n}|y-v|^{\zeta-n}\left\|f(u,v,\cdot)\right\|_{\mathbf{L}^p(\mathbb{R})}dudv\right\}^q dxdy\right\}^{\frac{1}{q}}.
\end{aligned} \tag{4. 81}$$

Define

$$\mathbf{II}_\zeta g(x,y) = \iint_{\mathbb{R}^{2n}} g(u,v)|x-u|^{\zeta-n}|y-v|^{\zeta-n} du dv, \qquad 0<\zeta<n. \tag{4. 82}$$

In summary of the previous subsection, we have

$$\gamma < \frac{2n}{q}, \qquad \delta < 2n\left(\frac{p-1}{p}\right), \qquad \gamma+\delta \geq 0;$$

$$\frac{\zeta}{n} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{2n}; \tag{4. 83}$$

$$\zeta - \frac{n}{p} < \delta \quad \text{for} \quad \gamma \geq 0,\ \delta \leq 0; \qquad \zeta - n\left(\frac{q-1}{q}\right) < \gamma \quad \text{for} \quad \gamma \leq 0,\ \delta \geq 0.$$

Recall **Theorem Two** and **Remark 1.2.3**. Take into account $\alpha = \beta = \zeta$ and $n = m$. We find that (4. 83) implies

$$\left\|\omega \mathbf{II}_\zeta g\right\|_{\mathbf{L}^q(\mathbb{R}^{2n})} \leq \mathfrak{C}_{p\ q\ \gamma\ \delta} \left\|g\sigma\right\|_{\mathbf{L}^p(\mathbb{R}^{2n})}, \qquad 1<p\leq q<\infty. \tag{4. 84}$$

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