# Strong maximal function revisit on Heisenberg group

**Abstract**

We prove the $\mathbf{L}^p$-boundedness of the strong maximal operator defined on a Heisenberg group *w.r.t* an absolutely continuous measure satisfying the product $\mathbf{A}_\infty$-property.

## 1 Introduction

The study of certain operators that commute with a multi-parameter family of dilations, dates back to the time of Jessen, Marcinkiewicz and Zygmund. A number of pioneering results have been accomplished, for example by Córdoba and Fefferman [1], Fefferman and Stein [7], Fefferman [5], Müller, Ricci and Stein [10], Journé [11] and Pipher [12].

In this paper, we consider the strong maximal function operator defined on a Heisenberg group with a multiplication law:

$$(u, v, t) \odot (\xi, \eta, \tau) = \Big[u + \xi, v + \eta, t + \tau + \mu\Big(u \cdot \eta - v \cdot \xi\Big)\Big], \qquad \mu \in \mathbb{R} \tag{1. 1}$$

for every $(u, v, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and $(\xi, \eta, \tau)^{-1} = (-\xi, -\eta, -\tau) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$.

Denote $\mathbf{R}$ to be a rectangle in $\mathbb{R}^{2n+1}$ parallel to the coordinates. Moreover,

$$\mathbf{R} = \bigotimes_{i=1}^{m} \mathbf{Q}_i \times I \subset \bigotimes_{i=1}^{m} \mathbb{R}^{\mathbf{N}_i} \times \mathbb{R}, \qquad \mathbf{N}_1 + \mathbf{N}_2 + \cdots + \mathbf{N}_m = 2n \tag{1. 2}$$

where $\mathbf{Q}_i \subset \mathbb{R}^{\mathbf{N}_i}, i = 1, 2, \ldots, m$ are cubes and $I \subset \mathbb{R}$ is an interval.

A strong maximal operator $\mathbf{M}$ is initially defined on a Heisenberg group as

$$\mathbf{M}f(u, v, t) = \sup_{\mathbf{R} \ni (0,0,0)} \mathbf{vol}\{\mathbf{R}\}^{-1} \iiint_{\mathbf{R}} \Big|f[(u, v, t) \odot (\xi, \eta, \tau)^{-1}]\Big| d\xi d\eta d\tau. \tag{1. 3}$$

Let $\xi \longrightarrow u - \xi$, $\eta \longrightarrow v - \eta$ and $\tau \longrightarrow t - \tau$, $\mathbf{M}$ can be equivalently defined as

$$\mathbf{M}f(u, v, t) = \sup_{\mathbf{R} \ni (u,v,t)} \mathbf{vol}\{\mathbf{R}\}^{-1} \iiint_{\mathbf{R}} \Big|f\left(\xi, \eta, \tau + \mu\left(u \cdot \eta - v \cdot \xi\right)\right)\Big| d\xi d\eta d\tau. \tag{1. 4}$$

$\diamond$ Throughout, $\mathfrak{B} > 0$ is regarded as a generic constant depending on its sub-indices.

**Theorem A: Christ, 1992** *Let* $\mathbf{M}$ *be defined in (1. 3). We have*

$$\Big\|\mathbf{M}f\Big\|_{\mathbf{L}^q(\mathbb{R}^{2n+1})} \leq \mathfrak{B}_p \Big\|f\Big\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})}, \qquad 1 < p < \infty. \tag{1. 5}$$

The $\mathbf{L}^p$-boundedness of $\mathbf{M}$ defined on more general Nilpotent Lie groups can be found in the paper of Michael Christ [4], in which the elegant argument is carried out using a number of 'ingredients' developed previously by Ricci and Stein [9] and Christ [2]-[3].

Our aim is to give a generalization of **Theorem A** by defining **M** *w.r.t* some appropriate absolutely continuous measure: $\omega(u,v)dudvdt$. Given any subset $E \subset \mathbb{R}^{2n+1}$, we write

$$\mathbf{vol}_\omega\{E\} \;=\; \iiint_E \omega(u,v)dudvdt.$$

Define

$$\mathbf{M}_\omega f(u,v,t) \;=\; \sup_{\mathbf{R}\ni(u,v,t)} \mathbf{vol}_\omega\{\mathbf{R}\}^{-1} \iiint_{\mathbf{R}} \Big|f\left(\xi,\eta,\tau+\mu\left(u\cdot\xi-v\cdot\eta\right)\right)\Big|\,\omega(\xi,\eta)d\xi d\eta d\tau. \tag{1. 6}$$

Let $(x_i, x_i^\dagger) \in \mathbb{R}^{\mathbf{N}_i} \times \mathbb{R}^{2n-\mathbf{N}_i}$ for every $i = 1, 2, \ldots, m$ where $\mathbf{N}_1 + \mathbf{N}_2 + \cdots + \mathbf{N}_m = 2n$. We say

$$\omega \;\in\; \bigotimes_{i=1}^m \mathbf{A}_\infty\left(\mathbb{R}^{2n}\right), \tag{1. 7}$$

if $\omega(\cdot, x_i^\dagger)$ satisfies the $\mathbf{A}_\infty$-property in $\mathbb{R}^{\mathbf{N}_i}$ for every $x_i^\dagger \in \mathbb{R}^{2n-\mathbf{N}_i}$.

◇ For brevity, we abbreviate $\left\|f\right\|^p_{\mathbf{L}^p(\mathbb{R}^{2n+1},\omega)} = \iiint_{\mathbb{R}^{2n+1}} |f(u,v,t)|^p \omega(u,v)dudvdt$, $1 < p < \infty$.

**Theorem A*** *Let* $\mathbf{M}_\omega$ *be defined in (1. 6). Suppose* $\omega \in \bigotimes_{i=1}^m \mathbf{A}_\infty\left(\mathbb{R}^{2n}\right)$*. We have*

$$\left\|\mathbf{M}_\omega f\right\|_{\mathbf{L}^p\left(\mathbb{R}^{2n+1},\omega\right)} \;\le\; \mathfrak{B}_{p\ \omega}\ \left\|f\right\|_{\mathbf{L}^p\left(\mathbb{R}^{2n+1},\omega\right)}, \qquad 1<p<\infty. \tag{1. 8}$$

The proof of **Theorem A*** is an application of a multi-parameter covering lemma due to Córdoba and Fefferman [1]: Unlike the Vitali-type covering lemma for cubes, there is no mutually disjointness between rectangles. Instead, this is replaced by the $\mathbf{L}^p$-norm of a summation of indicator functions supported on a sequence of selected rectangles. We observe that Córdoba-Fefferman covering lemma is particularly useful to handle $\mathbf{M}_\omega$ defined on a Heisenberg group.

We prove **Theorem A*** in Section 2. For the sake of self-containedness, we give a proof of the covering lemma within the required setting in Section 3.

## 2 Proof of Theorem A*

Consider $(u,v,t) = (x_i, x_i^\dagger) \in \mathbb{R}^{\mathbf{N}_i} \times \mathbb{R}^{2n+1-\mathbf{N}_i}$ for every $i = 1, 2, \ldots, k$. We define

$$\mathbf{w} \in \bigotimes_{i=1}^k \mathbf{A}_\infty\left(\mathbb{R}^{2n+1}\right), \tag{2. 1}$$

where $\mathbf{w}(\cdot, x_i^\dagger)$ satisfies the $\mathbf{A}_\infty$-property in $\mathbb{R}^{\mathbf{N}_i}$ for every $x_i^\dagger \in \mathbb{R}^{2n+1-\mathbf{N}_i}$, $i = 1, 2, \cdots, k$.

For every subset $E \subset \mathbb{R}^{2n+1}$, we write

$$\mathbf{vol}_{\mathbf{w}}(E) \;=\; \iiint_E \mathbf{w}(u,v,t)dudvdt. \tag{2. 2}$$

Especially, if $\mathbf{w} = 1$, then the measure becomes the Lebesgue measure, and

$$\mathbf{vol}(E) = \mathbf{vol_1}(E) = \iiint_E \mathbf{1} du dv dt. \tag{2. 3}$$

Abbreviate

$$\left\|f\right\|^p_{\mathbf{L}^p(\mathbb{R}^{2n+1},\mathbf{w})} = \iiint_{\mathbb{R}^{2n+1}} \left|f(u,v,t)\right|^p \mathbf{w}(u,v,t) du dv dt, \qquad 1 < p < \infty.$$

**Córdoba-Fefferman covering lemma** *Let $\left\{\mathbf{R}_j\right\}_{j=1}^{\infty}$ be a collection of rectangles in $\mathbb{R}^{2n+1}$ parallel to the coordinates. Suppose $\mathbf{w} \in \bigotimes_{i=1}^k \mathbf{A}_\infty\left(\mathbb{R}^{2n+1}\right)$. There is a subsequence $\left\{\widehat{\mathbf{R}}_k\right\}_{k=1}^{\infty}$ such that*

$$\mathbf{vol_w}\left\{\bigcup_j \mathbf{R}_j\right\} \lesssim \mathbf{vol_w}\left\{\bigcup_k \widehat{\mathbf{R}}_k\right\} \tag{2. 4}$$

*and*

$$\left\|\sum_k \chi_{\widehat{\mathbf{R}}_k}\right\|^p_{\mathbf{L}^p(\mathbb{R}^{2n+1},\mathbf{w})} \lesssim \mathbf{vol_w}\left\{\bigcup_k \widehat{\mathbf{R}}_k\right\}, \qquad 1 < p < \infty \tag{2. 5}$$

*where $\chi$ is an indicator function.*

We take $\mathbf{w}(u,v,t) = \omega(u,v)\cdot 1$, where $\omega \in \bigotimes_{i=1}^m \mathbf{A}_\infty\left(\mathbb{R}^{2n}\right)$. Then (2. 4)-(2. 5) become

$$\mathbf{vol}_\omega\left\{\bigcup_j \mathbf{R}_j\right\} \lesssim \mathbf{vol}_\omega\left\{\bigcup_k \widehat{\mathbf{R}}_k\right\} \tag{2. 6}$$

and

$$\left\|\sum_k \chi_{\widehat{\mathbf{R}}_k}\right\|^p_{\mathbf{L}^p(\mathbb{R}^{2n+1},\omega)} \lesssim \mathbf{vol}_\omega\left\{\bigcup_k \widehat{\mathbf{R}}_k\right\}, \qquad 1 < p < \infty. \tag{2. 7}$$

Let

$$\mathbf{U}_\lambda = \left\{(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \colon \mathbf{M}_\omega f(u,v,t) > \lambda\right\}. \tag{2. 8}$$

Given any $(u,v,t) \in \mathbf{U}_\lambda$, there is a rectangle $\mathbf{R}_j \ni (u,v,t)$ such that

$$\mathbf{vol}_\omega\{\mathbf{R}_j\}^{-1} \iiint_{\mathbf{R}_j} \left|f(\xi,\eta,\tau + \mu(u\cdot\eta - v\cdot\xi))\right| \omega(\xi,\eta) d\xi d\eta d\tau > \frac{1}{2}\lambda. \tag{2. 9}$$

Let $(u,v,t)$ run through the set $\mathbf{U}_\lambda$. We have

$$\mathbf{U}_\lambda \subset \bigcup_j \mathbf{R}_j.$$

By applying the covering lemma, we select a subsequence $\{\widehat{\mathbf{R}}_k\}_{k=1}^{\infty}$ from the union above and

$$\begin{aligned}
\mathbf{vol}_\omega\Big\{\mathbf{U}_\lambda\Big\} \lesssim \mathbf{vol}_\omega\Big\{\bigcup_j \mathbf{R}_j\Big\} &\lesssim \mathbf{vol}_\omega\Big\{\bigcup_k \widehat{\mathbf{R}}_k\Big\} \qquad \text{by (2. 6)} \\
&\leq \sum_k \mathbf{vol}_\omega\left\{\widehat{\mathbf{R}}_k\right\} \\
&\leq \sum_k \frac{2}{\lambda} \iiint_{\widehat{\mathbf{R}}_k} \Big|f(\xi,\eta,\tau+\mu(u\cdot\eta-v\cdot\xi))\Big|\,\omega(\xi,\eta)d\xi d\eta d\tau \qquad \text{by (2. 9).}
\end{aligned} \tag{2. 10}$$

Furthermore, we find

$$\begin{aligned}
\mathbf{vol}_\omega\Big\{\bigcup_k \widehat{\mathbf{R}}_k\Big\} &\lesssim \lambda^{-1}\sum_k \iiint_{\widehat{\mathbf{R}}_k} \Big|f(\xi,\eta,\tau+\mu(u\cdot\eta-v\cdot\xi))\Big|\,\omega(\xi,\eta)d\xi d\eta d\tau \\
&= \lambda^{-1}\iiint_{\mathbb{R}^{2n+1}} \left| f(\xi,\eta,\tau+\mu(u\cdot\eta-v\cdot\xi))\sum_k \chi_{\widehat{\mathbf{R}}_k}(\xi,\eta,\tau)\right| \omega(\xi,\eta)d\xi d\eta d\tau \\
&\leq \lambda^{-1}\left\{\iiint_{\mathbb{R}^{2n+1}} \Big|f(\xi,\eta,\tau+\mu(u\cdot\eta-v\cdot\xi))\Big|^p \omega(\xi,\eta)d\xi d\eta d\tau\right\}^{\frac{1}{p}} \left\|\sum_k \chi_{\widehat{\mathbf{R}}_k}\right\|_{\mathbf{L}^{\frac{p}{p-1}}(\mathbb{R}^{2n+1},\omega)} \\
&\qquad\qquad \text{by Hölder inequality} \\
&= \lambda^{-1}\left\{\iint_{\mathbb{R}^{2n}} \Big\|f(\xi,\eta,\cdot)\Big\|^p_{\mathbf{L}^p(\mathbb{R})}\,\omega(\xi,\eta)d\xi d\eta\right\}^{\frac{1}{p}} \left\|\sum_k \chi_{\widehat{\mathbf{R}}_k}\right\|_{\mathbf{L}^{\frac{p}{p-1}}(\mathbb{R}^{2n+1},\omega)} \\
&\leq \lambda^{-1}\Big\|f\Big\|_{\mathbf{L}^p(\mathbb{R}^{2n+1},\omega)}\,\mathbf{vol}_\omega\Big\{\bigcup_k \widehat{\mathbf{R}}_k\Big\}^{\frac{p-1}{p}} \qquad \text{by (2. 7).}
\end{aligned} \tag{2. 11}$$

This implies

$$\mathbf{vol}_\omega\Big\{\bigcup_k \widehat{\mathbf{R}}_k\Big\}^{\frac{1}{p}} \lesssim \frac{1}{\lambda}\Big\|f\Big\|_{\mathbf{L}^p(\mathbb{R}^{2n+1},\omega)}. \tag{2. 12}$$

Let $\mathbf{U}_\lambda$ defined in (2. 8). We have

$$\begin{aligned}
\mathbf{vol}_\omega\Big\{(x,y,t)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}\colon \mathbf{M}_\omega f(u,v,t)>\lambda\Big\}^{\frac{1}{p}} &= \mathbf{vol}_\omega\Big\{\mathbf{U}_\lambda\Big\}^{\frac{1}{p}} \\
&\lesssim \mathbf{vol}_\omega\Big\{\bigcup_k \widehat{\mathbf{R}}_k\Big\}^{\frac{1}{p}} \qquad \text{by (2. 10)} \\
&\lesssim \frac{1}{\lambda}\Big\|f\Big\|_{\mathbf{L}^p(\mathbb{R}^{2n+1},\omega)} \qquad \text{by (2. 12)}\,.
\end{aligned} \tag{2. 13}$$

By using this weak type $(p,p)$-estimate and applying Marcinkiewicz interpolation theorem, we conclude that $\mathbf{M}_\omega$ is bounded on $\mathbf{L}^p(\mathbb{R}^{2n+1},\omega)$ for $1<p<\infty$.

## 3 Proof of the covering lemma

First, we need the following lemma proved by Fefferman [6]:

**Lemma 3.1.** *If* $\mathbf{w} \in \bigotimes_{i=1}^{k} \mathbf{A}_\infty\left(\mathbb{R}^{2n+1}\right)$*, then* $\mathbf{w}$ *satisfies the following: If* $\mathbf{R} \subset \mathbb{R}^{2n+1}$ *is any rectangle with its sides parallel to the axes and* $E \subset \mathbf{R}$ *is such that* $\mathbf{vol}\,(E) > \frac{1}{2}\mathbf{vol}\,(\mathbf{R})$*, then* $\mathbf{vol}_\mathbf{w}(E) > \eta\mathbf{vol}_\mathbf{w}(\mathbf{R})$*, for some* $\eta > 0$*.*

**Proof**. The proof is by induction on $k$. Assume the result is true for $k-1$. Consider a rectangle $\mathbf{R}$ as above, $\mathbf{R} = I \times J$ where $I$ is a rectangle in $\mathbb{R}^{\mathbf{N}_1} \times \mathbb{R}^{\mathbf{N}_2} \times \cdots \times \mathbb{R}^{\mathbf{N}_{k-1}}$, and $J$ is a cube in $\mathbb{R}^{\mathbf{N}_k}$, $\mathbf{N}_1 + \mathbf{N}_2 + \cdots + \mathbf{N}_k = 2n+1$.

Let $E \subset \mathbf{R}$ such that

$$\frac{\mathbf{vol}\,(E)}{\mathbf{vol}\,(\mathbf{R})} > \frac{1}{2}. \tag{3. 1}$$

For each $x_k^\dagger \in I$, let $J_{x_k^\dagger} = \left\{(x_k^\dagger, x_k) : x_k \in J\right\}$. We claim that there exists $I' \subset I$ satisfying $\mathbf{vol}(I') \geq \varepsilon\mathbf{vol}(I)$, where $\varepsilon > 0$ is small enough, such that for $x_k^\dagger \in I'$,

$$\mathbf{vol}\left(E \cap J_{x_k^\dagger}\right) > \varepsilon\mathbf{vol}(J_{x_k^\dagger}) = \varepsilon\mathbf{vol}(J). \tag{3. 2}$$

If not, we must have $\mathbf{vol}\left(\left\{x_k^\dagger \in I : \mathbf{vol}(E \cap J_{x_k^\dagger}) > \varepsilon\mathbf{vol}(J)\right\}\right) \leq \varepsilon\mathbf{vol}\,(I)$.

Divide $I$ into two parts:

$$G = \left\{x_k^\dagger \in I : \mathbf{vol}(E \cap J_{x_k^\dagger}) > \varepsilon\mathbf{vol}(J)\right\}, \qquad B = \left\{x_k^\dagger \in I : \mathbf{vol}(E \cap J_{x_k^\dagger}) \leq \varepsilon\mathbf{vol}(J)\right\}. \tag{3. 3}$$

Then

$$\begin{aligned} \mathbf{vol}(E) &= \int_B \mathbf{vol}(E \cap J_{x_k^\dagger})dx_k^\dagger + \int_G \mathbf{vol}(E \cap J_{x_k^\dagger})dx_k^\dagger \\ &\leq \mathbf{vol}(B) \cdot \varepsilon\mathbf{vol}(J) + \mathbf{vol}(G) \cdot \mathbf{vol}(J) \\ &= \left[\mathbf{vol}(I) - \mathbf{vol}(G)\right] \cdot \varepsilon\mathbf{vol}(J) + \mathbf{vol}(G) \cdot \mathbf{vol}(J) \\ &= \varepsilon\mathbf{vol}(I) \cdot \mathbf{vol}(J) + (1-\varepsilon)\mathbf{vol}(G) \cdot \mathbf{vol}(J) \end{aligned} \tag{3. 4}$$

Suppose $\mathbf{vol}(G) \leq \varepsilon\mathbf{vol}(I)$, then we further have

$$\mathbf{vol}(E) \leq \varepsilon\mathbf{vol}(I) \cdot \mathbf{vol}(J) + (1-\varepsilon) \cdot \varepsilon\mathbf{vol}(I) \cdot \mathbf{vol}(J) \leq \left(2\varepsilon - \varepsilon^2\right)\mathbf{vol}(\mathbf{R}). \tag{3. 5}$$

We can choose $\varepsilon > 0$ small enough such that $0 < 2\varepsilon - \varepsilon^2 < \frac{1}{2}$, then

$$\mathbf{vol}(E) < \frac{1}{2}\mathbf{vol}(\mathbf{R}), \tag{3. 6}$$

which is contradicted to $\frac{\mathbf{vol}(E)}{\mathbf{vol}(\mathbf{R})} > \frac{1}{2}$.

Now, from (3. 2) and since $\mathbf{w}(x_k^\dagger, \cdot)$ is $\mathbf{A}^\infty$ in the $x_k$ variable for every $x_k^\dagger \in \mathbb{R}^{2n+1-\mathbf{N}_k}$, we have

$$\int_{E\cap J_{x_k^\dagger}} \mathbf{w}(x_k^\dagger, x_k)dx_k \geq \eta \int_{J_{x_k^\dagger}} \mathbf{w}(x_k^\dagger, x_k)dx_k, \qquad x_k^\dagger \in I'. \tag{3. 7}$$

But also if we fix any $x_k \in J$, then

$$\int_{I'} \mathbf{w}(x_k^\dagger, x_k)dx_k^\dagger \geq \eta' \int_I \mathbf{w}(x_k^\dagger, x_k)dx_k^\dagger, \tag{3. 8}$$

by induction. It follows by integrating (3. 7) in $x_k^\dagger \in I'$ that

$$\int_E \mathbf{w}(x)dx \geq \eta \int_{I'\times J} \mathbf{w}(x)dx, \tag{3. 9}$$

and integrating (3. 8) in $x_k \in J$ gives

$$\int_{I'\times J} \mathbf{w}(x)dx \geq \eta' \int_{I\times J} \mathbf{w}(x)dx = \int_R \mathbf{w}(x)dx. \tag{3. 10}$$

By combing (3. 9)-(3. 10), we have

$$\int_E \mathbf{w}(x)dx \geq \eta\eta' \int_R \mathbf{w}(x)dx. \tag{3. 11}$$

Now we finish the proof of Lemma 3.1.

Then we continue to prove the covering lemma. We re-arrange the order of $\{\mathbf{R}_j\}_{j=1}^\infty$ if necessary so that the cross-section volume of $\mathbf{R}_j$ in $\mathbb{R}^{\mathbf{N}_k}$ is decreasing as $j \longrightarrow \infty$. For brevity, we call it $x_k$-cross section. Denote $\mathbf{R}_j^*$ to be the rectangle co-centered with $\mathbf{R}_j$ having its $x_k$-cross section tripled and keeping the others same. We select $\widehat{\mathbf{R}}_k$ from $\left\{\mathbf{R}_j\right\}_{j=1}^\infty$ as follows.

Let $\widehat{\mathbf{R}}_1 = \mathbf{R}_1$. Having chosen $\widehat{\mathbf{R}}_1, \widehat{\mathbf{R}}_2, \ldots, \widehat{\mathbf{R}}_{N-1}$, we pick $\widehat{\mathbf{R}}_N$ as the first rectangle $\mathbf{R}$ on the list of $\mathbf{R}_j$'s after $\widehat{\mathbf{R}}_{N-1}$ so that

$$\mathbf{vol}\left\{ \mathbf{R} \cap \left[ \bigcup_{\substack{k=1 \\ \widehat{\mathbf{R}}_k^* \cap \mathbf{R} \neq \emptyset}}^{N-1} \widehat{\mathbf{R}}_k^* \right] \right\} < \frac{1}{2}\mathbf{vol}\left\{\mathbf{R}\right\}. \tag{3. 12}$$

Suppose $\mathbf{R}$ is an unselected rectangle. There is a positive number $M$ such that $\mathbf{R}$ is on the list of $\mathbf{R}_j$'s after $\widehat{\mathbf{R}}_M$ and

$$\mathbf{vol}\left\{ \mathbf{R} \cap \left[ \bigcup_{\substack{k=1 \\ \widehat{\mathbf{R}}_k^* \cap \mathbf{R} \neq \emptyset}}^{M} \widehat{\mathbf{R}}_k^* \right] \right\} \geq \frac{1}{2}\mathbf{vol}\left\{\mathbf{R}\right\}. \tag{3. 13}$$

Recall $\widehat{\mathbf{R}}_k^*$ whose $x_k$-cross section is tripled. Moreover, the $t$-side length of $\{\mathbf{R}_j\}_{j=1}^\infty$ is decreasing as $j \longrightarrow \infty$. On the $x_k$-cross section, the projection of $\mathbf{R}$ is covered by the projection of the union inside (3. 13).

Let $(x_1, x_2, \cdots, x_k) \in \mathbf{R}$. Then slice all rectangles with a plane through $(x_1, x_2, \cdots, x_k)$ perpendicular to the $x_k$-cross section. Denote $\mathbf{S}$, $\widehat{\mathbf{S}}_k$ and $\widehat{\mathbf{S}}_k^*$ to be the slices regarding to $\mathbf{R}$, $\widehat{\mathbf{R}}_k$ and $\widehat{\mathbf{R}}_k^*$. Consequently, (3. 13) implies

$$\mathbf{vol}\left\{ \mathbf{S} \cap \left[ \bigcup_{k=1}^{M} \widehat{\mathbf{S}}_k^* \right] \right\} \geq \frac{1}{2}\mathbf{vol}\left\{\mathbf{S}\right\}. \tag{3. 14}$$

Since $\mathbf{w} \in \bigotimes_{i=1}^{k-1} \mathbf{A}_\infty\left(\mathbb{R}^{2n+1}\right)$, then $\mathbf{w}(\cdot, x_i^\dagger)$ satisfies the $\mathbf{A}_\infty$-property in $\mathbb{R}^{\mathbf{N}_i}$ for every $x_i^\dagger \in \mathbb{R}^{2n+1-\mathbf{N}_i}$, $i = 1, 2, \cdots, k$. This implies $\mathbf{w}(\cdot, x_k) \in \bigotimes_{i=1}^{k-1} \mathbf{A}_\infty\left(\mathbb{R}^{2n+1-\mathbf{N}_k}\right)$ for any fixed $x_k \in \mathbb{R}^{\mathbf{N}_k}$.

For every subset $F \subset \mathbb{R}^{2n+1-\mathbf{N}_k}$, we define

$$\mathbf{vol}_{\mathbf{w}(\cdot,x_k)}^{2n+1-\mathbf{N}_k}(F) = \int_F \mathbf{w}(x_k^\dagger, x_k)dx_k^\dagger, \qquad \text{for every } x_k \in \mathbb{R}^{\mathbf{N}_k}. \tag{3. 15}$$

Then by Lemma 3.1, we have

$$\mathbf{vol}_{\mathbf{w}(\cdot,x_k)}^{2n+1-\mathbf{N}_k}\left\{ \mathbf{S} \cap \left[ \bigcup_{k=1}^{M} \widehat{\mathbf{S}}_k^* \right] \right\} \geq \eta\mathbf{vol}_{\mathbf{w}(\cdot,x_k)}^{2n+1-\mathbf{N}_k}\left\{\mathbf{S}\right\}, \qquad \eta > 0. \tag{3. 16}$$

Let $\mathbf{M}_{\mathbf{w}(\cdot,x_k)}^{2n+1-\mathbf{N}_k}$ be the strong maximal operator defined in $\mathbb{R}^{2n+1-\mathbf{N}_k}$. Observe that (3. 16) further implies

$$\mathbf{M}_{\mathbf{w}(\cdot,x_k)}^{2n+1-\mathbf{N}_k}(\chi_{\bigcup_k \widehat{\mathbf{S}}_k^*})(x_k^\dagger) > \eta, \qquad x_k^\dagger \in \bigcup\nolimits_j \mathbf{S}_j. \tag{3. 17}$$

It is well-known that $\mathbf{M}_{\mathbf{w}(\cdot,x_1^\dagger)}^{\mathbf{N}_1}$ is the Hardy-Littlewood maximal operator bounded on $\mathbf{L}^p(\mathbb{R}^{\mathbf{N}_1}, \mathbf{w}(\cdot, x_1^\dagger))$, for $1 < p < \infty$. By induction, we may assume that $\mathbf{M}_{\mathbf{w}(\cdot,x_k)}^{2n+1-\mathbf{N}_k}$ is bounded on all $\mathbf{L}^p(\mathbb{R}^{2n+1-\mathbf{N}_k}, \mathbf{w}(\cdot, x_k))$, $1 < p < \infty$. From (3. 16)-(3. 17), by applying the $\mathbf{L}^p$-boundedness of $\mathbf{M}_{\mathbf{w}(\cdot,x_k)}^{2n+1-\mathbf{N}_k}$, we find

$$\mathbf{vol}_{\mathbf{w}(\cdot,x_k)}^{2n+1-\mathbf{N}_k}\left\{\bigcup\nolimits_j \mathbf{S}_j\right\} \lesssim \mathbf{vol}_{\mathbf{w}(\cdot,x_k)}^{2n+1-\mathbf{N}_k}\left\{\bigcup\nolimits_k \widehat{\mathbf{S}}_k^*\right\}. \tag{3. 18}$$

By using (3. 18) and integrating in the $x_k$-coordinate, we have

$$\mathbf{vol}_\mathbf{w}\left\{\bigcup\nolimits_j \mathbf{R}_j\right\} \lesssim \mathbf{vol}_\mathbf{w}\left\{\bigcup\nolimits_k \widehat{\mathbf{R}}_k^*\right\} \lesssim \mathbf{vol}_\mathbf{w}\left\{\bigcup\nolimits_k \widehat{\mathbf{R}}_k\right\} \tag{3. 19}$$

which is (2. 4).

On the other hand, (3. 12) implies

$$\mathbf{vol}\left\{ \widehat{\mathbf{S}}_N \cap \left[ \bigcup_{k=1}^{N-1} \widehat{\mathbf{S}}_k^* \right] \right\} < \frac{1}{2}\mathbf{vol}\left\{\widehat{\mathbf{S}}_N\right\}. \tag{3. 20}$$

We are given that the measure $\mathbf{w}(\cdot, x_k)$ as above on the hyperplane ( if the hyperplane in $x_k = \mathbf{c}$, then $d\mathbf{w}(\cdot, \mathbf{c}) = \mathbf{w}(x_1, x_2, \cdots, x_{k-1}, \mathbf{c})dx_1dx_2 \cdots dx_{k-1}$ ) belongs to $\bigotimes_{i=1}^{k-1} \mathbf{A}_\infty\left(\mathbb{R}^{2n+1-\mathbf{N}_k}\right)$ for every $x_k \in \mathbb{R}^{\mathbf{N}_k}$, so that for some $\eta > 0$,

$$\mathbf{vol}_{\mathbf{w}(\cdot,x_k)}^{2n+1-\mathbf{N}_k}\left\{ \widehat{\mathbf{S}}_N \cap \left[ \bigcup_{k=1}^{N-1} \widehat{\mathbf{S}}_k^* \right] \right\} < (1-\eta)\mathbf{vol}_{\mathbf{w}(\cdot,x_k)}^{2n+1-\mathbf{N}_k}\left\{\widehat{\mathbf{S}}_N\right\}. \tag{3. 21}$$

Denote $\widehat{\mathbf{E}}_N = \widehat{\mathbf{S}}_N \setminus \bigcup_{k<N}\widehat{\mathbf{S}}_k$. From (3. 21), we find

$$\mathbf{vol}_{\mathbf{w}(\cdot,x_k)}^{2n+1-\mathbf{N}_k}\left\{\widehat{\mathbf{E}}_N\right\} \geq \eta\mathbf{vol}_{\mathbf{w}(\cdot,x_k)}^{2n+1-\mathbf{N}_k}\left\{\widehat{\mathbf{S}}_N\right\}. \tag{3. 22}$$

Let $\phi \in \mathbf{L}^{\frac{p}{p-1}}(\mathbb{R}^{2n+1-\mathbf{N}_k}, \mathbf{w}(\cdot, x_k))$ and $\left\|\phi\right\|_{\mathbf{L}^{\frac{p}{p-1}}(\mathbb{R}^{2n+1-\mathbf{N}_k},\mathbf{w}(\cdot,x_k))} = 1$. We have

$$\begin{aligned}
\int_{\mathbb{R}^{2n+1-\mathbf{N}_k}} \phi \cdot \sum_k \chi_{\widehat{\mathbf{S}}_k} d\mathbf{w}(\cdot, x_k) &= \sum_k \int_{\widehat{\mathbf{S}}_k} \phi d\mathbf{w}(\cdot, x_k) \\
&= \sum_k \left\{\frac{1}{\mathbf{vol}_{\mathbf{w}(\cdot,x_k)}^{2n+1-\mathbf{N}_k}\{\widehat{\mathbf{S}}_k\}} \int_{\widehat{\mathbf{S}}_k} \phi d\mathbf{w}(\cdot, x_k)\right\} \mathbf{vol}_{\mathbf{w}(\cdot,x_k)}^{2n+1-\mathbf{N}_k}\left\{\widehat{\mathbf{S}}_k\right\} \\
&\leq \sum_k \left\{\frac{1}{\mathbf{vol}_{\mathbf{w}(\cdot,x_k)}^{2n+1-\mathbf{N}_k}\{\widehat{\mathbf{S}}_k\}} \int_{\widehat{\mathbf{S}}_k} \phi d\mathbf{w}(\cdot, x_k)\right\} \cdot \frac{1}{\eta}\mathbf{vol}_{\mathbf{w}(\cdot,x_k)}^{2n+1-\mathbf{N}_k}\left\{\widehat{\mathbf{E}}_k\right\} \qquad \text{by (3. 22)} \\
&\lesssim \sum_k \int_{\widehat{\mathbf{E}}_k} \mathbf{M}_{\mathbf{w}(\cdot,x_k)}^{2n+1-\mathbf{N}_k}(\phi) d\mathbf{w}(\cdot, x_k) \\
&= \int_{\bigcup_k \widehat{\mathbf{S}}_k} \mathbf{M}_{\mathbf{w}(\cdot,x_k)}^{2n+1-\mathbf{N}_k}(\phi) d\mathbf{w}(\cdot, x_k).
\end{aligned} \tag{3. 23}$$

By applying Hölder inequality and the $\mathbf{L}^p$-boundedness of $\mathbf{M}_{\mathbf{w}(\cdot,x_k)}^{2n+1-\mathbf{N}_k}$, we find

$$\begin{aligned}
\int_{\bigcup_k \widehat{\mathbf{S}}_k} \mathbf{M}_{\mathbf{w}(\cdot,x_k)}^{2n+1-\mathbf{N}_k}(\phi) d\mathbf{w}(\cdot, x_k) &\leq \left\|\mathbf{M}_{\mathbf{w}(\cdot,x_k)}^{2n+1-\mathbf{N}_k}(\phi)\right\|_{\mathbf{L}^{\frac{p}{p-1}}(\mathbb{R}^{2n+1-\mathbf{N}_k},\mathbf{w}(\cdot,x_k))} \mathbf{vol}_{\mathbf{w}(\cdot,x_k)}^{2n+1-\mathbf{N}_k}\left\{\bigcup_k \widehat{\mathbf{S}}_k\right\}^{\frac{1}{p}} \\
&\leq \mathfrak{B}_p\, \mathbf{vol}_{\mathbf{w}(\cdot,x_k)}^{2n+1-\mathbf{N}_k}\left\{\bigcup_k \widehat{\mathbf{S}}_k\right\}^{\frac{1}{p}}.
\end{aligned} \tag{3. 24}$$

By substituting (3. 24) to (3. 23) and taking the supremum of $\phi$, we arrive at

$$\left\|\sum_k \chi_{\widehat{\mathbf{S}}_k}\right\|_{\mathbf{L}^p(\mathbb{R}^{2n+1-\mathbf{N}_k},\mathbf{w}(\cdot,x_k))} \leq \mathfrak{B}_p\, \mathbf{vol}_{\mathbf{w}(\cdot,x_k)}^{2n+1-\mathbf{N}_k}\left\{\bigcup_k \widehat{\mathbf{S}}_k\right\}^{\frac{1}{p}}. \tag{3. 25}$$

Raising both sides of (3. 25) to the $p^{th}$ power and integrating over $x_k$ give us (2. 5).

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