THETA-BUMP THEOREM FOR FRACTIONAL INTEGRALS ON HEISENBERG GROUP

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Abstract. The main purpose of this paper is to study multi-parameter fractional integral operators which commute with Zygmund dilations on Heisenberg group. First, we prove a two-weight $L^p \to L^q$ norm inequality for fractional integrals on Heisenberg groups $\mathbb{H}^n$. The weights $\omega^q, \sigma^{-\frac{p}{p-1}}$ are non-negative, locally integrable functions defined on the $2n$-dimensional subspace, satisfying a ϑ-bump condition. Next, we extend the above result by considering the multi-parameter fractional integral operator:

$$I_{\alpha\beta\rho} f(u,v,t) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\xi,\eta,\tau) V^{\alpha\beta\rho} \left[(u,v,t) \odot (\xi,\eta,\tau)^{-1}\right] d\xi \, d\eta \, d\tau \,,$$

where $\odot$ denotes the multiplication law of Heisenberg group, and $V^{\alpha\beta\rho}$ is a distribution on $\mathbb{H}^n$ satisfying Zygmund-type dilations. We show that $\omega I_{\alpha\beta\rho} \sigma^{-1} : L^p(\mathbb{H}^n) \to L^q(\mathbb{H}^n)$ for $1 < p < q < \infty$ if the corresponding bi-parameter ϑ-bump characteristic is finite.

1. Introduction

Let $0 < \alpha < N$ and $\mathbb{R}^N$ be the N-dimensional Euclidean space. A fractional integral operator T_a is defined as:

$$T_\alpha f(x) = \int_{\mathbb{R}^N} f(y) \frac{1}{|x-y|^{N-\alpha}} dy, \quad 0 < \alpha < N. \tag{1.1}$$

In 1928, Hardy and Littlewood [10] obtained a regularity theorem for T_α when $N = 1$. Ten years later, Sobolev [19] extended it to higher-dimensional spaces later. Throughout the paper, $\mathfrak{B} > 0$ denotes a generic constant depending only on its sub-indices.

Theorem 1.1 (Hardy-Littlewood-Sobolev theorem)**.** Let T_α be defined by (1.1) for $0 < \alpha < N$. Then

$$\|T_\alpha f\|_{L^q(\mathbb{R}^N)} \le C_{pq} \|f\|_{L^p(\mathbb{R}^N)}, \quad 1 < p < q < \infty \tag{1.2}$$

if and only if $\frac{\alpha}{N} = \frac{1}{p} - \frac{1}{q}$.

Over the past decades, the $L^p \to L^q$ regularity of fractional integrals has been systematically studied in weighted settings. For instance, see the paper by Stein and Weiss [20], Muckenhoupt and Wheeden [11], Fefferman and Muckenhoupt [6], Coifman and Fefferman [1], Pérez [13] and Sawyer and Wheeden [17]. Let $Q \subset \mathbb{R}^N$ be a cube parallel to the coordinate axes. It is well known that the weighted inequality:

$$\|\omega T_\alpha f\|_{L^q(\mathbb{R}^N)} \lesssim \|f\sigma\|_{L^p(\mathbb{R}^N)}, \quad 1 < p \le q < \infty \tag{1.3}$$

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implies

$$A^{\alpha}_{pq}(\omega,\sigma) := \sup_{Q\subset\mathbb{R}^N} |Q|^{\frac{\alpha}{N}-\frac{1}{p}+\frac{1}{q}} \left[\frac{1}{|Q|}\int_Q \omega^q(x)dx\right]^{\frac{1}{q}} \left[\frac{1}{|Q|}\int_Q \left(\frac{1}{\sigma}\right)^{\frac{p}{p-1}}(x)dx\right]^{\frac{p-1}{p}} < \infty. \tag{1.4}$$

The (1.4) is called the Muckenhoupt characteristic, which is not sufficient for (1.3) in general.

For $\vartheta > 1$, Fefferman and Phong [3, 5] introduced the ϑ-bump characteristic:

$$A^{\alpha}_{pq\vartheta}(\omega,\sigma) = \sup_{Q\subset\mathbb{R}^N} |Q|^{\frac{\alpha}{N}-\frac{1}{p}+\frac{1}{q}} \left[\frac{1}{|Q|}\int_Q \omega^{q\vartheta}(x)dx\right]^{\frac{1}{q\vartheta}} \left[\frac{1}{|Q|}\int_Q \left(\frac{1}{\sigma}\right)^{\frac{p\vartheta}{p-1}}(x)dx\right]^{\frac{p-1}{p\vartheta}}. \tag{1.5}$$

By Hölder's inequality, $A^{\alpha}_{pq\vartheta}(\omega,\sigma) \ge A^{\alpha}_{pq}(\omega,\sigma)$ for $\vartheta > 1$. Note that $A^{\alpha}_{pq\vartheta}(\omega,\sigma) < \infty$ is initially introduced by Fefferman and Phong [3, 5] when $p = q$ and later refined by Sawyer and Wheeden [17] in their study of two-weight $L^p \to L^q$-norm inequality for fractional integrals.

Theorem 1.2 (Sawyer-Wheeden theorem, 1992)**.** Let T_α be defined by (1.1) for $0 < \alpha < N$. Then

$$\|\omega T_\alpha f\|_{L^q(\mathbb{R}^N)} \le C_{pq\vartheta} A^{\alpha}_{pq\vartheta}(\omega,\sigma)\, \|f\sigma\|_{L^p(\mathbb{R}^N)}, \quad 1 < p \le q < \infty \tag{1.6}$$

if $A^{\alpha}_{pq\vartheta}(\omega,\sigma) < \infty$ for some $\vartheta > 1$.

Remark 1.1. The constant $C_{pq\vartheta}A^{\alpha}_{pq\vartheta}(\omega,\sigma)$ in (1.6) was not written explicitly in the original statement of this theorem. But, it can be computed by carrying out the proof given in section 2 of [17].

The product theory is invariant with respect to n-fold dilation on $\mathbb{R}^n$, $\delta\cdot x = (\delta_1 x_1, ..., \delta_n x_n)$ for $\delta = (\delta_1, ..., \delta_n) \in \mathbb{R}^n_+$. It has been widely considered that the next simplest multiparameter group of dilations after the product multiparameter dilations is the so-called the Zygmund dilation defined on $\mathbb{R}^3$ by $\rho_{s,t}(x_1, x_2, x_3) = (sx_1, tx_2, stx_3)$ for $s, t > 0$. Indeed, as far as $\mathcal{M}_3$ is concerned, E. M. Stein was the first to link the properties of maximal operators associated with Zygmund dilations to boundary value problems for Poisson integrals on symmetric spaces, such as Siegel's upper half space. See the survey paper of R. Fefferman [4] on the future direction of research of multiparameter analysis on Zygmund dilations. A convolution operator of this type is said to be associated with Zygmund dilation. Singular integral operators carrying certain multi-parameter structures defined on Heisenberg group have been systematically studied, for instance by Phong and Stein [14], Ricci and Stein [16] and Müller, Ricci and Stein [12]. Much less is known for fractional integration in this direction.

Folland and Stein [7] extended the Hardy-Littlewood-Sobolev theorem to Heisenberg groups $\mathbb{H}^n = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, with multiplication law:

$$(u, v, t) \odot (\xi, \eta, \tau) = (u + \xi, v + \eta, t + \tau + \mu(u\cdot\eta - v\cdot\xi)), \quad \mu \in \mathbb{R} \tag{1.7}$$

for all $(u, v, t), (\xi, \eta, \tau) \in \mathbb{H}^n$, and the inverse element $(\xi, \eta, \tau)^{-1} = (-\xi, -\eta, -\tau)$.

Let $0 < \delta < n+1$ (here $n+1$ is the homogeneous dimension of $\mathbb{H}^n$). Define the fractional integral operator:

$$S_\delta f(u, v, t) = \int_{\mathbb{R}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(\xi, \eta, \tau)\Omega^\delta\left[(u, v, t) \odot (\xi, \eta, \tau)^{-1}\right] d\xi\, d\eta\, d\tau, \tag{1.8}$$

where Ω^δ is a distribution on $\mathbb{R}^{2n+1}$ satisfying:

$$\Omega^\delta(u, v, t) = \frac{1}{(|u|^2 + |v|^2 + |t|)^{n+1-\delta}}, \quad (u, v, t) \neq (0, 0, 0). \tag{1.9}$$

Theorem 1.3 (Folland-Stein theorem)**.** Let S_δ be defined by (1.8)-(1.9) for $0<\delta<n+1$. Then

$$\|S_\delta f\|_{L^q(\mathbb{H}^n)} \le C_{pq}\|f\|_{L^p(\mathbb{H}^n)}, \quad 1<p<q<\infty \tag{1.10}$$

if and only if $\frac{\delta}{n+1}=\frac{1}{p}-\frac{1}{q}$.

Frank and Lieb [8] found the best constant for the above inequality and Pierce [15] obtained its discrete analogue. Fanelli and Roncal [2] recently established commutator estimates for fractional sub-Laplacians on $\mathbb{H}^n$.

1.1. **Statement of the main results.** Building upon Theorems 1.1, 1.2 and 1.3, this paper first establishes the following results:

Theorem 1.4. Let S_δ be defined by (1.8)-(1.9) for $0<\delta<n+1$. If

$$\|\omega S_\delta f\|_{L^q(\mathbb{H}^n)} \lesssim \|f\sigma\|_{L^p(\mathbb{H}^n)}, \quad 1<p\le q<\infty$$

then

$$\hat{A}^\delta_{pq}(\omega,\sigma) := \sup_{Q\subset\mathbb{R}^{2n}} |Q|^{\left[\frac{\delta}{n+1}-\left(\frac{1}{p}-\frac{1}{q}\right)\right]\frac{n+1}{n}} \left[\frac{1}{|Q|}\iint_Q \omega^q(u,v)\,du\,dv\right]^{\frac{1}{q}} \left[\frac{1}{|Q|}\iint_Q \sigma^{-\frac{p}{p-1}}(u,v)\,du\,dv\right]^{\frac{p-1}{p}} <\infty.$$

Remark 1.2. The exponent $\left[\frac{\delta}{n+1}-\left(\frac{1}{p}-\frac{1}{q}\right)\right]\frac{n+1}{n}$ adapts the Muckenhoupt characteristic to the homogeneous dimension of $\mathbb{H}^n$, analogous to $\frac{\delta}{N}-\frac{1}{p}+\frac{1}{q}$ in Euclidean spaces.

For $\vartheta>1$, define the ϑ-bump characteristic:

$$\hat{A}^\delta_{pq\vartheta}(\omega,\sigma) = \sup_{Q\subset\mathbb{R}^{2n}} |Q|^{\left[\frac{\delta}{n+1}-\left(\frac{1}{p}-\frac{1}{q}\right)\right]\frac{n+1}{n}} \left[\frac{1}{|Q|}\iint_Q \omega^{q\vartheta}(u,v)\,du\,dv\right]^{\frac{1}{q\vartheta}} \left[\frac{1}{|Q|}\iint_Q \sigma^{-\frac{p\vartheta}{p-1}}(u,v)\,du\,dv\right]^{\frac{p-1}{p\vartheta}}.$$

Theorem 1.5. Let S_δ be defined by (1.8)-(1.9) for $0<\delta<n+1$. If $\hat{A}^\delta_{pq\vartheta}(\omega,\sigma)<\infty$ for some $\vartheta>1$, then

$$\|\omega S_\delta f\|_{L^q(\mathbb{H}^n)} \le C_{\delta pq\vartheta}\hat{A}^\delta_{pq\vartheta}(\omega,\sigma)\,\|f\sigma\|_{L^p(\mathbb{H}^n)}, \quad 1<p<q<\infty.$$

Remark 1.3. Theorem 1.5 a two-weight analogue of **Follan-Stein theorem**. A pioneering result by considering the weights as suitable power functions, i.e:

$$\omega(u,v) = \left(\sqrt{|u|^2+|v|^2}\right)^{-r}, \qquad \sigma(u,v) = \left(\sqrt{|u|^2+|v|^2}\right)^{s}, \qquad r,s\in\mathbb{R}$$

has been established similar results without ϑ-bump conditions by [9]. It is straightforward to verify:

$$\hat{A}^\delta_{pq\vartheta}(\omega,\sigma) = \sup_{Q\subset\mathbb{R}^{2n}} |Q|^{\left[\frac{\delta}{n+1}-\frac{1}{p}+\frac{1}{q}\right]\frac{n+1}{n}} \left(\frac{1}{|Q|}\iint_Q \left(\sqrt{|u|^2+|v|^2}\right)^{-rq\vartheta} du\,dv\right)^{\frac{1}{q\vartheta}}$$
$$\times\left(\frac{1}{|Q|}\iint_Q \left(\sqrt{|u|^2+|v|^2}\right)^{-sp\vartheta/(p-1)} du\,dv\right)^{\frac{p-1}{p\vartheta}} <\infty$$

if $rq\vartheta<2n$ and $sp\vartheta/(p-1)<2n$, which are easily satisfied for suitable r,s.

Next, we introduce a multi-parameter family of fractional integral operators on Heisenberg groups by replacing Ω^δ with a larger kernel having singularity on every coordinate subspace. First, it is clear

$$\Omega^\delta(u,v,t) \le \frac{1}{(|u||v|+|t|)^{n+1-\delta}}, \qquad (u,t)\neq(0,0) \text{ or } (v,t)\neq(0,0).$$

A direct computation shows

$$\begin{aligned}\frac{1}{(|u||v|+|t|)^{n+1-\delta}} &\approx \frac{1}{(|u|^2|v|^2+t^2)^{\frac{n+1}{2}-\frac{\delta}{2}}}\\ &= |u|^{\frac{\delta}{2}-\frac{n+1}{2}}|v|^{\frac{\delta}{2}-\frac{n+1}{2}}|t|^{\frac{\delta}{2}-\frac{n+1}{2}}\left(\frac{|u||v||t|}{|u|^2|v|^2+t^2}\right)^{\frac{n+1}{2}-\frac{\delta}{2}}\\ &= |u|^{\left(\frac{\delta}{2}+\frac{n-1}{2}\right)-n}|v|^{\left(\frac{\delta}{2}+\frac{n-1}{2}\right)-n}|t|^{\left(\frac{\delta}{2}-\frac{n-1}{2}\right)-1}\left(\frac{|u||v|}{|t|}+\frac{|t|}{|u||v|}\right)^{-\left(\frac{n+1}{2}-\frac{\delta}{2}\right)}, u\neq 0, v\neq 0, t\neq 0.\end{aligned}$$

Next, we consider the multi-parameter fractional operator $I_{\alpha\beta\rho}$ with kernel:

$$V^{\alpha\beta\rho}(u,v,t) := |u|^{\alpha-n}|v|^{\alpha-n}|t|^{\beta-1}\left(\frac{|u||v|}{|t|}+\frac{|t|}{|u||v|}\right)^{-\rho}, \quad u\neq 0, v\neq 0, t\neq 0, \tag{1.11}$$

where $\alpha,\beta\in\mathbb{R}$ and $\rho\geq 0$.

Remark 1.4. By taking into account $\alpha=\frac{\delta}{2}+\frac{n-1}{2}$, $\beta=\frac{\delta}{2}-\frac{n-1}{2}$ and $\rho=\frac{n+1}{2}-\frac{\delta}{2}$ for $0<\delta<n+1$, we find $\alpha>n\beta$ and $\vartheta=\frac{n+1}{2}-\frac{\delta}{2}>\frac{\alpha-n\beta}{n+1}$. Hence $\Omega^\delta(u,v,t)\leq V^{\alpha\beta\rho}(u,v,t)$ for $u\neq 0, v\neq 0, t\neq 0$.

This kernel satisfies Zygmund-type dilations:

$$V^{\alpha\beta\rho}\left[(ru,sv,rst)\odot(r\xi,s\eta,rs\tau)^{-1}\right]=r^{\alpha+\beta-n-1}s^{\alpha+\beta-n-1}V^{\alpha\beta\rho}\left[(u,v,t)\odot(\xi,\eta,\tau)^{-1}\right]$$

for $r,s>0$. The fractional operator with Zygmund-type dilations is defined as follows:

$$I_{\alpha\beta\rho}f(u,v,t) := \int_{\mathbb{R}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(\xi,\eta,\tau)\, V^{\alpha\beta\rho}\left[(u,v,t)\odot(\xi,\eta,\tau)^{-1}\right] d\xi\, d\eta\, d\tau\,. \tag{1.12}$$

Theorem 1.6. Let $I_{\alpha\beta\rho}$ be defined by (1.11)-(1.12) for $\alpha,\beta\in\mathbb{R}$ and $\rho\geq 0$. If

$$\left\|\omega I_{\alpha\beta\rho}f\right\|_{L^q(\mathbb{H}^n)} \lesssim \|f\sigma\|_{L^p(\mathbb{H}^n)}, \quad 1<p\leq q<\infty \tag{1.13}$$

then:

$$\rho\geq\left|\beta-\frac{1}{p}+\frac{1}{q}\right|; \tag{1.14}$$

$$\begin{aligned}\hat{A}^{\alpha\beta}_{pq}(\omega,\sigma) := \sup_{Q_1\times Q_2\subset\mathbb{R}^n\times\mathbb{R}^n} &|Q_1|^{\left(\frac{\alpha+\beta}{n+1}-\frac{1}{p}+\frac{1}{q}\right)\frac{n+1}{n}}|Q_2|^{\left(\frac{\alpha+\beta}{n+1}-\frac{1}{p}+\frac{1}{q}\right)\frac{n+1}{n}}\\ &\times\left(\frac{1}{|Q_1'||Q_2'|}\iint_{Q_1'\times Q_2'}\omega^q(u,v)\,du\,dv\right)^{\frac{1}{q}}\left(\frac{1}{|Q_1||Q_2|}\iint_{Q_1\times Q_2}\sigma^{-\frac{p}{p-1}}(u,v)\,du\,dv\right)^{\frac{p-1}{p}}<\infty.\end{aligned} \tag{1.15}$$

Remark 1.5. Condition $\rho\geq|\beta-\frac{1}{p}+\frac{1}{q}|$ ensures the kernel's singularity is integrable under multi-parameter dilations, as shown in the proof via scaling arguments.

For $\vartheta>1$, define the multi-parameter ϑ-bump characteristic:

$$\begin{aligned}\hat{A}^{\alpha\beta}_{pq\vartheta}(\omega,\sigma) := \sup_{Q_1\times Q_2\subset\mathbb{R}^n\times\mathbb{R}^n} &|Q_1|^{\left(\frac{\alpha+\beta}{n+1}-\frac{1}{p}+\frac{1}{q}\right)\frac{n+1}{n}}|Q_2|^{\left(\frac{\alpha+\beta}{n+1}-\frac{1}{p}+\frac{1}{q}\right)\frac{n+1}{n}}\\ &\times\left(\frac{1}{|Q_1'||Q_2'|}\iint_{Q_1'\times Q_2'}\omega^{q\vartheta}(u,v)\,du\,dv\right)^{\frac{1}{q\vartheta}}\left(\frac{1}{|Q_1||Q_2|}\iint_{Q_1\times Q_2}\sigma^{-\frac{p\vartheta}{p-1}}(u,v)\,du\,dv\right)^{\frac{p-1}{p\vartheta}}.\end{aligned} \tag{1.16}$$

Theorem 1.7. Let $I_{\alpha\beta\rho}$ be defined by (1.11)-(1.12) for $\alpha, \beta \in \mathbb{R}$ and $\rho \geq 0$. If $\rho \geq |\beta - \frac{1}{p} + \frac{1}{q}|$ and $\hat{A}^{\alpha\beta}_{pq\vartheta}(\omega, \sigma) < \infty$ for some $\vartheta > 1$, then

$$\left\|\omega I_{\alpha\beta\rho} f\right\|_{L^q(\mathbb{H}^n)} \leq C_{\alpha\beta\rho p q\vartheta} \hat{A}^{\alpha\beta}_{pq\vartheta}(\omega, \sigma) \left\|f\sigma\right\|_{L^p(\mathbb{H}^n)}, \quad 1 < p < q < \infty.$$

1.2. **Main contributions and structure.** Existing ϑ-bump theorems mainly focus on Euclidean spaces or single-parameter operators on Heisenberg groups. We now provide a detailed comparison between our research results and existing relevant works in the field. For the classic Sawyer-Wheeden theorem proposed in 1992, its research scope is limited to the N-dimensional Euclidean space $\mathbb{R}^N$, with the kernel type being single-parameter fractional integral, which is adapted to the single-parameter ϑ-bump condition, and its core innovation lies in being the first to establish the two-weight ϑ-bump theorem in the relevant research framework. In contrast, the Folland-Stein theorem published in 1974 shifts the research space to the Heisenberg group $\mathbb{H}^n$, still adopting the single-parameter fractional integral kernel, but without any adaptation to the ϑ-bump condition, and its key contribution is the establishment of the Hardy-Littlewood-Sobolev (HLS) theorem on the Heisenberg group. Our work further expands on these prior results: on the one hand, Theorems 1.4 and 1.5 of this paper are also set in the Heisenberg group $\mathbb{H}^n$ with single-parameter fractional integral kernels, which are adapted to the single-parameter ϑ-bump condition, and this part of the work is the first to establish the ϑ-bump theorem on the Heisenberg group, filling the research gap in this field; on the other hand, Theorems 1.6 and 1.7 of this paper still take $\mathbb{H}^n$ as the research space but adopt multi-parameter fractional integral kernels, which are adapted to the multi-parameter ϑ-bump condition, and the core innovation is the extension of the ϑ-bump theorem to the multi-parameter scenario with the assistance of Zygmund-type dilations. This work exhibits broad application potential, including the estimation of solutions for nonlinear integral equations on Heisenberg groups, the investigation of Sobolev embedding theorems pertaining to sub-elliptic equations, and the further extension to other nilpotent Lie groups that possess similar homogeneous structures.

The paper is organized in a logical structure as follows: Sections 2 to 5 are devoted to the proofs of Theorems 1.4 through 1.7 respectively, with each section focusing on the corresponding theoretical derivation. We conclude this section with some notational conventions. The symbol C denotes a positive constant that may vary from line to line but is independent of the main parameters. We write $C_{\alpha,\beta,\cdots}$ for constants depending explicitly on parameters $\alpha, \beta, \cdots$. The notation $A \lesssim B$ means $A \leq CB$ for some constant $C > 0$, and $A \sim B$ abbreviates $A \lesssim B \lesssim A$. Let $1 < p \leq q < \infty$. We denote ω^q and $\sigma^{-\frac{p}{p-1}}$ as non-negative, locally integrable functions defined on $\mathbb{R}^{2n}$. Denote $R = Q_1 \times Q_2 \times I \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, where Q_1, Q_2 are cubes parallel to the coordinates, I is an interval and $|I| = |Q_1|^{\frac{1}{n}} |Q_2|^{\frac{1}{n}}$ (multi-parameter volume matching). $R' = Q_1' \times Q_2' \times I'$ is a translation of R defined as

$$R' = \left\{(u, v, t) \colon u_i = \xi_i + 2|Q_1|^{\frac{1}{n}},\ v_i = \eta_i + 2|Q_2|^{\frac{1}{n}},\ i = 1, 2, \ldots, n, t = \tau + 2|I|, (\xi, \eta, \tau) \in R\right\}, \tag{1. 1}$$

where $u = (u_1, u_2, ..., u_n), v = (v_1, v_2, ..., v_n), \xi = (\xi_1, \xi_2, ..., \xi_n), \eta = (\eta_1, \eta_2, ..., \eta_n)$.

2. Proof of Theorem 1.4

By choosing a test function adapted to the cube Q and interval I, we derive the necessary condition for the weighted inequality, leveraging the homogeneous dimension of $\mathbb{H}^n$ to match the volume scaling.

Proof of Theorem 1.4. Let S_δ be defined by (1.8)-(1.9) for $0 < \delta < n+1$. By changing variable $\tau \to \tau - \mu(u\cdot\eta - v\cdot\xi)$, we rewrite $S_\delta f$ as:

$$\begin{aligned} S_\delta f(u,v,t) &= \int_{\mathbb{R}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f\left(\xi,\eta,\tau-\mu(u\cdot\eta-v\cdot\xi)\right)\Omega^\delta(u-\xi,v-\eta,t-\tau)\,d\xi\,d\eta\,d\tau \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f\left(\xi,\eta,\tau-\mu(u\cdot\eta-v\cdot\xi)\right)\frac{1}{(|u-\xi|^2+|v-\eta|^2+|t-\tau|)^{n+1-\delta}}\,d\xi\,d\eta\,d\tau\,. \end{aligned}$$

Let $Q\subset\mathbb{R}^{2n}$ be a cube parallel to coordinates, and $I\subset\mathbb{R}$ be an interval with $|I| = |Q|^{\frac{1}{n}}$. Define the test function:

$$f(u,v,t) = \sigma^{-\frac{p}{p-1}}(u,v)\mathbf{1}_Q(u,v)\mathbf{1}_I(t),$$

where $\mathbf{1}$ denotes the indicator function.

On the one hand, the right-hand side of $\|\omega S_\delta f\|_{L^q(\mathbb{H}^n)} \lesssim \|f\sigma\|_{L^p(\mathbb{H}^n)}$ satisfies:

$$\|f\sigma\|_{L^p(\mathbb{H}^n)} = \left(\int_{\mathbb{R}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|f(u,v,t)\sigma(u,v)|^p\,du\,dv\,dt\right)^{\frac{1}{p}} = |Q|^{\frac{1}{pn}}\left(\iint_Q\sigma^{-\frac{p}{p-1}}(u,v)\,du\,dv\right)^{\frac{1}{p}}.$$

On the other hand, the left-hand side of $\|\omega S_\delta f\|_{L^q(\mathbb{H}^n)} \lesssim \|f\sigma\|_{L^p(\mathbb{H}^n)}$ satisfies:

$$\begin{aligned} &\|\omega S_\delta f\|_{L^q(\mathbb{H}^n)} \\ &\geq \left\{\iiint_{Q\times I}\omega^q(u,v)\left[\iiint_{Q\times\mathbb{R}}\sigma^{-\frac{p}{p-1}}(\xi,\eta)\mathbf{1}_I(\tau-\mu(u\cdot\eta-v\cdot\xi))\right.\right. \\ &\qquad \left.\left.\times\ \frac{1}{(|u-\xi|^2+|v-\eta|^2+|t-\tau|)^{n+1-\delta}}d\xi d\eta d\tau\right]^q dudvdt\right\}^{\frac{1}{q}} \\ &\geq |Q|^{\frac{\delta}{n}-\frac{n+1}{n}}\left\{\iiint_{Q\times I}\omega^q(u,v)dudvdt\right\}^{\frac{1}{q}}\iint_Q\sigma^{-\frac{p}{p-1}}(\xi,\eta)\left\{\int_{\mathbb{R}}\mathbf{1}_I\left(\tau-\mu(u\cdot\eta-v\cdot\xi)\right)d\tau\right\}d\xi d\eta \\ &= |Q|^{\frac{\delta}{n}-\frac{n+1}{n}}\left\{\iiint_{Q\times I}\omega^q(u,v)dudvdt\right\}^{\frac{1}{q}}\iint_Q\sigma^{-\frac{p}{p-1}}(\xi,\eta)\left\{\int_{\mathbb{R}}\mathbf{1}_I(\tau)\,d\tau\right\}d\xi d\eta \\ &= |Q|^{\frac{\delta}{n}-\frac{n+1}{n}}|I|^{\frac{1}{q}+1}\left\{\iint_Q\omega^q(u,v)dudv\right\}^{\frac{1}{q}}\iint_Q\sigma^{-\frac{p}{p-1}}(u,v)dudv \\ &= |Q|^{\frac{\delta}{n}+\frac{1}{qn}-1}\left\{\iint_Q\omega^q(u,v)dudv\right\}^{\frac{1}{q}}\iint_Q\sigma^{-\frac{p}{p-1}}(u,v)dudv. \end{aligned}$$

This two-weight $L^p\to L^q$-norm inequality implies

$$\begin{aligned} |Q|^{\frac{\delta}{n}+\frac{1}{qn}-1}\left\{\iint_Q\omega^q(u,v)dudv\right\}^{\frac{1}{q}}\iint_Q\sigma^{-\frac{p}{p-1}}(u,v)dudv &\lesssim \left\{\iiint_{Q\times I}\sigma^{-\frac{p}{p-1}}(u,v)dudvdt\right\}^{\frac{1}{p}} \\ &= |Q|^{\frac{1}{pn}}\left\{\iint_Q\sigma^{-\frac{p}{p-1}}(u,v)dudv\right\}^{\frac{1}{p}} \end{aligned}$$

Combining the above two inequalities and canceling the common term $\iint_Q\sigma^{-\frac{p}{p-1}}(u,v)\,du\,dv$, we get:

$$|Q|^{\frac{\delta}{n}+\frac{1}{qn}-\frac{1}{pn}-1}\left\{\iint_Q\omega^q(u,v)dudv\right\}^{\frac{1}{q}}\left\{\iint_Q\sigma^{-\frac{p}{p-1}}(u,v)dudv\right\}^{\frac{p-1}{p}}$$

$$= |Q|^{\left[\frac{\delta}{n+1}-\frac{1}{p}+\frac{1}{q}\right]\frac{n+1}{n}} \left\{\frac{1}{|Q|}\iint_Q \omega^q(u,v)dudv\right\}^{\frac{1}{q}} \left\{\frac{1}{|Q|}\iint_Q \sigma^{-\frac{p}{p-1}}(u,v)dudv\right\}^{\frac{p-1}{p}} < \infty.$$

The above estimates hold for every $Q \subset \mathbb{R}^{2n}$. Substituting δ with the Folland-Stein condition $\frac{\delta}{n+1} = \frac{1}{p} - \frac{1}{q}$, we obtain $\hat{A}^{\delta}_{pq}(\omega,\sigma) < \infty$. □

3. Proof of Theorem 1.5

Decompose the kernel into spatial and temporal parts, apply the Hardy-Littlewood-Sobolev theorem to the temporal integral, and the Sawyer-Wheeden theorem to the spatial integral, linking the Euclidean ϑ-bump characteristic to the Heisenberg group setting.

Proof of Theorem 1.5. Recall the ϑ-bump characteristic $\hat{A}^{\delta}_{pq\vartheta}(\omega,\sigma)$. We first rewrite it as:

$$\begin{aligned} &\hat{A}^{\delta}_{pq\vartheta}(\omega,\sigma)\\ &= \sup_{Q\subset\mathbb{R}^{2n}} |Q|^{\frac{n+1}{n}\left[\frac{\delta}{n+1}-\left(\frac{1}{p}-\frac{1}{q}\right)\right]} \left(\frac{1}{|Q|}\iint_Q \omega^{q\vartheta}(u,v)\,du\,dv\right)^{\frac{1}{q\vartheta}} \left(\frac{1}{|Q|}\iint_Q \sigma^{-\frac{p\vartheta}{p-1}}(u,v)\,du\,dv\right)^{\frac{p-1}{p\vartheta}} \quad (3.\ 1)\\ &= \sup_{Q\subset\mathbb{R}^{2n}} |Q|^{\frac{1}{n}\left[\delta-\left(\frac{1}{p}-\frac{1}{q}\right)\right]-1+\left(1-\frac{1}{\vartheta}\right)\left[1-\frac{1}{p}+\frac{1}{q}\right]} \left(\iint_Q \omega^{q}(u,v)\,du\,dv\right)^{\frac{1}{q}} \left(\iint_Q \sigma^{-\frac{p}{p-1}}(u,v)\,du\,dv\right)^{\frac{p-1}{p}}. \end{aligned}$$

Remark 3.1. We essentially need to consider $\frac{\delta}{n+1} \geq \frac{1}{p} - \frac{1}{q}$. If not, suppose $\frac{\delta}{n+1} < \frac{1}{p} - \frac{1}{q}$. By shrinking Q to some (u_0,v_0) and applying Lebesgue differentiation theorem, the first line in (3. 1) equals $\lim_{|Q|\to 0} |Q|^{\left[\frac{\delta}{n+1}-\frac{1}{p}+\frac{1}{q}\right]\frac{n+1}{n}}\omega(u_0,v_0)\sigma^{-1}(u_0,v_0)$ which is finite only if $\omega(u_0,v_0)\sigma^{-1}(u_0,v_0) = 0$. Consequently, $\hat{A}^{\delta}_{pq\vartheta}(\omega,\sigma) < \infty$ requires that ω and σ^{-1} have disjoint support. This situation is excluded for which the weighted inequality in $\|\omega S_{\delta} f\|_{L^q(\mathbb{H}^n)} \lesssim \|f\sigma\|_{L^p(\mathbb{H}^n)}$ is trivially true.

Remark 3.2. We necessarily have $\frac{1}{n}\left[\delta - \frac{1}{p} + \frac{1}{q}\right] - 1 \leq -(1-\frac{1}{\vartheta})[1-\frac{1}{p}+\frac{1}{q}] < 0$. Indeed, this can be seen by taking $|Q| \to \infty$ in (3. 1).

Decompose $\delta = \delta_1 + \delta_2$ where $\delta_2 = \frac{1}{p} - \frac{1}{q}$. Now, we assert $\delta = \delta_1 + \delta_2$ where $\delta_2 = \frac{1}{p} - \frac{1}{q}$. For $1 < p < q < \infty$, we find $0 < \delta_2 < 1$. According to **Remark 3.1** and **Remark 3.2**, we have $(n+1)\left(\frac{1}{p}-\frac{1}{q}\right) \leq \delta < n + \left(\frac{1}{p}-\frac{1}{q}\right)$ which implies $0 < n\left(\frac{1}{p}-\frac{1}{q}\right) \leq \delta_1 = \delta - \left(\frac{1}{p}-\frac{1}{q}\right) < n$.

The kernel Ω^{δ} satisfies:

$$\begin{aligned} \Omega^{\delta}(u,v,t) &= \left(\frac{1}{|u|^2+|v|^2+|t|}\right)^{n+1-\delta}\\ &\leq \left(\frac{1}{\sqrt{|u|^2+|v|^2}}\right)^{2n-2\delta_1}\left(\frac{1}{|t|}\right)^{1-\delta_2}, \qquad (u,v)\neq(0,0) \text{ and } t\neq 0. \end{aligned}$$

Because Ω^{δ} is positively definite, it is suffice to assert $f \geq 0$. We have

$$\begin{aligned} \|\omega S_{\delta} f\|_{L^q(\mathbb{H}^n)} \leq \Bigg\{\iiint_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}} \omega^q(u,v)\Bigg[\iiint_{\mathbb{R}^{2n+1}} f(\xi,\eta,\tau-\mu(u\cdot\eta-v\cdot\xi))\\ \times\left(\frac{1}{\sqrt{|u-\xi|^2+|v-\eta|^2}}\right)^{2n-2\delta_1}\left(\frac{1}{|t-\tau|}\right)^{1-\delta_2} d\xi d\eta d\tau\Bigg]^q dudvdt\Bigg\}^{\frac{1}{q}} \end{aligned}$$

$$\leq \left\{ \iint_{\mathbb{R}^n\times\mathbb{R}^n} \omega^q(u,v) \left\{ \iint_{\mathbb{R}^n\times\mathbb{R}^n} \left\{ \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(\xi,\eta,\tau-\mu(u\cdot\eta-v\cdot\xi)) \left(\frac{1}{|t-\tau|}\right)^{1-\delta_2} d\tau \right]^q dt \right\}^{\frac{1}{q}} \right.\right.$$

$$\left.\left. \times \left(\frac{1}{\sqrt{|u-\xi|^2+|v-\eta|^2}} \right)^{2n-2\delta_1} d\xi d\eta \right\}^q dudv \right\}^{\frac{1}{q}} \text{ by Minkowski's integral inequality.}$$

By the Hardy-Littlewood-Sobolev theorem (for $N = 1$, $\alpha = \delta_2$):

$$\|\omega S_\delta f\|_{L^q(\mathbb{H}^n)}$$

$$\leq \left\{ \iint_{\mathbb{R}^n\times\mathbb{R}^n} \omega^q(u,v) \left\{ \iint_{\mathbb{R}^n\times\mathbb{R}^n} \left\{ \int_{\mathbb{R}} [f(\xi,\eta,t-\mu(u\cdot\eta-v\cdot\xi))]^p dt \right\}^{\frac{1}{p}} \right.\right.$$

$$\left.\left. \times \left(\frac{1}{\sqrt{|u-\xi|^2+|v-\eta|^2}} \right)^{2n-2\delta_1} d\xi d\eta \right\}^q dudv \right\}^{\frac{1}{q}} \tag{3. 2}$$

$$= \left\{ \iint_{\mathbb{R}^{2n}} \omega^q(u,v) \left[\iint_{\mathbb{R}^{2n}} \|f(\xi,\eta,\cdot)\|_{L^p(\mathbb{R})} \left(\frac{1}{\sqrt{|u-\xi|^2+|v-\eta|^2}} \right)^{2n-2\delta_1} d\xi\, d\eta \right]^q du\, dv \right\}^{\frac{1}{q}}.$$

Because $\delta_1 = \delta - \delta_2 = \delta - \left(\frac{1}{p} - \frac{1}{q}\right)$, we find

$$\frac{2\delta_1}{2n} - \frac{1}{p} + \frac{1}{q} = \frac{\delta}{n} - \left(\frac{1}{n}+1\right)\left(\frac{1}{p}-\frac{1}{q}\right) = \left(\frac{\delta}{n+1} - \frac{1}{p} + \frac{1}{q}\right)\frac{n+1}{n}. \tag{3. 3}$$

Therefore, $\hat{A}^{\delta}_{pq\vartheta}(\omega,\sigma)$ equals

$$\hat{A}^{2\delta_1}_{pq\vartheta}(\omega,\sigma) = \sup_{Q\subset\mathbb{R}^{2n}} |Q|^{\frac{2\delta_1}{2n}-\frac{1}{p}+\frac{1}{q}} \left[\frac{1}{|Q|} \iint_Q \omega^{q\vartheta}(u,v)dudv \right]^{\frac{1}{q\vartheta}} \left[\frac{1}{|Q|} \iint_Q \sigma^{-\frac{p\vartheta}{p-1}}(u,v)dudv \right]^{\frac{p-1}{p\vartheta}} < \infty. \tag{3. 4}$$

From (3. 2), (3. 3) and (3. 4), by applying the Sawyer-Wheeden theorem to the $2n$-dimensional Euclidean space (with $\alpha = 2\delta_1$):

$$\|\omega S_\delta f\|_{L^q(\mathbb{H}^n)} \leq C_{\delta pq\vartheta}\, \hat{A}^{2\delta_1}_{pq\vartheta}(\omega,\sigma)\, \|f\sigma\|_{L^p(\mathbb{H}^n)} = C_{\delta pq\vartheta}\, \hat{A}^{\delta}_{pq\vartheta}(\omega,\sigma)\, \|f\sigma\|_{L^p(\mathbb{H}^n)}, \quad 1 < p < q < \infty.$$

Note that $A^{2\delta_1}_{pq\vartheta}(\omega,\sigma) = \hat{A}^{\delta}_{pq\vartheta}(\omega,\sigma)$, hence the theorem holds. □

4. Proof of Theorem 1.6

The proof of Theorem 1.6 extends the single-parameter test function method to the multi-parameter framework, and leverages dilation invariance to derive the necessary condition imposed on ρ.

Proof of Theorem 1.6. By changing variable $\tau \to \tau - \mu(u\cdot\eta - v\cdot\xi)$, the operator $I_{\alpha\beta\rho}$ becomes:

$$I_{\alpha\beta\rho}f(u,v,t) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\xi,\eta,\tau-\mu(u\cdot\eta-v\cdot\xi))|u-\xi|^{\alpha-n}|v-\eta|^{\alpha-n}|t-\tau|^{\beta-1}$$

$$\times \left(\frac{|u-\xi||v-\eta|}{|t-\tau|} + \frac{|t-\tau|}{|u-\xi||v-\eta|} \right)^{-\rho} d\xi\, d\eta\, d\tau\,.$$

Define the test function:

$$f(u,v,t) = \sigma^{-\frac{p}{p-1}}(u,v)\mathbf{1}_{Q_1\times Q_2}(u,v)\chi_I(t),$$

where $\mathbf{1}$ is an indicator function.

Substitute f into (1.13) and estimate the left-hand side:

$$\begin{aligned}
&\left\|\omega I_{\alpha\beta\rho}f\right\|_{L^q(\mathbb{H}^n)}\\
\geq&\left\{\iiint_{Q_1'\times Q_2'\times\{I'+\mu(u\cdot\eta-v\cdot\xi)\}}\omega^q(u,v)\left[\iiint_{Q_1\times Q_2\times\mathbb{R}}\sigma^{-\frac{p}{p-1}}(\xi,\eta)\mathbf{1}_I(\tau-\mu(u\cdot\eta-v\cdot\xi))\right.\right.\\
&\left.\left.\times\ |u-\xi|^{\alpha-n}|v-\eta|^{\alpha-n}|t-\tau|^{\beta-1}\left(\frac{|u-\xi||v-\eta|}{|t-\tau|}+\frac{|t-\tau|}{|u-\xi||v-\eta|}\right)^{-\rho}d\xi d\eta d\tau\right]^q dudvdt\right\}^{\frac{1}{q}}\\
\geq&|Q_1|^{\frac{\alpha}{n}-1}|Q_2|^{\frac{\alpha}{n}-1}|I|^{\beta-1}\\
&\times\left[\iint_{Q_1'\times Q_2'}\omega^q(u,v)dudv\int_{I'+\mu(u\cdot\eta-v\cdot\xi)}dt\right]^{\frac{1}{q}}\iint_{Q_1\times Q_2}\sigma^{-\frac{p}{p-1}}(\xi,\eta)d\xi d\eta\int_{\mathbb{R}}\mathbf{1}_I(\tau-\mu(u\cdot\eta-v\cdot\mu))d\tau\\
\geq&|Q_1|^{\frac{\alpha}{n}-1}|Q_2|^{\frac{\alpha}{n}-1}|I|^{\beta+\frac{1}{q}}\times\left(\iint_{Q_1'\times Q_2'}\omega^q(u,v)\,du\ dv\right)^{\frac{1}{q}}\iint_{Q_1\times Q_2}\sigma^{-\frac{p}{p-1}}(\xi,\eta)\,d\xi\ d\eta\,.
\end{aligned}$$

Recall (1.13), the two-weight $L^p\to L^q$-norm inequality implies

$$\begin{aligned}
&|Q_1|^{\frac{\alpha}{n}-1}|Q_2|^{\frac{\alpha}{n}-1}|I|^{\beta-1+\frac{1}{q}+1}\left\{\iint_{Q_1'\times Q_2'}\omega^q(u,v)dudv\right\}^{\frac{1}{q}}\iint_{Q_1\times Q_2}\sigma^{-\frac{p}{p-1}}(u,v)dudv\\
\lesssim&\left\{\iiint_{Q_1\times Q_2\times I}\sigma^{-\frac{p}{p-1}}(u,v)dudvdt\right\}^{\frac{1}{p}}\\
=&|I|^{\frac{1}{p}}\left\{\iint_{Q_1\times Q_2}\sigma^{-\frac{p}{p-1}}(u,v)dudv\right\}^{\frac{1}{p}}.
\end{aligned}$$

By taking into account $|I|=|Q_1|^{\frac{1}{n}}|Q_2|^{\frac{1}{n}}$, we find

$$\begin{aligned}
&|Q_1|^{\frac{\alpha}{n}-1}|Q_2|^{\frac{\alpha}{n}-1}|I|^{\beta+\frac{1}{q}-\frac{1}{p}}\left[\iint_{Q_1'\times Q_2'}\omega^q(u,v)dudv\right]^{\frac{1}{q}}\left[\iint_{Q_1\times Q_2}\sigma^{-\frac{p}{p-1}}(u,v)dudv\right]^{\frac{p-1}{p}}\\
=&|Q_1|^{\left[\frac{\alpha+\beta}{n+1}-\left(\frac{1}{p}-\frac{1}{q}\right)\right]\frac{n+1}{n}}|Q_2|^{\left[\frac{\alpha+\beta}{n+1}-\left(\frac{1}{p}-\frac{1}{q}\right)\right]\frac{n+1}{n}}\\
&\times\left\{\frac{1}{|Q_1'||Q_2'|}\iint_{Q_1'\times Q_2'}\omega^q(u,v)dudv\right\}^{\frac{1}{q}}\left\{\frac{1}{|Q_1||Q_2|}\iint_{Q_1\times Q_2}\sigma^{-\frac{p}{p-1}}(u,v)dudv\right\}^{\frac{p-1}{p}}.
\end{aligned}$$

The above estimates hold for every $Q_1\times Q_2\subset\mathbb{R}^n\times\mathbb{R}^n$. This further implies (1.15).

Next, we show $\vartheta\geq\left|\beta-\frac{1}{p}+\frac{1}{q}\right|$. By changing dilations $(u,v,t)\to(u,v,\lambda t)$ and $(\xi,\eta,\tau)\to(\xi,\eta,\lambda\tau)$ for $0<\lambda<1$ or $\lambda>1$, we have

$$\begin{aligned}
&\left\{\iiint_{\mathbb{R}^{\times}\mathbb{R}^n\times\mathbb{R}}\omega^q(u,v)\left[\iiint_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}}f\left(\xi,\eta,\lambda^{-1}(\tau-\mu\lambda(u\cdot\eta-v\cdot\xi))\right)\right.\right.\\
&\left.\left.|u-\xi|^{\alpha-n}|v-\eta|^{\alpha-n}|t-\tau|^{\beta-1}\left(\frac{|u-\xi||v-\eta|}{|t-\tau|}+\frac{|t-\tau|}{|u-\xi||v-\eta|}\right)^{-\rho}d\xi d\eta d\tau\right]^q dudvdt\right\}^{\frac{1}{q}}\\
&=\lambda^{\beta}\lambda^{\frac{1}{q}}\left\{\iiint_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}}\omega^q(u,v)\left\{\iiint_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}}f\left(\xi,\eta,\tau-\mu(u\cdot\eta-v\cdot\xi)\right)\right.\right.
\end{aligned}\tag{4. 1}$$

$$|u-\xi|^{\alpha-n}|v-\eta|^{\alpha-n}|t-\tau|^{\beta-1}\left[\frac{|u-\xi||v-\eta|}{\lambda|t-\tau|}+\frac{\lambda|t-\tau|}{|u-\xi||v-\eta|}\right]^{-\vartheta}d\xi d\eta d\tau\Bigg\}^{q}dudvdt\Bigg\}^{\frac{1}{q}}.$$

For $0<\lambda<1$, the two-weight $L^p\to L^q$-norm inequality in (1.13) suggests that the preceding formula (4. 1) is greater than or equal to the subsequent formula.

$$\lambda^{\beta}\lambda^{\frac{1}{q}}\lambda^{\rho}\left\{\iiint_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}}\omega^q(u,v)\left[\iiint_{\mathbb{R}^{2n+1}}f(\xi,\eta,\tau-\mu(u\cdot\eta-v\cdot\xi))\right.\right.$$

$$\left.\left.|u-\xi|^{\alpha-n}|v-\eta|^{\alpha-n}|t-\tau|^{\beta-1}\Big(\frac{|u-\xi||v-\eta|}{|t-\tau|}+\frac{|t-\tau|}{|u-\xi||v-\eta|}\Big)^{-\rho}d\xi d\eta d\tau\right]^{q}dudvdt\right\}^{\frac{1}{q}}$$

$$\lesssim\lambda^{\beta}\lambda^{\frac{1}{q}}\lambda^{\rho}\lambda^{\frac{1}{p}}\left[\iiint_{\mathbb{R}^{2n+1}}\Big|f(u,v,t)\,\sigma(u,v)\Big|^{p}dudvdt\right]^{\frac{1}{p}}.$$

For $\lambda>1$, the two-weight $L^p\to L^q$-norm inequality in (1.13) suggests that the preceding formula (4. 1) is greater than or equal to the subsequent formula.

$$\lambda^{\beta}\lambda^{\frac{1}{q}}\lambda^{-\rho}\left\{\iiint_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}}\omega^q(u,v)\left[\iiint_{\mathbb{R}^{2n+1}}f(\xi,\eta,\tau-\mu(u\cdot\eta-v\cdot\xi))\right.\right.$$

$$\left.\left.|u-\xi|^{\alpha-n}|v-\eta|^{\alpha-n}|t-\tau|^{\beta-1}\Big(\frac{|u-\xi||v-\eta|}{|t-\tau|}+\frac{|t-\tau|}{|u-\xi||v-\eta|}\Big)^{-\rho}d\xi d\eta d\tau\right]^{q}dudvdt\right\}^{\frac{1}{q}}$$

$$\lesssim\lambda^{\beta}\lambda^{\frac{1}{q}}\lambda^{-\rho}\lambda^{\frac{1}{p}}\left[\iiint_{\mathbb{R}^{2n+1}}\Big|f(u,v,t)\,\sigma(u,v)\Big|^{p}dudvdt\right]^{\frac{1}{p}}.$$

This assertion holds for all $0<\lambda<1$ or $\lambda>1$. It then follows necessarily that

$$\beta+\rho\geq\frac{1}{p}-\frac{1}{q}\tag{4. 2}$$

or

$$\beta-\rho\leq\frac{1}{p}-\frac{1}{q}\tag{4. 3}$$

Combining the two inequalities (4. 2) and (4. 3), we deduce that

$$\rho\geq\left|\beta-\frac{1}{p}+\frac{1}{q}\right|.$$

□

5. Proof of Theorem 1.7

Our multi-parameter ϑ-bump characteristic is inspired by the product fractional integral results in [18]. To establish the multi-parameter ϑ-bump theorem, we first introduce several auxiliary lemmas for product operators and dyadic grids.

Let $\mathcal{D}$ denote the grid of dyadic cubes in $\mathbb{R}^n$, and let $\mathcal{R}^{2n}:=\mathcal{D}^n\times\mathcal{D}^n$ denote the partial grid of dyadic rectangles in $\mathbb{R}^n\times\mathbb{R}^n$ (which is not actually a grid since it fails the nested property). For a measure $d\mu=u(x)dx$ absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^n$, we define the following ϑ-bump functional for a cube Q and $\vartheta>1$:

$$\mathrm{vol}\{Q\}_{\mu,\vartheta}:=|Q|^{\frac{1}{\vartheta'}}\left(\int_Q u^{\vartheta}(x)dx\right)^{\frac{1}{\vartheta}},\tag{5. 1}$$

where $\frac{1}{\vartheta}+\frac{1}{\vartheta'}=1$. Note that $\mathrm{vol}\{Q\}_\mu=\mathrm{vol}\{Q\}_{\mu,1}\le \mathrm{vol}\{Q\}_{\mu,\vartheta}$. If $P=\bigcup_{i=1}^\infty Q_i$ is a pairwise disjoint union of cubes Q_i, then

$$
\begin{aligned}
\sum_{i=1}^{\infty}\mathrm{vol}\{Q_i\}_{\mu,\vartheta} &= \sum_{i=1}^{\infty}|Q_i|^{\frac{1}{\vartheta'}}\left(\int_{Q_i}u^{\vartheta}\right)^{\frac{1}{\vartheta}}\\
&\le \left(\sum_{i=1}^{\infty}|Q_i|\right)^{\frac{1}{\vartheta'}}\left(\sum_{i=1}^{\infty}\int_{Q_i}u^{\vartheta}\right)^{\frac{1}{\vartheta}}\\
&= |P|^{\frac{1}{\vartheta'}}\left(\int_{P}u^{\vartheta}\right)^{\frac{1}{\vartheta}}=\mathrm{vol}\{P\}_{\mu,\vartheta}.
\end{aligned}
\tag{5. 2}
$$

A key property of the ϑ-bump functional on cubes for our purposes is that, when raised to a power greater than 1, it automatically satisfies a Carleson condition over all dyadic subcubes. More precisely, if $r>1$, then

$$
\begin{aligned}
\sum_{\substack{Q\in\mathcal{D}^n\\ Q\subset P}}\mathrm{vol}\{Q\}_{\mu,\vartheta}^{r} &= \sum_{k=0}^{\infty}\sum_{\substack{Q\in\mathcal{D}^n\\ \ell(Q)=2^{-k}\ell(P)}}|Q|^{\frac{r-1}{\vartheta'}}\left(\int_{Q}u^{\vartheta}\right)^{\frac{r-1}{\vartheta}}\mathrm{vol}\{Q\}_{\mu,\vartheta}\\
&\le \sum_{k=0}^{\infty}\sum_{\substack{Q\in\mathcal{D}^n\\ \ell(Q)=2^{-k}\ell(P)}}\left(C2^{-kn}|P|\right)^{\frac{r-1}{\vartheta'}}\left(\int_{P}u^{\vartheta}\right)^{\frac{r-1}{\vartheta}}\mathrm{vol}\{Q\}_{\mu,\vartheta}\\
&\le \sum_{k=0}^{\infty}\left(C2^{-kn}|P|\right)^{\frac{r-1}{\vartheta'}}\left(\int_{P}u^{\vartheta}\right)^{\frac{r-1}{\vartheta}}\mathrm{vol}\{P\}_{\mu,\vartheta}\\
&= C_{r,\vartheta'}\mathrm{vol}\{P\}_{\mu,\vartheta}^{r},
\end{aligned}
\tag{5. 3}
$$

where C and $C_{r,\vartheta'}$ are positive constants depending only on their subscripts.

Lemma 5.1. *Suppose that $1<s<t<\infty$, and let $d\mu(x)=u(x)dx$ be a locally L^{ϑ}-absolutely continuous measure on $\mathbb{R}^n$. Then for all non-negative $f\in L^s(\mu)$, we have*

$$
\left[\sum_{Q\in\mathcal{D}^n} vol\{Q\}_{\mu,\vartheta}^{\frac{t}{s}}\left(\frac{1}{vol\{Q\}_{\mu,\vartheta}}\int_Q f d\mu\right)^{t}\right]^{\frac{1}{t}}\le C_{t,s,\vartheta}\|f\|_{L^s(\mu)},
\tag{5. 4}
$$

where $C_{t,s,\vartheta}$ is a positive constant depending only on t,s,ϑ and n.

Proof. Since $\mathcal{D}^n$ is a grid, for each integer $k\in\mathbb{Z}$, we consider the maximal dyadic cubes $\{\mathcal{M}_i^k\}_{i=1}^\infty\subset\mathcal{D}^n$ such that

$$
\frac{1}{\mathrm{vol}\{\mathcal{M}_i^k\}_{\mu,\vartheta}}\int_{\mathcal{M}_i^k} f d\mu>2^k.
\tag{5. 5}
$$

With the help of (5. 3), we proceed to estimate:

$$
\begin{aligned}
\sum_{Q\in\mathcal{D}^n} \mathrm{vol}\{Q\}_{\mu,\vartheta}^{\frac{t}{s}} \left(\frac{1}{\mathrm{vol}\{Q\}_{\mu,\vartheta}} \int_Q f d\mu\right)^t &\le \sum_{k=-\infty}^{\infty} \sum_{\substack{Q\in\mathcal{D}^n \\ 2^k < \frac{1}{\mathrm{vol}\{Q\}_{\mu,\vartheta}} \int_Q f d\mu \le 2^{k+1}}} \mathrm{vol}\{Q\}_{\mu,\vartheta}^{\frac{t}{s}} (2^{k+1})^t \\
&\le \sum_{k=-\infty}^{\infty} \sum_{i=1}^{\infty} \sum_{\substack{Q\in\mathcal{D}^n \\ Q\subset \mathcal{M}_i^k}} \mathrm{vol}\{Q\}_{\mu,\vartheta}^{\frac{t}{s}} 2^{t(k+1)} \\
&= 2^t \sum_{k=-\infty}^{\infty} \sum_{i=1}^{\infty} \left(\sum_{\substack{Q\in\mathcal{D}^n \\ Q\subset \mathcal{M}_i^k}} \mathrm{vol}\{Q\}_{\mu,\vartheta}^{\frac{t}{s}} \right) 2^{kt} \\
&\le 2^t C_{t,s,\vartheta'} \sum_{k=-\infty}^{\infty} \sum_{i=1}^{\infty} \mathrm{vol}\{\mathcal{M}_i^k\}_{\mu,\vartheta}^{\frac{t}{s}} 2^{kt},
\end{aligned}
\tag{5. 6}
$$

where $C_{t,s,\vartheta'}$ is the constant from (5. 3) with $r = \frac{t}{s}$.

Next, we note that

$$
\begin{aligned}
\frac{1}{\mathrm{vol}\{\mathcal{M}_i^k\}_{\mu,\vartheta}} \int_{\mathcal{M}_i^k \cap \{f > 2^{k-1}\}} f d\mu &= \frac{1}{\mathrm{vol}\{\mathcal{M}_i^k\}_{\mu,\vartheta}} \int_{\mathcal{M}_i^k} f d\mu - \frac{1}{\mathrm{vol}\{\mathcal{M}_i^k\}_{\mu,\vartheta}} \int_{\mathcal{M}_i^k \cap \{f \le 2^{k-1}\}} f d\mu \\
&\ge \frac{1}{\mathrm{vol}\{\mathcal{M}_i^k\}_{\mu,\vartheta}} \int_{\mathcal{M}_i^k} f d\mu - \frac{1}{\mathrm{vol}\{\mathcal{M}_i^k\}_{\mu,\vartheta}} \int_{\mathcal{M}_i^k} 2^{k-1} d\mu \\
&> 2^k - 2^{k-1} \cdot \frac{\mathrm{vol}\{\mathcal{M}_i^k\}_{\mu}}{\mathrm{vol}\{\mathcal{M}_i^k\}_{\mu,\vartheta}} \\
&\ge 2^{k-1},
\end{aligned}
\tag{5. 7}
$$

where the last inequality follows from $\mathrm{vol}\{\mathcal{M}_i^k\}_{\mu} \le \mathrm{vol}\{\mathcal{M}_i^k\}_{\mu,\vartheta}$.

Combining (5. 7) with the previous estimate, we obtain:

$$
\begin{aligned}
\sum_{Q\in\mathcal{D}^n} \mathrm{vol}\{Q\}_{\mu,\vartheta}^{\frac{t}{s}} \left(\frac{1}{\mathrm{vol}\{Q\}_{\mu,\vartheta}} \int_Q f d\mu\right)^t &\le 2^t C_{t,s,\vartheta'} \sum_{k=-\infty}^{\infty} \sum_{i=1}^{\infty} \mathrm{vol}\{\mathcal{M}_i^k\}_{\mu,\vartheta}^{\frac{t}{s}} 2^{kt} \\
&\le C_{t,s,\vartheta}^t \sum_{k=-\infty}^{\infty} \sum_{i=1}^{\infty} \left(2^{-k} \int_{\mathcal{M}_i^k \cap \{f > 2^{k-1}\}} f d\mu\right)^{\frac{t}{s}} 2^{kt} \\
&\le C_{t,s,\vartheta}^t \left(\sum_{k=-\infty}^{\infty} \sum_{i=1}^{\infty} 2^{k(s-1)} \int_{\mathcal{M}_i^k \cap \{f > 2^{k-1}\}} f d\mu \right)^{\frac{t}{s}}.
\end{aligned}
$$

Since the cubes $\{\mathcal{M}_i^k\}_{i=1}^{\infty}$ are pairwise disjoint for each k, we have:

$$
\begin{aligned}
\left(\sum_{k=-\infty}^{\infty}\sum_{i=1}^{\infty}2^{k(s-1)}\int_{\mathcal{M}_i^k\cap\{f>2^{k-1}\}}f\,d\mu\right)^{\frac{t}{s}} &\leq \left(\sum_{k=-\infty}^{\infty}2^{k(s-1)}\int_{\{f>2^{k-1}\}}f\,d\mu\right)^{\frac{t}{s}}\\
&=\left(\int_{\mathbb{R}^n}\Big(\sum_{\substack{k\in\mathbb{Z}\\ 2^k<2f(x)}}2^{k(s-1)}\Big)f(x)d\mu(x)\right)^{\frac{t}{s}}\\
&\leq C_s\left(\int_{\mathbb{R}^n}f(x)^{s-1}f(x)d\mu(x)\leq C_s\int_{\mathbb{R}^n}f(x)^s d\mu(x)\right)^{\frac{t}{s}}\\
&=C_s\|f\|_{L^s(\mu)}^t.
\end{aligned}
$$

Substituting back completes the proof. □

The following theorem is a variation of the Tanaka-Yabuta theorem involving general weights satisfying a ϑ-bump analogue of the "rectangle testing" condition in [21]. We extend the ϑ-bump functional to rectangles in $\mathbb{R}^n\times\mathbb{R}^n$ by

$$\mathrm{vol}\{R\}_{\mu,\vartheta}:=|R|^{\frac{1}{\vartheta'}}\left(\int_R u^{\vartheta}\right)^{\frac{1}{\vartheta}},$$

for a measure $d\mu(x,y)=u(x,y)dxdy$ absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^n\times\mathbb{R}^n$ and a rectangle $R\subset\mathbb{R}^n\times\mathbb{R}^n$.

Let $R:=Q_1\times Q_2\subset\mathbb{R}^n\times\mathbb{R}^n$, where Q_1,Q_2 are axis-aligned cubes. A translation of R is defined as

$$R':=\Big\{(x,y)\colon x_i=u_i+2|Q_1|^{\frac{1}{n}},\ y_i=v_i+2|Q_2|^{\frac{1}{n}},\ i=1,\ldots,n,\quad (u,v)\in R\Big\}. \tag{5. 8}$$

Proposition 5.2. *Suppose* $1<s<t<\infty$, $\vartheta>1$*, and let* μ *be a locally finite absolutely continuous measure on* $\mathbb{R}^n\times\mathbb{R}^n$*. Then*

$$\left\{\sum_{R\in\mathcal{R}^{2n}}\mathrm{vol}\{R\}_{\mu,\vartheta}^{\frac{t}{s}}\left(\frac{1}{\mathrm{vol}\{R\}_{\mu,\vartheta}}\int_R f\,d\mu\right)^t\right\}^{\frac{1}{t}}\leq C_{s,t,\vartheta}\|f\|_{L^s(\mu)},\qquad f\geq 0.$$

Proof. Let $d\mu(x,y)=\omega(x,y)dxdy$. For a.e. $x\in\mathbb{R}^n$, define $\omega^y(x):=\omega(x,y)$, $\quad d\mu^y(x):=\omega^y(x)dx$, and a.e. $y\in\mathbb{R}^n$, define $\omega_x(y):=\omega(x,y)$, $\quad d\mu_x(y):=\omega_x(y)dy$. Note that let

$$\mathrm{vol}\{Q_2\}_{\mu_x,\vartheta}:=|Q_2|^{\frac{1}{\vartheta'}}\left(\int_{Q_2}\omega_x(y)^{\vartheta}dy\right)^{\frac{1}{\vartheta}},\ \text{for a.e. } x\in\mathbb{R}^n;$$

$$\mathrm{vol}\{Q_1\}_{\mu^y,\vartheta}:=|Q_1|^{\frac{1}{\vartheta'}}\left(\int_{Q_1}\omega^y(x)^{\vartheta}dx\right)^{\frac{1}{\vartheta}},\ \text{for a.e. } y\in\mathbb{R}^n;.$$

Take $f\in L^s(\mathbb{R}^{2n},d\mu)$ and define for a.e. $x\in\mathbb{R}^n$,

$$F^{Q_2}(x):=\frac{1}{\mathrm{vol}\{Q_2\}_{\mu_x,\vartheta}}\int_{Q_2}f(x,y)\omega(x,y)dy.$$

First, we compute the volume of the product cube $Q_1 \times Q_2$ with respect to μ and ϑ:

$$\begin{aligned}\mathrm{vol}\{Q_1 \times Q_2\}_{\mu,\vartheta} &= |Q_1 \times Q_2|^{\frac{1}{\vartheta'}} \left(\int_{Q_1} \int_{Q_2} \omega(x,y)^{\vartheta} dydx \right)^{\frac{1}{\vartheta}} \\ &= |Q_1|^{\frac{1}{\vartheta'}} \left(\int_{Q_1} \left[|Q_2|^{\frac{1}{\vartheta'}} \left(\int_{Q_2} \omega(x,y)^{\vartheta} dy \right)^{\frac{1}{\vartheta}} \right]^{\vartheta} dx \right)^{\frac{1}{\vartheta}}.\end{aligned}$$

By the definition of $\mathrm{vol}\{Q_2\}_{\mu_x,\vartheta}$, the term in brackets equals $\mathrm{vol}\{Q_2\}_{\mu_x,\vartheta}$. Thus,

$$\mathrm{vol}\{Q_1 \times Q_2\}_{\mu,\vartheta} = \mathrm{vol}\{Q_1\}^{\frac{1}{\vartheta'}} \left(\int_{Q_1} \mathrm{vol}\{Q_2\}^{\vartheta}_{\mu_x,\vartheta} dx \right)^{\frac{1}{\vartheta}} =: \mathrm{vol}\{Q_1\}_{Q_{2\mu,\vartheta},\vartheta},$$

where $Q_{2\mu,\vartheta}$ is the absolutely continuous measure on $\mathbb{R}^n$ with density $Q_{2\mu,\vartheta}(x) := \mathrm{vol}\{Q_2\}_{\mu_x,\vartheta}$.

We now estimate the left-hand side of the desired inequality:

$$\begin{aligned}&\sum_{R \in \mathcal{R}^{2n}} \mathrm{vol}\{R\}^{\frac{t}{s}}_{\mu,\vartheta} \left(\frac{1}{\mathrm{vol}\{R\}_{\mu,\vartheta}} \int_R f(x,y)\omega(x,y)dxdy \right)^t \\ &= \sum_{Q_1 \times Q_2 \in \mathcal{R}^{2n}} \mathrm{vol}\{Q_1 \times Q_2\}^{\frac{t}{s}}_{\mu,\vartheta} \left(\frac{1}{\mathrm{vol}\{Q_1 \times Q_2\}_{\mu,\vartheta}} \iint_{Q_1 \times Q_2} f(x,y)\omega(x,y)dxdy \right)^t \\ &= \sum_{Q_2 \in \mathcal{D}^n} \sum_{Q_1 \in \mathcal{D}^n} \mathrm{vol}\{Q_1\}^{\frac{t}{s}}_{Q_{2\mu,\vartheta},\vartheta} \left(\frac{1}{\mathrm{vol}\{Q_1\}_{Q_{2\mu,\vartheta},\vartheta}} \int_{Q_1} \left(\frac{1}{Q_{2\mu,\vartheta}(x)} \int_{Q_2} f(x,y)\omega(x,y)dy \right) Q_{2\mu,\vartheta}(x)dx \right)^t \qquad (5.\ 9) \\ &= \sum_{Q_2 \in \mathcal{D}^n} \left[\sum_{Q_1 \in \mathcal{D}^n} \mathrm{vol}\{Q_1\}^{\frac{t}{s}}_{Q_{2\mu,\vartheta},\vartheta} \left(\frac{1}{\mathrm{vol}\{Q_1\}_{Q_{2\mu,\vartheta},\vartheta}} \int_{Q_1} F^{Q_2}(x) Q_{2\mu,\vartheta}(x)dx \right)^t \right] \\ &\lesssim \sum_{Q_2 \in \mathcal{D}^n} \left(\int_{\mathbb{R}^n} F^{Q_2}(x)^s Q_{2\mu,\vartheta}(x)dx \right)^{\frac{t}{s}},\end{aligned}$$

where the last step follows from Lemma 5.1 applied to the locally finite absolutely continuous measures $Q_{2\mu,\vartheta}$ on $\mathbb{R}^n$ and $Q_2 \in \mathcal{D}^n$.

By Minkowski's inequality, raising the above sum to the power $\frac{s}{t}$ gives

$$\begin{aligned}\left(\sum_{\mathbb{Q}_2 \in \mathcal{D}^n} \left(\int_{\mathbb{R}^n} F^{Q_2}(x)^s Q_{2\mu,\vartheta}(x)dx \right)^{\frac{t}{s}} \right)^{\frac{s}{t}} &\le \int_{\mathbb{R}^n} \left(\sum_{Q_2 \in \mathcal{D}^n} \left(F^{Q_2}(x)^s \right)^{\frac{t}{s}} \right)^{\frac{s}{t}} Q_{2\mu,\vartheta}(x)dx \\ &= \int_{\mathbb{R}^n} \left(\sum_{Q_2 \in \mathcal{D}^n} Q_{2\mu,\vartheta}(x)^{\frac{t}{s}} F^{Q_2}(x)^t \right)^{\frac{s}{t}} dx.\end{aligned}$$

Applying Lemma 5.1 again to the locally finite absolutely continuous measures μ_x on $\mathbb{R}^n$ (for a.e. $x \in \mathbb{R}^n$), we obtain

$$\begin{aligned}\sum_{Q_2 \in \mathcal{D}^n} Q_{2\mu,\vartheta}(x)^{\frac{t}{s}} F^{Q_2}(x)^t &= \sum_{Q_2 \in \mathcal{D}^n} \mathrm{vol}\{Q_2\}^{\frac{t}{s}}_{\mu_x,\vartheta} \left(\frac{1}{\mathrm{vol}\{Q_2\}_{\mu_x,\vartheta}} \int_{Q_2} f(x,y)\omega_x(y)dy \right)^t \\ &\lesssim \left(\int_{\mathbb{R}^n} f(x,y)^s \omega_x(y)dy \right)^{\frac{t}{s}},\end{aligned}$$

uniformly for a.e. $x \in \mathbb{R}^n$.

Substituting this into the previous inequality yields

$$\left(\sum_{Q_2\in\mathcal{D}^n}\left(\int_{\mathbb{R}^n}F^{Q_2}(x)^s Q_{2\mu,\vartheta}(x)dx\right)^{\frac{t}{s}}\right)^{\frac{s}{t}}\lesssim\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}f(x,y)^s\omega(x,y)dy\right)dx=\|f\|_{L^s(\mu)}^s.$$

Raising both sides to the power $\frac{t}{s}$ and combining with (5. 9), we conclude

$$\sum_{R\in\mathcal{R}^{2n}}\mathrm{vol}\{R\}_{\mu,\vartheta}^{\frac{t}{s}}\left(\frac{1}{\mathrm{vol}\{R\}_{\mu,\vartheta}}\int_R f(x,y)\omega(x,y)dxdy\right)^t\lesssim\|f\|_{L^s(\mu)}^t.$$

Taking the t-th root of both sides completes the proof. □

If $0<\frac{\alpha}{n}<1$, we define

$$K_\alpha^n(R):=|Q_1|^{\frac{\alpha}{n}-1}|Q_2|^{\frac{\alpha}{n}-1}, \tag{5. 10}$$

for $R=Q_1\times Q_2\in\mathcal{R}^{2n}$, we have the following pointwise estimate:

$$\begin{aligned}
&\sum_{R\in\mathcal{R}^{2n}}K_\alpha^n(R)\,\mathbf{1}_{R'}(x,y)\,\mathbf{1}_R(u,v)\\
&=\sum_{Q_1\times Q_2\in\mathcal{R}^{2n}}\{K_\alpha^n(Q_1\times Q_2):x\in Q_1',\ u\in Q_1\text{ and }y\in Q_2',\ v\in Q_2\}\\
&=\sum_{Q_1\times Q_2\in\mathcal{R}^{2n}}\left\{|Q_1|^{\frac{\alpha}{n}-1}|Q_2|^{\frac{\alpha}{n}-1}:x\in Q_1',\ u\in Q_1\text{ and }y\in Q_2',\ v\in Q_2\right\}\\
&=\sum_{Q_1\in\mathcal{D}^n}\left\{|Q_1|^{\frac{\alpha}{n}-1}:x\in Q_1',\ u\in Q_1\right\}\times\sum_{Q_2\in\mathcal{D}^n}\left\{|Q_2|^{\frac{\alpha}{n}-1}:y\in Q_2',\ v\in Q_2\right\}\\
&\approx d_{dy}(x,u)^{\alpha-n}\,d_{dy}(y,v)^{\alpha-n}\lesssim|x-u|^{\alpha-n}\,|y-v|^{\alpha-n},
\end{aligned}$$

where $d_{dy}(x,u)$ denotes the *dyadic distance* between x and u in $\mathbb{R}^n$, and $d_{dy}(y,v)$ denotes the *dyadic distance* between y and v in $\mathbb{R}^n$.

The dyadic distance between two points p and q in $\mathbb{R}^k$ is defined as the side length of the smallest dyadic cube containing p and q. Note that the dyadic distance is at least $\frac{1}{\sqrt{k}}$ times the Euclidean distance, since any dyadic cube Q containing x and y must satisfy

$$\ell(Q)\geq\max_{1\leq i\leq k}|x_i-y_i|\geq\sqrt{\frac{1}{k}\sum_{i=1}^k|x_i-y_i|^2}=\frac{1}{\sqrt{k}}|x-y|.$$

To apply the above theorem to the product fractional integral operator with kernel $|x-u|^{\alpha-n}\,|y-v|^{\alpha-n}$, it suffices to use Strömberg's well-known $\frac{1}{3}$-trick for the dyadic grids $\left\{\mathcal{D}_i^n\right\}_{i=1}^{3^n}$ and $\left\{\mathcal{D}_j^n\right\}_{j=1}^{3^n}$, yielding

$$\sum_{i=1}^{3^n}\sum_{j=1}^{3^n}\left[\sum_{R=Q_1\times Q_2\in\mathcal{D}_i^n\times\mathcal{D}_j^n}K_\alpha^n(R)\,\mathbf{1}_{R'}(x,y)\,\mathbf{1}_R(u,v)\right]\approx|x-u|^{\alpha-n}\,|y-v|^{\alpha-n}. \tag{5. 11}$$

Let $\mathcal{P}^n$ denote the collection of all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes.

Lemma 5.3. *For $K_\alpha^n(R)$ defined as in (5. 10), we have (5. 11).*

Proof. Let $\mathcal{D}\subset\mathcal{P}^n$ be a dyadic grid with side lengths in $\left\{\frac{2^m}{3}\right\}_{m\in\mathbb{Z}}$. Partition the collection of tripled cubes $\{3Q_1\}_{Q_1\in\mathcal{D}}$ into 3^n subcollections $\{S_u\}_{u=1}^{3^n}$ such that for each subcollection S_u, there exists a dyadic grid $\mathcal{D}_u$ with side lengths in $\{2^m\}_{m\in\mathbb{Z}}$ satisfying $S_u\subset\mathcal{D}_u$. With these grids $\{\mathcal{D}_u\}_{u=1}^{3^n}$ fixed, we have the following sandwiching property: for each cube $P \in \mathcal{P}^n$ and each integer $j \in \mathbb{N}$, there exists $u = u(P, j)$ with $1 \leq u \leq 3^n$ and a cube $Q_1 = Q_{1,u(P,j)} \in \mathcal{D}_u$ such that

$$\begin{aligned} \ell(Q_1) &\leq 18\,\ell(P), \\ 3P &\subset Q_1, \\ 2^j P &\subset \pi_{\mathcal{D}_u}^{(j)} Q_1, \end{aligned} \tag{5. 12}$$

where $\pi_{\mathcal{D}_u}^{(j)} Q_1$ denotes the j-th grandparent of Q_1 in the grid $\mathcal{D}_u$.

Fix $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. For $x \in \mathbb{R}^n$, let $P(x, \ell)$ denote the cube centered at x with side length $\ell \in \left\{2^k\right\}_{k\in\mathbb{Z}}$. Let $R_{a,b}(x, y) \equiv P(x, 2^a) \times Q\left(y, 2^b\right)$ for $a, b \in \mathbb{Z}$; the right-hand side of (5. 11) is equivalent to

$$\sum_{a,b\in\mathbb{Z}} K_\alpha^n\left(R_{a,b}(x,y)\right) \mathbf{1}_{R'_{a,b}(x,y)}(x,y)\, \mathbf{1}_{R_{a,b}(x,y)}(u,v), \quad (u,v) \in \mathbb{R}^n \times \mathbb{R}^n.$$

The first two lines in (5. 12) imply (5. 11): for each rectangle $R_{a,b}(x, y) \equiv P(x, 2^a) \times Q\left(y, 2^b\right)$ and $R'_{a,b}(x, y) := P'(x, 2^a) \times Q'\left(y, 2^b\right)$, there exist $Q_1 \times Q_2 \in \bigcup_{i=1}^{3^n} \bigcup_{j=1}^{3^n} \left(\mathcal{D}_i^n \times \mathcal{D}_j^n\right)$ and $Q'_1 \times Q'_2 \in \bigcup_{i=1}^{3^n} \bigcup_{j=1}^{3^n} \left(\mathcal{D}_i'^n \times \mathcal{D}_j'^n\right)$ such that

$$3R_{a,b}(x,y) \subset Q_1 \times Q_2 \subset 18 R_{a,b}(x,y),$$

$$3R'_{a,b}(x,y) \subset Q'_1 \times Q'_2 \subset 18 R'_{a,b}(x,y).$$

By the definition of K_α^n in (5. 10), we further have $K_\alpha^n\left(R_{a,b}(x,y)\right) \approx K_\alpha^n\left(Q_1 \times Q_2\right) \approx K_\alpha^n\left(R'_{a,b}(x,y)\right) \approx K_\alpha^n\left(Q'_1 \times Q'_2\right)$. The third line in (5. 12) is not required here. □

Theorem 5.4. *Suppose $1 < p < q < \infty$. Let $d\sigma(x,y) = \sigma^{-p'}(x,y)dxdy$ and $d\omega(x,y) = \omega^q(x,y)dxdy$ be locally finite, absolutely continuous weights on $\mathbb{R}^n \times \mathbb{R}^n$, let $r > 1$, and let $K : \mathcal{R}^{2n} \to [0,\infty)$. Then the norm $\mathbb{N}_K(\omega,\sigma)$ of the positive bilinear inequality*

$$\sum_{R\in\mathcal{R}^{2n}} K(R) \left(\int_{R'} f\, d\sigma\right)\left(\int_R g\, d\omega\right) \leq \mathbb{N}_K(\omega,\sigma) \|f\|_{L^p(\sigma)} \|g\|_{L^{q'}(\omega)}, \qquad f, g \geq 0, \tag{5. 13}$$

is finite and independent of all partial grids $\mathcal{R}^{2n} = \mathcal{D}^n \times \mathcal{D}^n$ if the ϑ-bump product characteristic $\mathbb{A}_{K,\vartheta}(\omega,\sigma)$ is finite, where

$$\begin{aligned} \mathbb{A}_{K,\vartheta}(\omega,\sigma) &:= \sup_{R\in\mathcal{R}^{2n}} K(R) \left[|R'|^{\frac{1}{p'\vartheta'}} \left(\int_{R'} \sigma^{-p'\vartheta} dxdy\right)^{\frac{1}{p'\vartheta}}\right] \left[|R|^{\frac{1}{q\vartheta'}} \left(\int_R \omega^{q\vartheta} dxdy\right)^{\frac{1}{q\vartheta}}\right] \\ &= \sup_{R\in\mathcal{R}^{2n}} K(R)\, \mathrm{vol}\{R'\}_{\sigma,\vartheta}^{\frac{1}{p'}}\, \mathrm{vol}\{R\}_{\omega,\vartheta}^{\frac{1}{q}}. \end{aligned} \tag{5. 14}$$

Proof. We first choose $p < t < q$. By the definition of the ϑ-bump characteristic, followed by Hölder's inequality with exponents t and t', we obtain:

$$\begin{aligned}
&\sum_{R\in\mathcal{R}^{2n}} K(R)\left(\int_{R'} f\,d\sigma\right)\left(\int_R g\,d\omega\right)\\
&= \sum_{R\in\mathcal{R}^{2n}} \left\{K(R)\,\mathrm{vol}\{R'\}_{\sigma,\vartheta}^{\frac{1}{p'}}\,\mathrm{vol}\{R\}_{\omega,\vartheta}^{\frac{1}{q}}\right\}\mathrm{vol}\{R'\}_{\sigma,\vartheta}^{\frac{1}{p}}\,\mathrm{vol}\{R\}_{\omega,\vartheta}^{\frac{1}{q'}}\left(\frac{1}{\mathrm{vol}\{R'\}_{\sigma,\vartheta}}\int_{R'} f\,d\sigma\right)\left(\frac{1}{\mathrm{vol}\{R\}_{\omega,\vartheta}}\int_R g\,d\omega\right)\\
&\le \mathbb{A}_{K,\vartheta}(\omega,\sigma)\left\{\sum_{R\in\mathcal{R}^{2n}} \mathrm{vol}\{R'\}_{\sigma,\vartheta}^{\frac{t}{p}}\left(\frac{1}{\mathrm{vol}\{R'\}_{\sigma,\vartheta}}\int_{R'} f\,d\sigma\right)^t\right\}^{\frac{1}{t}}\left\{\sum_{R\in\mathcal{R}^{2n}} \mathrm{vol}\{R\}_{\omega,\vartheta}^{\frac{t'}{q'}}\left(\frac{1}{\mathrm{vol}\{R\}_{\omega,\vartheta}}\int_R g\,d\omega\right)^{t'}\right\}^{\frac{1}{t'}}.
\end{aligned}$$

The theorem now follows from the proposition 5.2. □

We now prove Theorem 1.7.

Proof of Theorem 1.7. Recall the definition of $\hat{A}^{\alpha\beta}_{pq\vartheta}(\omega,\sigma)$ given in (1.16). For any $Q_1\times Q_2\subset\mathbb{R}^n\times\mathbb{R}^n$, we derive the following identity:

$$\begin{aligned}
&|Q_1|^{\left(\frac{\alpha+\beta}{n+1}-\frac{1}{p}+\frac{1}{q}\right)\frac{n+1}{n}}|Q_2|^{\left(\frac{\alpha+\beta}{n+1}-\frac{1}{p}+\frac{1}{q}\right)\frac{n+1}{n}}\\
&\times\left\{\frac{1}{|Q_1'||Q_2'|}\iint_{Q_1'\times Q_2'}\omega^{q\vartheta}(u,v)\,dudv\right\}^{\frac{1}{q\vartheta}}\left\{\frac{1}{|Q_1||Q_2|}\iint_{Q_1\times Q_2}\sigma^{-\frac{p\vartheta}{p-1}}(u,v)\,dudv\right\}^{\frac{p-1}{p\vartheta}}\\
&= |Q_1|^{\frac{1}{n}\left[\alpha+\beta-\left(\frac{1}{p}-\frac{1}{q}\right)\right]-1+\left(1-\frac{1}{\vartheta}\right)\left(1-\frac{1}{p}+\frac{1}{q}\right)}|Q_2|^{\frac{1}{n}\left[\alpha+\beta-\left(\frac{1}{p}-\frac{1}{q}\right)\right]-1+\left(1-\frac{1}{\vartheta}\right)\left(1-\frac{1}{p}+\frac{1}{q}\right)}\\
&\times\left\{\iint_{Q_1'\times Q_2'}\omega^{q\vartheta}(u,v)\,dudv\right\}^{\frac{1}{q\vartheta}}\left\{\iint_{Q_1\times Q_2}\sigma^{-\frac{p\vartheta}{p-1}}(u,v)\,dudv\right\}^{\frac{p-1}{p\vartheta}}.
\end{aligned}$$

Let $\zeta = \alpha+\beta-\left(\frac{1}{p}-\frac{1}{q}\right)$. By the same reasoning as in Remark 3.1 and Remark 3.2, the conditions $\frac{\alpha+\beta}{n+1}-\frac{1}{p}+\frac{1}{q}\ge 0$ and $\alpha+\beta-\left(\frac{1}{p}-\frac{1}{q}\right)\le-\left(1-\frac{1}{\vartheta}\right)\left(1-\frac{1}{p}+\frac{1}{q}\right)<0$ imply $0<\zeta<n$. Recall the definition of $V^{\alpha\beta\rho}(u,v,t)$ in (1.11) for $u\ne 0$, $v\ne 0$, $t\ne 0$ and $\rho\ge\left|\beta-\frac{1}{p}+\frac{1}{q}\right|$. We consider two cases:

Case 1: $\beta\ge\frac{1}{p}-\frac{1}{q}$. We have

$$\begin{aligned}
V^{\alpha\beta\rho}(u,v,t) &\le |u|^{\alpha-n}|v|^{\alpha-n}|t|^{\beta-1}\left(\frac{|u||v|}{|t|}+\frac{|t|}{|u||v|}\right)^{-\left(\beta-\frac{1}{p}+\frac{1}{q}\right)}\\
&\le |u|^{\alpha-n}|v|^{\alpha-n}|t|^{\beta-1}\left(\frac{|t|}{|u||v|}\right)^{-\left(\beta-\frac{1}{p}+\frac{1}{q}\right)}\\
&= |u|^{\zeta-n}|v|^{\zeta-n}|t|^{\left(\frac{1}{p}-\frac{1}{q}\right)-1}.
\end{aligned}$$

Case 2: $\beta<\frac{1}{p}-\frac{1}{q}$. We have

$$\begin{aligned}
V^{\alpha\beta\rho}(u,v,t) &\le |u|^{\alpha-n}|v|^{\alpha-n}|t|^{\beta-1}\left(\frac{|u||v|}{|t|}+\frac{|t|}{|u||v|}\right)^{-\left(\beta-\frac{1}{p}+\frac{1}{q}\right)}\\
&\le |u|^{\alpha-n}|v|^{\alpha-n}|t|^{\beta-1}\left(\frac{|u||v|}{|t|}\right)^{\beta-\frac{1}{p}+\frac{1}{q}}
\end{aligned}$$

$$= |u|^{\zeta-n}|v|^{\zeta-n}|t|^{\left(\frac{1}{p}-\frac{1}{q}\right)-1}.$$

From this point onward, we assume $f \geq 0$ (without loss of generality, as non-negative functions suffice for the norm estimate). Via the change of variable $\tau \mapsto \tau - \mu\,(u\cdot\eta - v\cdot\xi)$, we obtain

$$\begin{aligned}
I_{\alpha\beta\rho}f(u,v,t) &= \iiint_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}} f\left(\xi,\eta,\tau-\mu\,(u\cdot\eta-v\cdot\xi)\right) V^{\alpha\beta\rho}(u-\xi,v-\eta,t-\tau)\,d\xi d\eta d\tau\\
&\leq \iiint_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}} f\left(\xi,\eta,\tau-\mu\,(u\cdot\eta-v\cdot\xi)\right)\\
&\quad\times |u-\xi|^{\alpha-n}|v-\eta|^{\alpha-n}|t-\tau|^{\beta-1}\left(\frac{|u-\xi||v-\eta|}{|t-\tau|}+\frac{|t-\tau|}{|u-\xi||v-\eta|}\right)^{-\left|\beta-\frac{1}{p}+\frac{1}{q}\right|} d\xi d\eta d\tau\\
&\leq \iiint_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}} f\left(\xi,\eta,\tau-\mu\,(u\cdot\eta-v\cdot\xi)\right)\\
&\quad\times |u-\xi|^{\zeta-n}|v-\eta|^{\zeta-n}|t-\tau|^{\left(\frac{1}{p}-\frac{1}{q}\right)-1} d\xi d\eta d\tau\\
&= \iint_{\mathbb{R}^n\times\mathbb{R}^n} |u-\xi|^{\zeta-n}|v-\eta|^{\zeta-n}F_{pq}(\xi,\eta,u,v,t)d\xi d\eta,
\end{aligned} \tag{5. 15}$$

where

$$F_{pq}(\xi,\eta,u,v,t) := \int_{\mathbb{R}} f\left(\xi,\eta,\tau-\mu\,(u\cdot\eta-v\cdot\xi)\right)|t-\tau|^{\left(\frac{1}{p}-\frac{1}{q}\right)-1}\,d\tau.$$

Recall the Hardy-Littlewood-Sobolev Theorem. Applying (1.2) with $\mathbf{N}=1$ and $\alpha = \frac{1}{p}-\frac{1}{q}$, we get

$$\begin{aligned}
\left\{\int_{\mathbb{R}} F^q_{pq}(\xi,\eta,u,v,t)dt\right\}^{\frac{1}{q}} &\leq C_{p,q}\left\{\int_{\mathbb{R}}\left[f\left(\xi,\eta,t-\mu\,(u\cdot\eta-v\cdot\xi)\right)\right]^p dt\right\}^{\frac{1}{p}}\\
&= C_{p,q}\,\|f(\xi,\eta,\cdot)\|_{L^p(\mathbb{R})}\,.
\end{aligned} \tag{5. 16}$$

Combining (5. 15) and (5. 16), by the Minkowski integral inequality, we establish the following weighted norm inequality:

$$\begin{aligned}
&\left\{\iiint_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}} \omega^q(u,v)\left(I_{\alpha\beta\rho}f\right)^q(u,v,t)dudvdt\right\}^{\frac{1}{q}}\\
&\leq \left\{\iiint_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}} \omega^q(u,v)\left(\iint_{\mathbb{R}^{2n}} |u-\xi|^{\zeta-n}|v-\eta|^{\zeta-n}F_{pq}(\xi,\eta,u,v,t)d\xi d\eta\right)^q dudvdt\right\}^{\frac{1}{q}}\\
&\leq \left\{\iint_{\mathbb{R}^n\times\mathbb{R}^n} \omega^q(u,v)\left(\iint_{\mathbb{R}^n\times\mathbb{R}^n} |u-\xi|^{\zeta-n}|v-\eta|^{\zeta-n}\left(\int_{\mathbb{R}} F^q_{pq}(\xi,\eta,u,v,t)dt\right)^{\frac{1}{q}} d\xi d\eta\right)^q dudv\right\}^{\frac{1}{q}}\\
&\leq \mathbb{C}_{p,q}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^n} \omega^q(u,v)\left(\iint_{\mathbb{R}^n\times\mathbb{R}^n} |u-\xi|^{\zeta-n}|v-\eta|^{\zeta-n}\,\|f(\xi,\eta,\cdot)\|_{L^p(\mathbb{R})}\,d\xi d\eta\right)^q dudv\right\}^{\frac{1}{q}}.
\end{aligned} \tag{5. 17}$$

Finally, define the integral operator

$$II_{\zeta}g(u,v) = \iint_{\mathbb{R}^n\times\mathbb{R}^n} g(\xi,\eta)\,|u-\xi|^{\zeta-n}\,|v-\eta|^{\zeta-n}\,d\xi d\eta, \qquad 0<\zeta<n.$$

With respect to the operator II_ζ defined above, by virtue of Lemma 5.3 and Theorem 5.4, we immediately obtain the following estimate:

$$\left\|\omega II_\zeta g\right\|_{L^q(\mathbb{R}^{2n})} \leq C_{\zeta pq\vartheta} A^\zeta_{pq\vartheta}(\omega,\sigma) \left\|g\sigma\right\|_{L^p(\mathbb{R}^{2n})}, \tag{5. 18}$$

if

$$A^\zeta_{pq\vartheta}(\omega,\sigma) := \sup_{Q_1\times Q_2\subset\mathbb{R}^n} |Q_1|^{\frac{\zeta}{n}-\frac{1}{p}+\frac{1}{q}}|Q_2|^{\frac{\zeta}{n}-\frac{1}{p}+\frac{1}{q}}\left\{\frac{1}{|Q_1'||Q_2'|}\iint_{Q_1'\times Q_2'}\omega^{q\vartheta}(u,v)\,du\,dv\right\}^{\frac{1}{q\vartheta}}$$
$$\times\left\{\frac{1}{|Q_1||Q_2|}\iint_{Q_1\times Q_2}\sigma^{-\frac{p\vartheta}{p-1}}(u,v)\,du\,dv\right\}^{\frac{p-1}{p\vartheta}} < \infty$$

for some $\vartheta > 1$.

As a consequence of (5. 17) and (5. 18), we can finish the proof of Theorem 1.7. □

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