The non-trivial solution obeys the gap equation

$$\frac{3}{2U} = \int_{\Omega} \frac{d^2\vec{k}}{(2\pi)^2} \frac{1}{\sqrt{\epsilon^2(\vec{k}) + \frac{1}{4}|\vec{M}_0|^2}} \qquad (*)$$

In the case of the two dimensional square lattice near half filling, the integral of right-hand side of above equation is dominated by contributions with momenta close to the Fermi surface (FS). i.e. near $k_1 + k_2 = \pi$, or for single-particle states with energy ~~too~~ close to the Fermi energy ($E_F = 0$ in this case). We introduce the DOS $\rho(\epsilon)$ and rewrite the gap equation as

$$\frac{3}{2U} = \int_{E_{min}}^{E_F} d\epsilon\, \rho(\epsilon) \frac{1}{\sqrt{\epsilon^2 + \Delta^2/4}} .$$

$\rho(\epsilon)$ is the one-particle DOS (per unit volume) for the single particle dispersion $\epsilon(\vec{k}) = -2t(\cos k_x + \cos k_y)$. Here $E_F = 0$ and $E_{min} = -4t$.

$\boxed{\text{Box}}$ Density of States at 2 Dimension Case

First, we derive the volume that ~~the~~ one quantum state occupy at k-space. we suppose the measure of each dimension $L \equiv 1$. And the possible wavelength of each dimension at periodic boundary is

$$n\lambda = 1$$
$$\Rightarrow \lambda = \frac{1}{n} \quad (n = 1, 2, 3 \cdots)$$

where n is the number of period of the wave. The wave vector at each dimension is given by

$$k_i \lambda = 2\pi$$
$$\Rightarrow k_i = \frac{2\pi}{\lambda} = 2\pi n \quad (i = x, y).$$

So we get the interval of two adjacent wave vector

$$\Delta k_i = 2\pi n - 2\pi(n-1) = 2\pi .$$

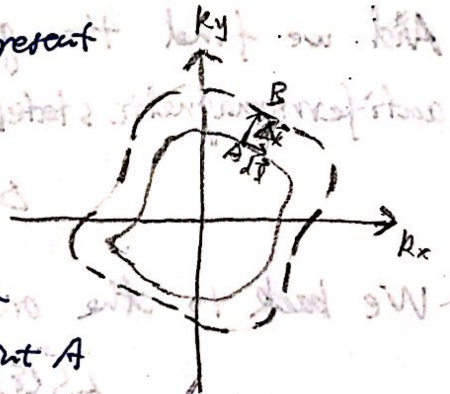
And the volume of one state at ~~k~~ k-space ν_k is given by

$$\boxed{\nu_k = \Delta k^2 = (2\pi)^2}.$$

Then we derive the expression from the ~~density~~ definition of density of state.

$$\rho(\varepsilon) = \frac{d\Omega(\varepsilon)}{d\varepsilon}$$

where $d\Omega(\varepsilon)$ is the difference of the number of states under the change of energy $\varepsilon \to \varepsilon + d\varepsilon$. The solid line represent the contour line of $\varepsilon(k) = \varepsilon$ and the dashed line represent the contour line of $\varepsilon(k) = \varepsilon + d\varepsilon$. And we suppose the solid line as P and the line element of P is $d\vec{l}$ which is a vector. If the energy is changed from ε to $\varepsilon + d\varepsilon$, the ~~A~~ point A will move to B. We suppose the small vector $\vec{AB}$ as $\Delta\vec{k}(P)$ which is a function of P. Now we can calculate the area that is caught between the solid line and dashed line.



$$\Delta V_k = \oint_P d\vec{l} \cdot \Delta\vec{k}(P).$$

And the number of state at this area is given by

$$\Delta\Omega = \frac{2 \cdot \Delta V_k}{\nu_k} = \frac{2}{(2\pi)^2} \oint_P d\vec{l} \cdot \Delta\vec{k}(P)$$

where the factor 2 is the freedom of spin. Finally, we derive the expression of DOS,

$$\rho(\varepsilon) = \frac{\Delta\Omega}{\Delta\varepsilon} = \frac{2}{(2\pi)^2} \oint_P \frac{d\vec{l} \cdot \hat{n}\,|\Delta\vec{k}(P)|}{\Delta\varepsilon}$$

$$= \frac{2}{(2\pi)^2} \oint_P \frac{d\vec{l} \cdot \hat{n}}{\frac{\Delta\varepsilon}{|\Delta\vec{k}|}} = \boxed{\frac{2}{(2\pi)^2} \oint_P \frac{d\vec{l} \cdot \hat{n}}{|\nabla\varepsilon|}}$$

where $\hat{n} = \Delta\vec{k}/|\Delta\vec{k}|$ and $\nabla\varepsilon = \vec{\nabla}_k \varepsilon(\vec{k})$.
$= \nabla\varepsilon / |\nabla\varepsilon|$

END Box