1 Fourier Transformation Forms in Quantum Field Theory

1.1 Discrete and finite size case

We consider a d dimensions discrete lattice system with lattice sites number $\mathcal{L}$ for each dimension and total lattice sites number $\mathcal{N}$. We consider two sets of basis. One is spatial basis, $\{|\boldsymbol{r}\rangle\}$ the other is momentum basis, $\{|\boldsymbol{k}\rangle\}$. Some useful expressions are listed here:

$$\begin{aligned}
\langle \boldsymbol{r}|\boldsymbol{k}\rangle &= \frac{1}{\sqrt{\mathcal{N}}}e^{i\boldsymbol{kr}} && (1)\\
\mathbb{1} &= \sum_{\boldsymbol{r}} |\boldsymbol{r}\rangle\langle \boldsymbol{r}| = \sum_{\boldsymbol{k}} |\boldsymbol{k}\rangle\langle \boldsymbol{k}| && (2)\\
\delta_{\boldsymbol{rr'}} &= \langle \boldsymbol{r}|\boldsymbol{r'}\rangle && (3)\\
\delta_{\boldsymbol{kk'}} &= \langle \boldsymbol{k}|\boldsymbol{k'}\rangle && (4)\\
\sum_{\boldsymbol{r}} e^{i\boldsymbol{r}(\boldsymbol{k}-\boldsymbol{k}')} &= \mathcal{N}\delta_{\boldsymbol{kk'}} && (5)\\
\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')} &= \boldsymbol{N}\delta_{\boldsymbol{rr'}} && (6)\\
\sum_{\boldsymbol{k}} \delta_{\boldsymbol{kk'}} &= 1 && (7)\\
\sum_{\boldsymbol{r}} \delta_{\boldsymbol{rr'}} &= 1. && (8)
\end{aligned}$$

And the dimensions of the δ function and the basis $\{|\boldsymbol{r}\rangle\}$, $\{|\boldsymbol{k}\rangle\}$ are

$$\begin{aligned}
[\delta] &= 1 && (9)\\
[|\boldsymbol{r}\rangle] &= 1 && (10)\\
[|\boldsymbol{k}\rangle] &= 1. && (11)
\end{aligned}$$

For field operators, we have

$$\begin{aligned}
a_{\boldsymbol{r}} &= \langle \boldsymbol{r}|a\rangle = \sum_{\boldsymbol{k}} \frac{1}{\sqrt{\mathcal{N}}}e^{i\boldsymbol{kr}}a_{\boldsymbol{k}} && (12)\\
a_{\boldsymbol{r}}^\dagger &= \sum_{\boldsymbol{k}} \frac{1}{\sqrt{\mathcal{N}}}e^{-i\boldsymbol{kr}}a_{\boldsymbol{k}}^\dagger && (13)\\
a_{\boldsymbol{k}} &= \sum_{\boldsymbol{r}} \frac{1}{\sqrt{\mathcal{N}}}e^{-i\boldsymbol{kr}}a_{\boldsymbol{r}} && (14)\\
a_{\boldsymbol{k}}^\dagger &= \sum_{\boldsymbol{r}} \frac{1}{\sqrt{\mathcal{N}}}e^{i\boldsymbol{kr}}a_{\boldsymbol{r}}^\dagger, && (15)
\end{aligned}$$

Now we can use these expressions to derive the matrix elements of physical operators, $\hat{\mathcal{O}}$. We must be notice that for any physical quantity, $\hat{o}$, if this quantity has translational symmetry (for lattice system, discrete translational symmetry), that means,

$$\langle \boldsymbol{r}|\hat{\mathcal{O}}|\boldsymbol{r}'\rangle = \mathcal{O}(\boldsymbol{r},\boldsymbol{r}') \to \hat{\mathcal{O}}(\boldsymbol{r}-\boldsymbol{r}'), \qquad (16)$$

at momentum space,

$$\mathcal{O}(\boldsymbol{r}-\boldsymbol{r}') \Leftrightarrow \mathcal{O}_{\boldsymbol{k}\boldsymbol{k}'} = \mathcal{O}_{\boldsymbol{k}}\delta_{\boldsymbol{k}\boldsymbol{k}'}. \tag{17}$$

For any two point operator, $\mathcal{M}$, the matrix element is given by

$$\langle\boldsymbol{r}|\mathcal{M}|\boldsymbol{r}'\rangle = \sum_{\boldsymbol{k}\boldsymbol{k}'}\frac{1}{\mathcal{N}}e^{i(\boldsymbol{k}\boldsymbol{r}-\boldsymbol{k}'\boldsymbol{r}')}\mathcal{M}_{\boldsymbol{k}\boldsymbol{k}'}. \tag{18}$$

If $\mathcal{M}(\boldsymbol{r},\boldsymbol{r}') = \mathcal{M}(\boldsymbol{r}-\boldsymbol{r}')$, we have

$$\mathcal{M}(\boldsymbol{r}-\boldsymbol{r}') = \sum_{\boldsymbol{k}}\frac{1}{\mathcal{N}}e^{i\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')}\mathcal{M}_{\boldsymbol{k}}. \tag{19}$$

For the inverse transform, we obtain

$$\langle\boldsymbol{k}|\mathcal{M}|\boldsymbol{k}'\rangle = \sum_{\boldsymbol{r}\boldsymbol{r}'}\frac{1}{\mathcal{N}}e^{-i(\boldsymbol{k}\boldsymbol{r}-\boldsymbol{k}'\boldsymbol{r}'}\mathcal{M}(\boldsymbol{r},\boldsymbol{r}'). \tag{20}$$

When the translational symmetry is satisfied, i.e., $\mathcal{M} \equiv \mathcal{M}(\boldsymbol{r}-\boldsymbol{r}')$, we have

$$\mathcal{M}_{\boldsymbol{k}} = \sum_{\boldsymbol{r}} e^{-i\boldsymbol{k}\boldsymbol{r}}\mathcal{M}(\boldsymbol{r}). \tag{21}$$

1.2 Continuous and finite size case

Now we consider the continuous space not a lattice with a finite length scale at each dimension. Some useful expressions are given here

$$\begin{aligned}
\langle\boldsymbol{r}|\boldsymbol{k}\rangle &= \frac{1}{L^{d/2}}e^{i\boldsymbol{r}\boldsymbol{k}} && (22)\\
\mathbb{1} &= \int \mathrm{d}^d\boldsymbol{r}|\boldsymbol{r}\rangle\langle\boldsymbol{r}| = \sum_{\boldsymbol{k}}|\boldsymbol{k}\rangle\langle\boldsymbol{k}| && (23)\\
\langle\boldsymbol{r}|\boldsymbol{r}'\rangle &= \frac{1}{L^d}\delta(\boldsymbol{r}-\boldsymbol{r}') && (24)\\
\langle\boldsymbol{k}|\boldsymbol{k}'\rangle &= \delta_{\boldsymbol{k}\boldsymbol{k}'} && (25)\\
\int \mathrm{d}^d\boldsymbol{r}e^{i\boldsymbol{r}(\boldsymbol{k}-\boldsymbol{k}')} &= L^d\delta_{\boldsymbol{k}\boldsymbol{k}'} && (26)\\
\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')} &= \delta(\boldsymbol{r}-\boldsymbol{r}') && (27)\\
\int \mathrm{d}^d\boldsymbol{r}\delta(\boldsymbol{r}-\boldsymbol{r}') &= L^d && (28)\\
\sum_{\boldsymbol{k}}\delta_{\boldsymbol{k}\boldsymbol{k}'} &= 1. && (29)
\end{aligned}$$

And the dimension analysis are showed:

$$\begin{aligned}
[\delta] &= 1 && (30)\\
[|\boldsymbol{r}\rangle] &= L^{d/2} && (31)\\
[|\boldsymbol{k}\rangle] &= 1. && (32)\\
& && (33)
\end{aligned}$$

$|\boldsymbol{r}\rangle$ is the spatial basis; $|\boldsymbol{q}\rangle$ is the momentum space basis; L represents the length of the system at one direction; d is the dimensions of the system and $\mathbb{1}$ is the identity operator at single particle Hilbert space.

For field operators in this framework, we have

$$\psi(\boldsymbol{r}) = \sum_{\boldsymbol{k}} \frac{e^{i\boldsymbol{k}\boldsymbol{r}}}{L^{d/2}}\psi_{\boldsymbol{k}} \qquad \text{with} \qquad k_i = \frac{2\pi}{L}n_i,\ n_i \in \mathbb{Z} \tag{34}$$

$$\psi_{\boldsymbol{k}} = \int \frac{\mathrm{d}^d\boldsymbol{r}}{L^{d/2}}e^{-i\boldsymbol{k}\boldsymbol{r}}\psi(\boldsymbol{r}) \qquad \text{with} \qquad r_i \in \left[-\frac{L}{2},\frac{L}{2}\right]. \tag{35}$$

$$\tag{36}$$

As in discrete case, the expressions about matrix elements of physical quantities is given by

$$\langle\boldsymbol{r}|\mathcal{M}|\boldsymbol{r}'\rangle = \sum_{\boldsymbol{k}\boldsymbol{k}'}\frac{1}{L^d}e^{i\boldsymbol{k}\boldsymbol{r}}e^{-i\boldsymbol{k}'\boldsymbol{r}'}\mathcal{M}_{\boldsymbol{k}\boldsymbol{k}'}. \tag{37}$$

If $\mathcal{M}(\boldsymbol{r},\boldsymbol{r}') \equiv \mathcal{M}(\boldsymbol{r}-\boldsymbol{r}')$, we have

$$\mathcal{M}(\boldsymbol{r}-\boldsymbol{r}') = \sum_{\boldsymbol{k}}\frac{1}{L^d}e^{i\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')}\mathcal{M}_k. \tag{38}$$

The inverse transforms are given by

$$\langle\boldsymbol{k}|\mathcal{M}|\boldsymbol{k}'\rangle = \int \mathrm{d}^d\boldsymbol{r}\mathrm{d}^d\boldsymbol{r}'\frac{1}{L^d}e^{-i\boldsymbol{k}\boldsymbol{r}}e^{i\boldsymbol{k}'\boldsymbol{r}'}\mathcal{M}(\boldsymbol{r},\boldsymbol{r}'). \tag{39}$$

If $\mathcal{M} \equiv \mathcal{M}(\boldsymbol{r}-\boldsymbol{r}')$, the inverse transform only contain one momentum label

$$\mathcal{M}_{\boldsymbol{k}\boldsymbol{k}'} \equiv \mathcal{M}_{\boldsymbol{k}}\delta_{\boldsymbol{k}\boldsymbol{k}'} = \left[\int \mathrm{d}^d\boldsymbol{r}e^{-i\boldsymbol{k}\boldsymbol{r}}\mathcal{M}(\boldsymbol{r})\right]\delta_{\boldsymbol{k}\boldsymbol{k}'}. \tag{40}$$

1.3 Continuous and infinite size case

In usual quantum field theory, the most common Fourier transforms involve continuous and infinite size 4 dimensions space-time. We start to introduce this kind of transform with some useful expressions

$$\langle\boldsymbol{r}|\boldsymbol{k}\rangle = \frac{1}{(2\pi)^d}e^{i\boldsymbol{k}\boldsymbol{r}} \tag{41}$$

$$\mathbb{1} = \int \mathrm{d}^d\boldsymbol{r}|\boldsymbol{r}\rangle\langle\boldsymbol{r}| = \int \mathrm{d}^d\boldsymbol{k}|\boldsymbol{k}\rangle\langle\boldsymbol{k}| \tag{42}$$

$$\langle\boldsymbol{k}|\boldsymbol{k}'\rangle = \int \frac{\mathrm{d}^d\boldsymbol{r}}{(2\pi)^d}e^{i\boldsymbol{r}(\boldsymbol{k}-\boldsymbol{k}')} \equiv \delta(\boldsymbol{k}-\boldsymbol{k}') \tag{43}$$

$$\langle\boldsymbol{r}|\boldsymbol{r}'\rangle = \int \frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d}e^{i\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')} \equiv \delta(\boldsymbol{r}-\boldsymbol{r}') \tag{44}$$

$$\int \mathrm{d}^d\boldsymbol{r}\delta(\boldsymbol{r}-\boldsymbol{r}') = 1 \tag{45}$$

$$\int \mathrm{d}^d\boldsymbol{k}\delta(\boldsymbol{k}-\boldsymbol{k}') = 1, \tag{46}$$

where $|\boldsymbol{r}\rangle$ is the spatial basis; $|\boldsymbol{k}\rangle$ is the momentum space basis; d is the spatial space dimensions; L is the length scale of one spatial dimension; $\mathbb{1}$ is the identity operator of single particle Hilbert space.The dimensions of Dirac delta function and basis are

$$\begin{aligned} [\delta(\boldsymbol{r})] &= L^d && (47)\\ [\delta(\boldsymbol{k})] &= L^{-d} && (48)\\ [|\boldsymbol{r}\rangle] &= L^{-d/2} && (49)\\ [|\boldsymbol{k}\rangle] &= L^{d/2}. && (50) \end{aligned}$$

The Fourier transformations of field operators are given by

$$\psi(\boldsymbol{r}) = \int \frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^{d/2}} e^{i\boldsymbol{kr}}\psi(\boldsymbol{k}) \tag{51}$$

$$\psi(\boldsymbol{k}) = \int \frac{\mathrm{d}^d\boldsymbol{r}}{(2\pi)^{d/2}} e^{-i\boldsymbol{kr}}\psi(\boldsymbol{r}). \tag{52}$$

For matrix element of physical operators, we have

$$\langle\boldsymbol{r}|\mathcal{M}|\boldsymbol{r}'\rangle = \int \frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^{d/2}}\frac{\mathrm{d}^d\boldsymbol{k}'}{(2\pi)^{d/2}} e^{i\boldsymbol{rk}}e^{-i\boldsymbol{r}'\boldsymbol{k}'}\mathcal{M}_{\boldsymbol{kk}'}. \tag{53}$$

If $\mathcal{M}(\boldsymbol{r},\boldsymbol{r}') \equiv \mathcal{M}(\boldsymbol{r}-\boldsymbol{r}')$, the expression can be simplify as

$$\mathcal{M}(\boldsymbol{r}-\boldsymbol{r}') = int\frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d} e^{i\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')}\mathcal{M}_{\boldsymbol{k}}. \tag{54}$$

The inverse transform is

$$\langle\boldsymbol{k}|\mathcal{M}|\boldsymbol{k}'\rangle = \int \frac{\mathrm{d}^d\boldsymbol{r}\mathrm{d}^d\boldsymbol{r}'}{(2\pi)^d} e^{-i\boldsymbol{kr}}e^{i\boldsymbol{k}'\boldsymbol{k}'}\mathcal{M}(\boldsymbol{r},\boldsymbol{r}'). \tag{55}$$

If $\mathcal{M}(\boldsymbol{r},\boldsymbol{r}') \equiv \mathcal{M}(\boldsymbol{r}-\boldsymbol{r}')$, we can simplify the form as

$$\mathcal{M}_{\boldsymbol{kk}'} = \mathcal{M}_{\boldsymbol{k}}\delta(\boldsymbol{k}-\boldsymbol{k}') = \left[\int \mathrm{d}^d\boldsymbol{r}e^{-i\boldsymbol{kr}}\mathcal{M}(\boldsymbol{r})\right]\delta(\boldsymbol{k}-\boldsymbol{k}'). \tag{56}$$

1.4 About real time and imaginary time variable

For the uniformity of form, the discussion above ignore the time dimension variable. The cases which include the time variables can be considered easily. For real time case, this variable is comprised in 1.3 case (i.e. $t \in (-\infty, +\infty)$). For imaginary time case, this variable is comprised in 1.2 case (i.e. $\tau \in [0, \beta]$).