MAP551 - PC6: Dynamics around critical points, hyperbolicity, continuation and bifurcations

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1 Some toy dynamical systems and course application

1.1 A vector field with a singular germ non $\mathcal{C}^0$-conjugated with the one related to its linearization

We consider the following vector field :

$$X(x1, x2) = (-x_2 - x_1(x_1^2 + x_2^2), x_1 - x_2(x_1^2 + x_2^2))$$

Therefore, at first order, the system can be linearized at (0,0) as :

$$X(x1, x2) = (-x_2, x_1)$$

One solution of the system is $(-A\sin(t), A\cos(t))\ t \in \mathbb{R}^+\ A \in \mathbb{R}$, which represents a cycle manifold. **2.1.1**

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}||\phi(t)||^2 &= 2 < \phi(t), \phi'(t) > \\ &= -2(x_1^2 + x_2^2)\end{aligned}$$

If $X(0) \neq (0,0)$, according to Cauthy theorem, the solution, the quantity of $\frac{\mathrm{d}}{\mathrm{d}t}||\phi(t)||^2$ is smaller than 0. Therefore, the norm of $\phi(t)$ decreases strictly in time. **2.1.2**

From both **2.1.1** and **2.1.2**, we can see the linearized system preserve the norm ϕ_1 while the real trajectory approaches the zero point. Consequently, they can not be $\mathcal{C}^0$-conjugated. **2.1.3**

1.2 Local / global stable/ unstable manifold

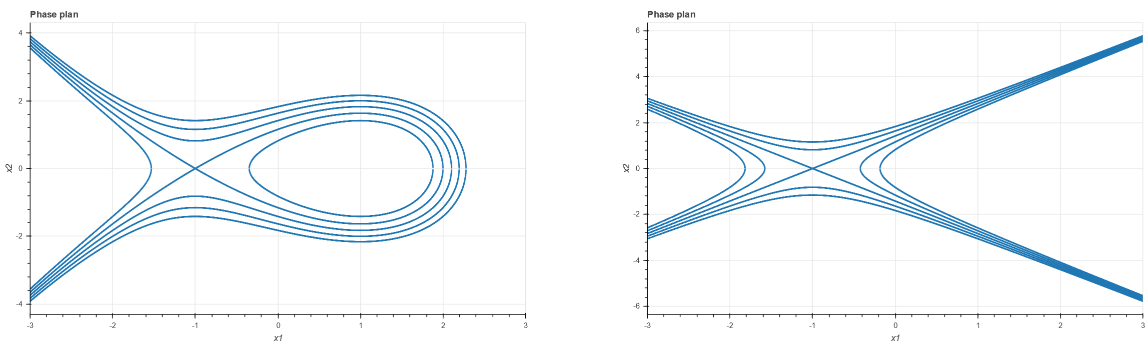
We consider the following vector field :

$$X(x_1, x_2) = (x_2, 1 - x_1^2)$$

which admits a singular hyperbolic germ at $a = (-1, 0)$. We try to compute the derivative of the given function : $f(X) = x_2^2/2 - x_1 + x_1^3/3$:

$$\mathrm{d}f(x_1, x_2)/\mathrm{d}t = x_2(1 - x_1^2) - x_2 + x_1^2 x_2 = 0$$

Therefore, the f is a first integral of X. **2.2.1**



Global solution　　　Local linearized solution

Figure 1 – Representation of manifold

Show as 1, we represent the phase portrait of X. At the point a=(-1,0), the left part where $x_1 < -1$ is unstable.

The local structure is a linearized system which is described as :

$$X(x_1, x_2) = (x_2, 1 - (x_1 + 1 - 1)^2) \approx (x_2, 2(x_1 + 1))$$

which gives us the analytic solution :

$$x_1 = A\exp(\sqrt{2}t) + B\exp(-\sqrt{2}t) - 1, \quad x_2 = \sqrt{2}(A\exp(\sqrt{2}t) - B\exp(-\sqrt{2}t))$$

If we suppose the system pass the point a at $t = 0$, $B = -A$. And the first integral for X_{linear} is $x_2^2/2 - x_1^2 - 2x_1$. The point a is hyperbolic singular point. So the global solution has a conjugation. Two structures are locally $\mathcal{C}^0$-conjugated. **2.2.2-2.2.3**

1.3 Stable/unstable manifold

We consider the following vector field :

$$X(x_1, x_2) = (-x_1, \alpha x_2^3)$$

which admits a singular germ at $O = (0, 0)$.

So the system can be solved as **2.3.1** :

$$x_1 = x_1(0)\exp(-t), \quad x_2 = \frac{x_2(0)}{\sqrt{1 - 2\alpha x_2(0)t}}$$

1. $\alpha < 0$. When $t \to \infty$, x_1 and x_2 tend to zero. System is stable for the positive direction time. However, when $t \to -\infty$, x_2 has no real solution and $|x_1|$ tends to ∞. But $|x_2|$ tends to ∞ when $t \to \frac{1}{2\alpha x_2(0)}$.

2. $\alpha = 0$, the system is stable with constant x_2 for all x_1.
3. the discussion of $\alpha < 0$ is similar to that of $\alpha > 0$. The system is unstable of the positive direction of t.

$$J = \begin{bmatrix} -1 & 0 \\ 0 & 3\alpha x_2^2 \end{bmatrix}$$

The eigenvalue of the Jacobean matrix is $(-1, 3\alpha x^2)$. If and only if $x^2 = 0$, we will have the zero eigenvalue. There, the value of x_2 remains 0 whatever the t. The manifold is the x_1 axis. **2.3.2-2.3.3**

2 Study of the ω-limit sets of the Brusselator model

The dynamics of the oscillating reaction discovered by Belousov and Zhabotinsky, can be modeled through the so-called Brusselator model depending on two parameters :

$$\begin{cases} \dfrac{dy_1}{dt} = a - (b+1)y_1 + y_1^2 y_2 \\ \dfrac{dy_2}{dt} = by_1 - y_1^2 y_2 \end{cases}$$

where a and b are two positive parameters.

2.1 Study of equilibria

The critical point of the dynamics is :

$$y_1 = a, \quad y_2 = \frac{b}{a}$$

If we fix $b = 3$, the critical points lie on the curve $y_1 y_2 = 3$. **3.1.1** The Jacobian matrix of the system is :

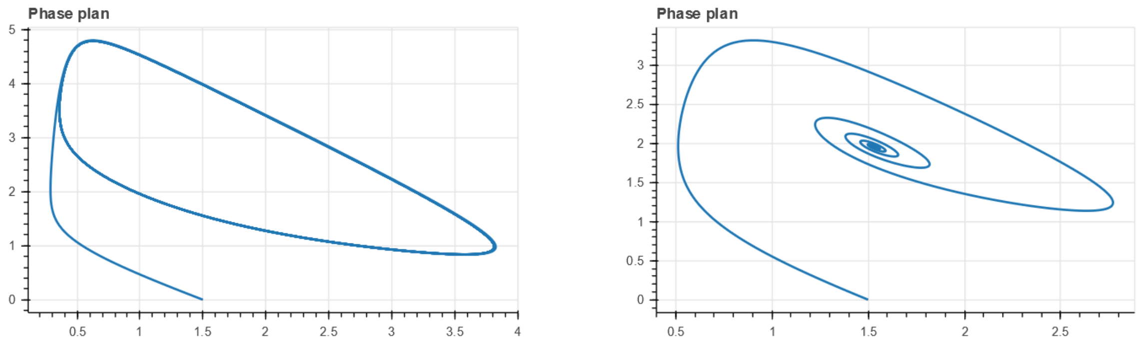
$$J = \begin{bmatrix} -(b+1) + 2y_1y_2 & y_1^2 \\ b - 2y_1y_2 & -y_1^2 \end{bmatrix}$$

We replace y_1, y_2 with the critical points, we have the characteristic equation of the matrix :

$$\lambda^2 + (a^2 + 1 - b)\lambda + a^2 = 0$$

So $\lambda_1 \times \lambda_2 > 0$, then the trace of the jacobian matrix is negative, the system has two eigenvalue of positive real part. And if $(a^2 + 1 - b) > 0$, the system is unstable. Else wise, the trace of Jacobian matrix is zero, then we have two different pure imaginary eigenvalues with opposite sign(if $a \neq 0$), and the system is unstable. **If $a \neq 0$, then neither of the eigenvalue should have a null real part. 3.1.1-3.1.2**

For b=3, and if $a = 0$, we will have two eigenvalues : $\lambda_1 = 0, \lambda_2 = 2$. The equilibrium is no longer hyperbolic. If this is the case, we can see one of the eigenvalue has a zero real part. Meanwhile, $a = \sqrt{2}$ also represents the non-hyperbolic propriety of the system, which is characterized with two purely imaginary solutions of eigenvalues. As for stability, the critical value for $|a|$ is $\sqrt{2}$, for which the system becomes unstable if $a < \sqrt{2}$ (because we have supposed a and b are two positive parameters). In this case, $\lambda_1 = \sqrt{2}i$, $\lambda_2 = -\sqrt{2}_i$, the system becomes oscillatory(Fig 2). **3.1.3**



$a = 0.98$ $a = 1.53$

Figure 2 – Two typical representations of manifold for the dynamics when $a < a_{critical}$ and $a > a_{critical}$

2.2 Proof of the existence of a limit Cycle -Poincaré-Bendixon Theorem

We focus in this part on the case $b = 3$. As we have discussed in the previous part, the parameters should be in the range of $0 < a < \sqrt{2}$ to guarantee the instability of the system. **3.2.1**.

If we let $a = 1$, the system becomes :

$$\begin{cases} \dfrac{dy_1}{dt} = 1 - 4y_1 + y_1^2 y_2 \\ \dfrac{dy_2}{dt} = 3y_1 - y_1^2 y_2 \end{cases}$$

The system will be attracted into the limit cycle after the initial perturbation. Here we set the initial point as $(1.5, 0)$ The invariant compact of the system can be identified as (Fig 3) :

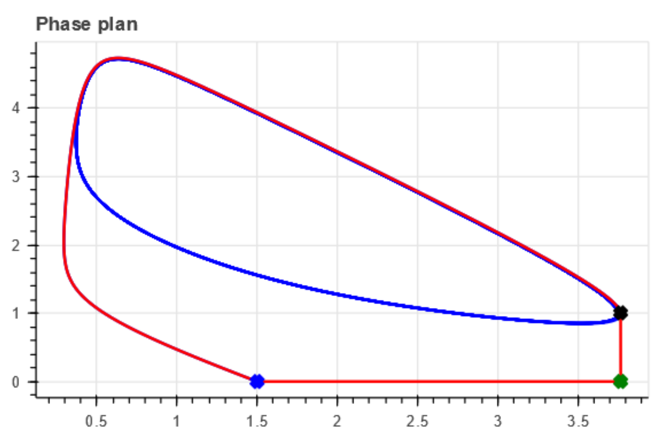


Figure 3 – Invariant compact set for a=1

The function being Lipschitz, then the solution is unique. If we follow the trajectory of the solution, we can never go out of the domain. As we reach the point $\max(y_1(t))$, we then follow a vertical line until $y_2 = 0$. And we close the curve with internal being the compact set.

We distinguish two cases : the vertical line and the horizontal line

1. Vertical : let's note the coordinate for black point $(y_1^{Bl}, y_2^{Bl}) \approx^{<} (3.77, 0.99)$, for green point (y_1^{Gr}, y_2^{Gr}) and for blue point (y_1^0, y_2^0). For $0 < y_2 < y_2^{Bl}$ and $y_1 = y_1^{Bl}$. $d_t\, y_1 = 1 - 4y_1^{Bl} + (y_1^{Bl})^2 y_2$,

$d_t y_2 = 3y_1^{Bl} - (y_1^{Bl})^2 y_2$. We can easily obtain the fact that $d_t\, y_1 < 0$ and $d_t\, y_2\Big|_{y_2=0} = 3y_1^{Gr} > 0$. Following the vector $(d_t\, y_1,\, d_t\, y_2)$ through time, it remains in the compact set.

2. Horizontal : for $y_1^0 < y_1 < y_1^{Gr}$ and $y_2 = 0$. $d_t\, y_1 = 1 - 4y_1 < 0$ and $d_t\, y_2 = 3y_1 > 0$. And according to the uniqueness of the solution, there will be no interior among different orbit, which means the solution will not go across the original curve. The orbit remains in the invariant compact.

As the system has a positive half-orbit that is in a compact set. According to the Poincaré Bendixon theorem, the system has a critical point, or a limit cycle. Since the system is not asymptotically stable (real part of the eigenvalues is positive), the system admits a limit cycle. **3.2.3**

We consider the Van der Pol problem :

$$\begin{cases} \mathrm{d}_t y_1 = y_2 \\ \mathrm{d}_t y_2 = (1 - y_1^2)\, y_2 - y_1 \end{cases} \tag{1}$$

The equilibrium lies at $(0, 0)$. With the same argument we can identify the compact set that is shown as Fig 4

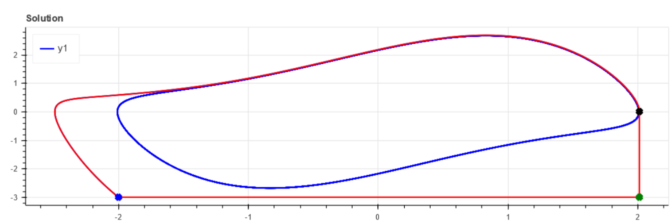


Figure 4 – Invariant compact set for ε=1

2.3 Dynamics around a Hopf bifurcation point

At this point, we try to inspect the efficiency of the numerical solution to represent the Hopf bifurcation with $b = 3$ and $a = \sqrt{2}$

At the Hopf bifurcation point $(\sqrt{2}, 3/\sqrt{2})$, the Jacobian matrix can be described as :

$$J = \begin{bmatrix} -(b+1) + 2y_1y_2 & y_1^2 \\ b - 2y_1y_2 & -y_1^2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -3 & -2 \end{bmatrix}$$

The eigenvalues are $\sqrt{2}i$, $-\sqrt{2}i$. The dynamics becomes critical. And the linearized system comes as :

$$\begin{cases} \dfrac{dy_1}{dt} = 2(y_1 - \sqrt{2}) + 2(y_2 - 3/\sqrt{2}) \\ \dfrac{dy_2}{dt} = -3(y_1 - \sqrt{2}) - 2(y_2 - 3/\sqrt{2}) \end{cases}$$

As we can see, two systems (original and the linearized ones) have the same Jacobian matrix at the bifurcation point. **3.2.1**

By simulating the system in the neighborhood of that critical point, the system seems to converge to the critical point but at a so slow speed that we can not have the absolute convergence with the numerical solution even we extend the studied time interval. **3.2.2**

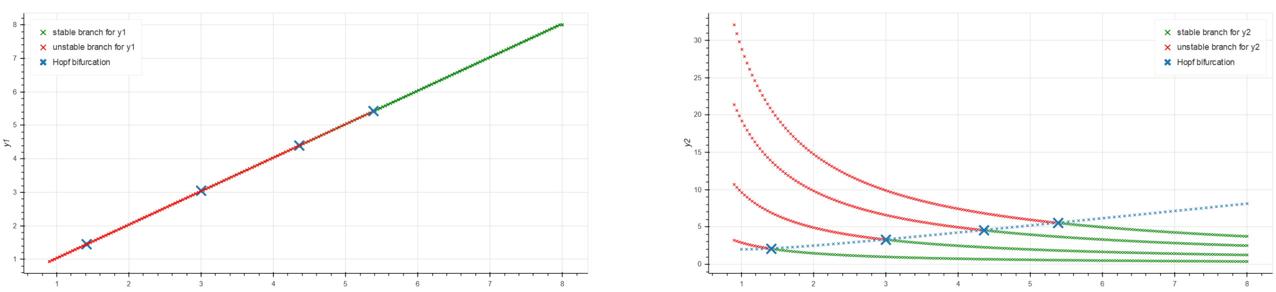
From the numerical interpretation, we have a huge difficulty to tell the stability of the system since it seems to converge but so slowly. And we can not observe the actual convergence within a finite time. Consequently, we can not draw any firm conclusion from the numerical simulations. **3.2.3**

3 Continuation of equilibria -limit points /Hopf /Pitchfork

3.1 Brusselartor model

Two algorithms is based on different methods to converge to the critical point. The main difference is the order of convergence for Newton resolution : we take advantage of the continuation and the derivability of the dynamical system for a. Then we iterate directly on the portrait phase plan. We implement the Newton solver to look for the critical point where $(d\,y_1, d\,y_2) = (0, 0)$. The order zero order method : firstly, we separate uniformly the space for "a" by N steps. to find the next candidate Y_{n+1}, we use the previous critical point Y_n as the initial departure point for Newton solver. However, the first order method : we use the $D\,Y_{n+1} - \mathrm{d}(D\,Y)/(d\,a) \times \mathrm{d}a$ (actually $\mathrm{d}(D\,Y)/(d\,a) = (1, 0)^T$)as the initialization for Newton solver. **4.1.1**

The Hopf bifurcation condition can be expressed as the $a^2 + 1 - b = 0$. As we know, the equilibrium point is $(a, b/a)$, therefore, the point $(\sqrt{b-1}, b/\sqrt{b-1})$.



equilibrium branches for y_1 — equilibrium branches for y_2

Figure 5 – Branch of equilibrium for different $b \in \{3, 10, 20, 30\}$ **4.1.2**

The convergence criterion that we use here is the $||\Delta D\,Y(a_n)||$ for the Newton resolution. If this quantity is smaller than the tolerance, we suppose that the convergence is reached. Therefore, the notion of accuracy for such algorithm is the norm of $(dy_1\,dy_2)$. **4.1.3**

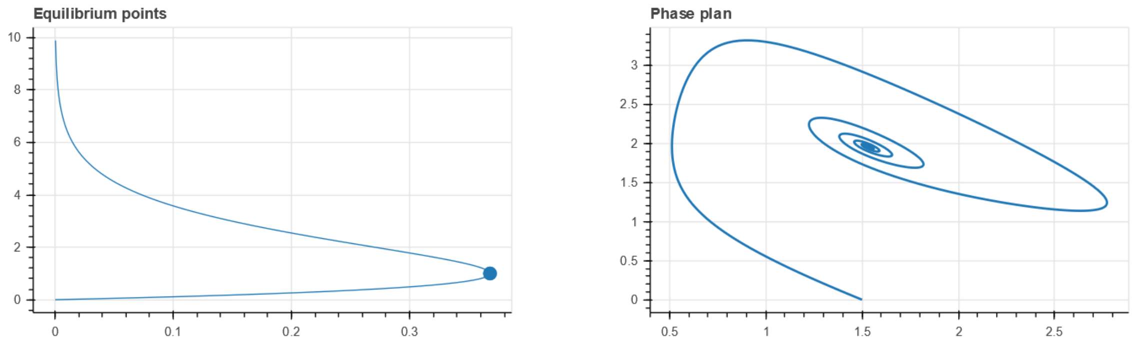
3.2 Thermal explosion

We consider the following system :

$$d_t\theta = F_k\,e^\theta - \theta$$

If $F_k < 1/e$, we will have two equilibrium point of which one is not stable, and another is stable. When $F_k = 1/e$, there is only one equilibrium point that is unstable. **4.2.1**

At that limit point, the eigenvalue of the jacobian is duplicated at zero.



equilibrium branches — eigenvalue of jacobian matrix

Figure 6 – Limit point of the equilibrium branches

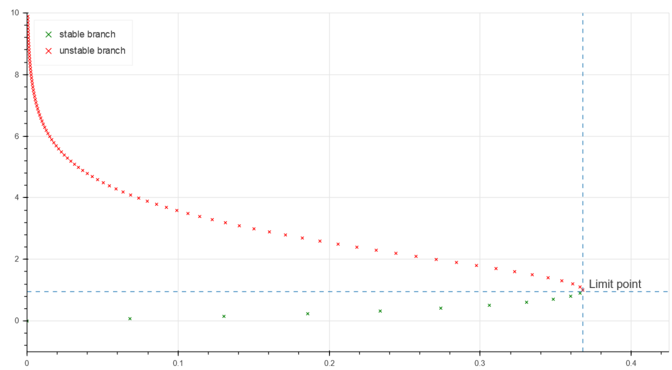


Figure 7 – Branches of equilibria

Rather than use the classical coordinate, we introduce another parameter s, the curvilinear through the trajectory. Both F_k and θ are parametered with this new variable s. Therefore, θ_{eq} : $\theta_{eq}(F_k(s), s)$. To increase step by step $s_{n+1} = s_n$, we calculate every time the increment of the system, then we can get the evolution of θ and F_k. (The numerical resolution is treated again with the Newton method, but the variable is s). By this means, we will continue to increase F_k by increasing s even though we have encountered the limit point, letting us to pass to the unstable branches. **4.2.3**.

Meanwhile, this limit point is a bifurcation point since the two eigenvalues possess null real part. **4.2.3**

3.3 Bead on a hoop configuration

A circular wire hoop rotates with constant angular velocity ω about a vertical diameter. A small bead moves, without friction, along the hoop.

The equation of motion can be shown to be (using the standard notation in classical mechanics) :

$$\ddot{\theta} = -\omega_c^2 \sin\theta + \omega^2 \sin\theta \, \cos\theta - \alpha\theta$$

with $\omega_c = \sqrt{g/R}$, where the gravity acceleration is denoted by g and the radius of the hoop is denoted R. The coefficient α is related to the friction in the system and can be idealized to be zero in the frictionless configuration.

Let $y_1 = \theta$ and $y_2 = \dot{\theta}$. Then, we can switch to a first order system of differential equations :

$$\begin{cases} d_t y_1 = y_2 \\ d_t y_2 = \sin y1 \left(\omega^2 \cos y_1 - \omega_c^2\right) - \alpha y_2 \end{cases}$$

In the friction less configuration, the system can be written :

$$\begin{cases} d_t y_1 = y_2 \\ d_t y_2 = \sin y1 \left(\omega^2 \cos y_1 - \omega_c^2\right) \end{cases}$$

Whatever the velocity, we can always guarantee one equilibrium of the system, which is $(0, 0)$. So if $\omega^2 \leq \omega_c^2$, there will be only one equilibrium of the system. And we will have three, of those two are symmetric to the axis. If we only work with y_1, the notion of stability of critical points can not be observed. We have to work in the (y_1, y_2) plane. **4.3.1**

We discuss the question **4.3.2-4.3.3** below : The Jacobian matrix of the system is presented as following :

$$J = \begin{bmatrix} 0 & 1 \\ cos(2y_1)\omega^2 - cos(y_1)\omega_c^2 & -\alpha \end{bmatrix}$$

The characteristic equation is : $x^2 + \alpha x - (cos(2y_1)\omega^2 - cos(y_1)\omega_c^2) = 0$.

First we consider a friction free dynamics $\alpha = 0$:

1. $\omega^2 < \omega_c^2$. There is only one equilibrium point $y_1 = 0$, we have two imaginary eigenvalues $\sqrt{\omega_c^2 - \omega^2 i}$, $-\sqrt{\omega_c^2 - \omega^2 i}$. The equilibrium is stable, but non-hyperbolic configuration ;
2. $\omega^2 = \omega_c^2$, one zero eigenvalue, stable and non-bifurcation point ;
3. $\omega^2 > \omega_c^2$, one positive eigenvalue and one negative. For $y_1 = 0$, the point is not stable. For $y_1 = \pm arccos(\omega_c^2/\omega^2)$, the equilibrium is stable, since we have two imaginary eigenvalues. It is not a hyperbolic point.

Then dynamics with $\alpha > 0$:

1. $\omega^2 < \omega_c^2$. There is only one equilibrium point $y_1 = 0$, the real part of eigenvalues is negative. Therefore, the critical point is stable, it is a hyperbolic point ;
2. $\omega^2 = \omega_c^2$, one equilibrium with one negative eigenvalue and one zero eigenvalue, it is a bifurcation point ;
3. $\omega^2 > \omega_c^2$: for equilibrium $y_1 = 0$, it has a positive and a negative eigenvalue, therefore the point is not stable ; Another case, $y_1 = \pm arccos(\omega_c^2/\omega^2)$, the real part of two eigenvalues is always negative, which guarantees the stability of the system. Therefore, they are hyperbolic points.

Since the friction dissipates the oscillation, the system will reach at the stable configuration after a sufficiently long duration. While the conservative system will accumulate the numerical error, the numerical resolution could be divergent because of this impact. The bifurcation take place when $|\omega| = \omega_c$. **The equilibrium point is not hyperbolic since it admits a zero eigenvalue. 4.3.4**

If we came from the direction ω^+, then at the bifurcation point :

(1)for the unstable branch, we should guarantee that the variable y_1 remains the previous direction : $\mathrm{d}y/\mathrm{d}\omega = 0$. In practice, we again use the curvilinear coordination to prevent from the singularity of the derivative $\mathrm{d}(\mathrm{d}_t\, y_1)/\mathrm{d}\omega$.

(2)Due to the numerical perturbation (imprecision of the type `float`), we find always one stable branch. Therefore, we can compute the usual derivative (increment) of $\mathrm{d}_t\, y_1$ at $\omega(s_{n_{critical}})$. And we use -$\mathrm{d}(\mathrm{d}_t\, y_1)/\mathrm{d}s_n$ as the initial condition for the other symmetric branches.

For vector expression, we set the initial vector expression as following (**4.3.5**) :

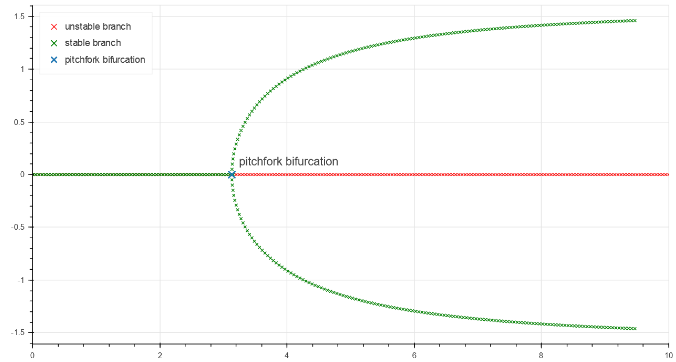


Figure 8 – different branches

$$(\omega, y_1)_+ = (\omega_c^- + \frac{\mathrm{d}\,\omega}{\mathrm{d}\,s}\Delta\, s, 0 + \frac{\mathrm{d}\,y_1}{\mathrm{d}\,s}\Delta\, s)$$
$$(\omega, y_1)_- = (\omega_c^- + \frac{\mathrm{d}\,\omega}{\mathrm{d}\,s}\Delta\, s, 0 - \frac{\mathrm{d}\,y_1}{\mathrm{d}\,s}\Delta\, s)$$
$$(\omega, y_1)_{unstable} = (\omega_c^- + \Delta\, s, 0)$$

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