MAP551 - PC2: Notions de base théroriques

Te SUN

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Systèmes Dissipatifs / Conservatifs

1 Fundamentals

1.1 Solution , integral curve and orbit

Consider the dynamical system on the pair $(u(t), v(t)) \in \mathbb{R}^2$:

$$d_t u = v, \quad d_t v = -v \tag{1}$$

We find that

$$u d_t u + v d_t v = d_t(u^2 + v^2) = 0$$

As a result, $u^2 + v^2$ is an invariant through the time.(**2.1.1**). If we make a substitution of variables as : $u = \rho cos(\theta)$, $v = \rho sin(\theta)$ where $(\rho, \theta) \in \mathbb{R}^+ \times \mathbb{R}$. Then, $d_t \rho^2 = 0$ which shows that ρ is a constant. On the other side, with the solution of 1 with initial condition $(\rho_0, \theta_0) = 0$, we have $\rho = 1, \theta = -t$. Therefore :

$$u = \rho cos(\theta) = cos(t), \quad v = -sin(t)$$

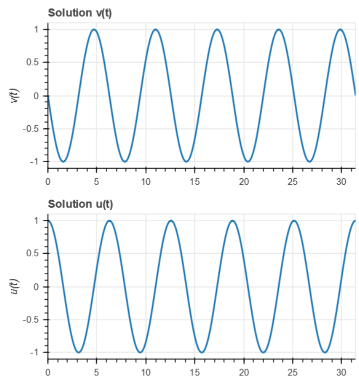
are the solutions for the dynamic system. (**2.1.2**)

If we look at the definition of "integral curve" and "orbit", we could illustrate two notions by the graph below 1 :

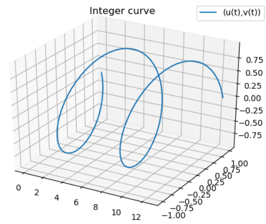
1.2 Autonomous versus non-autonomous dynamical systems

Consider a general dynamical system :

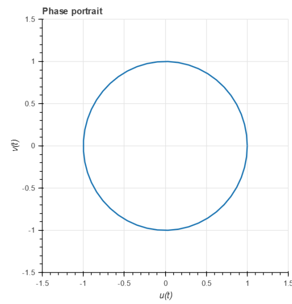
$$d_t x = f(x, t) \tag{2}$$



Solution



Integral curve



Orbit

Figure 1 – Interpretation of orbit and integral curve

with proper initial conditions $x(0) = x_0$. Suppose that there exists a frozen time equilibrium and $D_x f(\bar{x}, \bar{t}) \neq 0$. Then on neighborhood V of $\bar{t}$, the function f is locally invisible for x noted inverse function g with $D_x g(x,t) = \frac{1}{D_x f(x,t)}$. Therefore, according to the *Implicit function theorem*, there exists $\bar{x}_0(t)$ defined on V that $f(\bar{x}_0(t), t) = 0$, $\forall t \in V(\bar{t})$.

As an example, we consider $d_t x = -x + t$. Asymptotically, the solution behave as $x(t) = t$ since $-1 + \exp(-t)(x_0 + 1) \sim o(t)$. As for the frozen equilibrium, it happens once when $t = ln(x_0 + 1)$. However, if we use $x = t$ as the solution, it does never happen if $x_0 <= -1$ through time. (**2.2.2**)

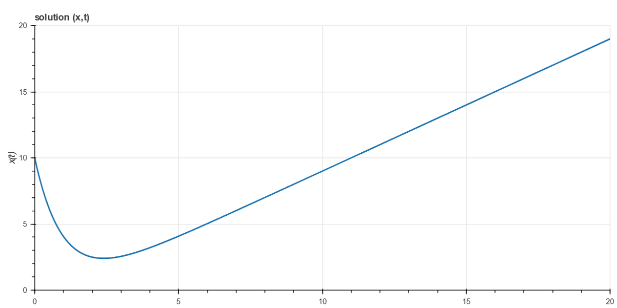


Figure 2 – Solution of $x(t)$

In general, a frozen equilibrium is a function $\bar{x}(t)$ verifying $d_t x(t) = 0$, which means x is a constant. Consequently, except if the frozen equilibrium function includes a constant solution that does not depend on time, this point is not a solution for the original system.(**2.2.3**)

2 Mechanical systems - Friction-less pendulum

We give an usual equation for the non-dimensional pendulum by :

$$\ddot{x} = -\sin(x) \tag{3}$$

Then kinetic energy $T(x,\dot{x})$ is defined as $T(x,\dot{x}) = \frac{\dot{x}^2}{2}$ which is a quadratic formulation. In the meanwhile, the potential term $V(x) = -\cos(x)$.

Then we could obtain the Lagrangian of the system(**3.1-3.2**) :

$$\mathcal{L}(x,\dot{x}) = \underbrace{\frac{1}{2}\dot{x}^2}_{\text{kinetic energy}} - \underbrace{(-\cos(x))}_{\text{potential}}$$

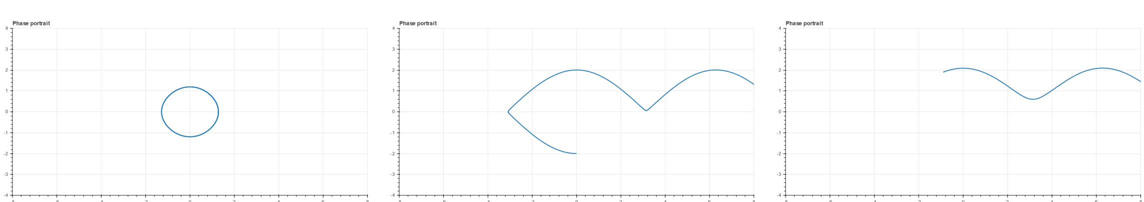
Without friction, the total energy $E = T + y = \dot{x}^2/2 - cos(x)$ is conservative through the time. If $E_0 > 1$, $|\dot{x}| > 0 \, \forall t > 0$. Therefore, if $|\dot{x}_0^2/2 - cos(x)| \geq 1$, the pendulum will never cease and the periodicity within 2π will be broken. Else wise, the pendulum will stay between $x \in [-\pi, \pi]$. (**3.3**)

Assume that $y = \dot{x}$, we will have two first-order ordinary differential equations below(**3.4**) :

$$\begin{cases} \dot{y} &= -\sin(x), \\ \dot{x} &= y \end{cases} \tag{4}$$

As discussed in **3.3**, if the absolute value of initial E is big enough (when E_0 is bigger than 1, because $|\cos(x)|$ can not be superior to 1, then the $|\dot{x}|$(speed) will never be zero), then we will encounter the case (1) (when $\dot{x}_0$ is positive) or (7) (other-wise). The curve (2) represent the case where $E = 1$, which is a critical case. (4) manifest the fact that the initial energy is too small, the system pendulum will retain within one "period" of x ($[-\pi, +\pi]$). We distinguish three different configurations : (**3.5-3.6**)

1. $E_0 = y_0^2/2 - \cos(x_0) < 1$, absorption case, then α-limit set equals to ω-limit set, which is the whole orbit since the solution is periodic. The sets are : $[-\sqrt{2(E_0+1)}, \sqrt{2(E_0+1)}] \times [-\arccos(E_0), \arccos(E_0)]$.
2. $E_0 = y_0^2/2 - \cos(x_0) =1$, the critical case, the α-limit set and ω-limit set depend on the initial datum.
3. $E_0 = y_0^2/2 - \cos(x_0) > 1$, α-limit set and ω-limit set depend again on the initial datum. For example, as shown in Figure 3, The α-limit set is $[\sqrt{2(E_0-1)}, \sqrt{2(E_0+1)}] \times \{-\infty\}$; The ω-limit set is $[\sqrt{2(E_0-1)}, \sqrt{2(E_0+1)}] \times \{+\infty\}$



$E_0 < 1$ $E_0 = 1$ $E_0 > 1$

Figure 3 – Different dynamics without friction

If we take friction into consideration, we could establish the dynamical system as below :

$$\begin{cases} \dot{y} &= -\sin(x) - \beta y, \quad \beta > 0 \\ \dot{x} &= y \end{cases} \tag{5}$$

In this case, there are always limits for x and y, the system becomes a dissipative dynamical system. ω-limit set becomes one point on the orbit which is also an equilibrium. However, if we trace back until $t = -\infty$, the limit for velocity will be ∞, and the limit for x wil be $sgn(\dot{x}_0) \times \infty$(**3.7**)

3 Population dynamics

3.1 The most elementary mode

For the following dynamical model :

$$\begin{cases} d_t u_1 & = u_1(1-u_2) \\ d_t u_2 & = u_2(-k+u_1) \end{cases} \tag{6}$$

Note that :

$$\begin{aligned} & d_t(u_1 + u_2 - kln(u_1) - ln(u_2)) \\ & \qquad = u_1(1-u_2) + u_2(-k+u_1) - k(1-u_2) - (-k+u_1) \\ & \qquad = 0 \end{aligned}$$

$\therefore \quad \mathcal{I} = u_1 + u_2 - k\ln(u_1) - \ln(u_2)$ is an invariant through time. (**4.1.1.a**)

Then $\mathcal{I}$ invariant limits the variation of u_1 and u_2. We can see that since $u_i > 0$, then $u_i - (k)\ln(u_i)$ has lower bound.

Suppose $u_2 \to 0$, then $\mathcal{I} - u_2 + \ln(u_2) \to -\infty$. However, we can easily prove that $u_1 - k\ln(u_1)$ is lower bounded. So comes the contradiction.

with the same argument, suppose that $u_2 \to \infty$, $u_1 - k\ln(u-1)$ should approach $-\infty$, which is contradictory.

As a result, u_1 and u_2 are have both upper and lower bounds. While the system is conservative, the solution appears to be periodic, as a result, α-limit and ω-limit set are equal and are presented by the orbit. (**4.1.1.b**)

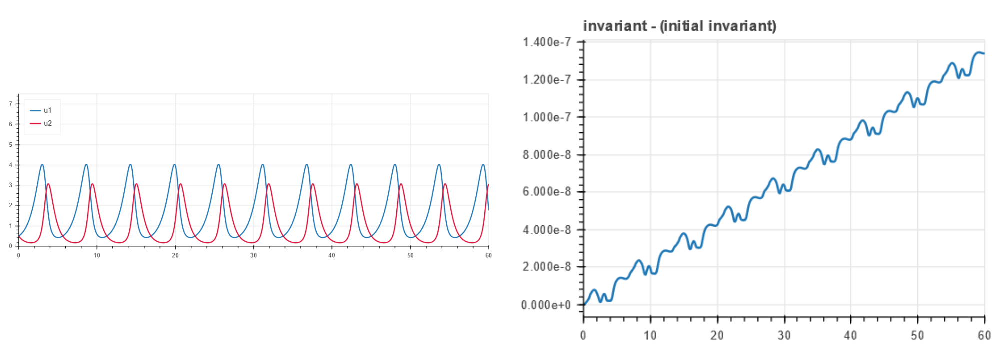


Figure 4 – Solution (u_1, u_2) and Variation of Invariant

With numerical integration, we get a rather proper resolution. However, due to the fact of a variation of 'Invariant', the resolution seems not to be correct. Firstly, the solutions don not converge, and this oscillatory evolution may accumulate solver's error. That may due to the accumulation of random (or deterministic) error during the integration through time. Though, we could limit the integration tolerance to restrict this effect. (**4.1.1.c**)

3.2 Adding intraspecific competition - change of mathematical behavior

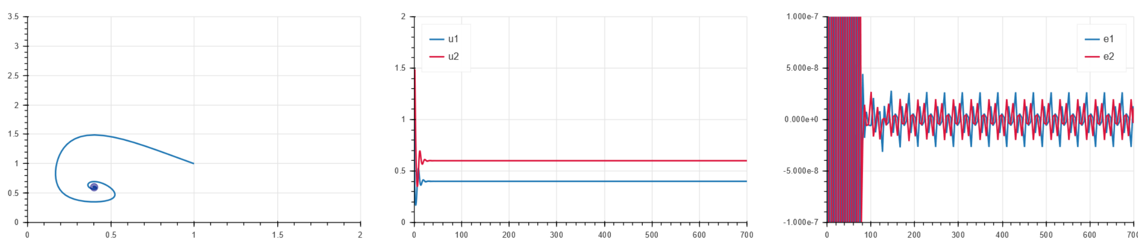
A modified Lotka-Volterra model is presented as below :

$$\begin{cases} d_t u_1 & = u_1(1 - u_1 - u_2) \\ d_t u_2 & = \beta(u_1 - \alpha)u_2 \end{cases} \tag{7}$$

From a numerical resolution with *Jupyter Notebook*, we observed an convergence of the solutions (u_1, u_2). With a few oscillations, population of two species stabilizes around their equilibrium points. For this system, the equilibrium points can be easily found :

$$u_1 = \alpha, \qquad u_2 = 1 - \alpha$$

They essentially depend on the parameter α, yet, they have no dependency of β (**4.2.1.a**)



phase plan (u_1, u_2) $(u_1, u_2; t)$ $(u_1 - \alpha, u_2 - (1 - \alpha); t)$

Figure 5 – Numerical resolution of 7

ω-limit set doesn't depend on the initial datum, it is represented as the equilibrium of the dynamical system.(**4.2.1.b**)

Here, we realize a numerical resolution with parameters :$\alpha = 0.40$, $\beta = 1.4$ From the 5, we can deduct that the integration error stabilize during long-term (between $[-5 \times 10^-8, 5 \times 10^-8]$). That may come from the fact that, in long-term, the solutions converge to the equilibrium points, which limits as well the accumulated error.(**4.2.1.c**)

3.3 Periodic solutions of the Rosenzweig-MacArthur model

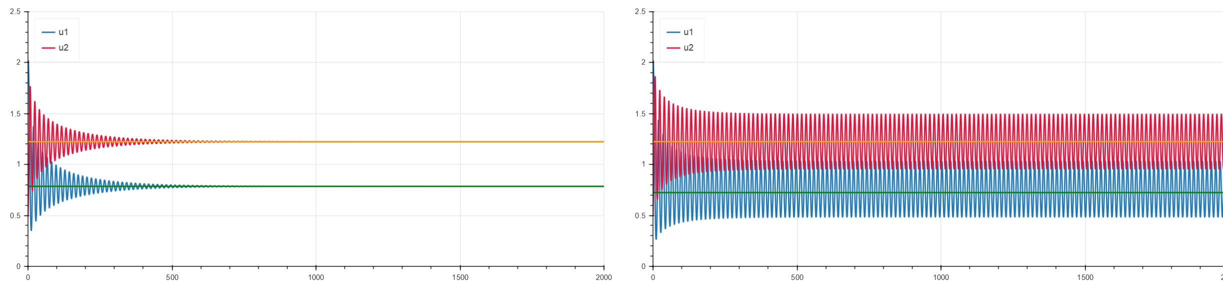
And again, we are not satisfied with the previous model which shows very few oscillation. In 1963, Rosenzweig-MacArthur proposed a finer model :

$$\begin{cases} \mathrm{d}_t u_1 = u_1 \left(1 - \dfrac{u_1}{\gamma}\right) - \dfrac{u_1 u_2}{1 + u_1} \\ \mathrm{d}_t u_2 = \beta u_2 \left(\dfrac{u_1}{1 + u_1} - \alpha\right) \end{cases} \tag{8}$$

Suppose $\beta = 1$, $\gamma = 2.5$,$(u_1, u_2; t = 0)) = (2, 0.5)$. We could see that when $\alpha > 0.43$, the system converges into one point.

In the meanwhile, we can easily find the equilibrium of the system, shown as below :

$$u_1 = \frac{\alpha}{1-\alpha}, \qquad u_2 = \frac{1}{1-\alpha}\Big(1 - \frac{\alpha}{\gamma(1-\alpha)}\Big)$$



$\alpha = 0.44$ $\qquad$ $\alpha = 0.42$

Figure 6 – Different configurations of system for different α

As I drew on the same time two equilibrium on the graph Fig 6, although the system doesn't posses literally a equilibrium, the oscillation is symmetrically distributed around those equilibrium point. We could therefore safely draw the conclusion that the oscillation is from the instability of the critical points.(**4.3.1.a**)

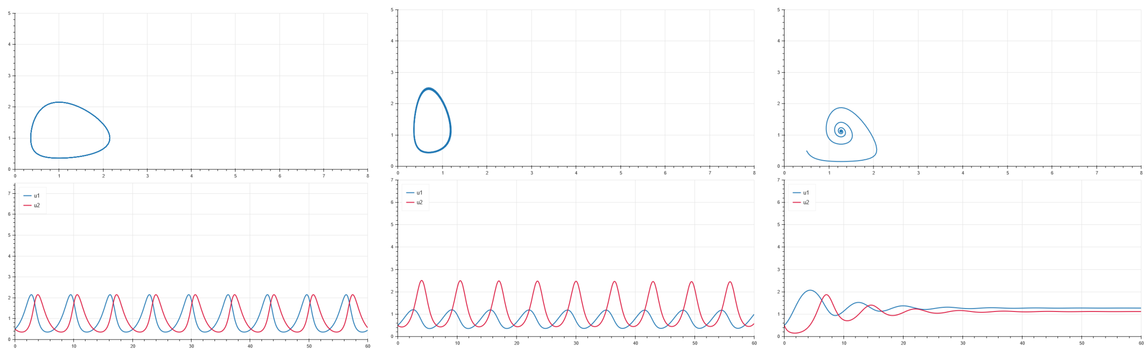
To discuss the stability around the equilibrium point, we can add a perturbation $(\delta u_1, \delta u_2)$ to the critical point. With a Taylor expansion we have :

$$\begin{aligned} d_t(u_1 + \delta u_1) &= (u_1 + \delta u_1)(1 - \frac{u_1 + \delta u_1}{\gamma}) - \frac{(u_1 + \delta u_1)(u_2 + \delta u_2)}{1 + (u_1 + \delta u_1)} \\ d_t(\delta u_1) &= \Big(\frac{\alpha}{1-\alpha} + \delta u_1)(-\frac{\delta u_1}{\gamma} - (1-\alpha)\delta u_2 + (1-\alpha)^2)(1 - \frac{\alpha}{\gamma(1-\alpha)})\delta u_1 \dots \\ &\qquad \dots - (1-\alpha)^2(1 - \frac{\alpha}{\gamma(1-\alpha)})\delta u_1^2 + \mathcal{O}(\delta u_1^3 + \delta u_2^3)\Big) \qquad (1) \end{aligned}$$

$$\begin{aligned} d_t(u_2 + \delta u_2) &= \beta(u_2 + \delta u_2)\Big(\frac{u_1 + \delta u_1}{1 + u_1 + \delta u_1} - \alpha\Big) \\ d_t(\delta u_2) &= \beta\Big(\frac{1}{1-\alpha}(1 - \frac{\alpha}{\gamma(1-\alpha)}) + \delta u_2\Big)\Big((1-\alpha)^2((1+\alpha)\delta u_1^2 - \delta u_1 + \mathcal{O}(\delta u_1^3)\Big) \qquad (2) \end{aligned}$$

If the system is stable, the direction of $(d_t\delta u_1, d_t\delta u_1)$ should be relatively opposite to $(\delta u_1, \delta u_2)$. However, there is an inner correlation between $(\delta u_1, \delta u_2)$, the calculation could not be feasible. Here, with the help of *Jupyter notebook*, we find a numerical solution for α_{cr} when $\gamma = 2.5$: $\alpha_{cr} \approx 0.430$(**4.3.1.a**) We note that, the discussion of stability doesn't depend on the value of β, in fact, only the symbol of β will exert an influence to the stability of the system. Although, β does impact the speed of convergence.

When $\alpha \in [0.34, \alpha_{cr}]$, the system doesn't converge to a point, we have then a conservative system. And the solution becomes periodic in long-term. The ω-limit set can be given by the



L.V. model | $\alpha = 0.41,\ \beta = 6.0$ | $\alpha = 0.56,\ \beta = 6.0$

Figure 7 – Numerical resolution for different model with initial condition $(u_1, u_2; t = 0) = (0.5, 0.5)$: L.V. model :Lotka-Volterra System ; $\alpha = 0.41,\ \beta = 6.0$:Rosenzweig-MacArthur model, of which the parameters are set as $\alpha = 0.41,\ \beta = 6.0$,$\gamma = 2.5$ $\alpha = 0.42,\ \beta = 6.0$: c.f. "$\alpha = 0.42,\ \beta = 6.0$"

maximum and minimum of u_1, u_2 of bifurcation circle. For $\alpha[\alpha_{cr}, 0.7]$, the ω-limit set is $\{u_{1,eq}\} \times \{u_{2,eq}\}$. In whichever the case, the ω-limit set doesn't depend on initial datum.

We present three graph of resolutions Figure 7.Compared with Lotka-Volterra system, the present system introduce a convergence towards a bifurcation or an equilibrium point. If we make a comparison with the pendulum system, the Lotka-Volterra system is just like a friction-less pendulum, while Rosenzweig-MacArthur model is like a pendulum with a filter friction that slow down a fast movement and accelerate a slow one but let a certain speed pass freely. Noted that when $\alpha > \alpha_{cr}$, the are not a friction-less area.(**4.3.1.b**)

With the same argument of **4.1.1.c**, we conjecture when $\alpha < \alpha_{cr}$(periodic bifurcation), the solver of integration accumulate the error that deviate the resolution in long-term. Yet, when $\alpha > \alpha_{cr}$, we can restrict the error within upper and lower bound.

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