Two gambler

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Suppose that we have two players A and B, with their initial number of chips *w.r.t* X and Y.

1 Steps towards solution

To begin with the results we had in the morning

1. Given that two players have the same number of chips, the game will end immediately.
2. If one of the player begins with 1 chips against another, then that "the sum of theirs chips equals to $2^n, n \in \mathbb{N}^+$" is a sufficient condition to put an end to game within finite rounds.

Then we could begin from the "end" of the game where both players should have the same number of chips, thus, $X = Y$. Otherwise, (let us suppose $X > Y$) the player who has more chips could lose the last round but will still have $X - Y$ chips left and the game will keep on going.

Thus, at the last round we will encounter necessarily the case $X = Y$. I put forward my conditions below to end the game within finite rounds.

$$\frac{2^n - k}{k} = \frac{X}{Y} \tag{1}$$

where $n \in \mathbb{N}^+$ and $k \in [1, 2^n - 1]$

2 Proof

Let us begin with the last Round (**instead of using the normal notation beginning from the first round, we denote the last round as** *Round 1*) before the ending. In fact, we can have an even stronger conclusion: **At Round j, we will have n=j, and k is strictly smaller than** 2^j **but larger than** 0.

We prove it in terms of *sufficiency* and *necessity*

2.1 sufficiency

Initialisation

$$\text{Round}\,1: \quad X = Y$$

Therefore, for the first round, our condition in Equation 1 is satisfied, since $n = 1$, $k = 1$.

Hypothesis Let us suppose, at the *Round j*, the condition still holds for the "finite-rounds' ending". That means, at *Round j*, if the remaining chips of player A and B: $\forall X_j, Y_j \in \mathbb{R}^2$ satisfy the condition below:

$$\frac{2^j - k}{k} = \frac{X_j}{Y_j}, \quad \forall k \in [1, 2^j - 1]$$

The game will end within j round.

Remark 1. If k is an even, it reduces to previous round. And the game will still end within finite rounds.

Recurrence We shall study the case for *Round j+1* now: Let player A has X_{j+1} chips and B has Y_{j+1} that satisfy the condition:

$$\frac{2^{j+1} - k}{k} = \frac{X_{j+1}}{Y_{j+1}}, \forall k \in [1, 2^{j+1} - 1]$$

Without loss of generality, we suppose $X_{j+1} \geq Y_{j+1}$ which denotes $k \leq 2^j$.

(a) if k is an even number, we come back to previous cases:

$$\frac{2^j - k/2}{k/2} = \frac{X_{j+1}}{Y_{j+1}},\ k/2 \in [1, 2^j - 1]$$

Therefore, the game will end within j rounds.

(b) if k is an odd number, let two player play for one round. The necessary condition that keeps the game going is the victory of player B, which doubles the chips of

B. So, after one round, the ratio between X and Y shows:

$$\frac{X}{Y} = \frac{2^{j+1} - 2k}{2k} = \frac{2^j - k}{k}$$

We can see that, after one round, we join the case of *Round j*. And the game will end after at most j rounds.

By recurrence, we have proved that, if the initial number of chips for two players satisfy the condition described in Equation 1, the game will end within finite rounds.

2.2 Necessity

$$\frac{X}{Y} = \frac{m - k}{k}$$

If m=2, the game will end in the next round because the only feasible case is $X = Y$.

If m and k are both integer and m is odd. The game can not end, since at the last round we have necessarily $X = Y$ however no integer m and k could satisfy this relation.

If m is even. Without loss of generality, we suppose $k < m - k$. After one round, to keep the game going, Y has to double to $2Y$ and we will have:

$$\frac{X - Y}{2Y} = \frac{m - 2k}{2k} = \frac{m/2 - k}{k}$$

. We go back to the beginning of the discussion whether $m/2$ is odd or even.

By this recurrence, we see that m has to be in the form of 2^n, $n \in \{1, 2, 3...\}$

3 What's more

1. Even X and Y are arbitrary real numbers, the conclusion still holds, as we never supposed X or Y is integer in the proof.
2. if we find the number n and k that satisfy

$$\frac{2^n - k}{k} = \frac{X}{Y}, \quad k \bmod 2 = 1$$

Then the game will end with at most n rounds from now on

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