Irreducible Representations of the Symmetric Group

classified over the field of complex numbers $\mathbb{C}$

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Table of Contents

Table of Contents

Foreword

We show a concise path to the classification of the irreducible representations of the symmetric group $\mathcal{S}_n$ over the field of complex numbers $\mathbb{C}$.

- mainly adapted from [FH04, section 4.2], minor references to [Sag01, chapter 2]
- Focus on the irreducibility of Specht modules
- Modest knowledge of group theory is enough to understand the combinatorial part, though familiarity of the basics of complex representations of groups [Ser77, chapter 1–2] and the group-algebra module perspective of representations is needed for other parts.

Young Tableaux

- Idea: Understand the group by its action on combinatorial objects

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & & \end{array}$$

(a) The Young diagram of shape $\lambda = (4, 3, 2)$

$$\begin{array}{cccc} 2 & 3 & 9 & 8 \\ 5 & 4 & 6 & \\ 7 & 1 & & \end{array}$$

(b) A Young tableau $T = (\lambda \vdash n, \sigma \in \mathcal{S}_n)$, where $\lambda = (4, 3, 2)$, $\sigma = 239854671$

Figure: Young diagrams and Young tableaux

$\mathcal{S}_n$ Acting on Young Tableaux

- $g \in \mathcal{S}_n$ acts on the Young tableau T of shape λ by replacing the number i with $g(i)$.
  - i.e. $gT := (\lambda, g\sigma)$.
  - The action is transitive.
- $P_T \leq \mathcal{S}_n$ consists of permutations that preserve the content of each row of T, similarly $Q_T \leq \mathcal{S}_n$ preserves the columns.
  - P_T and Q_T intersect trivially.
  - Their union generates the entire $\mathcal{S}_n$.
- Relationship between P_T, Q_T induced by different Young tableaux of the same shape:
  - The process $gT \to gpT$ can be understood as performing an "effect" equivalent to a row permutation $p \in P_T$ on T
  - $P_{gT} = gP_Tg^{-1}$
  - $Q_{gT} = gQ_Tg^{-1}$

Young Symmetrizers

Let $\mathbb{C}[\mathcal{S}_n]$ be the complex group algebra of S_n, i.e. its complex regular representation. Define:

- Row symmetrizer $a_T := \sum_{g \in P_T} g \in \mathbb{C}[\mathcal{S}_n]$
- Column symmetrizer $b_T := \sum_{g \in Q_T} \operatorname{sgn}(g) g \in \mathbb{C}[\mathcal{S}_n]$
- Young symmetrizer $c_T := a_T b_T$

It is evident that for any $p \in P_T$, $q \in Q_T$, we have:

- $p a_T = a_T p = a_T$
- $q b_T = b_T q = \operatorname{sgn}(q) b_T$
- $p c_T q = \operatorname{sgn}(q) c_T$
    - Though it's nontrivial that c_T is (almost) the only group algebra element satisfying this condition.

Tabloids

Define the $\mathbb{C}[\mathcal{S}_n]$-module

$$M_T := \mathbb{C}[\mathcal{S}_n]a_T$$

- Acting on Young tableaux, $a_T T$ forgets labelings inside the rows of T.
- It is the induced representation $1 \uparrow_{P_T}^{\mathcal{S}_n}$ obtained by lifting the trivial representation of P_T to $\mathcal{S}_n$.

Specht Modules

The Specht module induced by a Young tableau T is the $\mathbb{C}[\mathcal{S}_n]$-module

$$V_T := \mathbb{C}[\mathcal{S}_n]c_T$$

We shall show the following to classify the complex irreducible representations of $\mathcal{S}_n$:

- Each Specht module is an irreducible representation of $\mathcal{S}_n$.
- Two Specht modules are isomorphic if and only if the Young tableaux inducing them have the same shape.
- Every irreducible representation of $\mathcal{S}_n$ is isomorphic to some Specht module.

Table of Contents

S.R.D.C.[1] for Young Tableaux of the Same Shape

Below helps us to understand how $P_T Q_T$ acts on Young tableaux combinatorially.

Proposition

Let T, S be two Young tableaux of shape λ. If any pair of numbers in the same row of T falls into different columns in S, then there exist $p \in P_T, q' \in Q_S$ such that $pT = q'S$.

$$T \quad \begin{array}{cccc} i & j & k & l \\ * & * & * & \\ * & * & & \end{array} \xrightarrow{p_1 \in P_T} \begin{array}{cccc} l & k & i & j \\ * & * & * & \\ * & * & & \end{array} \xrightarrow{p_2 \in P_T} \cdots$$

$$S \quad \begin{array}{cccc} l & * & * & j \\ * & * & i & \\ * & k & & \end{array} \xrightarrow{q'_1 \in Q_S} \begin{array}{cccc} l & k & i & j \\ * & * & * & \\ * & * & & \end{array} \xrightarrow{q'_2 \in Q_S} \cdots$$

[1] Same Row, Different Column

Consequences

- Say $S = gT$, whence $q' = gqg^{-1}$ with $q \in Q_T$. Then $pT = q'S = gqT$, so $g = pq^{-1} \in P_T Q_T$.
- The converse is also true: for any $p \in P_T$, $q \in Q_T$, pT and qT are S.R.D.C.
- T, gT being S.R.D.C. is an iff condition for $g \in P_T Q_T$.

Uniqueness of the Young Symmetrizer

Proposition

Let $c \in \mathbb{C}[\mathcal{S}_n]$ *satisfy* $pcq = \mathrm{sgn}(q)c$ *for any* $p \in P_T$, $q \in Q_T$. *Then* $c \in \mathbb{C}c_T$, *i.e.* c *is unique up to scalar multiplication.*

Say $c = \sum_{g \in \mathcal{S}_n} n_g g$,

- $n_{pgq} = \mathrm{sgn}(q) n_g$
- n_g for $g \in P_T Q_T$ is fixed up to a scalar multiplication.
- n_g for $g \notin P_T Q_T$ is zero. Our combinatorial interpretation guarantees that there must be a pair of numbers (i, j) fall into the same row in T and the same column in gT. The alternating property of c results in $n_g = -n_g$ and thus $n_g = 0$.

Idempotenty of c_T

Corollary

$c_T x c_T \in \mathbb{C} c_T$ holds for any $x \in \mathbb{C}[\mathcal{S}_n]$.

- In particular, $c_T^2 = \alpha c_T$ for some $\alpha \in \mathbb{C}$.
- It can be shown that $\alpha = n!/\dim V_T$.

Irreducibility of Specht Modules

Proposition

The Specht module $V_T := \mathbb{C}[\mathcal{S}_n]c_T$ is an irreducible representation of $\mathcal{S}_n$.

Assume V_T has a subrepresentation $W \subseteq V_T$. Then the subspace:

$$c_T W \subset c_T V_T = c_T \mathbb{C}[\mathcal{S}_n] c_T \subset \mathbb{C} c_T$$

is a one-dimensional subspace. Thus, we consider the following two cases:

- $c_T W = \mathbb{C} c_T$. In this case:

$$V_T = \mathbb{C}[\mathcal{S}_n] c_T = \mathbb{C}[\mathcal{S}_n](\mathbb{C} c_T) = \mathbb{C}[\mathcal{S}_n] W = W$$

- $c_T W = 0$. In this case, for any $w = x c_T \in W$, we have $w^2 = x c_T w = 0$, i.e., $W^2 = 0$. We know that there are no non-zero nilpotent left ideals in the finite-dimensional semisimple group algebra $\mathbb{C}[\mathcal{S}_n]$, so $w = 0$, $W = 0$.

Table of Contents

A Brief Outline of Classification

Are Specht modules the only irreducible representations of $\mathcal{S}_n$?

$$\begin{aligned} \#\text{ irreducible representations} &= \#\text{ conjugacy classes of } \mathcal{S}_n \\ &= \#\text{ integer partitions of } n \end{aligned}$$

It suffices to show that:

- Two Specht modules are isomorphic if and only if the Young tableaux inducing them have the same shape.
  - $V_T \cong V_{gT}$ by direct construction $x \mapsto xg^{-1}$
  - For the converse, refine the S.R.D.C. condition to adapt Young tableaux of different shapes

This completes the classification of the irreducible representations of $\mathcal{S}_n$.

Wrap up

Classification of complex irreducible representations of $\mathcal{S}_n$:

- all Specht modules V_T, where T is a Young tableau of shape $\lambda \vdash n$.
- Two Specht modules are isomorphic if and only if the Young tableaux inducing them have the same shape.

Remark (What we've benefited from working over $\mathbb{C}$)

- We've used the semisimplicity of the group algebra $\mathbb{C}[\mathcal{S}_n]$ to show the irreducibility of Specht modules.
- We need the result over $\mathbb{C}$ that the number of complex irreducible representations of a group equals the number of conjugacy classes of the group.
- Other arguments do not rely on any special property of $\mathbb{C}$.

Problem Remains

- Dimension of Specht modules V_T
  - the hook-length formula for the number of standard Young tableaux of shape λ
- Explicit construction of a basis of Specht modules
- A more natural combinatorial interpretation of Specht modules
- Classification over other fields, e.g. $\mathbb{F}_p$

Thank you for your attention!

References I

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