

# Irreducible Representations of the Symmetric Group

classified over the field of complex numbers  $\mathbb{C}$

Xingyu Zhong

Beijing Institute of Technology

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# Table of Contents

- 1 Introduction & Constructions
- 2 Irreducibility of Specht Modules
- 3 Classification of the Irreducible Representations

# Table of Contents

## 1 Introduction & Constructions

## 2 Irreducibility of Specht Modules

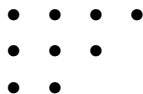
## 3 Classification of the Irreducible Representations

We show a concise path to the classification of the irreducible representations of the symmetric group  $\mathcal{S}_n$  over the field of complex numbers  $\mathbb{C}$ .

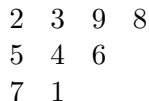
- mainly adapted from [FH04, section 4.2], minor references to [Sag01, chapter 2]
- Focus on the irreducibility of Specht modules
- Modest knowledge of group theory is enough to understand the combinatorial part, though familiarity of the basics of complex representations of groups [Ser77, chapter 1–2] and the group-algebra module perspective of representations is needed for other parts.

# Young Tableaux

- Idea: Understand the group by its action on combinatorial objects



(a) The Young diagram of shape  $\lambda = (4, 3, 2)$



(b) A Young tableau  $T = (\lambda \vdash n, \sigma \in \mathcal{S}_n)$ , where  $\lambda = (4, 3, 2)$ ,  $\sigma = 239854671$

Figure: Young diagrams and Young tableaux

# $\mathcal{S}_n$ Acting on Young Tableaux

- $g \in \mathcal{S}_n$  acts on the Young tableau  $T$  of shape  $\lambda$  by replacing the number  $i$  with  $g(i)$ .
  - i.e.  $gT := (\lambda, g\sigma)$ .
  - The action is transitive.
- $P_T \leq \mathcal{S}_n$  consists of permutations that preserve the content of each row of  $T$ , similarly  $Q_T \leq \mathcal{S}_n$  preserves the columns.
  - $P_T$  and  $Q_T$  intersect trivially.
  - Their union generates the entire  $\mathcal{S}_n$ .
- Relationship between  $P_T, Q_T$  induced by different Young tableaux of the same shape:
  - The process  $gT \rightarrow gpT$  can be understood as performing an “effect” equivalent to a row permutation  $p \in P_T$  on  $T$
  - $P_{gT} = gP_Tg^{-1}$
  - $Q_{gT} = gQ_Tg^{-1}$

# Young Symmetrizers

Let  $\mathbb{C}[\mathcal{S}_n]$  be the complex group algebra of  $\mathcal{S}_n$ , i.e. its complex regular representation. Define:

- Row symmetrizer  $a_T := \sum_{g \in P_T} g \in \mathbb{C}[\mathcal{S}_n]$
- Column symmetrizer  $b_T := \sum_{g \in Q_T} \text{sgn}(g)g \in \mathbb{C}[\mathcal{S}_n]$
- Young symmetrizer  $c_T := a_T b_T$

It is evident that for any  $p \in P_T$ ,  $q \in Q_T$ , we have:

- $pa_T = a_T p = a_T$
- $qb_T = b_T q = \text{sgn}(q)b_T$
- $pc_T q = \text{sgn}(q)c_T$ 
  - Though it's nontrivial that  $c_T$  is (almost) the only group algebra element satisfying this condition.

Define the  $\mathbb{C}[\mathcal{S}_n]$ -module

$$M_T := \mathbb{C}[\mathcal{S}_n]a_T$$

- Acting on Young tableaux,  $a_T T$  forgets labelings inside the rows of  $T$ .
- It is the induced representation  $1 \uparrow_{P_T}^{\mathcal{S}_n}$  obtained by lifting the trivial representation of  $P_T$  to  $\mathcal{S}_n$ .



The Specht module induced by a Young tableau  $T$  is the  $\mathbb{C}[\mathcal{S}_n]$ -module

$$V_T := \mathbb{C}[\mathcal{S}_n]c_T$$

We shall show the following to classify the complex irreducible representations of  $\mathcal{S}_n$ :

- Each Specht module is an irreducible representation of  $\mathcal{S}_n$ .
- Two Specht modules are isomorphic if and only if the Young tableaux inducing them have the same shape.
- Every irreducible representation of  $\mathcal{S}_n$  is isomorphic to some Specht module.

# Table of Contents

1 Introduction & Constructions

2 Irreducibility of Specht Modules

3 Classification of the Irreducible Representations

# S.R.D.C.<sup>1</sup> for Young Tableaux of the Same Shape

Below helps us to understand how  $P_T Q_T$  acts on Young tableaux combinatorially.

## Proposition

*Let  $T, S$  be two Young tableaux of shape  $\lambda$ . If any pair of numbers in the same row of  $T$  falls into different columns in  $S$ , then there exist  $p \in P_T, q' \in Q_S$  such that  $pT = q'S$ .*

$$\begin{array}{ccccccc}
 & i & j & k & l & & \\
 T & * & * & * & & \xrightarrow{p_1 \in P_T} & l \quad k \quad i \quad j \\
 & * & * & & & & * \quad * \quad * \\
 & & & & & & * \quad * & \xrightarrow{p_2 \in P_T} & \dots
 \end{array}$$
  

$$\begin{array}{ccccccc}
 & l & * & * & j & & \\
 S & * & * & i & & \xrightarrow{q'_1 \in Q_S} & l \quad k \quad i \quad j \\
 & * & k & & & & * \quad * \quad * \\
 & & & & & & * \quad * & \xrightarrow{q'_2 \in Q_S} & \dots
 \end{array}$$

<sup>1</sup>Same Row, Different Column

# Consequences

- Say  $S = gT$ , whence  $q' = gqg^{-1}$  with  $q \in Q_T$ . Then  $pT = q'S = gqT$ , so  $g = pq^{-1} \in P_TQ_T$ .
- The converse is also true: for any  $p \in P_T$ ,  $q \in Q_T$ ,  $pT$  and  $qT$  are S.R.D.C.
- $T, gT$  being S.R.D.C. is an iff condition for  $g \in P_TQ_T$ .

# Uniqueness of the Young Symmetrizer

## Proposition

*Let  $c \in \mathbb{C}[\mathcal{S}_n]$  satisfy  $pcq = \text{sgn}(q)c$  for any  $p \in P_T$ ,  $q \in Q_T$ . Then  $c \in \mathbb{C}c_T$ , i.e.  $c$  is unique up to scalar multiplication.*

Say  $c = \sum_{g \in \mathcal{S}_n} n_g g$ ,

- $n_{pgq} = \text{sgn}(q)n_g$
- $n_g$  for  $g \in P_T Q_T$  is fixed up to a scalar multiplication.
- $n_g$  for  $g \notin P_T Q_T$  is zero. Our combinatorial interpretation guarantees that there must be a pair of numbers  $(i, j)$  fall into the same row in  $T$  and the same column in  $gT$ . The alternating property of  $c$  results in  $n_g = -n_g$  and thus  $n_g = 0$ .

# Idempotency of $c_T$

## Corollary

$c_T x c_T \in \mathbb{C} c_T$  holds for any  $x \in \mathbb{C}[\mathcal{S}_n]$ .

- In particular,  $c_T^2 = \alpha c_T$  for some  $\alpha \in \mathbb{C}$ .
- It can be shown that  $\alpha = n! / \dim V_T$ .

# Irreducibility of Specht Modules

## Proposition

*The Specht module  $V_T := \mathbb{C}[\mathcal{S}_n]c_T$  is an irreducible representation of  $\mathcal{S}_n$ .*

Assume  $V_T$  has a subrepresentation  $W \subseteq V_T$ . Then the subspace:

$$c_T W \subset c_T V_T = c_T \mathbb{C}[\mathcal{S}_n] c_T \subset \mathbb{C} c_T$$

is a one-dimensional subspace. Thus, we consider the following two cases:

- $c_T W = \mathbb{C} c_T$ . In this case:

$$V_T = \mathbb{C}[\mathcal{S}_n] c_T = \mathbb{C}[\mathcal{S}_n] (\mathbb{C} c_T) = \mathbb{C}[\mathcal{S}_n] W = W$$

- $c_T W = 0$ . In this case, for any  $w = xc_T \in W$ , we have  $w^2 = xc_T w = 0$ , i.e.,  $W^2 = 0$ . We know that there are no non-zero nilpotent left ideals in the finite-dimensional semisimple group algebra  $\mathbb{C}[\mathcal{S}_n]$ , so  $w = 0$ ,  $W = 0$ .

# Table of Contents

1 Introduction & Constructions

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# A Brief Outline of Classification

Are Specht modules the only irreducible representations of  $\mathcal{S}_n$ ?

$$\begin{aligned}\# \text{ irreducible representations} &= \# \text{ conjugacy classes of } \mathcal{S}_n \\ &= \# \text{ integer partitions of } n\end{aligned}$$

It suffices to show that:

- Two Specht modules are isomorphic if and only if the Young tableaux inducing them have the same shape.
  - $V_T \cong V_{gT}$  by direct construction  $x \mapsto xg^{-1}$
  - For the converse, refine the S.R.D.C. condition to adapt Young tableaux of different shapes

This completes the classification of the irreducible representations of  $\mathcal{S}_n$ .

# Wrap up

Classification of complex irreducible representations of  $\mathcal{S}_n$ :

- all Specht modules  $V_T$ , where  $T$  is a Young tableau of shape  $\lambda \vdash n$ .
- Two Specht modules are isomorphic if and only if the Young tableaux inducing them have the same shape.

Remark (What we've benefited from working over  $\mathbb{C}$ )

- We've used the semisimplicity of the group algebra  $\mathbb{C}[\mathcal{S}_n]$  to show the irreducibility of Specht modules.
- We need the result over  $\mathbb{C}$  that the number of complex irreducible representations of a group equals the number of conjugacy classes of the group.
- Other arguments do not rely on any special property of  $\mathbb{C}$ .

# Problem Remains

- Dimension of Specht modules  $V_T$ 
  - the hook-length formula for the number of standard Young tableaux of shape  $\lambda$
- Explicit construction of a basis of Specht modules
- A more natural combinatorial interpretation of Specht modules
- Classification over other fields, e.g.  $\mathbb{F}_p$

Thank you for your attention!

# References I

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