

# SUNQUART<sub>EX</sub> Example - Lean Include

## Abstract

This file demonstrates the `lean-include` feature.

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```
import Mathlib
```

Logic (Part II)

## 1 And and Or

In Lean's dependent type theory,  $\wedge$  and  $\vee$  serve as the *direct product* and the *direct sum* in the universe of **Prop**.

Eagle-eyed readers may notice that  $\wedge$  and  $\vee$  act similarly to Cartesian product and disjoint union in set theory.

They are also constructed as inductive types.

```
section
```

```
variable (p q r : Prop)
```

## 2 And ( $\wedge$ )

The only constructor of **And** is **And.intro**, which takes a proof of **p** and a proof of **q** to produce a proof of **p**  $\wedge$  **q**.

Regard this as the *universal property of the direct product* if you like.

**And.intro** **hp** **hq** can be abbreviated as **<hp, hq>**, called the *anonymous constructor*.

**constructor** tactic applies **And.intro** to split the goal **p**  $\wedge$  **q** into subgoals **p** and **q**. You may also use the anonymous constructor notation **<hp, hq>** to mean **And.intro** **hp** **hq**.

**split\_and** tactic is like **constructor** but works for nested **And**s.

```
#print And
```

introducing And

```
#check And.intro
```

These examples, as introduction rules, are self-evidently true.

```
example (hp : p) (hq : q) : p ∧ q := And.intro hp hq
example (hp : p) (hq : q) : p ∧ q := ⟨hp, hq⟩
example (hp : p) (hq : q) : p ∧ q := by
  constructor
  · exact hp
  · exact hq
```

[EXR]  $\rightarrow$ - $\vee$  distribution. Universal property of the direct product.

```
example (hrp : r → p) (hrq : r → q) : r → p ∧ q := by
  intro hr
  exact ⟨hrp hr, hrq hr⟩
```

And.left and And.right are among the elimination rules of And, which extract the proofs of p and q.

rcases hpq with ⟨hp, hq⟩ is a tactic that breaks down the hypothesis hpq : p ∧ q into hp : p and hq : q. Equivalently you can use let ⟨hp, hq⟩ := hpq. eliminating And

```
#check And.left
#check And.right
example (hpq : p ∧ q) : p := hpq.left
example (hpq : p ∧ q) : p := by
  rcases hpq with ⟨hp, _⟩
  exact hp
example : p ∧ q → p := by
  intro ⟨hp, _⟩ -- implicit break-down in `intro`
  exact hp
```

[EXR] And is symmetric

```
example : p ∧ q → q ∧ p := by
  intro hpq
  exact ⟨hpq.right, hpq.left⟩
#check And.comm -- above has a name
```

[EXR]  $\rightarrow$ - $\vee$  distribution, in another direction.

```
example (hrpq : r → p ∧ q) : (r → p) ∧ (r → q) := by
  constructor
  · intro hr
    exact (hrpq hr).left
  · intro hr
    exact (hrpq hr).right
```

nested and

```

example (hpqr : p ∧ q ∧ r) : r := hpqr.right.right
example (hpqr : p ∧ q ∧ r) : r := by
  rcases hpqr with ⟨_, ⟨_, hr⟩⟩ -- anonymous constructor can be nested
  exact hr

example (hp : p) (hq : q) (hr : r) : p ∧ q ∧ r := by
  exact ⟨hp, ⟨hq, hr⟩⟩
example (hp : p) (hq : q) (hr : r) : p ∧ q ∧ r := by
  split_and
  · exact hp
  · exact hq
  · exact hr

```

The actual universal elimination rule of **And** is the so-called *decurrification*: From  $(p \rightarrow q \rightarrow r)$  we may deduce  $(p \wedge q \rightarrow r)$ . This is actually a logical equivalence.

Intuitively, requiring both  $p$  and  $q$  to deduce  $r$  is nothing but requiring  $p$  to deduce that  $q$  is sufficient to deduce  $r$ .

[IGNORE] Decurrification is also self-evidently true in Lean's dependent type theory.

Currification is heavily used in functional programming for its convenience, Lean is no exception.

You are no stranger to decurrification even if you are not a functional programmer: The *universal property of the tensor product of modules* says exactly the same:

$$\text{Hom}(M \otimes N, P) \cong \text{Hom}(M, \text{Hom}(N, P))$$

[EXR] currification

```

example (h : p ∧ q → r) : (p → q → r) := by
  intro hp hq
  exact h ⟨hp, hq⟩

```

[EXR] decurrification

```

example (h : p → q → r) : (p ∧ q → r) := by
  intro hpq
  exact h hpq.left hpq.right

example (h : p → q → r) : (p ∧ q → r) := by
  intro ⟨hp, hq⟩ -- 'intro' is smart enough to destructure 'And'
  exact h hp hq

example (h : p → q → r) : (p ∧ q → r) := by
  intro ⟨hp, hq⟩
  apply h -- 'apply' is smart enough to auto-decurrify and generate two subgoals
  · exact hp
  · exact hq

```

[IGNORE] decurrification actually originates from **And.rec**, which is self-evident

```

#check And.rec
theorem decurrify (h : p → q → r) : (p ∧ q → r) := And.rec h

```

[EXR] **And.left** is actually a consequence of decurrification

```
example : p ∧ q → p := by
  apply decurrify
  intro hp _
  exact hp
```

## 2.1 Iff ( $\leftrightarrow$ ), first visit

It's high time to introduce **Iff** here for the first time.

**Iff** ( $\leftrightarrow$ ) contains two side of implications: **Iff.mp** and **Iff.mpr**.

Though it is defined as a distinct inductive type, **Iff** is very similar to **And** in that you may, somehow, even use it like a  $(p \rightarrow q) \wedge (q \rightarrow p)$ . The only major difference is the name of the two components.

```
#check Iff.intro
#check Iff.mp
#check Iff.mpr

example : (p ↔ q) ↔ (p → q) ∧ (q → p) := by
  constructor
  · intro h
    exact ⟨h.mp, h.mpr⟩
  · intro ⟨hpq, hqp⟩
    exact ⟨hpq, hqp⟩
```

## 2.2 Or ( $\vee$ )

**Or** has two constructors **Or.inl** and **Or.inr**. Either a proof of **p** or a proof of **q** produces a proof of  $p \vee q$ .

[TODO]

```
#print Or
#check Or.inl
#check Or.inr
#check Or.elim
#check Or.rec
```

introducing **Or**

```
example (hp : p) : p ∨ q := Or.inl hp
example (hq : q) : p ∨ q := by
  right
  exact hq
```

elimination rule of **Or**, universal property of the direct sum

```
example (hpr : p → r) (hqr : q → r) : (p ∨ q → r) := fun hpq ↦ (Or.elim hpq hpr hqr)
example (hpr : p → r) (hqr : q → r) : (p ∨ q → r) := (Or.elim · hpr hqr) -- note the use of `·`
example (hpr : p → r) (hqr : q → r) (hpq : p ∨ q) : r := by
  apply Or.elim hpq
  · exact hpr
  · exact hqr
```

```

example (hpr : p → r) (hqr : q → r) : (p ∨ q → r) := fun
  | Or.inl hp ⇒ hpr hp
  | Or.inr hq ⇒ hqr hq
example (hpr : p → r) (hqr : q → r) (hpq : p ∨ q) : r :=
  match hpq with
  | Or.inl hp ⇒ hpr hp
  | Or.inr hq ⇒ hqr hq
example (hpr : p → r) (hqr : q → r) (hpq : p ∨ q) : r := by
  match hpq with
  | Or.inl hp ⇒ exact hpr hp
  | Or.inr hq ⇒ exact hqr hq
example (hpr : p → r) (hqr : q → r) (hpq : p ∨ q) : r := by
  cases hpq with
  | inl hp ⇒ exact hpr hp
  | inr hq ⇒ exact hqr hq
example (hpr : p → r) (hqr : q → r) (hpq : p ∨ q) : r := by
  rcases hpq with (hp | hq) -- 'rcases' can also destructure 'Or'
  · exact hpr hp
  · exact hqr hq
example (hpr : p → r) (hqr : q → r) : p ∨ q → r := by
  rintro (hp | hq) -- 'rintro' is a combination of 'intro' and 'rcases'
  · exact hpr hp
  · exact hqr hq

```

## 2.3 Comprehensive exercises for And and Or

[EXR] distributive laws

```

example : p ∧ (q ∨ r) ↔ (p ∧ q) ∨ (p ∧ r) := by sorry
example : p ∨ (q ∧ r) ↔ (p ∨ q) ∧ (p ∨ r) := by sorry

end

```

Forall and Exists

## 3 Forall (∀)

As you may have already noticed,  $\forall$  is just an alternative way of writing  $\rightarrow$ . Say  $p$  is a predicate on a type  $X$ , i.e. of type  $X \rightarrow \text{Prop}$ , then  $\forall x : X, p\ x$  is exactly the same as  $(x : X) \rightarrow p\ x$ .

Though  $\rightarrow$  is primitive in Lean's dependent type theory, we may still (perhaps awkwardly) state the introduction and elimination rules of  $\forall$ :

- Introduction: `fun (x : X) => (h x : p x)` produces a proof of  $\forall x : X, p\ x$ .
- Elimination: Given a proof  $h$  of  $\forall x : X, p\ x$ , we can obtain a proof of  $p\ a$  for any specific  $a : X$ . It is exactly `h a`.

section

```

variable {X : Type} (p q : X → Prop) (r s : Prop) (a b : X)

```

```
#check ∀ x : X, p x
#check ∀ x, p x -- Lean is smart enough to infer the type of 'x'
```

[IGNORE] Writing  $\forall$  emphasizes that the arrow  $\rightarrow$  is of dependent type, and the domain  $X$  is a type, not a proposition. But they are just purely psychological, as the following examples show.

```
example : (hrs : r → s) → (∀ _ : r, s) := by
  intro hrs
  exact hrs
```

## 4 Exists ( $\exists$ )

$\exists$  is a bit more complicated.

Slogan:  $\forall$  is a dependent  $\rightarrow$ ,  $\exists$  is a dependent  $\times$  (or  $\wedge$  in **Prop** universe)

```
#check ∃ x : X, p x
#check ∃ x, p x -- Lean is smart enough to infer the type of 'x'
```

$\exists x : X, p x$  means that we have the following data:

- an element  $a : X$ ;
- a proof  $h : p a$ .

So a pair  $(a, h)$  would suffice to construct a proof of  $\exists x : X, p x$ .

This is the defining introduction rule of **Exists** as an inductive type.

```
#check Exists.intro
```

As like **And**, you may use the anonymous constructor notation  $\langle a, h \rangle$  to mean **Exists.intro**  $a h$ .

In tactic mode, use **a** make use of **Exists.intro a** to reduce the goal  $\exists x : X, p x$  to  $p a$ .

```
example (a : X) (h : p a) : ∃ x, p x := Exists.intro a h
example (a : X) (h : p a) : ∃ x, p x := ⟨a, h⟩
example (a : X) (h : p a) : ∃ x, p x := by use a

-- [EXR]
example (x y z : ℕ) (hxy : x < y) (hyz : y < z) : ∃ w, x < w ∧ w < z :=
  ⟨y, ⟨hxy, hyz⟩⟩
```

Note that in the defining pair  $(a, h)$ ,  $h$  is a proof of  $p a$ , whose type depends on  $a$ . Thus psychologically, you may view  $\exists x : X, p x$  as a dependent pair type  $(x : X) \times (p x)$ .

Have writing **Exists** as a dependent pair type reminded you of the currfication process?

Elimination rule: To construct the implication  $(\exists x : X, p x) \rightarrow q$ , it suffices to have a proof of  $(\forall x : X, p x \rightarrow q)$ , i.e.  $(x : X) \rightarrow p x \rightarrow q$ .

In tactic mode, **rcases h with ⟨a, ha⟩** make use of this elimination rule to break down a hypothesis  $h : \exists x : X, p x$  into a witness  $a : X$  and a proof  $ha : p a$ .

```

#check Exists.elim

example : (∀ x, p x → r) → ((∃ x, p x) → r) := by
  intro hf he
  exact Exists.elim he hf

example : (∀ x, p x → r) → ((∃ x, p x) → r) := by
  intro hf he
  rcases he with ⟨a, hpa⟩
  exact hf a hpa

example : (∀ x, p x → r) → ((∃ x, p x) → r) := by
  intro h ⟨a, hpa⟩ -- you may also `rcases` explicitly
  exact h a hpa

-- [EXR] reverse direction is also true
example : ((∃ x, p x) → r) → (∀ x, p x → r) := by
  intro h a hpa
  apply h
  use a

-- [EXR]
example : (∃ x, r ∧ p x) → r ∧ (∃ x, r ∧ p x) := by
  intro ⟨a, ⟨hr, hpa⟩⟩
  exact ⟨hr, ⟨a, ⟨hr, hpa⟩⟩⟩

-- [EXR]
example : (∃ x, p x ∨ q x) ↔ (∃ x, p x) ∨ (∃ x, q x) := by
  constructor
  · rintro ⟨a, (hpa | hqa)⟩
    · left; use a
    · right; use a
  · rintro (⟨a, hpa⟩ | ⟨a, hqa⟩)
    · use a; left; exact hpa
    · use a; right; exact hqa

end

```

#### 4.1 [IGNORE] A cosmological remark

The pair  $(a, h)$  actually do not have type  $(x : X) \times (p\ x)$ . The latter notation is actually for the *dependent pair type* (or **Sigma** type), which lives in **Type\*** universe.

But **Exists** should live in **Prop**, and in **Prop** universe we admit *proof-irrelevance*, i.e. we do not save data. So **Exists** forget the exact witness **a** once it is proved.

This “forgetfulness” is revealed by the fact that there is no elimination rule **Exists.fst** to extract the witness **a** from a proof of  $\exists x : X, p\ x$ , as long as **X** lives in the **Type\*** universe. (Note that **Exists.elim** can only produce propositions in **Prop**)